Determination of nuclear spin states from measurement of the angular correlation function of emitted particles and γ rays

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A method of determining the spin states of nuclei is proposed and justified theoretically. It is based on measurement of the angular correlation function of particles and γ rays emitted by the final nucleus in an excited state with spin J_f on transition to the ground state, the γ rays being detected in different planes relative to the reaction plane. It is shown that this method gives a unique possibility for recovering the spin density matrix in transitions with $J_f = L$ (L is the multipolarity of the γ transition). The minimal number of planes in which it is necessary to measure the angular correlation function in order to determine the density matrix fully is found. Relationships between the components of the spin tensors of the density matrix and the polarization characteristics of even–even nuclei are derived. The experimental values obtained by this method for the density-matrix spin-tensor components, the substate populations, and the orientations of the tensor operators are compared with the theoretical spin characteristics calculated in the framework of the distorted-wave Born approximation with a finite-range interaction.

INTRODUCTION

The study of nuclear structure by means of nuclear reactions has yielded much information about the states of nuclei. In recent years, the experimental methods having being significantly improved, in particular through the increase in the intensity of beams of polarized particles, much attention has been devoted to the study of magnetic substates, since knowledge of these substates yields new information about nuclear properties at a qualitatively higher level.

It is usually assumed that the populations of unperturbed substates are described by the Boltzmann distribution:

$$P(M) \sim \exp\left[-\frac{\mu H}{kT}\frac{M}{J}\right],$$

where μ is the magnetic moment of the nucleus, H is the nominal external field, and T is the absolute temperature. Except in very strong fields (as in ferromagnets) or at very low temperatures, the magnetic energy $\mu HM/J$ is negligibly small compared with the energy kT of the thermal motion. As a result, the exponential factor in $P(M)$ is near unity, and the population of the substates with different M is nearly uniform. In the case of a nuclear reaction for which the angular distribution of the reaction products is anisotropic, the substate population of the final nucleus is not uniform, i.e., it is partly or completely aligned or polarized. Such a state of the final nucleus will also decay anisotropically, and the angular dependences of the cross sections for the emission of the reaction products, and also their polarization characteristics, will be determined by the substate populations of the final nucleus.

The traditional method for studying populations is to investigate the polarization of the reaction products. Questions related to the study of polarization phenomena in nuclear physics have been considered in detail in Ref. 1. Just as much information about the magnetic substates can in principle be obtained by studying the angular correlation function of the final particles (i.e., the probability of emission of one of the particles in the direction $\mathbf{n}_1$ when the other is emitted in the direction $\mathbf{n}_2$) even if the final nucleus formed in the reaction is not determined. In the study of such processes, the nuclear reaction itself plays a double role. On the one hand, it may be the polarizer of the final particles or nuclei, so that by investigating the magnetic substates we study this polarization. On the other hand, the reaction may be the analyzer or detector of polarization of the incident particles if it gives an angular correlation that depends on this polarization.

In an experimental study of the angular correlations, one can detect coincidences in a definite angular interval of either cascade γ rays emitted by the final nucleus on the transition to the ground state (γ–γ correlations), or of the final particles that are the reaction products (particle–particle correlations), or of the final particles and the γ rays (particle–γ-ray correlations). The theoretical formalism for the analysis of all types of angular correlations is practically the same apart from some small modifications in the detection efficiency tensor of the reaction products. This theoretical formalism for analyzing the angular correlations and polarizations in nuclear reactions has been presented in a review by Goldfarb.[2]

The fullest investigation has been of the γ–γ correlations, the experimental study of which has led to the determination of the properties of many excited nuclear states (spins, parities, probabilities of electromagnetic transitions, etc.). The theory of γ–γ correlations in cascade processes has been reviewed by Biedenharn and Rose[3] and Frauenfelder.[4] There have been some studies[5,6] of the distortion of the γ–γ correlations by intranuclear perturbing fields. These distortions make it possible to obtain additional information about the electric moments of the excited states of the nuclei.

In principle, study of the angular correlations between the products of nuclear reactions gives much more information about the properties of the excited nuclear states than can be extracted from γ–γ correlations. The reason for this is that even at moderate energies the long-wave approximation is not valid in nuclear reactions, i.e., the transition matrix element is, in contrast to the case of γ–γ correlations, sensitive to both the state structure of the final nucleus and the reaction mechanism.

Particle–particle correlations have been studied experi-

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mentally in detail by Ogloblin *et al.*[7] These investigations made it possible to establish with confidence the existence of cluster states in nuclei at sufficiently high excitation energies.

Many studies, both theoretical and experimental, have been devoted to the angular correlations between the particle reaction products and the γ rays emitted by the final nucleus on the transition from its excited state to the ground state. The experimental study of these correlations has mainly developed in a number of directions: correlation experiments with protons, neutrons, and deuterons of relatively low energies (up to 10 MeV),[8-15] with protons produced in (d,p) stripping reactions,[16,17] and through the experimental study of the angular correlation function and the populations of the magnetic substates in reactions in which unpolarized particles of relatively high energies (tens of mega-electron-volts) participate.[18-32] The theoretical analysis of such experiments, in particular, of the substate populations, is of great interest, since *a priori* such quantities must be more sensitive to the reaction mechanism and the structure of the nuclear state than the differential cross section.

In the majority of theoretical analyses of particle–γ-ray correlation experiments the reaction matrix element is calculated in the compound-nucleus model. There is a beautiful review of Sheldon[33] in which a number of general theorems are formulated in the framework of this model about the substate populations for inelastic scattering of unpolarized particles with spin $\frac{1}{2}$ (for example, in spin-flip reactions) and spinless particles by even–even nuclei with excitation of a 2^+ level. In Ref. 12, this model was used to calculate the parameters of the angular correlation function in the $^{12}C(\alpha, \alpha'\gamma)^{12}C$ reaction. In recent theoretical studies[34] made at the Leningrad Institute of Nuclear Physics in the framework of the compound-nucleus model the substate populations of the 2^+ level of the ^{12}C nucleus formed in inelastic scattering of polarized neutrons by carbon nuclei have been calculated.

A significant number of the particle–γ-ray correlation experiments[18-32] have been made for a fairly high energy of the incident particles, so that for their interpretation it is necessary to use the model of direct nuclear interactions when calculating the reaction matrix element. Unfortunately, there have been no reviews of the analysis of the angular correlations of the products of direct nuclear reactions. In some papers, the correlation experiments in direct reactions have been analyzed theoretically in the framework of the coupled-channel method. In Ref. 26, it was used to analyze the parameters of the α–γ angular correlation function in the $^{12}C(\alpha,\alpha'\gamma)^{12}C$ reaction at $E_\alpha = 42$ MeV. In Ref. 35, a group of physicists at the University of Leningrad used the method to calculate the substate populations of the 2^+ level of the ^{12}C nucleus found experimentally in Refs. 34 and 35. In Ref. 36, the method was used to calculate the p–γ correlation function in elastic scattering of $E_p = 13$-MeV protons by ^{112}Ga. Many papers have been devoted to the theoretical analysis of angular correlations in direct reactions in the framework of the standard distorted-wave Born approximation (DWBA) with a zero-range interaction. The p–γ angular correlations in (d,p) stripping reactions were discussed in the DWBA in Ref. 37. In Ref. 38 the DWBA with a finite-range interaction was used to analyze p–γ correlations in the $^{12}C(p,p'\gamma)^{12}C$ reaction; in Ref. 39, a detailed analysis for protons of higher energies was made in the impulse approximation. In Refs. 40 and 41, the parameters of the angular correlation function were considered for $(\alpha, y'\gamma)$ reactions in the plane-wave approximation for the cluster stripping, replacement, and heavy-particle stripping mechanisms, and it was shown that such mechanisms cannot explain all the behavior of the parameters of the angular correlation function, for example, the angle of its symmetry as a function of the emission angle of the final particles y. To explain such features, it was proposed in Refs. 29 and 42 to use reaction mechanisms that take into account the retardation in the interaction, in particular, the simplest of them, illustrated by the box diagram.

However, not one of the listed studies of the angular correlation function has investigated the question of the number of parameters that completely determines the spin properties of the excited nuclear state and whether these parameters can in principle be found by studying the final-particle–γ-ray angular correlation function. The present review, which is based on the studies[43-50] made in recent years at the Institute of Nuclear Physics at Moscow State University, is devoted to study of the angular correlation function from precisely this point of view. Specifically, the following questions are examined.

1. The determination of the density matrix of the excited state of the final nucleus formed in a nuclear reaction $A(x, y)B^*$ and of the number of independent parameters that characterize the density matrix for particular types of reaction.

2. The determination of the angular correlation function, its parametrization, and the connection between the parameters of the density matrix and the angular correlation function.

3. The recovery of the density matrix of a given nuclear state by measurement of the angular correlation function in different planes of γ-ray detection relative to the reaction plane.

4. The determination of the polarization characteristics of excited nuclear states and of the orientations of the various tensor operators.

5. Theoretical calculations of the density matrix and of the polarization characteristics of excited nuclear states in direct reactions by the DWBA with a finite-range interaction.

1. SPIN DENSITY MATRIX AND ITS PARAMETRIZATION FOR DIFFERENT SYSTEMS

The density-matrix concept was first introduced in theoretical physics[51] to describe statistical concepts in quantum mechanics, since the density-matrix method makes it possible to consider all quantum-mechanical states, both fully and incompletely determined. The general theory of the density matrix and its application have been studied by many authors. Here, we shall not consider all aspects of the density-matrix concept, referring the interested readers to Blum's monograph,[52] which contains a detailed analysis of the density-matrix formalism. Moreover, we shall limit ourselves here to the density matrix of specific systems, omitting the general quantum-mechanical definitions of the density matrix.

We formulate here only those definitions that will be important for us in what follows.[52-54]

1. The density matrix $\rho(M, M')$ is a positive-definite Hermitian matrix,

$$\rho(M, M')^* = \rho(M', M), \quad (1)$$

and it satisfies the normalization condition

$$\mathrm{Sp}\,\rho(M, M') = \mathrm{const}. \quad (2)$$

2. The average value of any physical quantity $\mathbf{T}$ in the state described by the density matrix $\rho(M, M')$ is given by

$$T_{\mathrm{av}} = \mathrm{Sp}\,(\rho\mathbf{T})/\mathrm{Sp}\,\rho. \quad (3)$$

3. The positive definiteness of the operator $\rho(M, M')$ means that all the eigenvalues $\rho(M, M)$ of the density matrix are non-negative; at the same time,

$$\sum_M \rho(M, M) = \mathrm{Sp}\,\rho. \quad (4)$$

We shall consider processes of the type

$$x + A \to y + B^* \to y + B + \gamma \quad (5)$$

and study the angular correlation function $W(\mathbf{\Omega}_y, \mathbf{\Omega}_\gamma)$ of the final particles y and γ rays emitted by the final nucleus B^* on the transition from the state with spin J_f to the ground state with spin J_0. Even if the initial system had no orientation (polarization of all initial particles equal to zero), after emission of particle y the nucleus B^* is oriented relative to the direction of emission $\mathbf{\Omega}_y$ of the final particles. As a result, the angular distribution of the γ rays, and with it $W(\mathbf{\Omega}_y, \mathbf{\Omega}_\gamma)$, is anisotropic with respect to $\mathbf{\Omega}_y$.

In investigating the angular correlation function, we detect coincidences of only the final particles and the γ rays. In other words, only some of the large number of degrees of freedom that characterize the system as a whole are of interest to us. In the general case,[52] it is impossible to construct a wave function that depends on the variables of only the particles that are detected experimentally (i.e., a wave function that does not depend on the variables of the remaining particles). To describe such systems, it is necessary to use the density-matrix formalism.

In accordance with the general theory,[52–54] the spin density matrix $\rho_{J_f}(M_f, M'_f)$ of the state of the nucleus B^* with spin J_f is determined by

$$\rho_{J_f}(M_f, M'_f) = \sum_{\substack{M_x, M'_x, M_A, M'_A \\ M_y, M'_y}} \rho_{J_x}(M_x, M'_x) \times \rho_{J_A}(M_A, M'_A)\, \mathbf{T}^{J_xJ_AJ_yJ_f}_{ij, M_xM_AM_yM_f}(\mathbf{\Omega}_y)\, \mathbf{T}^{*J_xJ_AJ_yJ_f}_{ij, M'_xM'_AM'_yM'_f}(\mathbf{\Omega}_y), \quad (6)$$

where $\rho_{J_x}(M_x, M'_x)$ and $\rho_{J_A}(M_A, M'_A)$ are the spin density matrices of the incident particles and the initial nucleus, respectively, and $\mathbf{T}^{J_xJ_AJ_yJ_f}_{ifM_xM_AM_yM_f}(\mathbf{\Omega}_\gamma)$ is the matrix element of the transition in the reaction $A(x, y)B^*$. Sometimes, the matrix element of the transition to the state $|J_fM_f|$ will be denoted simply by $\mathbf{T}_{M_f}(\mathbf{\Omega}_y)$, and the indices $J_x, J_A, J_y, J_f, M_x, M_A, M_y$ will be omitted.

If the spin density matrices of the initial particles and the initial nucleus satisfy the normalization condition

$$\mathrm{Sp}\,\rho_{J_j}(M_x, M'_x) = 1;\ \mathrm{Sp}\,\rho_{J_A}(M_A, M'_A) = 1, \quad (7)$$

then the density matrix (6) is normalized by the differential reaction cross section:

$$\mathrm{Sp}\,\rho_{J_f}(M_f, M'_f) = d\sigma/d\Omega. \quad (8)$$

The spin density matrices of the initial states are determined by their polarization state. In the case of unpolarized incident particles and targets,

$$\rho_{J_x}(M_x, M'_x) = \frac{\delta_{M_xM'_x}}{2J_x+1};\ \rho_{J_A}(M_A, M'_A) = \frac{\delta_{M_AM'_A}}{2J_A+1}. \quad (9)$$

If the initial system has polarization properties, the corresponding density matrix is expressed in terms of the polarization tensors of the system. In particular, if the spin of the particle (or initial nucleus) is $\frac{1}{2}$, then

$$\rho = 1/2\,(\hat{\mathbf{I}} + \mathbf{P}\hat{\sigma}), \quad (10)$$

where $\hat{\sigma}$ are the Pauli spin matrices, and P characterizes the degree of polarization of the particles, this satisfying, in accordance with the normalization condition (7),

$$0 \leqslant P \leqslant 1, \quad (11)$$

where for $P = 1$ the state is completely polarized, while for $P = 0$ there is no polarization.

We consider how many independent parameters characterize the density matrix (6) of the nucleus in the state with spin J_f. It is obvious that $\rho_{J_f}(M_f, M'_f; \mathbf{\Omega}_y)$ is a square matrix of dimension $(2J_f + 1)$ with total number of complex elements $(2J_f + 1)^2$. It follows from the definition (3) that the diagonal elements of the density matrix are real and that the real and imaginary parts of the nondiagonal elements are related by the symmetry

$$\mathrm{Re}\,\rho_{J_f}(M_f, M'_f) = \mathrm{Re}\,\rho_{J_f}(M'_f, M_f); \quad (11a)$$

$$\mathrm{Im}\,\rho_{J_f}(M_f, M'_f) = -\mathrm{Im}\,\rho_{J_f}(M'_f, M_f). \quad (11b)$$

As a result, the number of real parameters that characterize the density matrix in the state with spin J_f is determined in the most general case by the expression $(2J_f + 1)^2$.[1)] However, in real physical systems possessing symmetry to some degree not all these parameters are independent.

To clarify the connection between the independent parameters of the density matrix and the symmetry properties of the system, it is convenient to consider, not the density matrix itself, but its spin tensors $\rho_{k\varkappa}(\mathbf{\Omega}_y)$, the coefficients of the decomposition of the density matrix $\rho_{J_f}(M_f, M'_f; \mathbf{\Omega}_y)$ with respect to the complete set of irreducible tensor operators $\mathbf{T}_{k\varkappa}(J_f)$ of rank k $(0 \leqslant k \leqslant 2J_f)$:

$$\rho_{J_f}(M_f, M'_f; \mathbf{\Omega}_y) = \sum_{k=0}^{2J_f} \sum_{\varkappa=-k}^{k} \rho_{k\varkappa}(\mathbf{\Omega}_y) \langle M_f \mid \hat{\mathbf{T}}_{k\varkappa}(J_f) \mid M'_f \rangle. \quad (12)$$

In accordance with the general definitions (see, for example, Refs. 1, 52, and 54), the matrix element of the irreducible tensor operator $\hat{\mathbf{T}}_{k\varkappa}(J_f)$ in the state with spin J_f is given by

$$\langle M_f \mid \mathbf{T}_{k\varkappa}(J_f) \mid M'_f \rangle = \frac{1}{\sqrt{2J_f+1}}(-1)^{J_f-M_f} \times \langle J_fM_fJ_f - M'_f \mid k\varkappa \rangle. \quad (13)$$

Substituting the definition (13) in (12), we obtain

$$\rho_{J_f}(M_f, M'_f; \mathbf{\Omega}_y) = \frac{1}{\sqrt{2J_f+1}} \sum_{k\varkappa} (-1)^{J_f-M_f}$$

$$\times \langle J_f M_f J_f - M'_f \mid k\varkappa \rangle \rho_{k\varkappa}(\Omega_y). \tag{14}$$

We multiply Eq. (14) by $(-1)^{J_f - M_f}\langle J_f\, M_f\, J_f - M'_f | k\varkappa\rangle$ and sum over M_f, M'_f. As a result, we obtain for the spin tensors the definition

$$\rho_{k\varkappa}(\Omega_y) = \sqrt{2J_f+1} \sum_{M_f M'_f} (-1)^{J_f - M_f} \langle J_f M_f J_f - M'_f \mid k\varkappa \rangle \rho_{J_f}(M_f,\, M'_f,\, \Omega_y), \tag{15}$$

$$\mathrm{Sp}\ \rho_{J_f}(M_f,\, M'_f) = \rho_{00}(\Omega_y). \tag{15a}$$

The spin tensors of odd rank ($k = 1, 3,...$) are usually called polarization tensors, and those of even rank ($k = 0, 2, 4,...$) are called alignment tensors.

Because the density matrix (1) is Hermitian,

$$\rho^*_{k\varkappa}(\Omega_y) = (-1)^\varkappa \rho_{k-\varkappa}(\Omega_y), \tag{16}$$

and this means that in the most general case, i.e., in the case of nuclear reactions with the participation of polarized particles and with detection of the polarization of the final particles, the components $\rho_{k\varkappa}(\Omega_y)$ with $\varkappa = 0$ are real, while the real and imaginary parts of the $\rho_{k\varkappa}(\Omega_y)$ with $\varkappa \neq 0$ are related by [cf. (11)]

$$\mathrm{Re}\, \rho_{k\varkappa}(\Omega_y) = (-1)^\varkappa\, \mathrm{Re}\, \rho_{k-\varkappa}(\Omega_y); \tag{17a}$$

$$\mathrm{Im}\, \rho_{k\varkappa}(\Omega_y) = -(-1)^\varkappa\, \mathrm{Im}\, \rho_{k-\varkappa}(\Omega_y). \tag{17b}$$

As a result, for each k the number of *real* parameters that determine the spin tensors $\rho_{k\varkappa}(\Omega_y)$ is $2k+1$. The total number of parameters that characterize the spin tensors $\rho_{k\varkappa}(\Omega_y)$ with all possible k is obviously given by the sum

$$N_\rho = \sum_{k=0,\,1,\,2}^{2J_f} (2k+1) = (2J_f+1)^2. \tag{18}$$

If the system of interacting particles is not in an external field, it is invariant with respect to time reversal (unpolarized particles and unpolarized targets participate in the reaction). In this case, the number of parameters that characterize the density matrix or its spin tensors can be determined from the following considerations.[36] If the quantization axis z is directed along the beam of the incident particles, the matrix element of a reaction with unpolarized initial particles satisfies the condition

$$\mathbf{T}^{J_x J_A J_y J_f}_{if,\, -M_x - M_A - M_y - M_f}(\vartheta_y) = (-1)^{J_A - J_f + J_x - J_y} \times (-1)^{M_A - M_f + M_x - M_y} \mathbf{T}^{J_x J_A J_y J_f}_{if,\, M_x, M_A, M_y, M_f}(\vartheta_y). \tag{19}$$

The relation (19) obviously leads to the following restriction on $\rho_{k\varkappa}(\vartheta_y)$:

$$\rho_{k\varkappa}(\vartheta_y) = (-1)^k \rho^*_{k\varkappa}(\vartheta_y) \tag{20}$$

(the angle ϑ_y is measured in the reaction plane from the z axis).

The condition (20) means that in reactions with the participation of unpolarized particles when the quantization axis z coincides with the direction of the momentum of the incident particles the spin tensors of the density matrix with even k are real, and those with odd k are purely imaginary. As a result, for each k the number of parameters of the spin tensor is $k+1$, and the total number is

$$N_\rho = \sum_{k=1,\,2,\,3\,\ldots}^{2J_f} (k+1) = (2J_f+1)(J_f+1). \tag{21}$$

If the quantization axis z for such reactions is perpendicular to the reaction plane, the change in the spin projection of the nucleus, $M_f - M'_f = \varkappa$, must have parity equal to k (Bohr's theorem[55]). The number of complex components of $\rho_{k\varkappa}(\Omega_y)$ with such projections for each k is $k+1$. Taking into account the relation (11), and also the fact that the component $\rho_{k0}(\Omega_y)$ is always real, we can readily show that the number of real parameters that determine $\rho_{k\varkappa}(\Omega_y)$ with given k is again $k+1$, i.e., does not depend on the choice of the quantization axis. Hence, the total number of real parameters of the density matrix is again given by the expression (21).

