Asymptotic behavior of the pion form factor in QCD

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A perturbative approach to the analysis of hard exclusive processes in quantum chromodynamics is explained for the example of an investigation of the asymptotic behavior of the pion form factor. The main elements of the approach are presented, namely, the method of proof of a factorization theorem for the contributions of small and large distances for the amplitudes of hard processes in QCD, analysis of the evolution of the pion wave function in the single- and two-loop approximations, and methods of allowing for radiative corrections.

INTRODUCTION

The investigation of the asymptotic behavior of the elastic hadron form factors in the framework of quantum chromodynamics (QCD), which was begun in Refs. 1–11, was a necessary step toward the extension of the region of applicability of perturbative QCD and its extension to a new class of high-energy phenomena—exclusive hard processes. A specific feature of processes of hard elastic scattering is that the colliding particles penetrate deeply into each other without, however, "disintegrating" into a huge number of secondary particles. Therefore, the study of the characteristics of such processes gives important information about the deepest properties of the particles, in particular, not only about whether or not the constituents are points but also about their number. As the most striking example we can mention the quark counting rules,[12,13] which played an important part in establishing the quark picture of hadron structure. They directly relate the exponent of the power-law decrease of the hadron form factors to the number of valence quarks of the hadrons.

One of the main computational methods in QCD is perturbation theory, i.e., expansion with respect to the QCD coupling constant.[14,15] Since this expansion is justified only in the region of large momentum transfers (or small distances), the cornerstone of all applications of perturbative QCD to real processes, either inclusive or exclusive, is factorization of the contributions of small and large distances. Several approaches to factorization of the contributions, differing somewhat in their technical aspects, are currently known.[16-22] The fundamentals of the factorization technique that we have developed are presented in the review of Ref. 23 for the example of the simplest characteristics of the processes of e^+e^- annihilation into hadrons and deep inelastic scattering. This technique is also effective for the analysis of hard exclusive processes. In particular, its use has made it possible to obtain a complete proof of the factorization theorem for the asymptotic behavior of the simplest process of hard elastic scattering: $e\pi \rightarrow e\pi$, i.e., for the electromagnetic form factor of the pion in the spacelike region of momentum transfers. The use of the methods of perturbative QCD for more complicated problems (behavior of the form factors in the timelike region, decays of hadrons, elastic hadron–hadron scattering, etc.[10,24,25]) is based on a number of assumptions that have not been proved in the framework of perturbative QCD itself and may not follow from it. Therefore, in this review our aim will be to present the formalism of the perturbative QCD approach to hard exclusive processes for the example of the most fully studied problem of the asymptotic behavior of the electromagnetic pion form factor.

It must be emphasized here that for elastic form factors even the assertion that their asymptotic behavior at large momentum transfers is determined by the small-distance dynamics is not at all obvious. In field theory there are models in which this is not the case. A heuristic discussion (in the language of the parton model) of the connection between the various mechanisms that govern the behavior of the form factors of bound states in the region of large momentum transfers is given in Sec. 1, which is introductory in nature. In it, we discuss the formulation of the problem of the form factors of bound states in quantum field theory. The proof of the theorem on factorization of the contributions of small and large distances for the asymptotic behavior of the pion form factor in QCD is given in Sec. 2. Section 3 is devoted to study of the radiative corrections to the asymptotic behavior of the pion form factor, without allowance for which the predictions of perturbative QCD do not have the necessary rigor. Necessary results on the analysis of the asymptotic behavior of hard processes are briefly presented in the Appendix.

1. FORM FACTORS OF BOUND STATES

Asymptotic behavior of elastic form factors in the parton model

As we said in the Introduction, the asymptotic behavior of the form factor $F(Q^2)$ of a composite particle is determined by the number n of its pointlike constituents (valence quark–partons)—the greater the number of constituents, the more rapid the decrease of $F(Q^2)$ with increasing Q^2. According to the quark counting rules,[12,13] the connection between the asymptotic behavior of $F(Q^2)$ and n has the form

$$F(Q^2) \sim (1/Q^2)^{n-1}, \quad Q^2 \rightarrow \infty \tag{1}$$

(where $Q^2 = -q^2$, and q is the momentum transfer).

Hard rescattering

The quark counting rules were initially obtained[12] from general principles based on dimensional analysis. A definite dynamical mechanism—hard rescattering of the quarks—that ensures fulfillment of the quark counting rules in the limit $Q^2 \rightarrow \infty$ was found later.[13] In the picture of hard rescattering,[12] it is assumed that one of the valence quarks first takes up the large momentum transfer and then, through hard rescattering (Fig. 1), this momentum transfer is dis-

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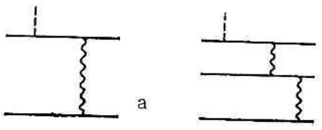

FIG. 1. Picture of hard scattering for the form factors of the pion (a) and proton (b).

tributed among all the valence quarks. It is assumed that in the infinite-momentum frame all the valence quarks in the initial and final states carry finite fractions of the momenta of the corresponding hadrons. As a consequence, all the momenta that flow along the lines of the parton subprocess have "virtuality" of order Q^2. In such an interpretation, the exponent $n-1$ in Eq. (1) is simply the minimal number of gluon exchanges.

The Feynman mechanism

To explain the power-law behavior of the form factors Feynman had earlier proposed a different mechanism,[26] in which the main contribution to the asymptotic behavior of the form factor is made by a configuration in which a valence quark that absorbs a large momentum transfer carries the entire momentum of the hadron. The remaining quarks are assumed to be soft and can be associated with the hadron of either the initial or the final state (Fig. 2). In this case, as we see, the small-distance dynamics plays no part at all.

Connection between the mechanisms

It is obvious that the Feynman mechanism works only if the amplitude of the probability for finding the hadron in a state in which just one quark carries all its momentum is sufficiently large. The two pictures preclude each other, and therefore either the Feynman mechanism or the hard-rescattering picture is dominant. We shall attempt to give an intuitive picture of the connection between these two mechanisms.[27]

We assume that the contribution of the diagram in Fig. 1 can be expressed in the form suggested by the parton model:

$$F(Q^2) \sim \int \varphi(y, k'_\perp)\, \varphi(x, k_\perp)\, V(Q^2(1-x)(1-y) + (k_\perp - k'_\perp)^2)\, dx\, dy\, d^2k_\perp d^2k'_\perp, \qquad (2)$$

where φ, φ^* are certain wave functions that describe the separation from the hadron of one of the valence quarks with momentum $k=(xP,k_\perp)$ $(k'=(yP',k'_\perp))$, and $V((k-k')^2)$ describes the interaction between the separated quark and the remainder of the hadron (Fig. 3). We also assume that the wave functions decrease rapidly with increasing transverse momentum $k_\perp$, i.e., that we can replace $(k_\perp - k'_\perp)^2$ by a certain mean value $M^2 \simeq 2\langle k_\perp^2\rangle$. If we take $V(t)\sim t^{-\alpha}$, then

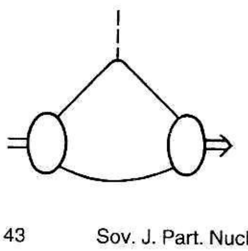

FIG. 2. Feynman mechanism for the pion form factor.

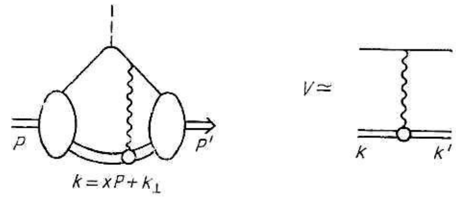

FIG. 3. Parton representation for the hard-scattering expression.

$$F(Q^2) \sim \int dx\, dy\, \varphi^*(y)\, \varphi(x) \times [Q^2(1-x)(1-y) + M^2]^{-\alpha}. \qquad (3)$$

It follows that M^2 can be ignored only when the integral

$$\int_0^1 \frac{dx}{(1-x)^\alpha}\varphi(x) \qquad (4)$$

converges at the upper limit. Otherwise the behavior of the form factor in (3) is determined as $Q^2 \to \infty$ by the region $1-x \sim M^2/Q^2$, and the result depends on the form of the function $\varphi(x)$. It follows from the quark-counting expression (1) that for theories with a dimensionless coupling constant [which was assumed in the derivation of (1) in Ref. 12] the relation $\alpha = n-1$ holds. Therefore, the contribution of the hard subprocess is dominant in the asymptotic region if $\varphi(x)$ behaves in the limit $x\to 1$ as $(1-x)^\beta$ with $\beta > \alpha - 1$, i.e., $\beta > n-2$. Otherwise, when $\beta < \alpha - 1$, the behavior of $F_\pi(Q^2)$ at large Q^2 is determined by the form of the wave function $\varphi(x)$ as $x \to 1$, and as a result

$$F(Q^2) \sim (Q^2)^{-\beta-1}. \qquad (5)$$

In the old (non-QCD) parton model[26] it was assumed that $V(t)$ decreases very rapidly as $t\to\infty$, i.e., that $\alpha \sim \infty$. In this case the Feynman mechanism is always dominant, but then fulfillment of the law (1) is also possible. For this, β must satisfy the condition $\beta = n-1$, i.e., the hadron wave functions must behave in the limit $x \to 1$ in the manner $\varphi_\pi(x) \sim$ const, $\varphi_N(x) \sim (1-x)$, etc. If the wave functions $\varphi(x)$ decrease faster as $x\to 1$, then ultimately, in the limit $Q^2 \to \infty$, the hard-scattering mechanism is dominant. However, at moderately large Q^2 the main contribution may well be due to the Feynman mechanism if the contribution of the hard-scattering diagrams is numerically small for any reason. In other words, if $V(t)$ is a sum of a rapidly (say, exponentially) decreasing soft contribution $V_1(t) \sim A\exp(at)$ and a hard contribution that behaves as a power as $|t| \to \infty$, $V_2(t) \sim B(at)^{1-n}$, with $B \ll A$, then in a fairly large region of Q^2 values the contribution due to $V_1(t)$ may be dominant, and only at very large Q^2 will the asymptotic contribution due to $V_2(t)$ be dominant.

Form factors of bound states in quantum field theory

The foregoing arguments are qualitative and must, of course, be supported by a more serious field-theoretical analysis. Usually, the Bethe–Salpeter formalism is taken as the basis for describing composite particles in quantum field theory.[28] In this formalism a particle with momentum P consisting of a particle ψ_1 and an antiparticle $\overline{\psi}_2$ is described by the Bethe–Salpeter wave function

$$\chi_P(x) = \left\langle 0 \left| T\left\{\psi_1\left(\frac{x}{2}\right)\bar{\psi}_2\left(-\frac{x}{2}\right)\right\}\right| P\right\rangle, \quad (6)$$

which depends not only on the relative distance **x** but also on the relative time x_0 of the constituents. In the quasipotential approach,[29] which is a three-dimensional formalism, one that uses the value of the Bethe–Salpeter function on some hypersurface, for example, on $x_0 = 0$ (Ref. 29), on the null plane $x_0 + x_3 = 0$ (Ref. 30), or on the surface $(xP) = 0$ (Ref. 31) (covariant generalization of the condition $x_0 = 0$).

The dynamical variables (form factors, scattering amplitudes) of the composite particles can be expressed in terms of the Bethe–Salpeter functions[32] or quasipotential functions.[33] The form factors of constituent particles have been considered by a number of authors, who used, in particular, the ladder approximation for the Bethe–Salpeter equation[34] and the idea of conformal invariance[35]; some results were obtained in the framework of three-dimensional formalisms.[36] In Refs. 37 and 38 the first attempts were made to use renormalization-group methods and operator expansions to investigate the asymptotic behavior of form factors. In our paper of Ref. 39 the pion form factor was investigated in the ladder approximation of a quark model with scalar gluons by a method based on analysis of the asymptotic behaviors of Feynman diagrams in the α representation.

Further generalization of the methods used in Ref. 39 served as the basis of our approach[1,7,8] to the factorization of the contributions of small and large distances for the asymptotic behavior of the pion form factor in QCD. Other approaches (Refs. 2, 3–6, 9–11, 20, and 25) have also been proposed for the investigation of the asymptotic behavior of the hadron form factors in QCD. They have been based on operator expansions (Refs. 2, 3, 6, 9, 20, and 25), the Bethe–Salpeter formalism,[4] the quasipotential formalism in "light-front" variables,[11] and the formulation, intimately related to it, of perturbation theory on the light cone.[5,10]

Factorization and form factors of bound states in perturbation theory

Bound states are absent in any finite order of perturbation theory. Therefore, it is necessary to consider the total amplitudes obtained by summing over all perturbation orders. The investigation of the electromagnetic form factor of the pion, treated as a bound state of a quark and an antiquark in QCD, is therefore based on analysis of the total amplitude $T(P, P')$ that describes the process $q\bar{q}\gamma \to q'\bar{q}'$ (P and P' are the total momenta of the initial and final states). Of course, it is necessary to take a combination of the quark fields $C(q, \bar{q})$ that has a nonzero projection onto the pion state $|P\rangle$:

$$\langle 0 \mid C(q, \bar{q}) \mid P\rangle \equiv \chi_P \neq 0. \quad (7)$$

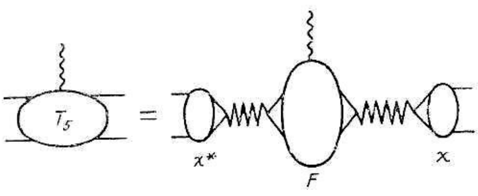

FIG. 4. Pole structure of the five-point function.

In this case the auxiliary amplitude T_5 will have two poles (cf. Ref. 32) corresponding to pion bound states (Fig. 4):

$$T(P, P' \ldots) = i^2 \frac{\chi_{P'} F_\pi(q) \chi_P^*}{(P^2 - m_\pi^2)(P'^2 - m_\pi^2)} + \ldots, \quad (8)$$

where $q = P' - P$, and $F_\pi(q)$ is the pion form factor:

$$\langle P' | J_\mu(0) | P\rangle = (P_\mu + P'_\mu) F_\pi(q). \quad (9)$$

We assume further that we have succeeded in showing that in any finite order of perturbation theory the amplitude T is given by an expression in which the contributions of the small and large distances factorize (Fig. 5a):

$$T(P', P) = f_{P'}^* \otimes E \otimes f_P \{1 + O(1/Q^2)\}, \quad (10)$$

where the functions $f_P, f_{P'}^*$ describe the interaction at large distances in the initial and final states, respectively, and the function E describes the interaction at small distances. If the asymptotic behavior of the total amplitude is the sum of the asymptotic behaviors of all the diagrams (in the framework of perturbation theory we cannot assume otherwise), the expression (10) will also hold for the total amplitude. In this case, the functions f, f^* are given by Green's functions of the form

$$f_P = \langle 0 \mid \mathcal{O} C_P(q, \bar{q}) \mid 0\rangle, \quad (11)$$

where $\mathcal{O}$ is some operator constructed from the quark and gluon fields. Obviously, the functions f, f^* must also have poles corresponding to the pion states (Fig. 5b):

$$f_P = i \frac{\chi_P}{P^2 - m_\pi^2} \langle 0 | \mathcal{O} | P\rangle. \quad (12)$$

Comparing now the expressions (8) and (10), (12), we conclude that

$$F_\pi(q) = \langle P' | \mathcal{O} | 0\rangle \otimes E \otimes \langle 0 | \mathcal{O}' | P'\rangle \{1 + O(1/Q^2)\} \quad (13)$$

(Fig. 5c). In other words, if the factorization (10) has been established for the auxiliary amplitude T in each order of perturbation theory, then it is also valid for the form factor of the bound state (pion), and the particular form of the amplitude T, i.e., the projections $\chi_P, \chi_{P'}^*$ has no influence on the final result—it is sufficient for these projections to be zero. This means that we can choose any combination of the quark, q and $\bar{q}$, and gluon, A_μ, fields that appears to us convenient, provided, of course, that $\langle 0 | C(q, \bar{q}, A) | P\rangle \neq 0$. This

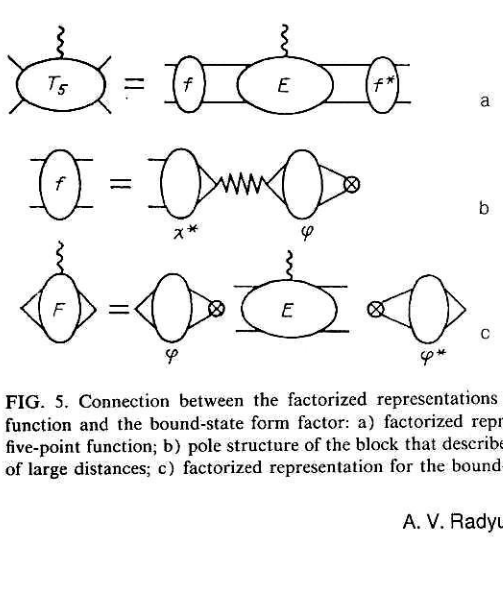

FIG. 5. Connection between the factorized representations for the five-point function and the bound-state form factor: a) factorized representation for the five-point function; b) pole structure of the block that describes the contribution of large distances; c) factorized representation for the bound-state form factor.

circumstance is very important in QCD, since the simplest combination $q(p_1)\bar{q}(P-p_1)$ [or, in the coordinate representation, $q(x)\bar{q}(y)$] is not gauge invariant and therefore cannot, strictly speaking, be regarded as colorless. As a result of this, its properties (particularly in the infrared region) may be quite different from the properties of the colorless state that the pion undoubtedly is. To obtain a gauge-invariant combination, we can take as $C(q, \bar{q}, A)$, for example,

$$C = \bar{q}(x)\gamma_5 \hat{E}(x, y; A) q(y), \tag{14}$$

where $\hat{E}(x, y; A)$ is the P exponential

$$\hat{E}(x, y; A) = P\exp\left(ig(x^\mu - y^\mu)\int_0^1 dt A_\mu(xt + y(1-t))\right). \tag{15}$$

However, the simplest solution is to take the fields q and $\bar{q}$ at the same point,

$$C(q, \bar{q}) = \bar{q}(x)\gamma_5 q(x) \equiv j_5(x),$$

i.e., to investigate the three-point Green's function

$$T_3 = \int \langle 0 \mid j_5^*(x) J^\mu(0) j_5(y) \mid 0 \rangle \times \exp(iPx - iP'y)\, d^4x\, d^4y \tag{16}$$

in the region of large momentum transfers $q^2 \doteq -Q^2$.

2. FACTORIZATION OF THE CONTRIBUTIONS OF SMALL AND LARGE DISTANCES

Alpha representation for the three-point function

Our approach to hard processes in QCD is based on analysis of the asymptotic behavior of the diagrams of perturbation theory in the α representation[40–42] as generalized to the case of gauge theories in Ref. 43. The reviews of Refs. 17 and 23 are devoted to an exposition of this analysis. Details of the analysis in the α representation that are needed for what follows can be found in the Appendix.

The three-point function $T_3(P, P')$ depends on three momentum invariants: P^2, P'^2, and $q^2 = (P' - P)^2$. In the α representation it can be written in the form (see Refs. 17 and 23)

$$T_3(P, P') = \sum_{\substack{\text{perturbative}\\ \text{diagrams}}} \frac{(g^2)^{z-1}}{(4\pi)^2} \int_0^\infty \prod_\sigma d\alpha_\sigma \frac{1}{D^2(\alpha)} G(\alpha; P, P') \times \exp\left\{-iQ^2\frac{A(\alpha)}{D(\alpha)} + iP^2\frac{A_1(\alpha)}{D(\alpha)} + iP'^2\frac{A_2(\alpha)}{D(\alpha)} - i\sum_\sigma (m_\sigma^2 - i\varepsilon)\alpha_\sigma\right\}, \tag{17}$$

where z is the number of loops of the corresponding Feynman diagram, the pre-exponential factor $G(\alpha, P, P')$ is some polynomial in P and P', and $A(\alpha)$, $A_1(\alpha)$, $A_2(\alpha)$, and $D(\alpha)$ are positive functions of the parameters α uniquely associated with the structure of the considered diagram.

