New tendencies in the development and application of the method of hyperspherical functions

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The main results obtained during the last ten years in the framework of the method of hyperspherical functions are reviewed. New developments of the mathematical formalism of the method are presented. The main results on the application of the method in the theory of many-particle reactions are given. The results of application of the method of hyperspherical functions in the investigation of collective excited states of light nuclei, and also of elastic and inelastic cross sections for reactions with ions are reviewed. An approach to the solution of the Coulomb problem in a hyperspherical basis is described, and a method for taking into account simultaneously the nuclear and Coulomb interactions is formulated. The main results of the application of the method of hyperspherical functions for the study of the structure of multiquark systems are presented. A review of results on the development of the hyperspherical approach to the solution of the Faddeev equations is given.

INTRODUCTION

The last two decades in nuclear physics have been characterized by the rapid development of few-body theory, and this has stimulated experimental investigation of various properties of few-nucleon nuclei and of nuclear reactions on such nuclei. In this theory, one of the methods often employed is the method of expansion with respect to a basis of hyperspherical function, which became popular quite quickly after the publication of the well-known papers of Simonov and Badalyan.[1] The results of the first decade of the development and application of the method of hyperspherical functions (MHF) was summarized in Refs. 2–5. During the last decade new trends in the development of the MHF have occurred, making possible further theoretical development of its applications.

For example, the Raynal–Revai transformation has begun to play an important part in the theory of a hyperspherical basis, and its use has led to an important simplification of the computational technique. From the point of view of application of the method, successes have been achieved in the description of the continuous spectrum of many-particle systems and processes. In recent years the method has been widely used in atomic and molecular physics and in elementary-particle physics (multiquark systems). A way of "hybridizing" the methods of the Faddeev equations and hyperspherical functions has been found.

This has made it necessary to review the progress made during the decade and to demonstrate the wide applicability of the hyperspherical basis in different fields of development of theoretical physics.

In the review we present new developments of the mathematical formalism of the method of hyperspherical functions: the Raynal–Revai transformation for three and four bodies, the algebra of hyperspherical functions, and the problem of symmetrization in the hyperbasis not only for systems with identical particles but also for systems containing a particle with different mass.

We then give the main results on the application of the method of hyperspherical functions in the theory of many-particle reactions. We present a method for describing "truly" three- and four-particle scatterings, and we give the main results on the description of many-particle reactions of decay type. We give results on the application of the MHF in investigations of collective excited states of light nuclei and of the elastic and inelastic cross sections for reactions with heavy ions.

We then present the solution of the Coulomb problem in the hyperspherical basis. The bound states of charged particles, scattering, and many-particle Coulomb functions are described, and a method for taking into account simultaneously the nuclear and Coulomb interactions is formulated.

The main results of application of the MHF to the study of the structure of multiquark systems are presented.

Finally, we discuss a "hybrid" method for investigating few-particle systems, namely, a partial expansion in the theory of the Faddeev integral equations in the hyperspherical basis, and application of the hybrid method to the investigation of three-particle processes.

1. DEVELOPMENT OF THE MATHEMATICAL FORMALISM OF THE METHOD OF HYPERSPHERICAL FUNCTIONS

Unitary transformation in the method of hyperspherical functions

During the last decade the mathematical formalism of the MHF has been further developed. Some of the new developments in this area were presented in Ref. 6. An important aspect of these developments is the use of the Raynal–Revai transformation. This transformation was introduced for three bodies,[7] and was subsequently generalized for four bodies.[8] We shall consider briefly these transformations.

We consider a system of three particles with different masses. The corresponding hyperspherical functions form a complete set of orthonormalized basis functions and have the form

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$$\Phi_K^{l_{x_i} l_{y_i} m_{x_i} m_{y_i}}(\Omega_i) = N_K^{l_{x_i} l_{y_i}} (\cos\alpha_i)^{l_{x_i}} (\sin\alpha_i)^{l_{y_i}} P_n^{l_{y_i}+\frac{1}{2},\, l_{x_i}+\frac{1}{2}} (\cos 2\alpha_i) \times Y_{l_{x_i} m_{x_i}}(\hat{\mathbf{x}}_i)\, Y_{l_{y_i} m_{y_i}}(\hat{\mathbf{y}}_i). \quad (1)$$

where K is the "grand" orbital (hypermoment), l_{x_i}, l_{y_i}, m_{x_i}, m_{y_i} are the orbital angular momenta corresponding to the three-particle Jacobi coordinates $\mathbf{x}_i$, $\mathbf{y}_i$ and their projections; $P_n^{l_1 l_2}$ are Jacobi polynomials, $\Omega_i \equiv (\alpha_i, \hat{\mathbf{x}}_i, \hat{\mathbf{y}}_i)$ is a set of five angles, four of which determine the directions of the Jacobi vectors, while the fifth is introduced by the relations $x_i = \rho \cos\alpha_i$, $y_i = \rho \sin\alpha_i$; ρ is the hyperradius, and

$$N_K^{l_{x_i} l_{y_i}} = \sqrt{\frac{2n!\,(K+2)\,(n+l_{x_i}+l_{y_i}+1)!}{\Gamma(n+l_{x_i}+3/2)\,\Gamma(n+l_{y_i}+3/2)}},$$

$$n = \frac{1}{2}(K - l_{x_i} - l_{y_i}). \quad (2)$$

In Eqs. (1) and (2) the index i takes the values 1, 2, 3 corresponding to the three sets of Jacobi coordinates. If the hyperspherical functions are taken in the form (1), the matrix elements of the interaction of particles j and k, namely, matrix elements of the type $\langle \Phi_K(\Omega_i) | V(\mathbf{x}_k) | \Phi_K(\Omega_i) \rangle$, can be calculated comparatively easily. However, in a calculation with the same functions of the matrix element of interaction of a different pair of particles, for example, the matrix element $\langle \Phi_K(\Omega_i)\ V(\mathbf{x}_k)\ \Phi_K(\Omega_i) \rangle$, great difficulties arise. Raynal and Revai[7] introduced coefficients of a unitary transformation between the three-particle hyperspherical functions (1) defined on the different sets of Jacobi coordinates:

$$\Phi_{KLM}^{l_{x_i} l_{y_i}}(\Omega_i) = \sum_{l_{x_k} l_{y_k}} \langle l_{x_k} l_{y_k} | l_{x_i} l_{y_i} \rangle_{KL}\, \Phi_{KLM}^{l_{x_i} l_{y_i}}(\Omega_k), \quad (3)$$

where $\langle l_{x_k} l_{y_k} | l_{x_i} l_{y_i} \rangle_{KL}$ are the Raynal–Revai coefficients, and

$$\Phi_{KLM}^{l_{x_i} l_{y_i}}(\Omega_i) = \sum_{m_{x_i} m_{y_i}} (l_{x_i} l_{y_i} m_{x_i} m_{y_i} | LM)\, \Phi_K^{l_{x_i} l_{y_i} m_{x_i} m_{y_i}}(\Omega_i). \quad (4)$$

The introduction of this transformation was an important stage in the development of the theory of hyperspherical functions. In Ref. 7 Raynal and Revai obtained an analytic expression for these coefficients and found their connection with the Talmi–Moshinsky coefficients. Calculation using the analytic expression is a rather laborious problem, particularly for harmonics with large values of the grand orbital K. The computational technique can be greatly simplified by using recursion relations. A number of such relations were obtained by Smorodinskiĭ and Éfros[9] by using the connection with the Talmi–Moshinsky coefficients. In Ref. 10 a set of recursion relations was obtained that made it possible to find all possible Raynal–Revai coefficients without recourse to the cumbersome general formula. The use of the Raynal–Revai coefficients leads to an important simplification of the scheme for constructing hyperspherical functions with a given symmetry.[11]

We now consider a system of four particles with different masses. The corresponding hyperspherical functions have the form[8]

$$\Psi_{\mu K}^{l_x l_y l_z m_x m_y m_z}(\omega) = N_K^{l_x l_y} (\cos\alpha)^{l_x} (\sin\alpha)^{l_y} P_n^{l_y+\frac{1}{2},\, l_z+\frac{1}{2}} (\cos 2\alpha) \times N_{\mu+3/2}^{l_z K+\frac{3}{2}} (\cos\beta)^{l_z} (\sin\beta)^{K} P_m^{K+2,\, l_z+\frac{1}{2}} (\cos 2\beta) \times Y_{l_x m_x}(\hat{\mathbf{x}})\, Y_{l_y m_y}(\hat{\mathbf{y}})\, Y_{l_z m_z}(\hat{\mathbf{z}}), \quad (5)$$

where μ and K are the grand orbitals for four and three bodies, respectively, l_x, l_y, l_z, m_x, m_y, m_z are the orbital angular momenta corresponding to the four-particle Jacobi coordinates $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ and their projections, and $\omega \equiv (\alpha, \beta, \hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$ is a set of eight angles, six of which determine the directions of the Jacobi vectors, while two are introduced by the relations

$$\left.\begin{aligned} &x = \rho \cos\alpha \sin\beta; \quad y = \rho \sin\alpha \sin\beta; \quad z = \rho \cos\beta; \\ &N_n^{bc} = \sqrt{\frac{2d!\,(a+2)\,\Gamma(d+c+b+2)}{\Gamma(d+b+3/2)\,\Gamma(d+c+3/2)}}; \\ &2d = a - b - c; \quad 2m = \mu - K - l_z; \quad 2n = K - l_x - l_y. \end{aligned}\right\} \quad (6)$$

In Eqs. (5) and (6) we have omitted the indices that fix the set of four-particle Jacobi coordinates. In Ref. 8 a unitary transformation of the four-particle hyperspherical functions defined on different sets of Jacobi coordinates was introduced:

$$\Psi_{\mu KLM}^{l_x l_y l_{xy} l_z}(\omega) = \sum_{l'_x l'_y l'_{xy} l'_z K'} \langle l'_x l'_y l'_{xy} l'_z K' | l_x l_y l_{xy} l_z K \rangle_{\mu L}\, \Psi_{\mu K' LM}^{l'_x l'_y l'_{xy} l'_z}(\omega'), \quad (7)$$

where $\langle l'_x l'_y l'_{xy} l'_z K' | l_x l_y l_{xy} l_z K \rangle_{\mu L}$ are the four-particle Raynal–Revai coefficients, and

$$\Psi_{\mu KLM}^{l_x l_y l_{xy} l_z}(\omega) = \sum_{m_x m_y m_{xy} m_z} (l_x l_y m_x m_y | l_{xy} m_{xy})\, (l_{xy} l_z m_{xy} m_z | LM)\, \Psi_{\mu K}^{l_x l_y l_z m_x m_y m_z}(\omega). \quad (8)$$

Analytic expressions were found for the four-particle coefficients of the unitary transformations[8] and their connection with the three- and two-particle Talmi–Moshinsky coefficients.[12] In Ref. 13 some recursion relations that lead to a simplification of the computational technique were obtained.

This scheme for treating the three- and four-particle basis hyperspherical functions can be naturally generalized and used for systems with a larger number of particles, and, particularly importantly, it can be conveniently used not only when the investigated system consists solely of all identical particles or all particles with different masses but also when there is a particle with a different mass alongside identical particles. In Ref. 14 five-particle coefficients of unitary transformations of the hyperspherical functions were introduced, and their connection with the four- and two-particle

Talmi–Moshinsky coefficients was found. In Ref. 15 a simple expression was found for the coefficients of the transformations of the hyperspherical functions of N bodies in the case in which the grand orbital is equal to the total orbital angular momentum of the system. The coefficients found in accordance with this expression can be used as initial coefficients, and the coefficients with larger grand orbitals (for the previous value of the total orbital angular momentum) can be found by means of the recursion relations.

Algebra of few-body hyperspherical functions

Hyperspherical functions are a generalization, to multidimensional spaces, of ordinary spherical functions. Therefore, some (and perhaps all) of the relations that hold for the latter may have analogs in the space of hyperspherical functions. In Ref. 16 some results were obtained that facilitate the development of the algebra of few-body hyperspherical functions.

Let $\boldsymbol{\rho} = (\boldsymbol{\xi}_1, \boldsymbol{\xi}_1, ..., \boldsymbol{\xi}_n)$ be a set of vectors, on the basis of which we construct hyperspherical functions $\Psi_{\nu_n}(\Omega_{3n})$ that realize a representation of the group $O(3n)$, where ν_n is the set of the following quantum numbers: $l_1, l_2, ..., l_n$, the orbital angular momenta; $L_2, L_3, ..., L_n$, the total orbital angular momenta of the subsystem; M_n, the projection of the total orbital angular momentum; and $K_1, K_2, ..., K_{n-1}$, the grand orbitals of the subsystem.

From the expansion of an n-dimensional spherical wave with respect to the hyperspherical functions we obtain

$$(\rho\rho')^m = 2^{1-m}\pi^{\frac{3n}{2}}(\rho\rho')^m\, m! \times \sum_{(K_{n-1})} \left[\left(\frac{m-K_{n-1}}{2}\right)! \,\Gamma\left(\frac{m+K_{n-1}+3n}{2}\right)\right]^{-1} \times \sum_{(K_{n-1})} \Psi^*_{\nu_n}(\Omega_{3n})\,\Psi_{\nu_n}(\Omega'_{3n}). \tag{9}$$

In (9), K_{n-1} takes all non-negative values of the same parity as m that do not exceed m, while $\Sigma_{(\tau)}$ means that the summation is over all quantum numbers apart from τ and $\rho = [\xi_1^2 + \xi_2^2 + ... + \xi_n^2]^{1/2}$.

Solving the infinite system (9), we obtain

$$\sum_{(K_{n-1})} \Psi^*_{\nu_n}(\Omega_{3n})\,\Psi_{\nu_n}(\Omega'_{3n}) = \frac{\pi^{-3n/2}}{4}(2K_{n-1}+3n-2)\,\Gamma\left(\frac{3n-2}{2}\right) C^{\frac{3n-2}{2}}_{K_{n-1}}\left(\frac{\rho\rho'}{\rho\rho'}\right), \tag{10}$$

where $C^{\alpha}_{\beta}(x)$ are Gegenbauer polynomials.

In the limit $\rho' \to \rho$ we have

$$\overline{\sum_{(K_{n-1})}} |\Psi_{\nu_n}(\Omega_{3n})|^2 = \frac{3n-1}{K_{n-1}(K_{n-1}+3n-2)} Q_n(K_{n-1}), \tag{11}$$

where

$$Q_n(K) = \frac{\pi^{-3n/2}}{4}(2K+3n-2)(3n-2) \times \binom{K+3n-2}{K-1}\Gamma\left(\frac{3n-2}{2}\right). \tag{12}$$

Applying to both sides of (10) different differential operators and going to the limit $\boldsymbol{\rho}' \to \boldsymbol{\rho}$, we obtain the relation

$$\sum_{(K_{n-1})} M_n |\Psi_{\nu_n}(\Omega_{3n})|^2 = 0; \tag{13}$$

$$\sum_{(K_{n-1})} M_n^2 |\Psi_{\nu_n}(\Omega_{3n})|^2 = Q_n(K_{n-1}) \sum_{i=1}^{n} \left(\frac{\xi_i}{\rho}\sin\theta_i\right)^2; \tag{14}$$

$$\overline{\sum_{(K_{n-1})}} l_i(l_i+1) |\Psi_{\nu_n}(\Omega_{3n})|^2 = 2Q_n(K_{n-1})\left(\frac{\xi_i}{\rho}\right)^2; \tag{15}$$

$$\sum_{(K_{n-1})} K_m(K_m+3m+1) |\Psi_{\nu_n}(\Omega_{3n})|^2 = Q_n(K_{n-1})(3m+2)\sum_{i=1}^{m+1}\left(\frac{\xi_i}{\rho}\right)^2. \tag{16}$$

In Ref. 16 some other formulas were obtained. As one would expect, Eqs. (10)–(16) for $n = 1$ go over into the well-known expressions for the spherical functions.

Hyperspherical basis for a system of three identical particles and one particle with a different mass

In numerous physical four-particle problems it is necessary to separate the degrees of freedom of one of the particles (investigation of four-particle hypernuclei, form factors, nuclear reactions with emission of one particle, etc.). When one is considering systems that consist of three identical particles and one particle with a different mass, the symmetrization procedure represents a three-particle problem with the choice of a (3 + 1) configuration. Possessing a basis of four-particle hyperspherical functions symmetrized with respect to three particles, one can construct by means of Young operators a basis of functions symmetrized with respect to the four particles and find the corresponding coefficients of fractional parentage. This program was implemented in Ref. 17.

For the four-particle hyperspherical function symmetrized with respect to three particles, we have

$$\Psi^{[\bar{f}]\lambda_{[\bar{f}]}\nu_{[\bar{f}]}l_{12}l_3}_{\mu LK}(\omega) = \sum_{l_1 l_2} C^{[\bar{f}]\lambda_{[\bar{f}]}\nu_{[\bar{f}]}}_{Kl_{12}}(l_1 l_2)\,\Psi^{l_1 l_2 l_3 l_4}_{\mu LK}(\omega), \tag{17}$$

where $C^{[\bar{f}]\lambda_{[\bar{f}]}\nu_{[\bar{f}]}}_{Kl_{12}}(l_1 l_2)$ are the three-particle symmetrization coefficients (Ref. 6 gives relations between these coefficients and the three-particle Raynal–Revai coefficients and also tables for these coefficients for several principal values of the quantum numbers); $[\bar{f}]$ is the Young diagram of the three-particle system; $\lambda_{[\bar{f}]}$ is a row of the representation $[\bar{f}]$; and $\nu_{[\bar{f}]}$ is the number of the appearance of the representation $[\bar{f}]$ for given K and l_{12}. We introduce the coefficients of the transformation of the function (17) under permutations of the particles:

$$P_{i4}\Psi^{[\bar{f}]\lambda_{[\bar{f}]}\nu_{[\bar{f}]}l_{12}l_3}_{\mu LK}(\omega) = \sum_{[\bar{f}']\lambda_{[\bar{f}']}\nu_{[\bar{f}']}} \sum_{l'_{12}l'_3K'} \Psi^{[\bar{f}']\lambda_{[\bar{f}']}\nu_{[\bar{f}']}l'_{12}l'_3}_{\mu LK'}(\omega) \times \langle[\bar{f}']\lambda_{[\bar{f}']}\nu_{[\bar{f}']}l'_{12}l'_3K' \,|\, [\bar{f}]\lambda_{[\bar{f}]}\nu_{[\bar{f}]}l_{12}l_3K\rangle_{\mu L}, \quad i = 1, 2, 3. \tag{18}$$

On the other hand, the functions $\Psi^{l_1 l_2 l_3 l_4}_{\mu LM}(\omega)$ transform un-

der permutations of the particles by means of the four-particle Raynal–Revai coefficients[6,8]:

$$P_{ii}\Psi^{l_1l_2l_{12}l_3}_{\mu LK}(\omega) = \sum_{l_1'l_2'l_{12}'l_3'K'} \langle l_1'l_2'l_{12}'l_3'K' \mid l_1l_2l_{12}l_3K\rangle^{P_{ii}}_{\mu L}\, \Psi^{l_1'l_2'l_{12}'l_3'}_{\mu LK'}(\omega). \tag{19}$$

From Eqs. (17)–(19) we obtain for the transformation coefficients

$$\langle[\bar f']\lambda_{[\bar f']}\nu_{[\bar f']}l_{12}'l_3'K' \mid [\bar f]\lambda_{[\bar f]}\nu_{[\bar f]}l_{12}l_3K\rangle_{\mu L} = \sum_{l_1l_2l_1'l_2'} C^{[\bar f']\lambda_{[\bar f']}\nu_{[\bar f']}}_{K'l_{12}'}(l_1'l_2') \times \langle l_1'l_2'l_{12}'l_3'K' \mid l_1l_2l_{12}l_3K\rangle_{\mu L} C^{[\bar f]\lambda_{[\bar f]}\nu_{[\bar f]}}_{Kl_{12}}(l_1l_2). \tag{20}$$

By means of the coefficients (20) it is possible to find the coefficients of fractional parentage for the four-particle symmetrized basis, i.e., one can construct a basis of symmetrized four-particle hyperspherical functions using the basis (17).

The coefficients of fractional parentage for the hyperspherical four-particle symmetrized functions are determined by

$$\Psi^{[f]\lambda_{[f]}\nu_{[f]}}_{\mu L}(\omega) = \sum_{\alpha\nu_{[\bar f]}} B^{[f]\lambda_{[f]}\nu_{[f]}}_{\mu L}(\nu_{[\bar f]}\alpha)\, \Psi^{[\bar f]\lambda_{[\bar f]}\nu_{[\bar f]}\alpha}_{\mu L}(\omega), \quad \alpha \equiv Kl_{12}l_3. \tag{21}$$

where $[f]$, $\lambda_{[f]}$, $\nu_{[f]}$ are, respectively, the Young diagram, the row of the representation, and the number of the representation $[f]$ for given μ and L relating to the group of permutations of the four particles; $[\bar f]$ is obtained from $[f]$ by removing the cell corresponding to the fourth particle.

In Ref. 17 Young operators were constructed and used to find the connection between the coefficients of fractional parentage and the coefficients (20).

2. THE METHOD OF HYPERSPHERICAL FUNCTIONS IN THE THEORY OF MANY-PARTICLE REACTIONS

Truly three- and four-particle scattering

Hyperspherical functions are eigenfunctions of the kinetic-energy operators. It is therefore natural that the wave functions of the final states of many-particle reactions (continuum functions) can be expanded with respect to a complete set of corresponding hyperspherical functions. Such an approach offers hope of a unified description of the initial and final states in reactions in which several particles participate. Use of expansions of the three- and four-particle functions of the continuum with respect to hyperspherical functions has made it possible[18,19] to generalize the variable-phase method[20,21] for quantum-mechanical three- and four-body problems. Such a generalization is possible only for truly many-particle scatterings.

We consider the process $3 \to 3$ when three free particles are present at the beginning and end of the process. The process of three-particle scattering is rather complicated, and fundamental difficulties arise when the theory of two-particle scattering is generalized to the case of three or more particles. In the coordinate representation these difficulties are associated with the boundary conditions. The corresponding questions are discussed in more detail in the review of Ref. 19. If in three-particle scattering two-particle bound states arise, or if two-particle scattering occurs on the energy shell, a hyperspherical basis for solution of the Schrödinger equations cannot ensure the necessary asymptotic behavior of the three-particle wave function. This is due to the fact that hypercoordinates are collective, and a transition to them is convenient when there is "democracy" in the system (no pair is distinguished in the sense of the formation of bound states or scattering on the energy shell). If such democracy is observed, we shall say that the scattering is truly three-particle scattering. In the final section of this review we shall show that in the solution of the Faddeev equations the possibilities of the MHF can in principle be extended.

We expand the three-particle continuum wave function with respect to hyperspherical functions (the case of nonidentical particles):

$$\Psi_{\mathbf{pq}}(\mathbf{x},\mathbf{y}) = \sum_{K l_x l_y LM} \frac{U^{l_x l_y}_{KL}(\rho)}{\rho^2}\, \Phi^{l_x l_y}_{KLM}(\Omega_\rho)\, \Phi^{l_x l_y}_{KLM}(\Omega_{\varkappa_0}). \tag{22}$$

where

$$p^2 + q^2 = \varkappa_0^2 = 2mE/\hbar^2; \quad q = \varkappa_0 \cos\alpha_{\varkappa_0}; \quad p = \varkappa_0 \sin\alpha_{\varkappa_0};$$
$$\Omega_{\varkappa_0} \equiv (\alpha_{\varkappa_0}, \hat{\mathbf{p}}, \hat{\mathbf{q}}).$$

For the hyperradial functions we obtain from the three-particle Schrödinger equation a system of coupled one-dimensional differential equations:

$$\frac{d^2U^{l_x l_y}_{KL}(\rho)}{d\rho^2} + \frac{1}{\rho}\frac{dU^{l_x l_y}_{KL}(\rho)}{d\rho} + \left[\varkappa_0^2 - \frac{(K+2)^2}{\rho^2}\right] U^{l_x l_y}_{KL}(\rho) = \sum_{K'l_x'l_y'} W^{l_x l_y; l_x' l_y'}_{KK'LM}(\rho)\, U^{l_x' l_y'}_{K'L}(\rho), \tag{23}$$

where

$$W^{l_x l_y; l_x' l_y'}_{KK'LM}(\rho) = \frac{2m}{\hbar^2}\int \Phi^{l_x l_y L}_{KLM}(\Omega)\, U_{123}\, \Phi^{l_x' l_y'}_{K'LM}(\Omega)\, d\Omega; \tag{24}$$

here U_{123} is the potential energy of the three-particle system.

The solution of one equation of the system (23) (the theory can be readily generalized to the case of any number of equations[6,19]) can be represented in the form

$$U^{l_x l_y}_{KL}(\rho) = A^{l_x l_y}_{KL}(\rho)\,[\cos\delta^{l_x l_y}_{KL}(\rho)\, J_{K+2}(\varkappa_0\rho) - \sin\delta^{l_x l_y}_{KL}(\rho)\, N_{K+2}(\varkappa_0\rho)], \tag{25}$$

where $j_{K+2}(\kappa_0\rho)$ and $N_{K+2}(\kappa_0\rho)$ are a Bessel function of the first kind and a Neumann function of order $K+2$; $\delta^{l_x l_y}_{KL}(\rho)$ is the phase function, which is obtained by solving a nonlinear equation of Riccati type; and $A^{l_x l_y}_{KL}(\rho)$ is the amplitude function, the equation for which can be solved by quadrature. The amplitude functions determine the normalization of the wave function and are associated with the boundary conditions.