If in reactions with unpolarized initial particles the polarization of the final particles is not detected, the system has a symmetry plane and the spin tensors $\rho_{k\varkappa}(\Omega_y)$ must be invariant with respect to reflection in this plane. Such a requirement leads to the condition[54]

$$\rho_{k\varkappa}(\Omega_y) = (-1)^k \rho_{k\varkappa}(\Omega_y), \tag{22}$$

i.e., in such systems only the $\rho_{k\varkappa}(\Omega_y)$ with even k are nonzero. Note that in coordinate systems with the z axis along the incident beam for reactions with unpolarized particles and without detection of the polarization of the final particles the $\rho_{k\varkappa}(\Omega_y)$ with even k are real and their projections $\varkappa$ take all values from $-k$ to k. In coordinate systems with the z axis perpendicular to the reaction plane, the $\rho_{k\varkappa}(\Omega_y)$ with even k are complex for such reactions, and the projection $\varkappa$ can take only even values. As a result, the total number of parameters that characterize the density matrix or its spin tensors is determined for such reactions by the following rules:

$$\text{Integral } J_f\text{:}\ N_\rho = \sum_{k=0,\,2,\,4\,\ldots}^{2J_f} (k+1) = (J_f+1)^2. \tag{23}$$

$$\text{Half-integral } J_f\text{:}\ N_\rho = \sum_{k=0,\,2,\,4\,\ldots}^{2J_f - 1} (k+1) = (J_f+1/2)^2. \tag{24}$$

In Table I, we give the number of real parameters N_ρ that determine the density matrix of the nucleus (or its spin tensors) in the state with spin J_f as a function of the methods of formation and detection of this state. It can be seen that N_ρ increases sharply with increasing spin J_f. This means that for nuclear states with large spins the measurement of just one quantity—the differential cross section—gives a small amount of the information about the properties of this state that can in principle be obtained, so that further experiments are needed for complete recovery of the density matrix and, thus, the spin characteristics of the nucleus. It is important to note that the experimental measurement of the populations of the magnetic substates also does not enable one to obtain a complete set of information about the density matrix of the excited state. Such measurements [with allowance for the symmetry relations (11) and (17)] enable one to find the diagonal elements of the density matrix (total number J_f+1), which are further related by the normalization condition (8). As follows from Table I, it is only for spin $s = \frac{1}{2}$ that a polarization experiment of this kind permits

TABLE I. Number of real parameters N_ρ of the density matrix of a nuclear state with spin J_f as a function of J_f and the methods of formation of this state.[50]

J_f	1/2	1	3/2	2	5/2	3	7/2
Reactions with polarized particles with detection of polarization	4	9	16	25	36	49	64
Reactions with unpolarized particles with detection of polarization	3	5	10	15	21	28	36
Reactions with unpolarized particles without detection of polarization	1	4	4	9	9	16	16

complete determination of the density matrix. For spins $s \geqslant 1$, it is necessary to seek other experimental methods in order to find the nondiagonal elements of the density matrix or the corresponding spin tensors. We shall show below that measurement of the particle–γ-ray angular correlation function gives, in a number of important special cases, a unique possibility for finding all parameters of the density matrix of the excited state of the final nucleus, so that one can determine its various spin characteristics, including the populations of the magnetic substates.

2. DETERMINATION OF THE ANGULAR CORRELATION FUNCTION OF THE FINAL PARTICLES AND γ RAYS. THE γ-RAY DETECTION EFFICIENCY TENSOR

To determine the angular correlation function $Q(\boldsymbol{\Omega}_y, \boldsymbol{\Omega}_\gamma)$, we must find the probability that the final system $B(J_f) \to B(J_0) + \gamma$ will be detected by an ideal instrument characterized by efficiency matrix $\boldsymbol{\varepsilon}$. In accordance with the general definitions (2),

$$W(\Omega_y, \Omega_\gamma) = \mathrm{Sp}\,(\rho_{J_f}\varepsilon). \qquad (25)$$

The definition of the ideal detector contains the following requirements:

1. The sensitivity of the detector to one of the properties of the detected particles (energy, momentum, spin, etc.) does not depend on its sensitivity to another property.

2. The detector efficiency does not depend on the energy of the particles.

3. The detector detects particles moving in a strictly defined direction.[2)]

We introduce the detector spin efficiency tensors[3)]:

$$\varepsilon_{k\varkappa}^{J_f}(\Omega_\gamma, \Omega_0) = \frac{1}{\sqrt{2J_f+1}} \sum_{M_f M_f'} (-1)^{J_f - M_f'} \times \langle J_f M_f J_f - M_f' \mid k\varkappa\rangle\, \varepsilon_{J_f}(M_f, M_f'). \qquad (26)$$

Then, obviously, the relation (25) can be rewritten in the form (Refs. 2, 36, 52, and 54)

$$W(\Omega_y, \Omega_\gamma) = \sum_{k\varkappa} \rho_{k\varkappa}(\Omega_y)\, \varepsilon_{k\varkappa}^{J_f}(\Omega_\gamma, \Omega_0). \qquad (27)$$

The expressions (25) and (27) can be interpreted as the probability that the system with density matrix $\boldsymbol{\rho}_{J_f}(M_f, M_f'; \boldsymbol{\Omega}_y)$ will be found by a detector that detects the final particles independently of their energies and spins in the directions $\boldsymbol{\Omega}_y$, $\boldsymbol{\Omega}_0$, $\boldsymbol{\Omega}_\gamma$. In other words, (25) and (27) give the most general definition of the angular correlation function of the final particles.

We obtain expressions for the efficiency tensors $\varepsilon_{k\varkappa}^{J_f}(\boldsymbol{\Omega}_\gamma, \boldsymbol{\Omega}_0)$. By property 1 of ideal detectors, the efficiency matrix $\boldsymbol{\varepsilon}_{J_f}(M_f, M_f')$ must factorize into parts associated with the matrices $\boldsymbol{\varepsilon}_{k_0\varkappa_0}^{J_0J_0'}(\boldsymbol{\Omega}_0)$ representing the efficiency of detection of the final nucleus and $\boldsymbol{\varepsilon}_{k_\gamma\varkappa_\gamma}^{LL'}(\boldsymbol{\Omega}_\gamma;\lambda p,p')$ of the γ-ray detection efficiency. Here, J_0 (M_0) is the spin (its projection) of the final nucleus in the ground state, L is the multipolarity of the emission of the γ ray, λ is its helicity, and the quantum number p distinguishes the electric and magnetic transitions:

$$\left.\begin{array}{l} p=0 \text{ for } E\lambda\text{-transitions;} \\ p=1 \text{ for } M\lambda\text{-transitions.} \end{array}\right\} \qquad (28)$$

Such factorization can be done by going over from the representation $|J_0, L(\lambda\, p, p'); J_f M_f\rangle$, in which the tensors $\boldsymbol{\varepsilon}_{k\varkappa}^{J_f}(\boldsymbol{\Omega}_\gamma,\boldsymbol{\Omega}_0)$ are defined, to the representation $|J_0M_0\rangle$, $|L\lambda\, p, p'\rangle$, in which $\boldsymbol{\varepsilon}_{k_0\varkappa_0}^{J_0J_0'}(\boldsymbol{\Omega}_0)$ and $\boldsymbol{\varepsilon}_{k_\gamma\varkappa_\gamma}^{LL'}(\boldsymbol{\Omega}_\gamma;\lambda p,p')$ are defined.

In other words, we must go over from the angular-momentum coupling scheme

$$(J_0 + L = J_f) + (J_0' + L' = J_f) = \mathbf{k}$$

to the other scheme

$$(J_0 + J_0' = \mathbf{k}_0) + (L + L' = \mathbf{k}_\gamma) = \mathbf{k}.$$

Such a transition is realized by means of the normalized $9j$ symbols. As a result, we obtain for the efficiency tensors[2,54]

$$\varepsilon_{k\varkappa}^{J_f}(\Omega_\gamma, \Omega_0) = \sum_{k_0\varkappa_0 k_\gamma \varkappa_\gamma} \langle k_0\varkappa_0 k_\gamma\varkappa_\gamma \mid k\varkappa\rangle \times \begin{pmatrix} J_0 & J_0' & k_0 \\ L & L' & k_\gamma \\ J_f & J_f & k \end{pmatrix} \varepsilon_{k_0\varkappa_0}^{J_0J_0'}(\Omega_0)\, \varepsilon_{k_\gamma\varkappa_\gamma}^{LL'}(\Omega_\gamma, \lambda p, p'). \qquad (29)$$

Since we do not observe the final nucleus, the efficiency tensor $\boldsymbol{\varepsilon}_{k_0\varkappa_0}^{J_0J_0'}(\boldsymbol{\Omega}_0)$ can be represented in the form

$$\varepsilon_{k_0\varkappa_0}^{J_0J_0'}(\Omega_0) = \sqrt{2J_0+1}\, \delta_{k_00}\delta_{\varkappa_00}\delta_{J_0J_0'}. \qquad (30)$$

By virtue of (30), the expression (29) simplifies to

$$\varepsilon_{k\varkappa}^{J_f}(\Omega_\gamma) = \sum_{LL'} \varepsilon_{k\varkappa}^{LL'}(\Omega_\gamma, \lambda p, p')\, w(J_f k J_0 L : J_f L')\sqrt{(2J_f+1)}. \qquad (31)$$

We find expressions for the γ-ray detection efficiency tensor $\boldsymbol{\varepsilon}_{k\varkappa}^{LL'}(\boldsymbol{\Omega}_\gamma,\lambda p,p')$ in terms of the characteristics L, λ, p of the photon emission. We first of all note that eigenvectors of the γ ray are states $|L\lambda\, p\rangle$ with definite projection λ onto the axis $\mathbf{p}_\gamma$. The orientation of this axis in the coordinate system of the detector is given by the angle $\mathbf{\Omega}_\gamma$. Therefore

$$\varepsilon_{k\varkappa}^{LL'}(\mathbf{\Omega}_\gamma, \lambda p, p') = \sum_\nu \mathbf{D}_{\varkappa\nu}^k(\mathbf{\Omega}_\gamma) C_{k\nu}(pL, p'L'), \quad (32)$$

where the quantities $C_{k\nu}(pL, p'L')$ are the radiation parameters of the photon detection efficiency tensor in the self-frame of the emitter. The photon radiation parameters can be expressed[56] in terms of the coefficients

$$\langle L\lambda p \mid 0\lambda\rangle = \lambda^p \sqrt{\frac{2L+1}{8\pi}} \quad (33)$$

in the expansion of the photon plane wave $|0\lambda\rangle$ with respect to multipoles L with helicity λ. The corresponding expression has the form[56]

$$C_{k\nu}(pL, p'L') = \sum_{\lambda\lambda'=\pm1} (-1)^{L'-\lambda'} \langle\lambda' \mid \rho_\gamma \mid \lambda\rangle \times \langle L\lambda p \mid 0\lambda\rangle \langle 0\lambda' \mid p'\lambda' L'\rangle \langle L\lambda L' - \lambda' \mid k\nu\rangle. \quad (34)$$

In (34), the matrix $\langle\lambda'|\rho_\gamma|\lambda\rangle$ is the density matrix of the photon in the helicity representation. Usually, this matrix is expressed in terms of Stokes parameters[52]:

$$\langle\lambda' \mid \rho_\gamma \mid \lambda\rangle = \frac{1}{2}\begin{pmatrix} 1+\eta_2 & -\eta_3+i\eta_1 \\ -\eta_3-i\eta_1 & 1-\eta_2 \end{pmatrix}. \quad (35)$$

The physical meaning of these parameters is as follows[52]:

1. The parameter η_3 determines the degree of linear polarization of the γ-ray beam with respect to the x and y axes:

$$\eta_3 = \frac{I(0^\circ) - I(90^\circ)}{I}. \quad (36)$$

2. The parameter η_1 specifies the degree of linear polarization with respect to two mutually perpendicular axes oriented at angle 45° to the x axis:

$$\eta_1 = \frac{I(45^\circ) - I(135^\circ)}{I}. \quad (37)$$

3. The parameter η_2 characterizes the degree of circular polarization of the γ rays:

$$\eta_2 = \frac{I(+) - I(-)}{I}. \quad (38)$$

It follows from the definitions (35)–(38) that if the beam of γ rays does not have linear polarization and is uniformly circularly polarized (there is no right–left asymmetry), then all three Stokes parameters vanish and the photon density matrix becomes the identity matrix.

We calculate explicitly the radiation parameters (34). Since the helicities are $\lambda, \lambda' = \pm 1$, two cases are possible for $C_{k\nu}(pL, p'L')$:

1) $\lambda = \lambda' = \pm 1$, $\nu = 0$; 2) $\lambda = 1$, $\lambda' = -1$ (or vice versa), $\nu = \pm 2$.

Combining the expressions (33)–(35), we can readily obtain for the photon radiation parameters the following results[54]:

1. $\nu = 0$:

$$C_{k0}(pL, p'L') = \frac{\sqrt{(2L+1)(2L'+1)}}{8\pi}(-1)^{L'-1} \times \langle L1L'-1 \mid k0\rangle \{1 + (-1)^f + \eta_2[1-(-1)^f]\}. \quad (39)$$

where $f = L + L' + p + p' + k$.

2. $\nu = \pm 2$:

$$C_{k2}(pL, p'L') = \frac{\sqrt{(2L+1)(2L'+1)}}{8\pi}(-1)^{L'+p'} \times \langle L1L'1 \mid k2\rangle \{(\eta_3 - i\eta_1) + (\eta_3 + i\eta_1)(-1)^{L+L'+p+p'-k}\}. \quad (40)$$

Substituting the expressions (39) and (40) in the definitions (31) and (32), we obtain for $\varepsilon_{k\varkappa}^{J_f}(\mathbf{\Omega}_\gamma)$

$$\varepsilon_{k\varkappa}^{J_f}(\mathbf{\Omega}_\gamma) = \frac{\sqrt{(2J_f+1)}}{2\sqrt{4\pi(2k+1)}} \sum_{LL'pp'} \sqrt{(2L+1)(2L'+1)} \times g_L(p) g_{L'}(p') w(J_f k J_0 L : J_f L') \times [(-1)^{L'+1}\langle L\,1\,L'-1 \mid k0\rangle \{1+(-1)^f + \eta_2[1-(-1)^f]\} \mathbf{Y}_{k\varkappa}^*(\mathbf{\Omega}_\gamma) + (-1)^{L'+p'}\langle L\,1\,L'\,1 \mid k2\rangle \sqrt{\frac{2k+1}{4\pi}} \mathbf{D}_{\varkappa 2}^k(\mathbf{\Omega}_\gamma) \times \{(\eta_3 - i\eta_1) + (\eta_3 + i\eta_1)(-1)^f\}], \quad (41)$$

where the quantities $g_L(p)$ are determined by the mixing coefficients of the electromagnetic transitions,[36] and $\Sigma_L g_L^2(p) = 1$.

It follows from analysis of the expression (41) that if we detect both linear and circular polarization of the γ rays, then the tensor $\varepsilon_{k\varkappa}^{J_f}(\mathbf{\Omega}_\gamma)$ is Hermitian, and the number of its parameters is given by the sum

$$N_e = \sum_{k=0,1,2}^{k_e} (2k+1) = (k_e+1)^2, \quad (42)$$

where $k_e = \max(2J_f, 2L)$.

If we detect only circular polarization of the γ rays ($\eta_1 = \eta_3 = 0$), then, as follows from (41), the tensor $\varepsilon_{k\varkappa}^{J_f}(\mathbf{\Omega}_\gamma)$ is real and is determined by the number of parameters

$$N_e = \sum_{k=0,1,2}^{k_e} (k+1) = \frac{1}{2}(k_e+1)(k_e+2). \quad (43)$$

Finally, if polarization of the γ rays is not detected at all (all $\eta_i = 0$), then, in accordance with (41), the tensor $\varepsilon_{k\varkappa}^{J_f}(\mathbf{\Omega}_\gamma)$ is nonzero only for even values of k and is determined by the number of parameters

$$N_e = \sum_{k=0,2,4}^{k_e} (k+1) = \frac{1}{4}(k_e+2)^2. \quad (44)$$

Comparing the expressions (18), (21), (23), and (24) for N_ρ with the corresponding expressions (42)–(44) for N_e, we readily see that under the condition

$$k_\rho = k_e; \quad J_f = L = L' \quad (45)$$

the spin tensor $\varepsilon_{k\varkappa}^{J_f}(\mathbf{\Omega}_\gamma)$ is characterized by the same number of parameters as the density matrix $\rho_{J_f}(M_f, M_f')$. This means that measurement of $W(\mathbf{\Omega}_y, \mathbf{\Omega}_\gamma)$ in reactions in which the condition (45) is satisfied makes it possible to find *all* real parameters of the density matrix of a nucleus in the given excited state J_f.

3. PARAMETRIZATION OF THE ANGULAR CORRELATION FUNCTION OF THE FINAL PARTICLES AND γ RAYS

In the previous section, we obtained expressions for the γ-ray detection efficiency tensor in the most general case. Combining the expressions (27) and (41), we can readily obtain the angular correlation function of the final particles and γ rays. Moreover, the analytic dependence of $W(\mathbf{\Omega}_y, \mathbf{\Omega}_\gamma)$ on either of the two angles that determine the direction of emission of the γ ray is given explicitly. This, in its turn, means that the form of the angular correlation function, i.e., its dependence on one of the angles of emission of the γ ray when the other is fixed, can be readily parametrized. At the same time, the minimal number of points at which it is necessary to measure the angular correlation function will be determined by the number of parameters of its shape. As a result, the time of measurement of the angular correlation function needed to accumulate the required statistics can be minimized and the labor of the correlation experiments accordingly reduced.

Hitherto, the polarization of γ rays has not been measured in experiments that have studied the angular correlation function, i.e., the experimental values of $W(\mathbf{\Omega}_y, \mathbf{\Omega}_\gamma)$ have related $\rho_{k\varkappa}(\mathbf{\Omega}_y)$ and $\varepsilon_{k\varkappa}(\mathbf{\Omega}_y)$ with even k, each of these spin tensors depending on $(J_f+1)^2$ parameters. As a result of such experiments, we can obtain only the orientational (and not the polarization) characteristics of the nucleus B^* in the state J_f. Nevertheless, such experiments do considerably raise the level of our knowledge of the properties of the excited nuclear states; for example, for $J_f = 2$ in a correlation experiment we can in principle obtain nine of the parameters that characterize the given state of the nucleus B^*, and for $J_f = 3$ we obtain 16. In what follows, we shall consider $W(\mathbf{\Omega}_y, \mathbf{\Omega}_\gamma)$ only for reactions without detection of the γ-ray polarization.

There have been numerous studies (Refs. 22, 28, 36, 37, 40, and 57–60) of the angular correlation function for the important special case of the scattering of spinless or spin-$\frac{1}{2}$ particles by even–even nuclei with excitation of a 2^+ level, and also for reactions with formation of an even–even nucleus in the same state. This state is de-excited by emission of a γ ray of multipolarity E2. For such reactions, the density matrix is obviously characterized by nine real parameters. However, if the γ ray is detected only in one plane relative to the reaction plane (with given azimuthal or polar angle), we are in principle unable to find all the parameters of the density matrix, irrespective of the number of points of measurement, since the form of the angular correlation function in each plane depends on only $2L+1$ parameters, where L is the multipolarity of the γ radiation. Therefore, it is necessary to solve in general form the problem of the connection between the parameters that determine the form of the angular correlation function and the parameters of the density matrix independently of model ideas about the reaction mechanism.

Using the expressions (27) and (41) and restricting the treatment to reactions with unpolarized particles when the condition (45) is satisfied, we obtain for the particle–γ-ray angular correlation function

$$W(\Omega_y, \Omega_\gamma) = \sum_{k\varkappa} \frac{1+(-1)^k}{2\sqrt{4\pi(2k+1)}} \sqrt{2L+1}\,(-1)^{L+1} \times \langle L1L-1 \mid k0\rangle\, u(LkJ_0L:LL)\, \mathrm{Y}^*_{k\varkappa}(\Omega_\gamma) \rho_{k\varkappa}(\Omega_y). \quad (46)$$

We introduce

$$\mathrm{A}_{k\varkappa}(\Omega_y) = (-1)^{L+1}\sqrt{2L+1}\,\langle L1L-1 \mid k0\rangle\, \rho_{k\varkappa}(\Omega_y). \quad (47)$$

For $\varkappa = 0$, for example, the relationship (47) takes the form

$$\mathrm{A}_{00}(\Omega_y) = \rho_{00}(\Omega_y);\ \mathrm{A}_{20}(\Omega_y) = -\sqrt{\frac{5}{14}}\,\rho_{20}(\Omega_y);$$
$$\mathrm{A}_{40}(\Omega_y) = -\sqrt{\frac{8}{7}}\,\rho_{40}(\Omega_y). \quad (48)$$

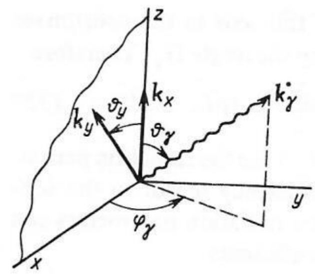

FIG. 1. Coordinate system with z axis along the beam of the incident particles.

Comparing (7), (15), and (48), we obtain

$$\mathrm{A}_{00}(\Omega_y) = d\sigma/d\Omega. \quad (49)$$

We note that the condition (45) is satisfied, as a rule, in transitions in which the spin of the final nucleus is $J_0 = 0$. Then the expression for the angular correlation function takes the form

$$W(\Omega_y, \Omega_\gamma) = \frac{1}{\sqrt{4\pi}} \sum_{k\varkappa} \frac{1+(-1)^k}{2\sqrt{2k+1}} \mathrm{A}_{k\varkappa}(\Omega_y)\, \mathrm{Y}^*_{k\varkappa}(\Omega_\gamma). \quad (50)$$

It is such transitions in reactions with unpolarized particles that we shall consider below.

For further transformation of the expression (50), it is necessary to choose a definite coordinate system. As a basis system (Fig. 1), we shall use a coordinate system with the z axis along the momentum of the incident particles and the reaction plane coincident with the (x, z) plane. In this coordinate system, the polar angle ϑ_y determines the angular distribution of the reaction product particles y, whereas the angular correlation function for a given plane of the γ detector (its direction is determined by the azimuthal angle φ_γ) depends on the polar angle ϑ_γ. This coordinate system is very convenient for comparing the ordinary and double differential cross sections, and also for obtaining the explicit dependence of the angular correlation function on the angle ϑ_y.

Some studies (Refs. 22, 36, 37, 59, and 60) use a different coordinate system, in which the z' axis is perpendicular to the reaction plane [the $(x'y')$ plane] (Fig. 2). In this case, the polar angle ϑ'_γ determines the orientation of the γ rays with respect to the reaction plane, the angular distribution of the reaction products is given by the azimuthal angle φ'_y, and the angular dependence of the angular correlation function for given ϑ'_γ is determined by the angle φ'_γ. Such a choice of the coordinate system is convenient for the direct determination of the polarization characteristics of the excited nucleus, since in this case the z' axis coincides with the direction of the spin of the nucleus.

