The Skyrme model: nucleons, dibaryons, and nuclei

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This review is devoted to a currently popular field-theoretic model of extended strongly interacting objects. The nonlinear nature of the Skyrme model leads to solutions of the Euler–Lagrange equations which already at the classical level have properties similar to those of baryons. Some details are given of the derivation of the effective quantum Hamiltonians using collective variables, which allow one to obtain numbers from the calculations. The Skyrme model is presently used to analyze a very broad class of problems in strong-interaction physics. For example, it has been used successfully to analyze the nucleon excitation spectrum and the static properties of baryons, nucleon–nucleon forces, meson–nucleon scattering, dibaryon and multibaryon states, meson exchange currents, photoproduction processes, and so on. An extremely attractive feature of the model is the unification of the techniques for studying both isolated baryons and the systems which they form, for example, nuclei.

INTRODUCTION

The Skyrme model is an example of a field-theoretic model of extended objects. The dynamical variables of the model are bosonic fields. The object of investigation is the baryon structure and interaction at low energies.

The large amount of recent interest in the Skyrme model[1] in the theory of strongly interacting particles is a consequence of the hope that meson effective Lagrangians can bridge the gulf between quantum chromodynamics (QCD) and the known theory of nuclear structure.

The dynamics of the elementary quark and gluon fields becomes difficult to analyze in low-energy quantum chromodynamics, owing to the large coupling constant. Although everyone believes that the physics of any known nucleus is also described by the QCD Lagrangian, no one has been able to obtain the basic properties of nuclei in terms of the quark and gluon fields.

The search for a small parameter in QCD led 't Hooft[2] to the idea of considering QCD with a large (tending to infinity) number of colors N_c. Witten[3] showed that if the limit $N_c \to \infty$ exists, then QCD will be a theory of effective local meson fields with local interactions of order N_c^{-1}. Moreover, in this limit the baryon masses prove to be of order N_c, while the number of colors completely drops out of the equations determining the size and structure of the baryons.

It is well known that nonlinear theories can have solutions corresponding to localized objects of finite size—solitons[4]—with the analogous dependences on the coupling constant. Therefore, the Witten result leads to the picture of baryons as the solitons of an effective meson theory. This picture does not require any further reference to the quark origin of the effective Lagrangian. A theory of just this type was proposed in the study by Skyrme in 1961–1962 (Ref. 1).

It should not be assumed that the effective theories (and the Skyrme model) verify QCD in the low-energy region. Rather, they only model that region of QCD where perturbative methods do not work.

At present, the problem of obtaining the effective meson Lagrangian directly from the QCD Lagrangian is being worked out. A large number of studies have been devoted to this problem (see, for example, Refs. 5 and 6 and references therein). The derivation of the effective meson Lagrangian inherently involves the study of the mechanism for spontaneous chiral symmetry breaking in QCD,[7] and this, in turn, involves the study of the vacuum structure of QCD.

On the other hand, we have the problem of studying the structure of baryons, baryon systems, and their interactions, which can be obtained theoretically by expanding the effective meson Lagrangian.

It has been found that some of the results are completely independent of the details of the effective Lagrangian, and only reflect its symmetries and the fact that baryons are the solitons of the nonlinear Euler–Lagrange equations derived from this Lagrangian. Symmetry considerations provide quite rigorous constraints on the possible form of the effective meson Lagrangian.

Restricting ourselves to the simplest model of this type—the Skyrme model, we probably cannot hope for good quantitative agreement with the experimental data, but we can obtain a qualitatively good description of the fundamental regularities characterizing a system of strongly interacting particles which would support the idea that baryons are the solitons of an effective meson Lagrangian. It is in this sense that the Skyrme model is an excellent theoretical laboratory for studying all the special features of nonlinear field theory.

CHIRAL SYMMETRY AND SOLITONS

The basic ideas of chiral dynamics are quite simple. In the infrared region of QCD, where the hadron spectrum is formed, one has collective degrees of freedom, which are observable fields, for describing the phenomena in a restricted energy range. Although the use of phenomenological chiral Lagrangians is usually restricted to the semiclassical approximation, the phenomenological success already at the tree level has caused chiral dynamics to be taken more seriously.[8] A systematic quantization scheme[9] and the superpropagator technique for regularizing the quantum chiral theory[10] have been developed.

Nonlinear chiral theories naturally lead to soliton sectors. Already at the classical level, chiral solitons are very similar to hadrons[11]:

1. They carry a definite, rigorously conserved topological charge, owing to vacuum degeneracy. This localized charge is a good candidate for the baryon number.

2. They are extended, strongly interacting objects.

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3. They are very massive compared with the masses of the fields involved in the Lagrangian.

These features plus the rich spectrum of states generated make chiral dynamics a very attractive theory for low-energy phenomena in strong-interaction physics.

Let us discuss some of the symmetries inherent in QCD which will be used later.

A FEW WORDS ABOUT THE SYMMETRIES OF QCD

The quark part of the QCD Lagrangian density (and, therefore, the entire $\mathcal{L}_{QCD}$)

$$\mathcal{L}^q_{QCD} = -\bar{q}_L\gamma_\mu D_\mu q_L - \bar{q}_R\gamma_\mu D_\mu q_R,$$

where D_μ is the covariant derivative, is invariant under separate unitary transformations of the flavor indices of left- and right-handed quarks q_L and q_R. We have the $U(N_f)\times U(N_f) \equiv U(N_f)_L \times U(N_f)_R = U(2)_L \times U(2)_R$ symmetry group, under which the quarks transform according to the rule

$$q_L \Rightarrow u_L q_L, \quad q_R \Rightarrow u_R q_R.$$

The vector $U(1)_V$ subgroup of this group, under which q_L and q_R are multiplied by the same phase, $q_L \Rightarrow \{\exp(i\theta)\}q_L$ and $q_R \Rightarrow \{\exp(i\theta)\}q_R$, is an exact symmetry and the charge corresponding to it is the baryon number. The axial $U(1)_A$ subgroup, under which the left- and right-handed quarks transform with opposite phases, $q_L \Rightarrow e^{i\theta}q_L$ and $q_R \Rightarrow e^{-i\theta}q_R$, is not a symmetry group of the quantized Lagrangian (the Adler–Bell–Jackiw anomaly). Therefore, there remains the chiral symmetry

$$SU(N_f)_L \times SU(N_f)_R = SU(N_f)_V \times SU(N_f)_A,$$

where the right- and left-handed transformations are restricted to matrices with unit determinant. Finally, the absence of parity doublets in the physical spectrum of particles with explicit isospin symmetry indicates that $SU(2)_L \times SU(2)_R$ is broken[11] down to $SU(2)_V$ by the spontaneous chiral symmetry-breaking mechanism with the appearance of massless pseudoscalar Goldstone excitations with the quantum numbers of the pions.

When speaking of the effective boson Lagrangian, one has in mind the Lagrangian describing the dynamics of the Goldstone bosons, while keeping these symmetries intact. It is simplest to construct this effective Lagrangian by taking as the fundamental variables the elements U of some SU(2) group, the local coordinates of which will be identified with the boson fields. Then the left- and right-handed transformations of the quarks can be associated with the left and right multiplication of elements of this group. The chiral $SU(2)_L \times SU(2)_R$ group of transformations of the quark fields will correspond to the direct product of left- and right-handed representations of the elements of SU(2), so that the transformation $U \Rightarrow AUB^+$ with an arbitrary constant A and $B \in SU(2)$ corresponds to an arbitrary element of the quark global group.

The chirally invariant action density is usually constructed using left-invariant Cartan forms $L_\mu = U^+\partial_\mu U$. Under right-handed transformations R they transform as $L_\mu \Rightarrow RL_\mu R^+$, but remain invariant under $SU(2)_L$. Analogously, the right-invariant forms $R_\mu = \partial_\mu U \cdot U^+$ can be used, which are, in turn, invariant under right-handed transformations, but transform under left-handed ones as $R_\mu \Rightarrow LR_\mu L^+$. An arbitrary chirally invariant polynomial $P(L_\mu)$ is equal to $P(R_\mu)$ and conversely, since $P(R_\mu) = P(AL_\mu A^+) = P(L_\mu)$.

THE SKYRME LAGRANGIAN

The SU(2) Skyrme model is defined by the Lagrangian density[12]

$$\mathcal{L} = \frac{F_\pi^2}{16}\,\mathrm{Tr}\,L_\mu L_\mu + \frac{1}{32e^2}\,\mathrm{Tr}\,[L_\mu, L_\nu]^2. \tag{1}$$

Here the currents $L_\mu = U^+\partial_\mu U$ are expressed in terms of SU(2) matrices $U(x) = \exp((2/F_\pi)i\tau\cdot\pi)$ defined by the isotriplet of pion fields π and the Pauli matrices τ. The Lagrangian is written in explicitly Lorentz-covariant form. It is easily verified that this Lagrangian defines a chirally invariant theory. It is invariant under the global $SU(2)_L SU(2)_R$ chiral group of transformations $U(x) \Rightarrow AU(x)B^+$ for any constant unitary 2×2 matrices A and B. This invariance corresponds to conservation of the axial and vector currents in the model. The single dimensional constant F_π in (1) can be determined in the bosonic sector from the pion weak-decay amplitude. (In this case F_π is equal to 186.4 MeV.) The first term of the Lagrangian generalizes the kinetic energy of the pion field to a chirally invariant form (the nonlinear sigma model). The appearance of the term with a fourth-order derivative is extremely important for the entire model. Without this term there would be none of the solitons, the quantized version of which we hope to use for describing heavy extended objects such as nucleons, multibaryons, and nuclei.

From dimensional considerations we see that the contribution of the first term in the Lagrangian to the classical mass is proportional to some linear dimension, while that of the second term is inversely proportional to this dimension. Therefore, the presence of the second (the so-called Skyrme) term in the Lagrangian can ensure the existence of a nontrivial energy minimum for solitons of finite size. There is a somewhat more rigorous proof of the Derrick theorem, which states that the minimum space dimension for which we will have a nontrivial solution is equal to three. This is just what we need.

The trivial solution defining the vacuum state is obviously $U_0(x) = \mathbb{1}$. Since, in general, $AU_0(x)B^+ \neq \mathbb{1}$, we are dealing with spontaneous breakdown of the chiral symmetry (the ground state is less symmetric than the Lagrangian).

It was pointed out in Ref. 13 that the fourth-order term can be viewed as the result of a specific choice of counterterms at the one-loop level in the Lagrangian regularized by the superpropagator method[14]:

$$\mathcal{L}^{\mathrm{Ren}} = \frac{1}{2}(1+Z_1\lambda^2)\,L_\mu L_\mu + \lambda^4 Z_2\,(L_\mu L_\mu)^2 + \lambda^3 Z_3\,(\partial_\mu L_\mu \partial_\nu L_\nu) + \lambda^4 Z_4\,([L_\mu, L_\nu]^2) + \frac{1}{2}\Lambda^{-4}\partial_\mu L_\mu \,\Box\, \partial_\nu L_\nu. \tag{2}$$

Here Z_1 diverges quadratically, and the other quantities diverge logarithmically. The last term is a regulator which removes the divergences in Z_2, Z_3, and Z_4, and Z_1 and Z_3 are wave-function renormalizations. The quantities Z_2 and Z_4 remain as free parameters. In the limit of infinite cutoff parameter $\Lambda \to \infty$, the term involving Z_3 vanishes. Then the choice $Z_4 - Z_2 = 1/32e^2$ gives the Skyrme model. It is well known that (2) satisfies the current algebra and partial conservation of the axial-vector current, and that it gives reasonable threshold behavior and unitarity up to energies somewhat below the cutoff.

THE TOPOLOGICAL CHARGE AND MASS OF THE CLASSICAL SOLITON

Now, among all the possible field configurations we shall restrict ourselves to only those satisfying the condition $U(x) \Rightarrow 1$ for $|\mathbf{x}| \to \infty$ at any arbitrary time t (otherwise the energy would be infinite). Therefore, $U(x)$ for any fixed t maps the physical space R^3 into the SU(2) group, and all points at spatial infinity from R^3 are mapped to the unit element of the SU(2) group. The space R^3 with identified points at infinity is compactified to the S^3 sphere. We are therefore dealing with a mapping $U(x)$ from S^3 to the S^3 sphere of the parameter space of the SU(2) group. Such mappings split up into equivalence classes, characterized by a certain integer, the degree of the mapping. Any two mappings belonging to the same class can be continuously deformed into each other while those belonging to different classes cannot. Examples of such deformations (homotopies) are global SU(2) rotations and time evolution. Therefore, the degree of the mapping (topological charge), being a homotopic invariant, is conserved independently of the dynamics of the system. The mapping degree is interpreted geometrically as the number of times the S^3 sphere of the SU(2)-matrix parameters is covered in a given mapping.

It is possible to construct[1,11,15] the trivially conserved topological current

$$J_\mu^B = -\frac{1}{24\pi^2}\varepsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\,(L_\nu L_\rho L_\sigma), \tag{3}$$

satisfying the continuity condition

$$\partial_\mu J_\mu^B = 0 \tag{4}$$

with charge corresponding to the mapping degree:

$$B = \int d^3x\, J_0^B(x). \tag{5}$$

The topological charge B is additive. In fact, an arbitrary configuration $U(x)$ can be factorized as

$$U(x) = U_1(x)\cdot U_2(x)$$

in accordance with the group multiplication rule. It is easily shown that

$$B(U(x)) = B(U_1(x)) + B(U_2(x)).$$

Moreover, if $B[U(x)] = 1$, then $B[U^+(x)] = -1$. In addition, $J_B^\mu \sim v^\mu$, where v^μ is the 4-velocity of the center of the soliton, which for $B = 1$ is determined by the condition $U(x_0) = -1$.

