Collective nuclear models based on dynamical SU(6) symmetry

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Different ways of constructing two popular nuclear models—the truncated quadrupole-phonon model (TQM) and the interacting-boson model (IBM)—are analyzed. The main results of studies of the possibility of establishing the equivalence of the two models are presented. The latest development and application of the TQM to the description of odd and odd–odd nuclei on the basis of the method of dynamical symmetries and supersymmetries are reviewed.

INTRODUCTION

In the theory of nuclear structure, it is now generally accepted that the properties of the low-lying states of medium and heavy nuclei are basically determined by quadrupole vibrations of the average field.[1] This is the basis of the phenomenological model of Bohr and Mottelson,[2,3] in the framework of which the quadrupole excitations of nuclei have traditionally been studied.

During the last ten years, the very successful description of the properties of the collective states of medium and heavy nuclei has made the interacting-boson model (IBM) very popular. The basic physical concept of the IBM is the same as in the collective model of Bohr and Mottelson—only the quadrupole degree of freedom is considered. One of the characteristic features of the IBM is the presence of a clear algebraic structure in the Hamiltonian, which is constructed from the generators of the unitary unimodular group SU(6) in six-dimensional space. Thus, SU(6) symmetry entered nuclear spectroscopy and began to play an important part when the IBM appeared. It needs to be emphasized that this symmetry is dynamical.[4] It is broken by the IBM Hamiltonian down to the symmetry group SO(3) (the physical group of rotations). Dynamical symmetry groups (or, more precisely, their unitary irreducible representations) are extremely helpful not only for explicit determination of the excitation spectra but also for the matrix elements of other physical operators (for example, the transition probabilities), i.e., for the entire dynamics of the system.

With regard to the history of the creation of the IBM, two different versions of the model were proposed by two different groups. First, a Dubna–Rossendorf collaboration[5-7] led to the construction of a microscopic variant of the IBM, this being known in the literature as the truncated quadrupole-phonon model (TQM), its Hamiltonian being expressed solely by means of operators of quadrupole bosons. Later, as a result of the Tokyo–Gröningen collaboration[8-10] the *s-d* version of the IBM was proposed, and this gave the name to the model (besides quadrupole *d* bosons, a scalar *s* boson was also introduced). This variant of the model (IBM-1) was introduced purely phenomenologically—it was postulated that the Hamiltonian is the most general rotational invariant constructed from the generators of the SU(6) algebra.

As regards microscopic justification of the Hamiltonian of the IBM-1, and also the IBM-2 (a variant in which distinctions are made between the proton and neutron bosons), such attempts were made in Refs. 11–14. However, these attempts at justification of the IBM-1 and IBM-2 are not consistently microscopic and, in our view, the criticism and objections made in Ref. 15 are entirely valid.

In such a situation, in which two models constructed independently to describe the same group of phenomena coexist in the theory of nuclear structure, it is clearly worthwhile comparing them, in the first place with one another, but also with the Bohr–Mottelson model. Indeed, as soon as the IBM-1 had appeared discussions began in the physics literature on its relationship to the truncated quadrupole-phonon model (Refs. 8, 9, 16, and 17) and the Bohr–Mottelson model.[18-22]

As already noted, the microscopic versions of the truncated quadrupole-phonon model and IBM-1 are radically different. But one can compare the phenomenological Hamiltonians of the two models, regarding the parameters of the models as absolutely free. The fact that the TQM and IBM-1 are identical in the most important respect—both are based on the same dynamical symmetry—suggests that at the phenomenological level the two models may be equivalent and can be regarded as two forms of the IBM. If the question of establishing the equivalence of the TQM and IBM-1 is rigorously approached, some complex questions requiring detailed mathematical investigation arise.

The equivalence of the TQM and IBM-1 was first pointed out by the authors of these models themselves. They showed[9,23] that the matrix elements of the Hamiltonians and the electric quadrupole operators (in appropriate bases) of the two models are identical. It was asserted in Ref. 16 that the Hamiltonians of the TQM and IBM-1 (as operators considered without regard to their basis) are exactly the same. The equivalence "at the level of operators" was rigorously proved in Ref. 24. It was noted in Ref. 17 that in the TQM the boson form of the Hamiltonian is constructed by means of the Holstein–Primakoff (HP) realization of the SU(6) algebra, whereas in the IBM-1 a boson representation of Schwinger type (S) is used. This circumstance and the fact that the TQM and IBM-1 are based on dynamical SU(6) symmetry evidently led the authors of Ref. 17 to regard the TQM and IBM-1 as two realizations of a single "phenomenological SU(6) boson model," i.e., the IBM in our terminology. Such a unification is, strictly speaking, justified if and only if it has been shown that the Holstein–Primakoff and Schwinger representations are isomorphic (equivalent) realizations of the SU(6) algebra. This was shown in Ref. 25

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by means of the formalism of the theory of representations of abstract algebras. The establishment of the equivalence of the TQM and the IBM-1, which required the complete series of studies of Refs. 9, 16, 17, and 23–25, can now be regarded as fully completed.

In our view, particularly serious attention needs to be paid to the results of the analysis of the different ways of constructing the collective nuclear model based on dynamical SU(6) symmetry. The importance of such analysis is explained by the fact that, first, it facilitates a deeper understanding of the content of both models, which have now firmly entered the arsenal of the theory of nuclear structure, second, the equivalence of the TQM and IBM-1 is used essentially in the consideration of the relationship between these models and the Bohr–Mottelson model,[21,26,27] and, third, the experience acquired from the use of the algebraic formalism of the theory of dynamical symmetries can be used to extend the IBM, the need for which is clearly indicated by experiments.

The aim of the present paper is to present the main results of studies of the possibility of establishing the equivalence of the TQM and IBM-1, and also some results of the latest development of the TQM as applied to the description of odd and odd–odd nuclei on the basis of the method of dynamical symmetries and supersymmetries.

Our review is arranged as follows.

The microscopic Hamiltonian of the TQM is constructed. The main attention is devoted to the appearance of dynamical SU(6) symmetry (Sec. 1). The basic propositions postulated as the basis of the phenomenological Hamiltonian of the IBM-1 are presented.

The algebraic and geometrical aspects of the IBM-1 are discussed. The equivalence of the TQM and IBM-1 are proved at the level of the matrix elements, operators, and representations. The justification for combining the TQM and IBM-1 is given in Sec. 2, in which we also prove the equivalence of the two in the classical limit. The relationship between them in the description of two-nucleon transfer reactions is discussed in Sec. 3. Section 4 is devoted to the most recent developments of the TQM as applied to odd and odd–odd nuclei. The conclusions are formulated in Sec. 5.

1. ANALYSIS OF DIFFERENT WAYS OF CONSTRUCTING THE COLLECTIVE HAMILTONIANS OF THE TQM AND IBM-1

The interacting-boson model[1)] is used with great success to describe the collective states of medium and heavy nuclei (see, for example, Ref. 28). The development of a mathematical formalism for the unified description of the collective states of nuclei is the important theoretical achievement of the IBM (Refs. 5–10, 29, and 30). The review of Ref. 31 is devoted to the physical justification of the IBM and its intensive applications. However, it should be emphasized that the achievements of the IBM are made up of the independent achievements of the two versions of it—the TQM and IBM-1—each of which developed in its own way.

Microscopic approach to construction of a collective quadrupole Hamiltonian of the TQM

The truncated quadrupole-phonon model was developed by Jolos, Donau, and Janssen.[5,6] The physical bases of the TQM and the explicit construction of its Hamiltonian were considered in detail in Refs. 5, 6, 29, and 31. Below, we shall follow Refs. 5 and 29, paying particular attention to the appearance of SU(6) as the dynamical symmetry group of the family of quadrupole excitations. As we noted in the Introduction, the quadrupole degrees of freedom play a decisive role in forming the properties of the low-lying collective states. It was for this reason that only these degrees of freedom were taken into account in the construction of the TQM, the main task then consisting of separating explicitly in the microscopic Hamiltonian of the nucleus the dependence on the collective quadrupole variables. In Refs. 5 and 29, this problem was solved by introducing generalized coordinates and momenta:

$$\left.\begin{aligned}\hat{q}_{\varkappa LM}&=\frac{1}{2}\sum_{ab} q_{ab}^{\varkappa L}\,[A_{LM}^{+}(ab)+(-1)^{L-M}A_{L-M}(ab)];\\ \hat{p}_{\varkappa LM}&=-\frac{\mathrm{i}}{2}\sum_{ab} p_{ab}^{\varkappa L}[A_{LM}(ab)-(-1)^{L-M}A_{L-M}^{+}(ab)],\end{aligned}\right\}\quad(1)$$

where

$$A_{LM}^{+}(ab)=\sum_{m_a m_b}\langle j_a m_a j_b m_b|LM\rangle\, a_{am_a}^{+}a_{bm_b}^{+};$$

$\varkappa$ is an additional quantum number that distinguishes operators with the same L and M.

By means of the operators $\hat{p}_{\varkappa LM}, \hat{q}_{\varkappa LM}$ and their commutators, we can obtain expressions for any bifermion operators $(a_\alpha^+ a_\beta^+, a_\alpha^+ a_\beta, a_\beta a_\alpha)$. The operators $\hat{q}, \hat{p}$ and all their commutators form the algebra SP(2Ω), where Ω is the number of single-particle states. The structure constants for this algebra are sums of products of four amplitudes, which determine the two-quasiparticle structure of the operators of the generalized coordinates and momenta.

Among the operators of the generalized coordinates and momenta, only the operators with $L = 2$ are of interest for constructing the TQM. They are subdivided into the collective operators ($\varkappa = 1$), which characterize the appreciable contribution of the large number of two-quasiparticle components, and the noncollective (nearly two-quasiparticle) operators. For the construction of the TQM, one uses (in the spirit of all collective models) only the collective operators $\hat{q}_{12\mu}, \hat{p}_{12\mu}$ and their commutators. But this set of operators does not form a closed algebra; for the double commutators of the collective operators $\hat{q}_{12\mu}$ and $\hat{p}_{12\mu}$ contain, in addition to the collective operators, terms that are proportional to noncollective operators. Since interest attaches to matrix elements of the operators and their commutators taken only between the collective states, and the matrix elements of the noncollective operators are small, it is possible in all the commutators of the collective operators to ignore the noncollective terms. This approximation is further justified by the smallness of the structure constants in front of the noncollective operators. As a result, we close the algebra of the collective operators. Since we thereby break the equivalence of the sets of operators $(a_\alpha^+, a_\beta, a_\alpha^+ a_\beta, a_\beta a_\alpha)$ and $(\hat{q}, \hat{p}, [\hat{q}, \hat{p}], [\hat{q}, \hat{q}], [\hat{p}, \hat{p}])$, the structure constants characterizing the algebra of the collective operators no longer automatically satisfy the Jacobi identities. If we require rigorous fulfillment of the Jacobi identities, then there are additional restrictions on the amplitudes that characterize the quasiparticle structure of the collective operators.