The connection between the two coordinate systems is given by

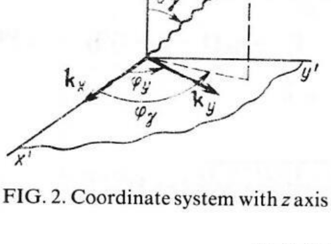

FIG. 2. Coordinate system with z axis perpendicular to the reaction plane.

$$\cos\vartheta_\gamma = \sin\vartheta'_\gamma \cos\varphi'_\gamma;\quad \cos\vartheta'_\gamma = \sin\vartheta_\gamma \sin\varphi_\gamma. \qquad (51)$$

In particular, in the reaction plane (in the coordinate system of Fig. 1, this plane is determined by the condition $\varphi_\gamma = 0$, and in the coordinate system of Fig. 2 by the condition $\vartheta'_\gamma = \pi/2$) the angle ϑ_γ is replaced by the angle φ'_γ.

We shall write down the explicit expressions for the function (50) in the different coordinate systems. As was shown in the previous sections, in the coordinate system of Fig. 1 the quantities $\mathbf{A}_{k\varkappa}(\vartheta_y)$ are real, and the sum (50) is calculated over even values of k and arbitrary $\varkappa$ (both even and odd). We then obtain

$$W(\vartheta_y,\ \vartheta_\gamma,\ \varphi_\gamma) = \frac{1}{4\pi} \sum_{k=0,\,2,\,\ldots;\,\varkappa} \mathbf{A}_{k\varkappa}(\vartheta_y) \times \sqrt{\frac{2}{2k+1}}\, \overline{\mathbf{P}}{}_k^\varkappa(\cos\vartheta_\gamma)\cos\varkappa\varphi_\gamma. \qquad (50a)$$

In the primed coordinate system, the sum in (50) is again over the even k, but the $A'_{k\varkappa}(\varphi'_y)$ with $\varkappa \neq 0$ are now complex, so that the angular correlation function in the primed variables takes the form

$$W(\varphi'_y,\ \vartheta'_\gamma,\ \varphi'_\gamma) = \frac{1}{4\pi} \sum_{k=0,\,2,\,\ldots;\,\varkappa=0,\,2,\,\ldots} \sqrt{\frac{2}{2k+1}} \times |\,\mathbf{A}'_{k\varkappa}(\varphi'_y)\,|\, \overline{\mathbf{P}}{}_k^\varkappa(\cos\vartheta'_\gamma)\cos\{\varkappa\varphi'_\gamma - \alpha_{k\varkappa}(\varphi'_y)\}, \qquad (50b)$$

in which

$$\mathbf{A}'_{k\varkappa}(\varphi'_y) = |\,\mathbf{A}'_{k\varkappa}(\varphi'_y)\,|\, e^{i\alpha_{k\varkappa}(\varphi'_y)}, \qquad (52)$$

where $\alpha_{k\varkappa}(\varphi'_y)$ is a real phase.

We shall obtain the parametrized form of the angular correlation function in the coordinate system of Fig. 1. Using the explicit form of the associated Legendre polynomials, we can represent the function (50a) for the 2^+–0^+ transition in the form

$$W(\vartheta_y,\ \vartheta_\gamma,\ \varphi_\gamma) = a - c_4\cos 4\vartheta_\gamma - s_4 \sin 4\vartheta_\gamma - c_2\cos 2\vartheta_\gamma - s_2 \sin 2\vartheta_\gamma = A + B\sin^2 2(\vartheta_\gamma - \Theta_0) + C\sin^2(\vartheta_\gamma - \Theta_{00}), \qquad (53)$$

where the real parameters a, c_4, s_4, c_2, s_2 $(A, B, C, \theta_0, \theta_{00})$ can be expressed in terms of the parameters of the spin tensors $A_{k\varkappa}(\vartheta_y)$ of the density matrix and the angle $\varphi\gamma$ as follows:

$$\left.\begin{aligned}
a &= A + \tfrac{1}{2}B + \tfrac{1}{2}C = \frac{1}{64\pi}\Big\{\Big(16\mathbf{A}_{00} + 4\mathbf{A}_{20} + \tfrac{9}{4}\mathbf{A}_{40}\Big) + (4\sqrt{6}\,\mathbf{A}_{22} + 3\sqrt{5/2}\,\mathbf{A}_{42}) \\
&\quad \times \cos 2\varphi_\gamma + 3/2\sqrt{\tfrac{35}{2}}\,\mathbf{A}_{44}\cos 4\varphi_\gamma\Big\};\\
c_4 &= \tfrac{1}{2}B\cos 4\Theta_0 = \frac{1}{64\pi}\Big\{-\tfrac{35}{4}\mathbf{A}_{40} + 7\sqrt{\tfrac{5}{2}}\,\mathbf{A}_{42}\cos 2\varphi_\gamma - \tfrac{1}{4}\sqrt{70}\,\mathbf{A}_{44}\cos 4\varphi_\gamma\Big\};\\
s_4 &= \tfrac{1}{2}B\sin 4\Theta_0 = \frac{1}{64\pi}\Big\{7\sqrt{5}\,\mathbf{A}_{41}\cos\varphi_\gamma - \sqrt{35}\,\mathbf{A}_{43}\cos 3\varphi_\gamma\Big\};\\
c_2 &= -\tfrac{1}{2}C\cos 2\Theta_{00} = \frac{1}{64\pi}\{-(12\mathbf{A}_{20} + 5\mathbf{A}_{40}) + (4\sqrt{6}\,\mathbf{A}_{22} - 2\sqrt{10}\,\mathbf{A}_{42})\cos 2\varphi_\gamma + \sqrt{70}\,\mathbf{A}_{44}\cos 4\varphi_\gamma\};\\
s_2 &= \tfrac{1}{2}C\sin 2\Theta_{00} = \frac{1}{64\pi}\{(8\sqrt{6}\,\mathbf{A}_{21} + 2\sqrt{5}\,\mathbf{A}_{41})\cos\varphi_\gamma + 2\sqrt{35}\,\mathbf{A}_{43}\cos 3\varphi_\gamma\}.
\end{aligned}\right\} \qquad (54)$$

It is clear that the parametrization (53) can be inverted to make the form of the angular correlation function determined by the angle φ_γ. Then the parameters of the angular correlation function (the number of which is not, of course, changed by this inversion) will depend on $\mathbf{A}_{k\varkappa}(\vartheta_y)$ and the angle ϑ_γ.

Thus, we have found that in the case of an arbitrary reaction with formation of an even–even nucleus in a 2^+ state with subsequent emission of a γ ray of multipolarity E2 the form of the angular correlation function of the final particles and the γ rays in an arbitrary plane can be expressed in terms of five linear combinations of the parameters of the density-matrix spin tensors; moreover, the dependence of the parameters of the form of the angular correlation function on the azimuthal angle of emission of the γ ray is given analytically.

Besides (54), some further constraints are imposed on the parameters $\mathbf{A}_{k\varkappa}(\vartheta_y)$ by the continuity of $W(\vartheta_y, \vartheta_\gamma, \varphi_\gamma)$ at the poles. They have the form

$$\left.\begin{aligned}
&W(\vartheta_y,\ \vartheta_\gamma = 0 = \pi,\ \varphi_\gamma) = a - c_2 - c_4 = \frac{1}{4\pi}(\mathbf{A}_{00} + \mathbf{A}_{20} + \mathbf{A}_{40});\\
&-\frac{1}{\cos\varphi_\gamma}\frac{\partial}{\partial\vartheta_\gamma} W(\vartheta_y,\ \vartheta_\gamma,\ \varphi_\gamma)\big|_{\vartheta_\gamma=0=\pi} = \frac{4s_4 + 2s_2}{\cos\varphi_\gamma} = \frac{1}{4\pi}(\sqrt{6}\,\mathbf{A}_{21} + 2\sqrt{5}\,\mathbf{A}_{41}).
\end{aligned}\right\} \qquad (55)$$

The relations (53)–(55) give the complete solution to our problem of parametrizing the form of the angular correlation function for $2^+ \to 0^+$ transitions and of the relationship between the corresponding parameters and the parameters of the density-matrix spin tensors. The generalization of our results to the case of other pure $E\lambda$ transitions in accordance with (50a) does not present any basic difficulties. However, since the number of independent parameters $\mathbf{A}_{k\varkappa}(\vartheta_y)$ (see Table I) increases sharply with increasing multipolarity of the transition, the corresponding expressions are rather cumbersome, and we shall not give them here.

For completeness, we give the analogous parametrization of the form of $W(\varphi'_y, \vartheta'_\gamma, \varphi'_\gamma)$ in the primed coordinate system. Using (50b) for a $2^+ \to 0^+$ transition, we can obtain

$$W(\varphi'_y,\ \vartheta'_\gamma,\ \varphi'_\gamma) = a' - c'_4\cos 4\varphi'_\gamma - s'_4\sin 4\varphi'_\gamma - c'_2\cos 2\varphi'_\gamma - s'_2\sin 2\varphi'_\gamma = A' + B'\sin^2 2(\varphi'_\gamma - \Phi_0) + C'\sin^2(\varphi'_\gamma - \Phi_{00}). \qquad (56)$$

The form parameters $a', c'_4, s'_4, c'_2, s'_2$ can be expressed in terms of nine quantities, namely, the six real amplitudes $\mathbf{A}'_{00}(\varphi'_y)$, $\mathbf{A}'_{20}(\varphi'_y)$, $\mathbf{A}'_{40}(\varphi'_y)$, $|\mathbf{A}'_{22}(\varphi'_y)|$, $|\mathbf{A}'_{42}(\varphi'_y)|$, $|\mathbf{A}'_{44}(\varphi'_y)|$ and the three real phases $\alpha_{k\varkappa}(\varphi'_y)$, as follows:

$$
\left.\begin{aligned}
a' &= A' + \frac{1}{2}B' + \frac{1}{2}C' = \frac{1}{32\pi}\Big\{8\mathbf{A}'_{00} + 2\mathbf{A}'_{20} + \frac{9}{8}\mathbf{A}'_{40} \\
&\quad + \Big(6\mathbf{A}'_{20} + \frac{5}{2}\mathbf{A}'_{40}\Big)\cos 2\vartheta'_\gamma + \frac{35}{8}\mathbf{A}'_{40}\cos 4\vartheta'_\gamma\Big\};\\
c'_4 &= \frac{1}{2}B'\cos 4\Phi_0 = \frac{1}{32\pi}\sqrt{70}\,|\mathbf{A}'_{44}|\cos\alpha_{44}\sin^4\vartheta'_\gamma;\\
s'_4 &= \frac{1}{2}B'\sin 4\Phi_0 = \frac{1}{32\pi}\sqrt{70}\,|\mathbf{A}'_{44}|\sin\alpha_{44}\sin^4\vartheta'_\gamma;\\
c'_2 &= \frac{1}{2}C'\cos 2\Phi_{00} = \frac{1}{32\pi}\Big\{2\sqrt{6}\,|\mathbf{A}'_{22}|\cos\alpha_{22}\\
&\quad + \sqrt{\frac{5}{2}}\,\frac{3}{2}\,|\mathbf{A}'_{42}|\cos\alpha_{42} + (\sqrt{10}\,|\mathbf{A}'_{42}|\cos\alpha_{42}\\
&\quad - 2\sqrt{6}\,|\mathbf{A}'_{22}|\cos\alpha_{22})\cos 2\vartheta'_\gamma - \frac{7}{2}\sqrt{5/2}\,|\mathbf{A}'_{42}|\\
&\quad \times\cos\alpha_{42}\cos 4\vartheta'_\gamma\Big\};\\
s'_2 &= \frac{1}{2}C'\sin 2\Phi_{00} = \frac{1}{32\pi}\Big\{2\sqrt{6}\,|\mathbf{A}'_{22}|\sin\alpha_{22}\\
&\quad + \sqrt{5/2}\,3/2\,|\mathbf{A}'_{42}|\sin\alpha_{42} + (\sqrt{10}\,|\mathbf{A}'_{42}|\sin\alpha_{42}\\
&\quad - 2\sqrt{6}\,|\mathbf{A}'_{22}|\sin\alpha_{22})\cos 2\vartheta'_\gamma\\
&\quad - \frac{7}{2}\sqrt{\frac{5}{2}}\,|\mathbf{A}'_{42}|\sin\alpha_{42}\cos 4\vartheta'_\gamma\Big\};
\end{aligned}\right\} \qquad (57)
$$

from which it follows directly that the symmetry angle Φ_0 is related, irrespective of the detection plane of the γ rays, to the phase α_{44} by

$$\Phi_0 = \frac{1}{4}\alpha_{44}. \qquad (58)$$

With allowance for the continuity of $W(\varphi'_y, \vartheta'_\gamma, \varphi'_\gamma)$ at the poles we obtain for the form parameters of the angular correlation function in the primed coordinate system the additional restrictions

$$
\left.\begin{aligned}
&W(\varphi'_y, \vartheta'_\gamma, \varphi'_\gamma = 0 = 2\pi) = a' - c'_4 - c'_2;\\
&\frac{1}{\cos\varphi'_\gamma}\frac{\partial}{\partial\varphi'_\gamma}W(\varphi'_y, \vartheta'_\gamma, \varphi'_\gamma)\big|_{\varphi'_\gamma=0=2\pi} = \frac{4s'_4 + 2s'_2}{\cos\varphi'_\gamma}.
\end{aligned}\right\} \qquad (59)
$$

4. POSSIBLE SIMPLIFICATIONS OF THE GENERAL PARAMETRIZATION OF THE FORM OF THE ANGULAR CORRELATION FUNCTION

We now consider various simplifications of the parametrization of the form of $W(\mathbf{\Omega}_y, \mathbf{\Omega}'_\gamma)$ associated with the introduction of additional conditions on the reaction matrix element $\mathbf{T}_{M_f}(\mathbf{\Omega}_y)$, these depending on the particular mechanism of the reaction.

We consider the case most often encountered in practice, when the reaction matrix element satisfies the condition

$$\mathbf{T}^*_{M_f}(\Omega_y) = (-1)^{M_f}\,\mathbf{T}_{-M_f}(\Omega_y), \qquad (60)$$

i.e., is Hermitian. In accordance with (19), this condition is always satisfied in reactions with spinless particles on even–even nuclei and in the scattering of spin-$\frac{1}{2}$ particles (without spin flip) by the same nuclei if mechanisms that take into account the retardation in the interaction are ignored.[42] It follows from the definitions (6) and (15) that if the condition (60) is satisfied, then the density matrix or its spin tensors are determined by the combination of amplitudes $\mathbf{T}_{M_f}\cdot\mathbf{T}^*_{M_f}$. For 2^+–0^+ transitions, there are obviously six such combinations:

$$\mathbf{T}_0^2,\ \mathbf{T}_1^2,\ \mathbf{T}_2^2,\ \mathbf{T}_{10} = \sqrt{6}\,(\mathbf{T}_1\mathbf{T}_0^* + \mathbf{T}_0\mathbf{T}_1^*);$$
$$\mathbf{T}_{20} = \sqrt{6}\,(\mathbf{T}_2\mathbf{T}_0^* + \mathbf{T}_0\mathbf{T}_2^*);\quad \mathbf{T}_{12} = (\mathbf{T}_1\mathbf{T}_2^* + \mathbf{T}_2\mathbf{T}_1^*).$$

For the state of a nucleus with arbitrary spin J_f, the spin tensors of this state are determined when (60) is satisfied by $2(J_f + 1)$ real parameters, i.e., on $\mathbf{A}_{k\varkappa}(\mathbf{\Omega}_y)$ a further $J_f^2 - 1$ conditions are imposed. It is not difficult to show that in this case too the form of the angular correlation function is determined by the expression (53), but the parameters of the form are expressed in terms of $\mathbf{T}_{M_f}\cdot\mathbf{T}^*_{M_f}$ combinations as follows:

$$
\left.\begin{aligned}
a &= A + \frac{1}{2}B + \frac{1}{2}C = \frac{5}{64\pi}\Big\{(3\mathbf{T}_0^2 + 8\mathbf{T}_1^2 + 5\mathbf{T}_2^2)\cdot\\
&\quad - 2\mathbf{T}_{20}\cos 2\varphi_\gamma - 3\mathbf{T}_2^2\cos 4\varphi_\gamma\Big\};\\
c_4 &= \frac{1}{2}B\cos 4\Theta_0 = \frac{5}{64\pi}\{(3\mathbf{T}_0^2 - 4\mathbf{T}_1^2 + \mathbf{T}_2^2)\\
&\quad - (4\mathbf{T}_1^2 + \mathbf{T}_{20})\cos 2\varphi_\gamma + \mathbf{T}_2^2\cos 4\varphi_\gamma\};\\
s_4 &= \frac{1}{2}B\sin 4\Theta_0 = \frac{5}{64\pi}\{(-2\mathbf{T}_{10} + 2\mathbf{T}_{12})\\
&\quad \times\cos\varphi_\gamma + 2\mathbf{T}_{12}\cos 3\varphi_\gamma\};\\
c_2 &= \frac{1}{2}C\cos 2\Theta_{00} = \frac{5}{64\pi}\{(-4\mathbf{T}_1^2 + 4\mathbf{T}_2^2)\\
&\quad + 4\mathbf{T}_1^2\cos 2\varphi_\gamma - 4\mathbf{T}_2^2\cos 4\varphi_\gamma\};\\
s_2 &= \frac{1}{2}C\sin 2\Theta_{00} = \frac{5}{64\pi}\{4\mathbf{T}_{12}\cos\varphi_\gamma\\
&\quad - 4\mathbf{T}_{12}\cos 3\varphi_\gamma\}.
\end{aligned}\right\} \qquad (61)
$$

If the relation (60) holds, the differential cross section has the form

$$d\sigma/d\Omega = \mathbf{T}_0^2 + 2\mathbf{T}_1^2 + 2\mathbf{T}_2^2, \qquad (62)$$

and the conditions of continuity of $W(\vartheta_y, \vartheta_\gamma, \varphi_\gamma)$ at the poles can be written as

$$a - c_4 - c_2 = \frac{5}{4\pi}\mathbf{T}_1^2;\quad \frac{4s_4 + 2s_2}{\cos\varphi_\gamma} = \frac{5}{4\pi}\Big(\mathbf{T}_{12} - \frac{1}{2}\mathbf{T}_{10}\Big).$$

Apart from normalizing factors, the expressions (61) are identical to the expression for the angular correlation function obtained in Ref. 58 for inelastic scattering of protons by an even–even nucleus in the L–S coupling approximation [for such a reaction, the condition (60) is obviously satisfied]. In the reaction plane ($\varphi_\gamma = 0$), the form parameters c_2 and s_2 vanish in accordance with (61), and the form of the angular correlation function is described by the three-parameter expression which is usually employed to analyze experimental data in the vast majority of studies:

$$W(\vartheta_y, \vartheta_\gamma, \varphi_\gamma = 0) = A(\vartheta_y) + B(\vartheta_y)\sin^2 2(\vartheta_\gamma - \Theta_0(\vartheta_y)). \qquad (63)$$

In this case, the expressions for $A(\vartheta_y)$, $B(\vartheta_y)$, and $\theta_0(\vartheta_y)$ also simplify:

$$
\left.\begin{aligned}
A + \frac{1}{2}B &= \frac{5}{64\pi}(3\mathbf{T}_0^2 + 8\mathbf{T}_1^2 + 2\mathbf{T}_2^2 - \mathbf{T}_{20});\\
\frac{1}{2}B\cos 4\Theta_0 &= \frac{5}{64\pi}(3\mathbf{T}_0^2 - 8\mathbf{T}_1^2 + 2\mathbf{T}_2^2 - \mathbf{T}_{20});\\
\frac{1}{2}B\sin 4\Theta_0 &= \frac{5}{64\pi}(-2\mathbf{T}_{10} + 4\mathbf{T}_{12}).
\end{aligned}\right\} \qquad (64)
$$

Apart from the normalizing factor, analogous expressions can be found in Ref. 57. We emphasize once more that the parametrization (63) holds only in the reaction plane for processes whose matrix element satisfies the condition (60).

For $2^+ \to 0^+$ transitions when the condition (60) is satisfied three additional restrictions are imposed on the parameters $\mathbf{A}_{k\varkappa}$. These can be taken, for example, in the form

$$\left.\begin{aligned} &\mathbf{A}_{00}+\mathbf{A}_{20}+\mathbf{A}_{40}-\sqrt{6}\,\mathbf{A}_{22}-\sqrt{\tfrac{5}{2}}\,\mathbf{A}_{42}=0;\\ &-4\mathbf{A}_{00}+8\mathbf{A}_{20}+\mathbf{A}_{40}-\sqrt{70}\,\mathbf{A}_{44}=0;\\ &4\sqrt{6}\,\mathbf{A}_{21}+\sqrt{5}\,\mathbf{A}_{41}+\sqrt{35}\,\mathbf{A}_{43}=0. \end{aligned}\right\}\quad(65)$$

These equations are equivalent to the vanishing of $W(\vartheta_\gamma, \vartheta_\gamma = \varphi_\gamma = \pi/2)$, i.e., if the condition (60) is satisfied, the isotropic part of the angular correlation function in the perpendicular plane vanishes. For completeness, we also give the expressions that relate $\mathbf{A}_{k\varkappa}$ to the $\mathbf{T}_{M_f}\mathbf{T}^*_{M'_f}$ combinations:

$$\left.\begin{aligned} &\mathbf{T}_0^2=-\tfrac{1}{4}\mathbf{A}_{40}+\tfrac{4}{5}\mathbf{A}_{20}-\tfrac{1}{2}\sqrt{\tfrac{7}{10}}\,\mathbf{A}_{44};\\ &\mathbf{T}_1^2=\tfrac{1}{5}\sqrt{6}\,\mathbf{A}_{22}-\frac{1}{\sqrt{10}}\mathbf{A}_{42}=\tfrac{1}{5}(\mathbf{A}_{00}+\mathbf{A}_{20}+\mathbf{A}_{40});\\ &\mathbf{T}_2^2=-\frac{\sqrt{7}}{2\sqrt{10}}\mathbf{A}_{44};\\ &\mathbf{T}_{10}=\tfrac{1}{2}\frac{7}{\sqrt{5}}\mathbf{A}_{41}-\tfrac{1}{2}\sqrt{\tfrac{7}{5}}\,\mathbf{A}_{43}=\\ &\quad=\tfrac{2}{5}\sqrt{6}\mathbf{A}_{21}-3\sqrt{\tfrac{1}{5}}\,\mathbf{A}_{41};\\ &\mathbf{T}_{20}=-\frac{3}{\sqrt{10}}\mathbf{A}_{42}-\tfrac{4}{5}\sqrt{6}\,\mathbf{A}_{22};\\ &\mathbf{T}_{21}=-\frac{\sqrt{7}}{2\sqrt{6}}\mathbf{A}_{43}. \end{aligned}\right\}\quad(65a)$$

It follows directly from (65a) that for reactions whose matrix element satisfies the condition (60) the value of $\mathbf{A}_{44}(\vartheta_y)$ is always negative.