Similarly, one can consider truly four-particle scattering by introducing four-particle phase functions $\delta^{l_1 l_2 l_{12} l_3}_{\mu KL}(\rho)$

and amplitude functions $A_{\mu KL}^{l_x l_y l_{xy} l_z}(\rho)$. Are there such processes in nature, and how important is their investigation? Of course, it is not possible to realize experimentally processes of truly three- and four-particle scattering with accelerators. Nevertheless, the detailed theoretical study of these processes may give very valuable information about many aspects of the problem of many-particle systems. Many results from the theory of two-particle scattering can be extended and generalized to the case of truly many-particle scattering. The variational principles of Hulthén and Kohn and also of Schwinger have already been generalized for the phase shifts of truly three-particle scattering, and the corresponding optical theorem has been formulated.[22] This direction of generalization can undoubtedly be taken further. Finally, in atomic and nuclear physics there is a large class of many-particle reactions in the final states of which truly many-particle scattering is dominant. These are complete breakup reactions.[19]

Complete breakup of three- and four-particle nuclei

The process of complete breakup is determined by the matrix element

$$\langle f \mid H' \mid i \rangle = \int \Psi_f^* H' \Psi_i d\tau. \tag{26}$$

The wave function Ψ_i of the initial state decreases rapidly with increasing hyperradius ρ (ρ is the radius of the hypersphere on which the interactions between the particles take place). Therefore, the terms in the wave function Ψ_f of the final state that contribute to the matrix element (26) are those that correspond to a small value of the hyperradius ρ and, ultimately, to short interparticle distances. Indeed, $\rho^2 = \Sigma_{i<j} \mathbf{r}_{ij}^2$, where $\mathbf{r}_{ij}^2$ is the relative radius vector of the pair i, j, and the situation in which all $\mathbf{r}_{ij}$ are small corresponds to a small value of ρ. Thus, for the matrix element (26) the only important part of the configuration space in the continuum is that in which the truly many-particle scatterings occur (democracy is not violated in complete breakup, in which the subsystems cannot rescatter on the energy shell because the remaining particles are also at short distances and contribute to the energy balance). In Ref. 19 the results of early investigations of complete breakup of light nuclei induced by elementary particles were discussed. Further investigations of this kind using the MHF have been made. For example, in Ref. 23 different potentials of the nucleon–nucleon interaction were used to investigate the reaction of pickup of a negative muon by the tritium nucleus:

$$\mu^- + {}^3\mathrm{H} \to \nu_\mu + 3n. \tag{27}$$

The wave functions of the initial and final nuclear states were obtained using the same nucleon–nucleon potentials (the S1 and S2 Afnan–Tang, Volkov, and Eikemaier–Hakenbroich potentials). The inital-state wave functions obtained by Mukhtarova and Éfros with allowance for $K \leqslant 24$ were chosen, while the final-state wave function was obtained by solving the system of equations (23) by the variable-phase method with allowance for two harmonics. It was shown that the final-state interaction between all the neutrons has a strong influence on the process—the energy distribution of the neutrinos is changed qualitatively, and the rate of μ capture is strongly increased. It seems evident that this is due to the fact that inclusion of the final-state interaction leads to spatial localization of the three neutrons, as a result of which the overlapping of the wave functions of the initial and final states for the given NN potential is increased, this leading to an increase of the rate of μ capture. It was shown further that allowance for the final-state interaction enhances the sensitivity of the rate of μ capture to the form of the NN potential. An analogous approach to the study of the initial and final nuclear states was used in Ref. 24 to investigate the reaction of double charge exchange of pions on three-particle nuclei with complete breakup:

$$\left.\begin{aligned} &\pi^- + {}^3\mathrm{He} \to \pi^+ + 3n; \\ &\pi^+ + {}^3\mathrm{H} \to \pi^- + 3p. \end{aligned}\right\} \tag{28}$$

It was shown that the effect of the final-state interaction between the products of the complete breakup of the nuclear system is decisive and leads to the appearance of a maximum in the differential cross section in the region of low energies of their relative motion (Fig. 1). In Ref. 24 good convergence of the reaction cross section with respect to the grand orbitals was demonstrated for both the initial and final states.

To study processes of truly many-particle scattering, it is convenient to use the hyperspherical formalism in the momentum representation.[11] In this representation the wave function of an unbound state of three interacting particles can be represented in the form of the expansion

$$\Psi_{\mathbf{q}_0\mathbf{p}_0}(\mathbf{q}_i, \mathbf{p}_i) = \sum_{K l_{q_i} l_{p_i} K_0 LM} \varphi_{KLK_0}^{l_{q_i} l_{p_i}}(\varkappa, \varkappa_0)\, \Phi_{KLM}^{* l_{q_i} l_{p_i}}(\Omega_i)\, \Phi_{K_0 LM}^{l_{q_i} l_{p_i}}(\Omega_0), \tag{29}$$

where $p_0^2 + q_0^2 = \kappa_0^2 = 2mE/\hbar^2$, $E > 0$ is the energy of the three particles in the center-of-mass system, Ω_i and Ω_0 are the sets of hyperangles corresponding to the six-dimensional momentum spaces $(\mathbf{q}_i, \mathbf{p}_i)$ and $(\mathbf{q}_0, \mathbf{p}_0)$, and the grand orbitals K_0 and K characterize the system before and after the truly three-particle scattering, respectively.

For the hyperradial functions $\varphi_{KLK_0}^{l_{q_i} l_{p_i}}(\kappa, \kappa_0)$ a system of coupled one-dimensional inhomogeneous singular integral equations is obtained:

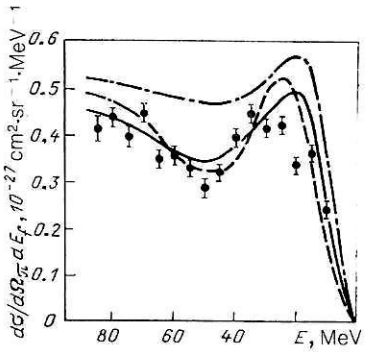

FIG. 1. Dependence of the differential cross section of the $\pi + {}^3\mathrm{He} \to \pi' + 3n$ reaction on the kinetic energy E of the three neutrons. The experimental data are taken from Ref. 25; the continuous curve is for a potential, the broken curve for a potential [Translator's note: the Russian does not state which potentials (words probably omitted)], and the chain curve for the Eikemaier–Hakenbroich potential.

$$\varphi^{l_q l_p}_{KLK_0}(\varkappa, \varkappa_0) = \frac{\delta(\varkappa-\varkappa_0)}{\varkappa_0^5}\delta_{KK_0} - \frac{2m}{\hbar^2}\varkappa^{-2}(\varkappa^2-\varkappa_0^2-i\varepsilon)^{-1}$$
$$\times \sum_{jK'}\sum_{l'_q l'_p \tilde{l}_q \tilde{l}_p} i^{K-K'}\langle \tilde{l}_q \tilde{l}_p \mid l_q l_p\rangle^{j}_{KL} i^{j}\langle \tilde{l}_q \tilde{l}_p \mid l'_q l'_p\rangle^{j}_{K'L}$$
$$\times \int \varkappa'^3 d\varkappa' \int \rho^5 d\rho J_{K+2}(\varkappa\rho) J_{K'+2}(\varkappa'\rho) \int d\Omega_{\rho_j} \Phi^{\tilde{l}_q \tilde{l}_p *}_{K'LM}(\Omega_{\rho_j})$$
$$\cdot U(x_j) \Phi^{l_q l_p}_{KLM}(\Omega_{\rho_j}) q^{l'_q l'_p}_{K'LK_0}(\varkappa', \varkappa_0). \qquad (30)$$

Expressions analogous to (29) and (30) hold for a bound state of the three particles. A difference is that an expansion of the type (29) contains one hyperspherical function in the momentum representation, and instead of the system (30) there is a system of coupled one-dimensional homogeneous integral equations.

The theory can be readily generalized to the case of four bodies.[13] In particular, for a four-particle bound system in the momentum representation we have the expansion

$$\Psi(\mathbf{p}, \mathbf{q}, \mathbf{r}) = \sum_{\mu K l_1 l_2 l_{12} l_3} q^{l_1 l_2 l_{12} l_3}_{\mu KL}(\varkappa) \Psi^{l_1 l_2 l_{12} l_3}_{\mu KLM}(\omega_\varkappa), \qquad (31)$$

where $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$ are Jacobi momenta; $\omega_\varkappa \equiv (\alpha, \beta, \hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{r}})$, $p = \varkappa \cos\alpha \sin\beta$; $q = \varkappa \sin\alpha \sin\beta$; $r = \varkappa \times \cos\beta$; $\varkappa^2 = p^2 + q^2 + r^2$.

Such an approach has often been used in investigations of the structure and complete-breakup reactions of light nuclei and hypernuclei. For example, the problem of hypertritium was investigated in Ref. 26 in a basis of hyperspherical functions in the momentum representation. The binding energy and wave function of hypertritium were calculated. For the first time an investigation of mesonic and mesonless decays of this hypernucleus with allowance for the final-state interaction was made. It was concluded that the MHF in the momentum representation makes it possible, without invoking model representations, to describe satisfactorily in a single scheme both the bound state of the ${}^{3}_{\lambda}$H hypernucleus and the continuum states. It was shown further that the effect of the final-state interaction between the nucleons of the reaction ${}^{9}_{\lambda}\mathrm{H}\pi^- + p + p + n$ has a strong influence on the probability of the process. Only with allowance for this effect can one obtain a reasonable value for the decay probability. It was also concluded that the channel of mesonless decay ${}^{3}_{\lambda}\mathrm{H} \to p + n + n$ is strongly suppressed because of the fragility of hypertritium.

In Ref. 27 the MHF in the momentum representation and the assumption of a truly many-particle nature of the rescattering of the decay products were used to develop a unified approach to investigation of the structure of the hypernucleus ${}^{9}_{\lambda}$Be and the nuclear reaction ${}^{9}_{\lambda}\mathrm{Be} \to \alpha + \alpha + p + \pi^-$. It was shown that such an approach makes possible a comparatively comprehensive investigation of the physics of the hypernucleus ${}^{9}_{\lambda}$Be in the framework of a single formalism. If the potentials of the $\alpha\alpha$ and $\lambda\alpha$ interactions are comparatively soft, the three-particle cluster structure of this hypernucleus can be reproduced basically by the minimal harmonic, although the following harmonic makes a significant contribution to the binding energy. The effect of the final-state interaction between all the particles of the $\alpha + \alpha + p$ system is decisive and leads to both a quantitative and a qualitative change of all the considered characteristics of the ${}^{9}_{\lambda}\mathrm{Be} \to \alpha + \alpha + p + \pi^-$ reaction. The energy spectrum of the pions in this reaction has a pronounced peak in the region of high energies of the pions, while in the distribution over the relative energies of the α particles there is a narrow peak at low energies, due primarily to the effect of the interaction between the α particles in the final state. It was shown further that the α particles from the reaction ${}^{9}_{\lambda}\mathrm{Be} \to \alpha + \alpha + p + \pi^-$ must be omitted mainly in opposite directions. The investigated characteristics were found to be fairly sensitive to the chosen form of the potentials of the $\alpha\alpha$ and $\alpha\lambda$ interactions, and qualitative features of these characteristics were established. All this makes the problem of detailed experimental investigation of the ${}^{9}_{\lambda}\mathrm{Be} \to \alpha + \alpha + p + \pi^-$ reaction very interesting. The noticeable quickening of interest in recent years in the development of hypernuclear physics offers hope that this problem will be solved in the near future.

In Ref. 28 an iterative solution of the three- and four-particle equations of the MHF in the momentum representation [of the type (30)] was proposed, and double hypernuclei and charmed supernuclei were studied on its basis.

3. THE METHOD OF HYPERSPHERICAL FUNCTIONS IN INVESTIGATIONS OF COLLECTIVE EXCITED STATES OF LIGHT NUCLEI AND OF PROCESSES INVOLVING LIGHT IONS

In this section we present the results of investigations on the application of the MHF to the study of the structure of light nuclei revealed in elastic and inelastic scattering of ions. Particular attention is devoted to the effect of the change of the nuclear properties with the excitation energy, which is automatically taken into account in the MHF.

Effect of the change of the nuclear properties with the excitation energy in the method of hyperspherical functions[29]

In recent years there have been many experimental investigations with heavy nuclei in which the nuclei are in highly excited states in the process of deep inelastic collisions. One of the results of these investigations is the fact that the properties of the nuclei change with the excitation energy, and these changes influence the reaction process (fission). On the other hand, various theoretical attempts have been made to describe the temperature dependence of such nuclear properties as the density distribution, size, and shape of nuclei. In Ref. 30 a realistic microscopic effective Hamiltonian was used in the framework of the spherical Hartree–Fock approximation with finite temperature in order to study the thermodynamic properties of the nuclei ^{16}O and ^{40}Ca. It was shown that the temperature dependence of the properties of these nuclei is much stronger than for the corresponding results obtained with phenomenological zero-range forces. This effect is more pronounced for light than for heavy ions. In Ref. 29 a different approach was proposed for studying the dependence of the density distribution and sizes of nuclei on the excitation energy without the direct introduction of a temperature parameter T in the method of calculation. A collective Hamiltonian with an effective NN interaction was used to describe the nucleus as a many-body system consisting of A nucleons. This makes it possible to study the change in the collective properties of nuclei with increasing excitation energy.

The system of equations for finding the eigenvalues and radial eigenfunctions is written in the form

$$\left\{\frac{d^2}{d\rho^2}-\frac{L_K(L_K+1)}{\rho^2}+\frac{2m}{\hbar^2}(E+W^{K\gamma}_{K\gamma}(\rho))\right\}\chi_{K\gamma}(\rho)$$
$$=\frac{2m}{\hbar^2}\sum_{K'\gamma'\neq K\gamma}W^{K'\gamma'}_{K\gamma}(\rho)\,\chi_{K'\gamma'}(\rho). \qquad (32)$$

Figure 2 shows the effective potential

$$V_{\text{eff}}=\frac{\hbar^2}{2m}\frac{L_K(L_K+1)}{\rho^2}+W^{K\gamma}_{K\gamma}(\rho),$$

the eigenvalues, and the first three radial wave functions obtained as solutions of Eq. (32) for the ^{16}O nucleus using the Brink–Boeker B4 potential.[31] It can be seen that the effective potential automatically expands for states with higher energy. Because of this, the formalism correctly reproduces the expansion of the radial wave functions with increasing excitation energy of the nucleus. In the theory of the average field, in which an oscillator potential is used to describe the interaction, this effect is created by changing the number of basis states and the value of the oscillator radius with increasing parameter T, which corresponds to high excitations in the nuclei. Figure 3 shows the dependence of the rms radius on the excitation energy for the ^{12}C, ^{15}O, and ^{16}O nuclei[29] in a comparison with the results of Ref. 30 for ^{16}O in which the expression for calculating the rms radius has the form

$$r_{RMS}(E^*)$$
$$=2.74\,(1.0056+6.51\cdot10^{-3}/\sqrt{E^*}+1.89\cdot10^{-3}E^*). \qquad (33)$$

The numbers 1 (B1) and 1′ (B4) correspond to the nucleus ^{12}C, 2 (B1) and 2′ (B4) to ^{15}O, and 3 (B1) and 3′ (B4) to ^{16}O. The brackets describe the set of parameters of the NN interaction.

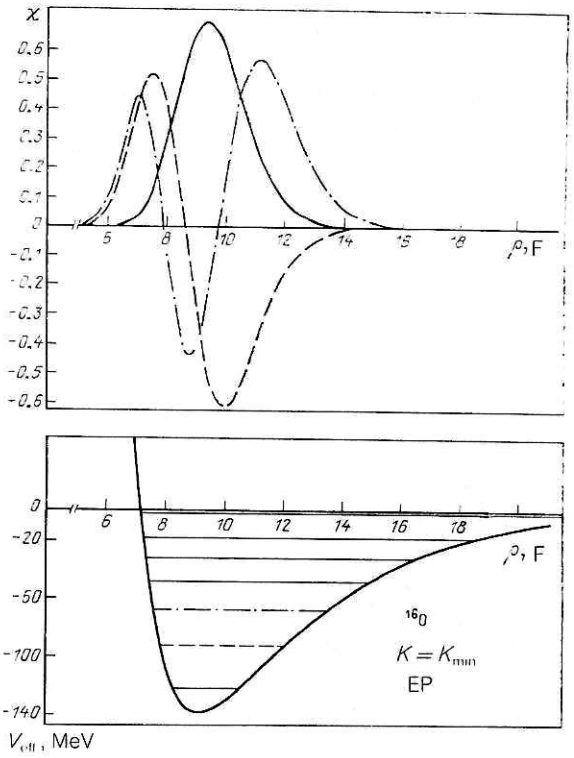

FIG. 2. Effective potential, eigenvalues, and the first three radial wave functions obtained as solutions of Eq. (32) for the ^{16}O nucleus.

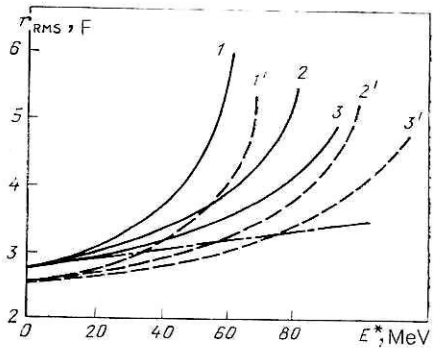

FIG. 3. Dependence of the rms radius on the excitation energy for the nuclei ^{12}C (1,1′), ^{15}O (2,2′), and ^{16}O (3,3′). The chain line shows the results of Ref. 30.

Thus, in Ref. 29 the dependence of the properties of light nuclei as functions of the excitation energy was studied. The results of the calculation were compared with the corresponding results in field theory in which one uses a realistic microscopic effective Hamiltonian at finite temperature.

The predictions for the change in the properties of the nuclei with increasing excitation energy that are obtained in the MHF differ from the results given by the theory of the average field. In particular, the calculations made in Ref. 29 showed that the rms radius as a function of the excitation energy tends to infinity at an excitation energy near the binding energy, whereas the theory of the average field predicts a finite value of the radius in this case. This discrepancy may be due to the fact that in the MHF the entire excitation energy of the nucleus is concentrated on a single degree of freedom, which is associated with the collective variable ρ, whereas in the Hartree–Fock method the excitation energy of the nucleus is distributed over a large number of particle–hole excitations of different natures. In agreement with the results obtained in the Hartree–Fock method at finite temperature using a realistic microscopic effective Hamiltonian, the method proposed here indicates a stronger temperature dependence for light nuclei than for heavy ones.

In Ref. 32 a study was made of the influence of a change of the nuclear properties with excitation energy on the results of calculations for giant multipole resonances.

Giant monopole resonances

Study of the nature of giant monopole resonances in light nuclei is stimulated by the fact that the problem of experimental detection of giant monopole resonances is not trivial and for the majority of light nuclei has not yet been finally solved. An MHF basis is extremely convenient for the microscopic description of monopole vibrations. Excitation with respect to the collective variable ρ corresponds to monopole vibrations of the nucleus as a whole, i.e., the density is a dynamical variable.

The results of the calculation of the excitation energy of monopole states and of the distribution of the energy-weighted monopole sum for light nuclei with $4\leqslant A\leqslant16$ (Ref. 5) showed that the shape of the distribution of the monopole sum is the same for all the studied nuclei, namely, the first monopole-excited state takes 70–80% of the monopole sum rule, while the second takes about 10%. These results are not

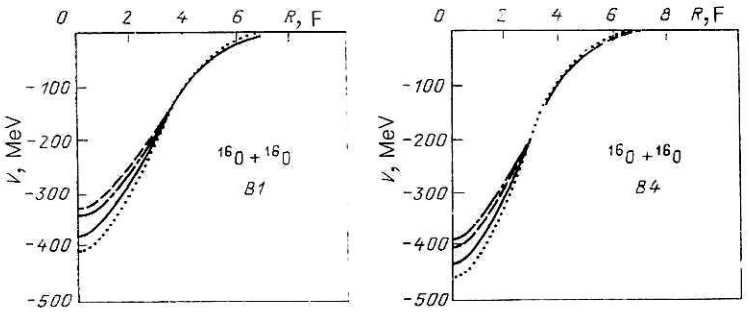

FIG. 4. Folding potential with Skyrme forces for the $^{16}O + ^{16}O$ system; the dotted curve is for $V_{0,0}$, the continuous curve for $V_{0,0_1'}$, the chain curve for $V_{0',0_1'}$, and the broken curve for $V_{0,0_2'}$.

very sensitive to the choice of the *NN* potential, as follows from a comparison of these values for the ^{16}O nucleus with the corresponding results obtained in Refs. 34 and 35.

As the next step it is interesting to consider the influence of a change in the properties of nuclei with the excitation energy on the interaction potential of two heavy ions and on the cross section for nucleus–nucleus inelastic scattering. It can be expected that this effect plays an important part for the description of processes in which giant resonances are excited in the region of 20 MeV in the inelastic scattering of heavy ions. Such an investigation was made in Ref. 36. The nuclear densities obtained in the method of hyperspherical functions were used to construct folding potentials for the $^{16}O + ^{16}O$ system. The properties of the interaction potentials of two heavy particles were studied in both the ground state and monopole-excited states with energy 20–40 MeV. The results of the calculation are shown in Fig. 4. It can be seen that the interaction potential for both nuclei in the ground state is the deepest. In principle, it becomes broader with increasing excitation energy of the system, but the potential for both fragments when in the first excited state is deeper than the corresponding potential for such a combination in which one nucleus is in the ground state and the other is in the second excited state, despite the fact that the total excitation energy of the system is higher in the first case. This fact is explained by the different structure of the wave functions that characterize the different nuclear states determining the radial distribution of the nuclear densities and, therefore, the shape of the folding potential.

A first attempt at a microscopic description of the angular distribution in inelastic nucleus–nucleus scattering with excitation of one of the nuclei was undertaken for the system ^{12}C (^3He,^3He)$^{12}C^{0+}$, $E^*_{C^{0+}} = 20.3$ MeV at energy $E_{^3He} = 108$ MeV of the incident particles and for ^{12}C (^4He, $^4He^{0+}$)^{12}C, $E^*_{He^{0+}} = 20.1$ MeV at $E_{^4He} = 65$ MeV (Refs. 37 and 38). The results shown in Figs. 5 and 6 are in good agreement with the experimental data. This means that in the calculation of the cross section of the inelastic scattering process it is necessary to take into account the expansion of the nucleus with increasing excitation energy.

Giant dipole resonances

Figure 7 gives the results of calculations of isovector states of the ^{16}O nucleus with $J^\pi = 1^-$, $T = 1$, which are known as the giant dipole resonance.[32] In the shell model, in which the change in the nuclear properties is not taken into account, the total dipole sum is concentrated in the region of excitation energies up to 30 MeV. In the MHF the dipole sum is redistributed to the region of higher excitation ener-

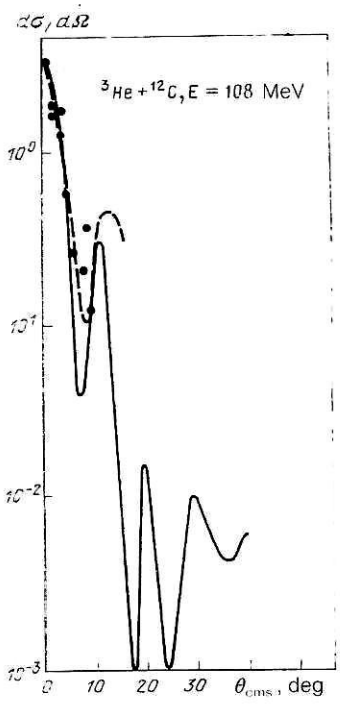

FIG. 5. Angular distribution of $^3He + ^{12}C$ inelastic scattering at $E_{^3He} = 108$ MeV: the continuous curve gives the calculation of Ref. 38, the broken curve is the phenomenological description of Ref. 39, and the points are the experimental data.

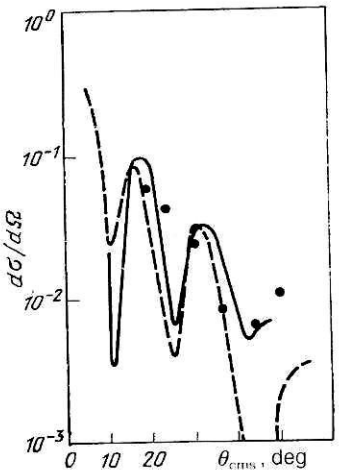

FIG. 6. Cross section of $^4He + ^{12}C$ inelastic scattering at $E_{^4He} = 65$ MeV (Ref. 40). The curves have the same significance as in Fig. 5.

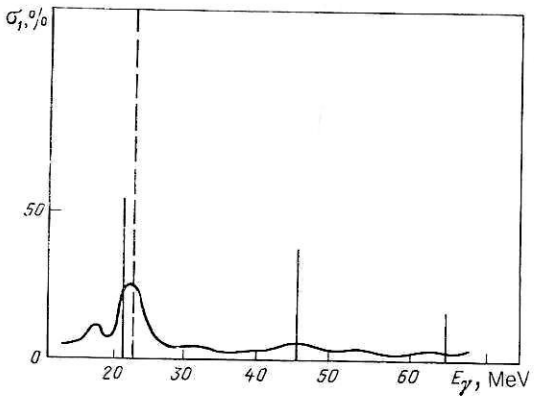

FIG. 7. Giant dipole resonance in the ^{16}O nucleus.

gies. This fact is in good agreement with the experimental data of Ref. 33, which show that about 50% of the dipole sum in ^{16}O is beyond the giant resonance.

Thus, allowance for the change in the properties of the nuclei with increasing excitation energy makes it possible to give a description of the properties of giant multipole resonances in good agreement with the experimental data.

Study of the structure of the ^{12}C nucleus in ion scattering

The excited states of the ^{12}C nucleus have been investigated in a number of theoretical[41–44] and experimental[48] studies. The spectroscopic characteristics of this nucleus have been investigated, and the problems of elastic and inelastic scattering by ^{12}C have been solved.

In the theoretical investigations the nature of the low-lying excited states has been studied in various models. For example, in Ref. 41 the MHF was used to make detailed calculations of four-particle excitations in the ^{12}C nucleus. It proved possible to describe the lowest excited 0^+ state (energy 7.65 MeV). It was shown that this state has an internal wave function corresponding to four-particle excitation, and that it can be satisfactorily described by one $K_{\min}+4$ harmonic. Later, in Ref. 42 the structure of the states of positive and negative parity in ^{12}C was described successfully in the framework of a microscopic 3α-particle model.

In Ref. 43 an attempt was made to calculate the binding energy and characteristics of various types of 0^+ levels of the ^{12}C nucleus in an orthogonal scheme with the most symmetric Young diagrams having $K=K_{\min}$, $K_{\min}+2$, $K_{\min}+4$. Finally, in Ref. 44 a detailed study was made of the transition density for 0^+ (ground state) $\to 2^+$ (4.44 MeV) in ^{12}C in an intermediate model. In this case states of the giant isoscalar quadrupole resonance are mixed with the states of the shell model.

In Ref. 45 various types of excited states of the ^{12}C nucleus were studied in the MHF. Calculations were made of the spectrum of excited 0^+ and 2^+ states in this nucleus, the densities, the rms radii, and the reduced probabilities $B(E2)$ for the investigated transitions. Folding potentials and cross sections for elastic and inelastic scattering with participation of ^{12}C ions were found. The investigation made it possible to give an interpretation of the structural features of excited 0^+ and 2^+ states in the ^{12}C nucleus.