Thus, we arrive at the following algebra of collective quadrupole operators (ACQO):

$$\left.\begin{aligned}
&[\hat{q}_\mu,\ \hat{p}_{\mu'}]=(-1)^{\mu+\mu'}[\hat{q}_{-\mu'},\ \hat{p}_{-\mu}];\\
&[\hat{p}_\mu,\ \hat{p}_{\mu'}]=(-1)^{\mu+\mu'}[\hat{q}_{-\mu},\ \hat{q}_{-\mu'}];\\
&[[\hat{q}_\mu,\ \hat{p}_{\mu'}],\ \hat{q}_{\mu''}]=-2\delta_{\mu\mu'}(-1)^{\mu''}\hat{p}_{-\mu''}\\
&\quad-\delta_{\mu''-\mu}(-1)^{\mu}p_{-\mu'}-\delta_{\mu'\mu''}(-1)^{\mu}\hat{p}_{-\mu};\\
&[[\hat{q}_\mu,\ \hat{p}_{\mu'}],\ \hat{p}_{\mu''}]=2\delta_{\mu\mu'}(-1)^{\mu''}\hat{q}_{-\mu''}\\
&\quad+\delta_{\mu'-\mu''}(-1)^{\mu'}\hat{q}_{\mu}+\delta_{\mu\mu''}(-1)^{\mu'}\hat{q}_{-\mu'};\\
&[[\hat{q}_\mu,\ \hat{q}_{\mu'}],\ \hat{q}_{\mu''}]=\delta_{\mu'-\mu''}(-1)^{\mu'}\hat{q}_{\mu}-\delta_{\mu''-\mu}(-1)^{\mu}\hat{q}_{\mu'};\\
&[[\hat{q}_\mu,\ \hat{q}_{\mu'}],\ \hat{p}_{\mu''}]=\delta_{\mu'\mu''}(-1)^{\mu}\hat{p}_{-\mu}-\delta_{\mu\mu''}(-1)^{-\mu'}\hat{p}_{-\mu'};
\end{aligned}\right\}\tag{2}$$

$$\left.\begin{aligned}
&[[\hat{q}_\mu,\ \hat{p}_{\mu'}],\ [\hat{q}_{\mu''},\ \hat{p}_{\mu'''}]]=\delta_{\mu-\mu''}(-1)^{\mu'+\mu''+\mu'''}[\hat{q}_{-\mu'''},\ \hat{q}_{-\mu'}]\\
&\quad+\delta_{\mu'''-\mu'}(-1)^{\mu'}[\hat{q}_{\mu''},\ \hat{q}_{\mu}]+\delta_{\mu'\mu''}(-1)^{\mu'''}[\hat{q}_{-\mu'''},\ \hat{q}_{\mu}]\\
&\quad+\delta_{\mu\mu'''}(-1)^{\mu'}[\hat{q}_{\mu''},\ \hat{q}_{-\mu'}];\\
&[[\hat{q}_\mu,\ \hat{p}_{\mu'}],\ [\hat{q}_{\mu''},\ \hat{q}_{\mu'''}]]=\delta_{\mu-\mu''}(-1)^{\mu}[\hat{q}_{\mu'''},\ \hat{p}_{\mu'}]\\
&\quad-\delta_{\mu-\mu'''}(-1)^{\mu}[\hat{q}_{\mu''},\ \hat{p}_{-\mu'}]+\delta_{\mu'\mu''}(-1)^{\mu'''}[\hat{q}_{\mu},\ \hat{p}_{-\mu'''}]\\
&\quad-\delta_{\mu'\mu'''}(-1)^{\mu''}[\hat{q}_{\mu},\ \hat{p}_{-\mu''}];\\
&[[\hat{q}_\mu,\ \hat{q}_{\mu'}],\ [\hat{q}_{\mu''},\ \hat{q}_{\mu'''}]]\\
&=\delta_{\mu'-\mu''}(-1)^{\mu'}[\hat{q}_{\mu},\ \hat{q}_{\mu'''}]+\delta_{\mu'-\mu'''}(-1)^{\mu}[\hat{q}_{\mu'},\ \hat{q}_{\mu''}]\\
&\quad-\delta_{\mu-\mu''}(-1)^{-\mu}[\hat{q}_{\mu'},\ \hat{q}_{\mu'''}]-\delta_{\mu'-\mu'''}(-1)^{\mu'}[\hat{q}_{\mu},\ \hat{q}_{\mu''}],
\end{aligned}\right\}$$

where $\hat{q}_\mu\equiv\hat{q}_{12\mu}/\sqrt{L}$, $\hat{p}_\mu\equiv\hat{p}_{12\mu}/\sqrt{K}$. The constants K and L are determined in Refs. 5 and 29 and are not needed in what follows. One can show that among the operators in (2) there are 35 that are linearly independent and have vanishing trace. Therefore, the ACQO is isomorphic to the Lie algebra of the group SU(6).

Thus, under the assumption that the collective branch of excitations is weakly coupled to the other degrees of freedom we have shown that the hidden symmetry of the quadrupole mode is the group SU(6).

Using the equivalence of the sets of operators (a^+a^+, aa, a^+a) and ($\hat{q},\hat{p},[\hat{q},\hat{p}],[\hat{q},\hat{q}],[\hat{p},\hat{p}]$), we can express the microscopic Hamiltonian of the nucleus in terms of the operators $\hat{q},\hat{p}$ and their commutators. Then, retaining only the collective operators $\hat{q}$ and $\hat{p}$, we can separate the collective part of the microscopic Hamiltonian (H_{coll}):

$$\begin{aligned}
H_{\text{coll}}={}&\text{i}e\sum_\mu[\hat{q}_\mu,\ \hat{p}_\mu]+u\sum_\mu(-1)^\mu\hat{q}_\mu\hat{q}_{-\mu}+v\sum_\mu(-1)^\mu\hat{p}_\mu\hat{p}_{-\mu}\\
&+\frac{1}{2}\text{i}w\sum_\mu(-1)^\mu\hat{q}_\mu([\hat{q},\ \hat{p}])_{(2-\mu)}\\
&-\frac{1}{4}\sum_{L=0}^{4}\frac{1}{2}(1+(-1)^L)\,t_L\\
&\times\sum_{M_L=-L}^{L}(-1)^M([\hat{q},\ \hat{\tilde{p}}])_{(LM)}([\hat{q},\ \hat{\tilde{p}}])_{(L-M)}\\
&+\frac{1}{4}\sum_{L=0}^{4}\frac{1}{2}(1-(-1)^L)t_L\\
&\times\sum_{M=-L}^{L}(-1)^M([\hat{q},\ \hat{q}])_{(LM)}([\hat{q},\ \hat{q}])_{(L-M)}.
\end{aligned}\tag{3}$$

The symbol $(\ )_{(LM)}$ denotes vector coupling, and $\hat{\tilde{p}}_\mu\equiv(-1)^\mu\hat{p}_{-\mu}$. One can show that H_{coll} is rotationally invariant and invariant with respect to time reversal. Information about the average field and the residual interaction is contained in the nine constants e,u,v,w,t_L.

For the generators of the ACQO, the Holstein–Primakoff realization is valid:

$$\left.\begin{aligned}
q_\mu^{\text{HP}}&=\sqrt{N-\sum_\nu d_\nu^+d_\nu}\,(-1)^\mu d_{-\mu}+d_\mu^+\sqrt{N-\sum_\nu d_\nu^+d_\nu};\\
p_\mu^{\text{HP}}&=\text{i}\left[(-1)^\mu d_{-\mu}^+\sqrt{N-\sum_\nu d_\nu^+d_\nu}-\sqrt{N-\sum_\nu d_\nu^+d_\nu}\,d_\mu\right];\\
\text{i}\,[q_\mu,\ P_{\mu'}]^{\text{HP}}&=d_\mu^+d_{\mu'}+(-1)^{\mu+\mu'}d_{-\mu'}^+d_{-\mu}-2\delta_{\mu\mu'}(N-\sum_\nu d_\nu^+d_\nu);\\
[q_\mu,\ q_{\mu'}]^{\text{HP}}&=(-1)^{\mu'}d_\mu^+d_{-\mu'}-(-1)^\mu d_{\mu'}^+d_{-\mu},
\end{aligned}\right\}\tag{4}$$

where N is the quantum number of the linear Casimir operator of SU(6), which characterizes the completely symmetric representation of SU(6); d_μ^+ and d_μ are the operators of creation and annihilation of a quadrupole boson with projection μ onto the z axis of the laboratory system.

Using (4), we can readily obtain from (3) the familiar form of the TQM Hamiltonian[5,29]:

$$\begin{aligned}
H_{\text{TQM}}={}&h_0+h_1\sum_\mu d_\mu^+d_\mu+h_2\sum_\mu(-1)^\mu\left(d_\mu^+d_{-\mu}^+\sqrt{(N-\sum_\nu d_\nu^+d_\nu)(N-1-\sum_\nu d_\nu^+d_\nu)}+\text{h.c.}\right)\\
&+h_3\sum_\mu(-1)^\mu\left(d_\mu^+(d_2^+\tilde{d}_2)_{(2-\mu)}\sqrt{N-\sum_\nu d_\nu^+d_\nu}+\text{h.c.}\right)\\
&+\sum_{\substack{L=0,\,2,\,4\\ M=-L}}^{L}h_{4L}(-1)^M(d_2^+d_2^+)_{(LM)}(\tilde{d}_2\tilde{d}_2)_{(L-M)}.
\end{aligned}\tag{5}$$

Here, $\tilde{d}_\mu \equiv (-1)^{-\mu} d_{-\mu}$. The set of coefficients $\{h_0, h_1, h_2, h_3, h_{4L}\}$ and the quantum number N are related to the coefficients in (3).[6] Explicit expressions for these quantities in terms of the single-particle energies and matrix elements of the residual interaction are given in Ref. 5.

In the framework of the TQM, the electric quadrupole operator is defined as a second-rank tensor constructed from the generators of the ACQO (2):

$$\hat{Q}_{\rm coll} = m_1 \hat{q}_\mu + m_2 [\hat{q}_2, \hat{p}_2]_{(2\mu)}, \tag{6}$$

where m_1 and m_2 can be uniquely expressed[5] in terms of the amplitudes q^{12}_{ab} and p^{12}_{ab}.

After substitution of (4) in (6), we obtain for $\hat{\rm Q}_{\rm coll}$ the expression

$$\begin{aligned} \hat{Q}_{\rm TQM} = m_1 \Big(& d^+_\mu \sqrt{N - \sum_\nu d^+_\nu d_\nu} \\ & + (-1)^\mu \sqrt{N - \sum_\nu d^+_\nu d_\nu}\, d_{-\mu} \Big) \\ & + m_2 (d^+ \tilde{d})_{(2\mu)}. \end{aligned} \tag{7}$$

Thus, we have constructed $H_{\rm TQM}$ and $Q_{\rm TQM}$ microscopically; they have a closed form that is convenient for practical calculations. The appearance of the square root $\sqrt{N - \Sigma_\nu d_\nu^+ d_\nu}$ is due to the use of the HP realization of the ACQO.

The main difference between the microscopic TQM and the other microscopic group-theoretical approaches to the description of collective motion in nuclei[32-36] is that in the other approaches the problem is, as a rule, formulated in terms of the coordinates of the nucleons, and not in the second-quantization formalism, and the conservation laws are satisfied more rigorously.

The properties of a number of transition nuclei have been investigated on the basis of the Hamiltonian (5) and the quadrupole operator (7). The SU(3) and SU(5) limits of $H_{\rm TQM}$ have also been obtained.[7,29]

Phenomenological description of collective states in the framework of the IBM-1

The IBM-1 was proposed by Arima and Iachello. On the basis of an analysis of experimental data, they proposed that a fundamental role in the model must be played by the group SU(6), which contains a number of important subgroups, such as SU(5), SU(3), and SO(6), which could be related to a definite shape of the nucleus. A characteristic difference of the IBM-1 Hamiltonian from the TQM Hamiltonian is the introduction of not only the quadrupole bosons (d_μ^+, d_μ) but also the scalar s boson (s^+, s). The IBM-1 Hamiltonian was postulated as the most general rotational invariant that can be constructed from the generators of the SU(6) group.[8,9]:

$$\begin{aligned} H_{\rm IBM\text{-}1} = {} & \epsilon_s s^+ s + \epsilon_d \sum_\mu d^+_\mu d_\mu + v_2 [((d^+_2 d^+_2)_2 \tilde{d})_{(00)} s + {\rm h.c.}] \\ & + \frac{v_0}{\sqrt{2}} [(d^+_2 d^+_2)_{(00)} ss + {\rm h.c.}] + \frac{1}{2} u_0 s^+ s^+ ss + u_2 (d^+ \tilde{d})_{(00)} s^+ s \\ & + \sum_{L=0,\,2,\,4} \frac{1}{2} \sqrt{2L+1}\, c_L ((d^+ d^+)_L (\tilde{d}\tilde{d})_L)_{(00)}. \end{aligned} \tag{8}$$

The Hamiltonian (8) contains nine free parameters: $\in_s, \in_d$, v_2, v_0, u_0, u_2, and c_L $(L = 0,2,4)$.