In the primed coordinate system, the relation (14) is the mathematical formulation of A. Bohr's theorem,[55] which can be stated as follows: If in an even–even nucleus a state of natural parity is excited, then only the populations for which $J_f - M_f$ is even are nonzero. In other words, if the condition (60) is satisfied, the matrix element $\mathbf{T}'_{\pm 1}(\varphi'_y)$ vanishes. As a result, the angular correlation function in the primed coordinate system will be expressed in terms of six parameters: $|\mathbf{T}'_0|$, $|\mathbf{T}'_2|$, $|\mathbf{T}'_{-2}|$ and the three phases δ_0, δ_2, δ_{-2}. The form of the angular correlation function in any γ-ray detection plane will again be determined by the expression (56), and the form parameters will be

$$\left.\begin{aligned} &a'=\frac{5}{64\pi}\Big\{3\,|\mathbf{T}'_0|^2+\tfrac{5}{2}|\mathbf{T}'_2|^2+\tfrac{5}{2}|\mathbf{T}'_{-2}|^2\\ &-2(|\mathbf{T}'_2|^2+|\mathbf{T}'_{-2}|^2)\cos 2\vartheta'_\gamma-\tfrac{1}{2}(|\mathbf{T}'_0|^2+|\mathbf{T}'_2|^2\\ &\qquad+|\mathbf{T}'_{-2}|^2)\cos 4\vartheta'_\gamma\Big\};\\ &c'_4=\frac{5}{8\pi}|\mathbf{T}'_2|^2|\mathbf{T}'_{-2}|^2\cos(\delta_2-\delta_{-2})\sin^4\vartheta'_\gamma;\\ &s'_4=\frac{5}{8\pi}|\mathbf{T}'_2|^2|\mathbf{T}_{-2}|^2\sin(\delta_2-\delta_{-2})\sin^4\vartheta'_\gamma;\\ &c'_2=\frac{5\sqrt{6}}{64\pi}|\mathbf{T}'_0|\{|\mathbf{T}'_2|\cos(\delta_2-\delta_0)\\ &\quad+|\mathbf{T}'_{-2}|\cos(\delta_0-\delta_{-2})\}(1-\cos 4\vartheta'_\gamma);\\ &s'_2=\frac{5\sqrt{6}}{64\pi}|\mathbf{T}'_0|\{|\mathbf{T}'_2|\sin(\delta_2-\delta_0)\\ &\quad+|\mathbf{T}'_{-2}|\sin(\delta_0-\delta_{-2})\}(1-\cos 4\vartheta'_\gamma). \end{aligned}\right\}\quad(66)$$

If we introduce the additional parameters $a_0 = |\mathbf{T}'_0|/|\mathbf{T}_{-2}|$ and $a_2 = |\mathbf{T}'_2|/|\mathbf{T}'_{-2}|$, then apart from a constant factor the parametrization (66) will be identical to the parametrization presented in Ref. 22 by introduction of the additional condition $\delta_{-2} = 0$ [which is evidently analogous to the normalization conditions (49) and (62)].

For completeness, we also give the restrictions on $\mathbf{A}'_{k\varkappa}$ that hold when the condition (60) is satisfied,

$$\left.\begin{aligned} &\mathbf{A}'_{00}+\mathbf{A}'_{20}+\mathbf{A}'_{40}=0;\\ &|\mathbf{A}'_{42}|=\tfrac{2}{5}\sqrt{3}\,|\mathbf{A}'_{22}|;\quad \alpha_{42}=\alpha_{22}; \end{aligned}\right\}\quad(67)$$

and the expressions for the moduli and phases of $\mathbf{A}'_{k\varkappa}$ in terms of $|\mathbf{T}'_0|$, $|\mathbf{T}'_2|$, $|\mathbf{T}'_{-2}|$, δ_0, δ_2, δ_{-2}:

$$\left.\begin{aligned} &\mathbf{A}'_{00}=|\mathbf{T}'_0|^2+|\mathbf{T}'_2|^2+|\mathbf{T}'_{-2}|^2;\\ &\mathbf{A}'_{20}=\tfrac{5}{7}\{|\mathbf{T}'_0|^2-|\mathbf{T}'_2|^2-|\mathbf{T}'_{-2}|^2\};\\ &\mathbf{A}'_{40}=-\tfrac{2}{7}\{6\,|\mathbf{T}'_0|^2+|\mathbf{T}'_2|^2+|\mathbf{T}'_{-2}|^2\};\\ &|\mathbf{A}'_{22}|\cos\alpha_{22}=-\tfrac{5}{7}|\mathbf{T}'_0|\{|\mathbf{T}'_2|\cos(\delta_2-\delta_0)\\ &\qquad+|\mathbf{T}'_{-2}|\cos(\delta_0-\delta_{-2})\};\\ &|\mathbf{A}'_{22}|\sin\alpha_{42}=-\tfrac{5}{7}\mathbf{T}'_0\{|\mathbf{T}'_2|\sin(\delta_2-\delta_0)\\ &\qquad+|\mathbf{T}'_{-2}|\sin(\delta_0-\delta_{-2})\}, \end{aligned}\right\}\quad(67a)$$

from which it follows immediately that when the condition (60) holds A'_{40} is always negative and the differential cross section is, as one would expect, determined by the sum of the squares of the moduli of the matrix elements. If the condition (60) is satisfied, then in the reaction plane in the primed coordinate system the parameters c'_2 and s'_2 vanish, and W again acquires a three-parameter form. The values of these parameters are

$$\left.\begin{aligned} &a'=\frac{5}{64\pi}\left(\tfrac{5}{2}|\mathbf{T}'_0|^2+4\,|\mathbf{T}'_2|^2+4\,|\mathbf{T}'_{-2}|^2\right);\\ &c'_4=\frac{5}{8\pi}|\mathbf{T}'_2|\,|\mathbf{T}'_{-2}|\cos(\delta_2-\delta_{-2});\\ &s'_4=\frac{5}{8\pi}|\mathbf{T}'_2||\mathbf{T}_{-2}|'\sin(\delta_2-\delta_{-2});\ \Phi_0=\tfrac{1}{4}(\delta_2-\delta_{-2}). \end{aligned}\right\}\quad(67b)$$

The form of the angular correlation function is most readily parametrized if the system has a symmetry axis, i.e., the reaction matrix element can be represented in the form

$$\mathbf{T}_{M_f}(\vartheta_y)=\sum_l a_l(q)\,\mathbf{Y}_{lM_f}(\mathbf{q}),\qquad(68)$$

where q is the momentum transfer. It is clear that this condition is satisfied in all direct nuclear reactions when the plane-wave approximation is used. In this case, the angular correlation function has the form

$$W(\vartheta_y,\ \vartheta_\gamma,\ \varphi_\gamma)=\sum_k A_k(q)\,P_k(\cos\{\mathbf{p}_\gamma\cdot\mathbf{q}\}),\qquad(69)$$

i.e., it is determined by the three independent parameters A_0, A_2, A_4. The symmetry angle of the angular correlation function is uniquely determined by the angles ϑ_γ and φ_γ and by the direction of the momentum of the recoil nucleus. If the selection rules in the reaction allow several values of the angular-momentum transfer l, then the form of $W(\vartheta_y, \vartheta_\gamma, \varphi_\gamma)$ does not reduce to the three-parameter form (63) even when (68) is satisfied. But if the angular-momentum transfer l has a unique value, then the form of $W(\vartheta_y, \vartheta_\gamma, \varphi_\gamma)$ is described by the expression (63). For the inelastic scattering of spinless particles by even–even nuclei when (68) is satisfied, $W(\vartheta_y, \vartheta_\gamma, \varphi_\gamma)$ acquires a two-parameter form, since the isotropic part in (63) vanishes.

5. RECOVERY OF THE DENSITY MATRIX OF A GIVEN NUCLEAR STATE BY MEASUREMENT OF THE ANGULAR CORRELATION FUNCTION IN DIFFERENT γ-RAY DETECTION PLANES

In Refs. 43 and 44, we proposed a method for finding all the real parameters of the density matrix of excited nuclear states by measurement of the angular correlation function of the final reaction product particles and γ rays emitted by the excited nucleus on transition to the ground state in γ-ray detection planes at different angles relative to the reaction plane.

It is obvious that for transitions with $J_f = L$ such a method actually enables one in a single experiment, without modification of its method, to obtain all components of the density matrix and thus recover the spin characteristics of the excited nuclear states, in particular those whose direct experimental determination is difficult.

It is not difficult to find in general form the minimal number of planes in which it is necessary to measure $W(\mathbf{\Omega}_{[SB::y]}, \mathbf{\Omega}_\gamma)$ in order to find all the parameters of $\mathbf{A}_{k\varkappa}(\mathbf{\Omega}_y)$. To be definite, we shall give all of the following treatment in a coordinate system whose z axis passes along the beam of the incident particles (see Fig. 1). As was shown in the previous section [see Eq. (53)], in each γ-ray detection plane measurement of $W(\vartheta_y, \vartheta_\gamma, \varphi_\gamma)$ permits the determination of only $2J_f + 1$ of the restrictions imposed on the $\mathbf{A}_{k\varkappa}(\vartheta_y)$ parameters. In the case of measurement in n planes, the conditions of continuity at the poles impose $2n - 2$ further restrictions. As a result, the minimal number of planes in which it is necessary to measure $W(\vartheta_y, \vartheta_\gamma, \varphi_\gamma)$ in order to obtain all the parameters of the density matrix is given by the following expressions[50]:

1. In the case of detection of linear and circular polarization of the γ rays (the number of parameters of the density matrix corresponds to the upper row of Table I)

$$n_{\min} \geqslant \frac{(2J_f+1)^2-2}{2J_f-1}. \tag{70}$$

2. In the case of detection of only circular polarization of the γ rays (the number of parameters of the density matrix corresponds to the central row of Table I)

$$n_{\min} \geqslant \frac{(J_f+1)(2J_f+1)-2}{2J_f-1}. \tag{71}$$

3. In the absence of detection of polarization (the number of parameters of the density matrix corresponds to the lower row of Table I)

$$n_{\min} \geqslant \frac{(J_f+1)^2-2}{2J_f-1}. \tag{72}$$

In particular, for $2^+ \to 0^+$ transitions the recovery of all the real parameters of the density matrix requires measurements in eight γ-ray detection planes; if the 15 parameters of the real spin tensors with all k are to be recovered, measurements must be made in five; and if the nine parameters of the real spin tensors with even k are to be found, measurements must be made in three planes.

Hitherto, the polarization of the γ rays in experiments associated with measurement of the angular correlation function has not been measured, i.e., from the experimental values of $W(\vartheta_y, \vartheta_\gamma, \varphi_\gamma)$ one can recover the $A_{k\varkappa}$ with even k and one can obtain only the orientational and not the polarization characteristics of the nucleus B^* in the state J_f. Nevertheless, even these experiments significantly raise the level of our knowledge of the properties of the excited nuclear states; for example, for $J_f = 2$ and $J_f = 3$ one can obtain in an experiment nine and 16 parameters characterizing the given nuclear state, respectively.

When a definite experiment is made, it is necessary to bear in mind that different γ-ray detection planes are not equivalent from the point of view of recovering different $A_{k\varkappa}(\vartheta_y)$. This inequivalence is a consequence of the fact that the angular correlation function at certain angles φ_γ does not depend on certain $\mathbf{A}_{k\varkappa}(\vartheta_y)$ [see (54)]. For example, a dependence on $\mathbf{A}_{44}(\vartheta_y)$ disappears in the case of a 2^+–0^+ transition in the first quadrant $0 \leqslant \varphi_\gamma \leqslant \pi/2$ in the planes with $\varphi_\gamma = 22.5°$ and $\varphi_\gamma = 67.5°$. The dependence on A_{43} disappears in a plane with $\varphi_\gamma = 30°$; the dependence on $\mathbf{A}_{k\varkappa}$ with $\varkappa = 2$, in the plane with $\varphi_\gamma = 45°$; and the dependence on $\mathbf{A}_{k\varkappa}$ with odd $\varkappa$, in the plane with $\varphi_\gamma = 90°$. For simplified variants of the parametrization [for example, if the condition (60) is satisfied], the dependence of $W(\vartheta_y, \vartheta_\gamma, \varphi_\gamma)$ on some $\mathbf{T}_{M_f} \cdot \mathbf{T}^*_{M'_f}$ combinations disappears in accordance with (61) in the planes with $\varphi_\gamma = 45°$ and $\varphi_\gamma = 90°$.

In Table II, we give a list of the number of form parameters of the angular correlation function and the number of $\mathbf{A}_{k\varkappa}(\vartheta_y)$ parameters (or $\mathbf{T}_{M_f} \cdot \mathbf{T}_{M'_f}$ combinations) on which these form parameters depend. As can be seen from the table, the plane with $\varphi_\gamma = 90°$ gives the least possibilities for experimental determination of the form parameters. Indeed, it can be readily seen from the expressions (54) that in this plane the parameters Θ_0 and Θ_{00} vanish, so that irrespective of the reaction mechanism the angular correlation function has at $\varphi_\gamma = 90°$ a maximum (or minimum), depending on the sign relationship between A and B. If the condition (60) is satisfied, the angular correlation function at the minimum in the plane with $\varphi_\gamma = 90°$ is strictly zero, since in this case

TABLE II. Table of form parameters of the angular correlation function.[43]

φ_γ, deg	Arbitrary reaction mechanisms		Reaction mechanisms whose amplitude satisfies the condition (60)	
	Number of form parameters of the angular correlation function	Number of $\mathbf{A}_{k\varkappa}$ components on which these parameters depend	Number of form parameters of the angular correlation function	Number of $\mathbf{T}_{M_f} \cdot \mathbf{T}^*_{M'f}$ combinations on which these parameters depend
0	5	9	3	6
22,5	5	8	5	6
30	5	8	5	6
45	5	7	5	5
90	3	6	2	4
Remaining planes	5	9	5	6

$A = 0$. If at the same time the minimum of the angular correlation function is at $\vartheta_\gamma = 90°$ (and this is evidently the situation for inelastic scattering of spinless particles[30]), the experimental determination of the form parameters of the function is greatly complicated, since the maxima of the angular correlation function are in this case in regions of angles not readily accessible to measurement ($\vartheta_\gamma \leqslant 30°$, $\vartheta_\gamma \geqslant 150°$).

The reaction plane also gives comparatively little information about the form parameters when the condition (60) is satisfied. In this case, the parameter C vanishes, and Θ_{00} becomes indefinite. As was shown in Ref. 29, in the case of inelastic scattering of spinless particles the angular correlation function is close to the three-parameter function (63) even for mechanisms that take into account retardation in the interaction of the particles, even though the amplitude of these mechanisms does not in general satisfy the condition (60). Therefore, the study of the angular correlation function of spinless particles in the reaction plane effectively enables one to find only three of its form parameters (and not five).

The recovery of $\mathbf{A}_{k\varkappa}$ or the combinations $\mathbf{T}_{M_f}\cdot\mathbf{T}^*_{M'_f}$ from experimental data necessarily entails the introduction of corrections for the finite resolution of the γ detector in the correlation experiments (see the definition of ideal detectors given at the beginning of Sec. 2). The angles ϑ_γ and φ_γ which occur in (50a) are not in fact known. In an experiment, we can measure only the angles $\vartheta_{\gamma 0}$ and $\varphi_{\gamma 0}$, which characterize the direction of the axis of the γ detector. In order to obtain the experimental angular correlation function $W(\vartheta_y, \vartheta_{\gamma 0}, \varphi_{\gamma 0})$, we must integrate the expression (50a) over the complete angular interval of the γ detector with a weight factor that determines the dependence of the detector efficiency on the direction of the plane in which the γ ray is absorbed by the detector. Denoting this efficiency by $\mathbf{E}(\Theta, \Phi)$ (the angles Θ and Φ determine the direction of emission of the γ ray relative to the detector axis), we can readily obtain in transitions with $J_f = L$ the following result for the experimentally observed angular correlation function in the case of cylindrically symmetric detectors[36]:

$$W(\vartheta_y, \vartheta_{\gamma 0}, \varphi_{\gamma 0}) = \frac{1}{\sqrt{4\pi}} \sum_{k\varkappa} \frac{1+(-1)^k}{2\sqrt{2k+1}} Q_k \mathbf{A}_{k\varkappa}(\vartheta_y) \mathbf{Y}^*_{k\varkappa}(\Omega_{\gamma 0}), \tag{73}$$

where

$$Q_k = \frac{\int P_k(\cos\Theta)\,\mathbf{E}(\Theta, \Phi) \sin\Theta\, d\Theta\, d\Phi}{\int \mathbf{E}(\Theta, \Phi) \sin\Theta\, d\Theta\, d\Phi}. \tag{74}$$

Comparison of the expressions (50) and (74) shows that the inclusion of corrections for the finite resolution of γ detectors possessing cylindrical symmetry does not change the form of the angular correlation (50) except for the fact that the $\mathbf{A}_{k\varkappa}(\vartheta_y)$ are replaced by the corrected $\widehat{\mathbf{A}}_{k\varkappa}(\vartheta_y) = Q_k \mathbf{A}_{k\varkappa}(\vartheta_y)$. The correction coefficients Q_k do not depend on $\varkappa$,[61] and, in addition, $Q_0 = 1$, i.e., $\widehat{\mathbf{A}}_{00}(\vartheta_y) = \mathbf{A}_{00}(\vartheta_y)$. The physics of this result is clear, since the finite size of the γ detector must not influence the differential cross section of the reaction. At the same time, since $\mathbf{A}_{00}(\vartheta_y)$ determines the reaction differential cross section [see (49)], measurement of $d\sigma/d\Omega$ makes it possible to realize correctly the normalization of the experimental data on the cross sections and angular correlations.

Because the corrections for the finite angular resolution of the γ detector do not influence the form of the angular correlation function (50), all the relations (54) that connect the form parameters of the angular correlation function to the $\widehat{\mathbf{A}}_{k\varkappa}(\vartheta_y)$ remain valid. Moreover, for the same reasons we still have the relations (70)–(72), which determine the minimal number of planes in which it is necessary to make measurements of the angular correlation function in order to find all the independent parameters of the density matrix. However, for the simplified variants of the form parametrization of the angular correlation the situation may be different. Thus, if the condition (60) is satisfied, then for a 2^+–0^+ transition we can extract all six linearly independent combinations $\mathbf{T}_{M_f}\cdot\mathbf{T}^*_{M'_f}$ by measuring the angular correlation function in two γ-detection planes relative to the reaction plane. In reality, however, such a procedure can be realized only if the corrections for the finite resolution of the γ detector are ignored. But if these corrections must be taken into account, then for a 2^+–0^+ transition it is necessary to make measurements in three γ-detection planes in order to obtain correct values, corrected for the finite resolution of the γ detector, of the parameters $\mathbf{A}_{k\varkappa}(\vartheta_y)$, and then, using the relations (65a), which connect the combinations $\mathbf{T}_{M_f}\cdot\mathbf{T}^*_{M'_f}$ to the $\mathbf{A}_{k\varkappa}$ with *different* k, obtain the true $\mathbf{T}_{M_f}\cdot\mathbf{T}^*_{M'_f}$ values.[4)]

Such an experimental program to recover the real $\mathbf{A}_{k\varkappa}(\vartheta_y)$ parameters with even k for the density matrix of an excited nuclear state with spin J_f was realized for the first time at the Institute of Nuclear Physics at Moscow State University.[45–49] Various experiments gave values of $\mathbf{A}_{k\varkappa}(\vartheta_y)$ with $k = 0, 2, 4$ for the 2^+ state of the ^{12}C nucleus formed in the $^{12}C(\alpha, \alpha')^{12}C(2^+)$ reaction at $E_\alpha = 25$ MeV (Refs. 45 and 46) and 30 MeV (Refs. 48 and 49), in $^{13}C(^3He, \alpha)^{12}C(2^+)$ reactions at $E_{^3He} = 22.5$ MeV (Refs. 47 and 48), and in some other reactions.

The most complete experimental data have been obtained for inelastic scattering of α particles by ^{12}C. It is here necessary to emphasize the following important circumstance. In the scattering of spinless particles by spinless nuclei, there are no physically distinguished directions, and therefore there is no polarization of the final particles (or other tensors of odd rank). Similarly, for inelastic scattering of spinless particles the polarization of the γ rays (both linear and circular) vanishes, so that the experiments of Refs. 45, 46, 48, and 49 yielded *all possible* information about the density matrix of the 2^+ state of the ^{12}C nucleus that can be obtained in elastic scattering of α particles.

In order to recover *all* the $\mathbf{A}_{k\varkappa}(\vartheta_\alpha)$ of the 2^+ level of the ^{12}C nucleus, measurements of $W(\vartheta_\alpha, \vartheta_\gamma, \varphi_\gamma)$ were made in three planes, $\varphi_\gamma = 0$, $\varphi_\gamma = 45°$, and $\varphi_\gamma = 90°$, in the interval of angles ϑ_γ from 27 to 149° and for 43 values of the angles ϑ_α for $E_\alpha = 25$ MeV.[45] It was found that the experimental angular correlation functions for different φ_γ have very different forms (Fig. 3). The values of the parameters $\mathbf{A}_{k\varkappa}(\vartheta_\alpha)$ were found by solving the system of equations (53)–(54) by the method of least squares with allowance for corrections for the finite resolution of the γ detector. It was found that introduction of the correction coefficients had little influence on the $\mathbf{A}_{k\varkappa}(\vartheta_\alpha)$ parameters[45] and, therefore, on the form of the angular correlation function.