General structure of the contributions for the three-point function

In accordance with the general prescription,[23] to study the asymptotic behavior of the amplitude it is necessary to find the regions of the α space in which the factor $A(\alpha)/D(\alpha)$ multiplying the large momentum invariant Q^2 (the only one in the given case) vanishes. The small-distance regime (SDR) ensures this in four cases (Fig. 6):

1) the entire diagram can be contracted to a point (Fig. 6a);

2) one can contract to a point a certain subgraph that contains the vertices 0 and x (Fig. 6b);

3) the same but for the vertices 0 and y (Fig. 6c);

4) one can contract to a point a subgraph containing only the vertex 0 (Figs. 6d and 6e).

It is only in the fourth case that the reduced diagram contains components describing an interaction at large distances in both the initial and the final states and capable, therefore, of giving poles $(P^2 - m_\pi^2)^{-1}$ and $(P'^2 - m_\pi^2)^{-1}$ after summation over all orders. The coefficient functions of the configurations of Figs. 6a–6c can be calculated at zero values of the invariants P^2 and/or P'^2 (see Ref. 23) and, naturally, do not contain any poles of the type $1/(P^2 - m_\pi^2)$ with respect to the variables.

"Double contraction"

We note that the fourth case has two variants (Figs. 6d and 6e). The configuration of Fig. 6e means that the two graphs, V_L and V_R, simultaneously give a leading pole, say $(J+1)^{-1}$, to the Mellin transform of the form factor $\Phi(J)$. It is really necessary to consider this configuration in some simple scalar theories.[8] In theories with spinor quarks, this configuration does not in reality work (this will be demonstrated later), since if the subgraph V_L has the pole $(J+1)^{-1}$, then the subgraph V_R can generate a pole only at $J = -2$ or to the left, while at $J = -1$ its contribution is regular.

Hierarchy of contributions

Estimates of the contributions of the SDR subgraphs can be obtained from the expression (A.7). Since $T_\mu = (P_\mu + P'_\mu)F$, it is convenient to consider the contraction $T = P_\mu T^\mu$. Then

$$F = 2T/Q^2 \tag{18}$$

and the expression (A.7) will give an estimate directly for T, and for the contribution to F we have

$$F_V^{\text{SDR}} \backsim Q^{2-\sum t_i}. \tag{19}$$

Thus, subgraphs with four external quark lines give the leading SDR contribution $F_V^{\text{SDR}} \propto 1/Q^2$.

The configuration of Fig. 6c, in which the lower quark is in the infrared regime ($\alpha \to \infty$), gives with allowance for the expression (18) the contribution

$$F^{\text{IR}} \backsim Q^{-2-\sum t_i} \to 1/Q^4, \tag{20}$$

since the subgraph S has at least two external quark lines.

The combined SDR–IR regime in the case when the infrared subgraph S has only gluon external lines makes the

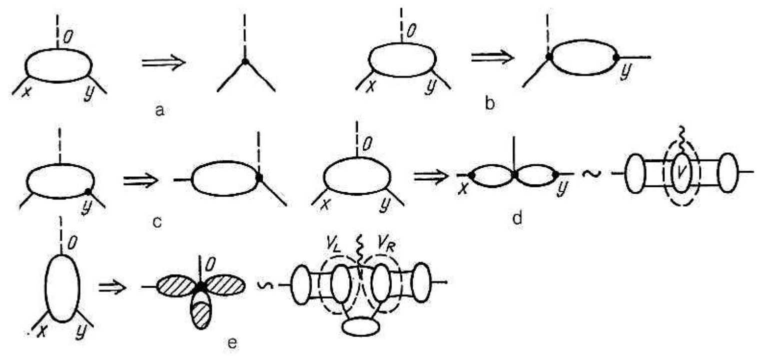

FIG. 6. Structure of the contributions corresponding to the small-distance regime for the three-point function.

same contribution as the SDR regime for the configuration without S. This follows from the estimate of (A.9), which in our case gives

$$F_{V\ S}^{\mathrm{SDR\ IR}}(Q^2) \leqslant Q^{2-\sum\limits_V t_i - \sum\limits_S t_j}. \quad (21)$$

Soft contributions

In reality all the leading SDR–IR contributions cancel after summation over all diagrams of a given order because the soft long-wave gluons "feel" only the total color charge of the system, and their interaction with the colorless systems is suppressed. The presence of the leading contributions $F_{V\ S}^{\mathrm{SDR\ IR}} \sim 1/Q^2$ in the individual diagrams merely reflects the fact that the investigated colorless system consists of objects with color charge—an individual diagram "knows nothing" about the fact that the complete system is colorless.

Technically, the summation over all possible sets of soft gluons can be done by the method described in Ref. 23. Since all the external lines in the considered case are colorless, the colored lines of the corresponding diagrams form closed loops. Therefore, as a result of summation we obtain a P-exponential taken around a closed contour. We must then bear in mind that

$$P\exp\left(ig\oint A_\mu(z)\,dz^\mu\right) = 1 + O(G).$$

The unity corresponds to a configuration without soft exhanges, i.e., to the "pure" SDR case, and the contribution $O(G)$ means that the total contribution of all the configurations that contain soft exhanges is suppressed in a power-law manner by virtue of (21), since the field $G_{\mu\nu}$ has twist 1.

It should be noted that in problems of form-factor type the SDR contributions are also somewhat special, namely, there are doubly logarithmic contributions in all contributions of the individual diagrams. To illustrate this, we consider the behavior of the quark form factor in QCD.

Quark form factor in the single-loop approximation

We consider the single-loop contribution to the Dirac quark form factor $f(Q^2)$ (Fig. 7). We introduce the Mellin transform for $f(Q^2)$:

$$f(Q)^2 = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Gamma(1-J)\,(Q^2)^J\,\Phi(J)\,dJ. \quad (22)$$

Of interest to us here is only the ultraviolet-finite part of the contribution, the Mellin transform of which has the α representation

$$\Phi^{\mathrm{con}}(J) = -\frac{g^2}{8\pi^2}C_F\int_0^{i\infty}\frac{d\alpha_1\,d\alpha_2\,d\alpha_3}{D^2(\alpha)}\left(\frac{\alpha_1\alpha_2}{D(\alpha)}\right)^{J-1}\left(1-\frac{\alpha_1}{D}\right)$$
$$\times\left(1-\frac{\alpha_2}{D}\right)\exp\left\{p_1^2\frac{\alpha_1\alpha_3}{D}+p_2^2\frac{\alpha_2\alpha_3}{D}-m_g^2\alpha_3\right.$$
$$\left.-m_q^2(\alpha_1+\alpha_2)\right\}, \quad (23)$$

where $D(\alpha) = \alpha_1 + \alpha_2 + \alpha_3$, m_q is the mass of the quark, and m_g is the (fictitious) mass of the gluon.

We first investigate the behavior of this contribution in the Sudakov regime: $-p_1^2 \sim -p_2^2 \gg m_q^2$. Note that for such a choice of the kinematics the integrals over α_3 have the necessary infrared cutoff, and, therefore, the gluon mass m_g can be kept equal to zero. However, in the given case the limits $Q^2 \to \infty$ and $m_g \to 0$ do not commute, and from the methodological point of view it is helpful to investigate both cases $m_g \neq 0$ and $m_g = 0$.

The considered diagram has four leading SDR subgraphs:

$$V_1 = \{\sigma_1\},\ V_2 = \{\sigma_2\},\ V_3 = \{\sigma_1,\ \sigma_2\},$$
$$V_4 = \{\sigma_1,\ \sigma_2,\ \sigma_3\}.$$

Integrating over the small λ ($\lambda = \alpha_1 + \alpha_2 + \alpha_3$), and then over the small β_1 ($\beta_1 = \alpha_1/\lambda$) and the small β_2 ($\beta_2 = \alpha_2/\lambda$), we obtain the maximal singularity J^{-3}, corresponding to the doubly logarithmic contribution $\sim g^2\ln^2(Q^2/P^2)$. Note that the doubly logarithmic contribution is due to integration over the region $\lambda \gg \beta_1\lambda \sim \beta_2\lambda$, i.e., $\lambda_3 \gg \alpha_1 \sim \alpha_2$. The main contribution is made by the region in which $Q^2\alpha_1\alpha_2/\lambda \sim 1$, i.e., the region $\alpha_1 \sim \alpha_2 \sim c/Q^2$, $\alpha_3 \sim c^2/Q^2$, where $c \ll 1$. In the momentum representation this corresponds to integration over the region $P^2 \ll k_3^2 \ll Q^2$.

For massless gluons there is also an infrared regime, $\alpha_3 \to \infty$, which also gives a pole J^{-1}. In this case the main contribution is made by the region $\alpha_3 \sim Q^2/P^4$ or, in the momentum representation, $k_3^2 \sim P^4/Q^2$. In this case the singularity J^{-3} can also be obtained by integration over the region $\alpha_1 \sim \alpha_2 \to 0$, $\alpha_3 \to \infty$.

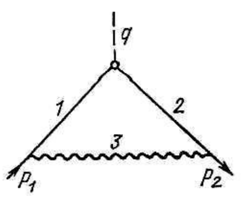

FIG. 7. Single-loop contribution to the quark form factor.

Thus, in the Sudakov regime there are two types of doubly logarithmic contribution. Those of the first type, due to integration over small α, are present both when $m_g \neq 0$ and when $m_g = 0$, while those of the second type, which arise as a result of integration over the region $\alpha_3 \to \infty$ (infrared regime), are present only when $m_g = 0$. As a result, for $m_g = 0$ the doubly logarithmic contribution is twice as large as for $m_g \neq 0$:

$$f(Q^2)\,|_{m_g=0} = -\frac{g^2}{8\pi^2} \ln\left(\frac{Q^2}{-p_1^2}\right) \ln\left(\frac{Q^2}{-p_2^2}\right). \tag{24}$$

For $m_g \neq 0$ the contribution obtained from the region $\alpha_1 \to 0$, $\alpha_2 \to 0$, $\lambda > 1/p^2$ depends logarithmically on the gluon mass:

$$f(Q^2)|_{m_g\neq 0} = -\frac{g^2}{16\pi^2} \ln^2\left(\frac{Q^2}{-p^2}\right) - \frac{g^2}{8\pi^2} \ln\left(\frac{Q^2}{-p^2}\right) \ln\left(\frac{p^2}{m_g^2}\right), \tag{25}$$

i.e., it is singular as $m_g \to 0$. This infinity signals, as it were, the fact that for $m_g = 0$ it is necessary to take into account the "infrared" pole J^{-1} which arises as a result of integration over the region $\alpha \sim Q^2/p^4$. For $m_g \neq 0$, however, there is a cutoff factor $\exp(-i\alpha m_g^2)$, and this leads to exponential suppression of the infrared contribution if $Q^2 m_g^2/p^4 \gg 1$. Thus, the expression (25) is valid only for $Q^2 \gg p_1^2 p_2^2/m_g^2$. It follows from this that passage to the limit $m_g \to 0$ in (25) is impossible, i.e., in this case the limits $Q^2 \to \infty$ and $m_g \to 0$ do not commute.

For the form factor on the mass shell $p_1^2 = p_2^2 = m_q^2$ the coefficients of p_1^2 and p_2^2 in the exponential of the expression (23) in the limit $\alpha_3 \to \infty$ are equal to α_1 and α_2 and cancel against the term $m_q^2(\alpha_1 + \alpha_2)$. As a result, the integral (23) is infrared-divergent if $m_g = 0$. In this kinematics it is necessary to introduce a nonzero gluon mass for infrared regularization of the form factor. For $m_g \neq 0$ the limit $p^2 \to m_q^2$ in (25) is smooth, and (25) for $p^2 = m_q^2$ gives the expression for the single-loop contribution to $f(Q^2, p^2 = m_q^2)$, the quark form factor on the mass shell.

For colorless systems (or, and this is the same thing, for amplitudes with colorless external lines) the leading infrared contributions cancel after summation over all diagrams of a given order. In the case $m_g = 0$ this is equivalent to cancellation of the leading infrared poles with respect to J, and in the case $m_g \neq 0$ to cancellation of the logarithmic singularities $\ln(m_g^2)$. Therefore, we shall now make a more detailed investigation of the SDR contributions.

Structure of the SDR contributions to the quark form factor

In the theory with $m_g \neq 0$ there are no infrared poles with respect to J, and the behavior of the quark form factor at $Q^2 \gg p^4/m_g^2$ is due to the small-α regime. A typical configuration that gives a leading contribution is shown in Fig. 8. The subgraph v corresponding to integration over the small α has two spinor external lines and an arbitrary number of vector lines. The vector lines can be divided into three types:

a) lines corresponding to the initial quark (A lines);

b) lines corresponding to the final quark (B lines);

c) vacuum lines (C lines).

As in the analysis of the form factors of deep inelastic scattering in QCD,[23] it is sensible to consider directly the sum of all configurations obtained from the same primitive

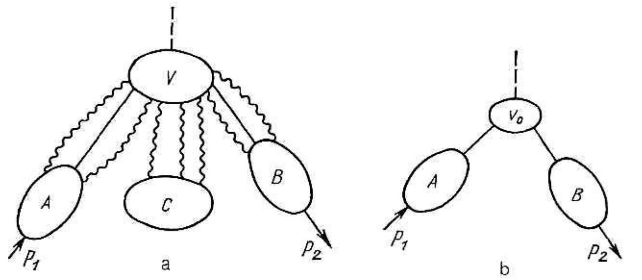

FIG. 8. Structure of the contributions for the quark form factor: a) general form; b) primitive configuration.

configuration (Fig. 8b). The contribution of the primitive configuration can be expressed in the form

$$f^{\rm pr}(Q^2) \backsim \int d^4\xi\, d^4\xi' \langle P' | \bar\psi(\xi') | 0\rangle \mathscr{E}_{v_0}(\xi, \xi') \langle 0 | \psi(\xi) | P\rangle. \tag{26}$$

However, in our problem there is a new possibility, namely, the added gluon lines may be connected not only to internal but also to external lines of the original subgraph v_0 (Fig. 9). In particular, in Fig. 9b the insertion into the external spinor line corresponds to the substitution

$$\psi(\xi) \to g \int d^4z S^c(\xi - z)\, \gamma_\mu A^\mu(z)\, \psi(z), \tag{27}$$

and the operator $\psi(z)$ replaces the operator $\psi(\xi)$ in the matrix element $\langle 0|...|P\rangle$, the operator $A_\mu^a(z)$ is added to the matrix element $\langle P'|...|0\rangle$, and the factor $S^c(\xi - z)\gamma_\mu$ is added to the coefficient function $\mathscr{E}(\xi, \xi', ...)$.

Modification of the coefficient function of the primitive subgraph v_0 by an external gluon field

The summation over the insertions into the internal lines of the subgraph v_0 can be done by means of the method explained in Ref. 23 (see also the Appendix), i.e., by means of the representation of the propagators $S^c(x, y; A)$ and $D^c(x, y; A)$ as products of the P exponentials $E(x, z_0; A)$ and $E(z_0, y; A)$, which absorb the "A dependence" (i.e., the contribution of the fields with zero twist), and of the propagators $\mathscr{S}^c(x, y; \mathscr{A})$ and $\mathscr{D}^c(x, y; \mathscr{A})$ in the gluon field $\mathscr{A}$ taken in the Fock–Schwinger gauge $(x^\mu - z_0^\mu)A_\mu(x) = 0$ (Refs. 44 and 45; see also Refs. 46–49), in which the twist of the field $\mathscr{A}_\mu$ is equal to unity: $\mathscr{A} = \mathscr{A}(G)$. We recall that the addition of an external line corresponding to a field with nonzero twist leads to a power-law suppression of the contribution of the SDR subgraph that then arises [see (19)]. Thus, allowance for the insertions into the internal lines reduces to a modification of the corresponding propagators by the factors $E(x, z_0; A)$ and $E(z_0, y; A)$. Combining them by means of the method of Ref. 23, we find that the total effect of the gluon insertions into the internal lines of the primitive subgraph v_0 reduces to the substitution

$$\bar\psi(\xi') E(\xi', \xi) \psi(\xi) \to \bar\psi(\xi') \hat{E}(\xi', z_0; A) \times \mathscr{E}_{v_0}(\xi, \xi') \hat{E}(z_0, \xi) \psi(\xi). \tag{28}$$

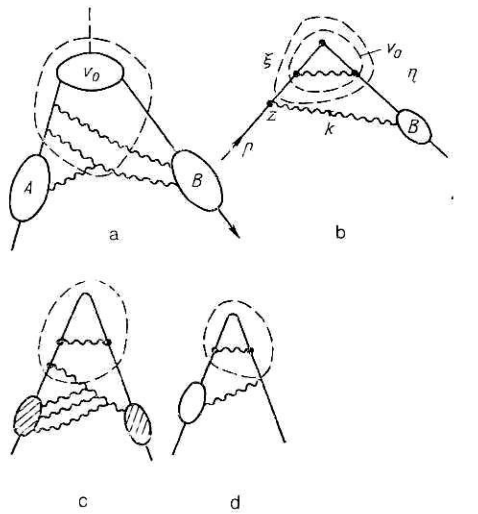

FIG. 9. Structure of the gluon insertions into the external lines.

Allowance for gluon insertions into external lines

The summation over the gluon insertions into an external quark line of the subgraph v_0 obviously leads to replacement of the operator $\psi(\xi)(\overline{\psi}(\xi'))$ by the operator $\Psi(\xi, A)\overline{\Psi}(\xi', A)$, which describes the quark fields in the external gluon field A, i.e., its replacement by the solution of the Dirac equation in the external gluon field:

$$i\gamma^\mu\left(\frac{\partial}{\partial\xi^\mu} - igA_\mu\right)\Psi(\xi, A) = 0. \qquad (29)$$

The problem now is to find for the function $\Psi(\xi, A)$ a representation that correctly (for the problem under consideration) reflects its dependence on the fields A_μ with nonzero twist. By analogy with the propagator, we can represent $\Psi(\xi, A)$ in the form

$$\Psi(\xi, A) = \hat{E}(\xi, z_0)\Psi(\xi, \mathscr{A}; z_0). \qquad (30)$$

The function $\Psi(\xi, \mathscr{A}; z_0)$ will then satisfy Eq. (28), in which we now have not A_μ but the gluon field $\mathscr{A}$ taken in the Fock–Schwinger gauge $(x^\mu - z_0^\mu)A_\mu = 0$. Since $\mathscr{A} = \mathscr{A}(G)$, we can, ignoring contributions $O(G)$, replace $\Psi(\xi, \mathscr{A}; z_0)$ by the original operator $\Psi(\xi)$. In other words, allowance for the insertions into the external quark line has been reduced to addition of the P exponential $\hat{E}(\xi, z_0)$.

We also point out that the gluon lines added to the primitive subgraph v_0 are divided into three types, depending on the block, A, B, or C (Fig. 8a), from which they emanate. In accordance with this, the operators of the gluon fields occur in the bracket $\langle 0|...|P\rangle$, or in $\langle P'|...|0\rangle$, or in $\langle 0|...|0\rangle$. This means that the summation over the gluon insertions must be made successively, say, first over the A insertions for fixed B and C insertions, then over the B insertions for fixed C insertions, and only then over the C insertions.

The primitive subgraphs for the intermediate stages have not only quark but also gluon external lines, into which insertions must also be made (see Fig. 9c). The gluon field operator $A_\mu(\xi)$ in the external gluon field B obviously acquires the factor $\widetilde{E}(\xi, B; z_0)$. (To distinguish the gluon fields according to their type, we shall, when necessary, denote them by the symbols A, B, or C, respectively.) Here it is also important to emphasize that A and B insertions cannot be made into all external lines but only into "opposite" ones, i.e., lines that enter an opposite block.