We briefly describe the scheme for calculating the various properties of the excited states of nuclei with allowance for three harmonics: $K_{\min}$, $K_{\min}+2$, $K_{\min}+4$. The wave function of nucleus A is sought in the form of an expansion with respect to K-harmonic polynomials,

$$\Psi(1, 2, \ldots, A) = \rho^{-\frac{1}{2}(3A-4)} \sum_{K\gamma} \chi_{K\gamma}(\rho) \mid AK\gamma\rangle, \qquad (34)$$

where $\gamma=[f]\epsilon LST$. In the nucleus ^{12}C, in the approximation $K=K_{\min}, K_{\min}+2, K_{\min}+4$, the determination of the eigenvalues and radial eigenfunctions involves solution of the system of coupled differential equations

$$\begin{aligned}
&\left\{\frac{d^2}{d\rho^2}-\frac{23\cdot 24}{\rho^2}-\frac{2m}{\hbar^2}[E+W^{K\gamma}_{K\gamma}(\rho)]\right\}\chi_{K\gamma}(\rho)\\
&=\sum_{\gamma'}\frac{2m}{\hbar^2}[W^{K+2\gamma'}_{K\gamma}(\rho)\chi_{K+2\gamma'}(\rho)+W^{K\gamma'}_{K\gamma}(\rho)\chi_{K\gamma'}(\rho)];\\
&\left\{\frac{d^2}{d\rho^2}-\frac{25\cdot 26}{\rho^2}-\frac{2m}{\hbar^2}[E+W^{K+2\gamma}_{K+2\gamma}(\rho)]\right\}\chi_{K+2\gamma}(\rho)\\
&=\frac{2m}{\hbar^2}\sum_{\gamma'}[W^{K\gamma'}_{K+2\gamma}(\rho)\chi_{K\gamma'}(\rho)+W^{K+2\gamma'}_{K+2\gamma}(\rho)\chi_{K+2\gamma'}(\rho)\\
&+W^{K+4\gamma'}_{K+2\gamma}(\rho)\chi_{K+4\gamma'}(\rho)];\\
&\left\{\frac{d^2}{d\rho^2}-\frac{27\cdot 28}{\rho^2}-\frac{2m}{\hbar^2}[E+W^{K+4\gamma}_{K+4\gamma}(\rho)]\right\}\chi_{K+4\gamma}(\rho)\\
&=\frac{2m}{\hbar^2}\sum_{\gamma'}[W^{K+2\gamma'}_{K+4\gamma}(\rho)\chi_{K+2\gamma'}(\rho)+W^{K+4\gamma'}_{K+4\gamma}(\rho)\chi_{K+4\gamma'}(\rho)],
\end{aligned} \qquad (35)$$

where $W^{K\gamma'}_{K\gamma}(\rho)$ are the matrix elements of the potential energy of the nucleon–nucleon interaction,

$$V=\sum_{i<j}^{A}V(r_{ij}), \quad V(r_{ij})=f(r_{ij})W_{\sigma\tau}, \qquad (36)$$

which can be expressed in terms of the two-particle coefficients of fractional parentage in the form

$$\begin{aligned}
W^{K'\gamma'}_{K\gamma}(\rho) &= \langle AK[f]\varepsilon LSTM_LM_SM_T|\hat{V}|AK'[f']\\
&\quad \varepsilon'L'S'T'M_{L'}M_{S'}M_{T'}\rangle\\
&= \frac{A(A-1)}{2}\sum\langle AK[f]\varepsilon LST|A-2K_2[f_2]\\
&\quad \varepsilon_2L_2S_2T_2,\ \Lambda(L''K'');\ L_0S_0T_0\rangle\\
&\quad \times\langle AK'[f']\varepsilon'L'S'T'|A-2K_2[f_2]\\
&\quad \varepsilon_2L_2S_2T_2,\ \Lambda(L''K'');\\
&\quad L_0S_0T_0\rangle\langle S_0T_0|W_{\sigma\tau}|S_0T_0\rangle R^{KK'}_{K''L_0}(\rho),
\end{aligned} \qquad (37)$$

where

$$\begin{aligned}
&R^{KK'}_{K''L_0}(\rho)\\
&\int d\theta_1(\sin\theta_1)^{3A-7}(\cos\theta_1)^2N_{KK''L_0}N_{K'K''L_0}f(\rho\cos\theta_1)\\
&\times(\sin\theta_1)^{2K''}(\cos\theta_1)^{2L_0}P^{K''+\frac{1}{2}(3A-6)-1,\ L_0+\frac{1}{2}}_{K-K''-L_0}(\cos 2\theta_1)\\
&\quad P^{K''+\frac{1}{2}(3A-6)-1,\ L_0+\frac{1}{2}}_{K'-K''-L_0}(\cos 2\theta_1).
\end{aligned} \qquad (38)$$

For the ground state of the ^{12}C nucleus, $LST=000$. If one includes in the scheme the most symmetric Young diagram [444], then for the states $K_{\min}$, $K_{\min}+2$, $K_{\min}+4$ the following configurations will be considered:

$K_{\min}$: $|s^4p^8\rangle = \gamma_1$;
$K_{\min}+2$: $|s^2p^{10}\rangle = \gamma_2$, $|s^4p^6\cdot 2S^2\rangle = \gamma_3$, $|s^4p^6|d^2\rangle = \gamma_4$;
$K_{\min}+4$: $|s^4p^4\cdot 2S^4\rangle = \gamma_5$, $|s^4p^4|d^2\cdot 2S^2\rangle = \gamma_6$, $|s^4p^4|d^4\rangle = \gamma_7$.

Thus, in the approximation $K = K_{\min}$ a restriction is made to the consideration of one equation, in the approximation $K = K_{\min}+2$ a system of four coupled differential equations is solved, and, finally, in the approximation $K = K_{\min}+4$ there is a system of seven coupled differential equations. This system is written as follows:

$$
\begin{aligned}
&\left\{\frac{d^2}{d\rho^2}-\frac{23\cdot 24}{\rho^2}-\frac{2m}{\hbar^2}[E+W^{K\gamma_1}_{K\gamma_1}(\rho)]\right\}\chi_{K\gamma_1}(\rho)=\frac{2m}{\hbar^2}\\
&\cdot[W^{K+2\gamma_2}_{K\gamma_1}(\rho)\,\chi_{K+2\gamma_2}(\rho)+W^{K+2\gamma_3}_{K\gamma_1}(\rho)\,\chi_{K+2\gamma_3}(\rho)\\
&+W^{K+2\gamma_4}_{K\gamma_1}(\rho)\,\chi_{K+2\gamma_4}(\rho)];\\
&\left\{\frac{d^2}{d\rho^2}-\frac{25\cdot 26}{\rho^2}-\frac{2m}{\hbar^2}[E+W^{K+2\gamma_3}_{K+2\gamma_2}(\rho)]\right\}\chi_{K+2\gamma_2}(\rho)\\
&=\frac{2m}{\hbar^2}W^{K\gamma_1}_{K+2\gamma_1}(\rho)\,\chi_{K\gamma_1}(\rho);\\
&\left\{\frac{d^2}{d\rho^2}-\frac{25\cdot 26}{\rho^2}-\frac{2m}{\hbar^2}[E+W^{K+2\gamma_3}_{K+2\gamma_3}(\rho)]\right\}\chi_{K+2\gamma_3}(\rho)\\
&=\frac{2m}{\hbar^2}[W^{K\gamma_1}_{K+2\gamma_3}(\rho)\,\chi_{K\gamma_1}(\rho)+W^{K+4\gamma_5}_{K+2\gamma_3}(\rho)\,\chi_{K+4\gamma_5}(\rho)\\
&+W^{K+2\gamma_4}_{K+2\gamma_3}(\rho)\,\chi_{K+2\gamma_4}(\rho)+W^{K+4\gamma_6}_{K+2\gamma_3}(\rho)\,\chi_{K+4\gamma_6}(\rho)];\\
&\left\{\frac{d^2}{d\rho^2}-\frac{25\cdot 26}{\rho^2}-\frac{2m}{\hbar^2}[E+W^{K+2\gamma_4}_{K+2\gamma_4}(\rho)]\right\}\chi_{K+2\gamma_4}(\rho)\\
&=\frac{2m}{\hbar^2}[W^{K\gamma_1}_{K+2\gamma_4}(\rho)\,\chi_{K\gamma_1}(\rho)+W^{K+2\gamma_3}_{K+2\gamma_4}(\rho)\,\chi_{K+2\gamma_3}(\rho)\\
&+W^{K+4\gamma_6}_{K+2\gamma_4}(\rho)\,\chi_{K+4\gamma_6}(\rho)+W^{K+4\gamma_7}_{K+2\gamma_4}(\rho)\,\chi_{K+4\gamma_7}(\rho)];\\
&\left\{\frac{d^2}{d\rho^2}-\frac{27\cdot 28}{\rho^2}-\frac{2m}{\hbar^2}[E+W^{K+4\gamma_5}_{K+4\gamma_5}(\rho)]\right\}\chi_{K+4\gamma_5}(\rho)\\
&=\frac{2m}{\hbar^2}[W^{K+2\gamma_3}_{K+4\gamma_5}(\rho)\,\chi_{K+2\gamma_3}(\rho)+W^{K+4\gamma_6}_{K+4\gamma_5}(\rho)\,\chi_{K+4\gamma_6}(\rho)];\\
&\left\{\frac{d^2}{d\rho^2}-\frac{27\cdot 28}{\rho^2}-\frac{2m}{\hbar^2}[E+W^{K+4\gamma_6}_{K+4\gamma_6}(\rho)]\right\}\chi_{K+4\gamma_6}(\rho)\\
&=\frac{2m}{\hbar^2}[W^{K+2\gamma_3}_{K+4\gamma_6}(\rho)\,\chi_{K+2\gamma_3}(\rho)+W^{K+2\gamma_4}_{K+4\gamma_6}(\rho)\,\chi_{K+2\gamma_4}(\rho)\\
&+W^{K+4\gamma_5}_{K+4\gamma_6}(\rho)\,\chi_{K+4\gamma_5}(\rho)+W^{K+4\gamma_7}_{K+4\gamma_6}(\rho)\,\chi_{K+4\gamma_7}(\rho)];\\
&\left\{\frac{d^2}{d\rho^2}-\frac{27\cdot 28}{\rho^2}-\frac{2m}{\hbar^2}[E+W^{K+4\gamma_7}_{K+4\gamma_7}(\rho)]\right\}\chi_{K+4\gamma_7}(\rho)\\
&=\frac{2m}{\hbar^2}[W^{K+2\gamma_4}_{K+4\gamma_7}(\rho)\,\chi_{K+2\gamma_4}(\rho)+W^{K+4\gamma_6}_{K+4\gamma_7}(\rho)\,\chi_{K+4\gamma_6}(\rho)].
\end{aligned}
\tag{39}
$$

Solving the system of equations (38), one finds the binding energy and spectrum of excited states of the ^{12}C nucleus, and also the corresponding wave functions.

Figure 8 shows the effective potential and the first two solutions in it for the ^{12}C nucleus in the approximations $K = K_{\min}$, $K_{\min}+4$. The computational scheme included the *NN* potentials from Ref. 41. Note that the depth of the effective potential increases sharply with increasing global moment K, this being due to the allowance for the Pauli principle.

Using the functions $\chi_K(\rho)$ that were obtained by solving the system (39), we can construct densities of the ground state and excited states of the nuclei, and also transition densities[46,47] in accordance with the general formula

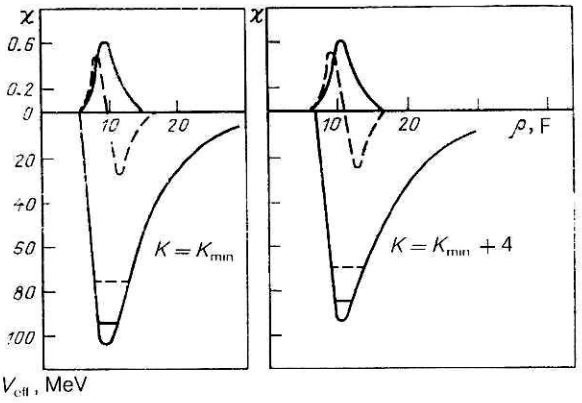

FIG. 8. Effective potential V_{eff} and the first two solutions χ in it for the ^{12}C nucleus.

$$
n_{J^\pi J'^{\pi'}}(\mathbf{r})=\left\langle J^\pi\left|\sum_K\delta(\mathbf{r}-\mathbf{r}_K)\right|J'^{\pi'}\right\rangle \tag{40}
$$

where J^π and $J'^{\pi'}$ characterize the initial and final states of the nucleus, and the multipole expansion of the density has the form

$$
n_{J^\pi J'^{\pi'}}(\mathbf{r})=\sum_{\lambda\mu}n^\lambda_{J^\pi J'^{\pi'}}(\mathbf{r})\,Y^*_{\lambda\mu}(\theta,\varphi),
$$

where the radial components $n^\lambda_{J^\pi J'^{\pi'}}(\mathbf{r})$ are expressed as follows:

$$
n^\lambda_{J^\pi J'^{\pi'}}(\mathbf{r})=\left\langle J^\pi\left|\left|\sum_K r_K^{-2}\delta(\mathbf{r}-\mathbf{r}_K)i^\lambda Y_\lambda(\theta_K,\Phi_K)\right|\right|J'^{\pi'}\right\rangle. \tag{41}
$$

Thus, the ground-state density of the ^{12}C nucleus is obtained as

$$
n_{0^+0^+}(\mathbf{r})=\frac{4}{\pi^{3/2}}\Gamma\left(\frac{49}{2}\right)\times\int_r^\infty\frac{\chi^2_{0^+}(\rho)\,d\rho}{\rho^{47}}\left[\frac{(\rho^2-r^2)^{22}}{\Gamma(23)}+\frac{4}{3}\frac{r^2(\rho^2-r^2)^{21}}{\Gamma(22)}\right]. \tag{42}
$$

where $\chi_0{}^+(\rho)$ is the radial wave function of the ground state of ^{12}C, and the density is normalized by

$$
4\pi\int n_{0^+0^+}(r)\,r^2\,dr=A. \tag{43}
$$

The rms radius of the nucleus is

$$
r_{RMS}=\frac{\int n_{J^\pi J^\pi}(r)\,r^4\,dr}{\int n_{J^\pi J^\pi}(r)\,r^2\,dr} \tag{44}
$$

The density of the excited 2^+ state is expressed as

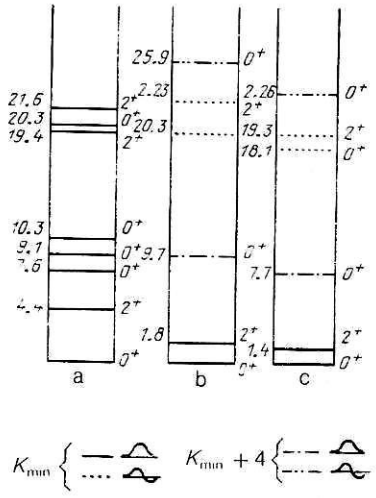

FIG. 9. Excitation spectrum of 0^+ and 2^+ states of the ^{12}C nucleus in the method of hyperspherical functions.

$$n_{2+2+}(\mathbf{r}) = Y_{00}(\theta,\ \varphi)\, n^0_{2+2+}(r) + \sum_{\mu} Y^*_{2\mu}(\theta,\ \varphi)\, n^2_{2+2+}(r);$$

$$n^0_{2+2+}(r) = \frac{8}{\pi}\,\frac{\Gamma\left(\frac{49}{2}\right)}{\Gamma(23)} \int_r^{\infty} \frac{(\rho^2 - r^2)^{22}}{\rho^{47}}\, \chi^2_2(\rho)\, d\rho + \frac{32}{3\pi}\,\frac{\Gamma\left(\frac{49}{2}\right)}{\Gamma(22)} \times \int_r^{\infty} \frac{r^2(\rho^2 - r^2)^{21}}{\rho^{47}}\, \chi^2_2(\rho)\, d\rho;$$

$$n^2_{2+2+}(r) = \frac{32\,|\,\bar{2}}{3\pi}\,\frac{\Gamma\left(\frac{49}{2}\right)}{\Gamma(22)} \int_r^{\infty} \frac{r^2(\rho^2 - r^2)^{21}}{\rho^{47}}\, \chi^2_2(\rho)\, d\rho. \tag{45}$$

The transition density with excitation of the 2^+ state is expressed in the form

$$n_{0+2+}(\mathbf{r}) = \sum_{\mu} Y^*_{2\mu}(\theta,\ \varphi)\, n^2_{0+2+}(r);$$

$$n^2_{0+2+}(r) = \frac{16\,|\,\bar{7}}{3\,|\,\bar{5}}\,\frac{\Gamma\left(\frac{49}{2}\right)}{\Gamma(22)}\,\frac{1}{\pi} \times \int_r^{\infty} \frac{r^2(\rho^2 - r^2)^{21}}{\rho^{47}}\, \chi_0(\rho)\, \chi_2(\rho)\, d\rho. \tag{46}$$

The reduced probability of the multipole transition is given by the expression

$$B(EL,\ J^{\pi} \to J'^{\pi'}) = \frac{e^2}{2J+1}\left(\int_0^{\infty} n_{J^{\pi}J'^{\pi'}}(r)\, r^{L+2}\, dr\right)^2 \tag{47}$$

and the energy-weighted sum rule is expressed in the form

$$S(EL) = \frac{L(2L+1)^2}{16\pi}\left(\frac{\hbar^2}{2m}\right) e^2 A \langle r^{2L-2}\rangle = \Delta EB(EL,\ 0^+ \to L^+), \tag{48}$$

where J^{π} and $J'^{\pi'}$ are the initial and final states of the nucleus, and L is the angular-momentum transfer.

Figure 9 shows the results of calculation of the excitation spectrum of the 0^+ and 2^+ states of the ^{12}C nucleus in the MHF with the *NN* potentials of Refs. 31 and 41. The experimental data are given on the left. It can be seen that the lowest excited 2^+ state with inclusion in the computational scheme of the potentials of Refs. 31 and 41 is in the region of energies 1.4–1.8 MeV and has as its main component a wave function without a node with $K = K_{min}$. The excited 2^+ levels in the region of energies 18–22 MeV have wave functions with a node and $K = K_{min}$. The 0^+ levels with excitation energy 7–10 MeV have as their main component a radial wave function without a node and with $K = K_{min} + 4$. The excited 0^+ levels in the region of energies ~ 20 MeV have radial wave functions with a node with the main components $K = K_{min}$, $K_{min} + 4$. Thus, when we use the MHF to calculate the ground and lowest excited states of nuclei we obtain, as a second solution of the system (39), resonances of corresponding multipolarity.

Figure 10 gives the densities of the ground state and various types of excited 0^+ states of the ^{12}C nucleus. In the left-hand part of the figure the continuous curve shows the ground-state density of the ^{12}C nucleus, the broken curve shows the density of the monopole resonance lying in the region of excitation energies of order 20 MeV, and the chain curve shows the transition density with excitation of the giant monopole resonance. The right-hand part of the figure gives the density of the "mysterious" 0^+ state in the region of energies 7–10 MeV.

Figure 11 shows three different density distributions of the ^{12}C nucleus: for the ground state [$n^0_{0^+0^+}(r)$, continuous curve], monopole resonance at energy 20.3 MeV [$n^0_{0^+0^+}(r)$, broken curve], and transition density with excitation of the monopole resonance [$n^0_{00^+}(r)$, broken curve with crosses]; we also give the components of the density for the lowest 2^+ state ($n^0_{2^+2^+}(r)$, $n^2_{2^+2^+}(r)$, chain curve) and of the transition density with excitation of the lowest 2^+ state [$n^2_{02^+}(r)$, chain curve with two dots].

Table I gives the rms radii for various types of excited states of the ^{12}C nucleus, calculated with the *NN* potential of Ref. 31. The results given in Figs. 10 and 11 and in Table I reflect the structural features of the studied states. Indeed, there is a clearly expressed effect of expansion of the nucleus

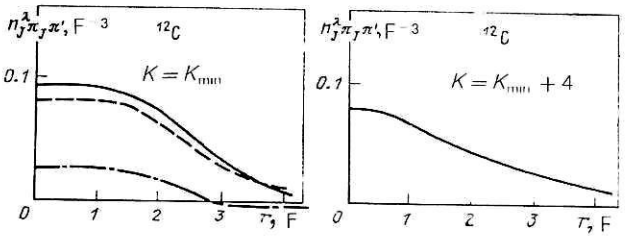

FIG. 10. Densities of the ground and excited 0^+ states of different nature of the ^{12}C nucleus.

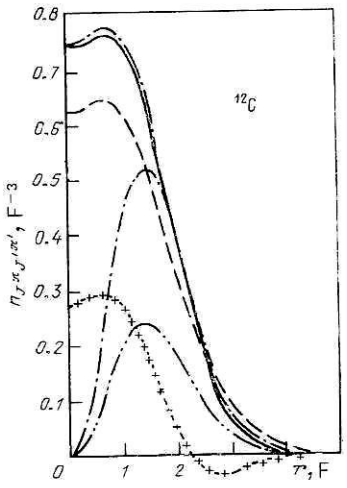

FIG. 11. Densities of excited 0^+ and 2^+ states of the ^{12}C nucleus.

with increasing excitation energy. On the other hand, the 0^+ state at excitation energy 9.7 MeV has an rms radius greater than the higher-lying 0^+ state at 20.3 MeV. This is explained by the specific nature of the function of the "mysterious" low-lying 0^+ state.

Table II gives the probabilities of E2 transitions with excitation of a 2^+ state, obtained with different *NN* potentials,[31] the results of calculation in the shell model,[44] and also experimental data. It can be seen that the values of these quantities depend strongly on the choice of the *NN* potential.

One of the interesting problems in the investigation of the structure of the ^{12}C nucleus is the distribution of the energy-weighted quadrupole sum $B(E2)$. This question was studied in Ref. 44, where it was found that $B(E2) = 1412$ $e^2 \cdot F^4 \cdot MeV$ for the ^{12}C nucleus. The lowest 2^+ state takes 13% of the energy-weighted quadrupole sum. In experimental investigations it was shown that in the C nucleus in the region of energies below 30 MeV only 20% of the energy-weighted quadrupole sum is observed. Moreover, in different investigations the position of the quadrupole resonance in the ^{12}C nucleus has been predicted differently. In Ref. 44 it was estimated at 56 MeV, but in Ref. 48 the value 28 MeV was found. In Ref. 45 an estimate was also made of the distribution of the energy-weighted quadrupole sum of the ^{12}C nucleus. The calculations showed that the lowest level takes 8.6% of the quadrupole sum, and a higher-lying level at 21–22 MeV takes about 8.1%. Thus, in the region of excitation energies up to 30 MeV about 16.7% of the energy-weighted quadrupole sum is exhausted. This result is in good agreement with experiment.

TABLE I. Excitation energies and rms radii for states of different nature of the ^{12}C nucleus.

E, MeV	r_{RMS}, F	$J^\pi T$
0	2.63	0^+0
1.8	2.89	2^+0
9.7	3.27	0^+0
20.3	3.05	0^+0
22.0	2.28	2^+0
25.9	3.69	0^+0

TABLE II. Probabilities of E2 transitions with excitation of the lowest 2^+ state in ^{12}C.

Potential	$B(E2)$
V3 (Ref. 31)	71.6
V7 (Ref. 31)	85.9
Shell model (Ref. 44)	17.5
Experiment	42.2

The densities of the ground state and excited states of various natures found in the MHF was subsequently used to construct double-folding potentials.[36]

Figure 12 shows the folding potential for the ^3He–^{12}C system calculated with the densities obtained in the MHF (Ref. 36) with different *NN* potentials [Figs. 12a (Ref. 41) and 12b (Ref. 31)]. The thin continuous curve (sic) shows the results of the calculation when both nuclei are in the ground state, the broken curve shows the corresponding results when the ^{12}C nucleus is in the monopole resonance (20.3 MeV), and, finally, the thick continuous curve shows the folding potential for the case when the ^{12}C nucleus is in the "mysterious" 0^+ state (at energy 7–10 MeV).

As one would expect (see Table I), the folding potential with excitation of the lowest 0^+ state is higher and broader than with excitation of the more highly excited 0^+ state. Comparison of the results of the calculations shown in Figs. 12a and 12b demonstrates the dependence of the shape of the folding potential on the choice of the *NN* potential in the calculation of the nuclear density. For the deeper variant of the *NN* potential of Brink type, which gives a smaller nuclear radius, the folding potential is narrower and deeper (for example, at the origin by 20 MeV).

These folding potentials were then used to calculate the elastic scattering cross sections. The angular distributions of the scattering were calculated in the coupled-channel method. It was assumed that the imaginary potential must have the same form as the real potential, so that

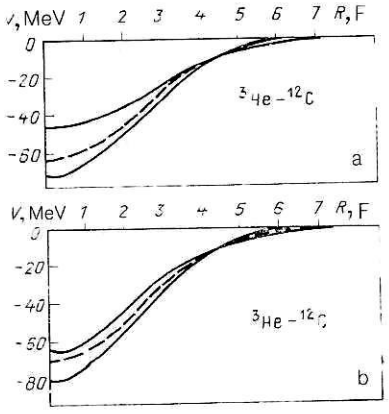

FIG. 12. Folding potentials for the ^3He–^{12}C system in the ground state and monopole-excited states of different nature in the ^{12}C nucleus.

$$U = V_{0,0}(1 + \alpha). \tag{49}$$

The parameter α was found in the elastic channel by fitting the theoretical differential cross section to the experimental data. The fitting criterion was usually minimization of

$$\chi^2 = \sum_i \left[\frac{\sigma_{\text{exp}}(\theta_i) - \sigma_{\text{theor}}(\theta_i)}{\Delta\sigma_{\text{exp}}(\theta_i)}\right]^2, \tag{50}$$

where $\sigma_{\text{theor}}(\theta_i)$ are the calculated differential cross sections, and $\Delta\sigma_{\text{exp}}(\theta_i)$ are the measured experimental errors.

Figure 13 shows the angular distributions of the cross section for ^{6}Li + ^{12}C elastic scattering at energies 30.6, 90, 99, and 156 MeV of the incident particles.[49] Despite the simplicity of the potential, which was used in this case with just one free parameter, the agreement with experiment was good and the parameter was found to be 0.7–0.9. Comparison with the results of the phenomenological description, in which six free parameters are used, illustrates the advantage of the proposed microscopic approach, which has just one free parameter.

Nuclear interaction potentials of light nuclei in the ground state and in monopole-excited states

In Ref. 50 the interaction potentials were investigated in the energy-density formalism for the nuclei ^{12}C–^{12}C and ^{16}O-^{16}O in both the ground state and the 0^+ excited state.

In the energy-density formalism, the nuclear part of the interaction potential has the form

$$V_N(\mathbf{R}) = \int \{\varepsilon[\rho^A(\mathbf{r} + \mathbf{R}/2) - \rho^B(\mathbf{r} - \mathbf{R}/2)] - \varepsilon[\rho^A(\mathbf{r})] - \varepsilon[\rho^B(\mathbf{r})]\}\, d\mathbf{r}, \tag{51}$$

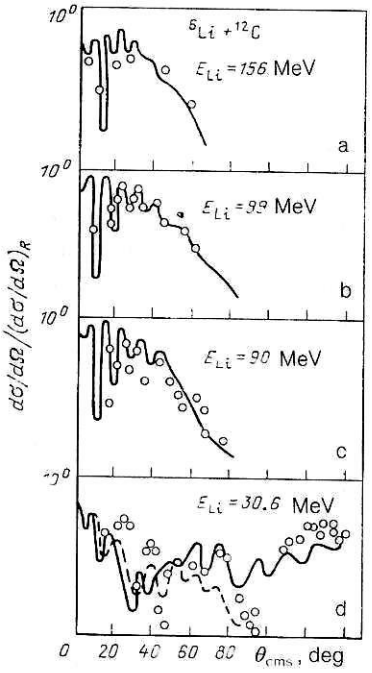

FIG. 13. Angular distributions of the cross section for ^{6}Li + ^{12}C elastic scattering at energies 30.6, 90, 99, and 156 MeV of the incident particles.

where ρ^A and ρ^B are the distributions of the nuclear density. The energy-density functional is written in the form

$$\varepsilon(\rho) = \tau_{TF} + V(\rho, \alpha) + \eta(\Delta\rho)^2, \tag{52}$$

where α is the neutron excess, ρ is the total density, and

$$\alpha = (\rho_n - \rho_p)/(\rho_n + \rho_p).$$

The kinetic-energy density is determined in the Thomas–Fermi approximation:

$$\tau_{TF} = \frac{3}{5}\frac{\hbar^2}{2m}\left(\frac{3}{2}\pi^2\right)^{2/3}\frac{1}{2}[(1-\alpha)^{5/3} + (1+\alpha)^{5/3}]\rho^{5/3}, \tag{53}$$

where m is the nucleon mass. The functional $V(\rho,\alpha)$ is the potential-energy density for infinite nuclear matter:

$$V(\rho, \alpha) = b_1(1 - a_1\alpha^2)\rho + b_2(1 - a_2\alpha^2)\rho^{4/3} + b_3(1 - a_3\alpha^2)\rho^{5/3}. \tag{54}$$

This scheme, in the energy-density formalism and using the nuclear densities obtained in the MHF, was used to calculate the nuclear potentials for the ^{12}C–^{12}C and ^{16}O–^{16}O systems when one or both of the nuclei are in the 0^+ excited state.