The parameters $\mathbf{A}_{k\varkappa}(\vartheta_\alpha)$ obtained from the experi-

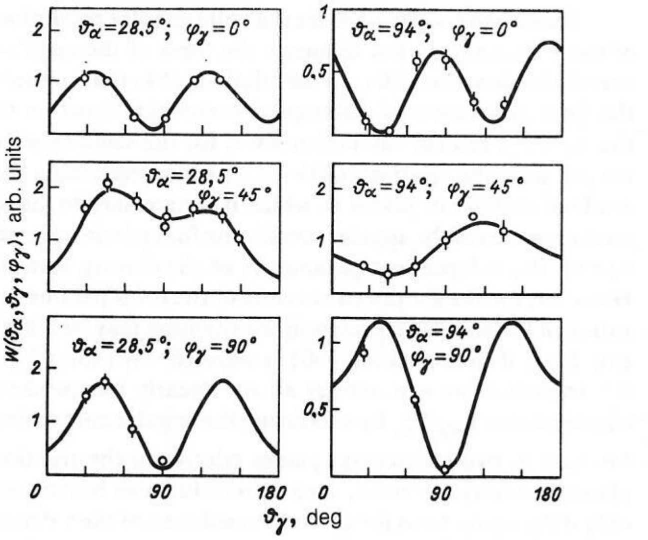

FIG. 3. Experimental angular correlation functions $W(\vartheta_\alpha, \vartheta_\gamma, \varphi_\gamma)$ (Ref. 45) as a function of ϑ_γ for different ϑ_α and φ_γ, $E_\alpha = 25$ MeV.

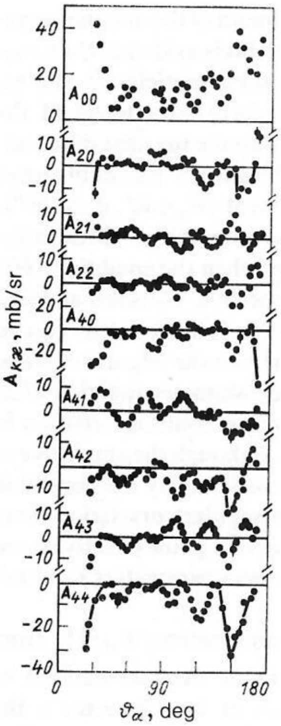

FIG. 4. Angular dependences of the experimental values of $\mathbf{A}_{k\varkappa}(\vartheta_\alpha)$ of the 2^+ state of the ^{12}C nucleus formed in inelastic scattering of α particles with $E_\alpha = 25$ MeV.[45]

mental values of $W(\vartheta_\alpha, \vartheta_\gamma, \varphi_\gamma)$ are given in Fig. 4. To within the experimental errors, the dependence of $\mathbf{A}_{00}$ on ϑ_α agrees, as it must, with the angular distribution of the differential cross section of inelastic scattering. It can be seen from the figure that the angular dependences of all the $\mathbf{A}_{k\varkappa}(\vartheta_\alpha)$, and not only $\mathbf{A}_{00}$, have an oscillatory form. However, the nature of the dependence of the different $A_{k\varkappa}$ does not repeat the dependence of the differential cross section, and in the majority of cases not only the value of $\mathbf{A}_{k\varkappa}(\vartheta_\alpha)$ but also the sign changes. It is also important that all the $A_{k\varkappa}(\vartheta_\alpha)$ are comparable in value.

In Fig. 5, we give the analogous dependences of the $\mathbf{A}_{k\varkappa}(\vartheta_\alpha)$ for the 2^+ level of the ^{12}C nucleus formed in the ^{13}C(^{3}He, α')^{12}C(2^+) reaction. We note first of all the appreciable differences between the spin tensors of the same state of the final nucleus formed in different ways. Indeed, the parameters $\mathbf{A}_{k\varkappa}(\vartheta_\alpha)$ differ radically both in the nature of the angular dependence on ϑ_α and in the relationship between the values of the components with different k. In other words, the spin tensors of a definite state of the nucleus exhibit a strong dependence on the method of formation of this state, and it is to be hoped that the experimentally determined spin tensors will give new information about the mechanisms of the particular reaction. We shall consider these questions in more detail in the following section.

To conclude this section, we note that, in their turn, the obtained $\mathbf{A}_{k\varkappa}(\vartheta_y)$ enable us to calculate the angular correlation function itself by using the relations (53) and (54) for any ϑ_γ and φ_γ. As an example, Fig. 6 shows the angular correlation function recovered in this manner in the ^{13}C(^{3}He, $\alpha'\gamma$)^{12}C reaction for $\vartheta_\alpha = 20°$. The values of the angular

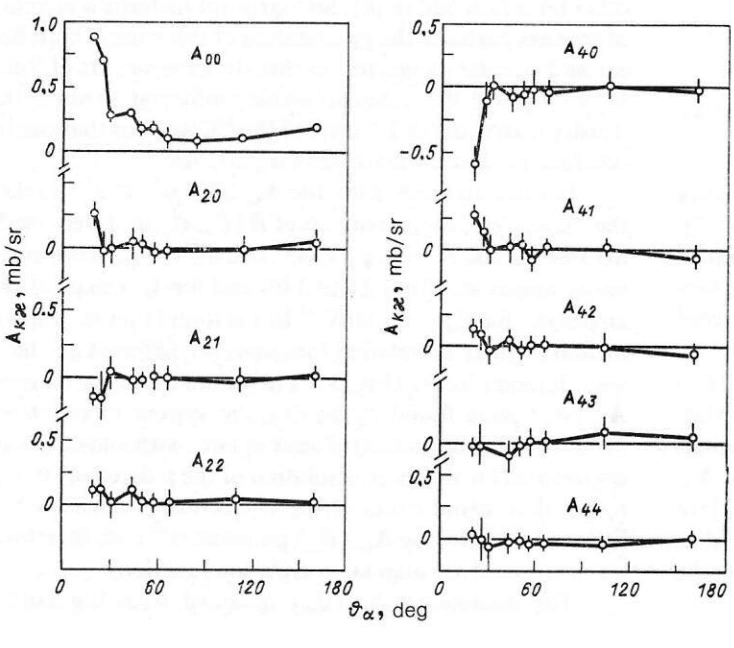

FIG. 5. The same as in Fig. 4 for the 2^+ state of the ^{12}C nucleus formed in the ^{13}C(^{3}He, α')^{12}C(2^+) reaction at $E_{^3\mathrm{He}} = 22.5$ MeV.[47,48]

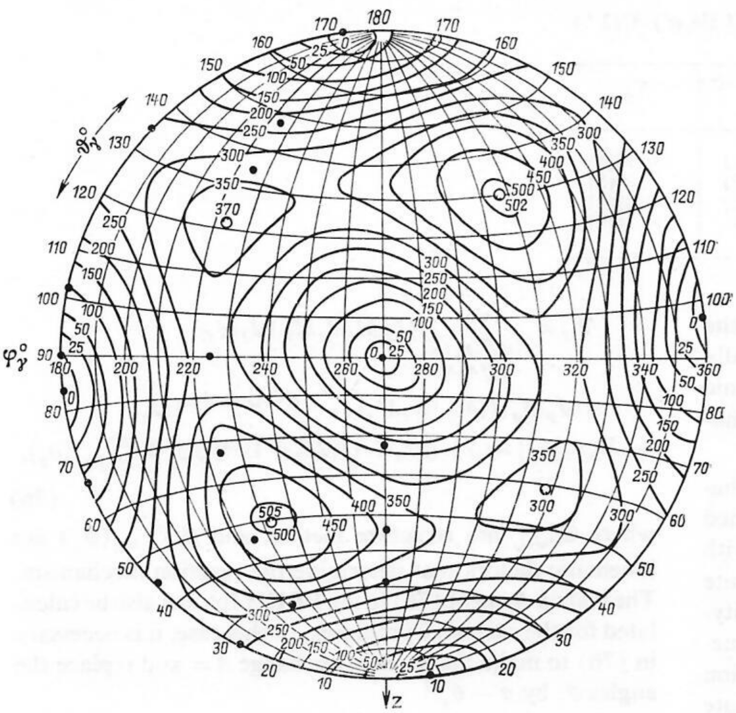

FIG. 6. Azimuthal projection of the α–γ angular correlation function in the $^{13}C(^3He, \alpha'\gamma)^{12}C$ reaction calculated in accordance with Eq. (50a) (Ref. 47) with the $\mathbf{A}_{k\varkappa}(\vartheta_\alpha)$ given in Fig. 5 for $\vartheta_\alpha = 20°$. The large symbols indicate the angles φ_γ and ϑ_γ at which $W(\vartheta_\alpha, \vartheta_\gamma, \varphi_\gamma)$ was measured. The numbers next to the corresponding curves give the angular correlation functions.

correlation function are given in the form of contours on the hemisphere $180 \leqslant \varphi_\gamma \leqslant 360°$. The direction of the beam of incident helium ions coincides with the z axis (see the figure), while the reaction plane coincides with the plane $\varphi_\gamma = 180°$.

6. THEORETICAL ANALYSIS OF THE DENSITY-MATRIX SPIN TENSORS IN THE DISTORTED-WAVE BORN APPROXIMATION WITH A FINITE-RANGE INTERACTION

It was shown earlier that particle–γ-ray correlation experiments can, even without detection of the polarization of the γ rays (or final particles), enable one to obtain appreciably more information about the properties of excited nuclear states than by measurement of the differential cross sections. In particular, in $2^+ \to 0^+$ transitions we obtain eight independent additional quantities that characterize the given nuclear state. In a number of cases, even qualitative analysis of

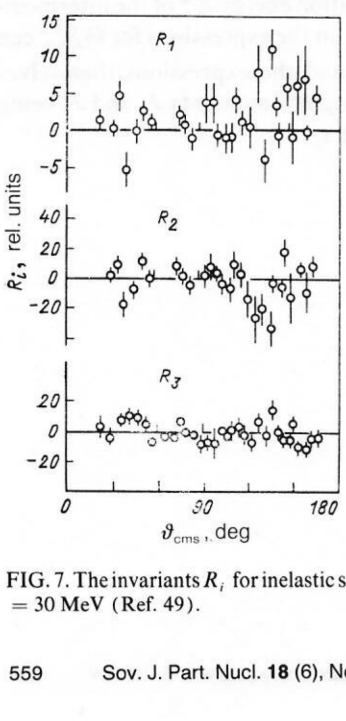

FIG. 7. The invariants R_i for inelastic scattering of α particles by ^{12}C at $E_\alpha = 30$ MeV (Ref. 49).

these data permit certain conclusions to be drawn about the reaction mechanism. We can give at least three qualitative examples that illustrate the significant sensitivitiy of the spin tensors $\mathbf{A}_{k\varkappa}(\vartheta_y)$ to the reaction mechanism.

1. If the condition (60) is satisfied, i.e., the reaction matrix element is Hermitian, then the spin tensors are described by six parameters, the invariants R_i (65) vanish, and it follows from the relations (65a) that $\mathbf{A}_{44}$ is always negative. As can be seen from Figs. 4 and 7, in inelastic α-particle scattering $\mathbf{A}_{44} < 0$, while the deviations of the invariants R_i from zero exceed the statistical errors only in some angular regions (as a rule, at the minima of the differential cross section). For comparison, in Table III we give the invariants R_i in the $^{13}C(^3He, \alpha')^{12}C^*$ reaction, for which the condition (60) is not satisfied, irrespective of the mechanism. Note also that in this reaction $\mathbf{A}_{44}$ may be positive (see Fig. 5).

2. If we have a single-nucleon transfer reaction on nuclei of the 1p shell, then the maximal rank of the spin tensor of a 2^+ state of the final nucleus obviously cannot be greater than 2 (it cannot exceed twice the angular-momentum transfer). Therefore, if for such reactions $\mathbf{A}_{4\varkappa} \neq 0$, then their mechanism cannot be reduced to ordinary stripping or pickup of a $1p$ nucleon. Table IV gives the $\mathbf{A}_{k\varkappa}$ obtained from experimental data on the angular correlation functions for the $^{13}C(^3He, \alpha\gamma)^{12}C$ reaction. It can be seen that in the entire region of angles the components $\mathbf{A}_{4\varkappa}$ are significantly nonzero. The comparatively large values of $\mathbf{A}_{k\varkappa}$ with $k = 4$ for $\vartheta_\alpha = 160°$ may be due to the contribution of exchange processes (the importance of this contribution in the analysis of the angular correlation function was first pointed out in Ref. 62). The appreciable deviations of $\mathbf{A}_{k\varkappa}$ from zero with $k = 4$ at $\vartheta_\alpha = 20$, 43, and 57° show that in the forward hemisphere too the mechanism of this reaction is not exhausted by the mechanism of 1p-nucleon pickup.

3. Finally, the symmetry angle of the form of the angular correlation function for simple single-step mechanisms

TABLE III. Values of $d\sigma/d\Omega$ and the invariants R_i, mb/sr, in the $^{13}C(^3He, \alpha')^{12}C(2^+)$ reaction for different $\vartheta_{\alpha lab}$ at $E_{^3He} = 22.5$ MeV (Ref. 47).

$d\sigma/d\Omega$, R_i	20°	43°	57°	160°
$d\sigma/d\Omega$	0.90(2)	0.30(1)	0.20(1)	0.18(1)
R_1	0.62(9)	0.06(5)	0.09(5)	0.08(3)
R_2	−1.60(3)	−0.96(6)	−0.84(5)	−0.44(4)
R_3	−4.0(1)	−6.0(5)	−2.0(3)	0.34(8)

must depend monotonically on the kinematic variables (the angles ϑ_y, and the energies E_x and E_y). If the experimentally determined parameters Θ_0 and Θ_{00} exhibit a nonmonotonic dependence on ϑ_y, E_x, and E_y,[29] then the reaction mechanism must be fairly complicated.

More detailed information about the reaction mechanism and the structure of the nuclear states can be obtained by comparing the experimentally deduced $\mathbf{A}_{k\varkappa}(\vartheta_y)$ with theoretical values obtained in the framework of definite theoretical approaches. We shall analyze here the density-matrix spin tensors [the parameters $\mathbf{A}_{k\varkappa}(\vartheta_y)$] in the framework of the DWBA with a finite range of the interaction (FRDWBA), which was developed in detail at the Institute of Nuclear Physics[63,64] on the basis of the pioneering studies of Satchler *et al.*[65]

According to Refs. 63 and 64, the matrix element of the reaction $A(x, y)B$ in the FRDWBA is treated in the approximation of a three-body problem, i.e., the way in which the reaction takes place is fixed as follows:

$$x + A \to (y + c) + A \to y + (c + A) \to y + B \quad \text{for direct processes} \tag{75a}$$

and

$$x + A \to x + (y + C) \to y + (x + C) \to y + B \quad \text{for exchange processes.} \tag{75b}$$

Since the direct and exchange processes in the three-body problem correspond to different cluster decompositions, interference does not arise between them, and the total reaction cross section is determined by the sum of the cross sections of the direct and exchange processes.

In accordance with Refs. 63 and 64, the matrix element of the reaction $A(x, y)B$ in the FRDWBA for direct processes has the form

$$\mathbf{T}_{if} = \sum_{\substack{J_1M_1J_2M_2lm_l \\ J_cE_c^*\Lambda_1\Lambda_2}} \langle J_AM_AJ_1M_1 \mid J_fM_f\rangle \times \langle J_yM_y\, J_2M_2 \mid J_xM_x\rangle (-1)^{l+M_2+\Lambda_1+\Lambda_2} \langle J_2 - M_2J_1M_1 \mid lm_l\rangle \sqrt{(2J_1+1)(2J_2+1)}\, \Theta_{J_1J_2J_c}^{\Lambda_1\Lambda_2l}\, \beta_{\Lambda_1\Lambda_2lm_l}^{J_cE_c^*}(\vartheta_y), \tag{76}$$

where $\Theta_{J_1J_2J_c}^{\Lambda_1\Lambda_2l}$ are structure factors, and $\beta_{\Lambda_2\Lambda_1lm_l}^{J_cE_c^*}(\vartheta_y)$ are kinematic factors that determine the reaction mechanism. The matrix element (76) in the FRDWBA can also be calculated for the exchange processes. In this case, it is necessary in (76) to make the index interchange $A \rightleftarrows$ and replace the angles ϑ_y by $\pi - \vartheta_y$.

Substituting the expression (76) in the definitions (6) and (15), we obtain for the spin tensors of the state with spin J_f formed in the reaction $A(x, y)B(J_f)$ in the FRDWBA the expression

$$\rho_{k\varkappa}(\vartheta_y) = \frac{(2J_f+1)}{(2J_A+1)} \frac{\mu_{xA}\mu_{yB}}{(2\pi\hbar^2)^2} \frac{k_y}{k_x} \times \sum_{J_1J_1'J_2} \sqrt{(2J_1+1)(2J_1'+1)}\, w(J_1J_fJ_1'J_f : J_Ak) \times \sum_{\substack{ll'm_lm_l'J_c \\ \Lambda_1\Lambda_1'\Lambda_2\Lambda_2'J_c'}} (-1)^{J_2+J_A+J_f+m_l'} \langle lm_ll' - m_l' \mid k\varkappa\rangle \times \sqrt{(2l+1)(2l'+1)}\, W(lJ_1l'J_1' : J_2k) \times \Theta_{J_1J_2J_c}^{\Lambda_1\Lambda_2l}\, \Theta_{J_1'J_2J_c'}^{\Lambda_1'\Lambda_2'l'}\, \beta_{\Lambda_1\Lambda_2lm_l}^{J_cE_c^*}(\vartheta_y)\, \beta_{\Lambda_1'\Lambda_2'l'm_l'}^{*J_c'E_c^*}(\vartheta_y). \tag{77}$$

Expressions for calculating the structure factors $\Theta_{J_1J_2J_c}^{\Lambda_1\Lambda_2l}$ are given in Ref. 66. For reactions with the participation of light particles that do not have their own excited states ($x, y \leqslant {}^4$He) and with neglect of the dependence of the bound-state wave functions on the excitation energy E_c^* of the intermediate system, the sums over J_c in the expressions for $\Theta_{J_1J_2J_c}^{\Lambda_1\Lambda_2l}$ can be calculated explicitly,[66] and the expressions themselves greatly simplify, the total angular momenta J_x and J_y being replaced by the spins s_x and s_y.

TABLE IV. Values of the spin tensors $\mathbf{A}_{k\varkappa}(\vartheta_\alpha)$, mb/sr, in the $^{13}C(^3He, \alpha')^{12}C(2^+)$ reaction at $E_{^3He} = 22.5$ MeV (Ref. 47) for different $\vartheta_{\alpha lab}$.

Spin tensor	20°	43°	57°	160°
$\mathbf{A}_{00}$	0.90(2)	0.30(1)	0.200(8)	0.180(7)
$\mathbf{A}_{20}$	0.27(4)	0.006(20)	−0.01(2)	0.04(1)
$\mathbf{A}_{21}$	−0.13(3)	−0.02(2)	0.007(12)	0.01(1)
$\mathbf{A}_{22}$	0.11(2)	0.11(1)	0.05(1)	0.02(1)
$\mathbf{A}_{40}$	−0.54(5)	0.08(3)	0.00(2)	0.02(2)
$\mathbf{A}_{41}$	0.28(4)	0.03(2)	−0.06(1)	0.05(1)
$\mathbf{A}_{42}$	0.15(3)	0.06(1)	0.02(1)	−0.05(1)
$\mathbf{A}_{43}$	−0.07(4)	−0.01(2)	0.02(1)	0.06(1)
$\mathbf{A}_{44}$	−0.04(3)	−0.035(16)	−0.03(1)	0.00(1)
χ^2	18.5	10	11	10

For such reactions the intermediate angular momenta J_1 and J_2 in the matrix element (76) can be replaced by other intermediate angular momenta, namely, the total spin transfer s and the channel spin j:

$$\mathbf{s} = \mathbf{s}_x + \mathbf{s}_y, \ \mathbf{j} = \mathbf{J}_f + \mathbf{l} = \mathbf{J}_A + \mathbf{s}.$$

Then the structure factors $\Theta_{J_1J_2}^{\Lambda_1\Lambda_2l}$ determined by Eq. (10) in Ref. 66 for reactions with particles not heavier than ^{4}He take the following form:

1. Direct processes:

$$\Theta_{J_1J_2}^{\Lambda_1\Lambda_2l} = \delta_{\Lambda_2 0}\delta_{\Lambda_1 l}\delta_{J_2 s_c}\sum_j (-1)^{J_f+s_c+J_1+j} \times \sqrt{2l+1}\, u(J_A J_f s_c l : J_1 f)\, \Theta_{ls=s_c j}^{B\to A+c}, \quad (78)$$

where

$$\Theta_{lsj}^{B\to A+c} = \sum_{L_A S_A L_f S_f} (-1)^{L_f+L_A+l}\, \Theta_{lsL_A L_f S_A S_f}^{B\to L+c} \times u(lL_A J_f S_f : L_f j)\, u(L_A S_A j s_c : J_A S_f). \quad (78a)$$

2. Exchange processes:

$$\Theta_{J_1J_2}^{\Lambda_1\Lambda_2 l} = \frac{1}{\sqrt{2J_A+1}}(-1)^{J_1-J_2}\sum_{sj}\Theta_{sj}^{\Lambda_1\Lambda_2 l} \times u(lJ_f J_2 s_x : jJ_1)\, u(s_x s_y j J_A : sJ_2), \quad (79)$$

where

$$\Theta_{sj}^{\Lambda_1\Lambda_2 l} = \sum_{L_A S_A L_f S_f L_C S_C} \Theta_{\Lambda_1 L_f S_f L_C S_C}^{B\to C+x}\Theta_{\Lambda_2 L_A S_A L_C S_C}^{A\to C+y} \times \sqrt{\frac{(2L_A+1)(2S_A+1)}{(2L_C+1)(2S_C+1)}}(-1)^{L_C-L_f+S_A-S_C+\Lambda_1+\Lambda_2} \times u(L_f\Lambda_1 L_A\Lambda_2 : L_C l) \times u(S_f S_x S_A S_y : S_C s)\, u(lL_A J_f S_f : L_f j) \times u(L_A S_A js : J_A S_f). \quad (79a)$$

In the expressions (78a) and (79a), $\Theta_{lL_A L_f S_A S_f}^{B\to A+c}$, $\Theta_{\Lambda_1 L_f S_f L_C S_C}^{B\to C+x}$, $\Theta_{\Lambda_2 L_A S_A L_C S_C}^{A\to C+y}$ are the amplitudes of the reduced widths in L–C coupling.[67]

When the angular momenta s and j are used, the density-matrix spin tensors for direct processes can be expressed in the FRDWBA as follows:

$$\rho_{k\varkappa}(\vartheta_y) = \frac{(2J_f+1)^{3/2}}{(2J_A+1)}\frac{\mu_{xA}\mu_{yB}}{(2\pi\hbar^2)^2}\frac{k_y}{k_x}\sum_{sjll'm_l m_l'}(-1)^{l+l'+m_l'+j-k} \times w(lJ_f l' J_f : jk)\sqrt{(2l+1)(2l'+1)}\langle l-m_l l' m_l' | k\varkappa\rangle \frac{1}{(2s+1)} \times \Theta_{lsj}^{B\to A+c}\Theta_{l'sj}^{B\to A+c}\beta_{lm_l}(\vartheta_y)\,\beta_{l'm_l'}^{*}(\vartheta_y). \quad (80)$$

For the exchange processes the analogous spin tensors $\rho_{k\varkappa}(\vartheta_y)$ have the form

$$\rho_{k\varkappa}(\vartheta_y) = \frac{(2J_f+1)^{3/2}}{(2J_A+1)(2s_x+1)}\frac{\mu_{xA}\mu_{yB}}{(2\pi\hbar^2)^2}\frac{k_y}{k_x} \times \sum_{sjll'm_l m_l'}(-1)^{l+l'+m_l+j-k} \times w(lJ_f l' J_f : jk)\sqrt{(2l+1)(2l'+1)} \times \langle l-m_l l' m_l' | k\varkappa\rangle \sum_{\Lambda_1\Lambda_2\Lambda_1'\Lambda_2'}\Theta_{sj}^{\Lambda_1\Lambda_1 l}\Theta_{sj}^{\Lambda_1'\Lambda_2' l'} \times \beta_{\Lambda_1\Lambda_2 lm_l}(\vartheta_y)\,\beta_{\Lambda_1'\Lambda_2' l' m_l'}^{*}(\vartheta). \quad (81)$$

The expressions (80) and (81) clearly demonstrate the advantage of choosing the intermediate angular momenta s and j; for over them there are only incoherent summations.