Skyrme concluded that this set of properties allows the topological charge to be identified with the baryon number.

In order to verify that this interpretation of the SU(2) model is correct, we can show that the quark Dirac sea always includes a new orbital when the topological charge of the chiral field is increased by one unit. Although the detailed behavior of the solutions of the Euler–Lagrange equations is not important here, we shall discuss a few of the properties of spherically symmetric stationary field configurations of the form $U(\mathbf{r}) = \exp[i\boldsymbol{\tau}\cdot\mathbf{n}F(r)]$, where $\mathbf{n} = \boldsymbol{\pi}/|\boldsymbol{\pi}| = \mathbf{r}/r$ and $F(r) = |\boldsymbol{\pi}(\mathbf{r})|$. This configuration is called the "Skyrme–Witten ansatz." It is precisely these configurations which ensure that the energy functional of the Skyrme model has an absolute minimum in the sector with unit topological charge.[16] For the topological charge density we easily find

$$\rho^B(r) = -\frac{1}{2\pi r^2}\sin^2 F(r)\,\frac{dF(r)}{dr}. \tag{6}$$

The profile function $F(r)$ satisfies the differential equation

$$\left(\frac{x^2}{4} + 2\sin^2 F\right)F'' + \frac{xF'}{2} + \sin 2F$$
$$(F')^2 - \frac{\sin 2F}{4} - \frac{\sin^2 F\sin 2F}{x^2} = 0, \tag{7}$$

where $x = eF_\pi r$ is a dimensionless variable.

The solution of this equation satisfying the conditions $F(0) = \pi n$, where n is an integer and $F(\infty) = 0$, ensures the finiteness and minimum of the energy functional determining the soliton mass:

$$M = \pi\frac{F_\pi}{e}\left[\frac{1}{2}\int\left[(F')^2 + \frac{2}{x^2}\sin^2 F\right]x^2\,dx \right.$$
$$\left. + 4\int \sin^2 F\left[(F')^2 + \frac{1}{2}\frac{\sin^2 F}{x^2}\right]dx\right] \tag{8}$$

in the sector with topological charge

$$B = \int d^3r\rho^B(r) = n.$$

Returning to the Lagrangian, let us take out the part of it which is independent of the time derivatives:

$$-M = \frac{F_\pi}{16e}\left\{\int d^3x\,\mathrm{Tr}\,L_i(x)\,L_i(x) + \frac{1}{2}\int d^3x\,\mathrm{Tr}\,[L_i, L_j]^2\right\}. \tag{9}$$

It is precisely this part of the Lagrangian, taken with the opposite sign, which determines the energy (mass) of the stationary configuration. After a trivial "adding and subtracting" procedure, we can write the mass as

$$M = -\frac{F_\pi}{16e}\left\{\int d^3x\,\mathrm{Tr}\,(L_i(x) \pm \varepsilon_{ijk}L_iL_k)^2\right.$$
$$\pm\frac{F_\pi}{16e}2\cdot 24\pi^2\int \mathrm{Tr}\,\frac{1}{24\pi^2}\varepsilon_{ijk}L_i(x)\,L_j(x)\,L_k(x), \tag{10}$$

from which we obtain the following estimate for the mass of the classical soliton[17]:

$$M \geqslant F_\pi\cdot 3\pi^2\,|B|/e. \tag{11}$$

Using the Skyrme–Witten ansatz, we can write M as[18]

$$M = \frac{F_\pi}{e}\cdot\frac{\pi}{2}\left\{\int\left[\frac{dF(x)}{dx} + 2\frac{\sin^2 F(x)}{x^2}\right]^2 x^2\,dx\right.$$
$$+ 2\int\sin^2 F(x)\left(1 + 2\frac{dF}{dx}\right)^2 dx + \frac{F_\pi}{e}3\pi^2\int\rho_B(x)\,d^3x. \tag{12}$$

Therefore, the lower limit in the topological estimate of Faddeev is unattainable because the conditions for the first two integrals on the right-hand side of this expression to vanish

$$\left.\begin{aligned}&\frac{dF(x)}{dx} + 2\frac{\sin^2 F(x)}{x^2} = 0;\\ &1 + 2\frac{dF(x)}{dx} = 0\end{aligned}\right\} \tag{13}$$

are incompatible.

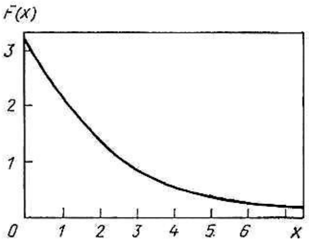


FIG. 1. Solution of Eq. (7) for the chiral angle $F(x)$ in the sector with $B = 1$.

Exact solution of Eq. (7) in the sector $B = 1$ gives the soliton mass $M \approx 36.5F_\pi/e$, which is roughly 23% greater than the value corresponding to the lower limit in the estimate (11).

In Fig. 1 we show a graph of the numerical solution of Eq. (7) with the boundary conditions $F(0) = \pi$ and $F(\infty) = 0$. The solution is a monotonic function. It is easily checked that no continuous deformation can transform the soliton into the trivial field $\boldsymbol{\pi} = 0$, $U = \mathbf{1}$. In fact, in any attempt to "switch off" the soliton, we must discontinuously jump from $F(0) = \pi$ to $F(0) = 0$. However, when $F(0) \neq \pi$, the soliton energy becomes infinite [see Eq. (8)]. For $x \to \infty$ Eq. (7) becomes

$$F'' + \frac{2}{x}F' - \frac{2F}{x^2} = 0 \tag{14}$$

and gives the asymptotic form

$$F(x) \sim \beta/x^2.$$

It is easy to see that near the origin

$$F(x) \sim B\pi - \alpha x.$$

The last two equations are sometimes used as an analytic approximation of the solution[18] after the matching points are chosen (uneven matching). Another common approximation is the Padé approximation used in Ref. 19.

QUARKS IN THE CHIRAL FIELD. THE BARYON NUMBER

Let us turn to the question of the interpretation of the topological charge as the baryon number. For this we consider the model problem of the motion of massive quarks in the field of a chiral soliton.[20,21] Let the quark orbitals $|\lambda\rangle$ satisfy the Dirac equation

$$\left[-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta\left(\frac{1+\gamma_5}{2}\right)U + \beta\left(\frac{1-\gamma_5}{2}\right)U^+\right]|\lambda\rangle$$

$$= [-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta(\cos F(r) + i\gamma_5\boldsymbol{\tau}\mathbf{n}\sin F(r))]\,|\lambda\rangle = \varepsilon_\lambda\,|\lambda\rangle, \tag{15}$$

where ε_λ is the orbital energy in units of the quark mass in the physical vacuum. The Hamiltonian of the problem commutes with the parity and "grand spin" $\mathbf{G} = \mathbf{j} + \mathbf{t}$ (the sum of the spin and isospin) operators and is invariant under simultaneous rotations in space and isospin space. The states are classified according to the grand spin and parity G^π. In Ref. 20 this problem was solved for a linear profile function $F(x)$ (or chiral angle) in a finite region of space $(0, X)$. The chiral angle varies from $F(0) = n\pi$ to $F(X) = 0$. The spectrum of quark orbitals thus obtained is shown in Fig. 2. For

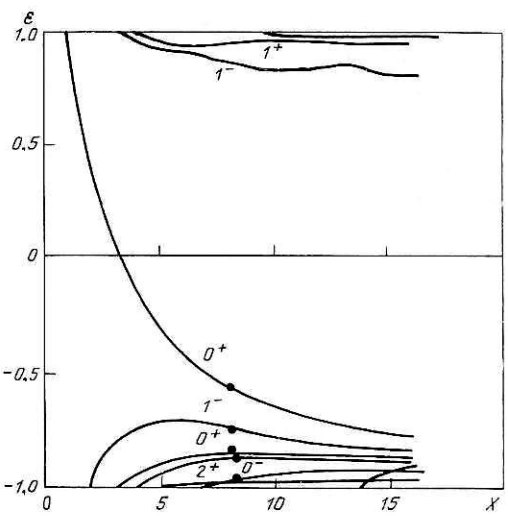


FIG. 2. Spectrum of quantum orbitals as a function of the soliton size X.

small X the chiral field is too weak to bind the quarks, and the spectrum displays only the free-quark mass gap. For $1.5 \leqslant X \leqslant 3$ the bound 0^+ orbital leaves the upper continuum and for $X \geqslant 3$ has negative energy. A system with baryon number $B = 1$ relative to the physical vacuum is obtained by the filling (and color saturation) of this 0^+ orbital and orbitals coming out of the lower continuum. For sufficiently large X, the 0^+ orbital joins the levels of the Dirac sea and we have a new mass gap of a new Dirac sea, which has baryon number 1 relative to the old one. The new sea acquires integer or half-integer spin, depending on whether the number of colors N_c required for saturation of the 0^+ orbital is even or odd. Therefore, for large X the ground state of the multiparticle Dirac Hamiltonian

$$H_D = \int d^3x\Psi^+ [-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta(\cos F + i\boldsymbol{\tau}\cdot\mathbf{n}\gamma_5\sin F)]\,\Psi \tag{16}$$

corresponds to the situation where all levels with negative energies (including the 0^+ level coming from the upper continuum) are filled. This state can be identified with the Skyrmion, and the vacuum rearrangement energy corresponds to the Skyrmion mass.

In the case of soliton topological charge $B = 2$ an additional 0^- orbital leaves the upper continuum and joins the sea of negative energy levels, and so on. This behavior suggests that the topological charge can be interpreted as the baryon number.

Goldstone and Wilczek[22] were the first to calculate the baryon current $\langle\bar\psi\gamma_\mu\psi\rangle$ of Dirac particles in a chiral field using perturbation theory. The baryon density $\langle\bar\psi\gamma_0\psi\rangle = \langle\psi^+\psi\rangle$ which they found is identical to the density found in the Skyrme model.[23]

The baryon density at the point x relative to the unperturbed vacuum can be written as

$$\rho(\mathbf{x}) = \mathrm{tr}\,|\,\mathbf{x}\rangle\langle\mathbf{x}\,|\,\{\sum_\lambda{}'|\,\lambda\rangle\langle\lambda\,| - \sum_k{}'|\,\mathbf{k}-\rangle\langle\mathbf{k}-|\}, \tag{17}$$

where the summation is restricted to only filled quark orbitals, which include all negative-energy orbitals (and, possibly, a positive-energy 0^+ orbital). The sum Σ'_λ includes the negative-frequency plane-wave orbitals $|\mathbf{k}-\rangle$ of the unperturbed vacuum. We stress that when the soliton is sufficiently large, the valence orbital has negative energy. Moreover, MacKenzie and Wilczek have noted, on the basis of study of the squared Dirac equation, that this orbital cannot lie far from the continuum.

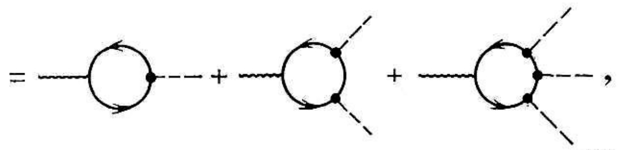
The perturbative expansion of the density is obtained by rewriting the density as

$$\rho(\mathbf{x}) = -\,\mathrm{tr}\,\frac{1}{2\pi i}\int\limits_C d\omega\,|\,\mathbf{x}\rangle\langle\mathbf{x}\,|\left[\frac{1}{h-\omega}-\frac{1}{h_0-\omega}\right]$$

$$= -\,\mathrm{tr}\,\frac{1}{2\pi i}\int\limits_C d\omega\,|\,\mathbf{x}\rangle\langle\mathbf{x}\,|\,\gamma_0\left[\frac{1}{1-\hat{p}+V}-\frac{1}{1-\hat{p}}\right]$$

$$= \mathrm{tr}\,\frac{1}{2\pi i}\int\limits_C d\omega\,|\,\mathbf{x}\rangle$$

$$\langle\mathbf{x}\,|\,\gamma_0\left[\frac{1}{1-\hat{p}}V\frac{1}{1-\hat{p}}-\frac{1}{1-\hat{p}}V\frac{1}{1-\hat{p}}V\frac{1}{1-\hat{p}}+\ldots\right]$$



(18)

where

$$\left.\begin{aligned} &h=-i\boldsymbol{\alpha}\boldsymbol{\nabla}+\beta(\sigma+i\tau\varphi\gamma_5);\quad h_0=-i\boldsymbol{\alpha}\boldsymbol{\nabla}+\beta;\\ &\hat{p}=\gamma_0\omega+i\boldsymbol{\gamma}\boldsymbol{\nabla};\quad V=\sigma+i\tau\varphi\gamma_5-1;\\ &\sigma=\cos F(r);\quad \varphi^i=n^i\sin F(r).\end{aligned}\right\} \qquad (19)$$

The wavy lines correspond to the density operator $|\mathbf{x}\rangle\langle\mathbf{x}|$, the dashed lines correspond to the interaction V, and the oriented lines correspond to the free quark propagator $(1-\hat{p})^{-1}$. The energy integral runs along a contour surrounding the points corresponding to the energies of occupied orbitals in the sum. The density converges when the number of negative-energy states included grows. The lowest-order contributions to the baryon density come from third- and fourth-order graphs. This gives

$$\rho(x) = -\frac{1}{2\pi x^2}\sin^2 F(x)\,\frac{dF(x)}{dx}, \qquad (20)$$

which coincides with the density of the Skyrmion topological charge. In Fig. 3 we show the baryon density of quarks for a field with topological charge $B=1$ and linear dimension $X=5$. The density was obtained by adding the contributions of the individual levels. The perturbation-theory estimate (which coincides with the result of the Skyrme model) practically reproduces the exact result. Moreover, we see from this figure that the density differs greatly from the density of the 0^+ orbital taken separately. Therefore, the one-loop contributions to the density cannot be omitted (except for solitons of small size). A detailed study of the baryon number of quark states in the field of a chiral soliton can be found in Refs. 21 and 24.