In other words, the summation over the A insertions into the primitive single-loop subgraph (Fig. 9a) gives the factor $E(z_0, \xi')$ for the field $\overline{\psi}(\xi')$ but in no way affects the field $\psi(\xi)$. Therefore, the result of summation over all the A insertions (into both the external and the internal lines) leads with allowance for (28) in the given case to the substitution

$$\overline{\psi}(\xi')\mathscr{E}_{v_0}(\xi', \xi)\psi(\xi) \to \overline{\psi}(\xi')\mathscr{E}_{v_0}(\xi', \xi)\hat{E}(z_0, \xi; A)\overline{\psi}(\xi). \qquad (31)$$

Optimization of the choice of the P exponential

We shall now consider what choice of the parameter z_0 most adequately reflects the structure of the investigated amplitude, in particular, its behavior in the momentum representation. The factors $E(x, z_0)$ and $E(z_0, y)$ [whose product is equal to $E(x, y; A)$ apart from $O(G)$ terms], which occur in the expression for the propagator $\mathscr{S}(x, y; A)$, mean that the field A acts on the quark in its motion from z_0 to x. Similarly, the factor $E(x, z_0; A)$ in the expression (30) for $\Psi(x, A)$ means that the field A acts on the quark in its motion from z_0 to x. In the example considered above, the summation over the A insertions corresponded to allowance for the effects associated with the motion of the final quark in the field A_μ of the gluons emitted by the initial quark. It is natural to expect that the final quark will move from the point ξ' to infinity in the direction specified by the momentum P', and therefore as the contour (x, z_0) it is most natural to take the line $z_\mu = \xi + P'_\mu s$. This choice of the contour corresponds to the P exponential

$$E_{P'}(\xi, \infty) = P\exp\left(igP'_\mu\int_0^\infty ds A_\mu(\xi + sP')e^{-\varepsilon s}\right)\Big|_{\varepsilon\to 0}, \qquad (32)$$

which is identical to the operator of the gauge transformation to the axial gauge $P'_\mu A^\mu = 0$, which, like the Fock–Schwinger gauge,[44,45] belongs to the class of physical gauges in which the field A has twist 1, i.e., $A^{\text{axial}} = A(G)$ (Ref. 77). Therefore, in all the expressions given above the P exponentials $E(x, z_0; A)$ can be understood as P exponentials $E_{P,P'}(x, A)$.

The choice of the P exponential in the form (32) for the A insertions is in reality uniquely dictated by the analysis in the momentum representation. Let us consider the configuration shown in Fig. 9b. To it there corresponds addition of the factor $\gamma^\mu(\hat{P}' - \hat{k})/(P'^2 - m_q^2 - 2(P'k) + k^2)$ to the coefficient function of the primitive subgraph v_0. It should here be noted that in accordance with the prescription formulated in Ref. 23 we must, in the construction of the coefficient function, set all the small momentum invariants (the masses m_q^2, the "virtualities" of the external lines P^2, P'^2, k^2, etc.) equal to zero, since allowance for them is equivalent to including in the coefficient function contributions corresponding to higher twists. In our example it is also necessary to discard the term $\hat{k}$ in the numerator of the additional factor, since the momentum $\hat{k}$ of the gluon does not have in this approximation components proportional to P':

$$i^n \langle 0 | \ldots \partial_{\mu_1} \ldots \partial_{\mu_n} A \quad \ldots | P\rangle = P_{\mu_1} \ldots P_{\mu_n} \int_0^1 \alpha^n f(\alpha)\, d\alpha \tag{33}$$

i.e., $k_\mu \approx \alpha P$, and the combination

$$\bar{u}(P')\gamma_\mu \hat{P} \ldots \langle 0 | \ldots A_\mu \ldots | P\rangle, \tag{34}$$

which corresponds to the configuration of Fig. 9b, is a quantity of order $O(P^2)$. Thus, the additional factor is proportional to the quark propagator in the eikonal approximation:

$$g\hat{A}_\mu(k)\gamma^\mu S(P'-k) \simeq g\frac{(P'A(k))}{(P'k)+i\varepsilon}. \tag{35}$$

The same will be true for the following A insertions into an external quark line, and the summation of the eikonal series gives precisely the P exponential (32).

We can also arrive at this result in a simpler way. We note that the configuration shown in Fig. 9b "works" only when the gluon external line introduces into the coefficient function the factor P_μ (it is this that corresponds to the fact that the field A_μ has zero twist), which in conjunction with P'_μ gives an $O(Q^2)$ contribution that compensates the denominator of the additional propagator. Thus, eveything reduces to the result that $P'_\mu\langle 0|...A^\mu...|P\rangle \sim (PP')$. But if we impose on A_μ the axial gauge $P'_\mu A^\mu = 0$, the configuration of Fig. 9b will not contribute. This means that in the given case all the necessary information about the contribution of the fields with zero twist is contained in the operator of the gauge transformation to the gauge $P'_\mu A^\mu = 0$, i.e., in the P exponential (32).

Structure of the final result

Thus, the summation over the A insertions has led to a modification of the matrix element $\langle 0|\psi(\xi)|P\rangle$ that describes the quark of the initial state. It has been transformed into $\langle 0|E_{P'}(\xi, A)\psi(\xi)|P\rangle$. Similarly, the summation over the B insertions [with allowance for the fact that the operator of the gauge transformation which "kills" the contribution of the configuration of Fig. 9d will in this case be the P exponential $E_P(\xi, B)$] transforms $\langle P'|\bar{\psi}(\xi')|0\rangle$ into $\langle P'|\bar{\psi}(\xi')E_P^{-1}(\xi', B)|0\rangle$.

The situation with regard to the C insertions is somewhat more complicated, since in this case gluons must be inserted in both the left and the right external lines. For the left external lines it is necessary to take E_P, and for the right $E_{P'}$, but what should we take for the insertions into the internal lines? The answer is very simple: For the C insertions the propagators of the internal lines must be taken in the form

$$\mathcal{S}^c(x, y; C) = E_P(x, C)\mathcal{S}^c(x, y; C)E_{P'}^{-1}(y, C) \rightarrow E_P(x, C)S^c(x-y)E_{P'}^{-1}(y, C) \tag{36}$$

for the quarks and similarly for the gluons. As a result, after all the necessary commutations, the C insertions give the factor $E_P^{-1}(0, C)E_{P'}(0, C)$, where 0 is the coordinate of the photon vertex. Thus, the C insertions give the P exponential calculated for a contour "traced" by the quark: from infinity along P_μ to the point 0, and from the point 0 along P'_μ to infinity. It is true that we must here be a bit more precise. The gluons must not be too soft, since the contribution of the infrared regime is taken into account separately. Therefore, in the α representation the values of the parameters α associated with the additional gluon lines must be bounded above: $\alpha_\sigma < 1/\lambda^2$, where λ is the energy boundary that separates the soft gluons from those that are not soft. In the language of P exponentials, such a bound corresponds to a cutoff of the contour length, i.e., replacement of the upper limit of integration over s in (32) by some finite value s_0.

We emphasize that this cutoff applies to the objects $\langle 0|E_{P'}(\xi, B)\psi(\xi)|P\rangle$ and $\langle 0|E_{P'}^{-1}(0, C)E_P(0, C)|0\rangle$, which describe the contribution of large distances and therefore have the usual (for such objects) ultraviolet lower bound on the parameters α (of the type $\alpha > 1/\mu^2$, where μ is the boundary between the small and large but not infrared-large distances). The infrared cutoff $\alpha_V < 1/\mu^2$ corresponding to it applies already to the coefficient function $\mathscr{E}(\xi, \xi')$. As we have already noted, in the momentum representation the coefficient functions $E(P,P',k,...)$ contain the logarithms $\ln(Q^2/k^2)$ and $\ln(Q^2/P^2)$, which are singular ($\rightarrow\infty$) in the limit $P^2, k^2 \rightarrow 0$. The bound $\alpha_V < 1/\mu^2$ ensures smoothness of the limit $P^2, k^2 \rightarrow 0$ and transforms $\ln(Q^2/P^2)$ into $\ln(Q^2/\mu^2)$.

Structure of the factorization at the single-loop level

We illustrate what we have said by the example of the single-loop diagram (see Fig. 7). Its contribution in the case $m_g \neq 0$ is [see (25)]

$$-C_F\frac{g^2}{16\pi^2}\left\{\ln^2\left(\frac{Q^2}{-p^2}\right) - 2\ln\left(\frac{Q^2}{-p^2}\right)\ln\left(-\frac{p^2}{m_g^2}\right)\right\}.$$

We represent $\ln(Q^2/(-p^2))$ in the form of the sum $\ln(Q^2/\mu^2) + \ln(\mu^2/(-p^2))$. Then

$$\ln^2\left(\frac{Q^2}{-p^2}\right) = \ln^2\frac{Q^2}{\mu^2} + 2\ln\frac{Q^2}{\mu^2}\ln\left(\frac{\mu^2}{-p^2}\right) + \ln^2\left(\frac{\mu^2}{-p^2}\right). \tag{37}$$

The first contribution, $\ln^2(Q^2/\mu^2)$, obviously corresponds to the regime in which all α are small (Fig. 10a). The second contribution corresponds to the C insertions in the order α_s:

$$\mathrm{Reg}_{\mu^2}^{UV}\langle 0|\hat{E}_{p_2}^{-1}(0, C)|_{-p_1^2=\mu^2}\hat{E}_{p_1}(0, C)|_{-p_2^2=\mu^2}|0\rangle = 1 - \frac{g^2}{8\pi^2}C_F\ln\left(\frac{Q^2}{\mu^2}\right)\ln\left(\frac{\mu^2}{-p^2}\right). \tag{38}$$

Finally, the third contribution is the sum of the matrix elements

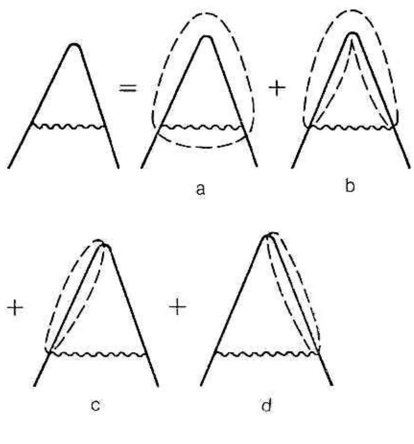

FIG. 10. Types of contributions for the single-loop diagram.

$$\langle 0|\hat{E}_{p_2}(0,\ A)\,\psi(0)\,|p_1\rangle\,|_{1\ \mathrm{loop}}$$

$$\simeq \frac{g^2}{16\pi^2}C_F\int\limits_{\substack{\alpha_2>1/\mu^2\\ \alpha_2+\alpha_3>1/\mu^2}}\frac{d\alpha_1\,d\alpha_3}{\alpha_2(\alpha_2+\alpha_3)}$$

$$\times\left(1-\frac{\alpha_2}{\alpha_2+\alpha_3}\right)\exp\left(\mathrm{i}p^2\frac{\alpha_2\alpha_3}{\alpha_2+\alpha_3}\right)$$

$$\times\left[1-\exp\left(\mathrm{i}(\varphi_1 p_2)\,s_0\frac{\alpha_2}{\alpha_2+\alpha_3}\right)\right]\sim\frac{g^2}{32\pi^2}C_F\ln^2\mu^2, \tag{39}$$

which have a doubly logarithmic dependence on the ultraviolet cutoff parameter μ. Note that only the presence of the second term in the square brackets ensures finiteness of the integral in the infrared region.

Factorization for the asymptotic behavior of the pion form factor in QCD

The structure of the SDR contributions for the three-point amplitude $T^{\mu}_{3}(P,P')$ (17) is completely analogous to the structure of the contributions for the quark form factor. The main difference is that the primitive configuration corresponds to subgraphs with four quark external lines (Fig. 11). We write its contribution in the form

$$T^{\mu}_{v_0}(P,\ P')=\int d^4\xi\,d^4\eta\,d^4\xi'\,d^4\eta'\int d^4y\,e^{-\mathrm{i}P'y}\,\langle 0|j_5(y)$$

$$\times\ \bar{\psi}_C(\eta')\,\gamma_5\gamma_\beta\psi_B(\xi')|0\rangle$$

$$\times\,\mathcal{E}^{\alpha\beta}_{ABCD}(\xi,\ \xi',\ \eta,\ \eta')\int d^4x\,e^{\mathrm{i}Px}\,\langle 0|\bar{\psi}_A(\xi)\,\gamma_5\gamma_\alpha\psi_D(\eta)\,j_5(x)|0\rangle, \tag{40}$$

where A, B, C, D are the color indices of the quarks. The gluon A, B, and C insertions must again be made in both the internal and the external lines of the primitive subgraph v_0. Since on the transition to the gauge $P_\mu A_\mu=0$ (or, respectively, $P'_\mu A_\mu=0$) the contributions with insertions into the external lines of the subgraph v_0 acquire a power-law suppression $O(1/Q^2)$, it follows that, as in the case of the quark form factor, their summation gives P exponentials of the form (32). Therefore, the technique of summation over the gluon insertions presented above is fully valid, and our results can be directly generalized to this case. Of course, it must be borne in mind that the antiquark lines run in the opposite direction.

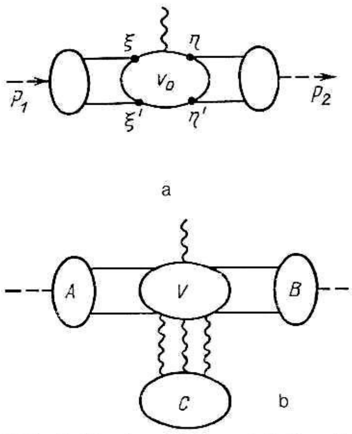

FIG. 11. Structure of the contributions for the pion form factor: a) primitive configuration; b) soft gluon exchanges.

The summation over the C gluons obviously gives a P exponential around a contour "traced" by a quark–antiquark pair. Since this contour is closed, the contributions of the fields with zero twist in it cancel, and this P exponential can be replaced by unity.

The summation over the B gluons adds two P exponentials (one for ψ, the other for $\bar{\psi}$) to the first matrix element, and it acquires the form

$$\int d^4x\,e^{\mathrm{i}Px}\,\langle 0|j_5(x)\,(\bar{\psi}(\xi)$$

$$\times\ E^{-1}_{P'}(\xi,\ A))_A\,\gamma_5\gamma_\alpha\,(E_{P'}(\eta,\ B)\,\psi(\eta))|_B|0\rangle. \tag{41}$$

Similarly, the summation over the A gluons gives

$$\int d^4y\,e^{-\mathrm{i}P'y}\,\langle 0|(\bar{\psi}(\eta')\,E^{-1}_{P}(\eta',\ B))$$

$$\times_C\,\gamma_5\gamma_\alpha\,(E_P(\xi',\ B)\,\psi(\xi'))_D\,j_5(y)|0\rangle. \tag{42}$$

In order to "decouple" the summation over the color indices A, B, C, D, it is necessary to use the analog of the Fierz identity for the group $SU(3)_C$:

$$\delta^A_{A'}\delta^{C'}_{C}=\frac{1}{3}\,\delta^{C'}_{A'}\delta^{A}_{C}+2\sum_a(\tau^a)^A_C\,(\tau_a)^{C'}_{A'}. \tag{43}$$

Cancellation of the doubly logarithmic contributions

Since the currents j_5 are colorless, only the singlet projections will be nonzero. As a result, we obtain in (41) and (42) colorless bilocal operators of the form

$$\bar{\psi}(\xi)\,E^{-1}_n(\xi,\ A)\,E_n(\eta,\ A)\,\gamma_5\gamma_\alpha\psi(\eta). \tag{44}$$

We now note that the product

$$E^{-1}_n(\xi;\ A)\,E_n(\eta;\ A)\,E(\eta,\ \xi;\ A)$$

is a P exponential calculated for a closed contour $(\eta\to\xi\to\infty\to\eta)$, and therefore, apart from terms $O(G)$, it can be replaced by unity. For the same reason, the operator (44) can be replaced by the bilocal operator

$$\mathcal{O}_{5\alpha}(\xi,\ \eta)=\bar{\psi}(\xi)\,E(\xi,\ \eta;\ A)\,\gamma_5\gamma_\alpha\psi(\eta). \tag{45}$$

The operators (45) can be expanded in Taylor series with respect to gauge-invariant local operators:

$$\mathcal{O}_{5\alpha}(\xi,\ \eta)=\sum_{N=0}^{\infty}\frac{1}{N!}\Delta_{\mu_1}\ \ldots\ \Delta_{\mu_N}\bar{\psi}\gamma_5\gamma_\alpha D^{\mu_1}\ \ldots\ D^{\mu_N}\psi, \tag{46}$$

where $\Delta_\mu=\xi_\mu-\eta_\mu$. The matrix elements of these operators have a singly logarithmic dependence on the ultraviolet cutoff parameter, and this means that all the doubly logarithmic contributions to $T^{\mu}(P,P')$ cancel. This cancellation again occurs because the initial and final states are colorless. If the currents j_5 were colored, the octet projections to the Fierz identity (43) would make a nonvanishing contribution, and instead of the operators (44) we would have operators of the form

$$\bar{\psi}(\xi)\,E^{-1}_n(\xi;\ A)\,\tau^a E_n(\eta;\ A)\,\gamma_5\gamma_\alpha\psi(\eta), \tag{47}$$

which cannot be represented in the form (45) by any amount of effort. We can rewrite (47) as

$$(\bar{\psi}(\xi)\hat{E}(\xi,\ \eta;\ A)\gamma_5\gamma_\alpha\tau^b\psi(\eta))(\tilde{E}_n(\eta;\ A))_{ba}(1+O(G)), \quad (48)$$

but in no way can we get rid of the P exponential $\tilde{E}_n(\eta; A)$ around the contour that goes to infinity.

Of course, the bilocal operator in (48) can be expanded in a series with respect to local operators, as in (46), but the matrix elements of the operators $O_N^b(\eta)(E_n(\eta, A))^{ab}$ will have a doubly logarithmic dependence on the parameter μ^2. In this case complete cancellation of the double logarithms does not occur. For example, at the single-loop level the doubly logarithmic contributions having the color factor C_F of the fundamental representation cancel, but doubly logarithmic contributions proportional to the octet factor $C_G (= N_c)$ remain.

Factorized representation for the form factor

Thus, for the asymptotic behavior of the auxiliary amplitude we have obtained a factorized representation of the form (10), in which the contributions of the small and large distances are separated. In accordance with the discussion in Sec. 1, this means that for the asymptotic behavior of the form factor there is also a factorization in the form (13):

$$F_\pi(Q^2) = \int d^4\xi\, d^4\eta\, d^4\xi'\, d^4\eta'\, \langle P'|\mathrm{Reg}_{\mu^2}^{UV}\mathcal{O}_{5\alpha}(\xi',\ \eta')|0\rangle$$
$$\times \mathrm{Reg}_{\mu^2}^{IR}(\mathscr{E}_{\alpha\beta}(\xi,\ \xi',\ \eta,\ \eta'))\langle 0|\mathrm{Reg}_{\mu^2}^{UV}\mathcal{O}_{5\beta}(\xi,\ \eta)|P\rangle$$
$$\times\{1+O(1/Q^2)\}. \quad (49)$$

The coefficient function $\mathscr{E}_{\alpha\beta}$ in the momentum representation corresponds to the quark $\gamma^* q\bar{q} \to q'\bar{q}'$ amplitude on the mass shell $k_i^2 = m_q^2 = 0$, and therefore, formally, it is gauge-invariant. The operators $\mathcal{O}_{5\alpha}$ and $\mathcal{O}_{5\beta}^*$ are also gauge invariant. If the procedures $\mathrm{Reg}_{\mu^2}^{UV}$, and $\mathrm{Reg}_{\mu^2}^{IR}$, do not destroy the gauge invariance, then in (49) the regularized expressions for the matrix elements and for the coefficient function will also be gauge-invariant. This can be achieved by defining the operation $\mathrm{Reg}_{\mu^2}^{UV}$ as a dimensional renormalization of composite operators.

Parton picture for the asymptotic behavior of the pion form factor in QCD

As in the case of inclusive processes, the expressions for the asymptotic behavior of the form factor simplify appreciably and acquire a transparent interpretation if they are rewritten in a parton form in which the reduced matrix elements of the local operators are identified with the moments of the parton functions, which in the given case have the meaning of hadron wave functions (and not of distribution functions that occur in the expressions for the cross sections of inclusive hard processes).