The operator of the quadrupole moment was expressed in the form

$$\hat{Q}_{\rm IBM\text{-}1} = m_1 [d^+_{2\mu} s + {\rm h.c}] + m_2 (d^+ \tilde{d})_{(2\mu)}. \tag{9}$$

An important step in the establishment of the IBM-1 was the systematic and intensive application of the group-theoretical approach to the description of the spectra of complex nuclei by the authors of the IBM-1 (Refs. 8–10 and 30) as well as by other authors.[29,36-38] A complete analysis of the group structure of the IBM-1 Hamiltonian is given in Ref. 30, where it is shown that breaking of the SU(6) symmetry down to the physical rotation group SO(3) is possible through only three reduction chains:

$$\begin{tikzcd} & SU(5) \supset SO(5) \supset SO(3) \supset O(2); \quad \text{(I)} \\ SU(6) \arrow[ur] \arrow[r] \arrow[dr] & SU(3) \supset SO(3) \supset O(2); \quad \text{(II)} \\ & SO(6) \supset SO(5) \supset SO(3) \supset O(2). \quad \text{(III)} \end{tikzcd}$$

Each of these chains determines a complete basis, which is characterized by the quantum numbers of the "embedded" subgroups. It was shown in Ref. 37 that the Hamiltonian (8) can be expressed as a sum of the Casimir operators of the groups SU(5), SO(5), SO(3), and SU(3):

$$\begin{aligned} H_{\rm IBM\text{-}1} = {} & \varepsilon C_{1v5} + \alpha C_{2v5} + \beta C_{2SO(5)} + \gamma C_{2SO(3)} + \delta C_{2SU(3)} \\ & + \eta C_{2SO(6)}. \end{aligned} \tag{10}$$

The expression (10) contains not all the invariants but only those of spectroscopic interest. Here, ε, α, β, γ, δ, and η are free parameters. The indices 1 and 2 (in front of the group symbol) indicate which Casimir operator is used—the linear or the quadratic one. This form of expression of the Hamiltonian is convenient in that by means of it one can find all possible particular cases for which the eigenvalue problem can be solved analytically. When this occurs, one says[30] that a dynamical symmetry arises.[2)] Such cases occur when $H_{\rm IBM\text{-}1}$ can be expressed in terms of the Casimir operators of just one of the reduction chains (I)–(III). They are associated with the vanishing of certain of the coefficients in (10). The existence of a connection between the reduction chains (I)–(III) and the nature of the phase transition from the spherical to the deformed nuclei was pointed out in Ref. 36.

One of the first successes of the IBM-1 was associated with the prediction that the SO(6) limit is realized in nuclei.[42] In this limit, $\varepsilon = \alpha = \delta = 0$ and

$$\langle SO(6) \mid H_{\rm IBM\text{-}1} \mid SO(6) \rangle = \beta 2\tau (\tau + 3) + \gamma 2L (L + 1) + \eta 2\sigma (\sigma + 4),$$

where τ, σ, and L are quantum numbers charactering the SO(6) representations. Such a spectrum was found in ^{196}Pt.[43] One can find other nuclei too whose spectra are close to those realized in the limiting cases (I) (^{110}Cd) and (II) (^{156}Gd), but the majority of nuclei have spectra intermediate between these limiting cases.

Besides the algebraic properties of the IBM-1, consideration was also given in Ref. 30 to the geometrical properties of the model, which can be described in terms of shape variables of the nuclear surface. Study of the geometrical properties of the IBM-1 made it possible to relate the limiting cases (I)–(III) to definite nuclear shapes.[3)] It was shown that a spherical shape corresponds to the limiting case (I), an axially deformed elongated shape to (II), and a deformed nucleus with γ-independent potential energy to (III).

2. ESTABLISHMENT OF THE EQUIVALENCE OF THE TQM AND IBM-1

Establishment of the equivalence of the TQM and IBM-1 by comparing the matrix elements of the Hamiltonian and the quadrupole operator

We consider the matrix elements of the Hamiltonian and the quadrupole operator in the TQM and IBM-1 bases, respectively.

The $H_{\text{IBM-1}}$ eigenfunctions can be expanded with respect to the states $|s^{n_s}d^{n_d}[N]\varkappa LM\rangle$, which form a complete basis in the six-dimensional Hilbert space. Here N is the eigenvalue of the operator of the total number of bosons:

$$\hat{N}=s^+s+\sum_{\mu}d_\mu^+d_\mu\equiv\hat{n}_s+\hat{n}_d,$$

and $\varkappa$ is an additional quantum number needed for the complete characterization of the basis states. The Hamiltonian H_{TQM} is diagonalized in the basis of quadrupole bosons $|d^{n_d}\varkappa LM\rangle$, where $n\leqslant N$.

Since H_{TQM} contains seven parameters (if N is eliminated) it is convenient, for finding the matrix elements, to transform $H_{\text{IBM-1}}$ as well from the standard nine-parameter form to a seven-parameter form. Since the operator $\hat{N}$ is the linear Casimir operator for the algebra SU(6),

$$n_s+n_d=N=\text{const.} \tag{11}$$

With allowance for this fact, and also using the identity

$$\hat{n}_d^2\equiv\sum_{L=0,\,2,\,4}\sqrt{2L+1}\,((d_2^+d_2^+)_L(\tilde{d}_2\tilde{d}_2)_L)_{(00)}+\hat{n}_d,$$

we can transform $H_{\text{IBM-1}}$ to the form[30,47]

$$\begin{aligned}H_{\text{IBM-1}}=&\epsilon_0+\epsilon_d'\hat{n}_d+v_2\,[((d_2^+d_2^+)_{(2\mu)}\,(\tilde{d}_2\tilde{s})_{(2-\mu)})_{(00)}+\text{h.c}]\\&+\frac{v_0}{\sqrt{2}}\,[(d_2^+d_2^+)_{(00)}\,(\tilde{s}\tilde{s})_{(00)}+\text{h.c}]\\&+\sum_{L=0,\,2,\,4}\frac{1}{2}\sqrt{2L+1}\,c_L'\,((d_2^+d_2^+)_{(L,\,M)}\,(\tilde{d}_2\tilde{d}_2)_{(L-M)})_{(00)},\end{aligned} \tag{12}$$

where

$$\left.\begin{aligned}\epsilon_0&\equiv\epsilon_sN+\frac{1}{2}u_0N\,(N-1);\\\epsilon'&\equiv\epsilon_d-\epsilon_s+\left(\frac{1}{\sqrt{5}}u_2-u_0\right)(N-1);\\c_L'&=c_L+\frac{1}{2}u_0-u_2/\sqrt{5}.\end{aligned}\right\} \tag{13}$$

We turn to the comparison of the H_{TQM} and $H_{\text{IBM-1}}$ matrix elements in the corresponding bases. To shorten the expressions, we shall retain only the symbols that characterize the numbers of s and d bosons in the basis vectors:

$$\begin{aligned}\langle d^{n_d'}|\,H_{\text{TQM}}\,|d^{n_d}\rangle=&h_0\delta_{n_dn_d'}+h_1n_d\delta_{n_dn_d'}\\&+h_2\,[\sqrt{(N-n_d)\,(N-n_d-1)}\,\langle d^{n_d'}|(d_2^+d_2^+)_{(00)}|d^{n_d}\rangle+\text{h.c}]\\&+h_3\,[\sqrt{N-n_d}\,\langle d^{n_d'}|(d_2^+\,(d_2^+\tilde{d}_2)_2)_{(00)}\,|d^{n_d}\rangle+\text{h.c.}]\\&+\sum_{L=0,\,2,\,4}h_{4L}\sqrt{2L+1}\,\langle d^{n_d'}|((d_2^+d_2^+)_3\,(\tilde{d}_2\tilde{d}_2)_2)_{(00)}|d^{n_d}\rangle.\end{aligned} \tag{14}$$

Further,

$$\begin{aligned}\langle d^{n_d'}s^{N-n_d'}|\,H_{\text{IBM-1}}\,|s^{N-n_d}d^{n_d}\rangle=&\epsilon_0\delta_{n_dn_d'}+\epsilon_d'n_d\delta_{n_dn_d'}\\&+\frac{v_0}{\sqrt{2}}\,[\sqrt{(N-n_d)\,(N-n_d-1)}\\&\times\langle d^{n_d'}s^{N-n_d'}|(d_2^+d_2^+)_{(00)}|\,s^{N-n_d-2}d^{n_d}\rangle\\&+\text{h.c.}]+v_2\,[\sqrt{N-n_d}\,\langle d^{n_d'}s^{N-n_d'}|\\&\times((d_2^+d_2^+)\,d_2)_{(00)}|s^{N-n_d-1}d^{n_d}\rangle+\text{h.c.}]\\&+\sum_{L=0,\,2,\,4}\frac{1}{2}\sqrt{2L+1}\,c_L'\,\langle d^{n_d'}s^{N-n_d'}|\\&\times((d_2^+d_2^+)_{(2)}\,(d_2d_2)_L)_{00}|s^{N-n_d}d^{n_d}\rangle.\end{aligned} \tag{15}$$

It can be seen that because the basis is orthonormal the dependence on the s bosons disappears in (15) and under the condition

$$\left.\begin{aligned}h_0&=\epsilon_sN+\frac{1}{2}v_0N\,(N-1);\\h_1&=\epsilon_d-\epsilon_s+\left(\frac{1}{\sqrt{5}}u_2-v_0\right)(N-1);\\h_2&=v_0/\sqrt{10};\\h_3&=v_2/\sqrt{5};\\h_{4L}&=c_L+\frac{1}{2}u_0-u_2/\sqrt{5}\end{aligned}\right\} \tag{16}$$

the right-hand sides of (14) and (15) are identical. One can prove similarly the identity of the matrix elements of the operators Q_{TQM} and $Q_{\text{IBM-1}}$ for equal values of m_1 and m_2. Thus, the equivalence of the TQM and IBM-1 "in the weak sense" can be regarded as proved.

Operator equivalence of the TQM and IBM-1

In Ref. 16, a stronger assertion was made, namely, that the Hamiltonians of the TQM and IBM-1 are as operators (irrespective of the basis) identical. To prove this, it is necessary to eliminate the cause that masks the equivalence of the operators of the two models (they have different boson forms). One of the possible ways of solving this problem is to obtain the Schwinger representation for the operators of the TQM and then compare them with the corresponding operators of the IBM-1.[24] (The possibility of using the Schwinger representation for the generators of the SU(6) algebra by introducing monopole bosons β^+,β was pointed out by the authors of the TQM in Refs. 6 and 23.)

In what follows, we shall for brevity denote the generators of the collective quadrupole algebra (ACQO) $\{\hat{q}_\mu,\hat{p}_\mu,i[\hat{q}_{\mu'},\hat{p}_{\mu'}],[\hat{q}_\mu,\hat{q}_{\mu'}]\}$ by $\{\hat{q}_\lambda\}$. The SU(6) algebra has 35 generators, and its fundamental representation is six-dimensional. As for any Lie algebra

$$[\hat{g}_\lambda,\ \hat{g}_\rho]=\gamma_{\lambda\rho}^{\tau}\hat{g}_\tau \tag{17}$$

(summation over τ), where λ_τ,ρ,τ take the values 1, 2,...,35, and $\gamma_{\lambda\rho}^{\tau}$ are the ACQO structure constants.