The results for ^{16}O–^{16}O are shown in Fig. 14, which includes a comparison with the nuclear potential of the system in the ground state.

In both cases in which the 0^+ excitation is encountered the depth of the potential increases, the minimum of the potential well is slightly shifted to smaller distances, and the repulsive core disappears. In the surface region, where the nuclear forces become attractive, all the potentials are very close together. The calculated potentials (see Fig. 14) differ from the similar ones constructed in the folding model. They are deeper and do not have a repulsive core. This happens because the Pauli principle is not taken into account in the folding procedure. The use of the hyperspherical density of monopole-excited nuclei in the folding procedure leads to a

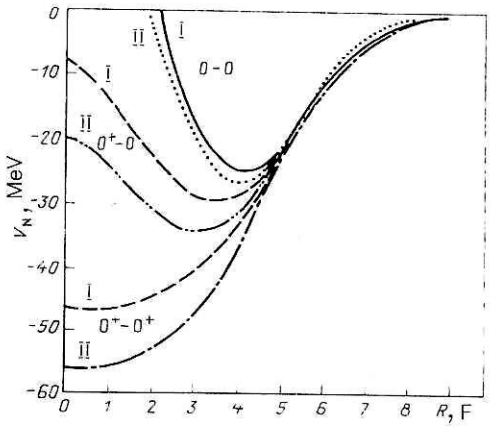

FIG. 14. Nuclear potentials for the ^{16}O + ^{16}O system when both nuclei are in the ground state (00), one of the nuclei is in a 0^+ excited state (0^+0), and both nuclei are in an excited state (0^+0^+), calculated with two sets of parameters: I (Ref. 51) and II (Ref. 52).

decrease in the depth of the potential, and the potential becomes broader. Thus, in the peripheral region the potentials for the monopole-excited states behave in the same way in the two different methods of calculation. A difference appears in the internal region through the effect of the antisymmetrization in the energy-density formalism.

Recently interesting results have been obtained on the use of the MHF in the cluster model of light nuclei. In particular, cluster hyperharmonics were introduced in Refs. 53–55, and excitations of monopole degrees of freedom in nuclear collisions were investigated. The approach makes it possible to generalize the algebraic version of the resonating-group method, and this makes it possible to take into account the formation of compound states in nuclear processes with the participation of clusters. In addition, "cluster" hyperharmonics make it possible to establish a certain hierarchy of internal functions of the nucleon systems.

4. THE METHOD OF HYPERSPHERICAL FUNCTIONS IN COULOMB FEW-BODY PROBLEMS

In the quantum mechanics of few-body systems Coulomb problems have come to the fore in recent years. Many studies have been devoted to investigation of Coulomb effects by means of the Faddeev integral equations, various approximate schemes for taking into account the Coulomb forces having been proposed and used in the face of some fundamental difficulties. The MHF has been little used to investigate Coulomb scattering in a system of three or four particles. In Ref. 56 truly three- and four-particle Coulomb scattering was considered using the MHF.

Truly three- and four-particle Coulomb scattering

We consider truly three-particle Coulomb scattering of particles with charges Z_1e, Z_2e, Z_3e and masses m_1m, m_2m, m_3m. After elimination of the center-of-mass motion, the corresponding Schrödinger equation takes the form

$$-\frac{\hbar^2}{2m}(\Delta_{\mathbf{x}}+\Delta_{\mathbf{y}})\Psi_{\mathbf{qp}}(\mathbf{x},\mathbf{y})+U_{123}(\mathbf{x},\mathbf{y})\Psi_{\mathbf{qp}}(\mathbf{x},\mathbf{y})=E\Psi_{\mathbf{qp}}(\mathbf{x},\mathbf{y}), \tag{55}$$

where

$$U_{123}(\mathbf{x},\mathbf{y})=\frac{a_1}{\mathbf{x}_1}+\frac{a_2}{\mathbf{x}_2}+\frac{a_3}{\mathbf{x}_3};$$
$$a_1=\sqrt{\frac{m_2m_3}{m_2+m_3}}Z_2Z_3e^2,\quad a_2=\sqrt{\frac{m_1m_3}{m_1+m_3}}Z_1Z_3e^2;$$
$$a_3=\sqrt{\frac{m_1m_2}{m_1+m_2}}Z_1Z_2e^2;$$
$$\mathbf{x}_i=\sqrt{\frac{m_jm_k}{m_j+m_k}}(\mathbf{r}_j-\mathbf{r}_k); \tag{56}$$

i, j, k form a cyclic permutation of (1, 2, 3); $\mathbf{q}$ and $\mathbf{p}$ are the Jacobi momenta corresponding to the Jacobi coordinates $\mathbf{x}_i$ and $\mathbf{y}_i$, where

$$\mathbf{y}_i=\sqrt{\frac{m_i(m_j+m_k)}{m_1+m_2+m_3}}\left(-\mathbf{r}_i+\frac{m_j\mathbf{r}_j+m_k\mathbf{r}_k}{m_j+m_k}\right).$$

In Eq. (55) we go over to the hypercoordinates ρ and $\Omega_i \equiv (\alpha_i, \mathbf{x}_i, \mathbf{y}_i)$ and we represent its solution in the form of the expansion (22). For the Coulomb hyperradial functions we finally obtain the system of coupled one-dimensional differential equations

$$\frac{d^2\chi_{KL}^{l_1l_2}(\varkappa\rho)}{d\rho^2}+\frac{1}{\rho}\frac{d\chi_{KL}^{l_1l_2}(\varkappa\rho)}{d\rho}+\left[\varkappa^2-\frac{(K+2)^2}{\rho^2}\right]\chi_{KL}^{l_1l_2}(\varkappa\rho)=\sum W_{KK'LL'MM'}^{l_1l_1'l_2l_2'}(\rho)\,\chi_{K'L}^{l_1'l_2'}(\varkappa\rho), \tag{57}$$

where

$$W_{KK'LL'MM'}^{l_1l_1'l_2l_2'}=\frac{2m}{\hbar^2}\int\Phi_{KLM}^{*l_1l_2}(\Omega_1^\rho)\left(\frac{a_1}{x_1}+\frac{a_2}{x_2}+\frac{a_3}{x_3}\right)\Phi_{K'L'M'}^{l_1'l_2'}(\Omega_1^\rho)\,d\Omega_1^\rho;$$
$$x_1=\rho\cos\alpha_1;\quad x_2=\rho\cos\alpha_2;\quad x_3=\rho\cos\alpha_3;$$
$$\varkappa^2=q_1^2+p_1^2=q_2^2+p_2^2=q_3^2+p_3^2=\frac{2mE}{\hbar^2}; \tag{58}$$

$E>0$ is the total c.m.s. energy.

Of the three terms of the angular integral (58) only the first can be calculated with comparative ease (the indices of the hyperspherical functions are equal to the index of the operator). Great difficulties arise in a direct calculation of the remaining two terms. It is here convenient to use the unitary transformation (3), which makes it possible to find the connection between different basis hyperspherical functions defined on different sets of Jacobi coordinates $\mathbf{x}_i$ and $\mathbf{y}_i$ ($i=1,2,3$). When this transformation is used, Eq. (58) takes the form

$$W_{KK'LL'MM'}^{l_1l_2l_1'l_2'}(\rho)=\frac{2m}{\hbar^2}\frac{1}{\rho}\Big\{a_1J_{KK'LL'MM'}^{l_1l_2l_1'l_2'}+a_2\sum_{\tilde{l}_1\tilde{l}_2\tilde{l}_1'\tilde{l}_2'}\langle\tilde{\tilde{l}}_1\tilde{\tilde{l}}_2|l_1l_2\rangle_{KL}\langle\tilde{\tilde{l}}_1'\tilde{\tilde{l}}_2'|l_1'l_2'\rangle_{K'L'}J_{KK'LL'MM'}^{\tilde{l}_1\tilde{l}_2\tilde{l}_1'\tilde{l}_2'}+a_3\sum_{\tilde{\tilde{l}}_1\tilde{\tilde{l}}_2\tilde{\tilde{l}}_1'\tilde{\tilde{l}}_2'}\langle\tilde{\tilde{\tilde{l}}}_1\tilde{\tilde{\tilde{l}}}_2|l_1l_2\rangle_{KL}\langle\tilde{\tilde{\tilde{l}}}_1'\tilde{\tilde{\tilde{l}}}_2'|l_1'l_2'\rangle_{K'L'}J_{KK'LL'MM'}^{\tilde{\tilde{l}}_1\tilde{\tilde{l}}_2\tilde{\tilde{l}}_1'\tilde{\tilde{l}}_2'}\Big\}, \tag{59}$$

where

$$J_{KK'LL'MM'}^{l_1l_2l_1'l_2'}=\int\Phi_{KLM}^{*l_1l_2}(\Omega^\rho)\,\Phi_{K'L'M}^{l_1'l_2'}(\Omega^\rho)\frac{d\Omega^\rho}{\cos\alpha} \tag{60}$$

The integral (60) can be solved analytically. Further, in the system of equations (57) we go over to new unknown functions and to a new variable in accordance with the expressions

$$\chi_{KL}^{l_1l_2}(\varkappa\rho)=\frac{q_{KL}^{l_1l_2}(\varkappa\rho)}{\sqrt{\rho}}; \tag{61a}$$

$$Z=-2i\varkappa\rho. \tag{61b}$$

Finally, the system (57) takes the form

$$\frac{d^2q_{KL}^{l_1l_2}(Z)}{dZ^2}+\left[-\frac{1}{4}+\frac{\frac{1}{4}-(K+2)^2}{Z^2}\right]q_{KL}^{l_1l_2}(Z)-\frac{i}{Z}\sum_{K'l_1'l_2'}A_{KK'L}^{l_1l_2l_1'l_2'}(\varkappa)\,q_{K'L}^{l_1'l_2'}(Z)=0, \tag{62}$$

where

$$A^{l_1l_2l_1'l_2'}_{KK'L}(\varkappa)=\frac{m}{\hbar^2\varkappa^2}\Big[a_1F(n,\ n';\ l_1,\ l_2)\,N^{l_1l_2}_{K}N^{l_1'l_2'}_{K'}\delta_{l_1l_1'}\delta_{l_2l_2'}$$
$$+a_2\sum_{\widetilde{l}_1\widetilde{l}_2}{}^{1}\langle\widetilde{l}_1\widetilde{l}_2|l_1l_2\rangle^{2}_{KL}\,{}^{1}\langle\widetilde{l}_1\widetilde{l}_2|l_1'l_2'\rangle^{2}_{K'L}F(\widetilde{n},\ \widetilde{n}';\ \widetilde{l}_1,\ \widetilde{l}_2)N^{\widetilde{l}_1\widetilde{l}_2}_{K}$$
$$\times N^{\widetilde{l}_1\widetilde{l}_2}_{K'}+a_3\sum_{\widetilde{\widetilde{l}}_1\widetilde{\widetilde{l}}_2}{}^{1}\langle\widetilde{\widetilde{l}}_1\widetilde{\widetilde{l}}_2|l_1l_2\rangle^{3}_{KL}\,{}^{1}\langle\widetilde{\widetilde{l}}_1\widetilde{\widetilde{l}}_2|l_1'l_2'\rangle^{3}_{K'L}$$
$$\times F(\widetilde{\widetilde{n}},\ \widetilde{\widetilde{n}}';\ \widetilde{\widetilde{l}}_1,\ \widetilde{\widetilde{l}}_2)N^{\widetilde{\widetilde{l}}_1\widetilde{\widetilde{l}}_2}_{K}N^{\widetilde{\widetilde{l}}_1\widetilde{\widetilde{l}}_2}_{K'}\Big];\qquad(63)$$

$$F(n,\ n';\ l_1,\ l_2)=\frac{1}{2}\Gamma\left(n+l_2+\frac{3}{2}\right)\Gamma\left(n+l_1+\frac{3}{2}\right)$$
$$\Gamma\left(n'+l_2+\frac{3}{2}\right)\Gamma\left(n'+l_1+\frac{3}{2}\right)$$
$$\times\sum_{\nu=0}^{n}\frac{(-)^{n-\nu}}{\Gamma(\nu+1)\,\Gamma(n+l_2-\nu+3/2)\,\Gamma(n-\nu+1)\,\Gamma\left(l_1+\nu+\frac{3}{2}\right)}$$
$$\times\sum_{\nu'=0}^{n'}\frac{(-)^{n'-\nu'}}{\Gamma(\nu'+1)\,\Gamma(n'+l_2-\nu'+3/2)\,\Gamma(n'-\nu'+1)\,\Gamma(l_1+\nu'+3/2)}$$
$$\times B[l_1+(\nu+\nu')+1,\ l_2+(n+n')-(\nu+\nu')+3/2];\qquad(64)$$

Γ is the gamma function, B is the beta function, $n=\frac{1}{2}(K-l_1-l_2)$, and $n'=\frac{1}{2}(K'-l_1-l_2)$.

In the approximation of the minimal harmonic for fixed values of the orbital quantum numbers l_1, l_2, and L there remains from the system (62) only one equation (diagonal approximation):

$$\frac{d^2\varphi^{l_1l_2}_{KL}(Z)}{dZ^2}+\left[-\frac{1}{4}+\frac{\frac{1}{4}-(K+2)^2}{Z^2}\right.$$
$$\left.+\frac{-iA^{l_1l_2}_{KKL}(\varkappa)}{Z}\right]\varphi^{l_1l_2}_{KL}(Z)=0,\qquad(65)$$

and this can be solved analytically. Solutions (irregular) are the Whittaker functions

$$\left.\begin{aligned}\varphi^{l_1l_2(1)}_{KL}(Z)&=W_{-iA^{l_1l_2}_{KKL},K+2}(-2i\varkappa\rho);\\ \varphi^{l_1l_2(2)}_{KL}(-Z)&=W_{iA^{l_1l_2}_{KKL},K+2}(2i\varkappa\rho).\end{aligned}\right\}\qquad(66)$$

Using the asymptotic expressions for these functions, we can readily find the asymptotic form of the Coulomb functions describing truly three-particle Coulomb scattering in the approximation of the minimal harmonic for fixed values of l_1, l_2, and L:

$$F^{l_1l_2}_{KL}(\varkappa\rho)=\sqrt{\frac{2}{\pi\varkappa\rho}}\cos\Big[\varkappa\rho-A^{l_1l_2L}_{KK}(\varkappa)\,l_n\cdot2\varkappa\rho$$
$$-\frac{1}{2}(K+2)\pi-\frac{\pi}{4}+\eta^{l_1l_2}_{KL}\Big];\qquad(67)$$

$$G^{l_1l_2}_{KL}(\varkappa\rho)=\sqrt{\frac{2}{\pi\varkappa\rho}}\sin\Big[\varkappa\rho-A^{l_1l_2L}_{KK}(\varkappa)\,l_n\cdot2\varkappa\rho$$
$$-\frac{1}{2}(K+2)\pi-\frac{\pi}{4}+\eta^{l_1l_2}_{KL}\Big],\qquad(68)$$

where $\eta^{l_1l_2}_{KL}$ is the phase shift of the Coulomb three-particle scattering.

It can be seen from (67) and (68) that $A^{l_1l_2L}_{KK}(\varkappa)$ plays the role of a parameter of the Coulomb scattering. If we consider S-wave scattering ($l_1=l_2=l_1'=l_2'=L=0$), then, using Eq. (63), we obtain for this parameter

$$A_{KK}(\varkappa)$$
$$=\frac{m}{\hbar^2\varkappa}\left[a_1+{}^{1}\langle00|00\rangle^{2}_{K0}|^2a_2+|^{1}\langle00|00\rangle^{3}_{K0}|^2a_3\right]J_{KK},\qquad(69)$$

where

$$J_{KK}=\left(\frac{K}{2}\right)!\,(K+2)\left(\frac{K}{2}+1\right)!\,\Gamma^2\left(\frac{K}{2}+\frac{3}{2}\right)$$
$$\times\sum_{\nu=0}^{K/2}\frac{(-)^{\frac{K}{2}-\nu}}{\Gamma(\nu+1)\,\Gamma(K/2-\nu+3/2)\,\Gamma(K/2-\nu+1)\,\Gamma\left(\nu+\frac{3}{2}\right)}$$
$$\times\sum_{\nu'=0}^{K/2}\frac{(-)^{\frac{K}{2}-\nu'}}{\Gamma(\nu'+1)\,\Gamma(K/2-\nu'+3/2)\,\Gamma(K/2-\nu'+1)\,\Gamma(\nu'+3/2)}$$
$$\times B[\nu+\nu'+1,\ K-(\nu+\nu')+3/2].\qquad(70)$$

In the special case with $K=0$ we obtain from (70)

$$A_{00}(\varkappa)=\frac{16}{3\pi}\frac{m}{\hbar^2\varkappa}(a_1+a_2+a_3).\qquad(71)$$

We can also readily find an expression for determining the phase $\eta^{l_1l_2}_{KL}$. To this end, we shall proceed from Eq. (65) and go over to a new function in accordance with

$$\varphi^{l_1l_2}_{KL}(Z)=Z^{\frac{2K+5}{2}}l^{-Z/2}\Phi^{l_1l_2}_{KL}(Z).\qquad(72)$$

For the function $\Phi^{l_1l_2}_{KL}(Z)$ we obtain the equation

$$Z\frac{d^2\Phi^{l_1l_2}_{KL}(Z)}{dZ^2}+(2K+5-Z)\frac{d\Phi^{l_1l_2}_{KL}(Z)}{dZ}$$
$$-\left[K+\frac{5}{2}+iA^{l_1l_2}_{KKL}(\varkappa)\right]\Phi^{l_1l_2}_{KL}(Z)=0.\qquad(73)$$

This equation is solved by the confluent hypergeometric function:

$$\Phi^{l_1l_2}_{KL}(Z)=F\left(K+\frac{5}{2}+iA^{l_1l_2}_{KK}(\varkappa),\ 2K+5,\ Z\right).\qquad(74)$$

Further, using the well-known representation of the hypergeometric function

$$F(a,\ c;\ Z)=\frac{\Gamma(c)}{\Gamma(c-a)}(-Z)^{-a}G(a,\ a-c+1;\ -Z)$$
$$+\frac{\Gamma(c)}{\Gamma(a)}l^{Z}Z^{a-c}G(c-a,\ 1-a;\ Z),\qquad(75)$$

where

$$G(a,\ b;\ z)=1+\frac{ab}{1!\,Z}+\frac{a(a+1)\,b(b+1)}{2!\,Z^2}+\ldots,\qquad(76)$$

we finally obtain for the phase of the truly three-particle Coulomb scattering

$$e^{2i\eta^{l_1l_2}_{KL}}=\frac{\Gamma\left(K+\frac{5}{2}+iA^{l_1l_2}_{KKL}(\varkappa)\right)}{\Gamma\left(K+\frac{5}{2}-iA^{l_1l_2}_{KKL}(\varkappa)\right)}.\qquad(77)$$

Having in mind (77), we can readily find the following representations for the regular and irregular Coulomb functions of truly three-particle Coulomb scattering in the approximation of the minimal harmonic for fixed values of l_1, l_2, and L:

$$F_{KL}^{l_1l_2}(\varkappa\rho)=\frac{1}{2}\sqrt{\frac{2}{\pi\varkappa\rho}}\,e^{A_{KKL}^{l_1l_2}(\varkappa)\frac{\pi}{2}}$$

$$\times\left\{\left[\frac{\Gamma\left(K+\frac{5}{2}+iA_{KKL}^{l_1l_2}(\varkappa)\right)}{\Gamma\left(K+\frac{5}{2}-iA_{KKL}^{l_1l_2}(\varkappa)\right)}\right]^{1/2}\right.$$

$$\times e^{i\left[-(K+2)\frac{\pi}{2}-\frac{\pi}{4}\right]}W_{-iA_{KKL}^{l_1l_2}(\varkappa),K+2}(-2i\varkappa\rho)$$

$$+\left[\frac{\Gamma\left(K+\frac{5}{2}-iA_{KKL}^{l_1l_2}(\varkappa)\right)}{\Gamma\left(K+\frac{5}{2}+iA_{KKL}^{l_1l_2}(\varkappa)\right)}\right]^{1/2}e^{-i\left[-(K+2)\frac{\pi}{2}-\frac{\pi}{4}\right]}$$

$$\left.\times W_{iA_{KKL}^{l_1l_2}(\varkappa),K+2}(2i\varkappa\rho)\right\}; \qquad (78)$$

$$G_{KL}^{l_1l_2}(\varkappa\rho)=\frac{1}{2i}\sqrt{\frac{2}{\pi\varkappa\rho}}\,e^{A_{KK}^{l_1l_2}(\varkappa)\frac{\pi}{2}}$$

$$\times\left\{\left[\frac{\Gamma\left(K+\frac{5}{2}+iA_{KKL}^{l_1l_2}(\varkappa)\right)}{\Gamma\left(K+\frac{5}{2}-iA_{KKL}^{l_1l_2}(\varkappa)\right)}\right]^{1/2}e^{i\left[-(K+2)\frac{\pi}{2}-\frac{\pi}{4}\right]}\right.$$

$$\times W_{iA_{KKL}^{l_1l_2}(\varkappa),K+2}(-2i\varkappa\rho)$$

$$-\left[\frac{\Gamma\left(K+\frac{5}{2}-iA_{KKL}^{l_1l_2}(\varkappa)\right)}{\Gamma\left(K+\frac{5}{2}+iA_{KKL}^{l_1l_2}(\varkappa)\right)}\right]^{1/2}e^{-i\left[-\frac{\pi}{2}(K+2)-\frac{\pi}{4}\right]}$$

$$\left.\times W_{iA_{KKL}^{l_1l_2}(\varkappa),K+2}(2i\varkappa\rho)\right\}. \qquad (79)$$

The approach presented here can be readily generalized to the case of truly four-particle Coulomb scattering. In particular, instead of (77) we obtain for the four-particle phase

$$e^{2i\delta_{\mu KL}^{l_1l_2l_{12}l_3}}=\frac{\Gamma(\mu+4-i\gamma_{\mu KL}^{l_1l_2l_{12}l_3}(\varkappa))}{\Gamma(\mu+4+i\gamma_{\mu KL}^{l_1l_2l_{12}l_3}(\varkappa))}, \qquad (80)$$

where $\gamma_{\mu KL}^{l_1l_2l_{12}l_3}(\kappa)$ can be specified in analytic form and plays the role of the parameter of the four-particle Coulomb scattering.

In particular, for the case of s-wave scattering we obtain for this parameter

$$\gamma_{\mu K}(\varkappa)=\frac{m}{\hbar^2\varkappa}[a_1+|{}^1\langle 00000|00000\rangle_{\mu0}^2|^2a_2$$

$$+|{}^1\langle 00000|00000\rangle_{\mu0}^3|^2a_3$$

$$+|{}^1\langle 00000|00000\rangle_{\mu0}^4|^2a_4+|{}^1\langle 00000|00000\rangle_{\mu0}^5|^2a_5$$

$$+|{}^1\langle 00000|00000\rangle_{\mu0}^6|^2a_6]J_{\mu K},$$

$$J_{\mu K}=\int\Phi_{\mu K}^{*00000}(\omega)\,\Phi_{\mu K}^{00000}(\omega)\,\frac{d\omega}{\cos\beta}, \qquad (81)$$

where ${}^1\langle|\rangle_{\mu0}^k$ are the four-particle Raynal–Revai coefficients [see Eq. (7)], and

$$\left.\begin{aligned}a_1&=\sqrt{\frac{m_2m_3}{m_2+m_3}}\,Z_2Z_3e^2;\quad a_2=\sqrt{\frac{m_1m_3}{m_1+m_3}}\,Z_1Z_3e^2;\\ a_3&=\sqrt{\frac{m_1m_2}{m_1+m_2}}\,Z_1Z_2e^2;\quad a_4=\sqrt{\frac{m_1m_4}{m_1+m_4}}\,Z_1Z_4e^2;\\ a_5&=\sqrt{\frac{m_2m_4}{m_2+m_4}}\,Z_2Z_4e^2;\quad a_6=\sqrt{\frac{m_3m_4}{m_3+m_4}}\,Z_3Z_4e^2.\end{aligned}\right\} \qquad (82)$$

The results obtained in the present section can be used in developing a method for taking into account simultaneously the Coulomb and nuclear interactions between particles in a microscopic description of three- and four-particle systems.

Method of simultaneous allowance for Coulomb and nuclear interactions in the investigation of systems of three and four particles

The investigation of systems of a few particles between which the interaction potential contains both a short-range part and a long-range Coulomb potential is a very complicated but also very important problem from both the fundamental and practical points of view. In recent years such problems have come to the fore in the theory of few-body systems.

It was shown in Ref. 57 that the use of a hyperspherical basis makes it possible to generalize the variable-phase method to the case of three and four particles when both nuclear and Coulomb interactions are present. Equations for the corresponding phase and amplitude functions were found. The partial amplitudes of truly three- and four-particle scattering were determined, and their connection with the corresponding phases of Coulomb and nuclear scattering was found. Equations were also found that permit determination of the binding energy of the bound states of three and four particles with allowance for the nuclear and Coulomb interactions. In particular, the three-particle phase function $\delta_{KL}^{l_1l_2}(\rho)$ and the amplitude function $A_{KL}^{l_1l_2}(\rho)$ in the approximation of the minimal harmonic with fixed values of the orbital angular momenta satisfy the equations

$$\frac{d\delta_{KL}^{l_1l_2}(\rho)}{d\rho}=-\frac{\pi}{2}N_{KLM}^{l_1l_2}(\rho)\,\rho$$

$$\times[\cos\delta_{KL}^{l_1l_2}\rho F_{KL}^{l_1l_2}(\varkappa\rho)-\sin\delta_{KL}^{l_1l_2}(\rho)\,G_{KL}^{l_1l_2}(\varkappa\rho)]^2; \qquad (83)$$

$$\frac{dA_{KL}^{l_1l_2}(\rho)}{d\rho}=-\frac{\pi}{2}N_{KLM}^{l_1l_2}(\rho)$$

$$\times\rho[\sin\delta_{KL}^{l_1l_2}(\rho)F_{KL}^{l_1l_2}(\varkappa\rho)+\cos\delta_{KL}^{l_1l_2}(\rho)\,G_{KL}^{l_1l_2}(\varkappa\rho)]$$

$$\times[\cos\delta_{KL}^{l_1l_2}(\rho)\,F_{KL}^{l_1l_2}(\varkappa\rho)$$

$$-\sin\delta_{KL}^{l_1l_2}(\rho)\,G_{KL}^{l_1l_2}(\varkappa\rho)]\,A_{KL}^{l_1l_2}(\rho), \qquad (84)$$

where

$$N_{KLM}^{l_1l_2}(\rho)=\frac{2m}{\hbar^2}\int\Phi_{KLM}^{*l_1l_2}(\Omega_1^\rho)$$

$$\times[V_{23}(\mathbf{x}_1)+V_{13}(\mathbf{x}_2)+V_{12}(\mathbf{x}_3)]\,\Phi_{KLM}^{l_1l_2}(\Omega_1^\rho)\,d\Omega_1^\rho; \qquad (85)$$

$V_{ij}(\mathbf{x})$ is the potential of the nuclear interaction; $F_{KL}^{l_1 l_2}(\kappa\rho)$ and $G_{KL}^{l_1 l_2}(\kappa\rho)$ are the regular and irregular Coulomb functions of truly three-particle scattering, which are expressed in analytic form by (78) and (79).