Parton wave functions

The bilocal gauge-invariant operators $\mathcal{O}(\xi, \eta)$ can be expanded in Taylor series with respect to local operators,

$$\langle 0|\mathcal{O}_{5\alpha}(\xi,\ \eta)|P\rangle = e^{iP\frac{\xi+\eta}{2}}\sum_{N=0}^{\infty}\frac{1}{N!}\Delta_{\mu_1}\ldots\Delta_{\mu_N}$$
$$\times\langle 0|\bar{d}\gamma_5\gamma_\alpha\overleftrightarrow{D}_{\alpha_1}\ldots\overleftrightarrow{D}_{\alpha_N}u|P\rangle, \quad (50)$$

with a two-sided, $\overleftrightarrow{D}_\mu = \overrightarrow{D}_\mu - \overleftarrow{D}_\mu$, or a single-sided derivative $\overrightarrow{D}_\mu = \overrightarrow{\partial}_\mu - igA_\mu$:

$$\langle 0|\mathcal{O}_{5\alpha}(\xi,\ \eta)|P\rangle$$
$$= \sum_{N=0}^{\infty}\frac{1}{N!}\Delta_{\mu_1}\ldots\Delta_{\mu_N}\langle 0|\bar{d}\gamma_5\gamma_\alpha\overrightarrow{D}_{\mu_1}\ldots\overrightarrow{D}_{\mu_N}u|P\rangle, \quad (51)$$

where $\Delta = \xi - \eta$. Determining the reduced matrix elements of their traceless parts by

$$\langle 0|\bar{d}\gamma_5\{\gamma_\alpha\overleftrightarrow{D}_{\alpha_1}\ldots\overleftrightarrow{D}_{\alpha_N}\}u|P\rangle|_{\mu^2}$$
$$= \frac{1+(-1)^N}{2}\Phi_N(\mu^2)\{P_\alpha P_{\alpha_1}\ldots P_{\alpha_N}\}, \quad (52)$$

$$\langle 0|\bar{d}\gamma_5\{\gamma_\alpha\overrightarrow{D}_{\alpha_1}\ldots\overrightarrow{D}_{\alpha_N}\}u|P\rangle|_{\mu^2} = \varphi_N(\mu^2)\{P_\alpha P_{\alpha_1}\ldots P_{\alpha_N}\}, \quad (53)$$

we can introduce gauge-invariant parton wave functions (Refs. 1–3) $\overline{\Phi}(\xi)$:

$$\int_{-1}^{1}\Phi(\xi)\xi^n d\xi = \Phi_n\frac{1+(-1)^n}{2} \quad (54)$$

and $\varphi(x)$:

$$\int_0^1\varphi(x)x^n dx = \varphi_n. \quad (55)$$

Between $\Phi(\xi; \mu^2)$ and $\varphi(\xi; \mu^2)$ there exists the obvious relation

$$\Phi(\xi;\ \mu^2) = \frac{1}{2}\varphi\left(\frac{1+\xi}{2};\ \mu^2\right). \quad (56)$$

The vanishing of the even moments in (52) and (54) is a consequence of the G-parity properties of the operators (52). The vanishing leads to symmetry of the wave function, $\Phi(\xi) = \Phi(-\xi)$ and $\varphi(x) = \varphi(1-x)$, and expresses the fact that the u and d quarks of the pion are on the same footing. The fulfillment of the spectral property $\varphi(x) = 0$ for $x \notin [0, 1]$ (or for $x<0$, $x<1$) for the wave function $\varphi(x)$ was proved in Ref. 50.

The wave function $\varphi(x; \mu^2)$ has an obvious parton interpretation, namely, it describes the amplitude for the probability of finding the pion π^+ with momentum P in a state for which the u quark in the infinite-momentum frame carries momentum $xP = ((1+\xi)/2)P$ and the d quark carries momentum $\bar{x}P = ((1-\xi)/2)P$. In fact, $\varphi(x; \mu^2)$ is a quasipotential wave function on the light front, $\varphi(x, k_\perp)$ (Ref. 30), integrated over the transverse momentum $k_\perp$ to, roughly speaking, μ (Refs. 5 and 10):

$$\varphi(x;\ \mu^2) = \int d^2k_\perp\varphi(x,\ k_\perp)\,\theta(k_\perp^2 < \mu^2). \quad (57)$$

The logarithmic divergences can be removed from the integral (57) by any other method, for example, by dimensional regularization, $d^2k_\perp \to d^{2-2\varepsilon}k_\perp$, with subsequent subtraction of the poles with respect to ε. The choice of such a method corresponds to the choice of the form of the procedure $\mathrm{Reg}_{\mu^2}^{UV}$ for the operators.

The parton wave function satisfies the specific normalization condition[1,3-5]

$$iP_\rho\int_{-1}^{1}\Phi(\xi,\ \mu^2)\,d\xi = \langle 0|\bar{d}\gamma_5\gamma_\rho u|P\rangle = iP_\rho f_\pi. \quad (58)$$

The matrix element of the axial current is known from the decay $\pi \to \mu\nu$, and by virtue of the conservation of the axial current in the chiral limit ($m_q = 0$) the condition (58) is satisfied for all μ^2.

Parton picture

Using for the operators (51) their expression in terms of the wave function (53), (55) and substituting everything in the factorized representation (40), we obtain for the asymptotic behavior of the pion form factor the expression

$$F_\pi(Q^2)=\int_0^1 dx\int_0^1 dy\varphi(x;\ \mu^2)\varphi^*(y;\ \mu^2)$$
$$\mathcal{M}(xP,\ \bar{x}P,\ yP',\ \bar{y}P';\ \mu^2), \qquad (59)$$

which has a simple parton interpretation in terms of the amplitude of the hard subprocess $q\bar{q}\gamma^*\to q'\bar{q}'$ (Fig. 12).

In the lowest approximation the amplitude $\mathcal{M}$ is

$$M_0(x,\ y;\ Q^2)=\frac{g^2}{2xyQ^2}\frac{C_F}{N_c}, \qquad (60)$$

where $C_F=4/3$ and $N_c=3$ are color factors. For the form factor $F_\pi(Q^2)$ we now obtain[1]

$$F_\pi^{(0)}(Q^2)=\frac{2\pi C_F\alpha_s(Q^2)}{N_cQ^2}\left|\int_0^1\frac{\varphi(x;\ \mu^2)}{x}dx\right|^2. \qquad (61)$$

It is natural to take the parameter μ^2 proportional to Q^2, but the coefficient of proportionality can be reliably determined only by calculating the next correction in α_s.

The wave functions $\varphi(x,\ \mu^2)$ are determined by the large-distance dynamics, and therefore their explicit form is not in general known. Perturbative QCD predicts only the law of their variation with increasing μ^2.

Evolution of the pion wave function in the leading logarithmic approximation

The dependence of the wave function $\varphi(x;\mu^2)$ on the parameter μ^2 is determined by the anomalous dimensions of the operators (52) and (53). A characteristic feature of our problem is the presence of mixing between the fundamental operators $O_{\mu_1\cdots\mu_N}(X)$ and their total derivatives $(\partial/\partial X)^kO_{\mu_1\ldots\mu_{N-k}}$:

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}\right)\langle0|\bar{\psi}D^n\psi|P\rangle$$
$$=\sum_{k=0}^{n}Z_{nk}\langle0|\partial^{n-k}(\bar{\psi}D^k\psi)|P\rangle. \qquad (62)$$

We note that the matrix elements of the derivatives do not contain new dynamical information,

$$\begin{aligned}&\mathrm{i}^n\left\langle0\left|\frac{\partial}{\partial X_{\alpha_1}}\cdots\frac{\partial}{\partial X_{\alpha_k}}O_{\mu_1\ldots\mu_N}\right|P\right\rangle\\&=P_{\alpha_1}\ldots P_{\alpha_k}\langle0|O_{\mu_1\ldots\mu_N}|P\rangle,\end{aligned} \qquad (63)$$

and therefore the expression (62) means that when the renormalizations are made there is mixing of the φ_n with different n:

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}\right)\varphi_n=\sum_{k=0}^{n}Z_{nk}\varphi_k. \qquad (64)$$

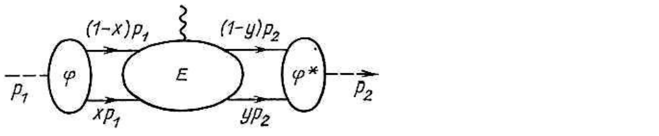

FIG. 12. Parton representation for the asymptotic behavior of the pion form factor.

Matrix of anomalous dimensions

A direct calculation of the diagrams shown in Fig. 13 gives for the matrix of anomalous dimensions the result

$$Z_{nk}^{(1)}=C_F\frac{g^2}{8\pi^2}\left\{-\delta_{nk}+\frac{2\theta_{nk}}{(n+1)(n+2)}\right\}; \qquad (65)$$

$$Z_{nk}^{(2)}=C_F\frac{g^2}{8\pi^2}\left\{-4\left(\sum_{j=2}^{n+1}\frac{1}{j}\right)\delta_{nk}+2\left(\frac{1}{n-k}-\frac{1}{n+1}\right)(\theta_{nk}-\delta_{nk})\right\}. \qquad (66)$$

The symbols δ_{nk} and θ_{nk} have the meaning

$$\delta_{nk}=1\ (n=k),\quad \delta_{nk}=0\ (n\neq k); \qquad (67)$$

$$\theta_{nk}=1\ (n\geqslant k),\quad \theta_{nk}=0\ (n<k). \qquad (68)$$

In the Feynman gauge the term $Z_{nk}^{(2)}$ corresponds to the contribution of the diagram of Fig. 13b, i.e., is due to the extension of the derivatives in the local operators.

Diagonalization of the matrix of anomalous dimensions

By a suitable choice of the operator basis one can make the matrix of anomalous dimensions diagonal. The diagonalization procedure is greatly simplified by noting that the matrices $Z_{nk}^{(1)}$ and $Z_{nk}^{(2)}$ commute. As a consequence, it is sufficient to find a basis in which the simpler matrix $Z_{nk}^{(1)}$ is diagonal; the more cumbersome matrix $Z_{nk}^{(2)}$ will automatically be diagonal in the same basis.

Since $Z_{nk}^{(1)}$ is a triangular matrix, its eigenvalues are given by the diagonal elements

$$\lambda^{(n)}=C_F\frac{g^2}{8\pi^2}\left(-1+\frac{2}{(n+1)(n+2)}\right). \qquad (69)$$

Therefore, it is necessary to find vectors k_n,

$$k_n=\sum_{m=0}^{\infty}d_{nm}a_m, \qquad (70)$$

that satisfy the equation

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}\right)k_n=\lambda^{(n)}k_n. \qquad (71)$$

Using the explicit form of $\lambda^{(n)}$, we obtain an equation for d_{nm}:

$$\sum_{l=m}^{\infty}\frac{d_{nl}}{(l+1)(l+2)}=\frac{d_{nm}}{(n+1)(n+2)}, \qquad (72)$$

the form of which depends only on the structure of the coefficient of θ_{nk} in (65). Subtracting from (72) the equation for $d_{n,m+1}$, we obtain a recursion relation, from which it follows that

$$d_{nm}=(-1)^m\frac{(m+n+2)!}{m!\,(m+1)!\,(n-m)!}d^{(n)}, \qquad (73)$$

where $d^{(n)}$ is an arbitrary normalization constant that it is convenient to take equal to $\frac{1}{2}$. In this case the multiplicatively renormalized combination has the form

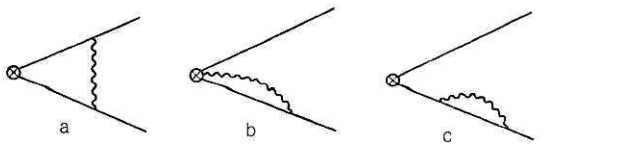

FIG. 13. Single-loop contributions to the evolution kernel for the pion wave function.

$$k_n = \sum_{m=0}^{n} \frac{(m+n+2)!\,(-1)^m}{2m!\,(m+1)!\,(n-m)!}\varphi_m. \quad (74)$$

With allowance for the definition of the coefficients φ_m (53) this means that there is a multiplicative renormalization of the matrix elements of the operators

$$K_{\mu_1\ldots\mu_n\mu} = \sum_{m=0}^{n} \frac{(m+n+2)!\,(-)1^m}{2m!\,(m+1)!\,(n-m)!} \{(\partial_+^{n-m}(\bar{d}\gamma_5\gamma_\mu \vec{D}^m u))_{\mu_1\ldots\mu_n}\}$$
$$= \bar{d}\gamma_5\{\gamma_\mu(\partial_+^n C_n^{3/2}(2\overleftrightarrow{D}/\partial_+))_{\mu_1\ldots\mu_n}\}u, \quad (75)$$

where $C_n^{3/2}(\xi)$ are Gegenbauer polynomials [see Ref. 51, Eq. (10.9.20)], and

$$2\overleftrightarrow{D} = \vec{D} - \overleftarrow{D},\ \partial_+ = \vec{\partial} + \overleftarrow{\partial} = \vec{D} + \overleftarrow{D},$$
$$\partial_+^n (D/\partial_+)^k \equiv \partial_+^{n-k} D^k.$$

It is interesting to note (as was emphasized in our papers of Refs. 8 and 27) that the tensors $K_{\mu\mu_1\ldots\mu_n}$ are conformal in the approximation of free fields. The connection between multiplicative invariance of the operators at the single-loop level and their conformal properties was then investigated in Refs. 52–54.

Expansion of the wave function with respect to multiplicatively renormalized combinations

In accordance with (75), the matrix elements of the operators $K_{\mu\mu_1\ldots\mu_n}$ are related to $\Phi(\xi)$ by

$$\{P_\mu P_{\mu_1} \ldots P_{\mu_n}\} \int_{-1}^{1} \Phi(\xi;\mu^2)\, C_n^{3/2}(\xi)\, d\xi$$
$$= \langle 0|K_{\mu\mu_1\ldots\mu_n}|P\rangle \equiv K_n(\mu^2)\{P_\mu P_{\mu_1}\ldots P_{\mu_n}\}.$$

The polynomials $C_n^{3/2}(\xi)$ are orthogonal on the interval $(-1,\ 1)$ with weight $(1-\xi^2)$; taking into account their normalization, we obtain

$$\Phi(\xi;\mu^2) = (1-\xi^2)\sum_{n=0}^{\infty} K_n(\mu^2)\frac{n+3\,2}{(n+1)(n+2)} C_n^{3/2}(\xi). \quad (76)$$

The coefficients $K_n(\mu^2)$ are multiplicatively renormalized with a change of μ^2. Taking into account the explicit form of the QCD β function in the single-loop approximation,[14,15] we find

$$K_N(\mu^2) = \left(\frac{\ln(\mu_0^2/\Lambda^2)}{\ln(\mu^2/\Lambda^2)}\right)^{\gamma_N/B} K_N(\mu_0^2), \quad (77)$$

where

$$\gamma_N = C_F\left(1 - \frac{2}{(N+1)(N+2)} + 4\sum_{j=2}^{N+1}\frac{1}{j}\right);$$
$$B = 11 - \frac{2}{3}N_f. \quad (78)$$

Asymptotic wave function

Since all the coefficients γ_N apart from γ_0 are positive, the contributions of the higher harmonics in (76) are damped with increasing N, and in the asymptotic limit $\mu^2 \to \infty$ the wave function $\Phi(\xi, \mu^2)$ takes the very simple form[7,10]

$$\Phi^{as}(\xi;\mu^2\to\infty) = \frac{3}{4}f_\pi(1-\xi^2). \quad (79)$$

Here we have used the fact that the constant K_0, which is determined by the matrix element of the axial current, is equal to f_π.

The limiting form of the wave function, $\Phi(\xi;\infty) = 3f_\pi(1-\xi^2)/4$, is analogous to the limiting form $f(x,\infty)\sim\delta(x)$ (Ref. 15) for parton distribution functions. However, it is well known that for the momentum transfers currently attained the functions $f(x, Q^2)$ differ appreciably from their limiting form and, if they do tend to it with increasing Q^2, they do so very slowly. Similarly, the wave function $\Phi(\xi, Q^2)$ in the region of moderately large Q^2 may in principle differ appreciably from $\frac{3}{4}(1-\xi^2)f_\pi$ and, moreover, may have a quite different law of behavior in the region $\xi^2\sim 1$, which is particularly important for the integral (61).

Each term of the sum (76) behaves when $\xi^2\sim 1$ as $1-\xi^2$, since

$$\frac{n+3\,2}{(n+1)(n+2)}(1-\xi^2)\,C_n^{3/2}(\xi)$$
$$\propto \sqrt{1-\xi^2}\,J_1\left(\left(n+\frac{3}{2}\right)\sqrt{1-\xi^2}\right) \sim (1-\xi^2), \quad (80)$$

where J_1 is the Bessel function, and $J_1(x)\sim x$ as $x\to 0$. But if the coefficients do not decrease sufficiently rapidly as $n\to\infty$, the infinite sum of the higher harmonics may radically change the behavior of $\Phi(\xi)$ for $\xi^2\sim 1$. Indeed, if $K_n\sim n^\alpha$, then in the region $n \gg 1/\sqrt{1-\xi^2}$ we find by means of the well-known asymptotic relation[51]

$$J_1(x) \propto \frac{\cos x}{\sqrt{x}} \quad (81)$$

that

$$\sum_{N_0}^{\infty} n^\alpha \sqrt{1-\xi^2}\,J_1\left(\left(n+\frac{3}{2}\right)\sqrt{1-\xi^2}\right) \sim (1-\xi^2)^{-\alpha/2}, \quad (82)$$

where $N_0 = A/\sqrt{1-\xi^2}$, with $A \gg 1$. As a result,

$$\Phi(\xi) = a(1-\xi^2) + b(1-\xi^2)^{-\alpha/2}, \quad (83)$$

from which it follows that the first term, which is given by the sum over $n < N_0$, is dominant only when $\alpha < -2$. The behavior of the coefficients $K_n(Q^2)$ at large Q^2 can be expressed in terms of the behavior of $K_n(Q_0^2)$ if we use the fact that at large n

$$\gamma_n = 4C_F \ln n + O(1) \quad (84)$$

[see (77)]. It follows that[7,27]

$$\alpha(Q^2) = \alpha(Q_0^2) - \frac{4C_F}{B}\left(\ln\ln\frac{Q^2}{\Lambda^2} - \ln\ln\frac{Q_0^2}{\Lambda^2}\right), \quad (85)$$

and, thus, for sufficiently large Q^2 the value of $\alpha(Q^2)$ becomes less than the "critical" value $\alpha_{crit} = -2$, as a consequence of which the wave function $\Phi(\xi, Q^2)$ behaves when $\xi^2\sim 1$ as $1-\xi^2$.

Asymptotic contribution to the form factor

Substituting the expressions (75) and (76) for $\Phi(\xi;\mu^2)$ in the expression for the form factor (61) and setting $\mu = Q$, we find that

$$F_\pi(Q^2) = 8\pi\alpha_s(Q^2)\frac{f_\pi^2}{Q^2}\frac{C_F}{N_c}(\gamma(Q^2))^2, \quad (86)$$

where $\gamma(Q^2)$ is the result of integration over ξ:

$$\gamma(Q^2)=\frac{3}{2}+\sum_{n=2,4\ldots} K_n(Q_0^2)\frac{2n+3}{(n+1)(n+2)}\left(\frac{\ln Q_0^2/\Lambda^2}{\ln Q^2/\Lambda^2}\right)^{\gamma_n/B}. \quad (87)$$

We recall that the Gegenbauer moments $K_n(Q_0^2)$ can be expressed in terms of the ordinary φ_n by means of the expression (79).

It is interesting to note that in the framework of a model with scalar gluons we obtained[39] for the asymptotic behavior of the pion form factor, using the method of direct summation in the ladder approximation, an expression with a structure analogous to (80). Namely, in Ref. 39 we obtained

$$F_{(\pi)}(Q^2)=\frac{c}{Q^2}\varphi^2\left(\frac{Q^2}{\mu^2}\right)\left(\frac{Q^2}{\mu^2}\right)^{-g_0^2}, \quad (88)$$

where

$$\varphi\left(\frac{Q^2}{\mu^2}\right)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left(\frac{Q^2}{\mu^2}\right)^{\tau}\frac{d\tau}{\tau}\xi(\tau); \quad (89)$$

$$\xi(\tau)=\sum_{n=0}^{\infty}\frac{\Gamma(n-\nu+1)\Gamma(n+\nu+2)}{n!(n+1)!}a_n, \quad (90)$$

and $\nu(\nu+1)=-g_0^2/\tau$, $g_0^2=g^2/16\pi^2$. Going over from the variable of integration τ in (89) to the variable ν and calculating the integral over ν by taking the residues at the points $\nu=n+k+1\equiv N$, we obtain for $\varphi(Q^2/\mu^2)$ the result

$$\varphi\left(\frac{Q^2}{\mu^2}\right)=\sum_{N=0}^{\infty}(-1)^N\frac{2N+3}{(N+1)(N+2)}\left(\frac{Q^2}{\mu^2}\right)^{-\frac{g_0^2}{(N+1)(N+2)}}\times\sum_{n=0}^{N}\frac{(-1)^n a_n}{n!(n+1)!(N-n)!}. \quad (91)$$

Since $\gamma_n=-(g^2/8\pi^2)(\frac{1}{2}+1/(n+1)(n+2))$ is the anomalous dimension of the operator $\bar\psi\gamma_5\gamma_\mu\partial^n\psi$ in this model,[81] the structure of the expression (91) is identical to the structure of the QCD expression;, moreover, in this case too the multiplicatively renormalized operators are expressed in terms of the same Gegenbauer polynomials $C_n^{3/2}$.