Let the set of six-dimensional matrices $\{g_\lambda\}$ generate the fundamental representation of the ACQO. By definition, the Schwinger (S) representation of this algebra is generated by the operators[48]

$$g_\lambda^{\text{S}}\equiv x^+g_\lambda x;$$

$$x^+=(s^+d_{-2}^+d_{-1}^+\ \dots\ d_2^+);\quad x=\begin{pmatrix}s\\d_{-2}\\d_{-1}\\\vdots\\d_2\end{pmatrix}. \tag{18}$$

Using (17), (18), and the Bose commutation relations of the operators s, s^+, d_μ, d_μ^+, we can show that

$$[\ g_\lambda^{\mathrm{S}},\ \ g_\rho^{\mathrm{S}}\ ] = \gamma_{\lambda\rho}^{\tau}\ g_\tau^{\mathrm{S}}\ ,$$

i.e., $\{g_\lambda^{\mathrm{S}}\}$ is a realization of the ACQO.

We shall show that if the parameters of H_{coll} in (3) and $H_{\mathrm{IBM\text{-}1}}$ in (8) are related by

$$\left.\begin{aligned}
&\epsilon_s = -10e + 5\,(u + v + t_0) + d;\\
&\epsilon_d = 2e + u + v + \frac{1}{5}\sum_{L=0}^{4}(2L+1)\,t_L + d;\\
&u_0 = 10t_0;\\
&u_2 = 2\sqrt{5}\,(u + v - t_0);\\
&v_0 = \sqrt{10}\,(u - v);\\
&v_2 = \sqrt{5}\,w;\\
&c_L = 2\sum_{L=0}^{4} t_l\,(-1)^l\,(2l+1)\begin{Bmatrix}22L\\22l\end{Bmatrix},
\end{aligned}\right\} \qquad (19)$$

then the following equations hold:

$$H_{\mathrm{coll}}^{\mathrm{S}} \equiv H_{TQM}^{\mathrm{S}} = H_{\mathrm{IBM\text{-}1}};\ Q_{\mu TQM}^{\mathrm{S}} = Q_{\mu(\mathrm{IBM\text{-}1})}\,. \qquad (20)$$

A connection between the parameters of the two models in the form (19) was first introduced in Ref. 49. It has certain advantages over the other forms of connection that include the quantum number N explicitly. The distinguished role of the quantum number N among the remaining free parameters of the TQM was emphasized in Ref. 49. In addition, the relations are between the same numbers of parameters of the two models [in contrast to the relations (16)] and the correspondence between them is one-to-one (see Ref. 49). The parameter d appears on the right-hand side of (19) if (3) is augmented by the ninth term $d\cdot\hat{N}$ [$\hat{N}$ is the 36th generator of U(6)].

The Lie algebra of the group SU(6) is one of the classical Lie algebras[50–52] (A_5 in the notation of Cartan). For the construction of $\{g_\lambda^{\mathrm{S}}\}$, we shall use the powerful formalism of the theory of classical Lie algebras.

It is well known that the SU(6) algebra has rank 5, i.e., among the 35 generators five commute with one another.[53] The canonical form of the commutation relations of the SU(6) algebra and the explicit form of the matrices of the defining representation are well known. We give some formulas that we shall use in what follows. The standard form of the SU(6) algebra is[51–53]

$$[H_k,\ H_l] = 0,\quad (k,\ l = 1,\ 2,\ \ldots,\ 5); \qquad (21)$$

$$[H_k,\ E_{\pm\alpha}] = r_k(\pm\alpha)\,E_{\pm\alpha},\quad (\alpha = 1,\ 2,\ \ldots,\ 15); \qquad (22)$$

$$[E_\alpha,\ E_{-\alpha}] = \sum_{k=1}^{5} r_k(\alpha)\,H_k; \qquad (23)$$

$$[E_\alpha,\ E_\beta] = N_{\alpha\beta}^{\delta} E_\delta. \qquad (24)$$

In (24), there are no summations.

We explain the notation. The six-dimensional matrices $\{H_k\}$ are elements of the maximal Cartan Abelian subalgebra. They are diagonal and can be chosen in the form[53]

$$\left.\begin{aligned}
&H_k = [12k\,(k+1)]^{-1/2}\,\mathrm{diag}\begin{pmatrix}\longleftarrow\!\!-\!\!-\!\!-\!\!-\ 6\ -\!\!-\!\!-\!\!-\!\!\longrightarrow\\ 1,\ \ldots,\ 1,\ -k,\ 0,\ \ldots,\ 0)\\ \leftarrow k \rightarrow\qquad\qquad\qquad\end{pmatrix};\\
&\qquad\qquad \mathrm{Tr}\,(H_k H_e) = \delta_{ke}.
\end{aligned}\right\} \qquad (25)$$

The raising and lowering canonical generators have the form

$$E_\alpha = \sqrt{\frac{1}{12}}\begin{pmatrix}0 & \overset{j}{} & 0\\ & 1 & \\ 0 & & 0\end{pmatrix} i;\quad E_{-\alpha} = E_\alpha^+, \qquad (26)$$

where $j > i = 1,\ldots,6$.

There is the following connection between α and $(10i + j)$:

α	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$10i+j$	12	13	14	15	16	23	24	25	26	34	35	36	45	46	56

The operators $\{H_k\}$ and $\{E_{\pm\alpha}\}$ are called the Cartan–Weyl canonical generators. The five-dimensional root vectors $\mathbf{r}(\alpha)$ satisfy the conditions

$$r(-\alpha) = -r(\alpha);\quad \sum_{\pm\alpha} r_k(\alpha)\,r_i(\alpha) = \delta_{ki}.$$

We turn to the problem of finding $\{g_\lambda^{\mathrm{S}}\}$. We use a purely algebraic method of finding 35 six-dimensional matrices satisfying the ACQO. It is based on a general theorem of Cartan, according to which the ACQO can be reduced to a canonical form for which a matrix realization is known: (25), (26). The essence of this method is as follows. We form linear combinations of the $\{g_\lambda\}$ that satisfy the relations (21)–(24). Therefore, these combinations can be associated with $\{H_k\}$ and $\{E_{\pm\alpha}\}$, and for them a representation in the form of matrices is known. Making the inverse transformation, we find in explicit form the matrices $\{q_\lambda\}$.

1. *We fix a basis in the ACQO*:

$$\begin{aligned}
\{g_\lambda\} = \{&[q_\mu q_{-\mu}],\ (\mu = 1,\ 2);\quad \mathrm{i}\,[q_\mu,\ p_\mu],\ (\mu = 0,\ 1,\ 2);\\
&q_\mu,\ p_\mu,\ (\mu = 0,\ \pm 1,\ \pm 2);\quad \mathrm{i}\,[q_\mu,\ p_{-\mu}],\ (\mu = \pm 1,\ \pm 2);\\
&\mathrm{i}\,[q_{-2},\ p_1],\ (\mathrm{i}\,[q_0,\ p_\mu],\ \mu = 1,\ 2);\quad \mathrm{i}\,[q_1,\ p_0],\ \mathrm{i}\,[q_1,\ p_2];\\
&(\mathrm{i}\,[q_2,\ p_\mu],\ \mu = 0,\ \pm 1);\quad ([q_{-2},\ q_\mu],\ \mu = 0,\ \pm 1);\\
&([q_2,\ q_\mu],\ \mu = 0,\ \pm 1);\quad [q_0,\ q_{-1}],\ [q_0,\ q_1]\}.
\end{aligned}$$

2. *We find the Cartan subalgebra in the ACQO*. Using the commutation relations (2), we can readily show that this subalgebra consists of the five generators

$$\{\mathrm{i}\,[q_\mu,\ p_\mu],\ (\mu = 0,\ 1,\ 2);\quad [q_\mu,\ q_{-\mu}],\ (\mu = 1,\ 2)\}.$$

They commute with each other and can be simultaneously reduced to diagonal form. We introduce the notation

$$H' \equiv \begin{pmatrix} [q_1,\ q_{-1}] \\ [q_2,\ q_{-2}] \\ \mathrm{i}\,[q_0,\ p_0] \\ \mathrm{i}\,[q_1,\ p_1] \\ \mathrm{i}\,[q_2,\ p_2] \end{pmatrix}; \quad H \equiv \begin{pmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \\ H_5 \end{pmatrix}. \tag{27}$$

It is clear that H' and H are connected by a linear transformation

$$H' = MH, \tag{28}$$

where M is a five-dimensional and as yet unknown matrix.

3. *As operators $E_{\pm\alpha}$ it is possible to choose the following 30 operators*:

$$\left.\begin{matrix} \{(\hat{q}_\mu \pm \mathrm{i}\hat{p}_{-\mu}),\ \mu=0,\ \pm1,\ \pm2;\ (\mathrm{i}\,[\hat{q}_\mu,\ \hat{p}_{-\mu}],\mu=\pm1,\ \pm2); \\ ([\hat{q}_\mu,\ \hat{q}_{\mu'}] \pm \mathrm{i}\,[\hat{q}_\mu,\ p_{-\mu'}])\}. \end{matrix}\right\} \tag{29}$$

One can show that the operators (29) satisfy the commutation relations (22)–(24).

4. *We find the exact correspondence* $\{\mu_1(\mu,\mu')\} \leftrightarrow \{\pm\alpha\}$. The solution of the problem is greatly facilitated by knowledge of the matrix M. Omitting the details,[24] we give the final result:

$$M^{-1} = \begin{pmatrix} 0 & \frac{1}{4}\sqrt{\frac{1}{6}} & 0 & 0 & -\frac{1}{4}\sqrt{\frac{1}{6}} \\ -\frac{1}{6}\sqrt{\frac{1}{2}} & \frac{1}{12}\sqrt{\frac{1}{2}} & 0 & \frac{1}{6}\sqrt{\frac{1}{2}} & \frac{1}{12}\sqrt{\frac{1}{2}} \\ \frac{1}{24} & -\frac{1}{24} & -\frac{1}{8} & \frac{1}{24} & \frac{1}{24} \\ \frac{5}{8}\sqrt{\frac{1}{15}} & -\frac{1}{8}\sqrt{\frac{1}{15}} & \frac{1}{8}\sqrt{\frac{1}{15}} & -\frac{3}{8}\sqrt{\frac{1}{15}} & \frac{1}{8}\sqrt{\frac{1}{15}} \\ 0 & -\frac{1}{2}\sqrt{\frac{1}{10}} & \frac{1}{12}\sqrt{\frac{1}{10}} & \frac{1}{6}\sqrt{\frac{1}{10}} & -\frac{1}{3}\sqrt{\frac{1}{10}} \end{pmatrix}; \tag{30}$$

$$M = \begin{pmatrix} 0 & -2\sqrt{2} & 1 & \sqrt{15} & 0 \\ \sqrt{6} & -\sqrt{2} & -1 & \frac{1}{15}\sqrt{15} & -\frac{6}{5}\sqrt{10} \\ -2\sqrt{6} & -2\sqrt{2} & -8 & 0 & 0 \\ -2\sqrt{6} & -4\sqrt{2} & -1 & -\sqrt{15} & 0 \\ -3\sqrt{6} & -\sqrt{2} & -1 & -\frac{1}{5}\sqrt{15} & -\frac{6}{5}\sqrt{10} \end{pmatrix}; \tag{31}$$

$$\left.\begin{aligned} q_2 + \mathrm{i}p_{-2} &= 2\sqrt{12}\,E_1; \\ q_1 - \mathrm{i}p_{-1} &= -2\sqrt{12}\,E_2; \\ q_0 + \mathrm{i}p_0 &= 2\sqrt{12}\,E_3; \\ q_{-1} - \mathrm{i}p_1 &= -2\sqrt{12}\,E_4; \\ q_{-2} + \mathrm{i}p_2 &= 2\sqrt{12}\,E_5. \end{aligned}\right\} \tag{32}$$

The other five combinations can be obtained by taking the Hermitian conjugates of (32).