Finally, we should note the following feature, which supports the interpretation of the topological charge as the baryon number. If we consider the SU(3) generalization of the Skyrme model including the entire octet of pseudoscalar mesons, the Lagrangian will necessarily contain an additional term, the so-called Wess–Zumino term[25]

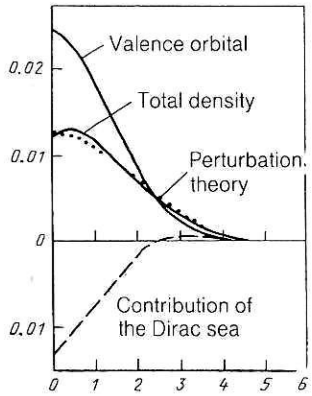


FIG. 3. Baryon density of quarks for a soliton with topological charge $B=1$ ($X=5$) from Ref. 20. The contributions of valence and sea quarks to the total density are shown. The points correspond to the results of perturbation calculations, which are identical to the results of Skyrmion calculations of the baryon density.

$$S_{WZ} = -\frac{iN_c}{240\pi^2}\int d^5x\,\varepsilon_{\alpha\beta\gamma\delta\varepsilon}\,\mathrm{Tr}\,(L_\alpha L_\beta L_\gamma L_\delta L_\varepsilon), \qquad (21)$$

which is written as an integral over five-dimensional space (where N_c is the number of colors, equal to three in the real world) or as an integral over the parameter σ (Refs. 26, 27):

$$\left.\begin{aligned} &S_{WZ}=\frac{N_c}{24\pi^2}\int d^4x\int\limits_0^1 d\sigma\,\varepsilon_{\alpha\beta\gamma\delta}\,\mathrm{Tr}\,2\,\frac{\pi^a\lambda^a}{F_\pi}\,(L_\alpha^{(\sigma)}L_\beta^{(\sigma)}L_\gamma^{(\sigma)}L_\delta^{(\sigma)});\\ &L_\alpha^{(\sigma)}=\exp(-i\pi^a\lambda^a\sigma)\,\partial_\alpha\exp(i\pi^a\lambda^a\sigma).\end{aligned}\right\} \qquad (22)$$

This term is needed, for example, to describe the amplitude of the process $K\bar{K}\to 3\pi$. In this model the conserved baryon current can be obtained as the Noether current corresponding to the invariance of the Lagrangian under gauge transformations. The explicit form of this current coincides with that of the topological current, although in the SU(2) variant of the model the corresponding Noether current is absent.

THE NUCLEON. COLLECTIVE VARIABLES

The Skyrme model is a type of collective model (in the terminology used in nuclear theory). The method of collective (group) variables was formulated by N. N. Bogolyubov in the polaron problem.[28] The collective variables of the Skyrme model define classical configurations of the pion field. It is assumed that a selected set of field configurations is, in some sense, the most important. This means that a sufficiently complete set of time-dependent collective variables defines a field which approximately satisfies the Euler–Lagrange equations.

Collective variables corresponding to nucleon rotation were first introduced in the model of the pion field coupled to a source.[29] In Ref. 12 they were used to quantize the Skyrme model. The authors of Ref. 30 have calculated the effective Hamiltonian, including rotational and monopole vibrational degrees of freedom, with and without the explicit breaking of chiral invariance by a meson mass term in the Lagrangian.

For quantizing the model we obviously need the time components of the currents entering into the Lagrangian. It should be noted that the energy of the system is not changed if the static solution $U_0(x)$ is subjected to the transformation $U_0 \Rightarrow A U_0 A^+$, where A is a unitary unimodular matrix of the form $a_0 \mathbf{1} + i\tau^a a^a$. The matrix A obviously corresponds to the SU(2) representation of the group of three-dimensional rotations in isotopic space. The parameters of the rotation matrix A can be chosen as the collective variables describing the rotational degrees of freedom. Therefore, restricting ourselves to field configurations

$$U(r, t) = A(t)\, U_0(\mathbf{r})\, A^+(t), \tag{23}$$

the Lagrangian, with the time components of the currents isolated,

$$\mathcal{L} = -M - \frac{F_\pi^2}{16} \int d^3r\, \mathrm{Tr}\,(L_0, L_0) - \frac{2}{32e^2} \int \mathrm{Tr}\,[L_k, L_0]^2\, d^3r \tag{24}$$

can be written as

$$\mathcal{L} = -M + 4\pi \widetilde{K}^2 \Lambda / 3e^2 F_\pi. \tag{25}$$

Here

$$\Lambda = \int x^3 \sin^2 F \left\{1 + 4\left((F')^2 + \frac{\sin^2 F}{x^2}\right)\right\} dx;$$

$$\widetilde{K}^2 = \mathrm{Tr}\,(\dot{A}(t)\, \dot{A}^+(t))/2. \tag{26}$$

In order to obtain a physical interpretation of this effective Lagrangian, let us consider the SU(2)×SU(2) group of left and right multiplications of the group SU(2), to which the matrices A belong. It is easily seen that the Lagrangian is invariant under multiplication of the matrix A on the right by a constant matrix h: $A \Rightarrow A \cdot h$. Here the field $U(\mathbf{r},t)$ becomes $\widetilde{U} = A(t) h U_0(\mathbf{r}) h^+ A^+(t)$. It is easily checked that the action of the matrix h on U_0 takes it into $\widetilde{U}_0(\mathbf{r}) = U_0(R_h^{-1}\mathbf{r})$, where R_h is the 3×3 matrix of spatial rotations. On the other hand, multiplication of the matrix A on the left by an SU(2) matrix g transforms the field configuration $U(\mathbf{r},t)$ into

$$\widetilde{U}(\mathbf{r}, t) = gA(t)\, U_0(\mathbf{r})\, A^+(t)\, g^+ = gU(\mathbf{r}, t)\, g^+,$$

i.e., it corresponds to an isotopic rotation. Therefore, we have the Cayley construction of the SU(2)×SU(2) group, formed using left and right transformations of the original SU(2) group. Study of the variation of the effective action under infinitesimal right and left time-dependent transformations shows that the generators $i\,\mathrm{Tr}\tau^j \dot{A}^+ A$ and $i\,\mathrm{Tr}\tau^j A \dot{A}^+$ will be conserved. Canonical quantization requires the correspondence $\dot{a}_i \Rightarrow -i\partial/\partial a_i$. Therefore, the spin and isospin operators take the form

$$\left.\begin{aligned} J_j &= \frac{1}{2} i\left(a_j \frac{\partial}{\partial a_0} - a_0 \frac{\partial}{\partial a_j} - \varepsilon_{jem} a_e \frac{\partial}{\partial a_m}\right); \\ I_j &= \frac{1}{2} i\left(a_0 \frac{\partial}{\partial a_j} - a_j \frac{\partial}{\partial a_0} - \varepsilon_{jlm} a_l \frac{\partial}{\partial a_m}\right). \end{aligned}\right\} \tag{27}$$

These operators act in the space of functions of the collective variables a_0 and a_i. Owing to the conditions $AA^+ = 1$ and $A^+A = 1$, the operators J_i and I_i are coupled to each other by an orthogonal transformation $g_{aj} I_j = -J_a$, where $g_{aj} = \mathrm{Tr}(A\tau^a A^+ \tau^j)/2i$. From this it follows that the ansatz under consideration generates states with equal values of the spin and isospin $\mathbf{I}^2 = \mathbf{J}^2$. The rotational part of the Hamiltonian is proportional to $\mathbf{J}^2$ or $\mathbf{I}^2$. We thus have the problem of a spherical top which is quantized with half-integer angular momenta, when we are describing Fermi particles, and with integer angular momenta when we are describing Bose particles. The eigenfunctions of the problem are obviously the finite-rotation matrices $D^{I=J}_{J_3 I_3}$, which are harmonic polynomials of degree $2J$ in the collective variables. The energy levels are degenerate in the projections J_3 and I_3, the full degree of degeneracy being $(2J+1)^2$.

CURRENTS AND EQUATIONS OF MOTION

The chiral invariance of the Skyrme-model Lagrangian corresponds to two conserved currents. They can be calculated by considering an infinitesimal chiral transformation

$$U(x) \Rightarrow \exp(-i\boldsymbol{\tau}\mathbf{a})\, U(x) \exp(+i\boldsymbol{\tau}\mathbf{b})$$

and the corresponding variation of the Lagrangian. The coefficient functions multiplying $\partial_\mu \mathbf{a}$ and $\partial_\mu \mathbf{b}$ specify left- and right-handed conserved currents:

$$\left.\begin{aligned} J_\mu^{Lk} &= i\frac{F_\pi^2}{8} \mathrm{Tr}\,\tau^k L_\mu + i\frac{1}{4e^2} \mathrm{Tr}\,\tau^k [L_\nu, L_\mu] L_\nu; \\ J_\mu^{Rk} &= i\frac{F_\pi^2}{8} \mathrm{Tr}\,\tau^k R_\mu + i\frac{1}{4e^2} \mathrm{Tr}\,\tau^k [R_\nu, R_\mu] R_\nu, \end{aligned}\right\} \tag{28}$$

where the Latin indices correspond to the isotopic components. Using the condition for conservation of the left-handed Noether current, $\partial_\mu J_\mu^{Lk} = 0$, it is easy to find the general equation of motion in the form given by Skyrme[1]:

$$\partial_\mu \left(\frac{\delta \mathcal{L}}{\delta L_\mu^k}\right) = 0. \tag{29}$$

Use of the Skyrme–Witten ansatz leads directly to Eq. (7) for the profile function (chiral angle). In the expressions (28) the right-handed current of the "free" Lagrangian $\mathcal{L}_2$ is $R_\mu = U\partial_\mu U^+$. In turn, the half-sum and half-difference of the left- and right-handed currents defines the axial and vector currents.

The isospin can be defined as the integral over the time component of the isovector current, $I_i = \int J_0^{Vi} d^3r$. For example, introducing the orthogonal 3×3 matrix of isotopic rotations $I_{ki}(t)$, which acts on the isovector components in the Euler–Rodrigues parametrization for the matrix $U(x) = \sigma(x) + i\tau^i \varphi^i(x)$, and defining the angular velocity ω' in the rotating coordinate frame via $\dot{I}_{ik} I_{kj}^{-1} = \varepsilon_{ijk}\omega'_k$, we arrive at the expression $I_k = (F_\pi e^3)^{-1} \lambda \omega'_k$ relating the isospin I_k to the angular velocity ω'_k and the moment of inertia:

$$\lambda = \frac{2\pi}{3} \int x^2 \sin^2 F \left[1 + 4\left((F')^2 + \frac{\sin^2 F}{x^2}\right)\right] dx. \tag{30}$$

In a similar manner, the spin can be defined in terms of the components of the energy–momentum tensor T_{0k}:

$$J_i = i \int \varepsilon_{ijk} x_j T_{0k}\, d^3r, \tag{31}$$

where

$$T_{0k} = -\frac{F_\pi^2}{4} (\mathbf{L}_k, \mathbf{L}_0) - \frac{1}{e^2} ([\mathbf{L}_k, \mathbf{L}_\nu], [\mathbf{L}_0, \mathbf{L}_\nu]). \tag{32}$$

Taking into account the explicit time dependence of the currents L_0 and the definition of the angular velocity ω in the stationary coordinate frame, we find

$$J_i = -\frac{2\pi}{3} (F_\pi e^3)^{-1} \omega_i \int x^2 \sin^2 F \left((F')^2 + \frac{\sin^2 F}{x^2}\right)\Big] dx. \tag{33}$$

The integral in this expression determines the moment of inertia in coordinate space. We see that it coincides with the moment of inertia in isotopic (internal) space.

The relation between the angular velocities ω_i and ω_j' specifies the relation between the spin and isospin mentioned above.

It is convenient to parametrize the rotation matrices I_{kl} in terms of the Euler angles.[31] Reference 31 also gives the representation of spin operators as differential operators acting on D functions of the Euler angles. The matrices I_{kl} and A used to derive the isospin and spin operators are related through the equation $I_{kl} = \mathrm{Tr}\tau_k A \tau_l A^+/2i$. This completes our interpretation of the operators I_i and J_k as the isospin and spin operators. Let us conclude this section with a remark about the structure of the collective wave function. If the vector $|A\rangle$ is understood as a state corresponding to a definite orientation in the internal space with matrix A and field configuration $U = AU_0A^+$, then the nucleon state $|N\rangle$ corresponds to the superposition given by the integral (over the group)

$$|N\rangle = \int dA\, \chi(A)\, |A\rangle, \tag{34}$$

where $\chi(A)$ is the wave function in collective-coordinate space. Since $|A\rangle$ corresponds to a state which is well defined in the internal space, it has poorly defined isospin and angular momentum. On the other hand, the state $|N\rangle$ has well-defined spin and isospin, but no definite direction in the internal space. Here we are dealing with a situation which is well known in the theory of nuclear rotations.