Returning to (87), we note that in the limit $Q^2\to\infty$ the function $\gamma(Q^2)$ tends to 3/2; this value obviously corresponds to the substitution in (61) of the asymptotic form $\Phi^{as}(\xi)$ (79) for the pion wave function. In other words, in the "rigorous" $Q^2\to\infty$ asymptotics the pion form factor can be expressed in terms of the constant f_π and the QCD running coupling constant $\alpha_s(Q^2)$ (Refs. 7, 9, and 10):

$$F_\pi^{as}(Q^2)\,|_{Q^2\to\infty}=\frac{8\pi f_\pi^2\alpha_s(Q^2)}{Q^2}. \quad (92)$$

Taking for $\alpha_s(Q^2)$ its value $\alpha_s\simeq 0.3$, which is typical for momentum transfers $Q^2=1$–$5\ \mathrm{GeV}^2$ (for $\Lambda_{QCD}\simeq 100$ MeV), we find from (92) that $Q^2F_\pi(Q^2)\simeq 0.15\ \mathrm{GeV}^2$. This is two and more times less than the experimentally observed value. This fact can be interpreted as an indication that the true low-energy wave function differs from the asymptotic function, or that at the currently achieved and rather small momentum transfers $Q^2\lesssim 5$ GeV the asymptotic analysis is not yet valid.

Evolution kernel for the pion wave function

Using the connection between φ_N and the wave function $\varphi(x,\mu^2)$, we can obtain an evolution equation directly for $\varphi(x,\mu^2)$ (Refs. 5 and 10):

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}\right)\varphi(x;\mu)^2=\int_0^1 V(x,y)\varphi(y;\mu^2)\,dy. \quad (93)$$

The evolution kernel $V(x,y;g)$ is related to the matrix Z_{nk} by

$$\int_0^1 V(x,y;g)x^n\,dx=\sum_{k=0}^{n}Z_{nk}(g)y^k \quad (94)$$

or, explicitly

$$V(x,y;g)=\frac{1}{2\pi i}\int_{-\sigma-i\infty}^{-\sigma+i\infty}dn\,x^{-n-1}\left(\sum_{k=0}^{N}Z_{nk}(g)y^k\right)\Big|_{AC}, \quad (95)$$

where the symbol AC means that the result of the summation over k must be analytically continued to the complex plane of n. For example, for the contribution $Z_{nk}^{(1)}$ (65) the sum over k in (95) is

$$\frac{g^2}{8\pi^2}\left\{-y^n+\frac{12}{(n+1)(n+2)}\frac{1-y^{n+1}}{1-y}\right\}, \quad (96)$$

and the integration over n gives

$$V_0^{(1)}(x,y)=\frac{g^2}{8\pi^2}\left\{-\delta(x-y)+\left[\frac{x}{y}\theta(x<y)+\frac{\bar x}{\bar y}\theta(\bar x<\bar y)\right]\right\}=\frac{g^2}{8\pi^2}\left\{\frac{x}{y}\theta(x<y)+(x\to\bar x,\ y\to\bar y)\right\}_+. \quad (97)$$

The operation $\{\ \}_+$ is defined as follows[55]:

$$\{V(x,y)\}_+=V(x,y)-\delta(x-y)\int_0^1 V(z,y)\,dz. \quad (98)$$

For the kernels "with plus" we have

$$\int_0^1\{V(x,y)\}_+\,dx=0. \quad (99)$$

Since the integral (99) for the evolution kernel of the pion wave function is $Z_{00}=\gamma_0$, i.e., is equal to the anomalous dimension of the axial current, which vanishes because the current is conserved, the evolution kernel of the pion wave function has "plus" form in all orders of perturbation theory.

For the contribution $Z_{NK}^{(2)}$ the calculation in accordance with (95) gives

$$V^{(2)}(x,y)=-\frac{g^2}{8\pi^2}\left\{\frac{x}{y}\frac{\theta(x<y)}{x-y}+(x\to\bar x,\ y\to\bar y)\right\}. \quad (100)$$

The use of evolution kernels considerably simplifies the problem of diagonalizing the matrix Z_{nk}, and this problem obviously reduces to finding the eigenfunctions of the kernel $V(x,y;g)$. But this problem can be solved trivially.[5,10] It is sufficient to note that the kernel $V(x,y)=V^{(1)}(x,y)+V^{(2)}(x,y)$ (like, in fact, each of its components $V^{(1)}$ and $V^{(2)}$) becomes symmetric with respect to the substitution $x\leftrightarrow y$ after multiplication by $y\bar y$. It immediately follows that the eigenfunctions $\varphi_N(x)$ of the kernel $V_0(x,y)$ must be orthogonal with weight $x(1-x)$. Now in the class of polynomials the Gegenbauer polynomials $C_n^{3/2}(x-\bar x)$ have precisely this property.

"Double-contraction" regime

From the very beginning of our analysis we systematically ignored the possibility of "double contraction" (see Fig. 6e) and promised to justify later such a course.

The question of the presence or absence of contributions due to double contraction (when in the SDR regime a leading pole is given simultaneously by two subgraphs V_L and V_R with neither contained in the other) is not at all academic, and an answer to it is required. In simple scalar models of the type $\varphi^3_{(4)}$, $\varphi^3_{(6)}$ it is particularly acute—the presence of these contributions has the consequence that the analysis of the form factor of the "pion," i.e., the two-particle bound state, in these simple models is in fact more complicated than in QCD.

To illustrate this thesis, we consider a very simple single-loop "two-sided" diagram (Fig. 14a) in the $\varphi^2_{(6)}$ model (this model, which is renormalizable, is closer to QCD than the super-renormalizable $\varphi^3_{(4)}$ model).

The Mellin transform of the contribution of this diagram has the α representation

$$\mathcal{M}(x, y; J) = \frac{eg^4}{(4\pi)^3} \int_0^\infty \prod_1^5 d\alpha_i \frac{(\alpha_1 + x\alpha_2)^J (\alpha_3 + y\alpha_4)^J}{(D(\alpha))^{J+3}} \exp\left(-m^2 \sum_1^5 \alpha_i\right), \quad (101)$$

where $D(\alpha) = \alpha_1 + ... + \alpha_5$. In this case the SDR subgraphs are $V_1 = \{\sigma_1, \sigma_2\}$, $V_2 = \{\sigma_3, \sigma_4\}$, $V_3 = \{\sigma_1, ..., \sigma_5\}$. Integrating first over the region $\lambda < 1/\mu^2 (\lambda \equiv \alpha_1 + ... + \alpha_5)$, next over the region $\beta_3 + \beta_4 \ll 1 (\beta_i = \alpha_i/\lambda)$, and then over the region $\beta_1 + \beta_2 \ll 1$, we obtain at the point $J = -2$ a pole of third order, and this corresponds to the contribution $Q^{-4} \ln^2 (Q^2/\mu^2)$ in the single-loop diagram!

To understand the origin of the "extra" logarithm, we write down the intermediate result obtained after only the integration over $\lambda < 1/\mu^2$ and $\beta_3 + \beta_4 \ll 1$ has been performed:

$$E(x, y; J) \simeq \frac{g^4 (1/\mu^2)^{J+2}}{(4\pi)^3 (J+2)^2} \int_0^1 d\varkappa (\varkappa + y(1-\varkappa))^J \times \int_0^1 \prod_{i=1}^3 d\beta_i (\beta_1 + x\beta_2)^J \delta(1 - \beta_1 - \beta_2 - \beta_5), \quad (102)$$

where $\varkappa = \beta_3/(\beta_3 + \beta_4)$. Denoting $\beta_1 + x\beta_2$ by z, we can represent E in the form

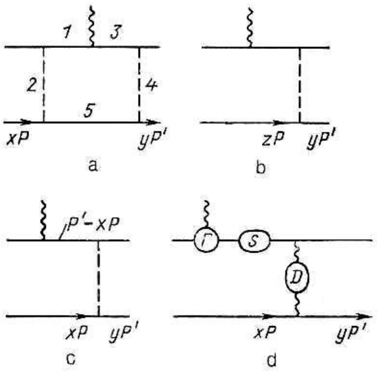

FIG. 14. Structure of contributions in the "double-contraction" regime.

$$E(x, y; J) = \left(\frac{1}{\mu^2}\right)^{J+2} \frac{1}{J+2} \int_0^1 dz z^{J+2} V(x, z) E_0(z, y; J), \quad (103)$$

where $E_0(z, y; J)$ is the Mellin transform of the tree diagram (Fig. 14b),

$$E_0(z, y; J) = \frac{g^2}{(J+2) yz^2}, \quad (104)$$

and $V(x, z)$ is the single-loop evolution kernel for the wave function,

$$V(x, z) = \frac{g^2}{(4\pi)^3} \int_0^1 \prod_{i=1, 2, 5} d\beta_i \delta(z - (\beta_1 + \beta_2 x)) \delta(1 - \beta_1 - \beta_2 - \beta_5) = \frac{g^2}{(4\pi)^3} \left\{\frac{z}{x} \theta(z < x) + \frac{1-z}{1-x} \theta(x < z)\right\} \quad (105)$$

[cf. (97)].

Thus, the appearance of the third pole at $J = -2$ is explained by the fact that the product $V(x, z) E_0(z, y)$ behaves at small z as $1/z$. We now note that convolution of the kernel $V(x, z)$ with the original wave function $\Phi_0(x, \mu_0^2)$ gives precisely the correction to Φ_0 due to the evolution:

$$\Phi(z, \mu^2) = \Phi(z, \mu_0^2) + \ln\left(\frac{\mu^2}{\mu_0^2}\right) \int_0^1 V(z, x) \Phi(x, \mu_0^2) dx \equiv \Phi_0 + \Delta\Phi. \quad (106)$$

Thus, if we write down the contribution of the configuration with the SDR subgraph of lowest order (Fig. 14b) in the parton form

$$F^{(14b)}(Q^2) = \int_0^1 \Phi(z) dz \int_0^1 \Phi(y) E(z, y; Q^2) dy \quad (107)$$

and take the evolving wave functions (106), then as a result of the high singularity of the amplitude $E(z, y; Q^2) \sim 1/(z^2 yQ^4)$ at small z the correction to Φ_0 due to the evolution will give a logarithmic divergence. In a more accurate approach an additional logarithm appears instead of the divergence. In QCD the contribution of the diagram of single-gluon exchange (Fig. 14c) is proportional to $1/(zyQ^2)$, and the evolution kernel $V(x, z)$ behaves at small z like z, and therefore no singularities at all arise in the integral over z; thus, in a diagram like that of Fig. 14a the double-contraction regime is impossible, and the diagram gives only a single logarithm.

Let us trace the reason for this difference in the behavior of the contributions $E_0(z, y)$. The denominators of the propagators in QCD give the same result $1/(z^2 yQ^4)$ as in the φ^3 model. However, in the numerator of the spinor propagator in the QCD diagram (Fig. 14b) there is the factor $(\hat{P}' - z\hat{P})$, and of its two terms only the second, proportional to z, gives a factor $O(Q^2)$, while the contribution of the first has a suppression $O(P'^2/Q^2)$ compared with it. As a result, $F_0^{QCD} \sim 1/(zyQ^2)$.

One can show that also in an arbitrary order of perturbation theory the leading contribution of the subgraph V_R (or V_L) does not have $1/z^2$ singularities as $z \to 0$. This can be shown by simple dimensional arguments. Indeed, the momentum P occurs in the amplitude T and the propagator S (Fig. 14d) only in the form of the product zP, and therefore

$$T \infty \gamma_5 \frac{aP'+b(z\hat{P})}{z(PP')}, \quad S \infty \frac{\tilde{a}\hat{P}'+\tilde{b}(z\hat{P})}{z(PP')}. \tag{108}$$

As a result, $TS\Gamma \sim 1/z$ as $z \to 0$, since the vertex function $\Gamma(q, p_1, p_2) = \Gamma(P'-P, (1-z)P', P'-zP)$ has not more than logarithmic singularities as $P_2^2/Q^2 \to 0$.

Thus, for the leading contribution to the asymptotic behavior of the pion form factor corresponding to the axial projection, i.e., to operators of the form $\bar{\psi}\gamma_5\gamma_\mu\psi$, the "double-contraction" regime can be ignored.

"Double-contraction" regime for nonleading contributions

The weakening of the singularity of the tree diagram (Fig. 14c) as $z \to 0$ due to the contribution of the numerator of the quark propagator was based essentially on the fact that the $\sim \hat{P}'$ terms of the numerator that do not weaken the singularity were "killed" by the projecting factor $\gamma_5 P'$. For pseudoscalar projection such suppression will not occur: $E_{PP} \propto g^2 C_F / x^2 y Q^4 N_c$. To settle whether or not there is convergence of the integrals with respect to x, it is necessary to use some information about the behavior of the pseudoscalar wave function $\varphi_P(x)$:

$$\langle 0 \mid \bar{d}(\xi)\gamma_5 \hat{E}(\xi, \eta; A) u(\eta) \mid P\rangle$$
$$= \int_0^1 dx \varphi_P(x) \exp[i(P\xi)x + i(P\eta)(1-x)]\{1 + O((\xi-\eta)^2)\} \tag{109}$$

[cf. (51), (55)] as $x \to 0$.

In Refs. 56 and 57 it is asserted that when the renormalization parameter μ^2 of the composite operators (which is taken equal to the external momentum transfer Q^2) tends to infinity the wave function $\varphi_P(x)$ becomes a constant, and, therefore, the integral over x diverges linearly at small x.

With allowance for the unavoidable cutoff at small x at $x_{\min} \propto \langle k_\perp^2 \rangle / Q^2$ due to the nonzero mean transverse momentum $\langle k_\perp^2 \rangle$ of the quarks, this divergence is transformed into an additional linear power of $Q^2/\langle k_\perp^2 \rangle$, and as a result $F_\pi^{PP}(Q^2)$ behaves in the limit $Q^2 \to \infty$ not as $1/Q^4$ but as $1/Q^2$, i.e., like the axial contribution $F_\pi^{AA}(Q^2)$, which in our analysis is assumed to be the leading contribution. Since this change in the power-law behavior is associated with the integration over the region of very small fractions $x \lesssim \langle k_\perp^2 \rangle / Q^2$ (i.e., over "virtualities" $k^2 \sim \langle k_\perp^2 \rangle$), where perturbations theory is unreliable, doubts may arise (see Ref. 58) concerning the validity of the factorized representation for the asymptotic behavior of the pion form factor in QCD. However, these doubts are based on two hurried conclusions. If they were correct, we should observe the behavior $E^{(PP)}(Q^2) \sim 1/Q^2$ already in the single-loop two-sided diagram (Fig. 14a). This can be shown by means of the arguments that follow below.[59]

The asymptotic wave function $\varphi^{as}(x)$ must satisfy the equation

$$\int_0^1 V(x, y)\varphi^{as}(y)\,dy = \gamma\varphi^{as}(x) \tag{110}$$

[where $V(x, y)$ is the evolution kernel], which reflects the fact that its form does not change in the process of evolution. Therefore, if the integral

$$I_\varphi = \int_0 dx E_0(x, y; Q^2)\varphi^{as}(x) \tag{111}$$

diverges linearly at small x, so will the integral

$$I_V = \int_0 E_0(x, y; Q^2) V(y, z)\,dy, \tag{112}$$

which occurs as a coefficient of $\ln Q^2$ in the single-loop contribution to the coefficient function:

$$E_1(x, z; Q^2) = \int_0^1 E_0(x, y) V(y, z)\,dy \ln Q^2 + \ldots \tag{113}$$

In other words, if for the asymptotic wave function $F_\pi^{(PP)}(Q^2) \sim 1/Q^2$, then also $E_1^{(P)} \sim 1/Q^2$. On the other hand, it follows categorically from dimensional analysis that $E_1^{(P)} \propto 1/Q^4$, i.e., the linear divergence of the integral over x discussed above is not in fact manifested at all for $E_1^{(P)}$. However, this is only an apparent paradox. The point is that the right-hand side of the expression (113) actually has the matrix form

$$E_1^{(PP)} = (E_0^{(PP)} \otimes V^{(PP)} + E_0^{(PT)} \otimes V^{(TP)}) \ln Q^2 + \ldots, \tag{114}$$

where T denotes the tensor Fierz projection. Therefore, the fact that $E_0^{(PP)} \otimes V^{(PP)} \sim 1/Q^2$, does not in principle contradict the relation $E_1^{(PP)} \sim 1/Q^4$. It is merely necessary for the $1/Q^2$ contributions to $E_0^{(PP)} \otimes V^{(PP)}$ and $E_0^{(PT)} \otimes V^{(TP)}$ to cancel. And since the relation $E_1^{(PP)} \sim 1/Q^4$ follows from dimensional analysis, the contributions are in reality simply bound to cancel.

Similarly, it is sensible to consider the sum of the pseudoscalar and tensor contributions to $F_\pi(Q^2)$, and not each of them separately. It was shown in Refs. 56 and 57 that if in the diagram of single-gluon exchange (Fig. 14c) the external lines are associated with free quarks, then for the sum there is indeed a cancellation of the contributions most singular with respect to x and y, so that the resulting contribution of the operators of twist 3 behaves as $1/Q^4$. A similar result can also be obtained in the framework of our approach.

In the Feynman gauge, by virtue of the property $\gamma^\alpha\sigma_{\mu\nu}\gamma_\alpha = 0$, the tensor ($T$) projection for the diagram (Fig. 14b) is nonzero only for the quark fields of the initial state, and for the final state it is necessary to take the P projection. For the remaining diagrams only the interference TP term "works." Using the definition of the tensor wave function

$$\langle 0 \mid \bar{d}(\xi)\hat{E}(\xi, 0; A)\gamma_5\sigma_{\mu\nu}u(0) \mid P\rangle$$
$$= (P_\mu\xi_\nu - P_\nu\xi_\mu)\int_0^1 \varphi_T(x)\, e^{i(P\xi)x}\{1 + O(\xi^2)\}, \tag{115}$$

we find that in the Feynman gauge[59]

$$E_\pi^{(TP)}(Q^2) = \frac{8\pi}{9Q^4}\int_0^1 dx \int_0^1 dy \varphi_T(x)\varphi_P(y)\left(\frac{1}{x^2 y} - \frac{2}{x^3 y}\right), \tag{116}$$

i.e., in the limit $x \to 0$ the coefficient function $E^{TP}(x, y)$ behaves as $1/x^3$.

The connection between the functions $\varphi_T(x)$ and $\varphi_P(x)$ can be found (see Refs. 59–61) by differentiating (116) with respect to ξ and using the equation of motion $\gamma^\mu(\partial_\mu - igA_\mu)d(\xi) = 0$ (the quarks are assumed to be massless). After simple manipulations we obtain

$$\varphi_T(x) = x^2 \int_x^1 \frac{\varphi_P(x)}{x^2} + (\bar{q}Gq), \tag{117}$$

where the term $\bar{q}Gq$ corresponds to the contribution of the wave function associated with the operator that arises when (115) is differentiated and contains the gluon field $G_{\mu\nu}$.

It is readily seen that if at small x the function $\varphi_P(x)$ behaves as $a+O(x)$, then $\varphi_T(x)=ax(1-x)+O(x^2)+(qGq)$, and in the purely two-quark sector the singularity $\varphi_P(x)/x^2$ in the PP contribution cancels with the singularity $\varphi_T(x)/x^3$ in the TP contribution, in accordance with the results of Refs. 56 and 57.