Since the explicit forms of the matrices M^{-1} and M are known, it is easy to find the correspondence between the operators in (29) and $\{E_{\pm\alpha}\}$.

$$\left.\begin{matrix} q_{-2} = \sqrt{12}\,(E_{-1} + E_5); \quad p_{-2} = \mathrm{i}\sqrt{12}\,(E_{-5} - E_1); \\ q_{-1} = \sqrt{12}\,(E_{-2} - E_4); \quad p_{-1} = -\mathrm{i}\sqrt{12}\,(E_{-4} + E_2); \\ q_0 = \sqrt{12}\,(E_3 + E_{-3}); \quad p_0 = \mathrm{i}\sqrt{12}\,(E_{-3} - E_3). \\ \mathrm{i}\,[q_1,\ p_{-1}] = 2\sqrt{12}\,E_{-11}; \\ \mathrm{i}\,[q_2,\ p_{-2}] = 2\sqrt{12}\,E_{-9}; \\ \mathrm{i}\,[q_{-2},\ p_1] = \sqrt{12}\,(E_8 - E_{12}), \quad \mathrm{i}\,[q_1,\ p_2] = \sqrt{12}\,(E_{15} - E_6); \\ \mathrm{i}\,[q_0,\ p_1] = \sqrt{12}\,(E_{13} - E_{10}); \quad \mathrm{i}\,[q_2,\ p_0] = \sqrt{12}\,(E_{-14} + E_{-7}); \\ \mathrm{i}\,[q_0,\ p_2] = \sqrt{12}\,(E_{14} + E_7); \quad \mathrm{i}\,[q_2,\ p_1] = \sqrt{12}\,(E_{-12} - E_{-8}); \\ \mathrm{i}\,[q_1,\ p_0] = \sqrt{12}\,(E_{-13} - E_{-10}); \quad \mathrm{i}\,[q_2,\ p_1] = \sqrt{12}\,(E_{-15} - E_{-6}); \\ [q_{-2},\ q_0] = \sqrt{12}\,(E_7 - E_{14}); \quad [q_2,\ q_{-1}] = -\sqrt{12}\,(E_{-6} + E_{-15}); \\ [q_{-2},\ q_{-1}] = -\sqrt{12}\,(E_8 + E_{12}); \quad [q_2,\ q_1] = -\sqrt{12}\,(E_{-8} + E_{-12}); \\ [q_{-2},\ q_1] = -\sqrt{12}\,(E_6 + E_{15}), \quad [q_0,\ q_{-1}] = -\sqrt{12}\,(E_{-10} + E_{-13}). \end{matrix}\right\} \tag{33}$$

5. *Making the inverse transformation, we find $\{g_\lambda\}$ expressed in terms of $\{H_k, E_{\pm\alpha}\}$.*

The final result is

The remaining operators can be obtained by using the properties of Hermitian conjugation of the operators.

The relations (33) give the required matrix realization of the ACQO. The corresponding Schwinger representation obtained by means of (18) has the form

$$\left.\begin{aligned} q_\mu^{\mathrm{s}} &= sd_\mu^+ + (-1)^\mu d_{-\mu}s^+; \\ p_\mu^{\mathrm{s}} &= \mathrm{i}\,((-1)^\mu sd_{-\mu}^+ - s^+d_\mu); \\ \mathrm{i}\,[q_\mu,\ p_{\mu'}]^{\mathrm{s}} &= d_\mu^+ d_{\mu'} + (-1)^{\mu+\mu'} d_{-\mu'}^+ d_{-\mu} - 2\delta_{\mu\mu'}s^+s; \\ [q_\mu,\ q_{\mu'}]^{\mathrm{s}} &= (-1)^{\mu'} d_\mu^+ d_{-\mu'} - (-)^\mu d_{\mu'}^+ d_{-\mu}. \end{aligned}\right\} \tag{34}$$

It follows from (10) and (34) that

$$[N^{\mathrm{S}}\ ,\ q_\lambda^{\mathrm{s}}\ ] = 0 \tag{35}$$

for all λ, i.e., N^{S} is a Casimir operator of the SU(6) algebra.

Substituting (34) in (3) and (6), we obtain expressions identical in their boson structure to the corresponding expressions of the IBM-1. The parameters of the two models are connected by the relations (19).

Thus, we have shown that the Hamiltonians and the quadrupole operators of the two models agree identically.

Unitary equivalence of the TQM and IBM-1 in the physical subspace

The question of the equivalence of the TQM and IBM-1 can be formulated as the question of whether or not the Schwinger and Holstein–Primakoff realizations of the ACQO are isomorphic[4)] and, in particular, unitarily equivalent. Questions of this kind have arisen in various branches of theoretical physics.[55-57] A central role in such questions is posed by a certain abstract algebra, the concrete physical theories corresponding to definite realizations of it.

This abstract algebra can have isomorphic ($\equiv$, i.e., equivalent) and nonisomorphic representations. Physical theories corresponding to inequivalent representations may differ strongly. However, if the representations are isomorphic, there is a similarity between the physical theories. Finally, if the representations are unitarily equivalent, the physical theories are identical.

We introduce the support spaces, i.e., the spaces on which act the generators $\{g_\lambda^{\mathrm{S}}\}$ and $\{g_\lambda^{\mathrm{HP}}\}$ [given by means of (4) and (34) as operators on the corresponding Hilbert spaces] of the collective quadrupole algebra, regarded as an abstract algebra. It is known[54,58] that the set of vectors

$$\left\{[N]\,n_s\{n_\nu\}\,\mathrm{s}\ \rangle = \prod_{y=-2}^{2}(n_\nu!)^{-1/2}\,(d_\nu^+)^{n_\nu}\,(n_s!)^{-\frac{1}{2}}\,(s^+)^{n_s}|0\rangle\right\}, \tag{36}$$

such that

$$n_d = \sum_{\nu=-2}^{2} n_\nu;\ \ n_s + n_d = N, \tag{37}$$

form a basis for the completely symmetric representation of SU(6) having dimension $(N+5)!/\mathrm{N}!5!$. In (36), $|0\rangle$ is the vacuum of the s and d bosons.

It can be seen from the explicit form of the ACQO generators in the Schwinger representation (34) that the set of vectors $\{|[N]n_s=\{n_\nu\}\rangle\}$ is invariant with respect to the action of $\{g_\lambda^{\mathrm{S}}\}$. By definition, the space formed by the vectors $\{|[N]n_s\{n_\nu\}\rangle_{\mathrm{S}}\}$ is called the support space.

Let $\{M\}$ be the set of vectors (36). The generators $\{g_\lambda^{\mathrm{S}}\}$ are well defined on $\{M\}$. Let N be fixed: $N = N_0$, i.e., we are in a given completely symmetric representation of SU(6). Any subset of vectors $\{|[N']n_s\{n_\nu\}\rangle_{\mathrm{S}}\}$ such that

$$n_s + n_d = N' < N_0, \tag{38}$$

forms a subspace in $\{M\}$ invariant under the action of $\{g_\lambda^{\mathrm{S}}\}$. In terms of $\{M\}$ and $\{|[N']n_s\{n_\nu\}\rangle_{\mathrm{S}}\}$, the support space of the Schwinger representation of the ACQO is the factor space[44]

$$\begin{aligned} &\{|\,[N_0]\,n_s\,\{n_\nu\}\,\rangle\ \mathrm{s}\ \ \} \\ &= \{M\}/\{|\,[N']\,n_s\ \ \{n_\nu\}\,\rangle\ \mathrm{s}\ \ \}, \end{aligned} \tag{39}$$

i.e., from the set of vectors (36) with $N = N_0, N_0 - 1, ..., 1$, or 0 all vectors for which (38) holds are eliminated, and only those for which $N = N_0$ remain. Since the space (39) is invariant under the action of $\{g_\lambda^{\mathrm{S}}\}$, this is the support space for the Schwinger representation.

We introduce the support space for the HP representation. The set of vectors

$$\left\{\,|[N_0]\{n_\nu\}\rangle_{\mathrm{HP}} = \prod_{y=-2}^{2}(n_\nu!)^{-\frac{1}{2}}\,(b_\nu^+)n^\nu|\widetilde{0}\rangle\right\}; \tag{40}$$

$$n_b = \sum_{\nu=-2} n_\nu \leqslant N_0 \tag{41}$$

forms a basis for the completely symmetric irreducible representation $[N_0]$ of the group SU(6).[59,60] Here, $|\widetilde{0}\rangle$ is the vacuum of the b bosons. For convenience when considering the generators in the HP representation, we shall use not the operators d_{μ^-} and d_μ^+, as in (4), but b_μ^+, b_μ. It is readily seen that when the generators $\{g_\lambda^{\mathrm{S}}\}$ are applied to the vectors (40) they do not carry them out of this subspace. The basis states (40) are to be regarded as physical states, and the vectors with $n_b > N_0$ as unphysical states, since the generators $q_\mu^{\mathrm{HP}}, p_\mu^{\mathrm{HP}}$ have a meaning only if the condition (41) is satisfied.

We introduce the operator that maps the support space of the Schwinger representation onto the support space of the Holstein–Primakoff representation:

$$P_{N_0} = \sum_{\{n_\nu\}} |[N_0]\{n_\nu\}\rangle\ _{\mathrm{HP\cdot S}}\ \langle\{n_\nu\}\,n_s\,[N_0]|. \tag{42}$$

We shall prove the unitary equivalence of the TQM and IBM-1 by using the intertwining operators. First, we recall some concepts from the theory of representations of Lie algebras.[63,64]

Let g^k and g^L be two representations of a certain abstract algebra A acting on the spaces M^k and M^L, respectively. One says that g^k and g^L are partly isomorphic (equivalent) if there exists a continuous linear mapping P: $M^k \to M^L$ such that

$$Pg_\lambda^k = g_\lambda^L P;\quad g_\lambda^k \in A;\ \ g_\lambda^L \in A. \tag{43}$$

A mapping with this property is called an intertwining operator for the representations g^k and g^L. If in addition P has a continuous inverse operator, the representations g^k and g^L are said to be isomorphic (equivalent). If P is isometric, i.e., preserves the norm, one says that g^k and g^L are unitarily equivalent. The set of points of M^k that are mapped to the null vector of M^L is called the kernel of the mapping P and denoted by the symbol ker P. The set of points $\{M^L\}$ obtained by applying P to $\{M^k\}$ is called the image of P.

It is readily verified that the operator P_{N_0} has the property (43), i.e., intertwines the S and HP representations of the collective quadrupole algebra:

$$P_{N_0} g^{\rm S} \quad = g^{\rm HP}\; P_{N_0} \tag{44}$$

for any generator in $\{g^{\rm S}\} \equiv (q_\mu^{\rm S}, p_\mu^{\rm S}, i[q_\mu, p_\mu]^{\rm S}, [q_\mu, q_{\mu'}])$ and $\{g^{\rm HP}\} \equiv (q_\mu^{\rm HP}, p_\mu^{\rm HP}, i[q_{\mu'}, p_{\mu'}]^{\rm HP}, [q_\mu, q_{\mu'}]^{\rm HP})$.

We investigate the zeros of P_{N_0}. It follows from (36), (37), (40), and (42) that outside the physical space where $\Sigma_\nu n_\nu > N_0$, $P_{N_0} \to 0$ and, therefore, an inverse operator does not exist. Hence, P_{N_0} is well defined only on the physical subspace.

We find the set of points mapped to the null vector of the set $\{|[N_0]\{n_\nu\}\rangle_{\rm HP}\}$. Using $n_s + n_d = N' < N_0$ and (42), we obtain

$$P_{N_0}|[N']\, n_s \{n_\nu\}\rangle\; _{\rm S} \; = 0. \tag{45}$$

It follows from this that P_{N_0} has a nontrivial kernel. In this case, it is natural to introduce the corresponding factor space

$$\{M\}/\{|[\dot{N}']\, n_s \{n_\nu\}\rangle\, {\rm s} \quad \equiv \{M\}/\ker P_{N_0}$$

and rewrite (39) as follows:

$$\{|[N_0]\, n_s \{n_\mu\}\rangle\} = \{M\}/\ker P_{N_0}.$$

The factor space consists of the classes $\{|m\rangle + \ker P_{N_0}\}$ of equivalent elements of the set $\{M\}$. [Two elements are said to be equivalent, i.e., to belong to one and the same class, if and only if $(|m_1\rangle - |m_2\rangle) \in \ker P_{N_0}$.] On this factor space, we can define[65] the operator $P^f_{N_0}$, setting

$$P^f_{N_0}(|m\rangle + \ker P_{N_0}) \equiv P_{N_0}|m\rangle, \tag{46}$$

where $|m\rangle \in \{M\}$.