The kinematic factors $\beta_{\Lambda_1\Lambda_2 lm_l}(\vartheta_y)$ in (76), (77), (80), and (81) include double integrals over the distorted waves of the initial and final channels of products of bound-state wave functions and interaction potentials (the actual expressions for them are given in Refs. 63 and 64). In the FRDWBA, as in other theoretical methods that take into account the finite range of the interaction, the bound-state wave functions are found by numerical calculations from the given binding energy and number of nodes.[68] As is shown by the results of calculations made in more rigorous theoretical approaches (few-body model,[69] resonating-group method[70]), these wave functions give a good description of the structure of the ground states of light nuclei, though the question of the description of the wave functions of excited states is to some extent still open.

For practical realization of the theoretical formalism for calculating the nuclear density matrix described above, a program has been written that is a modification of the program OLYMP-2,[71] designed for calculating differential cross sections of reactions. The new version of the program, OLYMP-3,[72] uses the main blocks of the program OLYMP-2, but since the calculation of the density-matrix spin tensors requires summation over the orbital angular momenta l and l' and the spins s and j, the structure of the main program has been partly changed in OLYMP-3 and an external memory has been connected to the computer. The program OLYMP-3 is written in FORTRAN-DUBNA and is adapted to the BÉSM-6 computer. The operative memory required for the program is 128 kbyte, and the external memory uses magnetic drums.

Actual results will be given below for two reactions: ^{12}C$(\alpha, \alpha')^{12}$C* and ^{11}B$(\alpha, t')^{12}$C*. The choice of precisely these reactions was dictated by the following considerations. For the reaction ^{12}C$(\alpha, \alpha')^{12}$C*, as was shown in the previous section, all the even-rank spin tensors of the density matrix of the 2^+ state have been determined,[45,46] so that one can hope that comparison of the theoretical and experimental spin tensors will make it possible to determine the contribution of the exchange processes to the inelastic α-particle scattering. The reaction ^{11}B$(\alpha, t')^{12}$C* has already been studied in the framework of the FRDWBA,[73] and it has been shown that the direct and exchange processes together enable one to describe its experimental cross section, i.e., the diagonal elements of the density matrix on the ^{12}C nucleus. Further, calculation shows that the cross section of this reaction is appreciably changed when allowance is made for the incoherent contribution of the different angular-momentum transfers l, and also the channel spins j. It is interesting to see to what extent too the nondiagonal elements of the density matrix are sensitive to the contribution of the different l and j.

^{12}C$(\alpha, \alpha')^{12}$C*. It is obvious in advance that the exchange processes (and it is they that determine the inelastic scattering cross section in the FRDWBA) cannot completely explain the angular distribution of the inelastically scattered α particles, particularly at forward angles, where the cross section is mainly determined by the coupling of the channels due to excitation of a rotational band of the ^{12}C nucleus.[26] In Ref. 26, the coupled-channel method was used

TABLE V. Parameters of optical potentials for elastic ($E_\alpha = 30$ MeV) and inelastic scattering of α particles by ^{12}C and ground-state potentials.[74]

Entrance and exit channels	V_0	r_{0V}	a_V	W_0	r_{0W}	a_W	$\alpha + {}^8Be$	
							r_{0c}	a_c
$\alpha + {}^{12}C\,(0^+)$	161.0	0.85	0.77	15.0	1.5	0.60	1.37	0.7
$\alpha + {}^{12}C\,(2^+)$	I. 149.0	1.20	0.69	4.8	1.75	0.76	1.25	0.50
	II. 149.0	1.20	0.69	4.8	1.75	0.76	1.00	0.65
	III. 140.0	1.24	0.69	4.8	1.75	0.76	1.60	0.70

to calculate the differential cross section of inelastic scattering [the parameter $\mathbf{A}_{00}(\vartheta_\alpha)$] in the forward hemisphere of α-particle emission and the parameters of the three-parameter form (63) for the angular correlation function in the reaction plane. The results showed that for a given set of parameters of the calculation it is not possible to describe simultaneously the cross section and the angular correlation function, i.e., even in the forward hemisphere the coupled-channel method does not completely determine the mechanism of inelastic scattering of α particles by the ^{12}C nucleus. At the same time, one can hope that at large emission angles of the α particles, where the contribution of the exchange processes must be fairly large, these processes will determine the behavior of not only the differential cross section but also the other spin tensors of the density matrix.

In Refs. 74 and 75, we calculated $\mathbf{A}_{k\varkappa}(\vartheta_\alpha)$ for the 2^+ level of the ^{12}C nucleus formed in inelastic scattering of α particles. The main parameters that determine the theoretical values of the spin tensors in the FRDWBA are the parameters of the optical potentials, and also the potentials of the bound states in the virtual vertices of the $^{12}C \rightarrow {}^8Be + \alpha$ decay. These parameters were fitted in the first place to describe the differential cross section of inelastic scattering, i.e., the quantity $\mathbf{A}_{00}$.

In the case of elastic scattering of α particles by ^{12}C several potentials are known, both phenomenological[76] and theoretical,[77] that enable one to reconcile the cross section with the experiments in a wide region of angles. For the entrance elastic channel we used a potential whose real part agrees with the theoretical potential calculated in the four-body problem[77] and whose imaginary part agrees with the phenomenological potential of Ref. 76 (the actual values of the parameters of this potential are given in Table V). For the exit channel (purely inelastic scattering), such information is completely absent. Therefore, in selecting the interaction potentials we were guided by qualitative physical arguments:

1. Since the ^{12}C nucleus in the 2^+ state is strongly deformed, the depth of the real part of the potential was taken to be less, and the radius greater, than for the potential in the elastic channel.

2. In order to ensure the experimentally observed rise of the cross section in the region of large angles, the depth of the imaginary part of the potential was reduced.

Table V gives values of the parameters of the optical potentials and the potentials of the bound states giving a perfectly satisfactory description of the experimental behavior of $\mathbf{A}_{00}$ as a function of ϑ_α in the complete interval of angles (Fig. 8). For all potentials in Table V, the dependence of $\mathbf{A}_{00}$ on ϑ_α hardly changes; only the normalizing factor is different.

The dependence of $A_{k\varkappa}(\vartheta_\alpha)$ with $k \neq 0$ on the parameters of the potential in the region of large angles is manifested mainly in a change in the values of the extrema rather than in a change of their positions (Fig. 9). The greatest sensitivity to a change in the parameters of the potential is exhibited by the component $\mathbf{A}_{k\varkappa}$ with $k = 4$. Figure 10 shows the ratio of $\mathbf{A}_{4\varkappa}$ for $\mathbf{A}_{00}$ as a function of ϑ_α for the potentials I and III. It can be seen from Fig. 10 that $\mathbf{A}_{4\varkappa}$, especially $\mathbf{A}_{40}$ and $\mathbf{A}_{41}$, are more sensitive to a change in the parameters of the potentials than $\mathbf{A}_{00}$ in the region of intermediate angles. In the region of backward angles, in which the contribution of the exchange mechanisms to the characteristics of the process must be decisive, not only the positions of the extrema but also their values coincide for the different potentials, i.e., in this region the sensitivity of all the $\mathbf{A}_{k\varkappa}$, including $\mathbf{A}_{00}$, to the choice of the parameters of the calculation is the same.

For comparison of the theoretical calculations with the experiments, we also give in Fig. 9 the experimental dependences of $\mathbf{A}_{k\varkappa}$ on ϑ_α. It can be seen that for all $\mathbf{A}_{k\varkappa}$ satisfactory agreement with the experiments cannot be obtained even in the region of backward angles, the components $\mathbf{A}_{22}$, $\mathbf{A}_{40}$, and $\mathbf{A}_{41}$ being described least well. Since in this region the sensitivity to a change in the parameters of the potentials is least, and the set of potentials of Table V can be regarded as realistic and giving a good description of the cross section of the process, the discrepancy between the theory and experiment is evidently of a basic nature and cannot be overcome by merely varying the parameters of the potential.

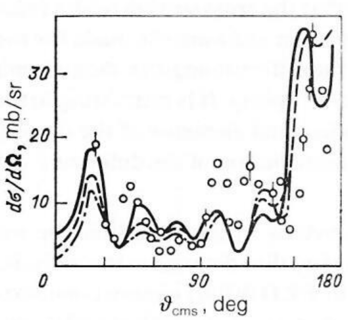

FIG. 8. Differential cross section of inelastic scattering of α particles by ^{12}C ($E_\alpha = 30$ MeV). The experimental data are from Ref. 49. The calculation was made in the FRDWBA[75] with the parameters of the potentials in Table V. The continuous curve is for potential I, the broken curve for potential II, and the chain curve for potential III.

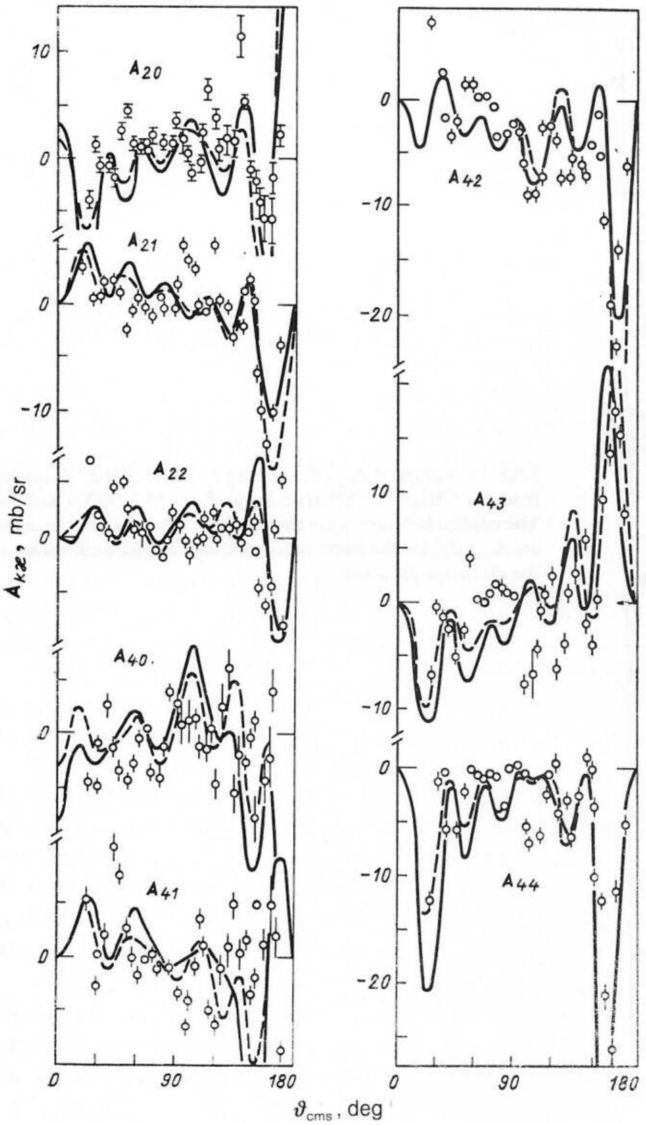

FIG. 9. Dependence of the theoretical $\mathbf{A}_{k\varkappa}$ for the 2^+ state of the ^{12}C nucleus on ϑ_α.[75] The continuous curve is for potential I in Table V, and the broken curve is for potential II. The experimental data are from Ref. 49.

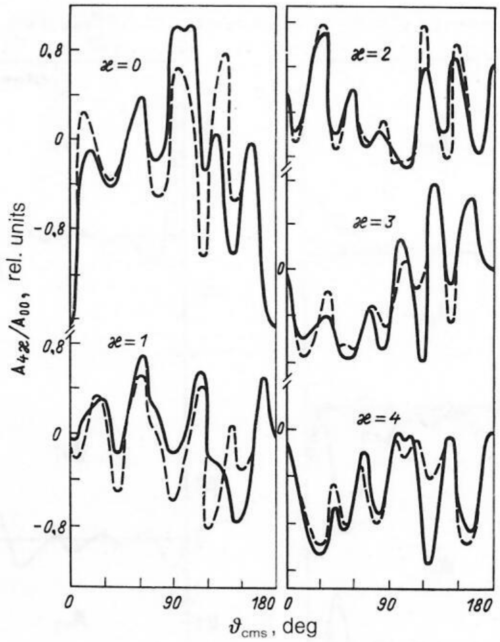

FIG. 10. Sensitivity of $\mathbf{A}_{4\varkappa}/\mathbf{A}_{00}$ to the choice of the parameters of the potentials.[75] The continuous curve is for potential I, and the broken curve for potential II.

^{11}B(α, t′)^{12}C*. In Ref. 73, an optimal set of parameters of the optical potentials in the entrance and exit channels of the reaction and of the potentials of the bound states was determined, giving the best agreement with experiment. In our calculations in Ref. 74 we used precisely these parameters of the optical potentials and the potentials of the bound states.

As follows from the general expressions (80) and (81), the spin tensors in the FRDWBA are determined by coherent sums over the angular-momentum transfers l and their projections, and over the angular momenta Λ_1 and Λ_2 at the vertices of virtual breakup of the nuclei A and B. In addition, the expressions for the spin tensors contain incoherent sums over the spin transfer s and the channel spin j. For the ^{11}B(α, t')^{12}C* reaction, these sums contain the following terms:

1. Direct processes: $\Lambda_1 = l = 1$, $\Lambda_2 = 0$, $s = \frac{1}{2}$, $j = \frac{1}{2}$.

2. Exchange processes: $\Lambda_1 = 4$, $\Lambda_2 = 3$ (Ref. 78), $l = 1,...,4$, $s = \frac{1}{2}$, $j = 1, 2$.

Numerical estimates showed that the contributions illustrated by the triangle diagrams (heavy substitution for direct processes and ordinary substitution for exchange processes) are suppressed in the spin tensors $\mathbf{A}_{k\varkappa}(\vartheta_t)$ by several orders of magnitude compared with the contributions of the pole mechanisms. Therefore, all the following calculations are given only for the pole mechanisms (stripping for direct processes, heavy stripping for the exchange processes).

Figure 11 shows the $\mathbf{A}_{k\varkappa}(\vartheta_t)$ as functions of the angle ϑ_t calculated by means of the program OLYMP-3 (Ref. 72) for the direct and exchange processes. It can be seen from the figure that $\mathbf{A}_{2\varkappa}(\vartheta_t)$ for the direct processes are mainly concentrated in the forward hemisphere, and for the exchange processes in the backward hemisphere. The only exception is $\mathbf{A}_{21}(\vartheta_t)$, for which the contribution of the stripping mechanism at angles equal to $-175°$ exceeds by a factor 2 the contribution of the heavy-stripping mechanism. The contribution of the direct processes to the $\mathbf{A}_{4\varkappa}(\vartheta_t)$ is strictly zero. Nevertheless, the $\mathbf{A}_{4\varkappa}(\vartheta_t)$ are not small even in the forward hemisphere; this is particularly true for $\mathbf{A}_{40}(\vartheta_t)$, which at small angles is comparable with $\mathbf{A}_{00}(\vartheta_t)$.

The value and nature of the behavior of $\mathbf{A}_{k\varkappa}(\vartheta_t)$ for the exchange processes depends strongly on the sums over the angular-momentum transfers l and the spins j. It is interesting to note that the components with different l make different contributions to $\mathbf{A}_{00}$ and $\mathbf{A}_{k\varkappa}$ ($k \neq 0$); for in $\mathbf{A}_{00}$ the main contribution is made by the components with $l = 1$ and 3, while in $\mathbf{A}_{k\varkappa}$ ($k \neq 0$) it is made by those with $l = 2$ and 3. The biggest change when allowance is made for incoherent summation over the spin j occurs in the $\mathbf{A}_{k\varkappa}(\vartheta_t)$. As can be seen from Fig. 12, the contribution of the component with spin $j = 2$ appreciably exceeds the contribution of the component with spin $j = 1$. Allowance for summation over j leads to a significant change in the nature of the dependence of $\mathbf{A}_{k\varkappa}$ on ϑ_t for both $\mathbf{A}_{2\varkappa}(\vartheta_y)$ and $\mathbf{A}_{4\varkappa}(\vartheta_t)$.

To conclude this section, we compare the parameters

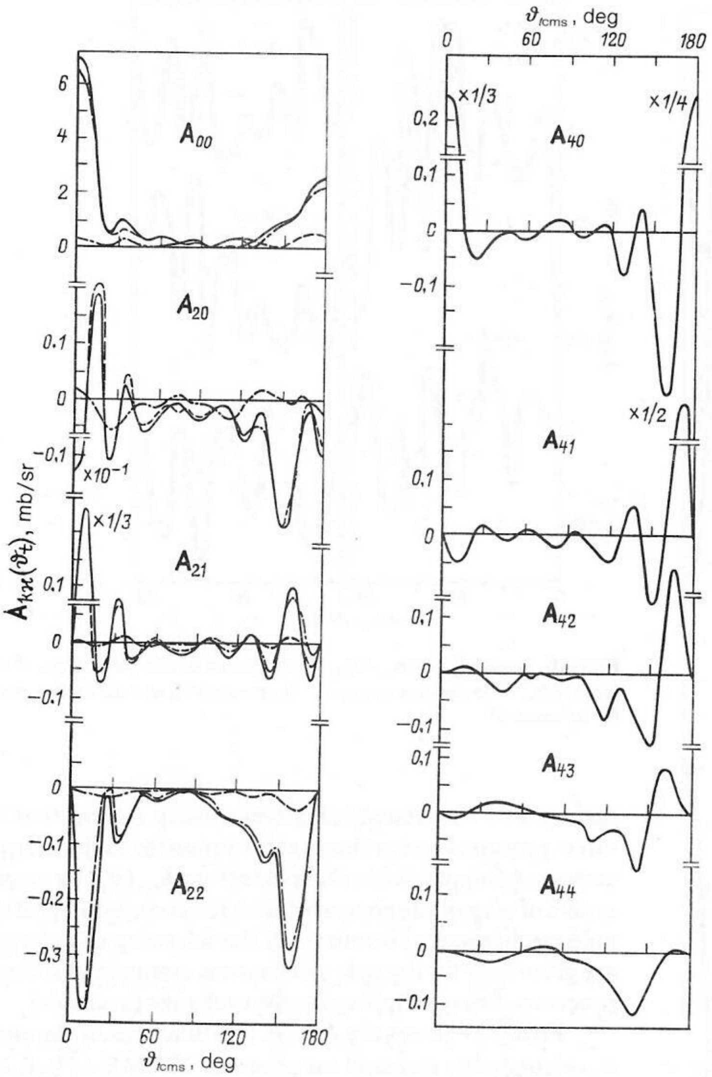

FIG. 11. Values of $\mathbf{A}_{k\varkappa}(\vartheta_t)$ for the 2^+ level of the ^{12}C nucleus from the ^{11}B$(\alpha, t')^{12}$C* reaction at $E_\alpha = 25.1$ MeV (Ref. 74). The continuous curves are the total $\mathbf{A}_{k\varkappa}(\vartheta_t)$, the broken curves are $\mathbf{A}_{k\varkappa}(\vartheta_t)$ for the direct processes, and the chain curves are for the exchange processes.