3. RADIATIVE CORRECTIONS TO THE ASYMPTOTIC BEHAVIOR OF THE PION FORM FACTOR IN QCD

The main result obtained in Sec. 2 is that for sufficiently large momentum transfers $q=P'-P$ the amplitude $T(P, P')$ corresponding to the pion form factor factorizes in all orders and for all logarithms of perturbation theory:

$$T(P, P')=\int_0^1 dx \int_0^1 dy\varphi^*(y, \mu, \mu_R)\varphi(x, \mu, \mu_R), \\ E(Q^2/\mu^2, Q^2/\mu_R^2, x, y, \alpha_s)/Q^2, \qquad (118)$$

where $Q^2=-q^2$, μ_R is the parameter of the R operation, φ is the wave function that describes the transition of the pion to the $q\bar{q}$ system, and E/Q^2 is the amplitude of the parton subprocess $q\bar{q}\gamma^*\to q'\bar{q}'$. The parameter $1/\mu$ is the boundary between "small" and "large" distances or, equivalently, μ is the renormalization parameter for the vertices $(\bar{\psi}\gamma_5\gamma_\nu D^n\psi)$, which correspond to the composite operators.

The product $\varphi^*\otimes E\otimes\varphi$ does not depend on the particular choice of the parameters μ and μ_R. However, this is ensured only by the summation over all orders. If a restriction is made to the first few terms of the series in α_s, the resulting expressions will depend on μ and μ_R. In addition, one can make the calculations in different renormalization schemes and use different prescriptions (schemes) for separating the contributions of small and large distances, and the truncated series will also depend on the chosen schemes.

In the problem considered here we shall see that these considerations are far from being merely of academic interest, and it is worth looking into the matter in more detail.

In the lowest approximation the amplitude E has the form

$$E\left(\frac{Q^2}{\mu^2}, \frac{Q^2}{\mu_R^2}, x, y, \alpha_s(\mu_R)\right)=\frac{2\pi\alpha_s(\mu_R)}{xy}\frac{C_F}{N_c}, \qquad (119)$$

where $C_F=4/3$ and $N_c=3$ are the usual color factors. For the form factor this gives

$$F_\pi^{(0)}(Q^2)=\frac{2\pi C_F\alpha_s(\mu_R)}{Q^2N_c}\int_0^1\frac{\varphi(x, \mu, \mu_R)\,dx}{x}\int_0^1\frac{\varphi^*(y, \mu, \mu_R)}{y}dy. \qquad (120)$$

In this approximation E does not depend on Q and μ, and it depends on μ_R only through α_s. A logarithmic dependence on Q, μ, and μ_R [in the form $\ln(Q^2/\mu^2)$ and $\ln(Q^2/\mu_R{}^2)$] appears only in the following order in α_s. Thus, the logarithmic contributions tend to compensate the dependence of $F_\pi^{(0)}$ on μ and μ_R. For a poor choice of μ and μ_R the lowest approximation of $F_\pi(Q, \mu, \mu_R)$ will differ strongly from the "true" value of $F_\pi(Q)$ (the sum of all orders of perturbation theory), and the corrections due to the higher perturbation orders will be large.

One is naturally led to ask whether μ and μ_R can be chosen in such a way as to make these corrections as small as possible. For example, if we take $\mu=\mu_R=Q$, then E no longer contains the logarithms $\ln(Q^2/\mu^2)$ and $\ln(Q^2/\mu_R^2)$, which for $Q\gg\mu$, μ_R (or $Q\ll\mu$, μ_R) lead to a growth of the coefficients of the series with respect to α_s. The logarithms $\ln(Q^2/\mu_R{}^2)$ indicate, as it were, that the "virtuality" of the particles in the subprocess is proportional to Q^2: $-\langle k^2\rangle\sim a^2Q^2$, and therefore $F_\pi(Q)$ must be expanded with respect to $\alpha_s(Q^2)$. Similarly, since the parameter μ in the argument of the wave function $\varphi(x, \mu^2)$ means that the pion structure is probed at distances $1/\mu$, the logarithms $\ln(Q^2/\mu^2)$ indicate that μ must also be taken of order Q If, however, the ratio $a^2\equiv\langle k^2\rangle/q^2$ is small (or large) compared with unity, then the coefficients in the expansion of $F_\pi(Q)$ in a series with respect to $\alpha_s(Q)$ will contain terms of the type $\ln a^2$, and the choice $\mu=\mu_R=aQ$ will be preferable. In particular, if the quark and antiquark in the pion have approximately equal momenta [i.e., if $\varphi(x)\sim\delta(x-\frac{1}{2})$], then the gluon momentum in this case is $q/2$, and a series expansion with respect to $\alpha_s(Q/2)$ is most natural. Although a is not very different from unity, at the experimentally attained momentum transfers ($Q\lesssim 2$ GeV) the difference between $a=1$ and $a=\frac{1}{2}$ is very important.

The wave functions $\varphi(x, \mu^2)$ describe the dynamics of the large-distance interaction, and therefore their explicit form is not in general known. Quantum chromodynamics predicts only their variation with increasing μ^2. In particular, as $\mu^2\to\infty$ we have[7,10,9]

$$\varphi(x, \mu^2)\to 6f_\pi x(1-x), \qquad (121)$$

where $f_\pi=133$ MeV is the $\pi\to\mu\nu$ decay constant. The appearance of the factor f_π is associated with the normalization condition

$$\int_0^1\varphi(x, \mu^2)\,dx=f_\pi. \qquad (122)$$

However, for $\mu^2\lesssim 1$ GeV the form of $\varphi(x, \mu^2)$ may differ strongly from $\varphi(x, \infty)$. For weakly interacting particles $\varphi(x)\sim\delta(x-\frac{1}{2})$, and the interaction obviously broadens the wave function. The width of the wave function $\varphi(x, M^2)$ for $M\propto 1/R_{\rm conf}\sim 200$–$500$ MeV (i.e., the width of the "soft" wave function) can be estimated as

$$\Gamma\sim(E_{\rm int}/m_q)^2, \qquad (123)$$

where $E_{\rm int}$ is the parameter that characterizes the interaction strength, and m_q is the mass of the constituents. Thus, for hadrons constructed from heavy quarks, for example, for the J/ψ and Υ particles, the wave function is rather narrow, since $E_{\rm int}\sim M\lesssim 500$ MeV and $m_q\lesssim 1$ GeV. The two-pion relation (123) gives $\Gamma>1$ for any reasonable choice of m_q, i.e., the pion wave function must be broad. Note that the amplitude (119) is singular at $x, y=0$. Therefore for sufficiently broad wave functions the main contribution to the integral will be made by the region $x, y\ll 1$, in which the "virtualities" of the gluon (xyQ^2) and the quark (xQ^2) (see Fig. 14c) are much less than the virtuality of the probing photon. In such a situation the choice $\mu^2, \mu_R^2\sim xyQ^2$ or xQ^2 is clearly preferable to the choice $\mu^2=\mu_R^2=Q^2$. In order to determine precisely which choice makes the series with respect to α_s converge most rapidly, it is necessary to calculate $E(x, y)$ in at least the single-loop approximation.

A correction of order α_s to the leading logarithmic approximation is also obtained when allowance is made for the two-loop contribution to the evolution kernel for the pion wave function. Allowance for such a contribution is all the more necessary because only in this case will the total $O(\alpha_s)$ correction be independent of the renormalization and factorization schemes.[62] In addition, calculation of the two-loop contribution to the kernel $V(x, y)$ gives the answer to the interesting question of whether in the higher orders the connection between the multiplicatively renormalized and the conformal operators found at the single-loop level survives or is destroyed by the radiative corrections.

Prescription for constructing the coefficient function

As a rule, the contributions of the many-loop diagrams contain renormaliztion-group, $(\alpha_s \ln Q^2/\mu_R^2)^N$, and mass, $(\alpha_s \ln Q^2/p^2)^N$ and $(\alpha_s \ln^2 Q^2/p^2)^N$, logarithms. The former arise as a result of the procedure for eliminating the ultraviolet divergences, while the latter appear as a result of the calculation of certain convergent (in the ultraviolet region) integrals. Here and in what follows all masses are assumed to be equal to zero, and p^2 is the parameter that ensures the infrared regularization, for example, the virtuality of the external particles.

The factorization procedure presented in Sec. 2 is essentially as follows. First we prove the cancellation of all the doubly logarithmic contributions $(\alpha_s \ln^2 Q^2/p^2)^N$. For the remaining singly logarithmic contributions $(\alpha_s \ln Q^2/p^2)^N$ we make the decomposition $\ln Q^2/p^2 = \ln Q^2/\mu^2 + \ln \mu^2/p^2$ into the contributions due to "small" and, respectively, "large" distances. We then prove that the logarithms $\ln Q^2/\mu^2$ and $\ln \mu^2/p^2$ are collected together into separate factors, i.e., that

$$\begin{aligned} T(Q, p) &\equiv \alpha_s t_0 + \alpha_s^2 (t_1 + t_0 \otimes V_1 \ln (Q^2/p^2)) + \dots \\ &= [\alpha_s t_0 + \alpha_s^2 ((t_1 - a_1) + t_0 \otimes V_1 \ln (Q^2/\mu^2)) + \dots] \\ &\otimes [1 + \alpha_s (V_1 \ln (\mu^2/p^2) + a_1) + \dots] \\ &\equiv E(Q^2/\mu^2) \otimes \Gamma(\mu^2/p^2), \end{aligned} \quad (124)$$

where V_1 is the single-loop evolution kernel. The expression (124) takes into account the fact that the factor $\Gamma(\mu^2/P^2)$ (which is usually associated with the matrix elements of certain local operators) in the single-loop approximation also contains in general the nonlogarithmic term $\alpha_s a_1$. Therefore, to find the coefficient of α_s^2 in the expansion of E with respect to α_s, it is necessary in accordance with (124) to calculate the contributions of the single-loop diagrams both for the $q\bar{q}\gamma^* \to q'\bar{q}'$ subprocess and for the matrix elements of the corresponding operators. Note that if the infrared cutoff is ensured by ascribing to the external particles a nonvanishing virtuality, neither t_1 nor a_1 will be gauge-invariant expressions, and only their difference $e_1 = t_1 - a_1$ will be independent of the choice of the gauge (see Ref. 62). In QCD the most convenient infrared cutoff is based on dimensional regularization,

$$\frac{d^4k}{(2\pi)^4} \to \left(\frac{d^{4+2\varepsilon}k}{(2\pi)^{4+2\varepsilon}}\right) (4\pi e^{-\gamma_E}) (\mu^2)^{-\varepsilon}, \quad (125)$$

with subsequent subtraction of the poles with respect to ε which formally correspond to $\ln(\mu^2/p^2)|_{p^2=0}$]. The choice (125) corresponds to the $\overline{\text{MS}}$ scheme,[62] in which the specially introduced factor $4\pi \exp(-\gamma_E)$ leads to cancellation of the contributions containing $\ln(4\pi)$ and Euler's constant γ_E. The virtualities of the external particles, and also all the masses, are in this case taken equal to zero ($p^2 = m^2 = 0$). Therefore, the initial (unregularized) amplitude is gauge-invariant. Since dimensional regularization does not destroy gauge invariance, the regularized quantities are now also gauge invariant.

When dimensional regularization is used,[63] the connection between the transition amplitude $T(Q^2, \mu^2, g, \varepsilon)$ and the coefficient function E in the single-loop approximation is given by[64]

$$\begin{aligned} &T_0(\varepsilon) + \left(\frac{B}{\varepsilon} + C\right) + \dots \\ &= \left(1 + \frac{V_1}{\varepsilon} + \dots\right) \otimes (E_0(\varepsilon) + E_1(\varepsilon) + \dots) \\ &\otimes \left(1 + \frac{V_1}{\varepsilon} + \dots\right). \end{aligned} \quad (126)$$

In the case when $E_0(\varepsilon)$ has a nontrivial dependence on ε, $E_0(\varepsilon) + \varepsilon E_0' + \dots$, the simple relation $E_1(0) = C$ no longer holds and we have the more complicated relation

$$E_1(0) = C - V_1 \otimes E_0' - E_0' \otimes V_1. \quad (127)$$

In the considered problem the contribution of the simplest diagram (Fig. 14c) is proportional to $1 - \varepsilon$, since

$$\gamma^\mu \gamma_\nu \gamma_\mu = -2(1 - \varepsilon)\gamma_\nu. \quad (128)$$

Thus, $E_0(\varepsilon) = E_0(1 - \varepsilon)$, and therefore

$$E_1(0) = C + V_1 \otimes E_0 + E_0 \otimes V_1. \quad (129)$$

Using the explicit form of E_0 (119) and V_1 (97), (100), we find that to obtain E_1 it is necessary to add to the sum of the finite parts C the expression

$$\Delta E^{(1)} = -\frac{\alpha_s}{2\pi} C_F (4 + \ln x + \ln y) E^{(0)}. \quad (130)$$

The result of calculations for the single-loop coefficient function

A direct calculation of the coefficient function was made originally in the Feynman gauge independently by two groups,[65,66] between the results of which there was a disagreement. Later and also independently made calculations in the isotropic gauge $n_\mu A_\mu = 0$ ($n^2 = 0$) (Refs. 64 and 66) confirmed the correctness of the result obtained in Ref. 65.

The particular computational scheme chosen in Ref. 65 corresponds to dimensional regularization of both the mass singularities and the ultraviolet divergences. Such a possibility is ensured by the fact that these two types of singularity do not arise in the considered problem simultaneously for the same integral, except for the trivial contributions corresponding to insertions into external lines. The dimensional regularization of the ultraviolet-divergent integrals,

$$d^4k \to d^{4-2\varepsilon}k, \quad (131)$$

was accompanied by 't Hooft renormalization,[63] i.e., by subtraction of the poles with respect to ϵ from the divergent subgraphs.

The results of the calculations for the individual diagrams are given in Ref. 65. The total contribution of all the diagrams is

$$E^{(1)}(x,\ y;\ \alpha_s;\ Q,\ \mu,\ \mu_R)$$
$$=\frac{2\pi C_F\alpha_s}{N_c xy}\left\{1+\frac{\alpha_s}{2\pi}\left[C_F\left((2+\ln x)\left(L(\sqrt{xy})-1\right)-\frac{1}{2}L(1)\right.\right.\right.$$
$$\left.+\frac{1}{2}\ln x\left(3-\frac{x}{\bar{x}}\right)-\frac{1}{3}\right)$$
$$-\frac{1}{4}\left(\frac{11}{3}N_c-\frac{2}{3}N_f\right)\left(L^{(R)}(xy)-\frac{5}{3}\right)$$
$$+\left(C_F-\frac{N_c}{2}\right)\left(\mathrm{Sp}(x)-\mathrm{Sp}(x)+\ln\bar{x}\ln\frac{y}{\bar{y}}-\frac{5}{3}\right)$$
$$+\left(C_F-\frac{N_c}{2}\right)\frac{1}{(x-y)^2}\left(\frac{y^2\bar{y}+x^2\bar{x}}{y-x}(\mathrm{Sp}(\bar{x})-\mathrm{Sp}(x)-\ln\bar{x}\ln y)\right.$$
$$\left.\left.\left.+2xy\ln x+(x+y-2xy)\ln\bar{x}\right)+\{x\leftrightarrow y\}\right]\right\},\tag{132}$$

where

$$\bar{x}=1-x,\quad \bar{y}=1-y;\tag{133}$$

$$C_F=4/3,\quad N_c=3;\tag{134}$$

$$L(a)\equiv\ln(aQ^2/\mu^2),\quad L^{(R)}(a)\equiv\ln(aQ^2/\mu_R^2),\tag{135}$$

and the Spence function $\mathrm{Sp}(a)$ is defined by

$$\mathrm{Sp}(a)=-\int_0^1\frac{dz}{z}\ln(1-az).\tag{136}$$

Note that despite the presence in (132) of terms containing $(y-x)^3$ in the denominator, $E'(x,y)$ does not have singularities on $x=y$.

Structure of the single-loop correction to the coefficient function

To obtain a clearer idea of the value and structure of the calculated corrections, we represent $F_\pi(Q)$ in the form

$$F_\pi(Q)=F_\pi^{(0)}(Q,\ \mu,\ \mu_R)\left\{1+\frac{\alpha_s(\mu_R)}{\pi}B(Q,\ \mu,\ \mu_R)+O(\alpha_s^2)\right\},\tag{137}$$

where the zeroth approximation $F_\pi^{(0)}$ is given by (120), and

$$B=\left[-A\ln\frac{Q^2}{\mu^2}\right]-\frac{1}{4}\left(11-\frac{2}{3}N_f\right)\ln\frac{Q^2}{\mu_R^2}+C.\tag{138}$$

The factor in the square brackets corresponds to contributions whose allowance leads to replacement of the argument μ^2 in the wave function $\varphi(x,\mu^2)$ by a quantity proportional to Q^2: $\mu^2\to a_1^2Q^2$, while the term containing $\ln(Q^2/\mu_R^2)$ leads to replacement of $g(\mu_R)$ by the effective coupling constant $\bar{g}(a_2Q)$. In principle, there are no grounds for taking a_1 and a_2 equal to unity.

The values of the coefficients A and C depend on the particular form of the wave functions $\varphi(x)$ and $\varphi(y)$. To analyze this dependence, we chose the simplest parametrization

$$\varphi_r(x)=f_\pi\frac{\Gamma(2+2r)}{\Gamma^2(1+r)}x^r(1-x)^r.\tag{139}$$

The numerical factor in (139) ensures the normalization (122). The values of A and C for different r are given in Table I.

If $a_1=a_2=1$, i.e., for $\mu=\mu_R=Q$, the correction is determined by the coefficient C. It can be seen that even for

TABLE I. Values of the coefficients A and C in the expression (138) for the functions $\varphi_r(x)$ specified by the parametrization (139), and the values of B in (137) for $\mu^2=\sqrt{xy}Q^2$ and different $\varphi_r(x)$.

r	A	C	$B\ (\mu^2=\sqrt{xy}\,Q^2)$
1	0	7.3	3.04
0.5	1.2	13.4	2.32
0.2	5.0	54.4	0.39
0.1	11.5	204	−0.63
0.05	24.8	803	−8.53
0.01	131	$2\cdot10^4$	−55.2

$r=1$, i.e., for a fairly narrow wave function, $B=C=7.25$, and this gives a 70% correction for $\alpha_s\simeq0.3$. As was pointed out in the introductory subsection, for narrow wave functions the choice $\mu^2=\mu_R^2=Q^2/4-Q^2/2$ seems more reasonable. For such a choice $B=4.1$–5.7, i.e., the correction is noticeably reduced. Strictly speaking, the choice $\mu^2=\mu_R^2=Q^2/2-Q^2/4$ must be optimal in the "physical" or MOM scheme,[66] in which $g(\mu_R)$ corresponds to a vertex at which the incoming momenta have virtuality μ_R^2. In the $\overline{\mathrm{MS}}$ scheme that we use the meaning of the parameter μ_R is less transparent. However, it is known that if the effective coupling constant $g_i(\mu)$ corresponding to the i-th scheme is expressed in the form

$$\frac{\bar{g}_i^2(\mu)}{(4\pi)^2}=\frac{1}{b_0\ln(\mu^2/\Lambda_i^2)}\left\{1-\frac{b_1}{b_0^2}\frac{\ln(\ln(\mu^2/\Lambda_i^2))}{\ln(\mu^2/\Lambda_i^2)}+\dots\right\},\tag{140}$$

then the results obtained in the different schemes differ only by a substitution $\Lambda_i=\varkappa_{ij}\Lambda_j$, where $\varkappa_{ij}$ is a numerical coefficient.[66] In particular, Λ_{phys} depends weakly on the vertex chosen to determine $\bar{g}(k)$: $\Lambda_{\mathrm{phys}}\simeq2\Lambda_{\overline{\mathrm{MS}}}$.[66] In other words, the choice $\mu^2=\mu_R^2=Q^2/2-Q^2/4$ in the physical scheme corresponds to the choice $\mu^2=\mu_R^2=Q^2/8-Q^2/16$ in the $\overline{\mathrm{MS}}$ scheme. For such a choice $B=1.0$–2.6 for $r=1$.