The relation (46) means that $P^f_{N_0}$ maps $\{M\}/\ker P_{N_0}$ isomorphically onto the physical subspace $\{|[N_0] \times \{n_\mu\}\rangle_{\rm HP}\}$. For the operator P_{N_0} this is not the case, since it has a nontrivial kernel.

Let $g_f^{\rm S}$ be the Schwinger representation of the ACQO induced by $g^{\rm S}$ in the factor space $\{M\}/\ker P_{N_0}$. We consider the expression

$$P^f_{N_0} g_f^{\rm S} \quad (|m\rangle + \ker P_{N_0}),$$

where $|m\rangle$ is any state in $\{M\}$. Bearing in mind that $\ker P_{N_0}$ is invariant with respect to the action ot $g_f^{\rm S}$, and taking into account (46), we obtain

$$P^f_{N_0}(\, g_{\rm S} \quad |m\rangle + \ker P_{N_0}) = P_{N_0}\, g_{\rm S} \quad |m\rangle.$$

We have used the fact that by virtue of (45) P_{N_0} "does not note" the elements of its kernel. Using the fact that P_{N_0} intertwines $g^{\rm S}$ and $\mathrm{g}^{\rm HP}$ [see (44)], we can write

$$P_{N_0}\dot{g}_{\rm S} \quad |m\rangle = g^{\rm HP} P_{N_0}|m\rangle \equiv g^{\rm HP} P^f_{N_0}(|m\rangle + \ker P_{N_0}). \tag{47}$$

To write down the second equation, we have again used the definition (46). In addition, we can write

$$P^f_{N_0}(|m\rangle + \ker P_{N_0}) \in |[N_0]\{n_\mu\}\rangle_{\rm HP},$$

since the image of the operator $P^f_{N_0}$ is simply the physical subspace. Thus, the right-hand side of (47) can be rewritten as

$$g^{\rm HP}_{\rm phys}\, P^f_{N_0}(|m\rangle + \ker P_{N_0}),$$

where $g^{\rm HP}_{\rm phys}$ is the restriction of the HP representation to the physical subspace. Thus, for any $|m\rangle \in \{M\}$

$$P^f_{N_0} g^{\rm S}_f \quad (|m\rangle + \ker P_{N_0}) = g^{\rm HP}_{\rm phys} P^f_{N_0}(|m\rangle + \ker P_{N_0}), \tag{48}$$

from which it follows that $P^f_{N_0} g_f^{\rm S} = g^{\rm HP}_{\rm phys} P^f_{N_0}$. Bearing in mind that $P^f_{N_0}$ has a well-defined inverse operator, the relation (48) means that the Schwinger realization of the ACQO $\{g_f^{\rm S}\}$ induced by the representation $\{g^{\rm S}\}$ in the factor space $\{M\}/\ker P_{N_0}$ is equivalent to the HP realization of the ACQO restricted to the physical subspace.

Note that if we work in the physical subspace (i.e., for $n_b < N_0$, $n_s > 0$), the operator $P^f_{N_0}$ preserves the norm in $\{|\rangle_{\rm S}\}$, and $(P^f_{N0})^+$ preserves it in $\{|\rangle_{\rm HP}\}$:

$$\begin{aligned}
&\| P^f_{N_0}|[N_0]\, n_s \{n_\nu\}\rangle\, {\rm s} \quad \| \\
&= {\rm s} \;\; \langle\{n_\nu\}\, n_s\, [N_0]|(P^f_{N0})^+ P^f_{N_0}|[N_0]\, n_s \{n_\nu\}\rangle\; {\rm s} \\
&= {}_{\rm HP}\langle\{n_\nu\}\, [N_0]|[N_0]\{n_\nu\}\rangle_{\rm HP} \\
&= 1 = {\rm s} \;\; \langle\{n_\nu\}\, n_s\, [N_0]|[N_0]\, n_s \{n_\nu\}\rangle\; _{\rm S} \\
&\qquad = \||[N_0]\, n_s \{n_\nu\}\rangle\, {\rm s} \quad \|; \\
&\| (P^f_{N0})^+\, |[N_0]\{n_\nu\}\rangle_{\rm HP} \| = \| [N_0]\{n_\nu\}\rangle_{\rm HP} \|.
\end{aligned}$$

The facts that $P^f_{N_0}$ is isometric and $g_f^{\rm S}$ and $g^{\rm HP}_{\rm phys}$ are isomorphic mean, taken together, that $g_f^{\rm S}$ and $g^{\rm HP}_{\rm phys}$ are unitarily equivalent:

$$\left.\begin{aligned}
g_f^{\rm S} &= (P^f_{N_0})^{-1}\; g^{\rm HP}_{\rm phys} P^f_{N_0}; \\
g^{\rm HP}_{\rm phys} &= P^f_{N_0}\, g_f^{\rm S} \quad (P^f_{N_0})^{-1}.
\end{aligned}\right\} \tag{49}$$

From the unitary equivalence of the Schwinger and Holstein–Primakoff realizations of the ACQO we obtain immediately the unitary equivalence of the IBM-1 and TQM, since

$$H_{\rm TQM} = P^f_{N_0}\, H_{\rm IBM\text{-}1}\, (P^f_{N_0})^{-1}; \tag{50}$$

$$Q_{\rm TQM} = P^f_{N_0}\, Q_{\rm IBM\text{-}1}\, (P_{N_0})^{-1}. \tag{51}$$

This circumstance was already pointed out in Refs. 66 and 67. The significance of the propositions (50) and (51) is indeed very transparent. However, the unitary equivalence of the two models holds only in the physical subspace. [Otherwise $\{g^{\rm S}_\lambda\}$ and $\{g^{\rm HP}_\lambda\}$ are only partly isomorphic, and the operator P_{N_0} is partly isometric (it makes the orthogonal complement to the physical subspace vanish).] We note that the fact that it is necessary to consider the TQM and IBM-1 in the physical space was previously assumed, and it was this that served as the basis for establishing the equivalence at the level of the matrix elements. But here, the region in which the unitary equivalence of the TQM and IBM-1 holds—the physical subspace—is obtained as a result of the derivation of the fundamental relation (48).

Unification of the TQM and IBM-1: phenomenological SU(6) boson model (IBM)

The unifying point of departure of the two models is the collective quadrupole algebra (2). Having shown that the Hamiltonian and the physical operators of the TQM and IBM-1 arise as a result of the Holstein–Primakoff and Schwinger bosonizations of the ACQO, respectively, we have ascribed its fundamental significance and studied these two representations of the ACQO in their interconnections. Since by virtue of (50) and (51) the TQM and IBM-1 are unitarily equivalent (if, of course, the operators from $\{g_\lambda^{\rm HP}\}$ and $\{g_\lambda^{\rm S}\}$ are given the same meaning), these models are indistinguishable. It has therefore been established that the TQM and IBM-1 are two unitarily equivalent forms of a single phenomenological SU(6) boson model—the interacting boson model—with the Hamiltonian (3) and quadrupole operator (6), i.e.,

$$\hat{H}_{\rm IBM} = \hat{H}_{\rm coll}; \; \hat{Q}_{\rm IBM} = \hat{Q}_{\rm coll}, \tag{52}$$

where the operators $\{\hat{q},\hat{p},i[\hat{q},\hat{p}],[\hat{q},\hat{q}]\}$ which occur in $\hat{H}_{\rm coll}$ and $Q_{\rm coll}$ close the SU(6) algebra (2) of the ACQO.

Thus, our approach to the establishment of the equivalence of the IBM-1 and TQM (the study of the representations of the ACQO in their mutual interconnections) eventually unifies the two models into one—the interacting-boson model (see footnote 1). One can also take the diametrically opposite point of view, namely, take as a basis the IBM defined by (52). We construct the Schwinger representation and, substituting it in (52), obtain the IBM-1 with the Hamiltonian $H_{\rm IBM\text{-}1}$ (8) and quadrupole operator $Q_{\rm IBM\text{-}1}$ (9). We then construct the HP realization of the ACQO (see Ref. 25) given by (4), substitute it in (52), and obtain $H_{\rm TQM}$ (5) and $Q_{\rm TQM}$ (7). Thus, the chosen approach to the study of the realizations of the ACQO and their interconnection permits a deductive derivation of the TQM and IBM-1. Moreover, in the framework of this method one can derive a new collective model [also based on dynamical SU(6) symmetry] corresponding to the Dyson boson realization of the ACQO. The derivation of this "finite quadrupole-phonon model" (FQM) is given in Ref. 68 (see also Ref. 25). The Dyson realization of the ACQO is constructed explicitly using generalized coherent states and the root vectors of the SU(6) algebra. The explicit form of the Dyson representation of the ACQO in terms of the quadrupole bosons is

$$\left.\begin{aligned}
q_\mu^{\rm D} &= b_\mu^+ (N - \sum_\nu b_\nu^+ b_\nu) + (-1)^\mu b_\mu;\\
p_\mu^{\rm D} &= {\rm i}\,[(-1)^\mu b_{-\mu}^+ (N - \sum_\nu b_\nu^+ b_\nu) - b_\mu];\\
{\rm i}\,[q_\mu, \; p_{\mu'}] &= b_\mu^+ b_{\mu'}\\
&\quad + (-1)^{\mu+\mu'} b_{-\mu'}^+ b_{-\mu} - 2\delta_{\mu\mu'} (N - \sum_\nu b_\nu^+ b_\nu);\\
[q_\mu, \; q_{\mu'}] &= (-1)^{\mu'} b_\mu^+ b_{-\mu'} - (-1)^\mu b_\mu^+ b_{-\mu}.
\end{aligned}\right\} \tag{53}$$

Substitution in (52) leads to the FQM Hamiltonian, which has the form

$$\begin{aligned}
H_{\rm FQM} &= 2e \sum_\nu b_\nu^+ b_\nu - 10e\,(N - \sum_\nu b_\nu^+ b_\nu)\\
&\quad + d_0 (5 - 2 \sum_\nu b_\nu^+ b_\nu)(N - \sum_\nu b_\nu^+ b_\nu)\\
&\quad + \sum_L d_L (-1)^{L-M} (b_2^+ b_2)_{LM} (b_2^+ b_2)_{L-M}\\
&+ (s - v)\,[\sum_\mu (-1)^\mu b_\mu^+ b_{-\mu}^+ (N - \sum_\nu b_\nu^+ b_\nu)(N - \sum_\nu b_\nu^+ b_\nu - 1)\\
&+ \sum_\mu (-1)^\mu b_\mu b_{-\mu}]\\
&+ (s + v)\,[(5 + 2 \sum_\nu b_\nu^+ b_\nu)(N - \sum_\nu b_\nu^+ b_\nu) + \sum_\nu b_\nu^+ b_\nu]\\
&+ w\,[\sum_\mu (b_2^+ b_2^+)_{2\mu} b_\mu (N - \sum_\nu b_\nu^+ b_\nu) + {\rm h.c.}].
\end{aligned} \tag{54}$$

Thus, in addition to the existing SU(6) Hamiltonians $H_{\rm TQM}$ and $H_{\rm IBM\text{-}1}$ we have obtained the SU(6) Hamiltonian $H_{\rm FQM}$, constructed as a rotational invariant from the generators in the Dyson representation (53) of the ACQO. A Hamiltonian of such type was discussed in Ref. 69 in connection with the group O(8), and not in the context of dynamical SU(6) symmetry.