Whereas in the SU(2) Skyrme model the quantized state of a Skyrmion with odd baryon number is a fermion only by construction, in the SU(3) generalization the fermionic nature of such states necessarily follows from the linearity of the Wess–Zumino term in the time derivative.[32]

STATIC PROPERTIES OF NUCLEONS IN THE SKYRME MODEL

The static properties of nucleons in the scheme described above have been obtained in Ref. 12, where the quantities F_π and e were treated as free parameters chosen such that the nucleon and delta-resonance masses were reproduced. Therefore, all static quantities become functions of F_π and e.

Expanding the expressions for the isovector current $J_\mu^{Va}(x)$ and the baryon current $J_\mu^B(x)$ in the Skyrme model, we can define the electromagnetic current

$$J_\mu^{em}(x) = J_\mu^{V3}(x) + \frac{1}{2} J_\mu^B(x). \tag{35}$$

Here $J_\mu^{V_3}$ is the third component of the isovector current and $J_\mu^B/2$ is half the baryon current determining the isoscalar part of the electromagnetic current. Equation (35) corresponds to the well-known expression relating the electric charge Q, the isospin projection T_3, and the baryon charge B: $Q = T_3 + B/2$. Using the Skyrme–Witten ansatz for the chiral angle F and the general representation for the isovector current, we can calculate the density of the third component of the vector current (averaged over angles):

$$J_0^{V3}(x) = \frac{\left[1+4\left((F')^2+\frac{\sin^2 F}{x^2}\right)\right]\sin^2 F}{\int x^2\left[1+4\left((F')^2+\frac{\sin^2 F}{x^2}\right)\right]\sin^2 F\, dr}\hat{T}_3, \tag{36}$$

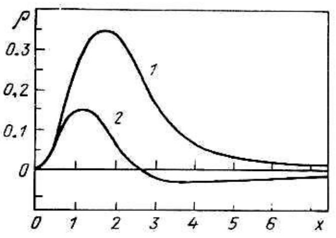


FIG. 4. Electric charge density $\rho_e x^2$ for the proton (curve 1) and neutron (curve 2) as a function of $x = F_\pi er$.

where $\hat{T}_3$ is the operator corresponding to the third component of the isospin. Adding to this half the baryon density $(1/\pi x^2)F'(x)\sin^2 F(x)$, we obtain the electric-charge density operator. The matrix element of this operator between the nucleon states of interest to us determines the spatial density of electric charge for the various isotopic components. For example, we find the electric-charge density distributions for the proton and neutron shown in Fig. 4. This distribution is difficult to obtain in the bag model (in the quark bag model the derivative of the electric-charge density is discontinuous at the bag surface, owing to the boundary conditions usually used).[33] The proton distribution is positive-definite everywhere, and the neutron distribution is characterized by a positively charged core and negatively charged shell.

For calculating the observables, we must first of all choose the values of F_π and e. Numerical computation of the integral (30) determining the value of the moment of inertia λ gives

$$\lambda \approx \frac{4\pi}{6}\left(\frac{1}{F_\pi e^3}\right)\cdot 51$$

and for the nucleon and Δ-isobar masses we find

$$M_N = M + \frac{1}{2\lambda}\frac{3}{4}; \quad M_\Delta = M + \frac{1}{2\lambda}\frac{5}{4}. \tag{37}$$

Using the experimental values of M_N and M_Δ, we obtain $F_\pi = 129$ MeV and $e = 5.45$.

The isoscalar rms radius, given by the integral

$$\langle r^2\rangle_{I=0} = \int_0^\infty r^2\rho^B(r)\, dr = 4.47/e^2F_\pi^2 \tag{38}$$

with this choice of the constants F_π and e, is $\langle r^2\rangle_{I=0}^{1/2} = 0.59$ F. The isoscalar and isovector magnetic moments have a clear physical interpretation and are given by the following integrals:

$$\left.\begin{aligned} \mu_{I=0} &= \frac{1}{2}\int \mathbf{r}\times\mathbf{B}\, d^3x; \\ \mu_{I=1} &= \frac{1}{2}\int \mathbf{r}\times\mathbf{J}^{V3}\, d^3x, \end{aligned}\right\} \tag{39}$$

where $\mathbf{B}$ and $\mathbf{J}^{V_3}$ are the spatial components of the baryon current and the third component of the isovector current. From this, for the isoscalar magnetic-moment density we have

$$\rho_M^{I=0} = \frac{x^2F'(x)\sin^2 F(x)}{\int x^2F'\sin^2 F(r)\, dr}.$$

This expression can be used to calculate the corresponding rms radius. However calculation of the proton and neutron magnetic moments μ_p and μ_n requires use of the explicit form of the collective wave function (see Ref. 12).

How can the axial constant g_A be estimated in this model? The standard definition of the matrix element of the axial current in the usual notation is

$$\langle N'(p_2) \mid A_\mu^a(0) \mid N(p_1)\rangle$$
$$= \bar{u}(p_2)\,\tau^a\,[\gamma^\mu\gamma^5 g_A(q^2) + q_\mu\gamma^5 h_A(q^2)]\,u(p_1). \quad (40)$$

Taking the symmetric nonrelativistic limit for $\mathbf{q}\to 0$, we find

$$\langle N'(p_2) \mid A_i^a(0) \mid N(p_1)\rangle = \frac{2}{3} g_A \langle N' \mid \sigma_i\tau^a \mid N\rangle,$$

where $g_A = g_A(0)$. This limit corresponds to calculation of the integral $\int d^3x A_i^a(x)$, where the integral over the angular variables is taken first and that over the radial variable afterwards. This procedure gives

$$\int d^3x A_i^a(x) = \frac{\pi}{3e^2} D\,\mathrm{Tr}\,[\tau_i A^{-1}\tau_a A]. \quad (41)$$

The constant D on the right-hand side of the last equation is given by an integral which is calculated using the solution of the stationary equation for the chiral angle. Using the fact that in the standard notation $\mathrm{Tr}[\tau_i A^+\tau^a A] = -\frac{2}{3}\langle\sigma_i\tau^a\rangle$, from this we find the quantity g_A.

Comparison of the traditional representation for the pion field in the asymptotic region

$$\langle\pi^a(x)\rangle = -\frac{g_{\pi NN}}{8\pi M_N}\frac{x_i}{r^3}\langle\sigma_i\tau^a\rangle \quad (42)$$

with the pion field π in the asymptotic region given by the Skyrme model

$$U \sim 1 + 2i\tau\pi F_\pi, \quad (43)$$

where $U = AU_0A^+$ and $U_0 \sim 1 + iC\boldsymbol{\tau}\cdot\mathbf{n}/r^2$, and C is a numerical coefficient found from the asymptote of the solution of the stationary Euler–Lagrange equation, verifies that the Goldberger–Treiman relation

$$g_A = \frac{F_\pi g_{\pi NN}}{2M_N} \quad (44)$$

is satisfied. This relation can be used to calculate $g_{\pi NN}$.

Some of the static quantities calculated in this scheme are in good agreement with the experimental values, while a few differ considerably. For example, the values obtained for F_π and g_A differ from the experimental values by 30 and 50%, respectively. Here it is often said that the Skyrme model can give a good description of quantities which are independent of the number of colors N_c of the initial gauge group SU(N_c) of the theory, but is completely incapable of describing quantities which depend significantly on N_c. Here we reproduce a table, taken from Ref. 34, of several quantities characterizing the nucleon in this model and their dependence on the number of colors. The numerical values are taken from Ref. 12. The calculated quantities in Table I which are independent of the number of colors N_c are, in fact, in better agreement with experiment. This is true of the isoscalar electric $\langle r^2\rangle_{I=0}^{1/2}$ and magnetic $\langle r^2\rangle_{M,I=0}^{1/2}$ rms radii, the ratio of the proton and neutron magnetic moments μ_p and μ_n, the ratio of the square of the pion constant F_π and the axial constant g_A, and the ratio of the interaction constants $g_{\pi N\Delta}$ and $g_{\pi NN}$.

The isovector rms radius diverges in the chirally invariant limit ($m_\pi = 0$). In order to avoid this, it is necessary to add to the Lagrangian a "mass" term which explicitly breaks the chiral symmetry:

$$\Delta\mathcal{L} = \frac{1}{8} m_\pi^2 F_\pi^2 (\mathrm{Tr}\,U - 2), \quad (45)$$

which leads to an extra term in the equation for the profile function

$$-\frac{1}{4}\frac{m_\pi^2}{e^2F_\pi^2} x^2 \sin x$$

and to a change in the numerical values of all the integrals determining the masses, moments of inertia, and other quantities, and, therefore, also the values of the coupling constants. For example, the determination of e and F_π from M_N and M_Δ now gives $e = 4.84$ and $F_\pi = 108$ MeV. It is obvious that the asymptote of the solution of the differential equation for the chiral angle also changes. Now

$$F(r) \sim \frac{1}{r} e^{-m_\pi r} \quad (46)$$

instead of $F(r) \sim 1/r^2$, as was the case for $m_\pi = 0$. Calculation of the pion field at large distances from the nucleon gives

TABLE I. Static properties of nucleons in the Skyrme model (including only rotational degrees of freedom) and their dependence on the number of colors.

Property	N_c dependence	Prediction	Experiment	Error, %
$\langle r^2\rangle_{I=0}^{1/2}$	N_c^0	0.59 F	0.72 F	18
$\langle r^2\rangle_{M,\ I=0}^{1/2}$	N_c^0	0.92 F	0.81 F	14
μ_p	N_c	1.87	2.79	33
μ_n	N_c	−1.31	−1.91	31
μ_p/μ_n	N_c^0	1.43	1.46	2
g_A	N_c	0.61	1.23	50
F_π	$N_c^{1/2}$	129 MeV	186 MeV	31
F_π^2/g_A	N_c^0	27 280 MeV²	28 127 MeV²	3
$g_{\pi NN}$	$N_c^{3/2}$	8.9	13.5	34
$g_{\pi N\Delta}$	$N_c^{3/2}$	13.2	20.3	35
$g_{\pi N\Delta}/g_{\pi NN}$	N_c^0	1.5	1.5	1

$g_{\pi NN} = 11.9$. The isovector electric and magnetic rms radii are finite. The isovector electric density is determined by the time component of the vector current, and the isovector magnetic density is determined by the spatial part of the vector current. The isovector electric and magnetic densities turn out to be equal to each other:

$$\rho_{I=1}(r) = \rho^{(r)}_{M,\ I=1}$$
$$= \frac{r^2 \sin^2 F \left\{F_\pi^2 + \frac{4}{e^2}\left[(F')^2 + \frac{\sin^2 F}{r^2}\right]\right\}}{\int_0^\infty r^2 \sin^2 F \left\{F_\pi^2 + \frac{4}{e^2}\left[(F')^2 + \frac{\sin^2 F}{r^2}\right]\right\} dr}; \qquad (47)$$

the corresponding rms radii are also equal:

$$\langle r^2\rangle^{1/2}_{I=1} = \langle r^2\rangle^{1/2}_{M,I=1} = 1.04\text{F}.$$

This equation, which can be written in terms of the proton and neutron magnetic moments as

$$[\langle r^2\rangle_p - \langle r^2\rangle_n]^{1/2} = \left[\frac{\mu_p \langle r^2\rangle_{M,p} - \mu_n \langle r^2\rangle_{M,n}}{\mu_p - \mu_n}\right]^{1/2}, \qquad (48)$$

is very well satisfied in reality (the left-hand side is 0.88 F, and the right-hand side is 0.80 F).

For small momentum transfers the electric form factor can be written as the Fourier transform of the density of the electric-charge distribution (36). Performing the integration over the angular variables and averaging over the wave functions of the states, the result can be written as[35-37]

$$F_e(Q) = -\frac{1}{\pi}\int F'(x)\sin^2 F j_0(Qx)\,dx$$
$$+ \frac{\int x^2 \sin^2 F\left(1 + 4\left[(F')^2 + \frac{\sin^2 F}{x^2}\right]\right)(j_0(Qx) + j_2(Qx))}{\int x^2 \sin^2 F\left(1 + 4\left[(F')^2 + \frac{\sin^2 F}{x^2}\right]\right) dx} T_3. \qquad (49)$$

In this expression $j_0(Qx)$ and $j_2(Qx)$ are the spherical Bessel functions. The dimensionless momentum transfer is $Q = q/F_\pi e$.

As has been emphasized repeatedly, the isoscalar part of the electromagnetic current in the Skyrme model is determined directly by the baryon current. This means that even in the absence of the equations of motion for the profile function F, the latter can be obtained from one of the empirical isoscalar form factors, and then used to calculate other observables. In this approach there are fewer constraints specified by the explicit form of the Lagrangian, and a larger likelihood of verifying the actual concept of the nucleon as a chiral soliton.

If we restrict ourselves to the dipole approximation for the isoscalar form factor,

$$F_{I=0}(q^2) = \frac{1}{(q^2 + \Lambda^2)^2},$$

then, as is easily seen from the preceding expression (in which the second term on the right-hand side should be dropped, since we are now dealing with the isoscalar form factor), the inverse Fourier transform leads to the following equation for the profile function:

$$F(r) - \frac{1}{2}\sin 2F(r) = \pi e^{-\Lambda r}\left(1 + \Lambda r + \frac{1}{2}\Lambda^2 r^2\right) \qquad (50)$$

with $\Lambda^2 = 0.71\ \text{GeV}^2$.

Therefore, the empirical form factor can be used to reconstruct the "empirical" profile function. The function thus obtained has been used[38] to analyze the deuteron electromagnetic form factors (see below).