The four-particle equations analogous to Eqs. (83) and (84) have the form

$$\frac{d\delta_{\mu KL}^{l_1 l_2 l_{12} l_3}(\rho)}{d\rho} = -\frac{1}{\varkappa} N_{\mu KL}^{l_1 l_2 l_{12} l_3}(\rho) \times [\cos_{\mu KL}^{l_1 l_2 l_{12} l_3}(\rho) F_{\mu KL}^{l_1 l_2 l_{12} l_3}(\varkappa\rho) - \sin \delta_{\mu KL}^{l_1 l_2 l_{12} l_3}(\rho) G_{\mu KL}^{l_1 l_2 l_{12} l_3}(\varkappa\rho)]^2; \quad (86)$$

$$\frac{dA_{\mu KL}^{l_1 l_2 l_{12} l_3}(\rho)}{d\rho} = -\frac{1}{\varkappa} N_{\mu KL}^{l_1 l_2 l_{12} l_3}(\rho) \times [\sin_{\mu KL}^{l_1 l_2 l_{12} l_3}(\rho) F_{\mu KL}^{l_1 l_2 l_{12} l_3}(\varkappa\rho) + \cos \delta_{\mu KL}^{l_1 l_2 l_{12} l_3}(\rho) G_{\mu KL}^{l_1 l_2 l_{12} l_3}(\varkappa\rho)] \times [\cos \delta_{\mu KL}^{l_1 l_2 l_{12} l_3}(\rho) F_{\mu KL}^{l_1 l_2 l_{12} l_3}(\varkappa\rho) - \sin \delta_{\mu KL}^{l_1 l_2 l_{12} l_3}(\rho) G_{\mu KL}^{l_1 l_2 l_{12} l_3}(\varkappa\rho)] A_{\mu KL}^{l_1 l_2 l_{12} l_3}(\rho); \quad (87)$$

$$N_{\mu KL}^{l_1 l_2 l_{12} l_3}(\rho) = \frac{2m}{\hbar^2} \int \Phi_{\mu KLM}^{*l_1 l_2 l_{12} l_3}(\omega_1^\rho) \times [V_{23}(\xi_1) + V_{34}(\xi_2) + V_{14}(\xi_3) + V_{12}(\xi_4) + V_{13}(\xi_5) + V_{24}(\xi_6)] \Phi_{\mu KLM}^{l_1 l_2 l_{12} l_3}(\omega_1^\rho)\, d\omega_1^\rho, \quad (88)$$

where $F_{\mu KL}^{l_1 l_2 l_{12} l_3}(\kappa\rho)$ and $G_{\mu KL}^{l_1 l_2 l_{12} l_3}(\kappa\rho)$ are the regular and irregular Coulomb functions of truly four-particle scattering, which are also given in analytic form.[56]

It can be seen from the results of the previous sections that in the few-body Coulomb problem the MHF has a very important advantage over other methods of treating such problems. When the MHF is used, a large proportion of the calculations can be made analytically. Analytic expressions are obtained for the basic characteristics in a definite approximation. These expressions can be used in a more accurate solution of the problem. The results of the following section confirm this conclusion.

Approximate analytic solution of the problem of a three-body bound state in atomic physics

In Ref. 58 one of the possibilities for developing a model-free approach to the investigation of the three-body problem in atomic physics was described. In the first approximation of this approach analytic expressions are obtained for not only the wave functions but also the energy spectra of the atoms.

If the wave function of the bound state of three charged particles is represented as an expansion with respect to hyperspherical functions,

$$\Psi(\mathbf{x}_i, \mathbf{y}_i) = \sum_{K l_1 l_2} \frac{\varphi_{KL}^{l_1 l_2}(\rho)}{\rho^{5/2}} \Phi_{KLM}^{l_1 l_2}(\Omega_i), \quad (89)$$

then for the hyperradial functions $\varphi_{KL}^{l_1 l_2}(\rho)$ a system of coupled differential equations is obtained:

$$\frac{d^2\varphi_{KL}^{l_1 l_2}(\rho)}{d\rho^2} - \left[\varkappa_0^2 + \frac{(K+2)^2 - \frac{1}{4}}{\rho^2}\right] \varphi_{KL}^{l_1 l_2}(\rho) = \frac{2}{\rho} \sum_{K' l_1' l_2'} I(K, K'; l_1, l_2; l_1' l_2') \varphi_{K'L}^{l_1' l_2'}(\rho). \quad (90)$$

where $I(K, K'; l_1 l_2, l_1', l_2')$ is an analytic function of its arguments:

$$I(K, K'; l_1, l_2; l_1', l_2') = \frac{\hbar^2 \varkappa^2}{m} A_{KK'}^{l_1 l_2 l_1' l_2'}(\varkappa); \quad (91)$$

$A_{KK'}^{l_1 l_2 l_1' l_2'}(\kappa)$ is given by Eq. (63).

We consider the problem of the helium atom and heliumlike ions. In the system (90) we equate all the orbital angular momenta to zero and retain the first equation with fixed K:

$$\frac{d^2\varphi_K}{d\rho^2} - \left[\varkappa_0^2 + \frac{(K+2)^2 - \frac{1}{4}}{\rho^2}\right] \varphi_K(\rho) = \frac{2}{\rho} I(K) \varphi_K(\rho). \quad (92)$$

Taking into account the asymptotic form of the solution of this equation, we seek the general solution in the form

$$\varphi_K(\rho) = C_K \rho^{K+\frac{5}{2}} \exp(-\varkappa_0 \rho) U_K(\rho). \quad (93)$$

Substituting (93) in (92), we obtain for $U_K(\rho)$ the equation

$$rU_K''(r) + [2(K+5/2) - r] U_K'(r) - \left[(K+5/2) + \frac{1}{\varkappa_0} I(K)\right] U_K(r) = 0, \quad (94)$$

where $r = 2\kappa_0 \rho$. The solution of this equation is a confluent hypergeometric function of the form

$$U_K(r) = F\left[K + \frac{5}{2} + \frac{1}{\varkappa_0} I(K),\ 2\left(K + \frac{5}{2}\right),\ r\right]. \quad (95)$$

Now, using the standard method, we shall take as our starting point the fact that the three-particle system is bound and impose on the function (95) the condition that it become a polynomial. We obtain a quantization condition for the energy:

$$K + \frac{5}{2} + (I(K)/\varkappa_0) = -N, \quad (96)$$

where $N = 0,1,2,\ldots$.

From (96) we obtain for the energy spectrum of the helium atom and heliumlike ions

$$E_J = -|J(K)|^2/2J(K)^2, \quad (97)$$

where $J(K) = N + K + 5/2$.

With allowance for (89), (93), and (95), and using the connection between the confluent hypergeometric function and the generalized Laguerre polynomials, we obtain the following general analytic expression for the wave functions of the helium atom and heliumlike ions:

$$\Psi(\mathbf{x}, \mathbf{y}) = (2\varkappa_0^J)^3 \left[\frac{(J-K-5/2)!}{2K+5}\right]^{1/2} \frac{1}{[(J+K+3/2)!]^{3/2}} \times \rho^K \exp(-\varkappa_0^J \rho) L_{J+K+3/2}^{2K+4}(2\varkappa_0^J \rho) \Phi_{K00}^{00}(\Omega). \quad (98)$$

We now consider definite atoms.

For n-helium we have a completely symmetric state and, therefore, grand orbital $K = 0$. From (98) for the wave function of n-helium we obtain

$$\Psi_n^J(\mathbf{x}, \mathbf{y}) = (2\varkappa_n^J)^3 \frac{[(J+5/2)!]^{1/2}}{\sqrt{5}\,[(J+3/2)!]^{3/2}} \times \exp(-\varkappa_n^J \rho) L_{J+3/2}^4(2\varkappa_n^J \rho) \Phi_{000}^{00}(\Omega). \quad (99)$$

where $(\kappa_n^J)^2 = -2mE_n^J$; and E_n^J is the energy of the atom corresponding to the level with quantum number J. The energy spectrum of n-helium is determined by

$$E_n^J = -(|I(0)|^2/2J^2), \qquad (100)$$

where $J = 5/2, 7/2, 9/2, \ldots$.

For the ground state $J = 5/2$, and from the expressions (63), (91), and (100) we obtain $E_n^{5/2} = -68.1$ eV, which is about 86% of the experimental value of this quantity. The wave functions of the excited states of n-helium are obtained from (99) for $J = 7/2, 9/2, \ldots$. Of course, they are all orthogonal to each other and to the ground-state function. Further, from the expression (100) we obtain for the first excited state of n-helium the value $E_n^{7/2} = -34.7$ eV, which is about 60% of the experimental value. With increasing excitation energy the contribution of the first equation to the energy of the atom decreases, and this indicates that in these spatial configurations of n-helium the electron states become so inequivalent that the restriction to one equation of the system (90) is a very rough approximation, and it is necessary to take into account more terms in the expansion (89) than in the case of the ground state. Such a result corresponds completely to the picture in n-helium. The excitations mentioned above are single-particle excitations, and only one electron participates in them, while the other electron together with the nucleus of the atom forms a hydrogen-like core.

In the case of 0-helium the two electrons in the atom are in an antisymmetric spatial state, since their spins are parallel. Therefore, the grand orbital $K = 0$ is forbidden for such a state, and the grand orbital $K = 2$ must make the main contribution. From (98) we obtain for the wave function of 0-helium

$$\Psi_0^J(x, y) = \frac{8}{3}(\kappa_0^J)^3 \frac{[(J-9/2)!]^{1/2}}{[(J-7/2)!]^{3/2}} \rho^2 \exp(-\kappa_0^J \rho) L_{J+7/2}^8(2\kappa_0^J \rho)\, \Phi_{200}^{00}(\Omega), \qquad (101)$$

and the energy spectrum of 0-helium is determined by

$$E_0^J = -(|I(2)|^2/2J^2), \qquad (102)$$

where $J = 9/2, 11/2, 13/2, \ldots$. For the ground state $(J = 9/2)$ we obtain from (102) the result -48.7 eV, which is about 86% of the experimental value.

It is interesting to note that in this scheme there is no connection between the ground states of para- and orthohelium, this being expressed by the fact that the function $I(K,K')$ vanishes identically for $K = 0$ and $K' = 2$. This leads naturally to an explanation of the existence of two forms of helium in nature. This fact, as is well known, is due to the symmetry of the wave function, which in this scheme is very different for $K = 0$ and $K' = 2$.

The energy spectrum of the levels of heliumlike ions with two electrons, H^-, Li^+, Be^{++}, also consists of two-systems of levels, one of which contains singlet levels (para-ions) and the other triplet levels (ortho-ions). Therefore, the states of the para-ions and ortho-ions can be described by the functions (99) and (101), respectively, and the energy spectrum can be calculated in accordance with (100) and (102) with the only difference that these expressions contain the values of κ and $I(K)$ corresponding to the ions H^-, Li^+, Be^{++}. One can easily show that here too the results reproduce the qualitative picture.

Adiabatic approximation in the method of hyperspherical functions

Recently use has often been made in the MHF of an adiabatic approximation of Born–Oppenheimer type, which was first applied in the helium problem in Ref. 59. Later, a similar approximation was used for other atomic systems, in particular for H^- (Ref. 60), for (pe^-e^+) (Ref. 61), and for $(e^+e^-e^-)$ (Ref. 62). The part played by correlations in calculations of autoionization states of the helium atom was investigated in the same approximation in Refs. 63 and 64. An original variant of the adiabatic approximation was proposed in Ref. 65. This approximation uses not an expansion with respect to the hyperspherical functions (89) but an expansion of the three-nucleon wave function with respect to a complete set of orthogonal basis functions $B_\lambda(\rho,\Omega)$ that are solutions of the equation

$$\left\{-\left[L^2(\Omega) - \frac{(D-1)(D-3)}{4}\right] \times \rho^{-2} + V(\rho, \Omega)\right\} B_\lambda(\rho, \Omega) = \omega_\lambda(\rho) B_\lambda(\rho, \Omega), \qquad (103)$$

where $L(\Omega)$ is the operator of the total orbital angular momentum in a space with dimension D (for three bodies, $D = 6$).

Equation (103) differs from the original three-particle Schrödinger equation in hyperspherical coordinates in that there is no operator term $d^2/d\rho^2$ and the eigenvalue $K^2 = 2mE/\hbar^2$ is replaced by the function $\omega_\lambda(\rho)$, which depends on ρ as a parameter. We expand the function $B_\lambda(\rho,\Omega)$ with respect to three-particle hyperspherical functions,

$$B_\lambda(\rho, \Omega) = \sum_{K\widetilde{l_x l_y}} \chi_{KL,\lambda}^{l_x l_y}(\rho)\, \Phi_{KLM}^{l_x l_y}(\Omega). \qquad (104)$$

We use the orthonormality of the basis functions B_λ and $\Phi_{KLM}^{l_x l_y}$ on the hypersphere, and from (104) we obtain for the function $\chi_{KL,\lambda}^{l_x l_y}(\rho)$ the condition of orthonormality

$$\sum_{K\widetilde{l_x l_y}} \chi_{KL,\lambda}^{l_x l_y} \chi_{KL,\lambda'}^{l_x l_y} = \delta_{\lambda\lambda'}.$$

We substitute (104) in Eq. (103), multiply the resulting expression by $\Phi_{KLM}^{l_x l_y}$, and integrate on the hypersphere of unit radius. As a result, we obtain the matrix equation

$$\sum_{K'\widetilde{l'_x l'_y}} \left[\frac{L_K(L_K+1)}{\rho^2}\delta_{KK'}\delta_{l_x l'_x}\delta_{l_y l'_y} + \langle K l_x l_y | V | K' l'_x l'_y\rangle\right] \chi_{K'L,\lambda}^{l'_x l'_y}(\rho) = \omega_\lambda(\rho)\, \chi_{KL,\lambda}^{l_x l_y}(\rho), \qquad (105)$$

where the eigenvalues $\omega_\lambda(\rho)$ and the eigenvectors $\chi_{KL,\lambda}^{l_x l_y}(\rho)$ are parametric functions of the variable ρ.

The solutions of the system (105) for the functions $\chi_{KL,\lambda}^{l_x l_y}(\rho)$, make it possible by means of Eq. (103) to obtain new basis functions $B_\lambda(\rho,\Omega)$. Expanding the three-particle wave function with respect to the new functions,

$$\Psi(\bar{x}, \bar{y}) = \rho^{-5/2} \sum_\lambda \zeta_\lambda(\rho) B_\lambda(\rho, \Omega), \qquad (106)$$

and introducing this expansion in the original three-particle Schrödinger equation for the hyperradial functions $\zeta_\lambda(\rho)$, we obtain a system of coupled equations. In the unbound adiabatic approximation[65] we restrict ourselves to a single

term in the expansion (106). In this approximation we ignore the change of the new basis functions $B_\lambda(\rho,\Omega)$ with ρ, this being equivalent to separation of the composite motion in the three-body system into hyperorbital and hyperradial motion. In Ref. 65 the fairly high accuracy of such an approximation was demonstrated for the example of some potentials with repulsion at short distances.

In Refs. 63, and 64 the part played by correlations in the autoionization states of the helium atom was investigated by means of the adiabatic procedure of Ref. 59, in which the point of departure is an equation of the type (103), but the determination of the wave function reduces to solution of the system of equations

$$\left[\left(\frac{d^2}{d\rho^2}-\frac{1}{\rho^2}U_\mu(\rho)+2E\right)\delta_{\mu\nu}+\left(\Phi_\mu,\frac{\partial\Phi_\nu}{\partial\rho}\right)\frac{d}{d\rho}+\left(\Phi_\mu,\frac{\partial^2\Phi_\nu}{\partial\rho^2}\right)\right]F_\nu(\rho)=0. \quad (107)$$

The channel functions Φ_ν are eigenfunctions of the operator $\lambda^2-\rho V(\omega)$ corresponding to the eigenvalues $U_\mu(\rho)$, which depend parametrically on ρ. When the eigenfunctions are found, the Sturm–Liouville problem is reduced to the solution of a system of homogeneous algebraic equations. For this, one uses the eigenfunctions of the operator $\hat{\lambda}^2$ of the total orbital angular momentum, which can be expressed in terms of Jacobi polynomials and bipolar harmonics. The extent to which the radial correlations are taken into account depends on the highest power of the Jacobi polynomials included in the complete set.

The determination of the potentials $U_\mu(R)$ is based on a procedure of extrapolation with respect to the dimension of the subspace with $l_1=l_2=0$, on which the operator $\lambda^2-\rho V(\omega)$ is diagonalized. The calculations show that the values of the potentials $U_\mu(\rho)$ obtained when the dimension of the subspace is increased tend to their asymptotic values but do not reach them. Knowing the potentials for several values of the dimension of the subspace, one can extrapolate their values to infinite dimension in accordance with the Padé formula

$$U_i(\rho)=\lim_{n\to\infty}U_i^{(n)}(\rho)=\lim_{n\to\infty}\frac{a_i(\rho)+n^2b_i(\rho)}{1+n^2c_i(\rho)}, \quad (108)$$

where a_i, b_i, and c_i are parameters that are to be determined.

The values obtained for the extrapolated potentials have good asymptotic behavior. The values of the energy obtained in the adiabatic approximation are given in Table III. Also given there are the results obtained by the authors of Refs. 59, 65, and 67 and the available experimental data.[68] It can be seen from the table that the values of the energy obtained in Refs. 63 and 64 (except for $2s^2$ and $3s^2$) are in the best agreement with experiment. This result shows that for inequivalent electrons an important part is played by the radial correlations, which to a certain degree are taken into account in these studies. With regard to the $2s^2$ and $3s^2$ states, it appears that the angular correlations, which were not taken into account in these studies, are important for them.

The problem of convergence in the method of hyperspherical functions for Coulomb problems

The system of MHF hyperradial equations (57) is infinite, and there is therefore a problem of convergence with respect to the number of equations taken into account. In the review of Ref. 69, devoted to the MHF in the microscopic theory of bound states and scattering of many-particle systems, this problem is discussed for nuclear systems. The problem of convergence in the MHF for Coulomb problems has been discussed in many studies.[70–74] When the number of necessary equations increases, the difficulties of numerical solution of such a system of coupled equations also naturally increase. To simplify the problem a variational approach to the solution of the hyperradial equations is often used. The realization of this approach involves the so-called physical basis of hyperspherical functions constructed by Éfros.[75] To solve the system of hyperradial equations, one uses a method of expansion with respect to a certain complete (on the interval $[0,\infty]$) hyperradial basis $\{R_n(\rho)\}$.

$$\chi_{KL}^{l_x l_y}(\rho)=\sum_n C_{nKL}^{l_x l_y}R_n(\rho), \quad (109)$$

where $R_n(\rho)$ are special hyperradial functions satisfying an orthogonality condition.

By means of the expansion (109) the problem of the bound state of a three-body system is reduced to the problem of finding the eigenvalues and eigenvectors of a symmetric matrix. The dimension of this matrix depends on the form chosen for the basis hyperradial functions $R_n(\rho)$. In Refs. 76 and 77 artificially chosen basis hyperradial functions were proposed for the first time, the free parameter in them being chosen to achieve the best convergence of the expansion. This parameter, and also the form of the basis functions do not depend on either the orbital angular momenta (l_x,l_y,L) or the grand orbital (K). For the Coulomb three-

TABLE III. Energies of the autoionization state of the ^{1}S helium atom converging on the $n=2$ and $n=3$ thresholds of the helium ion (He^+), eV.

State	GSK method			Configuration superposition method (Ref. 67)	Experiment (Ref. 68)
	Refs. 63 and 64	Ref. 59	Ref. 66		
$2s^2$	58.34	57.99	56.91	57.91	57.82
$2s3s$	62.93	63.10	62.64	63.00	62.94
$2s4s$	64.17	64.25	64.21	64.20	64.18
$2s5s$	64.68	64.71	64.70	—	64.67
$2s6s$	64.93	—	64.94	—	—
$3s^2$	69.71	—		69.78	—
$3s4s$	71.39	—		71.39	—
$3s5s$	72.06	—		—	—
$3s6s$	72.38			—	—

body problem such a basis was used in Ref. 73. In Ref. 72 a hyperradial basis of a different form, in which a dependence on the grand orbital is present, was proposed, but in this basis too the free parameter $\hbar\omega$ is selected. Of course, the successes of the MHF depend to a large degree on the choice of an optimal hyperradial basis. In the Coulomb three-body problem the most accurate values for the energy are obtained by applying variational methods. In particular, if we use an expansion with respect to exponential wave functions that depend on the interparticle distances, then, taking into account the large number of terms, we can ensure record accuracy from the point of view of atomic spectroscopy.[78] For the problem of a two-electron atom such a wave function arises, strictly speaking, only in the approximation in which the repulsive interaction between the electrons is ignored. On the other hand, it has frequently been emphasized[79] that allowance for the interaction between the electrons must significantly change the analytic form of the wave functions of three-particle atoms. This circumstance strengthens the interest in approximate analytic solutions of the initial three-particle equations.[58]

In Ref. 74 a method was developed for solving systems of one-dimensional coupled differential equations using power series. The possibility of using this method in few-body theory with a hyperspherical basis was studied. The helium atom and the ground states of the positron ion and negative hydrogen were considered. Power series of the argument and its logarithm were used to solve the coupled radial equations of the MHF even earlier in Ref. 81. In Ref. 74 it was shown that application of a power-law expansion in the Coulomb three-body problem ensures convergence with respect to the number of harmonics taken into account for the wave function, $1/K_m^2$, and for the energy, $1/K_m^4$.

The method of hyperspherical functions in molecular physics and chemistry

So far in this section we have considered problems of atomic physics that can be successfully solved by means of the MHF. This method is applied with no less success in molecular physics and, particularly, in chemistry. A comparatively detailed bibliography can be found in Refs. 82 and 83. A triatomic molecule has been studied by the MHF using two-body potentials of van der Waals type.[84] Molecular scattering has also been considered with these potentials.[85] Double excited states of molecular type were investigated by means of hyperspherical coordinates in Ref. 86.

5. INVESTIGATION OF THE STRUCTURE OF MULTIQUARK SYSTEMS USING THE METHOD OF HYPERSPHERICAL FUNCTIONS

In this section we shall present the main results obtained by using the MHF to investigate the structure of multiquark systems.

The structure of multiquark systems

In recent times the MHF has been used intensively in the nonrelativistic model of quark systems. The method has been used to make numerous calculations of the structure of heavy baryons and other multiquark systems using various potentials of the interquark interaction. The MHF was first used in the physics of heavy baryons in Ref. 87, in which the quark system CCC was considered, and the low-lying radial excitations of this system with spin and parity $3/2^+$ were investigated. Baryons made from quarks with the same masses were also investigated by the MHF in Ref. 88. In Refs. 89 and 90 the same method was used to study baryons formed from quarks with different masses. Recently, wide use has been made of the Cornell potential,[91] by means of which quark systems were very intensively studied in the framework of the MHF in the papers of Badalyan's group.[92,93] The same method was used in the semiclassical approximation in Ref. 94 to calculate the spectra of some $6q$, $9q$, and $12q$ states. A logarithmic potential of the quark–quark interaction was used in the MHF in Ref. 95. Earlier, the method was used in Ref. 105 to study the properties of heavy baryons.

In Ref. 96 the MHF was used to study the problem for a system of three quarks. Analytic expressions for the effective potential were obtained. It was shown that the contribution of the spin–spin interaction is nonadditive, this leading to a change of the width of the core and the depth of the well for the effective potential. The states corresponding to a breathing excitation mode were found by a second solution of the eigenvalue problem in the three-quark system. The spectrum of the $3q$ system, the rms radius, and the form factor of elastic eN scattering were studied. We shall dwell on the results of this paper in more detail.