Although $H_{\rm FQM}$ is non-Hermitian for $s \neq v$ and $w \neq 0$, it has the advantage that it is finite; it is regarded hopefully and methods of solution are being developed,[45] especially in connection with the microscopic justification of the boson models.[70]

Since the finite quadrupole-phonon model is still being developed, it has not achieved the popularity of the IBM. Therefore, the question of the relationship between these two models is not yet topical. A different question is of interest: Is the unitary equivalence of the TQM and IBM-1 that holds in the quantum case preserved in the classical limit or, in other words, will the classical limits of the TQM and IBM-1 give identical energy surfaces and, therefore, lead to the same classical trajectories? We shall show that the answer to this question is in the affirmative.

Establishment of the equivalence of the IBM-1 and TQM in the classical limit

We recall that the classical limit of a quantum-mechanical operator is defined as its mean value in a generalized coherent state.[44,71] Such a state is a convenient tool for constructing the classical limits of quantum observables, enabling one to remain nevertheless in the quantum domain.[71]

The generalized coherent state for the canonical SU(6) algebra (21)–(22) is determined[44,71] by the action of representatives of the factor space SU(6)/U(5) on the highest vector $|\mathbf{j}\rangle$ of a completely symmetric SU(6) representation:

$$|c, \; \mathbf{j}\rangle \equiv \exp\{\sum_{\mathbf{r}(\alpha)\mathbf{j}>0} c_\alpha^* E_\alpha\} \, |\mathbf{j}\rangle. \tag{55}$$

Here, $\{c_\alpha^*\}$ is a set of complex parameters with index corresponding to the number of the root vector $\mathbf{r}(\alpha)$, $\mathbf{j}$ is the highest weight, and the remaining notation is given after the commutation relations (21)–(24). To obtain the classical limits of $H_{\rm TQM}$ and $H_{\rm IBM\text{-}1}$, we must construct in explicit form the coherent states corresponding to the TQM and IBM-1. The generalized coherent states $|c, \mathbf{j}\rangle_{\rm TQM}$ and $|c, \boldsymbol{\alpha}\rangle_{\rm IBM\text{-}1}$ can be constructed by particularizing (55) with allowance for the following:

1) in the case of the SU(6) algebra (21)–(24), contributions to the exponential factor are made by only the five lowering operators $\{E_{-\beta}\}$ $(\beta = 1,...,5)$ corresponding to the simple root vectors $\mathbf{r}(\beta)$ (see Appendix A in Ref. 25);

2) the explicit form of $\{{\rm E}^{\rm S}_{-\beta}\}$ and $\{E_\beta^{\rm HP}\}$ can be readily found by means of the relations (32)–(34) and (4);

3) $|\mathbf{j}\rangle_{\rm S}$ and $|\mathbf{j}\rangle_{\rm HP}$ are the vacua for the d and b bosons, respectively, and have the form $|\mathbf{j}\rangle_{\rm S} = (N!)^{-1/2}(s^+)^N|0\rangle$ (condensate of N s bosons), $|\mathbf{j}\rangle_{\rm HP} = |\tilde{0}\rangle$ from (40). Introducing an additional a boson (Refs. 25 and 46), we can also

represent $|\mathbf{j}\rangle_{\mathrm{HP}}$ in the condensate form $|\mathbf{j}\rangle_{\mathrm{HP}} = (N)^{-1/2} (a^{+})^{N}|0\rangle$ (see also Sec. 3):

$$|\{c_\mu\},\, \mathbf{j}\rangle_{\mathrm{TQM}} = \Big(1+\frac{1}{12}\sum_{\mu=-2}^{2}|c_\mu|^2\Big)^{-N/2} \times \Big\{1+\sum_{l=1}^{N}(l!)^{-1}[N(N-1)\ldots \ldots (N-l+1)]^{1/2}\prod_{m=1}^{l}\Big(\frac{1}{\sqrt{12}}c^{*}_{\mu m}b^{+}_{\mu m}\Big)\Big\}(N!)^{-1/2}(a^{+})^{N}|0\rangle; \tag{56}$$

$$|\{c_\mu\},\, \mathbf{j}\rangle_{\mathrm{IBM\text{-}1}} = (N!)^{-1/2}\Big(1+\frac{1}{12}\sum_{\mu=-2}^{2}|c_\mu|^2\Big)^{-1/2}\Big(s^{+} + \sum_{\mu=-2}^{2}c^{*}_{\mu}\frac{1}{\sqrt{12}}d^{+}_{\mu}\Big)^{N}|0\rangle. \tag{57}$$

By means of these expressions, using the explicit expressions for H_{TQM} (5) and $H_{\mathrm{IBM\text{-}1}}$ (8), we can calculate the mean values of the latter in the states (56) and (57), respectively. It is readily verified that the equation

$${}_{\mathrm{TQM}}\langle\{c_\mu\}\,\mathbf{j}|H_{\mathrm{TQM}}|\{c_\mu\}\,\mathbf{j}\rangle_{\mathrm{TQM}} = {}_{\mathrm{IBM\text{-}1}}\langle\{c_\mu\}\,\mathbf{j}|H_{\mathrm{IBM\text{-}1}}|\{c_\mu\}\,\mathbf{j}\rangle_{\mathrm{IBM\text{-}1}} \tag{58}$$

will hold if the corresponding parameters are connected by the relations (16). Therefore, the equivalence of the TQM and IBM-1 also holds in the classical limit if there is the same relationship between the parameters that was used to establish the equivalence at the level of the matrix elements. Note that if we use the connection between the parameters H_{TQM} (5) and H_{coll} (3), which has the form

$$\left.\begin{aligned} h_0 &= 5N(u+v+Nt_0-2e);\\ h_1 &= 12e+(2N-6)(u+v)\\ &\quad + t_0\Big(\frac{36}{5}-12N\Big)+\frac{1}{5}\sum_{L=1}(2L+1)t_L;\\ h_2 &= u-v;\\ h_3 &= w;\\ h_{4L} &= c_L-2(u+v)+7t_0, \end{aligned}\right\} \tag{59}$$

we find that the equivalence in the classical limit expressed by (58) holds for a relationship between the parameters in the form (19).

3. CONSISTENCY OF THE IBM-1 AND TQM FOR THE DESCRIPTION OF TWO-NUCLEON TRANSFER REACTIONS

In a study of the relationship between the TQM and IBM-1, it is necessary to distinguish two situations.

1. The Hamiltonian and physical operators are constructed from the generators of the ACQO. This means that one is describing the physical properties and processes in a given nucleus, for example, the spectra and probabilities of electromagnetic transitions. In this situation, one calculates in the theoretical description only the matrix elements between states belonging to a fixed completely symmetric representation of the SU(6) algebra. As we showed in Sec. 2, in this case there is unitary equivalence. Physically, this means that all the calculated results relating to the spectra and electromagnetic transitions, as well as the analytic expressions for the dynamical symmetries [SU(3), SU(5), SO(6)], will be the same.

2. The physical operators are not tensor operators constructed from the generators of the ACQO.

This means that we are considering physical processes relating different nuclei, for example, two-nucleon transfers. The two-nucleon transfer operators that are used in the IBM-1 are not generators of the SU(6) algebra[72,73] (see below). In addition, in the TQM there is no s boson, and the quadrupole bosons do not change the nucleon number. One could get the superficial impression that two-nucleon transfer reactions cannot be described in the framework of the TQM.

In Ref. 46, two of the present authors, taking as their point of departure the corresponding operators used in the framework of the IBM-1, succeeded in constructing their TQM analogs and showing that the two-nucleon amplitudes calculated in the IBM-1 can be expressed in terms of matrix elements that are standard for the TQM. This means that in case (II) as well the TQM and IBM-1 agree.

In the framework of the IBM-1, the transfer of two nucleons is treated in a first approximation as the addition or removal of one boson. Accordingly, the operators for transfer of two bound nucleons ($p_{\tau}^{(L)}$ and $p_{-\tau}^{(L)}$) are assumed to be proportional to the basis operators of the IBM-1:

$$\left.\begin{aligned} &p^{(0)}_{-\tau}=A_0 s,\quad p^{(0)}_{+\tau}=A_0 s^{+}\ \text{(transfer with } \dot{L}=0);\\ &p^{(2)}_{-\tau}=A_2 d_\mu,\quad p^{(2)}_{+\tau}=A_2 d^{+}_{\mu}\ \text{(transfer with } L=2), \end{aligned}\right\} \tag{60}$$

where A_L are normalization constants. The superscript L denotes the angular-momentum transfer, and the subscript identifies protons ($\tau = p$) or neutrons ($\tau = n$).

The idea of constructing two-nucleon transfer operators in the framework of the TQM is very simple, namely, one attempts to express the s and d bosons in terms of the basis operators of the TQM, using the connection between the Schwinger and HP realizations of the ACQO. It is not difficult to express the quadrupole bosons of the TQM in terms of the s and d bosons of the IBM-1:

$$\left.\begin{aligned} b^{+}_{\mu} &= (s^{+}s+1)^{-1/2}\, s d^{+}_{\mu} \equiv E_s d^{+}_{\mu};\\ b_{\mu} &= d_\mu s^{+}(s^{+}s+1)^{-1/2} \equiv d_\mu E^{+}_{s}. \end{aligned}\right\} \tag{61}$$

One can show by direct verification that on the replacement of b_μ and b_μ^{+} in accordance with (61) the HP realization (4) goes over into the Schwinger realization. However, we need the inverse transformation. The problem of finding it is not entirely trivial. The difficulty is that in the TQM the degree of freedom associated with the conservation of N [see (11)] becomes unimportant and is frozen. Clearly, to construct the inverse transformation $\{d, d^{+}, s, s^{+}\} \to \{b, b^{+}\}$ it is necessary to introduce an additional a boson. In Ref. 46, this boson was introduced as follows:

$$\begin{aligned} a^{+} &= \Big(s^{+}s+\sum_{\mu} d^{+}_{\mu}d_{\mu}\Big)^{1/2} E^{+}_{s};\\ a &= E_s\Big(s^{+}s+\sum_{\mu} d^{+}_{\mu}d_{\mu}\Big)^{1/2}. \end{aligned} \tag{62}$$

The relations (62) are more rigorously justified in Ref. 25. We can now express $\{d, d^{+}, s, s^{+}\}$ in terms of $\{b, b^{+}, a, a^{+}\}$. The inverse transformation has the form[46]

$$\left.\begin{aligned} s^{+} &= E_{a}^{+}\left(a^{+}a-\sum_{\nu} b_{\nu}^{+}b_{\nu}+1\right)^{1/2} \\ s &= \left(a^{+}a-\sum_{\nu} b_{\nu}^{+}b_{\nu}+1\right)^{1/2} E_{a}; \\ d_{\mu}^{+} &= b_{\mu}^{+}E_{a}^{+}\, d_{\mu}=E_{a}b_{\mu}, \end{aligned}\right\} \qquad (63)$$

where $E_a^+ = a^+(a^+, a+1)^{1/2}, E_a = E_a^+)^+$.