As was pointed out in Ref. 38, in the Skyrme model there is a simple relation between the isoscalar electric $F_{I=0}$ and magnetic $F_{M,I=0}$ form factors of the nucleon:

$$F_{M,\ I=0}(q^2) = -\frac{2M_N}{\lambda}\frac{d}{dq^2}F_{I=0}(q^2). \qquad (51)$$

For small momentum transfers this gives

$$F^{(0)}_{M,\ I=0} = \frac{M_N}{3\lambda}\langle r^2\rangle_{I=0}, \qquad (52)$$

where $\langle r^2\rangle_{I=0}$ is the isoscalar rms radius. If we used the moment of inertia λ obtained from the mass spectrum, the right-hand side of the last equation gives an isoscalar magnetic moment of about $(0.82\text{–}0.86)\mu_{\text{nuc}}$, which corresponds to the range of uncertainty in the empirical isoscalar radius 0.72–0.79 F. This value is in good agreement with the empirical value $0.88\mu_{\text{nuc}}$.

Relation (52) can be used to determine the value of the moment of inertia λ without reference to the Δ resonance.

VIBRATIONAL DEGREES OF FREEDOM

The scheme for calculating the properties of baryons described above is restricted to the inclusion of only the rotational degrees of freedom. Even the first calculations including monopole vibrations showed that they have a very strong effect on the properties of baryons[39] (dibaryons; see below).

The collective variables describing monopole vibrations are introduced by substituting into (1) a more general ansatz for the form of the solution:

$$U(\mathbf{r},\ t) = A(t)\,U_0(e^{\lambda(t)}\mathbf{r})\,A^+(t). \qquad (53)$$

The time-dependent scalar parameter λ plays the role of the collective variable describing monopole vibrations corresponding to scale transformations of the solution of the stationary equation, $U_0(\mathbf{r})$, and uniform fluctuations of the densities of observables. After canonical quantization and diagonalization in the angular variables, we arrive at the effective Hamiltonian[40]

$$\hat{H} = \frac{\hat{p}_\lambda^2}{2A(\lambda)} + B(\lambda) + \frac{j(j+1)}{C(\lambda)}. \qquad (54)$$

Here $\hat{p}_\lambda$ is the momentum corresponding to monopole vibrations and j is the spin (equal to the isospin) of the state. The effective mass $A(\lambda)$, the potential $B(\lambda)$, and the moment of inertia $C(\lambda)$ are given by the following expressions:

$$\left.\begin{aligned} A(\lambda) &= e^{-3\lambda}Q_2 + e^{-\lambda}Q_4; \\ B(\lambda) &= e^{-\lambda}V_2 + e^{\lambda}V_4 + e^{-3\lambda}V_6; \\ C(\lambda) &= e^{-3\lambda}I_2 + e^{-\lambda}I_4. \end{aligned}\right\} \qquad (55)$$

The coefficients Q_i, V_i, and I_i are determined by the following integrals:

$$\left.\begin{aligned}
Q_2&=\frac{\pi}{e^3F_\pi}\int_0^\infty (F')^2x^4\,dx;\\
Q_4&=\frac{8\pi}{e^3F_\pi}\int_0^\infty (F')^2x^2\sin^2F\,dx;\\
V_2&=\frac{\pi F_\pi}{2e}\int_0^\infty x^2\left((F')^2+\frac{2\sin^2F}{x^2}\right)dx;\\
V_4&=\frac{2\pi F_\pi}{e}\int_0^\infty \sin^2F\left(2(F')^2+\frac{\sin^2F}{x^2}\right)dx;\\
V_6&=\frac{\pi m_\pi^2}{e^2F_\pi}\int_0^\infty x^2(1-\cos F)\,dx;\\
I_2&=\frac{4\pi}{3}\left(\frac{1}{F_\pi e^3}\right)\int_0^\infty x^2\sin^2F\,dx;\\
I_4&=\frac{16\pi}{3}\left(\frac{1}{e^3F_\pi}\right)\int_0^\infty\left((F')^2+\frac{\sin^2F}{x^2}\right)x^2\sin^2F\,dx.
\end{aligned}\right\}\qquad(56)$$

The integrands in these expressions are determined by the solution of the stationary equation with $m_\pi=0$ or $m_\pi\neq0$, depending on the case in question. In the case $m_\pi=0$, the integral V_6 obviously falls out of the expression for $B(\lambda)$.

The mass spectrum is determined by the solution of the Schrödinger equation with the Hamiltonian (54). If we neglect the coupling of vibrational and rotational motion, the spectrum is characterized by the following dependence on the number of colors:

$$E=aN_c+bN_c^0+dN_c^{-1}. \qquad (57)$$

The first term on the right-hand side corresponds to the classical soliton mass, the second corresponds to the contribution of vibrations of the soliton, and the third corresponds to the contribution of rotations. The dependence of the spectrum on N_c is reconstructed from the dependence of the constants F_π and e on N_c. The spectrum therefore has the form of rotational bands constructed on vibrational states. Analysis of the effective Hamiltonian shows that the rotational and vibrational motions are strongly coupled, which is consistent with experiment:

$$\frac{\Delta_{\rm rot}}{\Delta_{\rm vib}}=\frac{M_\Delta-M_N}{M_{1470}-M_N}=0.564.$$

Here the vibrational splitting between levels is defined relative to a reference resonance, and the rotational splitting is defined relative to the Δ resonance. It should be emphasized that the rigidity to λ deformations is large and is determined by the Skyrmion mass (the situation is analogous to that in bag models of baryons).

The calculated values of the masses show that the states $|1,1/2\rangle$ and $|0,3/2\rangle$ are degenerate (the first index gives the number of λ phonons, and the second gives the spin and isospin). This feature is practically independent of the coupling constant e. If we consider the experimentally observed states $|n+1,j\rangle$ and $|n,j+1\rangle$ lying above the Δ and reference resonances, they do actually confirm this degeneracy. For example, the states $|3/2\,(3/2^+)\rangle$ P''_{33} $[\Delta(1690)]$ and $|1/2(1/2^+)\rangle$ P''_{11} $[N(1710)]$, which in our scheme are the first vibrational excitations above the Δ and the second excited state with the nucleon quantum numbers, are practically degenerate. If the constant F_π is chosen to be equal to the experimental value 186 MeV and the constant e is taken equal to the experimental N–Δ mass splitting, the calculated P''_{33} and P''_{11} masses coincide with the experimental values.

In the course of the procedure of quantizing the vibrational degrees of freedom we encounter an operator ordering problem owing to the λ dependence of the effective mass.[42] On the basis of the results of Ref. 42, it can be concluded that, whereas the ordering of the operators involved in the kinetic energy has little effect on the energies (masses) of the lowest states, it strongly affects the wave functions. The latter, in turn, can significantly affect the calculated form factors including the vibrational degrees of freedom of the soliton. Similar difficulties also arise in a careful treatment of the quantization of the rotational degrees of freedom, but we shall not discuss this here (see Ref. 43).

In Ref. 44 it was found that the inclusion of local radial fluctuations of the pion field can lead to negative contributions to the mass of the quantized Skyrmion.

A richer excitation spectrum arises when spatially asymmetric fluctuations are considered.[45]

NONSTRANGE DIBARYONS AND MULTIBARYONS

After some success had been achieved in the study of the properties of nucleons and their excitations in the Skyrme model, investigations were begun in sectors with large baryon numbers: two-baryon,[46] three-baryon,[47] and multibaryon states,[48] and also nonstrange and strange dibaryons.[49]

For dibaryons (and for multibaryons in general),[2)] three types of stability conditions should be distinguished:

1. Topological stability. The classical evolution of an object corresponding to a deformation of the mapping $S^3\Rightarrow S^3$ does not change its equivalence class, which is specified by the index of the mapping—the topological charge. This (quite weak for multibaryons) condition ensures only baryon-number conservation. The multibaryons obtained using the Skyrme–Witten ansatz $\boldsymbol{\pi}/|\boldsymbol{\pi}|=\mathbf{n}=\mathbf{r}/|\mathbf{r}|$ are characterized by only topological stability. Such multibaryons have a large classical mass, which increases with increasing baryon number B as $B(B+0.87)$ (Ref. 50).

2. Quantum-mechanical stability. The mass of the lowest quantum state of a dibaryon is less than the sum of the masses of two quantum Skyrmions with $B=1$. This stronger condition is satisfied for the lowest states of the dibaryons computed using the ansatz in which, as the azimuthal angle φ is varied, the chiral field is rotated an integral number of times more rapidly about the symmetry axis[51]

$$\mathbf{n}=(\cos\alpha,\ \sin\alpha\cos k\varphi,\ \sin\alpha\sin k\varphi), \qquad (58)$$

where k is the index of the mapping. The classical mass of a Skyrmion with $B=2$ is equal to 2.14 times the mass of a Skyrmion with $B=1$. Quantum-mechanical stability occurs also for torus-like solitons.[53]

3. Classical stability. The classical mass of the dibaryon is less than the sum of the masses of Skyrmions with unit baryon number. The solutions found in Ref. 53 satisfy this condition.

The possibility of the appearance of nonstrange dibaryons of low mass and rotational bands in the dibaryon mass spectrum was first discussed in Refs. 51 and 52.

The $k\varphi$ ansatz (58) was used for the static solution U_0. Consideration of a fairly general time dependence of the field

$$U(\mathbf{r},\ t)=A(t)\,U_0\,(e^{\lambda(t)}R^{-1}_{\alpha\beta}r^{\beta})\,A^{+}(t), \qquad (59)$$

where $R_{\alpha\beta}$ is the 3×3 matrix of spatial rotations, $A(t)$ are the coordinates of isotopic rotations, and $\lambda(t)$ is the coordinate of monopole vibrations, leads to the quantum effective Hamiltonian[41]

$$\hat{H}=\frac{F_\pi}{e}\,(M_2e^{-\lambda}+M_4e^{\lambda})$$
$$+\Big\{-\frac{1}{2V_2(\lambda)}\frac{\partial^2}{\partial\lambda^2}+\frac{\hat{T}^2}{4\,(Q_2(\lambda)+7\Delta(\lambda))}$$
$$+\frac{\hat{S}^2}{4\left(\frac{7}{4}\,Q_2(\lambda)-19\Delta(\lambda)\right)}$$
$$+\frac{1}{16}\Big([Q_2-16\Delta]^{-1}-[Q_2-7\Delta]^{-1}$$
$$-4\left[\frac{7}{4}\,Q_2(\lambda)-19\Delta(\lambda)]^{-1}\right)\Big\}\,F_\pi\,e^3. \qquad (60)$$

Here $\hat{T}$ and $\hat{S}$ are the isospin and spin operators; see Ref. 52 for the functions M_2, M_4, V_2, Δ, and Q_2. Our ansatz is constructed such that a rotation about the 3-axis corresponds to an isospin rotation about the same axis by twice the angle. This leads to a relation between the 3-projections of the spin and isospin in the frame attached to the body (b.f.): $S_3^{\text{b.f}}=-2T_3^{\text{b.f}}$. This relation should be viewed as an operator constraint on the choice of wave functions. The rotational part of the Hamiltonian is diagonalized by a product of D functions corresponding to matrices of finite rotations in coordinate and isospin space:

$$\psi\sim D^{T}_{M_TL}D^{S}_{M_S-2L}. \qquad (61)$$

Here the indices have the usual meaning and take integral values, since in this case we are dealing with bosons. Solution of the Schrödinger equation for the vibrational degree of freedom gives the mass spectrum shown in Fig. 5. The selected values $T_3=0$ correspond to dibaryon electric charge $Q=+1$. It is easily seen that the slope and location of the bands depend significantly on the inclusion of the vibrational degree of freedom. The mass of the state with the quantum numbers of the deuteron is about 80 MeV smaller than the sum of the masses of the states with the nucleon quantum numbers. Study of the model with the $k\varphi$ ansatz leads to the following conclusions:

a) configurations with masses significantly smaller than allowed by the Skyrme–Witten ansatz can exist;

b) the quantum Hamiltonian leads to states with spin different from the isospin, and rotational bands similar to those predicted by the quark bag model appear;

c) λ vibrations can significantly change the slope of the bands;

d) the binding energy of a state is determined by the number of degrees of freedom involved in the motion (and taken into account in the theoretical calculation).

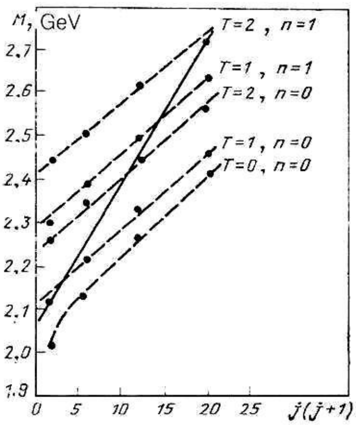


FIG. 5. Masses of dibaryon states: points—calculation including λ vibrations; solid line—calculation for $T=2$ without vibrations.

Finally, it should be stressed that the calculation of the absolute values of the masses requires the inclusion of the change of the energy of zero-point fluctuations of the pion field in the presence of the Skyrmion.

The authors of Ref. 53 studied configurations described by two functions—the profile function $F(z,\rho)$ and the angular function $\alpha(z,\rho)$, determining the direction of the field vector and depending on the cylindrical coordinates (z,ρ) of the system. Calculations show that the masses of solitons with small B are smaller than $BM_{B=1}$, i.e., such solitons are classically stable. The density of the mass distribution for these solitons given in Ref. 53 indicates that in this case we have torus-like objects.