With decreasing r, i.e., for even broader wave functions, the coefficient C and, therefore, the value of the correction for $\mu=\mu_R=Q$ increase as $O(1/r^2)$. Simultaneously, but somewhat more slowly [as $O(1/r)$] the coefficient A also increases. For small x and y, $E^{(1)}(x,y;\alpha_s)$ can be represented in the form

$$E^{(1)}(x,\ y;\ \alpha_s)=E^{(0)}(x,\ y)$$
$$\left\{1+\frac{\alpha_s(\mu_R)}{4\pi}\left[C_F\left(\ln^2\frac{xyQ^2}{\mu^2}-\ln^2\frac{Q^2}{\mu^2}\right)\right.\right.$$
$$+4C_F\ln\left(xy\left(\frac{Q^2}{\mu^2}\right)^2\right)-2C_F\ln\frac{Q^2}{\mu_R^2}+3C_F\ln(xy)$$
$$-\left(11-\frac{2}{3}N_f\right)\ln\left(\frac{xyQ^2}{\mu_R^2}\right)$$
$$\left.\left.-2\left(C_F-\frac{N_c}{2}\right)\ln(xy)+f(x,\ y)\right]\right\},\tag{141}$$

where $f(x,y)$ is regular for $x=y=0$ and does not depend on μ, μ_R, and Q. For small r the main contribution to A and C is made by the terms written out explicitly in (141), which for $x=y=0$ are more singular than $E^{(0)}(x,y)$. The appearance of the factors $\ln(x)$ in $\ln(y)$ in (141) is undoubtedly due to the presence of the invariants xQ^2 and xyQ^2.

Minimization of the single-loop corrections

The mass and renormalization-group logarithms have different natures, and we shall therefore consider them separately. The corrections to the gluon propagator have the simplest structure. They depend only on xyQ^2.

In Abelian theory (for example, in QED) the behavior of $\bar{g}(\mu_R)$ is completely determined by the corrections to the propagator of the vector particles and, therefore, the most expedient choice for μ_R in this case will be $\mu_R^2 = xyQ^2$ (in the physical scheme). The mean value of μ_R^2 for such a choice can be estimated by requiring that the correction due to the contribution $\ln(xyQ^2/\mu_R^2)$ for $\mu_R^2 = \langle\mu_R^2\rangle$ vanish. This gives

$$\langle\mu_R^2\rangle = \langle x\rangle \langle y\rangle Q^2, \tag{142}$$

where

$$\langle x\rangle \equiv \exp(\langle\ln x\rangle) = \exp\left\{\left[\int_0^1 \frac{\varphi(x)}{x}\ln x\,dx\right]\left[\int_0^1 \frac{\varphi(x)}{x}dx\right]^{-1}\right\}. \tag{143}$$

For the functions (139) for small r we have $\langle x\rangle = \exp(-1/r)$. In particular, $\langle x\rangle = 5\times 10$ for $r = 0.1$.

In QCD the behavior of the effective constant $\bar{g}(\mu_R)$ also depends on the value of the corrections to the quark–gluon vertex and to the quark propagator. If we take all the terms $\ln(Q^2a/\mu_R^2)$ that arise in the diagrams when the divergent integrals are calculated, then, as a simple calculation shows, vanishing of their contribution at small r requires that we take

$$\mu_R^2 = \langle\mu_R^2\rangle = Q^2\exp\left\{-\frac{1}{r}\frac{64-4N_f}{41-2N_f}\right\}. \tag{144}$$

For $N_f \lesssim 6$ the coefficient in the exponential is very close to $-3/2r$. In other words, these corrections are minimal for

$$\mu_R^2 = \langle x\rangle^{3/2}Q^2 = \sqrt{\langle xQ^2\rangle\langle x\rangle\langle y\rangle Q^2},$$

i.e., when μ_R^2 is equal to the geometric mean of the mean virtualities of the quark and gluon.

The renormalization-group logarithms make a contribution to C that increases at small r only as $1/r$. Much more important at small r are the contributions associated with the mass logarithms. The main contribution, equal to

$$E^{(0)}(x,y)\frac{\alpha_s(\mu_R)}{4\pi}\frac{N_c}{2}\left(\ln^2(xy) + 2\ln(xy)\ln\left(\frac{Q^2}{\mu^2}\right)\right), \tag{145}$$

is made in the Feynman gauge by the diagrams shown in Figs. 15a and 15b. There are also contributions with a similar structure in the "nonplanar" diagrams of Figs. 15c–15f. But these contributions have a color factor $C_F - N_c/2$ that is $N_c^2 = 9$ times smaller than the contribution of the diagrams of Figs. 15a and 15b in accordance with the rules of the $1/N_c$ expansion.

It is readily noted that the contribution (145) vanishes for $\mu^2 = \sqrt{xy}Q^2$. It is also easy to calculate the mean value of the parameter μ^2 at small r:

$$\langle\mu^2\rangle = Q^2\exp\left(-\frac{3}{2r}\right). \tag{146}$$

In particular, for the value $r = 0.5$, which ensures for $Q^2 F_\pi(Q^2)$ a value close to the experimental one (about 0.3 GeV^2), we have $\langle\mu^2\rangle = Q^2/20$, and at the existing momentum transfers $Q^2 \lesssim 4$ GeV^2 this is much less than the value $\mu^2 \sim 1$ GeV^2 at which one can hope for validity of perturbative QCD calculations.

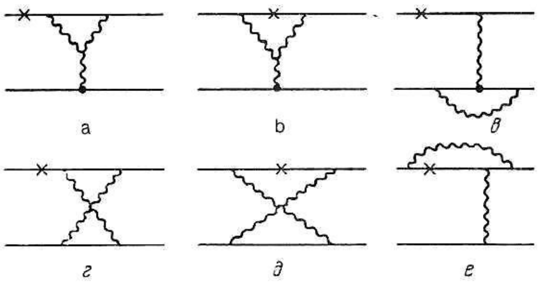

FIG. 15. The diagrams that make the main contribution to the single-loop correction to the coefficient function.

Two-loop evolution kernel

One further contribution to the $O(\alpha_s^2)$ correction to $F_\pi(Q^2)$ is due to the two-loop evolution of the pion wave function, i.e., the $O(\alpha_s^2)$ contribution to the kernel $V(x,y)$ or to the matrix of anomalous dimensions $Z_{nk}(g)$. The standard method of calculating the anomalous dimensions of composite operators is based on the expansion of the quantities $(nD)^N$ in accordance with the binomial theorem, and this makes it possible to find the dependence of an operator quantity on the quark and gluon fields:

$$\bar{d}\gamma_5\hat{n}(nD)^N u = \bar{d}\gamma_5\hat{n}u^{(N)} + \sum_{m=0}^{N-1} C_N^{m+1}\bar{d}\gamma_5\hat{n}\hat{A}^{(m)}u^{(N-1-m)}, \tag{147}$$

where $\varphi^{(k)} \equiv (n\partial)^k\varphi$, $\bar{A} = n_\nu A_\nu^a\tau_a g$.

The further calculation is made in the standard manner.[15,16] The single-loop calculation is simplified if one goes over to the isotropic axial gauge $(nA) = 0$, for which only the first contribution remains in (147). The first calculations[66,67] of the two-loop contribution to $V(x,y)$ were in fact made in such a gauge. However, because of the more complicated structure of the gluon propagator,

$$D_{\mu\nu}(k) = \frac{1}{k^2}\left(-g_{\mu\nu} + \frac{k_\mu n_\nu + k_\nu n_\mu}{(kn)}\right), \tag{148}$$

the calculations in this gauge are much more cumbersome. In addition, the gauge-fixing isotropic vector n_ν leads to the appearance of additional divergences, both of ultraviolet and infrared type.[68,69] The latter must cancel each other after summation of all diagrams of a given order, while the former are actually subtracted "by hand" (see, for example, Ref. 70), since no consistent method of working with them has yet been developed. This is due, in particular, to the fact that the additional ultraviolet divergences can be eliminated only by means of counterterms whose structure has no analogs in the original Lagrangian. Therefore complete confidence in the reliability of the results (at the given stage) can be ensured only for calculations in covariant gauges, in which the structure of the divergences is well understood. The two-loop contribution to the evolution kernel $V(x,y)$ was calculated in a covariant (Feynman) gauge in Ref. 71. The result confirmed the correctness of the calculations of Refs. 66 and 67. One of the most nontrivial contributions to $V(x,y)$, proportional to C_F^2, was calculated independently in the Feynman gauge in Ref. 72. The identity of the results of all these calculations is a sufficient guarantee of the correctness of the result for $V^{(2)}(x, y)$, which has the form

$$V(x, y; g) = \left\{ V_0(x, y; g) + \left(\frac{\alpha_s}{2\pi}\right)^2 \left[\frac{N_f C_F}{2} V_N(x, y) + C_F^2 V_F(x, y) + \frac{1}{2} C_F N_c V_G(x, y) \right] \right\}_+ ; \tag{149}$$

$$V_0 = \frac{\alpha_s}{2\pi} C_F \theta(x < y) F + (x \to \bar{x}, \ y \to \bar{y}); \tag{150}$$

$$V_N(x, y) = \theta(x < y) \left[-\frac{10}{9} F - \frac{2}{3} \frac{x}{y} - \frac{2}{3} F \ln\left(\frac{x}{y}\right) \right] + (x \to \bar{x}, \ y \to \bar{y}); \tag{151}$$

$$V_G(x, y) = \theta(x < y) \left[\frac{67}{9} F + \frac{17}{3} \frac{x}{y} + \frac{11}{3} \ln\left(\frac{x}{y}\right) - 2\bar{F} \ln y \ln \bar{x} \right] + G(x, y) + (x \to \bar{x}, \ y \to \bar{y}); \tag{152}$$

$$V_F(x, y) = \theta(x < y) \left[-\frac{\pi^2}{3} F + \frac{x}{y} - \left(\frac{3}{2} F - \frac{x}{2\bar{y}}\right) \ln\left(\frac{x}{y}\right) - (F - \bar{F}) \ln\left(\frac{x}{y}\right) \ln\left(1 - \frac{x}{y}\right) + \left(F + \frac{x}{2\bar{y}}\right) \ln^2\left(\frac{x}{y}\right) - 2\bar{F} \ln y \ln \bar{x} \right] - \frac{x}{2\bar{y}} \ln x (1 + \ln x - 2 \ln \bar{x}) - G(x, y) + (x \to \bar{x}, \ y \to \bar{x}); \tag{153}$$

$$\{V(x, y)\}_+ = V(x, y) - \delta(x - y) \int_0^1 V(z, y)\, dz, \tag{154}$$

where $F = f(x, y) = x/y[1 - 1/(x - y)]$, $\bar{F} = F(\bar{x}, \bar{y})$, and the "plus" operator is defined by the expression (154). The function $G(x, y)$ is given by

$$G(x, y) = \theta(x > \bar{y}) \left[2(F - \bar{F}) \mathrm{Li}_2\left(1 - \frac{x}{y}\right) + (F - \bar{F}) \ln^2 y - 2F \ln x \ln y \right] + 2F \mathrm{Li}_2(y) [\theta(x > \bar{y}) - \theta(x < y)] - 2F \mathrm{Li}_2(x) [\theta(x > \bar{y}) - \theta(x > y)], \tag{155}$$

and $\mathrm{Li}_2(x)$ is the Spence function

$$\mathrm{Li}_2(x) = -\int_0^x dt \frac{\ln(1 - t)}{t}. \tag{156}$$

Structure of the two-loop evolution operator

For the following applications it is also necessary to know the explicit form of the solutions of the evolution equation (93). In the lowest approximation this problem reduces to finding the eigenfunctions of the kernel $V(x, y; g)$, which, as we have seen, are the Gegenbauer polynomials $x\bar{x}C_n^{3/2}(x - \bar{x})$. The single-loop evolution kernel V_0 [see (97) and (100)] becomes a symmetric function of its arguments after multiplication by $y\bar{y}$: $V_0(x, y)y\bar{y} = V_0(y, x)x\bar{x}$. Therefore, the eigenfunctions $\varphi_n / y\bar{y}$ of the equation

$$\int_0^1 V_0(x, y) \varphi_n(y)\, dy = \lambda_n \varphi_n(y) \tag{157}$$

must be orthogonal to each other on the interval (0, 1) with weight $y\bar{y}$, i.e., they are proportional to Gegenbauer polynomials.

However, at the two-loop level the combination $y\bar{y}V(x, y)$ contains terms that violate the $x \leftrightarrow y$ symmetry. In V_G and V_N [see (151) and (152)] such terms are $(11/3)F \ln(x/y)$ and $-(2/3)F \ln(x/y)$, respectively. Taken together, with allowance for the group factors, these terms are proportional to the first coefficient β_0 in the Gell-Mann–Low β function:

$$C_F \left(\frac{\alpha_s}{2\pi}\right)^2 \left(\frac{11}{3} N_c - \frac{2}{3} N_f\right) F \ln \frac{x}{y} = C_F \beta_0 \ln\left(\frac{x}{y}\right) \ldots \tag{158}$$

The appearance of nonsymmetric terms at the two-loop level was already expected (see Ref. 73) before actual calculations were made. The motivation was based on the connection between the eigenfunctions of the kernel $V(x, y)$, i.e., between the multiplicatively renormalized operators and the conformal operators. Since the conformal invariance is violated by the renormalization procedure (the coupling constants, for example), there are no grounds for expecting operators that are conformally invariant for the free fields to be multiplicatively renormalized at the two-loop level.

We now establish why the β function arises in the final answer. Two different classes of diagram contribute to the function

$$F \ln\left(\frac{x}{y}\right) = \frac{x}{y} \ln \frac{x}{y} + \frac{x}{y} \frac{1}{y - x} \ln \frac{x}{y}$$

in V_N and V_G, and also to the term $(2/3)F \ln(x/y)$ in V_F. The coefficient β_0 of the structure $\theta(x < y)(x/y) \ln(x/y)$ is formed by the diagrams of Figs 15a–15e. Allowance for these diagrams actually leads to renormalization of the charge at the vertices of the single-loop diagram (see Fig. 13a). At the same time, the diagrams of Figs. 15a and 15b introduce a contribution proportional to the anomalous dimension of the gluon line; the diagrams of Figs. 15d and 15e contribute to the renormalization of the vertex; the diagram of Fig. 15a renormalizes the quark line that emanates from the vertex. Accordingly, the diagrams of Figs. 16g–16j form the coefficient β_0 of the structure $((x/y(y - x)) \ln(x/y)$. This class of diagrams is responsible for the renormalization

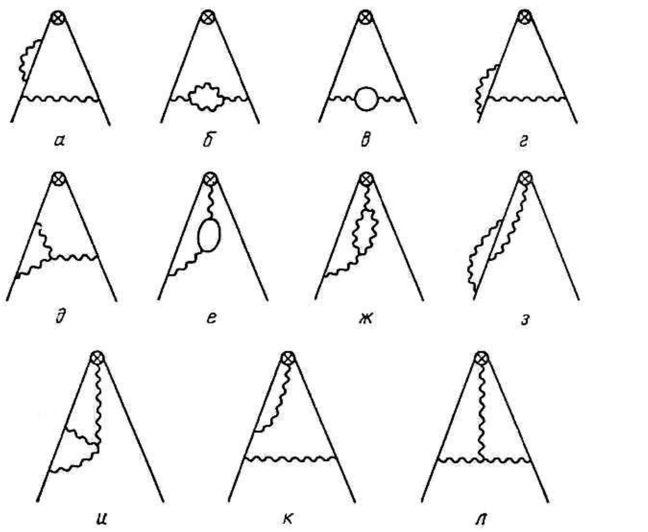

FIG. 16. The diagrams that make the two-loop correction to the evolution kernel.

of the charge at the vertex of the single-loop diagram of Fig. 13b.

However, in the complete answer in $y\bar{y}V_F(x, y)$ there are also contributions [for example, proportional to $\ln^2(x/y)$] that violate the $x\leftrightarrow y$ symmetry more fundamentally. Note that in the functions V_N and V_G all the contributions except those proportional to $F\ln(x/y)$ are $x\leftrightarrow y$ symmetric after multiplication by $y\bar{y}$.

Solution of the evolution equation for the pion wave function in the two-loop approximation

If the two-loop kernel had the symmetry property $V_2(x, y)y\bar{y} = V_2(y, x)x\bar{x}$, the solutions of the evolution equation (93) would have the form

$$\Phi_n^{(D)}(x, \mu^2) = K_n(\mu_0^2) \exp\left\{\int_{\mu_0^2}^{\mu^2} \gamma_n(\bar{g}(t)) \frac{dt}{t}\right\} x\bar{x} C_n^{3/2}(x), \tag{159}$$

where $\gamma_n(g) = -\alpha_s\gamma_n^{(1)} - \alpha_s\gamma_n^{(2)}$ is the diagonal element of the matrix $Z_{nn}(g)$ of anomalous dimensions. This situation corresponds to the fact that in the basis of Gegenbauer polynomials the matrix of anomalous dimensions

$$G_{nk} = \frac{4(2k+3)}{(k+1)(k+2)} \int_0^1 dx \int_0^1 dy\, C_n^{3/2}(x-\bar{x}) \times V(x, y)\, y\bar{y} C_k^{3/2}(y) \tag{160}$$

is diagonal: $G_{nk} = \mathrm{diag}(\gamma_n)$.

In the case when $G_{nk}^{(2)}$ contains nondiagonal terms the solutions of the evolution equation must differ from $\Phi_n^{(D)}$ by contributions of order α_s. We shall therefore seek solutions for this general case in the form

$$\Phi_n = (1 + \alpha_s W) \otimes \Phi_n^{(D)}, \tag{161}$$

where $W = W(x, y)$ and $\otimes$ denotes convolution.

The problem is now to find $W(x, y)$. Substituting (161) in the evolution equation, we obtain an equation for W:

$$b_0 W + [v_0, W] - V_2^{(\mathrm{ND})} = 0, \tag{162}$$

where $V_2^{(\mathrm{ND})}$ is the part of the kernel V_2 that does not commute with V_1 and, as a consequence of this, is responsible for the nondiagonal contributions to G_{nk}.

Formally, the solution of Eq. (162) is given by the expression

$$W = \int_0^\infty e^{-(b_0+v_0)t} \otimes V_2^{(\mathrm{ND})} \otimes e^{v_0 t}\, dt. \tag{163}$$

However, for applications this representation is not very convenient. It is more effective to use the matrix representation for W in the Gegenbauer basis $\psi_n = x\bar{x}C_n^{3/2}(x-\bar{x})$:

$$W \otimes \psi_n = \sum_k \frac{G_{kn}^{(\mathrm{ND})}\psi_k}{\gamma_k^{(1)} - \gamma_n^{(1)} - b_0}. \tag{164}$$

If the wave function $\varphi(x, \mu^2)$ on $\mu^2 = Q^2$ is to be determined directly from its form on $\mu^2 = Q_0^2$, it is also expedient to modify somewhat the scheme for solving the evolution equation, namely, to assume that the operator W depends on Q^2: $W \to \widetilde{W}(Q^2)$ with the boundary condition $\widetilde{W}(Q_0^2) = 0$.

In this case Eq. (162) is modified and takes the form

$$Q^2 \frac{\partial}{\partial Q^2} \widetilde{W} = \frac{\alpha_s(Q^2)}{4\pi} \left\{ b_0 \widetilde{W} + [V_0, W] + V_2^{(\mathrm{ND})} \right\}. \tag{165}$$

The formal solution of this equation is obtained in exactly the same way as (163):

$$\widetilde{W} = W - \left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{1+V_0/b_0} \otimes W \otimes \left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{-V_0/b_0}. \tag{166}$$

As a result, we obtain an expression for the correction in the form of the series

$$\left(1 + \frac{\alpha_s(Q^2)}{4\pi} \widetilde{W}\right) \otimes \psi_n = \psi_n + \frac{\alpha_s(Q^2)}{4\pi} \sum_{k\geqslant 0} d_n^k(Q^2)\psi_k, \tag{167}$$

where

$$d_n^k(Q^2) = \frac{(V_2^{(\mathrm{ND})})_{kn}}{\gamma_k^{(1)} - \gamma_n^{(1)} - b_0} S_{kn}(Q^2); \tag{168}$$

$$G_{kn}^{(2,\mathrm{ND})} = \frac{4(2k+3)}{(k+1)(k+2)} C_k^{3/2} \otimes G_2^{(\mathrm{ND})} \otimes \psi_n, \tag{169}$$

and $\gamma_n^{(1)}$ are the eigenvalues of the kernel $\{V_0\}_+$:

$$\gamma_n^{(1)} = C_F\left(1 + 4\sum_{j=2}^{n+1} \frac{1}{j} - \frac{2}{(n+1)(n+2)}\right); \tag{170}$$

$$\{V_0\}_+ \otimes \psi_n = -\gamma_n^{(1)}\psi_n, \tag{171}$$

and finally $S_{kn}(Q^2)$ is the factor that determines that Q^2 dependence of the operator $\widetilde{W}$:

$$S_{kn}(Q^2) = 1 - \left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{(b_0+\gamma_n^{(1)}-\gamma_k^{(1)})/b_0}. \tag{172}$$

Note that the only nonzero elements of the matrix $(V_2^{\mathrm{ND}})_{kn}$ are those for which k and n have the same parity and $k > n$. The first condition is a manifestation of the "geometrical" symmetry of the evolution kernel, while the second follows from the triangular nature of the renormalization matrix.