$\mathbf{A}_{k\varkappa}(\vartheta_t)$ of one and the same state of the ^{12}C nucleus formed in different ways. As can be seen from Figs. 9 and 11, the nature of the behavior of $\mathbf{A}_{k\varkappa}(\vartheta_y)$ as functions of the emission angles of the final particles differs fundamentally. To eliminate the common dependence on the angle associated with the dependence of the differential cross section, we compared the relative values $\mathbf{A}_{20}/\mathbf{A}_{00}$ and $\mathbf{A}_{44}/\mathbf{A}_{00}$ for the exchange processes in the (α, α') and (α, t') reactions (Fig. 13). We found that the ratios $\mathbf{A}_{k\varkappa}/\mathbf{A}_{00}$ corresponding to different ways of formation of the 2^+ state of the ^{12}C nucleus

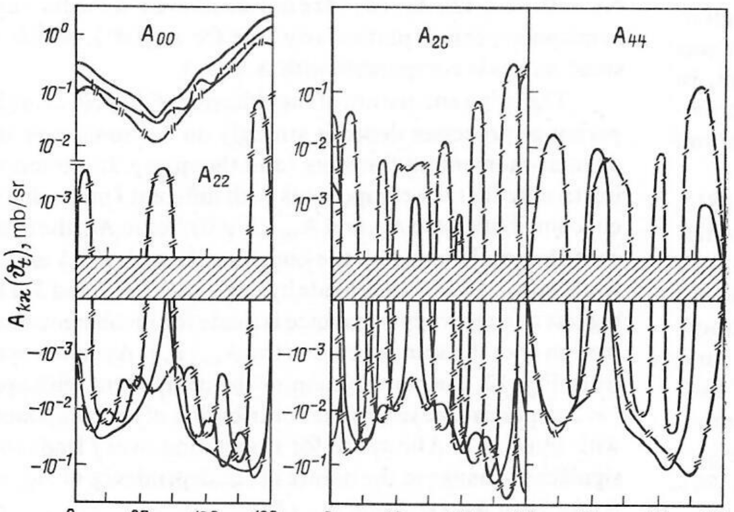

FIG. 12. Some $\mathbf{A}_{k\varkappa}(\vartheta_t)$ (continuous curves) for the 2^+ state of the ^{12}C nucleus from the ^{11}B$(\alpha, t')^{12}$C* reaction.[74] The curves with one and two cross strokes show the contributions of the components with j equal to 1 and 2, respectively.

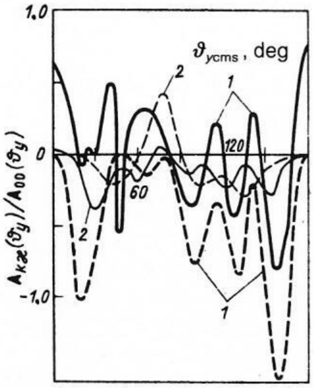

FIG. 13. The ratios $\mathbf{A}_{20}/\mathbf{A}_{00}$ (broken curves) and $\mathbf{A}_{44}/\mathbf{A}_{00}$ (continuous curves) for the 2^+ state of the ^{12}C nucleus formed in the reactions $^{12}\mathrm{C}(\alpha, \alpha')^{12}\mathrm{C}^*$ (heavy curves 1) and $^{11}\mathrm{B}(\alpha, t')^{12}\mathrm{C}^*$ (light curves 2).

differ strongly, and in their behavior as functions of ϑ_y it is difficult to find systematic correlations. This evidently shows [an analogous situation also holds, as we noted in the previous section, in the reaction $^{13}\mathrm{C}(^3\mathrm{He}, \alpha')^{12}\mathrm{C}^*$] that the spin tensors of the density matrix are mainly sensitive in the FRDWBA to the type of reaction and not to the structure of the given state of the nucleus. We have already noted above that in the FRDWBA the structure of the bound-state functions is determined only by the binding energy and the number of nodes. More detailed structural characteristics such as the deformation of the nucleus are not taken into account in the FRDWBA. It is this that evidently explains the low sensitivity of the theoretical spin tensors to the structure of the wave functions of the ^{12}C nucleus.

7. SPIN CHARACTERISTICS OF EXCITED NUCLEAR STATES AND THEIR THEORETICAL ANALYSIS

The experimentally determined model-independent components of the spin tensors $\boldsymbol{\rho}_{k\varkappa}(\vartheta_y)$ [or, equivalently, the parameters $\mathbf{A}_{k\varkappa}(\vartheta_y)$] can be used to determine other spin characteristics of the final nucleus without additional experimental measurements. Above all, the measured $\boldsymbol{\rho}_{k\varkappa}(\vartheta_y)$ can be used to obtain the diagonal elements of the density matrix itself, i.e., the populations of the substates of the given level of the final nucleus, this being equivalent to the determination of the spin orientation of this level. Obviously, to calculate the substate populations of the level it is necessary to use a coordinate system whose z axis is perpendicular to the reaction plane and along the direction of the spin of the nucleus (see Fig. 2). Since the parameters $\mathbf{A}_{k\varkappa}(\vartheta_y)$ are found in a coordinate system whose z axis is along the beam of the incident particles (see Fig. 1), the required populations of the substate with projection m are determined as follows:

$$p_m^{J_f} = \frac{\boldsymbol{\rho}_{J_f}(m, m)}{\mathrm{Sp}\,\boldsymbol{\rho}'_{J_f}(m, m)} = \frac{1}{\mathbf{A}_{00}} \sum_{M_f M'_f} \mathbf{D}^{*J_f}_{mM_f} \times \left(0, \frac{\pi}{2}, \frac{\pi}{2}\right) \boldsymbol{\rho}_{J_f}(M_f, M'_f)\, \mathbf{D}^{J_f}_{mM'_f}\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right), \quad (82)$$

where $\boldsymbol{\rho}_{J_f}(M_f, M'_f)$ is given by the expression (6). If we do not detect the polarization of the final particles, $\boldsymbol{\rho}_{J_f}(M_f, M'_f)$ is symmetric not only with respect to the principal diagonal but also with respect to the secondary diagonal. In this case, for transitions with $J_f = L = 2^+$ we can readily obtain from (82) the following expressions for the populations[5)]:

$$\left.\begin{aligned} p_0^{(2+)} &= \frac{1}{5}\left\{1 - \frac{\mathbf{A}_{20}}{\mathbf{A}_{00}} - \frac{9}{16}\frac{\mathbf{A}_{40}}{\mathbf{A}_{00}} - \sqrt{6}\,\frac{\mathbf{A}_{22}}{\mathbf{A}_{00}} - \frac{3}{4} \times \sqrt{\frac{5}{2}}\,\frac{\mathbf{A}_{42}}{\mathbf{A}_{00}} - \frac{3\sqrt{70}}{16}\frac{\mathbf{A}_{44}}{\mathbf{A}_{00}}\right\};\\ p_{\pm 1}^{(2+)} &= \frac{1}{5}\left\{1 - \frac{1}{2}\frac{\mathbf{A}_{20}}{\mathbf{A}_{00}} + \frac{3}{8}\frac{\mathbf{A}_{40}}{\mathbf{A}_{00}} - \frac{1}{2}\sqrt{6} \times \frac{\mathbf{A}_{22}}{\mathbf{A}_{00}} + \frac{1}{2}\sqrt{\frac{5}{2}}\,\frac{\mathbf{A}_{42}}{\mathbf{A}_{00}} + \frac{\sqrt{70}}{8}\frac{\mathbf{A}_{44}}{\mathbf{A}_{00}}\right\};\\ p_{\pm 2}^{(2+)} &= \frac{1}{5}\left\{1 + \frac{\mathbf{A}_{20}}{\mathbf{A}_{00}} - \frac{3}{32}\frac{\mathbf{A}_{40}}{\mathbf{A}_{00}} + \sqrt{6}\,\frac{\mathbf{A}_{22}}{\mathbf{A}_{00}} - \frac{1}{8}\sqrt{\frac{5}{2}}\,\frac{\mathbf{A}_{42}}{\mathbf{A}_{00}} - \frac{\sqrt{70}}{32}\frac{\mathbf{A}_{44}}{\mathbf{A}_{00}}\right\} \end{aligned}\right\} \quad (83)$$

[it is readily seen that the normalization condition $\Sigma_m p_m^{(J_f)} = 1$ is automatically satisfied]. Since $p_m^{(J_f)} \geqslant 0$, the expressions (83) impose subsidiary restrictions on the $\mathbf{A}_{k\varkappa}$ with even $\varkappa$.

If the condition (60) is satisfied, then in this case of simplified parametrization the substate populations can be expressed by the relations (see footnote 5)

$$\left.\begin{aligned} \mathbf{p}_0^{(2^+)} &= \frac{1}{4} - \frac{1}{2}\,\frac{\mathbf{T}_1^2 - 2\mathbf{T}_2^2 - \frac{1}{2}\mathbf{T}_{20}}{\mathbf{T}_0^2 + 2\mathbf{T}_1^2 + 2\mathbf{T}_2^2};\\ \mathbf{p}_{\pm 2}^{(2^+)} &= \frac{3}{8} + \frac{1}{4}\,\frac{\mathbf{T}_1^2 - 2\mathbf{T}_2^2 - \frac{1}{2}\mathbf{T}_{20}}{\mathbf{T}_0^2 + 2\mathbf{T}_1^2 + 2\mathbf{T}_2^2}. \end{aligned}\right\} \quad (84)$$

Some general propositions can be formulated for the substate populations. First, in reactions with spinless particles on even–even nuclei [the condition (60) is satisfied] the populations of the substates with $m = \pm 1$ are strictly zero. Second, it can be readily seen from (80) that in single-nucleon transfer reactions on 1p-shell nuclei for any mechanism of the process the quantities $p_{\pm m}^{(2+)}$ in the case of a $2^+ \to 0^+$ transition do not depend on the angle ϑ_y and are determined by

$$p_0^{(2+)} = \frac{1}{5}\left(1 + \frac{1}{2}\,\frac{\sum_j (-1)^j \Theta_j^2}{\sum_j \Theta_j^2}\right); \quad (85a)$$

$$p_{\pm 1}^{(2+)} = \frac{1}{5}\left(1 + \frac{1}{4}\,\frac{\sum_j (-1)^j \Theta_j^2}{\sum_j \Theta_j^2}\right); \quad (85b)$$

$$p_{\pm 2}^{(2+)} = \frac{1}{5}\left(1 - \frac{1}{2}\,\frac{\sum_j (-1)^2 \Theta_j^2}{\sum_j \Theta_j^2}\right), \quad (85c)$$

where the structure factor Θ_j^2 depends only on the channel spin j. For 1p-shell nuclei, the values of J_A can be 3/2 and 1/2, i.e., the channel spin has only two values: j equal to 1 and 2. If the selection rules allow one value of j, the populations $p_{\pm m}^{(2+)}$ do not depend on the structure of the wave function of the final nucleus. In Table VI, we give the values of $p_{\pm m}^{(2+)}$ calculated in accordance with (85) for different single-nucleon transfer reactions with $l = 1$ and $s = 1/2$. In the pickup reaction $^{13}\mathrm{C}(^3\mathrm{He}, \alpha')^{12}\mathrm{C}(2^+)$ the channel spin j has one value, equal to unity, and for this reaction $p_{\pm m}^{(2+)}$ have the values 0.10, 0.15, and 0.30 irrespective of the wave-function structure of the 2^+ state of the ^{12}C nucleus.

Figure 14 gives the populations of the substates of the 2^+ level of the ^{12}C nucleus formed in inelastic scattering of α particles with energy $E_\alpha = 30$ MeV. The experimental data

TABLE VI. Values of $p_{\pm m}^{(2+)}$ in single-nucleon transfer reactions with $s=\frac{1}{2}, l=1$ on 1p-shell nuclei.

Reaction	$^{11}B(\alpha, t')^{12}C(2+)$	$^{13}C(^3He, \alpha')^{12}C(2+)$	$^{11}B(^3He, d')^{12}C(2+)$
p_0	0.2424	0.1000	0.2424
$p_{\pm1}$	0.2212	0.1500	0.2212
$p_{\pm2}$	0.1576	0.3000	0.1576

on $A_{k\varkappa}$ were taken from Ref. 49, and the experimental values for the populations were obtained in accordance with the expressions (83) from these data. The theoretical curves were obtained for the exchange processes in the FRDWBA for potential I in Table V. It can be seen from the figure that as the parameters of the potentials are changed the behavior of the populations changes weakly, the main changes occurring in the region of intermediate angles $\vartheta_\alpha \simeq 90$–$120°$. It can be seen that the curves of the populations for the sublevels with $m=0$ and $m=\pm2$ have a well-defined oscillatory nature, and therefore it is difficult to speak of a preferred population of any sublevel, although the amplitudes of the oscillations are larger for the sublevel with $m=0$.

Comparison with experiment shows that the theoretical curves agree well with the experimental points, the positions of all the extrema being almost coincident. In the region of angles 90–120°, the calculation somewhat overestimates the population of the sublevel with $m=0$; for the sublevels with $m=\pm2$ the discrepancy with experiment is smaller.

In Ref. 35, the population p_0 of the 2^+ state of the ^{12}C nucleus determined in the experiments of Refs. 31 and 32 was analyzed by the coupled-channel method in a coordinate system whose z axis was directed along the momentum of the recoil nucleus (inelastic scattering of α particles was considered). It was shown in Ref. 35 that the coupled-channel method cannot simultaneously describe the differential cross section and the populations of the magnetic substates of the 2^+ level. Figure 15 shows the theoretical values[35] of p_0, for which the parameters of the calculation were chosen to give the best agreement between p_0 and experiment. In the same figure we give the theoretical values of p_0 calculated in

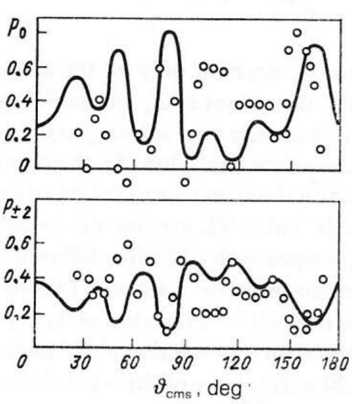

FIG. 14. Populations of the magnetic sublevels of the 2^+ state of the ^{12}C nucleus calculated in the FRDWBA[75] for inelastic scattering of α particles. The continuous curve is for potential I from Table V. The experimental data are from Ref. 49.

the FRDWBA (analogous to the p_0 values in Fig. 14 except for the change in the energy and the coordinate system). It can be seen that the FRDWBA gives a better description of the experimental data at large angles, and the coupled-channel method gives a better description at small and intermediate angles. We emphasize once more that the FRDWBA describes simultaneously both $d\sigma/d\Omega$ (see Fig. 8) and p_0, while the coupled-channel method cannot do this.[35]

Figure 16 shows the populations $p_{\pm m}$ calculated in the FRDWBA for the same ^{12}C level but formed in the $^{11}B(\alpha, t')^{12}C(2^+)$ reaction. It is important to emphasize that the behavior of $p_{\pm m}$ is fundamentally different in character from that of $p_{\pm m}$ of the same state formed in inelastic scattering. Further, the value of $p_{\pm1}$ for this reaction is determined for the direct processes solely by the structure factors. Its value is 0.22 and does not depend on ϑ_t. For the exchange processes in the case when the selection rules permit several values of l, the value of $p_{\pm m}$ is no longer constant, but nevertheless its dependence on the angle ϑ_t does not have a well-defined nature. We note also that the theoretically obtained value $p_{\pm1}=0.15$ for the $^{13}C(^3He, \alpha')^{12}C^*$ reaction is completely confirmed experimentally.[47]

Let us now consider what further spin characteristics of the excited states of nuclei can be obtained by means of the independent density-matrix parameters of these states. It is obvious that in general a knowledge of the irreducible spin tensors of the density matrix makes it possible to determine the orientation of the tensor operators of different ranks (up to rank k) that characterize a given state of a nucleus. Indeed, it follows from the general propositions of quantum mechanics that the expectation value of the spherical component of any tensor operator is determined by

$$\langle J_f | T_{k\varkappa}(\vartheta_y) | J_f \rangle = \frac{1}{\mathrm{Sp}\,\rho} \mathrm{Sp}(\rho T_{k\varkappa}). \qquad (86)$$

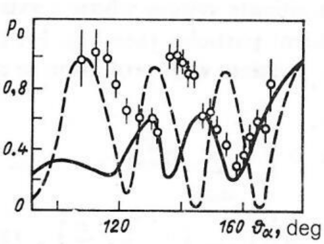

FIG. 15. Angular distribution of the population p_0 of the sublevel with $\alpha=0$ of the 2^+ state of the ^{12}C nucleus in inelastic α-particle scattering. The experimental data are taken from Refs. 31 and 32. The continuous curve is the calculation in the FRDWBA (it coincides with the continuous curve of Fig. 14 exactly, apart from a rotation of the coordinate system and a change of the energy); the broken curve is the calculation by the coupled-channel method.[35]

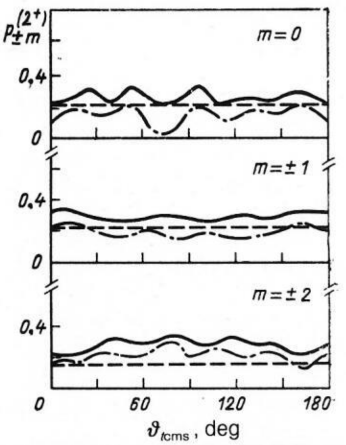

FIG. 16. Angular dependences of the sublevel populations of the 2$^+$ state of the ^{12}C nucleus formed in the ^{11}C(α, t')^{12}C* reaction, calculated in the FRDWBA.[50] The continuous curves are the total populations, the broken curves are the populations for the direct processes, and the chain curves are those for the exchange processes.

We write out (86) in more detail and note that in accordance with the normalization condition (15a) Sp $\rho = \rho_{00}$. As a result, we obtain

$$\rho_{00}(\vartheta_y)\langle J_fM_f|\mathbf{T}_{k\varkappa}(\vartheta_y)|J_fM_f\rangle = \sum_{M'_f}\langle J_fM_f|\rho_{J_f}(M_f, M'_f)|J_fM'_f\rangle\langle J_fM'_f|\mathbf{T}_{k\varkappa}(\vartheta_y)|J_fM_f\rangle. \quad (87)$$

Using the Wigner–Eckart theorem,[79] we can transform the expression (87) to

$$\rho_{00}(\vartheta_y)\langle J_fM_f|\mathbf{T}_{k\varkappa}(\vartheta_y)|J_fM_f\rangle = \frac{1}{\sqrt{2k+1}}\langle J_f\|\mathbf{T}_k\|J_f\rangle \times \sum_{M'_f}(-1)^{J_f-M'_f}\langle J_fM_fJ_f-M'_f|k\varkappa\rangle \times \langle J_fM_f|\rho_{J_f}(M_f, M'_f)|J_fM'_f\rangle = \frac{1}{\sqrt{(2k+1)(2J_f+1)}}\langle J_f\|\mathbf{T}_k(\vartheta_y)\|J_f\rangle\rho_{k\varkappa}(\vartheta_y), \quad (88)$$

or, in more perspicuous form,

$$\mathbf{t}^{(J_f)}_{k\varkappa}(\vartheta_y) \equiv \frac{\langle J_fM_f|\mathbf{T}_{k\varkappa}(\vartheta_y)|J_fM_f\rangle}{\langle J_f\|\mathbf{T}_k(\vartheta_y)\|J_f\rangle} = \frac{1}{\sqrt{(2k+1)(2J_f+1)}}\frac{\rho_{k\varkappa}(\vartheta_y)}{\rho_{00}(\vartheta_y)}. \quad (89)$$

The relation (89) connects the ratio of a component of a tensor operator of arbitrary rank to its reduced matrix element (which contains all information about the structure of the given state of the nucleus and reflects its dynamical characteristics) to the geometrical properties of this state, i.e., the ratio $\rho_{k\varkappa}(\vartheta_y)/\rho_{00}(\vartheta_y)$. The quantity $\mathbf{t}^{(J_f)}_{k\varkappa}(\vartheta_y)$ characterizes the orientation of the corresponding multipole moment relative to the symmetry axis of the nucleus.