Since in a number of hybrid models of the deuteron form factors the traditional form factor, determined by the nucleon–nucleon component of the semiphenomenological wave function, is augmented by the contribution of the form factors of dibaryon (six-quark) (Ref. 54) states, it is interesting to calculate the dibaryon form factor in the Skyrme model. It was shown in Ref. 37 that all the features of the $k\varphi$ ansatz which distinguish it from the Skyrme–Witten solution cancel in a remarkable manner in algebraic calculations of the isovector component of the electromagnetic current. A difference remains in the isoscalar part and, naturally, in the numerical values. It has been shown that the electric form factor for the dibaryon corresponding to the ansatz in the calculated region is closer to the form factor of the relativistic harmonic-oscillator model than that for the dibaryon corresponding to the Skyrme–Witten solution.

The Skyrme model is already being used to analyze some very hypothetical situations. For example, the authors of Ref. 55 have studied the problem of reaching densities in nuclear systems which are sufficient for the formation of a quark–gluon plasma. Since a new hypothetical nuclear state—a quark–gluon plasma—must be formed at large distances (on the QCD scale), one can attempt to use the Skyrme effective Lagrangian to see whether the necessary spatial energy density can, in principle, be attained. It is assumed that a Skyrmion with large baryon number (in a state corresponding to the Skyrme–Witten solution) can represent a crude model of the initial stage of the transition to a quark–gluon plasma, since here we are dealing with theoretical hadronic matter which is concentrated in a small spatial region. Of course, such a state would be unstable and would decay into nucleons. Those authors have calculated the Skyrmion energy density as a function of the baryon number. On the basis of ideas about the quark–gluon plasma,[56] it is suggested that, if the spatial density of the Skyrmion energy is larger than the energy density in the nucleon by a factor of two, then it is reasonable to assume that the Skyrme model allows a transition to a quark–gluon plasma. The method of adiabatic invariants, first applied to the Skyrme model in Ref. 50, can be used to show that the energy density grows rapidly with increasing baryon number. Calculations confirm the result of Bogomol'nyĭ and Fateev for the classical mass of a multibaryon state, $E = 18.46B(B + 0.8726)$. The increase of the rms radius can be approximated by the dependence $r_{RMS} = 2.17B^{1/2}$. Using this, it is easy to estimate the dependence of the energy density on the baryon number for large B. Already at $B = 13$, the energy density is 2 times larger than for $B = 1$. Therefore, this criterion does not prohibit the formation of a quark–gluon plasma even in nuclear systems with a small number of nucleons. This result can probably be extended to other modifications of the Skyrme model (in relation to this, see Ref. 57, where the asymptotic behavior of the mass at large baryon numbers was studied for models with a Lagrangian containing stabilizing terms of sixth order in the pion field).

THE SKYRMION–SKYRMION INTERACTION. COLLECTIVE COORDINATES

The study of nucleon–nucleon forces in the Skyrme model probably began with Ref. 58. It was shown that the dependence on the Euler angles fixing the orientation of the isotopic quantization axes relative to the spin axes in the Skyrmion–Skyrmion interaction is similar to the dependence obtained in rotations of quark hedgehog wave functions. This fact was used to analyze the central, spin–spin, and tensor components of the nucleon–nucleon potential. Comparison with the corresponding components of the one-boson exchange potential and the semiphenomenological Paris potential showed that there is qualitative agreement.

In most of the studies published up to now,[59] the study of the Skyrmion–Skyrmion interaction is based on the following representation for the field in the sector with baryon number $B = 2$:

$$U_B(\mathbf{x};\ \mathbf{r}_1,\ A_1;\ \mathbf{r}_2,\ A_2) = A_1U_0(\mathbf{x}-\mathbf{r}_1)A_1^+A_2U_0(\mathbf{x}-\mathbf{r}_2)A_2^+ = U_1U_2, \quad (62)$$

where $U_0(\mathbf{x}-\mathbf{r}_i)$ for $i = 1, 2$ is a stationary solution localized at $\mathbf{r}_i$ and A_i is a collective coordinate describing the rotation. The Skyrmion–Skyrmion potential is defined as

$$V_{SK-SK}(\mathbf{r}_{12}) = -\int d^3x\,\{\mathcal{L}(U_1U_2) - \mathcal{L}(U_1) - \mathcal{L}(U_2)\}, \quad (63)$$

where $\mathbf{r}_{12}$ is the relative position coordinate for the Skyrmions.

Other definitions of the Skyrmion–Skyrmion potential differ by the definition of the relative coordinate. For example, Skyrme proposed the introduction, in the $B = 2$ sector, of the coordinate determining the relative location of the points at which $U(\mathbf{r}_i) = -1$. It is easy to see that this definition does not coincide with that in (62), particularly at short distances. Moreover, at large relative separations both definitions lose the clear physical meaning which we usually associate with this concept for point particles. The Skyrmion separation can be defined using a constraint, as was done in Ref. 60. The entire set of possible definitions says only that the relative coordinate $\mathbf{r}_{12}$ plays the role of a collective field coordinate which can conveniently be used in variational calculations. For example, in the sector with $B = 2$ a set of variational configurations of the form (62) has been considered in Ref. 61. The static energy is calculated as a function of the relative isospin orientation $A = A_1^+A_2$ and the half-distance $s = |\mathbf{r}_1 - \mathbf{r}_2|/2$ between Skyrmions localized at $\mathbf{r}_1$ and $\mathbf{r}_2$. It is shown numerically that the minimum energy M_d is reached for a configuration $U_d(\mathbf{x})$ with $A = i\tau^2$. The explicit form of the minimizing static configuration is $U_d(\mathbf{x}) = U_1(\mathbf{x} + sd\hat{\mathbf{z}})\tau^2U_1(\mathbf{x} - sd\hat{\mathbf{z}})\tau^2$, where $s_d = 2.8/eF_\pi$. For our values of the parameters, $e = 4.84$ and $F_\pi = 108$ MeV, we find $s_d = 1.1$F and $M_d - 2M_1 = -24$ MeV. Next the collective variables $A(t)$ and $\mathbb{R}(t)$ are introduced: $U(\mathbf{x},t) = AU_d[\mathbb{R}(t)x]A^+$. The Hamiltonian reduces to the quadratic form

$$H = M_d + K_iX_{ij}K_j/2 + K_iZ_{ij}L_j + L_iY_{ij}L_j/2, \quad (64)$$

where the static energy M_d and the inertia tensors X, Y, and Z are functionals of $U_d(\mathbf{x})$. Here $\mathbf{K}$ and $\mathbf{L}$ are the body-fixed isospin and spin operators, related to the "stationary" operators $\mathbf{I}$ and $\mathbf{J}$ by orthogonal transformations. The solution of the problem reduces to diagonalization of a matrix with dimension $(2i + 1)(2j + 1)$ in the Hilbert space with basis formed by the vectors $|i, i_3, k_3\rangle |j, j_3, l_3\rangle$ $(-i \leqslant i_3, k_3 \leqslant +i$ and $-j \leqslant j_3, l_3 \leqslant +j)$. Two states with similar features and the deuteron quantum numbers $i = 0$ and $j = 1$ were found (Table II). The binding energy was not estimated, since the difference of the zero-point energies of the field in the presence of a soliton with $B = 2$ and of two separated solitons with $B = 1$ was not calculated. The appearance of two similar states with the deuteron quantum numbers may indicate that the solution is only approximate or, possibly, that the Skyrme Lagrangian is not a good enough approximation to low-energy QCD to correctly reproduce the ordering of the nuclear levels.

NUCLEON–NUCLEON FORCES

The product ansatz (62) has certain technical advantages. It allows the potential to be calculated directly from the total energy of a static configuration, and the different spin–isospin channels are easily separated. It automatically

TABLE II. Some static characteristics of states with the deuteron quantum numbers.

State	Energy, MeV	$\langle r^2\rangle^{1/2}$, F	μ^*, μ_{nuc}	Q, F^2
d	1764	1.3	0.69	0.22
d'	1767	1.3	0.66	0.21
Deuteron	1875.6	2.095	0.8574	0.2859

*μ is the magnetic moment and Q is the quadrupole moment of the states.

gives the correct asymptotic form of the one-pion exchange potential and permits a simple and qualitatively useful analysis of the potential to be carried out in terms of the quantum numbers exchanged between Skyrmions.

In addition to the one-pion nature of its asymptote, the phenomenological nucleon–nucleon potential is also, as is well known, characterized by a strong repulsive central core, which in traditional models arises from ω-meson exchange. Finally, there is yet another characteristic feature—central attraction in the intermediate region, arising from exchange of correlated pairs of pions (σ mesons). It is this attraction which ensures binding in nuclear systems.

It might be hoped that all these features will be reproduced in a systematic theory of nucleons as the solitons of a scalar field. At present, when calculating the components of the nucleon–nucleon potential V_{NN} it is no longer necessary to use quark models for comparison.

It is sufficient to use the projection theorems[62] for the matrix elements $R_{ia} = \mathrm{Tr}[A\tau_i A^+\tau_a]/2$

$$\langle B \mid R_{ia} \mid B' \rangle = \Delta(B, B')\, S_i T_a \qquad (65)$$

for an arbitrary baryon state $|B\rangle$, where S_i and T_a are the generalized spin and isospin operators and $\Lambda(B, B')$ are state-dependent geometrical coefficients, to verify that in general

$$V_{NN}(r) = V_C(r) + \mathbf{T}_1\mathbf{T}_2\,[\mathbf{S}_1\cdot\mathbf{S}_2 V_{SS}(r) + S_{12}V_T(r)] + \text{higher rank tensors}. \qquad (66)$$

Here we have used the standard component notation in the theory of the nucleon–nucleon potential: $V_C(r)$ is the central, $V_{SS}(r)$ the spin–spin, and $V_T(r)$ the tensor potential. Of course, instead of the generalized spin and isospin operators, we can use the ordinary spin and isospin Pauli matrices,[63,64] if we use relations analogous to

$$\langle N' \mid A_k \boldsymbol{\tau} A_k^+ \mid N \rangle = -\frac{1}{3}\langle N' \mid \boldsymbol{\sigma}^k(\boldsymbol{\tau}\cdot\boldsymbol{\tau}^k) \mid N \rangle, \qquad (67)$$

where the matrices $\boldsymbol{\sigma}^k$ and $\boldsymbol{\tau}^k$ operate on the variables of the k-th nucleon of the interacting pair. The G-parity structure is revealed by comparison with the nucleon–antinucleon potential.[65] The radial dependence of the potentials found in Ref. 66 is shown in Fig. 6 for the parameters $F_\pi = 186$ MeV and $e = 3.4$, which gives $g_{\pi NN} = 14.3$. The numerical calculations[58,66–68] can be summarized as follows:

1) V_{SS} and V_T are in good agreement with the corresponding components of the Paris potential[69] at large distances ($r > 2$ F);

2) the ρ-meson-like contributions, isolated using the G-parity dependence of the potentials V_{SS} and V_T, have the correct order of magnitude (these contributions mimic $m_\rho \sim 500$ MeV and $f^2_{\rho NN}/4\pi \approx 1.2$–$1.8$);

3) the central part V_C has a repulsive core of about 1 GeV.

Therefore, on the whole the qualitative agreement with the behavior of the nuclear forces is remarkable. However, the core in the nucleon–nucleon potential V_C is too wide and suppresses the intermediate region of the possible attractive potential. This makes it necessary to consider nonadiabatic contributions,[70] contributions arising from deformations of the Skyrmions during interaction,[71] and also contributions corresponding to pion-field fluctuations in the vicinity of the interacting Skyrmions [$\sim O(N_c)$] (Ref. 72). In spite of the notable success of investigations of the role of these contributions, an attraction in the intermediate region sufficient for nucleus formation in the traditional sense has not yet been obtained. It is probably necessary to make some modification of the model so as to organically include an additional scalar field, which would lead to additional attraction in the nucleon–nucleon forces.[73] Here mention should be made of the important role of one-loop corrections in weakening the strength of the repulsive core in the intermediate region.

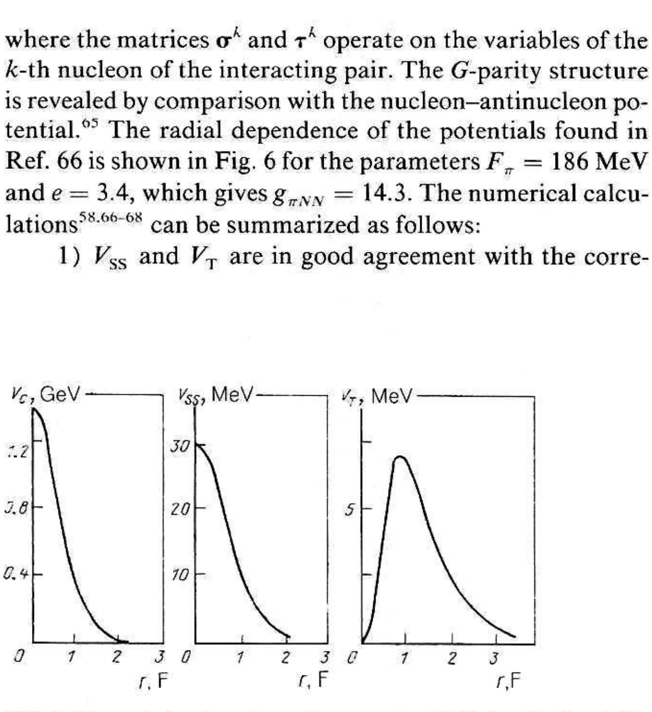


FIG. 6. The central, spin–spin, and tensor potentials[66] given by Eqs. (62), (63), and (66).