In the nonrelativistic potential model[97-106] the Hamiltonian of the system of quarks, $H = T + V$ (where $T = \Sigma_i (p_i^2/2m_i) + \Sigma_i m_i$), can be expressed, with the center-of-mass motion separated, as follows:

$$T = \frac{1}{2M}\mathbf{P}^2 + \frac{1}{2M}\sum_{i<j}\left(\frac{\mathbf{p}_i^2}{m_i} - \frac{\mathbf{p}_j^2}{m_j}\right) m_i m_j + Mc^2, \quad (110)$$

where $\mathbf{P} = \Sigma_i \mathbf{p}_i$ and $M = \Sigma_i m_i$. The potential energy $V = \Sigma_{i<j} \lambda_i \lambda_j V(r_{ij})$ of the system consists of a central term $V_c(r) = Br^2 + C$, which determines the confinement, and a spin–spin term $V_c = \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j \delta(\mathbf{r})$, or $V_b = \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j e^{-r^2/\Lambda^2}$, which determines the N–Δ mass difference. In Refs. 91–99 study of the properties of three- and six-quark systems in the framework of a nonrelativistic model with harmonic-oscillator functions led to the proposal of some forms of quark–quark potentials:

$$V_{ij} = (\lambda_i \lambda_j)\,[f(r_{ij}) + (\boldsymbol{\sigma}_i \boldsymbol{\sigma}_j)\, g(r_{ij})]; \quad (111)$$

$$f(r) = Ae^{-r^2/\alpha^2} - Br^2 + c + K\delta(r), \quad (112)$$

where

$$g(r) = \frac{2}{3} K\delta(r) \quad \text{or} \quad g(r) = \frac{2}{3} Ke^{-r^2/\Lambda};$$

$$f(r) = -ar^2 + \frac{\alpha_s \hbar c}{4r}, \quad g(r) = -\frac{\pi \alpha_s \hbar^3}{6m^2 c}\delta(r);$$

$$V_{ij} = (\lambda_i \lambda_j)\left[-ar_{ij} + \frac{\alpha_s}{4r_{ij}} - \frac{\pi\alpha_s}{4m_q^2}\left(1 + \frac{2}{3}\boldsymbol{\sigma}_i \boldsymbol{\sigma}_j\right)\delta(r_{ij})\right]. \quad (113)$$

This model was successfully developed and made it possible to obtain a number of important results both on the masses and on the decay properties of light baryonic systems. However, the use of harmonic-oscillator basis functions has the consequence that to describe the masses of the n-quark system and reproduce its size one must use different values of the oscillator radius b_0 ($b_0 = 0.4$ and $b_0 = 0.8$, respective-

ly). But the parameters of the employed quark–quark potential are very sensitive to the value of b_0. We illustrate this fact by the example of the parameter K, which determines the N–Δ mass difference:

$$V_b(r) = \sum_{i<j} \lambda_i \lambda_j K \left(1 + \frac{2}{3}\sigma_i\sigma_j\right)\delta(r_{ij}). \quad (114)$$

The mass difference $\Delta M = M_\Delta - M_N = 293$ MeV is determined as

$$\langle \Psi_\Delta | V_b(r) | \Psi_\Delta \rangle - \langle \Psi_N | V_b(r) | \Psi_N \rangle = \Delta M. \quad (115)$$

For the three-quark system the corresponding wave functions have the symmetry

$$\Psi_\Delta : [1^3]_c [3]_L \{[3]_T \times [3]_S\}_{[3]_{TS}};$$
$$\Psi_N : [1^3]_c [3]_L \{[21]_T \times [21]_S\}_{[3]_{TS}},$$

where for the spin functions χ_i we have

$$[3]_S = \left\{\left(\frac{1}{2}\frac{1}{2}\right)^1 \frac{1}{2}\right\}_{\frac{3}{2}} = \chi_1;$$

$$[21]_S = \begin{cases} \left|\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\right| = \left\{\left(\frac{1}{2}\frac{1}{2}\right)^1 \frac{1}{2}\right\}_{\frac{1}{2}} = \chi_1; \\ \left|\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\right| = \left\{\left(\frac{1}{2}\frac{1}{2}\right)^0 \frac{1}{2}\right\}_{\frac{1}{2}} = \chi_0; \end{cases}$$

then for the Δ particle the spin–isospin part of the wave function has the form $[3]_{TS} = \tau_1\chi_1$, and accordingly for the N particle $[3]_{TS} = (1/\sqrt{2})(\tau_1\chi_1 + \tau_0\chi_0)$. The mean value over the functions χ_1 and χ_0 of $\sigma_1\cdot\sigma_2$ for $S=1$ is $\langle\chi_1|\sigma_1\cdot\sigma_2|\chi_1\rangle = 1$, and for $S=0$ it is $\langle\chi_0|\sigma_1\cdot\sigma_2|\chi_0\rangle = -3$. Further, one can find the matrix elements of the spin–spin interaction for the N and Δ particles:

$$\left.\begin{aligned} &\langle \Psi_N | \sum_{ij} \lambda_i\lambda_j\sigma_i\sigma_j\delta(r_{ij}) | \Psi_N\rangle \\ &= 3\left(-\frac{2}{3}\right)\frac{1}{2}\langle \tau_1\chi_1 + \tau_0\chi_0 | \sigma_1\sigma_2 | \tau_1\chi_1 \\ &\qquad + \tau_0\chi_0\rangle \langle 0s|\delta(\mathbf{r})|0s\rangle = 2\bar{V}; \\ &\langle \Psi_\Delta | \sum_{ij} \lambda_i\lambda_j\sigma_i\sigma_j\delta(\mathbf{r}_{ij}) | \Psi_\Delta\rangle = -2\bar{V}, \end{aligned}\right\} \quad (116)$$

where

$$\bar{V} = \langle 0s|\delta(\mathbf{r})|0s\rangle = \frac{1}{(2\pi b^2)^{3/2}} \int e^{-\frac{r^2}{2b^2}} \delta(\mathbf{r})\, d\mathbf{r} = \frac{1}{(2\pi b^2)^{3/2}}. \quad (117)$$

We now find the spin–spin matrix element for N and Δ particles,

$$\langle \Psi_N | V_b | \Psi_N\rangle = K\left(-2 + \frac{2}{3}2\right)\bar{V} \quad \text{и} \quad \langle \Psi_\Delta | V_b | \Psi_\Delta\rangle = K\left(-2 + \frac{2}{3}(-2)\right)\bar{V}, \quad (118)$$

and the mass difference.

$$\Delta M = \langle \Psi_\Delta | V_b | \Psi_\Delta\rangle - \langle \Psi_N | V_b | \Psi_N\rangle = -\frac{8}{3} K \frac{1}{(2\pi)^{3/2} b^3}. \quad (119)$$

Then the constant K is determined by

$$K = -\frac{3}{8}(2\pi)^{3/2} b^3 \Delta M. \quad (120)$$

In the case, for example, when $b_N = b_\Delta = b_0 = 0.8$ F, we have $K = -886$ MeV·F^3.

One can determine the oscillator parameters for the wave functions of the N and Δ systems by analyzing the results of the calculation in the method of hyperspherical functions. In this case $b_N = 0.39$ F and $b_\Delta = 0.42$ F, and this leads to the value $K = (2\pi)^{3/2}\Delta M/(1/b_N^3 - 5/b_\Delta^3) = -173$ MeV·F^3. Thus, the estimates show that the parameter K is strongly changed when allowance is made for collective effects in the model of the three-quark system. It should be noted that in Ref. 104, in which the structure of baryons was investigated on the basis of the Faddeev equations, the need to take into account these effects was demonstrated. In Ref. 96 a method that makes it possible to determine more accurately the parameters of the quark–quark potentials in the nonrelativistic quark model was proposed. In this case one uses the method of hyperspherical functions in which the oscillators do not have a free parameter, the radius, but there is a collective variable ρ, so that as a result one obtains a self-adjusted quark system in which the parameters of the qq interaction determine the size of the system. In this approach the spectrum of the $3q$ system, the size, and the elastic-scattering form factor are studied in a unified manner. We shall dwell in somewhat more detail on the method of calculation.

In the MHF the wave function of the n-quark system is represented in the form of an expansion with respect to K-harmonic polynomials:

$$\Psi(1, 2, \ldots, n) = \rho^{-\frac{1}{2}(3n-1)} \sum_{K\gamma} \chi_{K\gamma}(\rho)\, |nK\gamma\rangle, \quad (121)$$

where $\gamma = [f]\varepsilon_c LST$ and $\int \chi^2_{K\gamma}(\rho)\, d\rho = 1$. The Hamiltonian has the form

$$H = -\frac{\hbar^2}{2m}\frac{1}{\rho^{3n-4}}\frac{d}{d\rho}\left(\rho^{3n-4}\frac{d}{d\rho}\right) - \frac{\hbar^2}{2m}\frac{1}{\rho^2}\Delta_\theta + V(\rho), \quad (122)$$

and the system of equations for finding the eigenfunctions $\chi(\rho)$ and the eigenvalues E is written as

$$\left\{\frac{d^2}{d\rho^2} - \frac{L(L+1)}{\rho^2} - \frac{2m}{\hbar^2}\left(E + W^{K\gamma}_{K\gamma}(\rho)\right)\right\}\chi_{K\gamma}(\rho) = \frac{2m}{\hbar^2}\sum_{K'\gamma' \neq K\gamma} W^{K'\gamma'}_{K\gamma}(\rho)\,\chi_{K'\gamma'}(\rho). \quad (123)$$

where $L = K + (3A-6)/2$, and $W^{K\gamma}_{K\gamma}(\rho)$ is an effective potential. The calculations are simplified appreciably if one uses the two-particle coefficients of fractional parentage, by means of which one can integrate over the coordinates of $n-2$ particles. In this case the effective potential $W(\rho)$ takes the form

$$W^{K\gamma}_{K\gamma}(\rho) = \langle nK[f]\varepsilon_c LST | \hat{V} | nK[f]\varepsilon_c LST\rangle = \frac{n(n-1)}{2}\sum_{K_2[f_2]\varepsilon_{c2}L_2S_2T_2L_0S_0T_0\varepsilon_0\Lambda L'K'} \times \langle nK[f]\varepsilon_c LST | n-2|K_2[f_2]\varepsilon_{2c}L_2S_2T_2,\ \Lambda(L''K'')\rangle$$
$$L_0S_0T_0\varepsilon_{0c}\rangle^2 \langle \varepsilon_{0c}S_0T_0 | W_{\lambda\sigma\tau} | \varepsilon_{0c}S_0T_0\rangle\, R^{K,K}_{K'',L_0}(\rho), \quad (124)$$

where $R^{K,K}_{K'',L_0}(\rho)$ is the orbital part and $\langle \epsilon_{0c} S_0 T_0 | W_{\lambda\sigma\tau} | \epsilon_{0c} S_0 T_0 \rangle$ is the color–spin–isospin part of the matrix elements of the two-particle quark–quark interaction. Further, one obtains analytic expressions for the effective potential in the MHF for the three-quark system with $L = 0$ (three particles in the s state) having symmetry $[3]_L$, $[3]_{ST}$, $[1]^3_C$. For the N isobar $ST = 1/2$ and $1/2$, and for the Δ isobar $ST = 3/2$ and $3/2$.

In accordance with Eq. (124), the effective potential for this case takes the form

$$W(\rho) = \frac{8}{\pi} \int_0^1 V(\rho\sqrt{2Z})\sqrt{Z(1-Z)}\,dZ. \qquad (125)$$

Making the simple integration over Z for the quark–quark potentials (111) and (113), we obtain the expressions

$V(r)$	$-ar$	$-a'r^2$	$b'r$	$\varkappa\delta(\mathbf{r})$	$V_1 e^{-r^2/\alpha^2}$
$W(\rho)$	$-0.96\,a\rho$	$-a'\rho^2$	$1.2\,b\rho$	$\frac{1}{\pi}\frac{2}{\pi}\frac{\varkappa}{\rho^3}$	$\frac{2V_1\alpha^2}{\rho^2} I_1(\rho^2/\alpha^2)$

where $I_1(\rho^2/\alpha^2)$ is a modified Bessel function.

An important feature of the δ-function spin–spin interaction is that its contribution to the effective potential acts in the complete region of the collective variable ρ.

In Ref. 96 calculations were made of the effective potential, the eigenvalues, eigenfunctions, densities, rms radii, and form factor of elastic eN scattering for the investigated three-quark system. The results of the calculations are given in Tables IV and V and in Figs. 15–22.

In the MHF calculations the constant C of the Harvey potential[97] (first row of Table IV) was renormalized in such a way as to reproduce the nucleon mass, and the parameter K of this quark–quark potential was changed in order to describe the mass difference ($M_\Delta - M_N = 293$ MeV). Figure 15 shows the effective potential, the first two solutions in it, and the corresponding wave functions for the three-quark system without allowance for spin–spin splitting ($K = 0$) (see also Tables IV and V).

It can be seen that the calculation in the method of hyperspherical functions leads to a significant increase of the constant C. In the three-quark system there appears a collective monopole level with energy ~ 1.8 GeV. In Figs. 16 and 17 and in Tables IV and V we show the results of calculation for the N and Δ particles with inclusion of the δ-function spin–spin interaction. In contrast to the results of the calculation with harmonic-oscillator functions, in which the spin–pin interaction enters additively, the inclusion of the spin–spin interaction in the MHF changes the width of the core and the depth of the well (the width of the core is less and the depth of the effective potential greater for the nucleon). Because of this, the energy of the monopole-excited state for the N particle is less (~ 1.7 GeV) than for Δ (1.9 GeV), and the radius for N ($r_N = 0.39$ F) is less than for Δ ($r_\Delta = 0.42$ F). In addition, the calculations show that there is a strong change (by about 1.6 times) in the radius of the system in the excited state: $r^*_N = 0.6$ F and $r^*_\Delta = 0.64$ F for the N and Δ particles, respectively. Calculations were made for the three-quark system with allowance for a Gaussian spin–spin interaction ($\Lambda = 0.2$ F) (Figs. 18 and 19). This leads to a slight renormalization of the constant C and to an increase, by 1.2 times, of the parameter K (namely, $K = -0.61$ MeV·F^3 for $\Lambda = 0$ and $K = -73$ MeV·F^3 for $\Lambda = 0.2$ F). Thus, the following effects are found when the three-quark problem is solved in the MHF:

1. A renormalization of the parameters C and K from the values proposed in the harmonic-oscillator model.

2. Nonadditivity of the contribution of the spin–spin interaction, this leading to a change of the width and depth of the level for the effective potential.

3. States corresponding to the breathing mode of excitation were found by a second solution of the eigenvalue problem in the three-quark system.

Study of the 6q system in a minimal approximation of the method of hyperspherical functions

In Ref. 111 calculations were made of the properties of the 6q system in a minimal approximation of the method of

TABLE IV. Parameters of the quark–quark potential in the method of hyperspherical functions.

$b = \alpha$, F	A, MeV	B, MeV·F^2	C, MeV	K, MeV·F^3	Λ, F	Model
0.8	3810.0	−12.5	−479.8	−911.0	0	Harvey
0.8	3810.0	−12.5	−1715	0	0	Fig. 15
0.8	3810.0	−12.5	−1614	−61	0	Fig. 16
0.8	3810.0	−12.5	−1614	−61	0	Fig. 17
0.8	3810.0	−12.5	−1618	−73	0.2	Fig. 18
0.8	3810.0	−12.5	−1618	−73	0.2	Fig. 19

TABLE V. Results of calculations of the mass and rms radii for the three-quark system.

M_N, MeV	M_Δ, MeV	E_0, MeV	E_1, MeV	r_0, F	r_1, F	Model
1086	1086	0	1858	0.39	0.60	Fig. 15
940.5	—	−145.5	1681	0.39	0.60	Fig. 16
—	1232	146.0	1889	0.42	0.64	Fig. 17
938.4	—	−147.6	1673	0.39	0.60	Fig. 18
—	1228.3	142.3	1869	0.42	0.63	Fig. 19

hyperspherical functions. The quark–quark potentials from Ref. 96 were used with allowance for the spin–spin interaction in Gaussian form. The mass and radius of the $6q$ system were studied in a unified manner, together with the properties of the highly excited monopole states. In the minimal approximation of the MHF the wave function of the six-quark system has the characteristics

$$|s^6\,[6]_L\,[2^3]_C\,[33]_{ST} L=0\ S=1\ T=0\rangle. \tag{126}$$

Using the two-particle coefficients of fractional parentage, we can express the matrix element of the effective potential in terms of the two-particle matrix elements of the quark–quark interaction in the form

$$\begin{aligned} W(\rho) &= \langle s^6\,[6]_L\,[2^3]_C\,[33]_{ST}\,LST \\ &\quad - 010\,|\hat{\Gamma}_{q_1}|\,s^6\,[6]_L\,[2^3]_C\,[33]_{ST}\,LST \\ &\quad - 010\rangle = -\frac{19}{6}\,W^{S_0=1}_{L_0=0}(\rho) - \frac{5}{6}\,W^{S_0=0}_{L_0=0}(\rho), \end{aligned} \tag{127}$$

where $W^{S_0}_{L_0}(\rho)$ is determined in accordance with (125).

Figure 20 shows the effective potential and the first two solutions in it for the investigated six-quark system.

To reproduce the mass of the six-quark system, the constant C was renormalized in Ref. 111 ($C = -1618$ MeV for the three-quark system and $C = -1994$ MeV for the six-quark system). Here, as for the three-quark system, a monopole-excited state appears at excitation energy 1.43 GeV. In the calculation the effect of the expansion of the system with increasing excitation energy is manifested. Thus, the rms radius for the ground state is $r_{NR} = 0.38$ F, and the rms radius for the monopole-excited state is $r^*_{NR} = 0.52$ F.

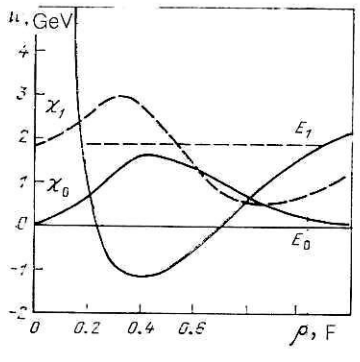

FIG. 15. Effective potential $W(\rho)$, the first two solutions E_0 and E_1 in it, and the corresponding wave functions $\chi_0(\rho)$ and $\chi_1(\rho)$ for a three-quark system without allowance for the spin–spin splitting ($K = 0$).

Elastic *N* scattering in the three-quark system

By means of the radial functions $\chi_i(\rho)$ it is possible to find the densities of the ground and excited states in the three-quark system:

$$n_{ij}(r) = N \int_r^\infty \frac{(\rho^2 - r^2)^{\frac{3n-8}{2}}}{\rho^{3n-5}}\,\chi_i(\rho)\,\chi_j(\rho)\,d\rho \tag{128}$$

and the rms radius

$$r_{NR} = \frac{\int n_{ii}(r)\,r^4\,dr}{\int n_{ii}(r)\,r^2\,dr}, \tag{129}$$

where $n_{ii}(r)$, the density, is normalized by

$$4\pi \int n_{ii}(r)\,r^2\,dr = n. \tag{130}$$

Figure 21 gives the results of a calculation of the nucleon densities in the ground state and monopole-excited state, and also the transition density. The calculation was made using the wave functions for the systems shown in Fig. 18 with parameters of the quark–quark potential from Table IV. Further, these densities were used to make nonrelativistic calculations of the form factor of elastic eN scattering in the three-quark system:

$$F_{NR}(q^2) = \frac{1}{n} \int n_{ii}(r)\,e^{i\mathbf{qr}}\,d\mathbf{r}. \tag{131}$$

Figure 22 shows the nonrelativistic form factor (q^2) (continuous curve). It can be seen that satisfactory agreement with the phenomenological dipole form factor

$$F_\Delta(q^2) = (1 + q^2/0.71)^{-2} \tag{132}$$

is observed only for $q^2 \ll M_N^2$. To describe the form factor of elastic scattering of the nucleon at momentum transfers $q^2 \gtrsim M_N^2$ it is necessary to take into account relativistic effects.[106,107]

To this end the method of relativizing the form factors

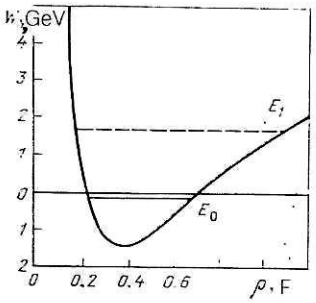

FIG. 16. Effective potential and the first two solutions in it for a nucleon with allowance for the spin–spin interaction in a δ-function form.

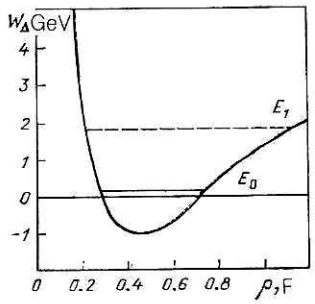

FIG. 17. Effective potential and the first two solutions in it for the Δ isobar with allowance for the spin–spin interaction in a δ-function form.

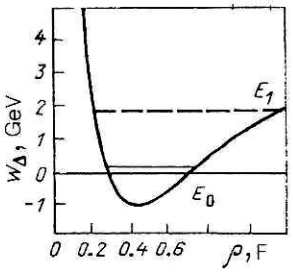

FIG. 19. Effective potential and the first two solutions in it for the Δ isobar with allowance for a Gaussian spin–spin interaction.

proposed in Refs. 106 and 107 was used. Essentially, it takes into account the effect of Lorentz transformation of the longitudinal distances on the transition from the Breit system to the laboratory system. Then the relativistic form factor can be determined fairly easily in terms of the nonrelativistic form factor:

$$F_R(q^2) = \frac{1}{(1+q^2/4M_A^2)^{n-1}} F_{NR}\left(\frac{q^2}{1+q^2/4M_A^2}\right), \qquad (133)$$

where n is the number of quarks in the system, and M_A is the effective parameter of the transformation, for which the following relation was proposed in Refs. 106–108:

$$M_A^2 = \sum_i (m_i^{\text{eff}}) = nm_q^2. \qquad (134)$$

Here, m_q is the quark mass, which in the calculations was chosen in the form $m_q = 1.086/3 = 0.362$ GeV, from which it follows that $M_A^2 = 0.393$ GeV^2. We note that the relativistic form factor for $q^2 \gg M_A^2$ satisfies the quark counting rules[109]:

$$F_R(q^2 \to \infty) \sim \left(\frac{4M_A^2}{q^2}\right)^{n-1} F_{RF}(4M_A). \qquad (135)$$

An important point in the calculations is the fact that allowance for the relativistic effects leads to an increase of the rms radius of the n-quark system:

$$r_R = r_{NR} - \frac{3(n-1)}{2M_A^2}. \qquad (136)$$

It can be seen from Fig. 22 that inclusion of the relativization effect leads to good agreement with the dipole form factor.

Very important here is the fact that the description of eN scattering in the complete region of measured momentum transfers $q^2 < 24$ GeV^2 requires information about the nonrelativistic form factor in the region of small momentum transfers $q^2 < 1.6$ GeV^2, i.e., the contribution of the so-called high-momentum component to the nonrelativistic wave function is negligibly small.

As we noted above, relativization leads to an increase in the size of the investigated system. Thus, for the nucleon $r = 0.66$ F, and this agrees with the experimental nucleon radius $r_{\text{exp}} = 0.78$ F.

The effect of relativization is manifested more strongly for the $6q$ system than for the three-quark system: $r_R = 0.72$ F and $r_R^* = 0.80$ F for the ground state and monopole-excited state, respectively.

In conclusion, we note the following results obtained in Refs. 96, 111, and 115:

1. In the MHF with a quark–quark potential of the type $V_{qq} = \Sigma\lambda_i\lambda_j V_{ij}(r)$, where $V_{ij} = (Ae^{-r^2/\alpha^2} + Br^2 + C) + K(1 + 2/3\sigma_i\sigma_j)e^{-r^2/\Lambda^2} = V_{ij}^c + V_{ij}^\sigma$, which leads to an analytic effective potential $W(\rho)$ of the $3q$ system, it is found that with the parameters given in Table IV one can describe:

a) the N–Δ mass difference;

b) the rms radius of the nucleon;

c) the form factor of elastic eN scattering.

2. The excitation energies and densities have been found for the ground state and monopole-excited states of the N particles.

3. It has been shown that inclusion of the spin–spin potential V_{ij}^σ significantly changes the effective potential $W(\rho)$ (the core and depth), i.e., V_{ij}^σ occurs in V_{qq} nonadditively.

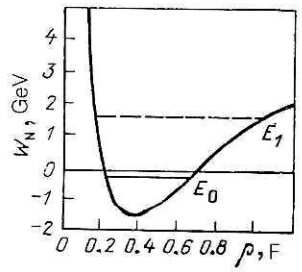

FIG. 18. Effective potential and the first two solutions in it for the nucleon with allowance for a Gaussian spin–spin interaction.

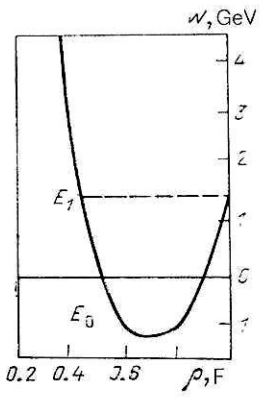

FIG. 20. Effective potential and the first two solutions in it for the six-quark system.

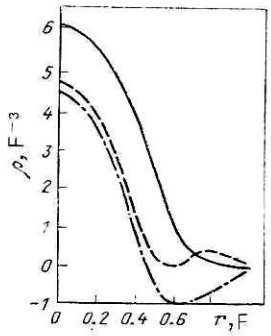

FIG. 21. Density distributions for the nucleon in the ground state (continuous curve) and the monopole-excited state (broken curve) and the transition density (chain curve).

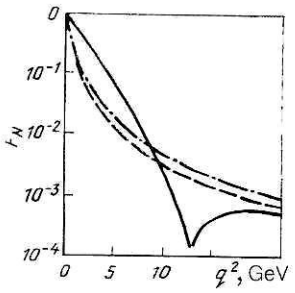

FIG. 22. Form factor of nucleon elastic scattering. The continuous and broken curves represent a nonrelativistic calculation, and the chain curve gives the dipole form factor.

Universal description of mesons and baryons in the ground state in the potential model of Refs. 120 and 130

In Ref. 120 the ground-state masses of all mesons and baryons were calculated using the Cornell potential in the approximation $V_{qq} = \frac{1}{2}V_{q\bar{q}}$. The results of the calculation for the meson and baryon masses are presented in Tables VI and VII and are in satisfactory agreement with experiment.

The spectra of all baryons from the lightest to the heavy baryons were studied in the nonrelativistic potential model. It was shown that the ground states of baryons can be described using a universal Coulomb potential in conjunction with a linear potential of the quark–quark interaction. The calculated baryon masses and the spin–spin splitting were compared experiment. Table VIII gives the results of calculations of the spin–spin splitting of (udf) baryons in comparison with experimental data. In Ref. 114 a simple potential model was proposed for investigation of the properties of baryons in the framework of the nonrelativistic quark model using the technique of hyperspherical functions. The quark–quark potential is expressed in the form

$$V_{ij} = -V_0 + Ar_{ij}^2 + \lambda_i\lambda_j\alpha_s\left\{r_{ij}^{-1} + c\frac{\pi}{2}\delta(r_{ij})\left(m_i^{-2} + m_j^{-2} + \frac{4}{3}\frac{\sigma_i\sigma_j}{m_i m_j}\right)\right\}. \quad (137)$$

The first two terms describe ordinary confinement. The third term derives from QCD. The additional parameter C was introduced into the term with the δ-function interaction in order to take into account approximately the terms absent in the standard Fermi–Breit potential. In this sense, the interaction used here is phenomenological. In Ref. 114 the value $\alpha_s = 0.4$ is retained, and V_0, A, and C are regarded as parameters.

The baryon wave function is expressed as follows:

$$\Psi = \{\sum_K U_K(r) B_K(\Omega)\}\Psi_{ST}\Psi_C, \quad (138)$$

where $U_K(r)$ are hyperspherical functions. They are expanded with respect to reduced Laguerre polynomials;

$$U_K(r) = \sum_n a_{nK}L_n(\beta r)\exp(-\beta r/2). \quad (139)$$

In this expression only the $n = 0$, $K = 0$ terms are retained; β is regarded as a variational parameter. Writing $E = \langle\Psi|H|\Psi\rangle$, we obtain, with the function Ψ given in (138), (139),

$$E = \beta^2\hbar^2/4m - 3V_0 + 42A(b_1^2 + b_2^2 + 1/2)/\beta^2 - 32\beta\alpha_s\left\{\sum_{i,j>1} C_{ij}^{-1}\right\}/45\pi + (\alpha_s c\beta^2/45\pi)\{(m_1^{-2} + m_2^{-2} + 4\langle\sigma_1\sigma_2\rangle/3m_1m_2) + 2(m_1^{-2} + m_3^{-2} + 4\langle\sigma_1\sigma_3\rangle/m_1m_3)/(b_1^2 + b_2^2)^{3/2}\}. \quad (140)$$

In this case it is possible to reproduce the ground-state masses of the baryons for the following values of the parameters of the potential: $V_0 = 132.02$ MeV, $A = 40.96$ MeV·F^{-2}, and $C = 3.4366$. For the quark masses the values $m_u = m_d = 336$ MeV and $m_s = 565$ MeV are adopted.

Table IX gives the results of the calculations of the masses of the baryons in the ground state, and also the corresponding experimental data. In (140),

TABLE VI. Masses of $^1S_0$ and $^3S_1$ mesons.

State	$u\bar{u}$	$u\bar{s}$	$u\bar{c}$	$u\bar{b}$	$s\bar{s}$	$s\bar{c}$	$c\bar{c}$
$M(^3S_1)$, MeV:							
theory	758	887	1982	5324	1022	2098	3101
experiment	770	892	2006	5325	1020	2110	3097
$M(^1S_0)$, MeV:							
theory	107	494	1832	5266	751	1970	2977
experiment	135	493	1865	5277	770	1970	2980

TABLE VII. Masses of baryon ground states, GeV.