We shall show concretely how, using the parameters $\mathbf{A}_{k\varkappa}(\vartheta_y)$ with even k found experimentally, one can calculate the orientations of the tensor operators of rank k equal to 2 and 4. First of all, it must be noted that the relation (89) is valid in the coordinate system whose z axis coincides with the symmetry axis of the nucleus (with the momentum of the recoil nucleus). But in Refs. 45–49 a coordinate system with the z axis along the direction of the incident beam was chosen. Therefore, when the experimental values of $\mathbf{A}_{k\varkappa}(\vartheta_y)$ are used it is necessary to convert them to a different coordinate system with the z axis along the axis of the nucleus. This transition is made as follows:

$$\widetilde{\mathbf{A}}_{k\varkappa}(\vartheta_y) = \sum_{\varkappa'}\mathbf{D}^{*k}_{\varkappa\varkappa'}(0, \vartheta_q, 0)\mathbf{A}_{k\varkappa'}(\vartheta_y), \quad (90)$$

where the angle ϑ_q is given by

$$\cos\vartheta_q = \frac{k_x - k_y\cos\vartheta_y}{\sqrt{k_x^2 + k_y^2 - 2k_xk_y\cos\vartheta_y}}.$$

Realizing the transformation (90), we can obtain for the tensor of the orientation of the quadrupole moment

$$\left.\begin{aligned}
\tilde{\mathbf{t}}^{(2^+)}_{20}(\vartheta_y) &= -\frac{\sqrt{7}}{5\sqrt{10}}\frac{1}{\mathbf{A}_{00}}[(3\cos^2\vartheta_q - 1)\mathbf{A}_{20} + \sqrt{6}\sin 2\vartheta_q\mathbf{A}_{21} + \sqrt{6}\sin^2\vartheta_q\mathbf{A}_{22}];\\
\tilde{\mathbf{t}}^{(2^+)}_{21}(\vartheta_y) &= -\frac{\sqrt{7}}{5\sqrt{10}}\frac{1}{\mathbf{A}_{00}}\Big[-\frac{\sqrt{6}}{2}\sin 2\vartheta_q\mathbf{A}_{20} + 2\cos 2\vartheta_q\mathbf{A}_{21} + \sin 2\vartheta_q\mathbf{A}_{22}\Big];\\
\tilde{\mathbf{t}}^{(2^+)}_{22}(\vartheta_y) &= -\frac{\sqrt{7}}{5\sqrt{10}}\frac{1}{\mathbf{A}_{00}}\Big[\frac{\sqrt{6}}{2}\sin^2\vartheta_q\mathbf{A}_{20} - \sin 2\vartheta_q\mathbf{A}_{21} + (1 + \cos^2\vartheta_q)\mathbf{A}_{22}\Big].
\end{aligned}\right\} \quad (91)$$

One can similarly determine the components of the tensor of the orientation of the hexadecupole moment:

$$\begin{aligned}
\tilde{\mathbf{t}}^{(2^+)}_{40}(\vartheta_y) &= -\frac{\sqrt{7}}{48\sqrt{10}}\frac{1}{\mathbf{A}_{00}}[(35\cos^4\vartheta_q - 30\cos^2\vartheta_q + 3)\mathbf{A}_{40} + 2\sqrt{5}\sin 2\vartheta_q(7\cos^2\vartheta_q - 3)\mathbf{A}_{41} + 2\sqrt{10}\sin^2\vartheta_q(7\cos^2\vartheta_q - 1)\mathbf{A}_{42}\\
&\quad + 2\sqrt{35}\sin 2\vartheta_q\sin^2\vartheta_q\mathbf{A}_{43} + \sqrt{70}\sin^4\vartheta_q\mathbf{A}_{44}];\\
\tilde{\mathbf{t}}^{(2^+)}_{41}(\vartheta_y) &= -\frac{\sqrt{7}}{48\sqrt{10}}\frac{1}{\mathbf{A}_{00}}[-\sqrt{5}\sin 2\vartheta_q(7\cos^2\vartheta_q - 3)\mathbf{A}_{40} + 2(28\cos^4\vartheta_q - 27\cos^2\vartheta_q + 3)\mathbf{A}_{41} + 2\sqrt{2}\sin 2\vartheta_q\\
&\quad \times(7\cos^2\vartheta_q - 4)\mathbf{A}_{42} + 2\sqrt{7}\sin^2\vartheta_q(4\cos^2\vartheta_q - 1)\mathbf{A}_{43} + \sqrt{14}\sin 2\vartheta_q\sin^2\vartheta_q\mathbf{A}_{44}];\\
\tilde{\mathbf{t}}^{(2^+)}_{42}(\vartheta_y) &= -\frac{\sqrt{7}}{48\sqrt{10}}\frac{1}{\mathbf{A}_{00}}[\sqrt{10}\sin^2\vartheta_q(7\cos^2\vartheta_q - 1)\mathbf{A}_{40}
\end{aligned}$$

$$
\begin{aligned}
&+2\sqrt{2}\sin 2\vartheta_q(4-7\cos^2\vartheta_q)\mathbf{A}_{41}+4(7\cos^4\vartheta_q-6\cos^2\vartheta_q+1)\\
&\times\mathbf{A}_{42}+2\sqrt{14}\sin 2\vartheta_q\cos^2\vartheta_q\mathbf{A}_{43}+2\sqrt{7}\sin^2\vartheta_q(1+\cos^2\vartheta_q)\mathbf{A}_{44}];\\
&\tilde{\mathbf{t}}^{(2^+)}_{43}(\vartheta_y)=-\frac{\sqrt{7}}{48\sqrt{10}}\frac{1}{\mathbf{A}_{00}}[-\sqrt{35}\sin 2\vartheta_q\sin^2\vartheta_q\mathbf{A}_{40}\\
&+2\sqrt{7}\sin^2\vartheta_q(4\cos^2\vartheta_q-1)\mathbf{A}_{41}-2\sqrt{14}\sin 2\vartheta_q\cos^2\vartheta_q\mathbf{A}_{42}\\
&+2(4\cos^4\vartheta_q+3\cos^2\vartheta_q-3)\mathbf{A}_{43}+\sqrt{2}\sin 2\vartheta_q(3+\cos^2\vartheta_q)\mathbf{A}_{44}];\\
&\tilde{\mathbf{t}}^{(2^+)}_{44}(\vartheta_y)=-\frac{\sqrt{7}}{48\sqrt{10}}\frac{1}{\mathbf{A}_{00}}\Big[\frac{\sqrt{35}}{\sqrt{2}}\sin^4\vartheta_q\mathbf{A}_{40}\\
&\quad-\sqrt{14}\sin 2\vartheta_q\sin^2\vartheta_q\mathbf{A}_{41}+2\sqrt{7}\sin^2\vartheta_q(1+\cos^2\vartheta_q)\mathbf{A}_{42}\\
&-\sqrt{2}\sin 2\vartheta_q(3+\cos^2\vartheta_q)\mathbf{A}_{43}+(\cos^4\vartheta_q+6\cos^2\vartheta_q+1)\mathbf{A}_{44}\Big].
\end{aligned}
\qquad (92)
$$

Figure 17 shows the values of the spherical components of the unit tensors of the orientation of the quadrupole $\tilde{\mathbf{t}}^{(2+)}_{2\varkappa}(\vartheta_\alpha)$ and hexadecupole $\tilde{\mathbf{t}}^{(2+)}_{4\varkappa}(\vartheta_\alpha)$ moments of the 2^+ state of the ^{12}C nucleus formed in inelastic scattering of α particles. We do not give the results of the calculations for the other potentials, since in the given scale they can hardly be distinguished. It can be seen from the figure that $\tilde{\mathbf{t}}^{(2+)}_{2\varkappa}(\vartheta_\alpha)$ and $\tilde{\mathbf{t}}^{(2+)}_{4\varkappa}(\vartheta_\alpha)$ oscillate with respect to the symmetry axis of the nucleus, the nature of the oscillations repeating the oscillations of the differential cross section. The frequency of the oscillations is practically independent of the rank of the operator and is constant in the complete region of angles for all components of the tensor of the quadrupole orientation, in good agreement with experiment. This circumstance evidently indicates that the oscillating behavior of the orientation tensor is due to a common mechanism of formation of the 2^+ state of the ^{12}C nucleus—the heavy-stripping mechanism. In absolute magnitude, $\tilde{\mathbf{t}}^{(2+)}_{20}(\vartheta_\alpha)$ is almost two times larger than the components $\tilde{\mathbf{t}}^{(2+)}_{21}(\vartheta_\alpha)$ and $\tilde{\mathbf{t}}^{(2+)}_{22}(\vartheta_\alpha)$. This means that the quadrupole moment of the ^{12}C nucleus in the 2^+ state is mainly oriented in the plane perpendicular to the symmetry axis of the nucleus. The same behavior is characteristic of the orientation tensor of the hexadecupole moment—with increasing projection, the absolute value of the component $\tilde{\mathbf{t}}^{(2+)}_{4\varkappa}$ decreases, and for $\tilde{\mathbf{t}}^{(2+)}_{43}(\vartheta_\alpha)$ and $\tilde{\mathbf{t}}^{(2+)}_{44}(\vartheta_\alpha)$ we also observe a decrease of the absolute values in the region of small angles, this being due to the decrease in the absolute values of the spin tensors

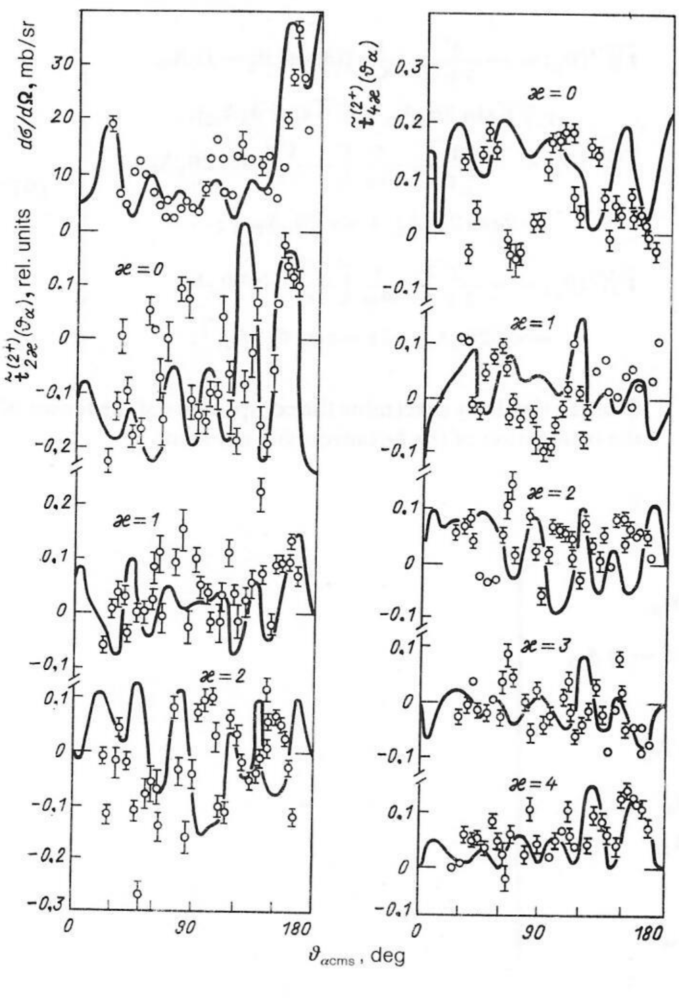

FIG. 17. Angular distribution of the orientation tensors of the quadrupole, $\tilde{\mathbf{t}}^{(2+)}_{2\varkappa}(\vartheta_\alpha)$, and hexadecupole, $\mathbf{t}^{(2+)}_{4\varkappa}(\vartheta_\alpha)$, moments of the 2^+ state of the ^{12}C nucleus calculated in the FRDWBA[75] for potential I in Table V. The experimental data are from Ref. 48.

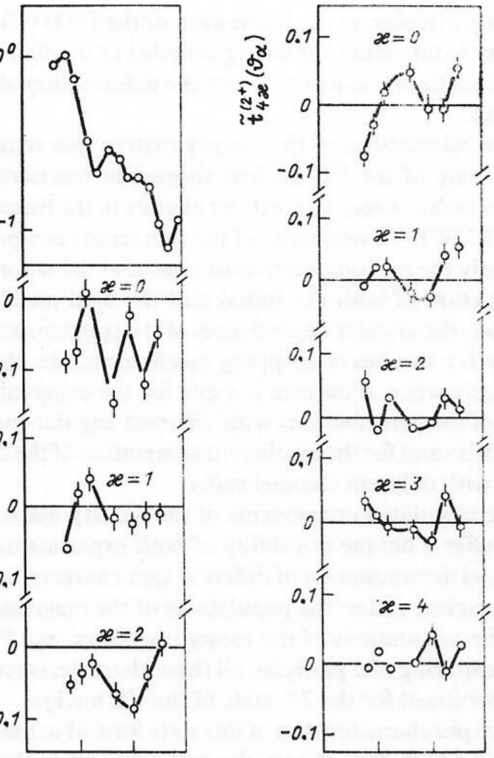

FIG. 18. Experimental angular dependences[48] of the orientation tensors of the quadrupole, $\tilde{t}_{2\varkappa}^{(2+)}(\vartheta_\alpha)$, and hexadecupole $\tilde{t}_{4\varkappa}^{(2+)}(\vartheta_\alpha)$ moments of the 2^+ state of the ^{12}C nucleus formed in the ^{13}C(^{3}He, α')^{12}C(2^+) reaction at $E_{^3\mathrm{He}} = 22.5$ MeV.

$A_{4\varkappa}(\vartheta_\alpha)$ in this region. In contrast to the components of the quadrupole-moment orientation tensor, the oscillations of the orientation tensor of the hexadecupole moment with even and odd projections occur almost in antiphase.

Such behavior of $\tilde{t}_{2\varkappa}^{(2+)}(\vartheta_\alpha)$ and $\tilde{t}_{4\varkappa}^{(2+)}(\vartheta_\alpha)$ indicates a complicated form of the ^{12}C nucleus in the 2^+ state, one that does not reduce to an ellipsoid of revolution. The components of the tensor $\tilde{t}_{2\varkappa}^{(2+)}(\vartheta_\alpha)$ agree better with the experiment of Ref. 48 than $\tilde{t}_{4\varkappa}^{(2+)}(\vartheta_\alpha)$. This is not surprising, since the $A_{4\varkappa}(\vartheta_\alpha)$ components that contribute to $\tilde{t}_{4\varkappa}^{(2+)}$ themselves agree poorly with the experiment (see Fig. 9).

Figure 18 gives the experimental values[48] of the orientation tensors for the same ^{12}C state obtained in the ^{13}C(^{3}He, α')^{12}C(2^+) reaction. For the orientation tensor $\tilde{t}_{4\varkappa}^{(2+)}(\vartheta_\alpha)$ of the hexadecupole moment at small ϑ_α the component with $\varkappa = 0$ is the largest, i.e., this moment is perpendicular to the symmetry axis of the nucleus and its precession is slight. With increasing ϑ_α, the predominance of the component with $\varkappa = 0$ does not become so appreciable, and the precession of the hexadecupole moment increases. For the orientation tensor $\tilde{t}_{2\varkappa}(\vartheta_\alpha)$ of the quadrupole moment the components with $\varkappa = 0$ and $\varkappa = 2$ in the region of intermediate angles ϑ_α are comparable, and this ensures appreciable precession of the quadrupole-moment orientation tensor when the nucleus ^{12}C(2^+) is formed in the ^{13}C(^{3}He, α')^{12}C(2^+) reaction.

Figure 19 shows the theoretical values of the orientation tensors of the quadrupole and hexadecupole moments of the ^{12}C nucleus in the 2^+ state formed in the ^{11}B(α, t')^{12}C* reaction. Comparison of Figs. 17–19 shows that the degree of precession of the quadrupole and hexadecupole moments depends strongly on the method of formation of the 2^+ level of this nucleus—whereas in inelastic scattering the precession of the hexadecupole moment is more strongly expressed than that of the quadrupole moment, for the (^{3}He, α') and (α, t') reactions this conclusion is not correct. In the ^{11}B(α, t')^{12}C* reaction, the nature of their oscillations as functions of the angle ϑ_t basically repeats the nature of the $d\sigma/d\Omega$ oscillations. It is also interesting to note that although $d\sigma/d\Omega$ is determined at small angles by direct processes and at large angles by exchange processes, the orientation of the quadrupole moment in the complete region of angles ϑ_t is due to the exchange processes. With regard to the orientation of the hexadecupole moment (only the exchange processes contribute to it), it is here important to

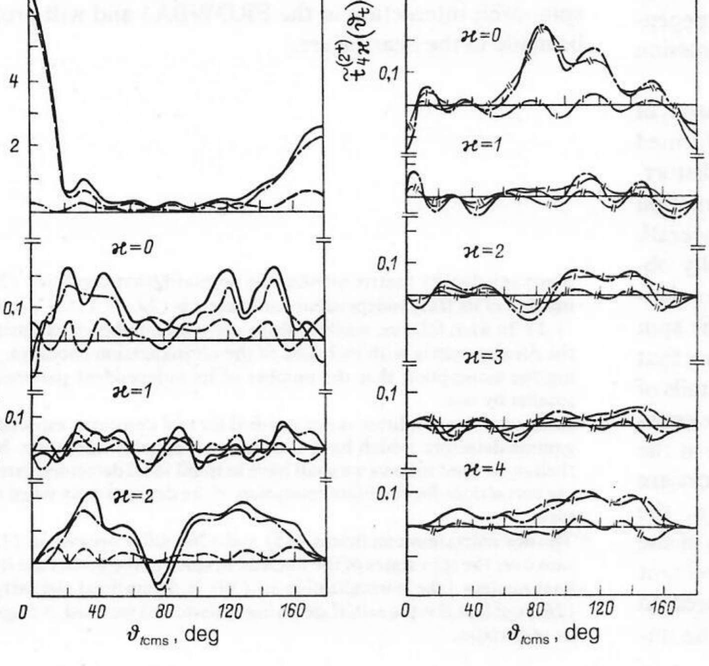

FIG. 19. Angular dependences, calculated in the FRDWBA,[50] of the differential cross section of the ^{11}B(α, t')^{12}C(2^+) reaction and of the orientation tensors $\tilde{t}_{2\varkappa}^{(2+)}(\vartheta_t)$ of the quadrupole moment (the continuous curves are the total values; the broken and chain curves are, respectively, the contributions of the direct and the exchange processes), and also of the orientation tensors $\tilde{t}_{4\varkappa}^{(2+)}(\vartheta_t)$ of the hexadecupole moment (the curves with one and two strokes across them correspond to the contributions of the components with channel spin j equal to 1 and 2).

emphasize the following circumstance. The orientation $\tilde{t}_{4\varkappa}^{(2+)}(\vartheta_t)$ in this reaction with different $\varkappa$ is differently sensitive to the contribution of the different channel spins j—whereas $\tilde{t}_{40}^{(2+)}(\vartheta_t)$ is determined by spin $j=2$, $\tilde{t}_{4\varkappa}^{(2+)}(\vartheta_t)$ with $\varkappa=2$ and 4 in the region of angles $\vartheta_t \simeq 100$–$140°$ is determined by spin $j=1$. In other words, by measuring the orientation of the hexadecupole moment of $^{12}C^*$ one can determine the structural characteristics of this state of the ^{12}C nucleus with given channel spin.

CONCLUSIONS

Thus, we have shown that measurement of the angular correlation function of the final reaction product particles and γ rays emitted by the final excited nucleus on the transition to the ground state in different γ-ray detection planes relative to the reaction plane enables one in principle to recover all the real parameters that determine the density matrix of the nucleus. The proposed method for recovering the density matrix is attractive above all by virtue of its relative simplicity, since it enables one, using results obtained in a single experiment, to find all parameters of the density matrix without changing the method.

The method was theoretically justified and developed at the Institute of Nuclear Physics at Moscow State University and was also realized there for the first time experimentally. In the case of inelastic scattering of α particles by ^{12}C, the spin density matrix of the 2^+ state of the ^{12}C nucleus has been *completely* recovered, i.e., the nine real components of its spin tensors of even rank have been determined. There have also been experimental determinations of the spin tensors of even rank of the density matrix of the same ^{12}C state formed in reactions with the participation of light particles with spins. However, for such reactions the measured values do not permit recovery of all the nonvanishing components of the density matrix, since in reactions with the participation of particles with nonzero spins the spin tensors of the density matrix of odd rank can, in principle, be nonzero. We emphasize that the experimental data on the spin tensors of even rank of the 2^+ state of the ^{12}C nucleus obtained from nuclear reactions are as yet limited and have a low statistical accuracy, and from them it is difficult to establish the dependence of the components of the spin tensors on the emission angle of the final particles.

The theoretical analysis of the even-rank spin tensors of the density matrix of the 2^+ state of the ^{12}C nucleus formed in inelastic scattering made in the framework of the distorted-wave Born approximation with a finite-range interaction for the heavy-stripping mechanism has shown that, overall, the method permits description of the experimentally obtained density-matrix spin tensors. The results of the calculations have revealed an appreciable sensitivity of the spin tensors of nonzero rank (at least, a sensitivity exceeding that of the differential cross section) to the choice of the details of the calculation, especially in the region of intermediate emission angles of the final particles. Nevertheless, even in the region of large angles, where the theoretical spin tensors are almost insensitive to the details of the calculation, the FRDWBA does not explain the experimental values of the spin tensors in a number of cases. To improve the agreement between the theory and experiment, we evidently need to take into account more accurately the structure of the nucleus; in particular, in the framework of the FRDWBA it is necessary to introduce the strong coupling of the channels in inelastic scattering associated with the deformation of the ^{12}C nucleus.

The calculations of the density-matrix spin tensors of the 2^+ state of the ^{12}C nucleus formed in reactions with particles with nonzero spin exhibit already in the framework of the FRDWBA a sensitivity of the spin tensor components to not only the reaction mechanism but also the wave-function structure of both the initial and the final nucleus. In particular, the angular dependences of the spin tensors, particularly for the heavy-stripping mechanism, are changed appreciably when allowance is made for the coherent summation of the contributions with different angular-momentum tranfers and for the incoherent summation of the contributions with different channel spins.

The calculated components of the density-matrix spin tensors offer a unique possibility of both experimental and theoretical determination of different spin characteristics of excited nuclear states: the populations of the magnetic sublevels, the orientations of the tensor operators, etc. For inelastic scattering of α particles, all these characteristics have been determined for the 2^+ state of the ^{12}C nucleus.

The spin characteristics of this state formed in reactions with the participation of particles with spin, when the components of the odd-rank spin tensors may also be nonzero, have been incompletely determined. To recover the odd-rank spin tensors it is necessary to measure either the circular polarization of the γ rays or the polarization tensors of the 2^+ state of the ^{12}C nucleus. It is clear that such experiments entail serious methodological problems. However, their realization will make it possible to advance significantly in the understanding of a complicated and subtle characteristic such as the induced polarization of even–even nuclei. Therefore, it is very desirable that such experiments should be made.

The theoretical estimates of both the polarization tensors and the polarization of the γ rays (at least in the framework of the FRDWBA) present no fundamental difficulties (to make such calculations, it is necessary to include the spin–orbit interaction in the FRDWBA) and will probably be made in the near future.

[1)]Since any density matrix satisfies the normalization condition (2), the number of its truly independent parameters is $(2J_f+1)^2-1=4J_f(J_f+1)$. In what follows, we shall always give the number of parameters of the density matrix with inclusion of the normalization condition, making the assumption that the number of its independent parameters is smaller by one.

[2)]Of course, this condition is not satisfied for real detectors, especially for gamma detectors, which have a rather poor angular resolution. Nevertheless, in what follows we shall have in mind ideal detectors, introducing corrections for the finite resolution of the detector only when necessary.

[3)]The normalization conditions (15) and (26) differ because in (15) we sum over the spin states of the nucleus B, since in the given case it is the final nucleus [the normalization in (30) is determined similarly]. In (26), nucleus B is the *initial* decaying system, and we must average over its spin states.

[4)]In the majority of experimental studies, $W(\Omega_y, \Omega_\gamma)$ is measured in one plane. Then corrections for the finite resolution of the gamma detector cannot be correctly introduced for all parameters of this function. As follows directly from the expressions (54), the correction coefficients (74) can be introduced only in the form parameters s_4 and c_4, since they depend on $A_{k\varkappa}(\Omega_y)$ with the unique value $k = 4$. It is in principle impossible to carry through an analogous procedure for the other form parameters of the angular correlation function in the case of measurement in one plane (including the plane with $\varphi_\gamma = 0$), since the relative contributions of the correction coefficients to $A_{k\varkappa}$ with different k are not known *a priori*. Therefore, it appears to us that the inclusion of the corrections for the finite resolution of the gamma detector made in Ref. 22 is not entirely correct, although in practice this error is evidently not too serious, since the correction coefficients themselves are small (see below).

[5)]Unfortunately, in Ref. 44 the expressions for the populations $p_0^{(2+)}$ and $p_{\pm 2}^{(2+)}$ are given with errors. As a result, the experimental values of the populations of these substates given in Refs. 45–47 and 49 must be corrected.

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