THE SPIN–ORBIT *NN* INTERACTION

It can be shown that the spin–orbit and quadratic spin–orbit *NN* forces arise naturally in the Skyrme model if the orbital degrees of freedom are quantized along with the internal degrees of freedom.

The Hamiltonian for a slowly moving particle is constructed by the introduction of time-dependent coordinates $\mathbf{R}_{SK}(t)$ for the center of the Skyrmion. For an individual soliton the field $U(\mathbf{r}) = A(t)U(\mathbf{r} - \mathbf{R}^{(t)}_{SK})A^+(t)$ can be used to study the translational motion of the Skyrmion and to calculate the effective Hamiltonian for the translational motion, $\hat{H} = \hat{p}^2/2M_{SK}$, where p is the Skyrmion momentum. This is a completely nontrivial result.[74] Substitution of the product ansatz into the Lagrangian density generates the spin–orbit and quadratic spin–orbit interactions by terms containing time derivatives. According to the quantization procedure described above, one-nucleon operators for noninteracting particles can be introduced. As a result, the contribution of the quadratic terms of the Lagrangian to the spin–orbit interaction, V^2_{LS}, has the form

$$V^{(2)}_{LS} = -\frac{\pi F_\pi^2}{18 M_N \lambda r}\,\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\,(\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)})\cdot\mathbf{l} \times \int_0^\infty R^2\,dR \int_{-1}^{+1} dz\,\frac{\sin^2 F(r_1)\sin^2 F(r_2)}{r_1^2}\left(\frac{1}{2}r - Rz\right). \qquad (68)$$

In this expression λ is the moment of inertia and

$$r_{1,2} = \sqrt{R^2 - \frac{r^2}{4} \mp Rrz}\,; \quad z = \hat{R}\cdot\hat{r}.$$

Here we are omitting the contribution of the Skyrme term. The resulting potentials at large distances are proportional to $F^2(r)$ and are associated with the exchange of two pions in a state with isospin 1 (ρ-meson exchange).

More details about spin–orbit nucleon–nucleon forces can be found in Ref. 75.

ELECTROMAGNETIC MESON EXCHANGE CURRENTS

We shall restrict ourselves to isoscalar meson exchange currents. As noted above, in the Skyrme model and all its generalizations, where baryons are the solitons of meson fields, the isoscalar-current operator is proportional to the operator for the anomalous baryon current. The baryon current is independent of the details of the dynamics in the model and is essentially determined by the mapping of the compactified space onto the chiral sphere. On the other hand, at the level of the SU(3) symmetry it is obtained as the Noether current of the Wess–Zumino interaction required by the chiral anomaly. In this sense, the isoscalar form factors of nucleons and nuclei are determined in an almost model-independent manner (in the approximation of topological solitons).

The product ansatz allows us to split the isoscalar current (half the baryon current) into two parts—the sum of two one-nucleon currents and a two-particle operator which should be identified as the exchange current:

$$J_\mu(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}) = J_\mu(\mathbf{r}-\mathbf{r}_1) + J_\mu(\mathbf{r}-\mathbf{r}_2) + J_{\mu,\mathrm{ex}}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}). \qquad (69)$$

Using the projection theorems, the Fourier transform of the spatial part of the exchange current can be written as[38]

$$\mathbf{J}_{\mathrm{ex}} = -\frac{i}{72\pi^2\lambda}\,\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 \exp\left[\frac{1}{2}\,i\mathbf{q}\cdot\mathbf{R}\right]\mathbf{q}\int d^3r\,\exp\,[i\mathbf{q}\cdot\mathbf{r}]\sin^2 F\,(|\mathbf{R}+\mathbf{r}|) \times\left\{\left[\alpha(r)+\frac{1}{3}\gamma(r)\right]\boldsymbol{\Sigma}+\gamma(r)\left[\boldsymbol{\Sigma}\hat{\mathbf{r}}\,\hat{\mathbf{r}}-\frac{1}{3}\boldsymbol{\Sigma}\right]\right\}, \qquad (70)$$

where $\boldsymbol{\Sigma} = \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2$ and the auxiliary functions $\alpha(R)$ and $\gamma(R)$ are determined by the chiral angle $F(R)$:

$$\left.\begin{aligned}\alpha(R) &= \frac{1}{2R}\sin 2F\,(R);\\ \gamma(R) &= F'\,(R) - \frac{1}{2R}\sin 2F\,(R).\end{aligned}\right\} \qquad (71)$$

The Fourier transform of the time component of the exchange current

$$\rho_{\mathrm{ex}} = \frac{i}{16\pi^2}\,\mathbf{q}\int d^3r\,\exp\,[i\mathbf{q}\cdot\mathbf{r}]\,\mathrm{Tr}\,\{U_1^+\nabla U_1 \times U_2\nabla U_2^+\} \qquad (72)$$

can be calculated in a similar manner. Here the indices 1, 2 correspond to the two Skyrmions and R is their separation. This exchange current operator can be verified in calculations of nuclear electromagnetic form factors. One such calculation was made in Ref. 76 for the deuteron. That calculation used the wave function of the Paris potential, which was also used for averaging the current operator (as if the Skyrme model reproduced the nucleon–nucleon potential leading to such a wave function). In Fig. 7 we give the deuteron magnetic form factor.[38] Use of the phenomenological profile function determined from a dipole fit of the isoscalar electric one-nucleon form factor leads to agreement between the calculated deuteron magnetic form factor and the experimental form factor up to $q \sim 1.7$ GeV/c. It should be noted that the calculation of the meson exchange-current contributions in this scheme is unrelated to perturbation theory, and it is almost impossible to indicate which Feynman diagrams correspond to the calculated quantity (however, see Ref. 77 in this regard).

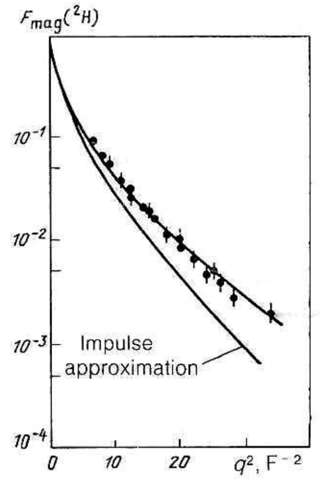


FIG. 7. Magnetic form factor of the deuteron.[38] The contribution from the total topological current and the impulse approximation are shown, and exchange-current contributions are absent. The chiral angle was determined from the isoscalar electric form factor of the nucleon [see (50)].

NUCLEAR MATTER

Nuclear matter is a system with large baryon number and constant spatial density of the baryon number. The average binding energy per nucleon in the ground state (16 MeV/nucleon) is small compared with the nucleon mass. The multibaryon states which we have considered do not possess such properties in the limit of large baryon number. A number of attempts have been made to obtain the nuclear-matter equation of state in the Wigner–Seitz approximation.[78] As was shown in Ref. 79, the result of such studies depends significantly on the details of the short-range repulsion (and, consequently, on the contribution of terms with higher derivatives in the Lagrangian) and does not ensure the saturation of the forces in the system.

A Skyrmion with a given fixed orientation in internal space corresponds to a coherent superposition of different rotational states: N, Δ, (5/2, 5/2),... . If the interaction between Skyrmions is smaller than the splitting in the rotational band, Skyrmions will mainly be found in the nucleon state with a small Δ admixture. In the opposite case, Skyrmions simply oscillate in the vicinity of the configuration with maximum attraction.

If we recall that, in general, the energy gain per nucleon due to optimal orientation in internal space is determined by the relative position of the Skyrmions in coordinate space, we see that it may be possible for ordered nuclear matter to be formed.

The authors of Ref. 80 considered the physically interesting case of cubic symmetry—a neutron crystal. The forces acting between Skyrmions, which strongly depend on the relative orientation in internal space, can cause the ordered crystalline configuration to be preferred. The main factor working against this model is the zero-point fluctuations of the Skyrmions near the localization points. For the density range studied in Ref. 80, $0.1 < \rho_B < 1.5$ F^{-3}, the result should not have a strong dependence on the form of the stabilizing term in the Lagrangian. Let us choose a variational field configuration in the form of a product of Skyrmions with some relative orientation and centered at points $\mathbf{R}_i$:

$$U_0(\mathbf{r}) = \prod_i A_i \exp\left(i F(|\mathbf{r}-\mathbf{R}_i|)\,\boldsymbol{\tau}\cdot\frac{\mathbf{r}-\mathbf{R}_i}{|\mathbf{r}-\mathbf{R}_i|}\right) A_i^+. \quad (73)$$

The SU(2) matrices $A_i = a_0^{(i)} + \mathbf{a}^{(i)}\cdot\boldsymbol{\tau}$ specify the orientation of a Skyrmion localized near $\mathbf{R}_i$. Using the fact that for a pair of well separated Skyrmions in the chiral limit the interaction potential is

$$\underset{r\to\infty}{V}(\mathbf{r},\ A) \sim e_{ij}(A)\frac{\partial}{\partial r_i}\frac{\partial}{\partial r_j}\frac{1}{r}, \quad (74)$$

where

$$e_{ij}(A) = \frac{1}{2}\operatorname{Tr}(A^+\tau_i A\tau_j);$$

$$\lim_{r\to\infty} V(r,\ A) \sim \frac{3(\mathbf{a}\cdot\mathbf{r}) - \mathbf{a}^2\mathbf{r}^2}{r^5}, \quad (75)$$

we conclude that the most attractive asymptotic potential satisfies the condition

$$\mathbf{a}\cdot\mathbf{r} = 0, \quad \mathbf{a}^2 = 1. \quad (76)$$

Among the configurations which are nondegenerate under translations, satisfy these conditions for each pair of neighboring Skyrmions, and carry unit topological charge, there is a configuration in which the points $\mathbf{R}_i$ obeying $U(\mathbf{R}_i) = -1$ form a cubic lattice.

The assumption (73) about $U_0(\mathbf{r})$ leads to a crystal whose classical energy is very close to the lower topological estimate. For example, the classical energy per baryon is $\sim 32F_\pi/e$ (the topological estimate is $29.6F_\pi/e$), which for our values $F_\pi = 129$ MeV and $e = 5.45$ gives a contribution to the binding energy of about 105 MeV (the mass of the free Skyrmion has been subtracted).

If we introduce collective variables $A(t)$ corresponding to the overall orientation of the crystal in isospin space, $U(\mathbf{r},t) = A(t)U_0(\mathbf{r})A^{-1}(t)$, then, just as in the one-nucleon case, we can determine the isospin by integrating the density of the isovector current and the angular momentum over the density and arrive at an expression for the total energy taking into account the rotational degrees of freedom for the n-baryon state:

$$E^{\text{tot}} = nM_{\text{cl}} + \frac{1}{2n\lambda} I^{\text{tot}}(I^{\text{tot}}+1), \quad (77)$$

where λ is the moment of inertia in isospin space. It should be emphasized that only the total isospin I_{tot} satisfies the angular-momentum algebra and is a quantum observable.

For finite nuclei this rotational energy leads to the well-known symmetry energy $25(N-Z)^2/A$ MeV and to obvious relations for the splittings between analog levels in nuclei.[81]

For the neutron crystal studied in Ref. 80, Eq. (77) gives the value $M_{\text{cl}} + 1/8\lambda$ for the energy per nucleon in the crystal, which increases the binding energy even more (!). However, an estimate of the kinetic energy of vibrations about the Skyrmion position shows that the crystal is very unlikely to be a stable state.

An attempt was made in Ref. 82 to theoretically calculate a cubic lattice, where the number of nearest-neighbor Skyrmions satisfying the optimal condition (76) was reduced to twelve. The variational estimate of the energy of this configuration indicates that it may be favored at densities significantly larger than the normal nuclear density ρ_0. Reference 83 is also devoted to detailed investigation of the symmetry of a Skyrmion crystal.

CONCLUSIONS

The richness of the structures presented by this model for investigation should be noted; most investigations have been carried out only in the semiclassical approximation. There is a clear manifestation of an analogy between the results obtained in quark models and in the Skyrme model (the stiffness of the states under uniform deformations, the type of spectrum—rotational bands in spin and isospin for multibaryon states, the form factors, the nature of the repulsive core in nucleon–nucleon forces, and so on). This analogy is probably not accidental and supports the fact that the Skyrme model is only an effective model representing degrees of freedom which are convenient to work with.

The model is useful for studying many hypothetical situations, including some which are difficult to study by perturbation theory (meson exchange currents).

The model-independent results which have been obtained so far are in very good agreement with experiment.

The model predicts relations between quantities which are not at all obvious *a priori* in other theories of the strong interaction (relations between the phase shifts in different partial waves of πN scattering[84]).

Of course, the broad spectrum of values of the constants F_π and e used in the literature indicates that it is impossible to obtain quantitative agreement with all known observables using a single set of constants. Moreover, the inclusion of a larger number of degrees of freedom in the quantization of the model and in the calculation of the effective collective Hamiltonian probably has a strong effect on the values of the fitted constants in the baryon sector. The choice of stabilizing term in the original form, as it was made in the studies by Skyrme, is very strong. A fundamental ingredient of any generalized Skyrme model might be the scalar dilaton field, which appears in the effective Lagrangian owing to the conformal anomaly in quantum chromodynamics.[85,86] Study of the two-current nucleon observables[87] (the nucleon polarizability and the nucleon structure function) gives information about the role of terms with higher derivatives in the effective Lagrangian.

The most attractive feature of the Skyrme model is the unification of the ideas and methods first used to study baryon and multibaryon nuclear states.

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[1)] Here and below we restrict ourselves to the number of flavors $N_f = 2$.