We now use the fact that V_N and V_F have only nondiagonal contributions to G_{nk} that are induced by renormalization of the coupling constant,

$$2(N_f C_F V_N + C_F C_A V_G) = b_0 F \ln\frac{x}{y} + u_D(x, y) + (x \to \bar{x},\ y \to \bar{y}), \tag{173}$$

where u_D makes only a diagonal contribution to G_{nk}. The matrix elements for the nondiagonal contribution $F\ln(x/y)$ can be calculated analytically. For this it must be borne in mind that $F\ln(x/y)$ can be represented as the derivative of the kernel $V_\delta(x, y)$:

$$V_\delta(x, y) = F\left(\frac{x}{y}\right)^\delta \theta(x < y) + (x \to \bar{x},\ y \to \bar{y}), \tag{174}$$

which becomes $x\leftrightarrow y$ symmetric if it is multiplied by $(y\bar{y})^{1+\delta}$

The eigenfunctions of the kernel $V_\delta(x, y)$ have the form

$$\psi_n^{(\delta)} = (x\bar{x})^{1+\delta} C_n^{3/2+\delta}(x-\bar{x}). \tag{175}$$

In other words, the combination $F\ln(x/y)$ is proportional to the generator of a shift of the upper index of the Gegenbauer polynomials $C_n^\nu(x-\bar{x})$. Differentiating the eigenvalue equation

$$V_\delta \otimes \psi_n^{(\delta)} = -\gamma_n^{(\delta)}\psi_n^{(\delta)} \tag{176}$$

with respect to δ at $\delta = 0$, we obtain the equation

$$\dot{V} \otimes \psi_n = -(\gamma_n + V_0) \otimes \dot{\psi}_n + \dot{\gamma}_n \psi_n. \quad (177)$$

We expand the derivative $\dot{\psi}_n$ in a series in ψ_n:

$$\dot{\psi}_n = \sum_{k>n} a_{nk} \psi_k, \quad (178)$$

the coefficients of which are

$$a_{nk} = -\frac{2(n+1)(n+2)(2k+3)}{(k+1)(k+2)(k-n)(k+n+3)}. \quad (179)$$

The expression (179) follows from the general expression for the Gegenbauer polynomials

$$C_n^{\nu}(x) = \frac{\Gamma(\lambda)}{\Gamma(\gamma)\Gamma(\nu-\lambda)} \sum_{\tau=0}^{[\frac{n}{2}]} \frac{\Gamma(\tau+\nu-\lambda)\Gamma(n+\nu-\tau)}{\tau!\,\Gamma(n-\tau)} \times (n-2\tau+\lambda)\, C_{n-2\tau}^{\lambda}(x), \quad (180)$$

which can be obtained by combining the expansion expressions for $C_n^{\nu}(x)$ with respect to x^k and for x^k with respect to C_l^{λ} given in the Appendix to the paper of Ref. 74. Thus, the coefficients $d_n^k(Q^2)$ (168) have the form

$$d_{nk}^{h} = \{b_0(\gamma_k^{(1)} - \gamma_n^{(1)})\, a_{nk} + C_F^2 (G_F^{(\mathrm{ND})})_{nk}\} \frac{1}{\gamma_k^{(1)} - \gamma_n^{(1)} - b_0}. \quad (181)$$

The matrix elements $(G_F^{\mathrm{ND}})_{nk}$ were found by numerical integration. The diagonal elements $(G_F)_{nn}$, which are equal to the two-loop anomalous dimensions, are given, for example, in Ref. 70.

The pion wave function, with allowance for its evolution in the two-loop approximation, is given as a result by the expression

$$\varphi(x, Q^2) = \sum_n b_n(Q_0^2) \exp\left\{\int_{Q_0^2}^{Q^2} \gamma_n(g(t)) \frac{dt}{t}\right\} \times \left(\psi_n(x) + \frac{\alpha_s}{4\pi} \sum d_n^h(Q^2)\, \psi_k(x)\right). \quad (182)$$

Numerical results for evolution of the pion wave function (Ref. 71)

Using Eqs. (181) and (182), we can calculate the QCD evolution with Q^2 of any wave functions defined at a certain Q_0^2. The general properties of the evolution are most conveniently formulated for the "partial" wave functions ψ_n that occur in the expansion (192) with weight $b_n(Q_0^2)$.

1. For $n > 2$ the corrections that derive from the nondiagonal part are approximately an order of magnitude smaller than the corrections from the diagonal part accumulated in the exponential factor, i.e., the kernel is "quasidiagonal" in the Gegenbauer basis (the corrections for $n = 0$, i.e., for the asymptotic function, are also very small and are considered below).

2. The corrections of the higher harmonics to ψ_n are determined mainly by the first term $d_n^{n+2}\psi_{n+2}$ in the sum over k in (182). The subsequent coefficients decrease rapidly with increasing k.

3. The contribution of the two-loop corrections increases with increasing index n. For $n = 6$ they already give a 6% correction at $x = 0.5$ and $Q^2 = 125$ GeV^2. (Here and below, $Q_0^2 = 1$ GeV^2, $\Lambda_{\mathrm{QCD}} = 0.1$ GeV.)

Thus, for the choice $\varphi(x, Q_0^2) = 6x\bar{x}f_\pi$, which corresponds to the asymptotic wave function, corrections arise only because of the nondiagonal terms:

$$\varphi(x, Q^2) = 6x\bar{x}f_\pi \left\{1 + \frac{\alpha_s}{4\pi} \sum_{k\geqslant 2} d_0^k C_k^{3/2}(x-\bar{x})\right\}. \quad (183)$$

The calculations show that the corrections are less than 0.5% up to $Q^2 = 6\times 10^3$ GeV^2.

Another well-known example of a low-energy pion wave function was proposed by Chernyak and Zhitnitskiĭ[26]:

$$\varphi^{CZ}(x, Q_0^2) = 30x\bar{x}f_\pi(1 - 4x\bar{x}). \quad (184)$$

In this case the relative contribution of the two-loop corrections is about 2% for $Q^2 = 125$ GeV^2. Using the data of Table II, we can calculate the corrections to any wave functions which for $Q_0^2 = 1$ GeV^2 can be represented by a sum of Gegenbauer polynomials ψ_n with $n \leqslant 8$.

Contribution of the two-loop evolution to the α_s corrections to the asymptotic behavior of the pion form factor

To analyze the part played by the $O(\alpha_s^2)$ corrections to $F_\pi(Q^2)$ due to the two-loop evolution of the pion wave function, a numerical estimate of the form factor was obtained for wave functions $\varphi_\pi(x)$ equal, for $Q_0^2 = 1$ GeV^2, to $\psi_n(x)$.[79] The results of the calculations are given in Table III, in which $T_1(n)$ corresponds to the $O(\alpha_s^2)$ correction due solely to the single-loop coefficient function, T_2 corresponds to the correction associated with the two-loop evolution and $A(n)$ characterizes the contribution of the lowest diagram:

$$Q^2 F_\pi^{(n)}(Q^2) = \frac{2\pi C_F}{N_c} f_\pi^2 \alpha_s(Q^2) A(n) \times \left\{1 + \frac{\alpha_s(Q^2)}{\pi}[T_1(n) + T_2(n)]\right\}. \quad (185)$$

TABLE II. The coefficients d_n^k (181) calculated for $Q_0^2 = 1$ GeV^2, $\Lambda_{\mathrm{QCD}} = 0.1$ GeV, $Q^2 = 125$ GeV^2.

k \ n	0	2	4	6	8
2	−0.277				
4	+0.012	−0.89			
6	0.032	−0.26	−1		
8	0.027	−0.087	−0.4	−1	
10	0.021	−0.03	−0.187	−0.47	−0.96
12	0.012	−0.007	−0.094	−0.25	−0.5
14	0.016	−0.002	−0.05	−0.15	−0.29
16	0.01	0.006	−0.025	−0.09	−0.18
18	0.009	0.007	−0.008	−0.05	−0.12

TABLE III. Values of the coefficients in Eq. (185) for $Q^2 = 33\ \mathrm{GeV}^2$ and $\alpha_s(Q^2) = 0.14$.

n	0	2	4	6	8
$A(n)$	0.25	0.124	0.08	0.073	0.062
$T_1(n)$	7.22	19.3	29.1	37.0	43.9
$T_2(n)$	−0.16	−0.4	−0.7	−2.8	11.6
T_2/T_1	0.02	0.02	0.025	0.03	0.035

It can be seen from Table III that allowance for the evolution corrections has little influence on the total result.

CONCLUSIONS

In this review, we have presented the fundamentals of the perturbative approach to hard exclusive hadron processes in QCD for the example of the analysis of the asymptotic behavior of the electromagnetic pion form factor. In principle, the perturbative approach can also be applied to the investigation of more complicated problems—to calculate the asymptotic behavior of nucleon form factors, the amplitudes of large-angle hadron scattering, the cross sections for the production of isolated hadrons, etc. (see, for example, Refs. 9, 10, 24, and 25). For the practical use of the results for the interpretation and analysis of existing experimental data the problem of determining the limits of the asymptotic region is the most important. As we have seen, the answer to the question of the momentum transfers from which the asymptotic expressions of perturbative QCD are valid depends on the form of the wave function—for narrow functions the asymptotic regime commences earlier than for broad functions. However, the explicit form of the wave function cannot be calculated in the framework of purely perturbative QCD without using information about the nonperturbative aspects of QCD dynamics. One of the most promising ways of solving this problem appears to be the QCD sum-rule method.[78] In its framework a number of results have already been obtained on the behavior of the hadron wave functions and of the form factors in the region of small and moderate momentum transfers. However, in this field there is still much to be done before one can say that we have a complete QCD picture of the behavior of the hadron form factors at all momentum transfers.

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APPENDIX. ANALYSIS OF ASYMPTOTIC BEHAVIOR OF HARD PROCESSES

A detailed exposition of the fundamentals of our approach to the analysis of hard processes in QCD is given in Ref. 23. Here we give a brief list of the conclusions from Ref. 23 that are helpful for understanding the proof of the factorization theorem in Sec. 2.

The approach developed in Refs. 23 and 40–43 is based on the α representation of Feynman diagrams (see, for example, Ref. 75), which arises if the denominator of the propagator of each line σ is expressed in the form

$$\frac{1}{m_\sigma^2 - k_\sigma^2 - i\varepsilon} = i \int_0^{i\infty} d\alpha_\sigma \exp\{\alpha_\sigma (k_\sigma^2 - m_\sigma^2 + i\varepsilon)\} \qquad (\mathrm{A.1})$$

and integrated over the virtual momenta k_σ. As a result, for the contribution of each diagram we obtain the expression

$$T(p_1, \ldots, p_n; m) = \frac{\Pi(\mathrm{c.c})}{(4\pi)^{zd/4}} \int_0^\infty \prod_\sigma d\alpha_\sigma D^{-d/2}(\alpha) \times G(\alpha_\sigma; p_i, m_\sigma)$$
$$\exp\left\{iQ(\alpha; p_1, \ldots, p_n)/D(\alpha) - i \sum_\sigma \alpha_\sigma (m_\sigma^2 - i\varepsilon)\right\}, \qquad (\mathrm{A.2})$$

where d is the number of space–time dimensions, $p_1,\ldots,p_n$ are the momenta corresponding to the external lines of the diagram, Π(c.c.) is the product of coupling constants, z is the number of loops of the diagram, and D, Q, G are functions that are uniquely determined by the structure of the diagram. In particular, $D(\alpha)$ is a definite sum of products of α parameters, and $Q(\alpha, \{p\})$ has the structure

$$Q(\alpha, \{p\}) = \sum_{Y_j} \alpha_{j_1} \ldots \alpha_{j_{z+1}} (p_{k_1} + \ldots p_{k_l})^2, \qquad (\mathrm{A.3})$$

where $\{pk_i\}$ is a certain set of external momenta, and $Y_j = \{\sigma_{j_l}\}$ is the corresponding set of internal lines (for more details, see Refs. 23 and 75).

In a situation in which some of the momentum invariants $(p_i p_k) = \omega_{ik} Q^2$ are much greater than the remainder $(p_{i'}, p_{k'}) = \nu_{i'j'} p^2$, the exponential factor in (A.2) can be represented in the form

$$\exp\left\{i\left[Q^2 \frac{A(\alpha, \omega)}{D(\alpha)} + p^2 \frac{A_{p^2}(\alpha, \nu)}{D(\alpha)} - \sum_\sigma \alpha_\sigma (m_\sigma^2 - i\varepsilon)\right]\right\}. \qquad (\mathrm{A.4})$$

The representation (A.2), (A.4) is very convenient for analysis of the limit $Q^2 \to \infty$. In particular, it follows from (A.2) that the region in which $A(\alpha, \omega) > \rho$ makes in the limit $Q^2 \to \infty$ an exponentially small contribution $O(\exp(-Q^2\rho))$. Therefore, all the contributions that have a power-law behavior with respect to Q^2 are due to integration over regions within which $A(\alpha)/D(\alpha)$ vanishes at a certain point.

There exist three main possibilities for making the function $A(\alpha, \{\omega\})/D(\alpha)$ vanish:.

1) The small-distance regime (SDR) (or the regime of small α), when the α parameters $\alpha_{\sigma_1},\ldots,\alpha_{\sigma_n}$ vanish simultaneously for a certain set of lines $\{\sigma\}$. In this case $A(\alpha) \to 0$ faster than $D(\alpha) \to 0$.

2) The infrared regime (IR) (or the regime $\alpha \to \infty$), when the α parameters $\alpha_{\sigma_1}, ..., \alpha_{\sigma_n}$ are infinite for a certain set $\{\sigma\}$. In this case $D(\alpha) \to \infty$ faster than $A(\alpha) \to \infty$.

3) The pinch regime, when $A(\alpha, \{\omega\})$ vanishes for nonvanishing finite α because A is the difference of two positive quantities. For the three-point function $T(q^2 = -Q^2, p_1^2, p_2^2)$ this regime is impossible.

Different combinations of the basic regimes are also allowed.

The simultaneous vanishing of a set of α parameters $\{\alpha_{\tau_1}, ..., \alpha_{\tau_k}\}$ can be conveniently described by means of the substitution

$$\lambda(V) = \sum_{j=1}^{k} \alpha_{\tau_j}; \quad \alpha_{\tau_j} = \lambda(V)\beta_{\tau_j}, \tag{A.5}$$

and the simultaneous tending of $\{\alpha_{\sigma_1}, ..., \alpha_{\sigma_L}\}$ to infinity by means of the substitution

$$\varkappa(S) = \sum_j \frac{1}{\alpha_{\sigma_j}}. \tag{A.6}$$

In the momentum representation the small-distance regime corresponds to integration over the region $k \sim Q$, the infrared regime to integration over the region $k \sim p^2/Q$, and the pinch regime to integration over the region $k \sim p$. Therefore, QCD perturbation theory can be applied only when the infrared and pinch regimes, and also the combined regimes, either do not contribute at all or have contributions which can be ignored in comparison with the SDR contribution.

In the case when the pinch regime does not work, a general analysis of the configurations responsible for a particular contribution can be made without recourse to explicit analysis of the structure of the function $A(\alpha)/D(\alpha)$ in the α representation. Instead of this it is sufficient to use the rule formulated below, which can be readily understood on the basis of the well-known analogy[76] between Feynman diagrams and electrical circuits. In this analogy the parameters α_σ are interpreted as the resistances of the corresponding lines σ. Since in accordance with (A.2) and (A.4) the amplitude T ceases to depend on Q^2 for $A/D = 0$, "it is necessary to find subgraphs V and S whose contraction to a point ($\alpha_{\sigma_1} = 0$) and/or whose elimination from the diagram ($\alpha_{\sigma_S} = \infty$) rids the amplitude of the dependence on Q^2."[23] To each such configuration there corresponds a certain power-law contribution $O(Q^{-N})$. By means of the rules $k_{\rm SDR} \sim Q$ and $k_{\rm IR} \sim p^2/Q$, we can obtain the following estimates of N:

$$T_V^{\rm SDR} \sim Q^{4-\sum_V t_i}; \tag{A.7}$$

$$T_S^{\rm IR} \lesssim Q^{-\sum_S t_j}; \tag{A.8}$$

$$T_{V,S}^{\rm SDR,IR} \sim Q^{4-\sum_V t_i - \sum_S t_j}. \tag{A.9}$$

where t_i (t_j) is the twist of the i-th (j-th) external line of the subgraph V (S) corresponding to integration over for the region $\{\alpha\} \to 0$ ($\{\alpha\} \to \infty$).

For the quark fields ψ, $\bar\psi$ and the gluon field $G_{\mu\nu}$ we have $t = 1$, whereas for the vector potential A_μ the twist is zero. Therefore, in QCD it is in general necessary to sum over the external gluon lines of the subgraphs V and S. Physically, this corresponds to the fact that the hard (parton) subprocess takes place not in a vacuum but in the gluon field of the hadrons. With respect to this subprocess this field can obviously be regarded as an external field.

If we write the quark propagator in the external gluon field A_μ in the form

$$\mathscr{S}^c(x, y; A) = \hat{E}(x, y; A)\{S^c(x-y) + \hat{R}(x, y; G)\}, \tag{A.10}$$

where E is the P exponential

$$\hat{E}(x, y; A) = P\exp\left\{ig \int_y^x \hat{A}_\mu(z)\, dz^\mu\right\}, \tag{A.11}$$

taken along the straight contour $z = y + t(x-y)$, then the dependence on the field A_μ (which has zero twist) is separated into the E factor, since the function R depends, as is readily seen (see Refs. 17, 23, and 43), only on the gluon field $G_{\mu\nu}$ and its covariant derivatives. Taking into account similarly the gluon insertions into the gluon lines and the lines of the fictitious Fadeev–Popov particles, one can show that for the contribution of any hard sub-block there is always a factorization of the A_μ dependence into corresponding $\hat{E}$ factors,[17,23,43] the role of which ultimately reduces to replacement of the ordinary derivatives ∂_μ in the local operators of the form $(\bar\psi ... \partial_\mu ... \psi)$ by the covariant derivatives $D_\mu = \partial_\mu - ig\hat{A}_\mu$.

For analysis of the contributions logarithm with respect to Q^2, it is very convenient to write the amplitude in the form of the Mellin integral

$$T(Q^2, p^2) = \frac{1}{2\pi i}\int_{\delta - i\infty}^{\delta + i\infty} \Gamma(-J)\left(\frac{Q^2}{p^2}\right)^J \Phi(J)\, dJ. \tag{A.12}$$

In this form the asymptotic behavior of the amplitude $T(Q^2, p^2)$ is determined by the extreme right-hand singularity of the function $\Phi(J)$ in the complex J plane. In particular, if $\Phi(J) \sim (J - J_0)^{-n}$, then $T(Q^2) \sim (Q^2)^{J_0}(\ln(Q^2/p^2))^{n-1}$. The power-law corrections to the leading contribution are due to the singularities of the function $\Phi(J)$ at $J = J_0 - 1$, $J_0 - 2$, $J_0 - 3$, and further to the left.

In the language of the Mellin representation the expression (A.7) means, for example, that the integration over the region $\lambda(V) \sim 0$ makes a pole contribution to $\Phi(J)$ at the point

$$J = J_0 = -\sum_{i \in V}(t_i/2). \tag{A.13}$$

Multiple poles $(J - J_0)^{-n}$, leading to logarithmic contributions, arise in the cases when the pole $(J - J_0)^{-1}$ can be obtained in several independent ways.

Actual examples of the use of the α representation are given in the main text.

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