Baryon	Theory	Experiment	Baryon	Theory	Experiment
N	0.931	0.939	Ξ_c^S	2.566	—
Δ	1.245	1.232	Ξ_c^*	2.628	—
Λ	1.117	1.115	Ξ_{cc}	3.605	—
Σ	1.200	1.193	Ξ_{cc}^*	3.680	—
Σ^*	1.386	1.383	$\Omega_c\left(\frac{1}{2}\right)$	2.725	2.740
Λ_c	2.258	2.282			
Σ_c	2.435	2.450	$\Omega_c\left(\frac{3}{2}\right)$	2.830	—
Σ_c^*	2.491	—			
Λ_b	5.595	5.50	$\Omega_{cc}\left(\frac{1}{2}\right)$	3.73	—
Σ_b	5.80	—			
Σ_b^*	5.82	—	$\Omega_{cc}\left(\frac{3}{2}\right)$	3.80	—
Ξ	1.330	1.318			
Ξ^*	1.528	1.533	Ω_{ccc}	4.793	—
Ω	1.664	1.672			
Ξ_c^A	2.462	2.460			

B (140)

$$m = \frac{\left(\sum_{j>i} m_i m_j\right)}{\sum_{j=1}^{3} m_i}; \quad C_{12} = 1; \quad C_{13} = C_{23} = (b_1^2 + b_2^2)^{1/2};$$

$$b_1^2 + b_2^2 = (m_1 + m_3)(m_1 + 2m_3)/(2m_3(2m_1 + m_3)).$$

The baryon rms radius was calculated in accordance with

$$\{\langle\Psi|r^2/6|\Psi\rangle\}^{1/2} = \sqrt{7}/\beta.$$

The proton rms radius was found to be 0.794 F, in good agreement with experiment (0.8 F). Also calculated was the baryon form factor (normalized to unity for angular-momentum transfer $q = 0$). The expression for the form factor with the wave function is obtained as

$$F(q) = \langle\Psi| \sum_{j=1}^{3} \times \exp(iq(\mathbf{r}_j - \mathbf{R}))|\Psi\rangle 3 = (1 + q^2/3\beta^2)^{-7/2}.$$

Figure 23 shows the proton form factor, and also the value of the experimental form factor. The two results agree well.

6. HYBRID METHOD FOR INVESTIGATING FEW-PARTICLE SYSTEMS

The MHF in the momentum representation[11,116] can be used to solve the Faddeev integral equations.[117] The connection between the Faddeev method and the method of K harmonics in the coordinate representation was discussed in Ref. 118. A unification of these two approaches appears very promising.

In Ref. 117 it was proposed that the Faddeev equations in the momentum representation should be solved by the method of partial expansion, though not, as is usually done, in a two-body Hilbert space but in a three-body Hilbert space, in which three-particle hyperspherical functions in the six-dimensional space of Jacobi momenta are introduced in place of spherical functions in three-dimensional space. The Faddeev functions and $\Psi^{(3)}(\mathbf{K}_{12},\mathbf{p}_3)$, $\Psi^{(1)}(\mathbf{K}_{23},\mathbf{p}_1)$ and $\Psi^{(2)}(\mathbf{K}_{31},\mathbf{p}_2)$ are expanded with respect to basis hyperspherical eigenfunctions, and the three-body bound-state problem is reduced to the solution of three infinite systems of one-dimensional homogeneous integral equations. These equations contain three-particle matrix elements of the two-particle S matrix, for the determination of which an infinite system of one-dimensional inhomogeneous integral equations is obtained. In such an approach to the solution of the Faddeev equations essential use is made of a Raynal–Revai unitary transformation. The method of "hybridization" of the Faddeev method and the method of hyperspherical functions proposed in Ref. 117 was realized for specific problems in Refs. 119 and 120. The corresponding questions were discussed in Refs. 69 and 121.

Bases of hyperspherical functions for the Faddeev wave functions

We consider a bound state of three particles with masses m_1, m_2, and m_3. As is well known, in the Faddeev method the total wave function is represented as a sum of three terms,

TABLE VIII. Spin–spin splitting of (udf) baryons.

Particle species	ΔE_{ss}^{cb}, MeV		Baryon	E_{ss}^{cb}, MeV
	theory	experiment		
$\Delta - N$	314	294	N	−157
$\Sigma - \Lambda$	83	78	Λ	−153
$\Sigma^* - \Sigma$	198	215	Λ_c	−162
$\Sigma_c - \Lambda_c$	177	168	Λ_b	−174
$\Xi^* - \Xi$	198	215		

TABLE IX. Masses in baryon ground states, MeV.

Baryons	N	Λ	$(\Sigma)_8$	$(\Xi)_8$	Δ	$(\Sigma)_{10}$	(Ξ)	Ω^-	$(\Lambda')_1$
Theory	938	1119.8	1184	1344	1232	1379.4	1534	1667	1406
Experiment	939	1115.6	1193	1317	1232	1385	1530	1672	1405

$$\Psi = \Psi^{(3)}(\mathbf{K}_{12}, \mathbf{p}_3) + \Psi^{(1)}(\mathbf{K}_{23}, \mathbf{p}_1) + \Psi^{(2)}(\mathbf{K}_{31}, \mathbf{p}_2), \tag{141}$$

where each term depends on its own set of Jacobi momenta. In accordance with this, we introduce three bases of three-particle hyperspherical functions in the momentum representation,

$$\Phi_{K_3LM}^{l_{\mathbf{K}_{12}} l_{\mathbf{p}_3}}(\Omega_3^{\varkappa}), \quad \Phi_{K_1LM}^{l_{\mathbf{K}_{23}} l_{\mathbf{p}_1}}(\Omega_1^{\varkappa}), \quad \Phi_{K_2LM}^{l_{\mathbf{K}_{31}} l_{\mathbf{p}_2}}(\Omega_2^{\varkappa}), \tag{142}$$

where K_1, K_2, and K_3 are the grand orbitals corresponding to the three different bases of hyperspherical functions, and $\Omega_i^{\varkappa}$ are the hyperangles in the momentum space for the i-th basis.

A connection between the functions (142) is established by the Raynal–Revai unitary transformation. The formula analogous to (3) corresponding to this transformation is written as

$$\Phi_{K_hLM}^{l_{\mathbf{K}_{ij}} l_{\mathbf{p}_k}}(\Omega_h^{\varkappa}) = \sum_{l_{\mathbf{K}_{ki}} l_{\mathbf{p}_j}} \langle l_{\mathbf{K}_{hi}} l_{\mathbf{p}_j} | l_{\mathbf{K}_{ij}} l_{\mathbf{p}_k} \rangle_{K_h L} \Phi_{K_hLM}^{l_{\mathbf{K}_{hi}} l_{\mathbf{p}_j}}(\Omega_j^{\varkappa}). \tag{143}$$

The indices $K_h L$ of the Raynal–Revai coefficients indicate that under the transformation (143) the grand orbital K_h and the total orbital angular momentum of the three-particle system do not change.

We represent the Faddeev functions in (141) in the form of expansions with respect to the corresponding basis hyperspherical functions (142):

$$\Psi^{(3)}(\mathbf{K}_{12}, \mathbf{p}_3) = \sum_{K_3 l_{\mathbf{K}_{12}} l_{\mathbf{p}_3}} \varphi_{K_3L}^{l_{\mathbf{K}_{12}} l_{\mathbf{p}_3}}(\varkappa)\, \Phi_{K_3LM}^{l_{\mathbf{K}_{12}} l_{\mathbf{p}_3}}(\Omega_3^{\varkappa}); \tag{144}$$

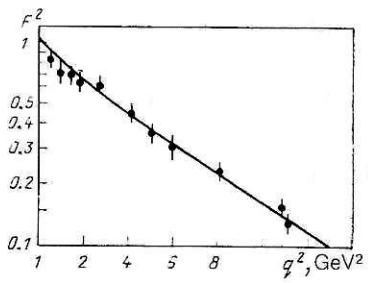

FIG. 23. Proton form factor (continuous curve); the points show the experimental results.

$$\Psi^{(1)}(\mathbf{K}_{23}, \mathbf{p}_1) = \sum_{K_1 l_{\mathbf{K}_{23}} l_{\mathbf{p}_1}} \varphi_{K_1L}^{l_{\mathbf{K}_{23}} l_{\mathbf{p}_1}}(\varkappa)\, \Phi_{K_1LM}^{l_{\mathbf{K}_{23}} l_{\mathbf{p}_1}}(\Omega_1^{\varkappa}); \tag{145}$$

$$\Psi^{(2)}(\mathbf{k}_3, \mathbf{p}_2) = \sum_{K_2 l_{\mathbf{K}_{31}} l_{\mathbf{p}_2}} \varphi_{K_2L}^{l_{\mathbf{K}_{31}} l_{\mathbf{p}_2}}(\varkappa)\, \Phi_{K_2LM}^{l_{\mathbf{K}_{31}} l_{\mathbf{p}_2}}(\Omega_2^{\varkappa}). \tag{146}$$

Using the expansions (144)–(146) we can readily find the connection between the Faddeev functions on the basis of the transformation (143). This fact plays a decisive part in the hyperspherical approach to the solution of the Faddeev equations.

Of course, using the direct connection between the different sets of Jacobi momenta, we can reduce the arguments of each of the Faddeev functions in (141) to a single set (standard approach). However, this introduces the angles between the Jacobi momenta, and this greatly complicates the calculation. The approach proposed in Ref. 117 makes it possible to avoid these complications and reduce the problem to the solution of infinite systems of one-dimensional integral equations for any form of the local two-particle potentials.

Faddeev equations in a hyperspherical basis

The system of Faddeev equations for the individual terms of Eq. (141) contains three coupled integral equations. These equations have the same form and can be obtained from each other by cyclic permutation of the indices. Therefore, one of the three equations can be considered in the hyperspherical basis, and to obtain the other two the indices in the final expression can be cyclically permuted. We consider, for example, the equation

$$\Psi^{(3)}(\mathbf{K}_{12}, \mathbf{p}_3) = D_{12}^{-1}(\mathbf{K}_{12}, \mathbf{p}_3, Z) \Big[\int \langle \mathbf{K}_{12}\mathbf{p}_3 | T_{12}(Z) | \mathbf{K}_{23}\mathbf{p}_1 \rangle \times \Psi^{(1)}(\mathbf{K}_{23}, \mathbf{p}_1)\, d\mathbf{K}_{23}\, d\mathbf{p}_1 + \int \langle \mathbf{K}_{12}\mathbf{p}_3 | T_{12}(Z) | \mathbf{K}_{31}\mathbf{p}_2 \rangle \times \Psi^{(2)}(\mathbf{K}_{31}, \mathbf{p}_2)\, d\mathbf{K}_{31}\, d\mathbf{p}_2 \Big], \tag{147}$$

where the Green's function in the hypermomenta can be represented in the form

$$D_{12}(\mathbf{K}_{12}, \mathbf{p}_3, Z) = -\frac{\hbar^2}{2m}(\varkappa^2 + \varkappa_0^2). \tag{148}$$

For a bound state we have here $Z = -\epsilon$, where $\epsilon = \hbar^2\kappa_0^2/2m$ is the binding energy of the three particles.

In (147) the integrand contains the matrix elements of the two-particle S matrix, which are taken in the three-body Hilbert space and depend on the three-particle energy Z. In contrast to the standard approach, to calculate them we shall not go over to the two-body Hilbert space (transition

from the matrices T_{12} to the matrices t_{12}) but will proceed from the original equation

$$T_{12}(Z) = v_{12} + v_{12}\sigma_0(Z)\, T_{12}(Z) \qquad (149)$$

and consider it in the basis of three-particle hyperspherical functions. In this equation, v_{12} is the potential of the two-particle interaction, and $\sigma_0(Z)$ is the same three-particle Green's function that is present in the Faddeev equation.

In the basis of hyperspherical functions in the momentum representation the solution of Eq. (149) in matrix form reduces to the solution of the infinite set of one-dimensional inhomogeneous integral equations[117,119]

$$T_{K_3K_1L}^{l_{K_{12}}l_{p_3};\, l'_{K_{12}}l'_{p_3}}(\varkappa,\ \varkappa',\ \varkappa_0) = V_{K_3K_1L}^{l_{K_{12}}l_{p_3};\, l'_{K_{12}}l'_{p_3}}(\varkappa,\ \varkappa')$$
$$- \frac{2m}{\hbar^2} \sum_{K'_3 \widetilde{l}'_k \widetilde{l}'_p \widetilde{\widetilde{l}}'_k \widetilde{\widetilde{l}}'_p} C^{\widetilde{l}'_k \widetilde{l}'_p;\, \widetilde{\widetilde{l}}'_k \widetilde{\widetilde{l}}'_p;\, K'_3}_{l_{K_{12}} l_{p_3} l'_{K_{12}} l'_{p_{31}};\, K_1}(l'_{K_{23}} l'_p,\ L)$$
$$\times \int \frac{\varkappa''\, d\varkappa''}{\varkappa_0^2 + \varkappa''^2} V_{K_3K_1L}^{l'_{K_{12}}l'_{p_3};\, \widetilde{l}'_k \widetilde{l}'_p}(\varkappa,\ \varkappa'')$$
$$\times T_{K_3K_2L}^{l_{K_{12}}l_{p_3};\, \widetilde{l}'_k \widetilde{l}'_p}(\varkappa'',\ \varkappa',\ \varkappa_0), \qquad (150)$$

where

$$V_{K_3K_1L}^{l_{K_{12}}l_{p_3};\, l'_{K_{12}}l'_{p_3}}(\varkappa,\ \varkappa') = \int_0^\infty \rho\, d\rho J_{K_3+2}(\varkappa\rho)$$
$$\times V_{K_3K_1L}^{l_{K_{12}}l_{p_3};\, l'_{K_{12}}l'_{p_3}}(\rho)\, J_{K_1+2}(\varkappa'\rho); \qquad (151)$$

$$V_{K_3K_1L}^{l_{K_{12}}l_{p_3};\, l'_{K_{12}}l'_{p_3}}(\rho) = \int_0^\infty \Phi_{K_3LM}^{*\, l_{K_{12}}l_{p_3}}(\Omega_3^\rho)$$
$$\times v(\rho\cos\alpha_3^\rho)\, \Phi_{K_3LM}^{l'_{K_{12}}l'_{p_3}}(\Omega_3^\rho)\, d\Omega_3^\rho; \qquad (152)$$

$$C^{\widetilde{l}'_K \widetilde{l}'_p;\, \widetilde{\widetilde{l}}'_K \widetilde{\widetilde{l}}'_p;\, K'_3}_{l_{K_{12}} l_{p_3};\, l'_{K_{12}} l'_{p_3};\, K_1}(l'_{K_{23}} l'_{p_1};\ L)$$
$$= \frac{\langle \widetilde{l}'_K \widetilde{l}'_p | l'_{K_{23}} l'_{p_1}\rangle_{K_3L}\, \langle \widetilde{\widetilde{l}}'_K \widetilde{\widetilde{l}}'_p | l'_{K_{23}} l'_{p_1}\rangle_{K'_3L}}{\langle l_{K_{23}} l_{p_1} | l'_{K_{23}} l'_{p_1}\rangle_{K_1L}\, \langle l'_{K_{12}} l'_{p_3} | l'_{K_{23}} l'_{p_1}\rangle_{K_1L}}. \qquad (153)$$

The matrices $T_{K_iK_jL}^{l_{K_i}l_{p_i};\, l'_{K_j}l'_{p_j}}(\kappa,\kappa',\kappa_0)$ are present in the Faddeev equations expressed in the hyperspherical basis, and therefore to calculate the binding energy and construct the hyperradial functions in the momentum representation by means of Eqs. (144)–(146) we must first solve the system of equations (150). The equations for the hyperradial functions are obtained after substitution of the expansions (144)–(146) in the Faddeev equation and the completion of the calculations that are usual for the scheme of the hyperspherical basis. As a result, we obtain for the functions $\varphi_{K_iL}^{l_{K_i}l_{p_i}}(\kappa)$, and $\varphi_{K_iL}^{l_{K_i}l_{p_i}}(\kappa)$ three infinite systems of coupled one-dimensional integral equations. We write down one of them[117,119]:

$$\varphi_{K_3L}^{l_{K_{12}}l_{p_3}}(\varkappa) = -\frac{2m}{\hbar^2}\varkappa^{-2}(\varkappa^2+\varkappa_0^2)^{-1}$$
$$\times \Big[\sum_{K_1} \sum_{l'_{K_{12}}l'_{p_1}} \sum_{l'_{K_{23}}l'_{p_1}} \sum_{\widetilde{l}'_{K_{12}}\widetilde{l}'_{p_3}} \sum_{l_{K_{23}}l_{p_1}} i^{K_1-K}$$
$$\times \langle \widetilde{l}'_{K_{12}}\widetilde{l}'_{p_3} | l'_{K_{23}}l'_{p_1}\rangle_{K_3L}\, \langle l_{K_{12}}l_{p_3} | l'_{K_{23}}l'_{p_1}\rangle_{K_3L}\, \langle l_{K_{23}}l_{p_1} | l'_{K_{12}}l'_{p_3}\rangle_{K_1L}$$
$$\times \int T_{K_1K_3L}^{l'_{K_{12}}l'_{p_3};\, \widetilde{l}'_{K_{12}}\widetilde{l}'_{p_3}}(\varkappa,\ \varkappa,\ \varkappa_0)\, \varphi_{K_1L}^{l_{K_{23}}l_{p_1}}(\varkappa')\, \varkappa'^3\, d\varkappa'$$
$$+ \sum_{K_2} \sum_{l'_{K_{12}}l'_{p_3}} \sum_{l'_{K_{31}}l'_{p_2}} \sum_{\widetilde{l}'_{K_{12}}\widetilde{l}'_{p'_3}} \sum_{l_{K_{31}}l_{p_2}} i^{K_3-K_2} \langle \widetilde{l}'_{K_{12}}\widetilde{l}'_{p_3} | l'_{K_{31}}l'_{p_2}\rangle_{K_3L}$$
$$\times \langle l_{K_{12}}l_{p_3} | l'_{K_{31}}l'_{p_2}\rangle_{K_3L}\, \langle l_{K_{31}}l_{p_2} | l'_{K_{12}}l'_{p_3}\rangle_{K_2L}$$
$$\times \int T_{K_2K_3L}^{l'_{K_{12}}l_{p_3};\, \widetilde{l}'_{K_{12}}\widetilde{l}'_{p_3}}(\varkappa,\ \varkappa',\ \varkappa_0)\, \varphi_{K_2L}^{l_{K_{31}}l_{p_2}}(\varkappa')\, \varkappa'^3\, d\varkappa'. \qquad (154)$$

The other two equations are obtained by cyclic permutation of the indices.

Thus, in the new approach of Refs. 117 and 119 to the solution of the Faddeev equations the problem is not divided into two stages as is usually done (the two-particle S matrix t_{12} is found from the two-particle problem on the energy shell, and then the analytically continued t_{12} matrix is inserted in the Faddeev equations). Here it is necessary to solve simultaneously the systems of one-dimensional integral equations (150) and (154) and the similar systems obtained from them by cyclic permutation of the indices. It is here particularly important that one-dimensional integral equations are obtained for any form of the potential without a preliminary separable expansion of it.

The variety of orbital quantum numbers give a complicated form to Eqs. (150) and (154), but this is inherent in the method of partial expansion for the Faddeev equations.[122] For a system of three bosons, when all the orbital angular momenta are zero, these equations take a simple form.

When allowance is made for the symmetrization of the Faddeev wave functions, the expansions must be made with respect to symmetrized basis hyperspherical functions. The remaining calculations must be repeated.

Hybrid method of investigating three-particle reactions

The MHF can also be used to solve the problem of the continuous spectrum using the Faddeev equations. This was first done in Ref. 120. It is necessary to proceed from the system of Faddeev integral equations for the continuous spectrum,

$$\Psi^{(3)}_{\mathbf{K}^0_{12},\, \mathbf{p}^0_3}(\mathbf{K}_{12},\ \mathbf{p}_3) = \delta(\mathbf{K}_{12} - \mathbf{K}^0_{12})\, \delta(\mathbf{p}_3 - \mathbf{p}^0_3)$$
$$+ D_{12}^{-1}(\mathbf{K}_{12},\ p_3,\ \mathbf{Z}) \Big[\int \langle \mathbf{K}_{12}\mathbf{p}_3 | T_{12}(\mathbf{Z}) | \mathbf{K}_{31}\mathbf{p}_2\rangle$$
$$\times \Psi^{(0)}_{\mathbf{K}^0_{31},\, \mathbf{p}^0_2}(\mathbf{K}_{31},\ \mathbf{p}_2)\, d\mathbf{K}_{31}\, d\mathbf{p}_2$$
$$+ \int \langle \mathbf{K}_{12}\mathbf{p}_3 | T_{12}(\mathbf{Z}) | \mathbf{K}_{23}\mathbf{p}_1\rangle$$
$$\times \Psi^{(1)}_{\mathbf{K}^0_{23},\, \mathbf{p}^0_1}(\mathbf{K}_{23},\ \mathbf{p}_1)\, d\mathbf{K}_{23}\, d\mathbf{p}_1 \Big], \qquad (155)$$

and from the other two equations obtained by cyclic permutation of the indices.

In (155), D_{12}^{-1} is the Green's function,

$$\left.\begin{aligned} D_{12}(\mathbf{K}_{12},\ \mathbf{p}_3,\ \mathbf{Z}) &= -\frac{\hbar^2}{2m}(\varkappa^2-\varkappa_0^2-i\varepsilon);\\ \varkappa_0^2 &= \mathbf{K}_{12}^{02}+\mathbf{p}_3^{02}=\mathbf{K}_{23}^{02}+\mathbf{p}_1^{02}=\mathbf{K}_{13}^{02}+\mathbf{p}_2^{02}=\frac{2m}{\hbar^2}Z, \end{aligned}\right\} \tag{156}$$

where $Z\equiv E>0$ is the total energy of the three particles in the center-of-mass system.

Using the integral representation of the function $\delta(K_{12}-K_{12}^0)\delta(\mathbf{p}_3-\mathbf{p}_3^0)$ and expanding the six-dimensional plane wave with respect to hyperspherical functions, we can readily obtain the representation

$$\delta(\mathbf{K}_{12}-\mathbf{K}_{12}^0)\,\delta(\mathbf{p}_3-\mathbf{p}_3^0)$$

$$=\frac{\delta(\varkappa-\varkappa_0)}{\varkappa_0^5}\sum_{K_3 l_{K_{12}} l_{p_3} L}\Phi^{*l_{K_{12}}l_{p_3}}_{K_3LM}(\Omega_3^{\varkappa})\,\Phi^{l_{K_{12}}l_{p_3}}_{K_3LM}(\Omega_3^{\varkappa_0}), \tag{157}$$

where $\Omega_3^{\varkappa}$ and $\Omega_3^{\varkappa_0}$ are the sets of hyperangles in the six-dimensional spaces of the Jacobi momentum $(\mathbf{K}_{12},\ \mathbf{p}_3)$ and $(\mathbf{K}_{12}^0,\mathbf{p}_3^0)$, respectively.

By analogy with the expansion (157), we represent the Faddeev functions that occur in Eq. (155) in the form of the expansions

$$\Psi^{(3)}_{K_{12}^0 p_3^0}(\mathbf{K}_{12},\ \mathbf{p}_3)$$

$$=\sum_{K_3K_0'l_{K_{12}}l_{p_3}}\varphi^{l_{K_{12}}l_{p_3}}_{K_3LK_0'}(\varkappa,\ \varkappa_0)\,\Phi^{*l_{K_{12}}l_{p_3}}_{K_3LM}(\Omega_3^{\varkappa})\,\Phi^{l_{K_{12}}l_{p_3}}_{K_0'LM}(\Omega_3^{\varkappa_0}), \tag{158}$$

$$\Psi^{(1)}_{K_{23}^0 p_1^0}(\mathbf{K}_{23},\ \mathbf{p}_1)$$

$$=\sum_{K_1K_0''l_{K_{23}}l_{p_1}}\varphi^{l_{K_{23}}l_{p_1}}_{K_1LK_0''}(\varkappa,\ \varkappa_0)\,\Phi^{*l_{K_{23}}l_{p_1}}_{K_1LM}(\Omega_1^{\varkappa})\,\Phi^{l_{K_{23}}l_{p_1}}_{K_0''LM}(\Omega_1^{\varkappa_0}), \tag{159}$$

$$\Psi^{(2)}_{K_{31}^0,\ p_2^0}(\mathbf{K}_{31},\ \mathbf{p}_2)$$

$$=\sum_{K_2K_0'''l_{K_{31}},\ l_{p_2}}\varphi^{l_{K_{31}}l_{p_2}}_{K_2LK_0'''}(\varkappa,\ \varkappa_0)\,\Phi^{*l_{K_{31}}l_{p_2}}_{K_2LM}(\Omega_2^{\varkappa})\,\Phi^{l_{K_{31}}l_{p_2}}_{K_0'''LM}(\Omega_2^{\varkappa_0}), \tag{160}$$

where φ are the hyperradial functions of the continuous spectrum in the momentum representation. In Eqs. (158)–(160) the grand orbitals K_3, K_2, and K_1 characterize the system after the three-particle scattering, while the grand orbitals K_0', K_0'', and K_0''' are for the same system before such scattering. In the general case these grand orbitals can be different.

When the MHF is used to solve the Faddeev integral equations, three different sets of grand orbitals occur in the expansions: (K_1, K_0'), (K_2,K_0''), and (K_3, K_0'''). When the same method is applied to solution of Schrödinger equations, only one set occurs. Therefore, when the Faddeev equations are solved it is in principle possible to describe all possible scatterings in the three-body system, whereas when Schrödinger equations are solved one can describe only truly three-particle scattering.

In Eq. (155) the integrand contains matrix elements of the two-particle S matrix that are taken in the three-body Hilbert space and depend on the same three-particle energy $Z=E$. To calculate them, as in the case of the discrete spectrum, we do not go over to the two-body Hilbert space (standard approach) but proceed from the original equation for the $T_{12}(Z)$ matrix, writing it in matrix form and calculating each matrix element in the resulting equation in the basis of three-particle hyperspherical functions. As a result, the problem of finding the two-particle S matrix is reduced to the solution of a system of one-dimensional inhomogeneous singular integral equations.[120] Further, using the representation of the matrix elements of the two-particle S matrix in the basis of three-particle hyperspherical functions and the expansions (158)–(160), we obtain from Eq. (155) a system of coupled one-dimensional inhomogeneous singular integral equations for the hyperradial functions $\varphi^{l_k l_p}_{KLK_0}(\kappa,\kappa_0)$.

This hybrid method was used to study double charge exchange of kaons on three-particle nuclei,[120] to investigate the $\gamma+{}^9\mathrm{Be}\to\alpha+\alpha+n$ reaction,[123] and to study kaon photoproduction on three-particle nuclei with complete breakup.[124] We describe briefly the results of these investigations.