[2)] In general, multibaryon states can have static characteristics which differ from those of nuclear states with the same quantum numbers.

[1] T. H. R. Skyrme, Proc. R. Soc. London, Ser. A **260**, 127 (1961); Nucl. Phys. **31**, 556 (1962).
[2] G. 't Hooft, Nucl. Phys. **B75**, 461 (1974).
[3] E. Witten, Nucl. Phys. **B160**, 57 (1979).
[4] L. D. Faddeev and V. E. Korepin, Phys. Rep. **42C**, 1 (1978); R. Rajara-

man, *Solitons and Instantons* (North-Holland, Amsterdam, 1982) [Russ. transl., Mir, Moscow, 1985].
[5]D. I. D'yakonov and M. I. Éides, Pis'ma Zh. Eksp. Teor. Fiz. **38**, 358 (1983) [JETP Lett. **38**, 433 (1983)]; A. Andrijanov and Yu. Novozhilov, Phys. Lett. **153B**, 422 (1985); N. I. Karchev and A. A. Slavnov, Preprint Imperial/TP/84-85/28, Imperial College, London (1985).
[6]I. J. R. Aitchison, Acta Phys. Pol. **B18**, 191 (1987).
[7]M. K. Volkov, Fiz. Elem. Chastits At. Yadra **17**, 433 (1986) [Sov. J. Part. Nucl. **17**, 186 (1986)].
[8]S. Weinberg, Phys. Rev. Lett. **18**, 188 (1967); S. Coleman, I. Wess, and B. Zumino, Phys. Rev. **177**, 2239 (1969); D. V. Volkov, Fiz. Elem. Chastits At. Yadra **4**, 3 (1973) [Sov. J. Part. Nucl. **4**, 1 (1973)].
[9]S. Gerstein, R. Jackiw, B. W. Lee, and S. Weinberg, Phys. Rev. D **3**, 2486 (1971); V. N. Pervushin, Teor. Mat. Fiz. **22**, 291 (1975); M. K. Volkov, D. I. Kazakov, and V. N. Pervushin, Teor. Mat. Fiz. **28**, 46 (1976).
[10]M. K. Volkov and V. N. Pervushin, Fiz. Elem. Chastits At. Yadra **6**, 632 (1975) [Sov. J. Part. Nucl. **6**, 254 (1975)]; D. I. Kazakov, V. N. Pervushin, and S. V. Pushkin, J. Phys. A **11**, 2093 (1978).
[11]L. D. Faddeev, Pis'ma Zh. Eksp. Teor. Fiz. **21**, 141 (1975) [JETP Lett. **21**, 64 (1975)].
[12]G. Adkins, C. Nappi, and E. Witten, Nucl. Phys. **B228**, 552 (1983).
[13]N. K. Pak and H. Ch. Tze, Ann. Phys. (N.Y.) **117**, 164 (1979).
[14]A. A. Slavnov, Nucl. Phys. **B31**, 301 (1971).
[15]L. D. Faddeev, Preprint TH-2188, CERN, Geneva (1976).
[16]Y. P. Rybakov and V. I. Sanyuk, Preprint NBI-HE-81-49.
[17]L. D. Faddeev, Lett. Math. Phys. **1**, 289 (1976).
[18]N. S. Manton and P. J. Ruback, Phys. Lett. **181B**, 137 (1986).
[19]V. A. Andrianov and V. Yu. Novozhilov, Yad. Fiz. **43**, 983 (1986) [Sov. J. Nucl. Phys. **43**, 628 (1986)].
[20]S. Kahana and G. Ripka, Nucl. Phys. **A429**, 462 (1984).
[21]D. I. Dyakonov, V. Yu. Petrov, and P. V. Pobylitsa, Preprint LNPI-1297, Leningrad Nuclear Physics Institute, Leningrad (1987).
[22]J. Goldstone and F. Wilczek, Phys. Rev. Lett. **47**, 986 (1981).
[23]A. P. Balachandran, V. P. Nair, S. G. Rajeev, and A. Stern, Phys. Rev. Lett. **49**, 1124 (1982); Phys. Rev. D **27**, 1153 (1983).
[24]R. MacKenzie, Preprint NSF-ITP-84-135 (1984); Preprint DAMTP/86-18 (1986).
[25]J. Wess and B. Zumino, Phys. Lett. **37B**, 95 (1971).
[26]A. P. Balachandran, Preprints COO-3533-314, SU-4222-314, Syracuse University, Syracuse, N.Y. (1985); I. Zahed and G. E. Brown, Phys. Rep. **142**, 1 (1986).
[27]D. I. D'yakonov and V. Yu. Petrov, Preprint No. 967 [in Russian], Leningrad Nuclear Physics Institute, Leningrad (1984).
[28]N. N. Bogolyubov, Ukr. Mat. Zh. **2**, 3 (1950).
[29]W. Pauli and S. M. Dancoff, Phys. Rev. **62**, 85 (1942).
[30]L. C. Biedenharn, Y. Dothan, and M. Tarlini, Phys. Rev. D **31**, 649 (1985); V. A. Nikolaev and É. Roka, Kratk. Soobshch. No. 14-86, JINR, Dubna (1986), p. 28.
[31]L. C. Biedenharn and J. D. Louck, *Angular Momentum in Quantum Physics* (Addison-Wesley, Reading, Mass., 1981) [Russ. transl., Mir, Moscow, 1984].
[32]E. Witten, Nucl. Phys. **B223**, 422 (1983).
[33]G. G. Bunatyan, Communication R2-86-408 [in Russian], JINR, Dubna (1986).
[34]M. Karliner, Preprint SLAC-PUB-4268, SLAC, California (1987).
[35]O. G. Tkachev, "Some properties of dibaryons in the Skyrme model," Thesis [in Russian], Far-East State University, Vladivostok (1986).
[36]E. Braaten, S. M. Tse, and Ch. Wilcox, Phys. Rev. Lett. **56**, 2008 (1986); Phys. Rev. D **34**, 1482 (1986).
[37]V. A. Nikolaev and O. G. Tkachev, Communication R4-87-422 [in Russian], JINR, Dubna (1987).
[38]E. M. Nyman and D. O. Riska, Preprint HU-TFT-88-2, Helsinki (1988).
[39]V. A. Nikolaev, in: *Proceedings of the International Conference on the Theory of Few-Particle and Quark–Hadron Systems* [in Russian], D4-87-692, JINR, Dubna (1987), p. 265.
[40]V. A. Nikolaev and É. K. Roka, Communication R4-86-514 [in Russian], JINR, Dubna (1986).
[41]V. A. Nikolaev, in: *Proceedings of the Eighth International Seminar on Problems of High Energy Physics* [in Russian], D1,2-86-688, JINR, Dubna (1986), Vol. 1, p. 78.
[42]J. Dey, H. Dey, and J. Le Tourneux, Nuovo Cimento **A91**, 15 (1986).
[43]K. Fujii and A. P. Kobushkin, Preprint ITP-87-1E, Kiev (1987); K. Fujii *et al.*, Phys. Rev. D **35**, 1896 (1987).
[44]M. I. Eremeev *et al.*, Preprint No. 77, ITEP, Moscow (1987).
[45]A. Hayashi and G. Holzwarth, Phys. Lett. **140B**, 175 (1984); Ch.-K. Lin, Nucl. Phys. **A449**, 673 (1986).
[46]J. M. Eisenberg, A. Erell, and R. R. Silbar, Phys. Rev. C **33**, 1531 (1986).
[47]U.-G. Meissner and U. B. Kaulfus, Phys. Rev. C **30**, 2058 (1984).
[48]M. Kutschera, C. J. Pethick, and D. G. Ravenhall, Phys. Rev. Lett. **53**, 1041 (1984).
[49]R. L. Jaffe and C. L. Korpa, Nucl. Phys. **B258**, 468 (1985); A. P. Balachandran *et al.*, Nucl. Phys. **B256**, 525 (1985); Phys. Rev. Lett. **52**, 887 (1984).
[50]E. B. Bogomol'nyĭ and V. A. Fateev, Yad. Fiz. **37**, 228 (1983) [Sov. J. Nucl. Phys. **37**, 134 (1983)].
[51]H. Weigel, B. Schwesinger, and G. Holzwart, Phys. Lett. **168B**, 321 (1986).
V. A. Nikolaev and O. G. Tkachev Communication R4-86-515 [in Russian], JINR, Dubna (1986).
[53]V. B. Kopeliovich and B. E. Shtern, Pis'ma Zh. Eksp. Teor. Fiz. **45**, 165 (1987) [JETP Lett. **45**, 203 (1987)].
[54]V. V. Burov, V. K. Luk'yanov, and A. I. Titov, Fiz. Elem. Chastits At. Yadra **15**, 1249 (1984) [Sov. J. Part. Nucl. **15**, 558 (1984)]; V. V. Burov and V. N. Dostovalov, Z. Phys. A **326**, 245 (1987).
[55]G. Kalbermann and J. M. Eisenberg, J. Phys. G **13**, 1029 (1987).
[56]G. Baym, in: *Common Problems in Low- and Medium-Energy Nuclear Physics*, edited by B. Castel *et al.* (Plenum, New York, 1979), p. 213; B. Muller, *The Physics of the Quark–Gluon Plasma, Lecture Notes in Physics* (Springer Verlag, Berlin, 1985), p. 225.
[57]V. I. Inozemtsev, Communication R2-86-317 [in Russian], JINR, Dubna (1986).
[58]A. Jackson, A. D. Jackson, and V. Pasquier, Nucl. Phys. **A432**, 567 (1985).
[59]S. Saito, in *Proceedings of the International Conference on the Theory of Few-Body and Quark–Hadronic Systems*, Dubna (1987), p. 244.
[60]J. J. M. Verbaarschot *et al.*, Preprint ILL-TH-86-56, University of Illinois (1986); Nucl. Phys. **A468**, 520 (1987).
[61]F. Braaten and L. Carson, Phys. Rev. Lett. **56**, 1897 (1986).
[62]T. Otofuji *et al.*, Preprint DPNU-86-19 (1986).
[63]E. M. Nyman and D. O. Riska, Preprint HU-TFT-85-47, Helsinki (1985).
[64]E. M. Nyman and D. O. Riska, Preprint HU-TFT-88-2, Helsinki (1988).
[65]H. Walliser, A. Hayashi, and G. Holzwarth, Nucl. Phys. **A456**, 717 (1986).
[66]T. Otofuji *et al.*, Preprint DPNU-85-26, Nagoya University (1985).
[67]R. Vinh Mau *et al.*, Phys. Lett. **150B**, 259 (1985).
[68]H. Yabu and K. Ando, Prog. Theor. Phys. **74**, 750 (1985).
[69]R. Vinh Mau, in: *Mesons in Nuclei*, edited by M. Rho and D. H. Wilkinson (North-Holland, Amsterdam, 1979), Vol. 1, p. 151.
[70]E. M. Nyman and D. O. Riska, Preprint HU-TFT-85-17, Helsinki (1985).
[71]A. De Pace *et al.*, Nucl. Phys. **A457**, 541 (1986); M. Oka, K. F. Liu, and Hong Yu, Phys. Rev. D **34**, 1575 (1986); M. Oka, Phys. Rev. C **36**, 720 (1987).
[72]S. Saito, Preprint DPNU-87-32, Nagoya University (1987).
[73]I. Zahed, A. Wirzba, and U.-G. Meissner, Phys. Rev. D **33**, 830 (1986); A. D. Jackson and A. Jackson, Nucl. Phys. **A457**, 687 (1986); P. Jain, R. Johnson, and J. Schechter, Phys. Rev. D **35**, 2230 (1987).
[74]M. Oka, Phys. Lett. **175B**, 15 (1986).
[75]D. O. Riska and K. Dannbom, Preprint HU-TFT-87-26, Helsinki (1987).
[76]M. Ebbe, E. M. Nyman, and D. O. Riska, Phys. Rev. Lett. **57**, 3007 (1986); Nucl. Phys. **A468**, 473 (1987).
[77]M. Wakamatsu and W. Weise, Preprint TPR-87-20, University of Regensburg (1987).
[78]M. Kutshera and C. J. Pethick, Nucl. Phys. **A440**, 670 (1985).
[79]E. Wüst, G. E. Brown, and A. D. Jackson, Nucl. Phys. **A468**, 450 (1987).
[80]I. Klebanov, Nucl. Phys. **B262**, 133 (1985).
[81]D. I. D'yakonov, Yad. Fiz. **45**, 1592 (1987) [Sov. J. Nucl. Phys. **45**, 946 (1987)].
[82]D. I. Dyakonov and A. D. Mirlin, Preprint No. 1327, Leningrad Nuclear Physics Institute, Leningrad (1987).
[83]A. S. Goldhaber and N. S. Manton, Phys. Lett. **198B**, 231 (1987).
[84]G. Holzwarth and B. Schwesinger, Rep. Prog. Phys. **49**, 825 (1986).
[85]A. A. Andrianov, V. A. Andrianov, V. Yu. Novozhilov, and Yu. V. Novozhilov, Pis'ma Zh. Eksp. Teor. Fiz. **43**, 557 (1986) [JETP Lett. **43**, 719 (1986)]; V. Andrianov and V. Novozhilov, Preprint IC/87/411, ICTP, Trieste (1987).
[86]H. Reinhardt and B. Kämpfer, Preprint NBF-HE-87-16, University of Copenhagen (1987); D. Ebert and H. Reinhardt, Communication E2-86-274, JINR, Dubna (1986).
[87]M. Chemtob, Preprint SPhT/87-100, Saclay (1987); Nucl. Phys. **A473**, 613 (1987).

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