Double charge exchange of kaons on three-particle nuclei

In Ref. 120 the hybrid method was used to investigate the reactions $K^-+{}^3\mathrm{H}\to K^++\Xi^-+n+n$ and $K^-+{}^3\mathrm{He}\to K^++\Xi^-+n+p$. The problem of investigating nuclear systems in which Ξ^- particles are present has recently become topical. The problem is interesting from two points of view. On the one hand, investigation of the interaction of Ξ^- with nucleons at low energies is needed to extend our ideas about the SU(3) structure of the baryon–baryon forces. On the other hand, it has been suggested that if appropriate conditions are satisfied, the two-particle Ξ^-+p system could go over to a six-quark H dibaryon. The main conclusions of Ref. 120 concerning the above-mentioned reactions are the following: a) the interaction between the baryons in the final state significantly influences the basic characteristics of these reactions; b) the experimental investigation of double charge exchange of kaons on three-particle nuclei may lead to a solution of the problem of the possibility of observing a six-quark H dibaryon; c) the momentum spectrum of the K^+ mesons from these reactions is fairly sensitive to the interaction between the baryons in the final state and can be used to investigate the Ξ^-N interaction.

Three-particle photodisintegration of the ^{9}Be nucleus

The process of three-particle photodisintegration of the ^{9}Be nucleus by low-energy γ photons has frequently been investigated both experimentally and theoretically. A few years ago interest in this problem was reawakened.[125] The reawakened interest is due to the fact that the ground state of the $^9\mathrm{Be}(3/2^-)$ nucleus in the three-particle cluster model is a state in which two α particles and a neutron are bound more weakly than the neutron and proton in the deuteron. Under the influence of the low-energy γ [$1.57\ \mathrm{MeV}\leqslant E_\gamma\leqslant\epsilon$, where ϵ is the threshold of the (γ,p) and (γ,n) photoreactions on the α particles] the nucleus can break up only into three composite particles. This is a typical three-particle photonuclear reaction. It can be investigated in "pure" form, and thus one can obtain more or less unambiguous

information about not only the structure of the ^{9}Be nucleus in the ground state but also the dynamics of the three-particle system $\alpha + \alpha + n$ in the continuum.

Interest in this reaction increased appreciably after distinctive behavior of the curve of the cross section near the threshold had been found in experiments. It was found that the assumption hitherto made, that an excited state ($1/2^+$) of the ^{9}He nucleus makes the main contribution to the final state, does not correspond to reality, and to fit the theoretical results to the experimental data it was necessary to assume the existence of a large contribution of the nonresonance reaction mechanism.[126] However, the procedure used in Ref. 126 to separate the transition amplitude into resonance and nonresonance parts and the use of "bare" model wave functions of the nuclear systems in their turn contain rather a lot of artificial devices and model assumptions. At the same time, one can develop a unified model-independent approach to the investigation of processes like three-particle breakup, in which one can, using the same realistic two-particle potentials in the same formalism, describe both the structure of the ground state of the initial nucleus and the dynamics of the three-body final state.[123]

In Ref. 123 the basic conclusion was obtained that the distinctive behavior of the cross section of three-particle photodisintegration of the ^{9}Be nucleus near the threshold can be described without the introduction of adjustable parameters and the artificial addition of the wave function of the resonant final state. This behavior is determined by the wave function of the final state, which is obtained by solving the Faddeev equations with allowance for all the interactions between the particles. Figure 24 demonstrates the degree of convergence with respect to the number of harmonics of the final state that are taken into account. The figure gives the results of calculations of the total cross section of the $\gamma + {}^9\text{Be} \to \alpha + \alpha + n$ reaction for the set of potentials of the $\alpha\alpha$ and αN interactions from Refs. 127 and 128, the results being obtained with allowance for the interaction in the final state between the three particles. The harmonics $K_f = 1, 3, 5$ were taken into account in the calculations to obtain the wave function of the initial state, and the final-state wave function was taken in the approximation of the minimal harmonic $K_f = 0$ (continuous curve) and with allowance for the first two harmonics: $K_f = 0, 2$ (broken curve). It can be seen that the addition of the second harmonic in the calculations of the final-state wave function changes the results for the reaction cross section only slightly.

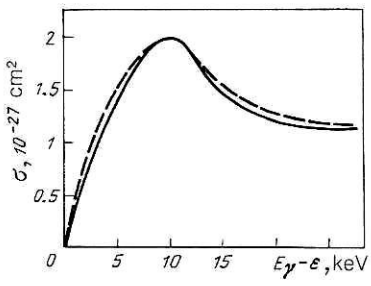

FIG. 24. Results of calculations of the total cross section of the reaction $\gamma + {}^9\text{Be} \to \alpha + \alpha + n$. The continuous curve is calculated with allowance for the minimal harmonic ($K_f = 0$), and the broken curve with allowance simultaneously for the first two harmonics ($K_f = 0,2$).

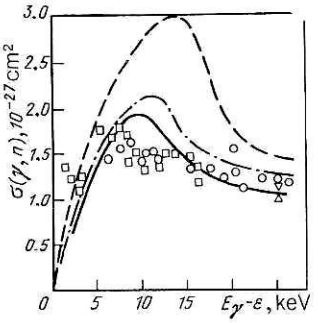

FIG. 25. Curves of the total cross section of the $\gamma + {}^9\text{Be} \to \alpha + \alpha + n$ reaction. The continuous curve is the calculation with allowance for the potentials from Refs. 127 and 128, the broken curve is the result of the plane-wave approximation (interaction potentials from Refs. 127 and 128), and the chain curve is with allowance for the interaction between all the particles in the final state (potentials from Refs. 129 and 130); the experimental points are as follows: ◇ from Ref. 131, ▽ from Ref. 132, and ○ and □ from Ref. 133.

The effect of the final-state interaction and a comparison with experiment are shown in Fig. 25, which gives the curves of the total cross section of the $\gamma + {}^9\text{Be} \to \alpha + \alpha + n$ reaction. It can be seen from the figure that the results for the different sets of two-particle potentials differ appreciably, and this offers hope that further work to determine the experimental data more accurately may make it possible to obtain unambiguous conclusions about these potentials. At the present time, as can be seen from Fig. 25, the experimental data differ strongly and are not supported by the necessary statistics. Thus, we have here an interesting experimental problem.

Photoproduction of kaons on three-particle nuclei

In Ref. 124 the hybrid method was used to investigate the (γ, K^+) reaction on three-particle nuclei. Such a reaction is interesting from the point of view of the development of hypernuclear physics. The point is that in hypernuclear physics the main reaction used is (K^-, π^-), which results in the production of the hypernucleus. However, the intensity of present-day K^- beams is very low, and this hinders the further development of hypernuclear physics in this direction. On the other hand, there is a possibility of investigating hypernuclei by means of other processes induced by other elementary particles for which the beam intensity greatly exceeds the K^- beam intensities. In particular, photoproduction of K^+ mesons on nuclei with the production of hypernuclei has recently become topical. In Ref. 124 the following basic conclusions were drawn about the (γ, K^+) reaction on three-particle nuclei: 1) the hybrid method can be successfully used to investigate the structure of the nuclei and hypernuclei, and also photoproduction of K^+ mesons on three-particle nuclei, and good convergence with respect to the number of harmonics taken into account is ensured for both the binding energy and the reaction cross sections; 2) the effect of the interaction in the final state between the baryons is significant for the processes of K^+ photoproduction; 3) the influence of the nuclear medium on the photoproduction processes is basically quantitative in nature; 4) in the case of photoproduction of K^+ mesons on three-particle nuclei it is much more advantageous for the produced λ particle to remain in an unbound state with the baryons. The

channel with production of the hypernucleus ${}^{3}_{\Lambda}\mathrm{H}$ in the final state is strongly suppressed.

CONCLUSIONS

The present paper, being too short, cannot pretend to give a complete account of the new tendencies in the development and application of the method of hyperspherical functions. Here we have considered only some aspects of the question, in the selection of which, naturally, the interests of the authors themselves were decisive. Of course, the literature that we have quoted also cannot pretend to completeness. Some new studies on the further development of the mathematical formalism of the method and its application in the physics of quarks, in nuclear physics, in atomic physics, in molecular physics, and in chemistry have not been considered. The number of such studies giving results of great interest is increasing rather rapidly, and it may become necessary to write reviews on the application of the method separately in quark physics, atomic physics, etc.

The last decade in this field has been characterized by the transition from calculations of the binding energy to studies of the continuous spectra of many-particle systems and of a diversity of processes using the method of hyperspherical functions. However, the method has largely been used to solve the many-particle Schrödinger equation, and this has undoubtedly restricted the class of continuum problems. On the transition to solution of the Faddeev equations this class will naturally be extended.

[1]Yu. A. Simonov, Yad. Fiz. **3**, 630 (1966) [Sov. J. Nucl. Phys. **3**, 461 (1966)]; A. M. Badalyan and Yu. A. Simonov, Yad. Fiz. **3**, 1032 (1966) [Sov. J. Nucl. Phys. **3**, 755 (1966)].
[2]A. I. Baz', Yu. T. Grin', V. F. Demin, and M. V. Zhukov, Fiz. Elem. Chastits At. Yadra **3**, 275 (1972) [Sov. J. Part. Nucl. **3**, 137 (1972)].
[3]G. F. Filippov, Fiz. Elem. Chastits At. Yadra **4**, 992 (1973) [Sov. J. Part. Nucl. **4**, 405 (1973)].
[4]A. I. Baz', V. F. Demin, and M. V. Zhukov, Fiz. Elem. Chastits At. Yadra **6**, 515 (1975) [Sov. J. Part. Nucl. **6**, 207 (1975)].
[5]Yu. F. Smirnov and K. V. Shitikova, Fiz. Elem. Chastits At. Yadra **8**, 847 (1977) [Sov. J. Part. Nucl. **8**, 344 (1977)].
[6]R. I. Dzhibuti (Jibuti) and N. B. Krupennikova, *The Method of Hyperspherical Functions in Few-Body Quantum Mechanics* [in Russian] (Metsniereba, Tbilisi, 1984).
[7]J. Raynal and J. Revai, Nuovo Cimento **68A**, 612 (1970).
[8]R. I. Dzhibuti (Jibuti), V. Yu. Tomchinskiĭ, and N. I. Shubitidze, Yad. Fiz. **18**, 1164 (1973) [Sov. J. Nucl. Phys. **18**, 597 (1974)].
[9]Ya. A. Smorodinskiĭ and V. D. Éfros Yad. Fiz. **17**, 210 (1973) [Sov. J. Nucl. Phys. **17**, 107 (1973)].
[10]R. I. Dzhibuti (Jibuti), N. B. Krupennikova, and V. Yu. Tomchinskii, Soobshch. Akad. Nauk Gruz. SSR **81**, 69 (1976).
[11]R. I. Jibuti, N. B. Krupennikova, and V. Yu. Tomchinsky, Nucl. Phys. **A276**, 421 (1977).
[12]R. I. Dzhibuti (Jibuti) and N. I. Shubitidze, Yad. Fiz. **20**, 666 (1974) [Sov. J. Nucl. Phys. **20**, 357 (1975)].
[13]R. I. Dzhibuti (Jibuti), N. B. Krupennikova, and N. I. Shubitidze, Teor. Mat. Fiz. **32**, 223 (1977).
[14]N. I. Shubitidze, in: *Many Particle Aspects of the Theory of Light Nuclei* [in Russian] (Metsniereba, Tbilisi, 1977), p. 65.
[15]R. I. Dzhibuti (Jibuti) and N. I. Shubitidze, Yad. Fiz. **32**, 940 (1980) [Sov. J. Nucl. Phys. **32**, 485 (1980)].
[16]R. I. Dzhibuti (Jibuti) and N. I. Shubitidze, Preprint YaF-1 [in Russian], Institute of Physics, Georgian Academy of Sciences, Tbilisi (1986).
[17]R. I. Dzhibuti (Jibuti), N. B. Krupennikova, and L. L. Sarkisyan, Preprint Ya-11 [in Russian], Institute of Physics, Georgian Academy of Sciences, Tbilisi (1986).
[18]R. I. Dzhibuti (Jibuti) and K. I. Sigua, Preprint YaF-31 [in Russian], Institute of Physics, Georgian Academy of Sciences, Tbilisi (1977).
[19]R. I. Dzhibuti (Jibuti), Fiz. Elem. Chastits At. Yadra **14**, 741 (1983) [Sov. J. Part. Nucl. **14**, 309 (1983)].
[20]V. V. Babikov, *Variable Phase Method in Quantum Mechanics* [in Russian] (Nauka, Moscow, 1976).
[21]F. Calogero, *Variable Phase Approach to Potential Scattering* (Academic Press, New York, 1967) [Russ. transl., Mir, Moscow, 1972].
[22]R. I. Jibuti and R. Ya. Kezerashvili, Czech. J. Phys. **B30**, 1090 (1980).
[23]R. I. Dzhibuti (Jibuti) and R. Ya. Kezerashvili, Yad. Fiz. **39**, 1109 (1984) [Sov. J. Nucl. Phys. **39**, 700 (1984)].
[24]R. I. Dzhibuti (Jibuti) and R. Ya. Kezerashvili, Yad. Fiz. **39**, 419 (1984) [Sov. J. Nucl. Phys. **39**, 264 (1984)].
[25]J. Sperinde, D. Fredrickson, *et al.*, Phys. Lett. **32B**, 185 (1970).
[26]R. I. Dzhibuti (Jibuti) and Sh. M. Tsiklauri, Yad. Fiz. **34**, 58 (1981) [Sov. J. Nucl. Phys. **34**, 33 (1981)].
[27]R. I. Dzhibuti (Jibuti) and Sh. M. Tsiklauri, Yad. Fiz. **36**, 1387 (1982) [Sov. J. Nucl. Phys. **36**, 805 (1982)].
[28]R. I. Dzhibuti (Jibuti) and Sh. M. Tsiklauri, Yad. Fiz. **39**, 1115 (1984) [Sov. J. Nucl. Phys. **39**, 704 (1984)].
[29]G. Saupe and K. V. Shitikova, Preprint E4-85-44, Dubna (1985).
[30]G. Bozzolo, O. Civitarese, and J. P. Vary, Phys. Rev. C 37, 1240 (1988).
[31]D. M. Brink and E. Boeker, Nucl. Phys. **A91**, 1 (1967).
[32]K. V. Shitikova, in: *Proc. of the Tenth European Symposium on the Dynamics of Few-Body Systems* (Balatonfüred, Hungary, 1985), p. 429.
[33]K. V. Shitikova, in: *Nuclear Structure in Heavy-Ion Collisions*, Vol. 1 (World Scientific, Singapore, 1987), p. 373.
[34]G. L. Strobel, Nucl. Phys. **A271**, 162 (1976).
[35]M. Fabre de la Ripelle and J. Navarro, Ann. Phys. (N.Y.) **147**, 281 (1983).
[36]R. Dymarz, J. L. Molina, and K. V. Shitikova, Z. Phys. **A299**, 245 (1981).
[37]K. V. Shitikova, Fiz. Elem. Chastits At. Yadra **16**, 824 (1985) [Sov. J. Part. Nucl. **16**, 364 (1985)].
[38]K. V. Shitikova, in: *First Topical Meeting on Giant Resonance Excitation in Heavy-Ion Collisions* (Italy, 1987), pp. 66, 67.
[39]D. Lebrun *et al.* Phys. Lett. **97B**, 358 (1980).
[40]H. P. Morsch and P. Decowski, Phys. Lett. **82B**, 1 (1979).
[41]Yu. N. Grin' and L. V. Levinson, Yad. Fiz. **14**, 536 (1971) [Sov. J. Nucl. Phys. **14**, 300 (1972)].
[42]E. Hegaki *et al.*, Prog. Theor. Phys. **57**, 1262 (1977).
[43]G.-P. P. Kamuntavichyus, L. Yu. Sabalyauskas, and K. I. Yankauskas, Liet. Fiz. Riuk. **469**, 319 (1971).
[44]R. H. Bassel *et al.*, J. Phys. G **8**, 1215 (1982).
[45]M. Kashiev, G. Saupe, and K. V. Shitikova, Preprint R4-85-319 [in Russian], Dubna (1985).
[46]Dao Tien Khoa and K. V. Shitikova, Preprint E4-85-143, Dubna (1985).
[47]Dao Tien Khoa and K. V. Shitikova, Preprint E4-85-384, Dubna (1985).
[48]M. W. Kirson, Nucl. Phys. **A337**, 194 (1980).
[49]G. Saupe, A. A. Shirokova, and K. V. Shitikova, Preprint E4-85-25, Dubna (1985).
[50]Chr. Christov, I. I. Delchev and K. V. Shitikova, Bulg. Fiz. Zh. **13**, 26 (1986).
[51]H. Ngo and Ch. Ngo, Nucl. Phys. **A348**, 140 (1980).
[52]R. G. Lombard, Ann. Phys. (N.Y.) **77**, 380 (1973).
[53]G. F. Filippov, V. S. Vasilevsky, and A. V. Nesterov, Nucl. Phys. **A426**, 327 (1984).
[54]G. F. Filippov, V. S. Vasilevsky, and A. V. Nesterov, Nucl. Phys. **38**, 584 (1983).
[55]G. F. Filippov, V. S. Vasilevskiĭ, and A. V. Nesterov, Izv. Akad. Nauk SSSR, Ser. Fiz. **49**, 173 (1985).
[56]R. I. Dzhibuti (Jibuti) and P. M. Imnadze, Preprint YaF-7 [in Russian], Institute of Physics, Georgian Academy of Sciences, Tbilisi (1985).
[57]R. I. Dzhibuti (Jibuti) and P. M. Imnadze, Preprint YaF-8 [in Russian], Institute of Physics, Georgian Academy of Sciences, Tbilisi (1985).
[58]R. I. Dzhibuti (Jibuti), T. I. Efremidze, and D. K. Tedoradze, in: *Chemical Physics*, Vol. 6 [in Russian] (1987), p. 34.
[59]J. H. Macek, J. Phys. B **1**, 831 (1968).
[60]U. Fano, Rep. Prog. Phys. **48**, 97 (1983).
[61]E. Pelikan and H. Klar, Z. Phys. **A310**, 153 (1983).
[62]J. Botero and C. H. Greene, Phys. Rev. A **32**, 1249 (1985).
[63]V. I. Lend'el, K. V. Shitikova, and M. I. Gaisak, in: *Tenth Intern. Conf. in Atomic Physics* (Tokyo, August, 1986), p. 74.
[64]V. I. Lend'el, K. V. Shitikova, and M. I. Gaĭsak, in: *International Symposium on the Theory of Few-Particle and Quark–Hadron Systems* [in Russian] (Dubna, June, 1987), p. 27.
[65]M. Fabre de la Ripelle, T. K. Das, and H. T. Coelho, Phys. Rev. **26**, 2281 (1982).

[66]H. Klar and M. Klar, J. Phys. B **13**, 1057 (1980).
[67]M. I. Gaĭsak, V. I. Lend'el, V. T. Navrotskiĭ, and E. P. Sabad, Czech. J. Phys. **2**, 1617 (1982).
[68]P. J. Hicks and J. Comer, J. Phys. B **8**, 1866 (1975).
[69]R. I. Dzhibuti (Jibuti), in: *Physics of Many-Particle Systems*, Vol. 8 [in Russian] (1985), p. 39.
[70]J. L. Ballot and J. Navarro, J. Phys. B **8**, 172 (1975).
[71]V. B. Mandelzweig, Ann. Phys. (N.Y.) **104**, 1 (1977).
[72]J. A. Micnaco and I. Raditi, J. Phys. B **14**, 161 (1981).
[73]V. D. Efors, A. M. Frolov, and M. I. Mukhtarova, J. Phys. B **15**, 819 (1982).
[74]M. L. Haftel and V. B. Mandelzweig, Ann. Phys. (N.Y.) **150**, 48 (1983).
[75]V. D. Éfros, Yad. Fiz. **27**, 845 (1978) [Sov. J. Nucl. Phys. **27**, 448 (1978)].
[76]G. Erens, J. L. Wischers, and R. Wan Wageningen, Ann. Phys. (N.Y.) **67**, 461 (1971).
[77]V. A. Fomin and V. D. Éfros, Yad. Fiz. **34**, 587 (1981) [Sov. J. Nucl. Phys. **34**, 327 (1981)].
[78]A. M. Frolov, Zh. Eksp. Teor. Phys. **92**, 1959 (1987) [Sov. Phys. JETP **65**, 1100 (1987)].
[79]V. A. Fock, Izv. Akad. Nauk SSSR, Ser. Fiz. **18**, 161 (1954).
[80]K. Frankowski and C. L. Pekeris, Phys. Rev. **146**, 46 (1966).
[81]D. L. Knirk, J. Chem. Phys. **60**, 760 (1974).
[82]T. K. Lim, Nucl. Phys. **A416**, 491 (1984).
[83]J. Manz, Comments At. Mol. Phys. **17**, 91 (1985).
[84]S. Nakaichi, T. K. Lim, I. Akaishi, and H. Tanaka, J. Chem. Phys. **78**, 2103 (1983).
[85]M. L. Haftel and T. K. Lim, J. Chem. Phys. **77**, 4515 (1982).
[86]S. Wanatabe and C. D. Lin, Phys. Rev. A **34**, 823 (1986).
[87]F. S. Moraes, T. Coelho, and R. Chanda, Lett. Nuovo Cimento **26**, 466 (1979).
[88]J. M. Richard, Phys. Lett. **100B**, 515 (1981).
[89]J. M. Richard and P. Taxil, Phys. Lett. **128B**, 453 (1983).
[90]J. M. Richard and P. Taxil, Ann. Phys. (N.Y.) **150**, 267 (1983).
[91]B. Richten *et al.*, Phys. Rev. D **21**, 203 (1984).
[92]A. M. Badalyan, L. I. Kitoroadzhe, and D. S. Pariĭskiĭ, Yad. Fiz. **46**, 226 (1987) [Sov. J. Nucl. Phys. **46**, 139 (1987)].
[93]A. M. Badalyan, B. L. Jaffe, and A. V. Smilga, Nucl. Phys. **B281**, 85 (1987).
[94]Yu. S. Kalashnikova, I. M. Narodetskiĭ, and Yu. A. Simonov, Yad. Fiz. **46**, 1181 (1987) [Sov. J. Nucl. Phys. **46**, 698 (1987)].
[95]A. A. Khelashvili, V. Yu. Khmaladze, and N. D. Chachava, Teor. Mat. Fiz. **62**, 136 (1985).
[96]V. V. Burov and K. V. Shitikova, Preprint R2-86-543 [in Russian], Dubna (1986).
[97]M. Harvey, Nucl. Phys. **A352**, 301 (1981).
[98]S. Ohta, M. Oka, A. Arima, and K. Yazaki, Phys. Lett. **119B**, 35 (1982).
[99]M. Oka and K. Yasaki, Nucl. Phys. **A402**, 477 (1983).
[100]A. Faessler *et al.*, Phys. Lett. **124A**, 145 (1983); **122B**, 201 (1982).
[101]Y. Suzuki *et al.*, Phys. Rev. C **27**, 299 (1983).
[102]V. G. Neudatchin *et al.*, Z. Phys. **A313**, 357 (1983).
[103]V. K. Lukyanov and A. I. Titov, in: *Few-Body Systems and Nuclear Forces* (1978), p. 397.
[104]A. A. Kvitsinskiĭ, Yu. A. Kuperin, S. P. Merkur'ev, and V. Yu. Novozhilov, Yad. Fiz. **38**, 702 (1983) [Sov. J. Nucl. Phys. **38**, 419 (1983)]; in: *Proc. of the Seventh International Seminar on Problems of High Energy Physics*, D1 2-84-599 [in Russian] (JINR, Dubna, 1984), p. 392.
[105]A. B. Guimaraes and H. T. Coelho, Phys. Rev. D **24**, 1343 (1981).
[106]D. P. Stanley and D. Robson, Phys. Rev. D **26**, 223 (1982).
[107]A. N. Mitra and I. Kumari, Phys. Rev. D **15**, 261 (1977).
[108]S. J. Brodsky and B. T. Chertok, Phys. Rev. D **14**, 3003 (1976).
[109]S. J. Brodsky and G. Farrar, Phys. Rev. Lett. **31**, 1153 (1973).
[110]S. Blantik and N. Zovko, Acta Phys. Austriaca **39**, 62 (1974).
[111]V. V. Burov and K. V. Shitikova, in: *Proc. of the Eighth International Seminar on Problems of High Energy Physics* (Dubna, 1986), p. 419.
[112]A. M. Badalyan, Preprint ITEP-87-21, Moscow (1987).
[113]A. M. Badalyan and D. I. Kitoroage, Preprint ITEP-87-33, Moscow (1987).
[114]R. Nag, S. Sanyal, and S. N. Mukherjee, in: *Eleventh Intern. Conf. on Few Body Systems in Particle and Nuclear Physics* (Sendai, Tokyo, 1987), p. 100.
[115]K. V. Shitikova, in: *Proc. of the Intern. Symposium on Symmetries and Nuclear Structure* (Yugoslavia, 1986), p. 469.
[116]R. I. Dzhibuti (Jibuti), N. B. Krupennikova, and V. Yu. Tomchinskiĭ, Yad. Fiz. **23**, 539 (1976) [Sov. J. Nucl. Phys. **23**, 285 (1976)].
[117]R. I. Dzhibuti (Jibuti), Preprint YaF-17 [in Russian], Institute of Physics, Georgian Academy of Sciences, Tbilisi (1983).
[118]H. I. Coelho, W. Glöckle, and A. Delfinov, Phys. Rev. **122**, 1796 (1986).
[119]R. I. Dzhibuti (Jibuti) and Sh. M. Tsiklauri, Yad. Fiz. **40**, 1171 (1984) [Sov. J. Nucl. Phys. **40**, 746 (1984)].
[120]R. I. Dzhibuti (Jibuti) and Sh. M. Tsiklauri, Yad. Fiz. **41**, 865 (1985) [Sov. J. Nucl. Phys. **41**, 554 (1985)].
[121]R. I. Dzhibuti (Jibuti), in: *Properties of Few-Particle and Quark–Hadron Systems, Part 1 (Abstracts of Lectures at the Third All-Union School on Few-Particle and Quark–Hadron Systems)* [in Russian] (Vilnius, 1986), p. 70.
[122]A. Ahmadzadeh and J. A. Tjon, Phys. Rev. **139**, B 1085 (1965).
[123]R. I. Dzhibuti (Jibuti), P. M. Imnadze, and Sh. M. Tsiklauri, Yad. Fiz. **43**, 834 (1986) [Sov. J. Nucl. Phys. **43**, 530 (1986)].
[124]R. I. Dzhibuti (Jibuti), P. M. Imnadze, and Sh. M. Tsiklauri, Yad. Fiz. **43**, 1169 (1986) [Sov. J. Nucl. Phys. **43**, 748 (1986)].
[125]M. Fujishiro, K. Okamoto, and T. Tsujimoto, Can. J. Phys. **61**, 1579 (1983).
[126]A. Salyers, Phys. Rev. C **2**, 1653 (1970).
[127]B. Buck, H. Friedrich, and C. Wheatley, Nucl. Phys. **A275**, 246 (1977).
[128]B. Tatischeff and I. Brussand, Nucl. Phys. **A155**, 89 (1970).
[129]J. Avishai, Phys. Rev. C **6**, 677 (1972).
[130]N. K. Glendening and M. Veneroni, Phys. Rev. **144**, 839 (1966).
[131]J. H. Gibborn, R. L. Machlin, J. B. Marion, and H. W. Schmidt, Phys. Rev. **114**, 1319 (1959).
[132]W. John and J. M. Prosser, Phys. Rev. **127**, 231 (1962).
[133]B. Berman, R. Van Hement, and C. Bowman, Phys. Rev. **163**, 958 (1967).

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