It should be noted that in the derivation of the inverse transformation we again come up against the existence of the unphysical orthogonal subspace in the domain of definition of the HP realization (4). Strictly speaking, the operators $\{E_s^+, E_s\}$ and $\{E_a^+, E_a\}$ are, as we have shown, only partly isometric, and as a result projection operators arise. This complicates the treatment, but, as was shown in Ref. 46, their effect is exactly equal to zero if we remain in the physical subspace. If we substitute the expressions for b_μ, b_μ^+ from (61) in (40) and take into account the fact that $|\widetilde{0}\rangle = (N!)^{-1/2}(a^+)^N|0\rangle$, then we can readily verify that (40) goes over into (36). Conversely, if in (36) we replace s, s^+, d_μ, d_μ^+ by means of (63), then (36) goes over into (40), i.e., the direct and inverse transformations are such that the basis spaces of the TQM and IBM-1 are carried into each other. This circumstance is used essentially in the calculation of the amplitudes of the two-nucleon transfer reactions. We now turn to the amplitudes. As was noted in Sec. 2, the eigenvalue problem is solved in the framework of the TQM by diagonalizing $H_{\rm TQM}$ in the basis of the quadrupole bosons $|d^{n_b}[N]\varkappa IM\rangle$, where $n_b \leqslant N$. This is a basis of the type (40), but the total angular momentum is a good quantum number. As we have noted, the IBM-1 eigenvectors can be expanded with respect to a basis of the type (36), in which the d bosons are coupled to angular momentum I. We shall denote the $H_{\rm TQM}$ and $H_{\rm IBM\text{-}1}$ eigenvectors by $|E, I; N\rangle_{\rm TQM}$ and $|E, I; N\rangle_{\rm IBM\text{-}1}$, respectively.

For the purposes of the following treatment, we separate explicitly the dependence on the a boson in the wave functions of the TQM:

$$|E, I; N\rangle_{\rm TQM} = (N!)^{-1/2}(a^+)^N\,|E, I; N\rangle_{\rm TQM}, \qquad (64)$$

where $|E, I; N\rangle_{\rm TQM}$ is the standard space of the TQM, i.e., without the a boson. Having in mind the relations (60), (63), and (64), we can readily express the matrix elements of $P_{+\tau}^{(L)}$ in the IBM-1 basis in terms of the standard TQM matrix elements. We have

$$\begin{aligned} &{}_{\rm IBM\text{-}1}\langle E_f, I_f; N+1|P_{+\tau}^{0}|E_i, I_i; N\rangle_{\rm IBM\text{-}1} \\ &= {}_{\rm TQM}^{\rm ext}\langle E_f, I_f; N+1|\widetilde{P}_{+\tau}^{0}|E_i, I_i; N\rangle_{\rm TQM}^{\rm ext} \\ &= A_0\langle 0|[(N+1)!]^{-1/2}a^{N+1}(a^+a)^{-1/2}a^+(N!)^{-1/2}(a^+)^N|0\rangle \\ &\times {}_{\rm TQM}\langle E_f, I_f; N|\left(N-\sum_{\nu} b_\nu^+ b_\nu+1\right)^{1/2}|E_i, I_i; N\rangle_{\rm TQM} \\ &= A_{0\rm TQM}\langle E_f, I_f; N|\left(N-\sum_{\nu} b_\nu^+ b_\nu+1\right)^{1/2}|E_i, I_i; N\rangle_{\rm TQM}. \end{aligned} \qquad (65)$$

As in the case of (65), we obtain

$$\begin{aligned} &{}_{\rm IBM\text{-}1}\langle E_f, I_f; N+1\,|\,P_{+\tau}^{(2)}\,|\,E_i, I_i; N\rangle_{\rm IBM\text{-}1} \\ &= A_{2\rm TQM}\,\langle E_f, I_f; N+1\,|\,b_\mu^+\,|\,E_i, I_i; N\rangle_{\rm TQM}. \end{aligned} \qquad (66)$$

We see that the matrix elements of the two-nucleon transfer operator in the IBM-1 basis can be expressed in terms of the matrix elements on the right-hand side of (65) and (66), which are standard for the TQM: They couple TQM wave functions calculated for different N ($\Delta N = 1$). In the case of two-nucleon transfer with $L = 0$, the corresponding amplitude (65) contains the "weight operator" $(N - \Sigma_\nu b_\nu^+ b_\nu + 1)^{1/2}$, which reflects the effect of the s boson. The only operator that changes the number of nucleons by two is the a boson [see (61) and (62)]. In the IBM-1, s and s^+ are treated as nucleon pairs bound to zero angular momentum. However, by virtue of the factorization (64) the a boson was eliminated in a natural manner from the amplitudes (65) and (66).

As an example, we consider the amplitude of $0_1^+ \to 0_1^+$ two-nucleon transfer in the SU(5) and SU(3) limits of the TQM.

In the framework of the TQM, the ground-state wave function in the SU(5) limit is a state without quadrupole bosons and, of course, with zero spin:

$$|E=0,\ I=0;\ N\rangle_{\rm TQM} = |n_b=0, I=0\rangle.$$

Then the amplitude of $0_1^+ \to 0_1^+$ two-nucleon transfer in the SU(5) limit is

$$\begin{aligned} &{}_{0\rm TQM}\langle E=0,\ I=0;\ N|\left(N-\sum_{\nu} b_\nu^+ b_\nu+1\right)^{1/2}|E=0, \\ &I=0;\ N\rangle_{\rm TQM} = A_0(N+1)^{1/2}. \end{aligned} \qquad (67)$$

In the SU(3) limit, the TQM ground state can be represented approximately in the form[74]

$$\begin{aligned} &|E=0,\ I=0;\ N\rangle_{\rm TQM} \\ &= \sum_{n_b}[\,N!/\,!\,(N-n_b)\,!]^{1/2}\left(\frac{2}{3}\right)^{n_b/2}3^{-(N-n_b)/2}|n_b,\ 0\rangle, \end{aligned}$$

where $|n_b, 0\rangle$ is a basis state of the type (40) with $I=0$ for which the seniority quantum number is minimal for the given n_b. In the considered limit, the amplitude of $0_1^+ \to 0_1^+$ two-nucleon transfer is

$$\begin{aligned} &A_{0\rm TQM}\langle E=0,\ I=0;\ N|\left(N-\sum_{\nu} b_\nu^+ b_\nu+1\right)^{1/2}|E=0, \\ &I=0;\ N\rangle_{\rm TQM} \\ &= A_0[(N+1)/3]^{1/2}\sum_{n_b}[N!/n_b!(N-n_b)!]\left(\frac{2}{3}\right)^{n_b}3^{-N+n_b} \\ &= A_0[(N+1)/3]^{1/2}. \end{aligned} \qquad (68)$$

It follows from (67) and (68) that the ratio of the squares of the amplitudes of two-nucleon transfers with $L=0$ in the SU(5) and SU(3) limits of the TQM is 3. Exactly the same result is obtained in the IBM-1.[72,73]

Thus, for the description of two-nucleon transfer reactions the TQM and IBM-1 again give identical numerical results and analytic expressions in the case of exact SU(3) and SU(5) symmetries.

4. LATEST DEVELOPMENTS AND APPLICATION OF THE TRUNCATED QUADRUPOLE-PHONON MODEL

The collective Hamiltonian (3) and quadrupole operator (6), based on dynamical SU(6) symmetry, have been widely used in their two equivalent forms [(5), (8) and (7), (9), respectively] to calculate specific properties of many nuclei belonging to different regions of the periodic table.[28,29,31] Because of the basically successful description of the experimental data, the IBM has recommended itself as a very felicitous form of phenomenology, the so-called new phenomenology. At the same time, the calculations have revealed a number of discrepancies between the theoretical predictions of the IBM and the experiments. These discrepancies are due, in the first place, to the initial limitations of the model. The creators of the IBM-1, and also other authors, have proposed a number of generalizations and modifications of the IBM making it possible to go beyond the limitations inherent in the IBM-1 in its original formulation. The need to introduce not only the quadrupole but also other collective degrees of freedom was recognized from the moment of creation of the IBM-1. Accordingly, additional bosons were introduced: dipole p, octupole f,[5)] and hexadecupole g. Additional s' and d' weakly collectivized bosons were also considered in order to take into account the effects of interaction of the collective and noncollective degrees of freedom. A review of studies on the generalization of the IBM-1 by the inclusion of new bosons (collective and weakly collectivized) is given in Ref. 75 (see also Ref. 49). The authors of the IBM-1 and their collaborators have developed a new variant of the model—the IBM-2 which contains proton and neutron bosons.[12,13] An interesting example of a modification of the IBM-2 associated with the abandonment of conservation of the total number of bosons[49] is the allowance for mixing of different boson configurations with a view to explaining coexisting collective structures in the framework of the IBM-2.[76]

It should be noted that not only the original versions of the TQM and IBM-1 but also the modifications of the IBM-1 listed above can be used only to describe even–even nuclei. An important step in the development of the IBM is its generalization to the case of odd nuclei, which has been implemented in the so-called interacting boson–fermion model (IBFM).[77] In recent years, a detailed experimental study has been made on a wide front of the structure of odd nuclei, and this has resulted in the accumulation of extensive material on their low-lying collective states. The development of the IBFM was due to the need to give a theoretical interpretation of the experimental data in the framework of a comparatively simple model, such as the IBFM is. The model was used to describe odd nuclei in Refs. 78–81, in which the IBFM Hamiltonian was diagonalized numerically. Somewhat later, different variants of the IBFM were developed on the basis of dynamical symmetries and supersymmetries. Reviews of the theoretical and experimental aspects of this direction are given in Refs. 82 and 83, respectively.

The modifications and improvements of the IBM discussed above relate to the IBM-1, i.e., to the IBM in the s, d representation. The results of these investigations have found a fairly complete reflection in Refs. 31, 75, 82, and 83.

At the same time and independently, a group of theoreticians from the University of Zagreb has made a series of studies on the further development and extension of the applicability of the TQM. This series of papers includes:

1) A formulation of a modification of the TQM suitable for describing odd nuclei, called the particle truncated quadrupole-phonon model (PTQM).[74] It has been shown[74] that the PTQM is equivalent to the IBFM.[74]

2) Application of the PTQM to the description of specific properties of odd nuclei.[84,85]

3) The development of a systematic model based on SU(3) boson–fermion dynamical symmetry, by means of which the rotational properties of even–even and odd nuclei can be treated on a unified basis.[86] In the framework of the SU(3) boson–fermion model [SU(3)-BFM],[86,87] limiting cases that are analogs of the weak and strong coupling of the IBM and intermediate coupling in the rotational-alignment model (aligned-coupling scheme)[88] have been found. An analytic solution of the SU(3)-BFM with supersymmetric properties and with no analog in the Bohr–Mottelson model has also been found.

4) The construction of a modification of the TQM suited for phenomenological description of odd–odd nuclei.[89–91]

5) The study of the part played by truncation of the phonon basis.[92]

This list does not include studies on the construction by the group at Zagreb of the physical operators of the IBM-1 and TQM for single-nucleon transfer reactions, since these studies go beyond the scope of the present paper. These studies on the development and application of the TQM have not yet been reflected in the review literature and will be discussed in the present section.

Calculation of the structure of low-lying states of $^{75}_{34}\mathrm{Se}_{41}$ and proton–hole states of $^{61}\mathrm{Co}$ on the basis of the PTQM

The explicit form of the PTQM Hamiltonian is given in Ref. 74. The nucleus studied in the $^{74}\mathrm{Se}(n, \gamma)$ reaction, the nucleus $^{74}_{34}\mathrm{Se}_{40}$, is treated[84] in the PTQM as an even–even core of $^{75}_{34}\mathrm{Se}_{41}$, to which a neutron quasiparticle is bound. Accordingly, in the first stage of the calculations in the framework of the TQM the spectrum of ^{74}Se was calculated with the Hamiltonian (5). The parameters $\{h\}$ in (5) were chosen to reproduce the experimental ^{74}Se spectrum as well as possible. It can be seen from Fig. 1 that the ^{74}Se spectrum can be described very satisfactorily. The PTQM Hamiltonian was then diagonalized in the basis $|jn\varkappa I; \mathscr{T}\rangle$, where j is the angular momentum of the neutron quasiparticle, $|n\varkappa I$ are the quantum numbers of the state of the quadrupole n phonons, and $\mathscr{T}$ is the total angular momentum. The results of the calculation of the spectrum of excited states of positive and negative parity of ^{75}Se are given in Fig. 2.

The additional (with respect to the even–even core) parameters used in the description of the odd system are given in Ref. 84. Despite a certain discrepancy in the energies of the states (reaching 300–450 keV for the positive-parity states) and in the succession of the levels, overall sat-

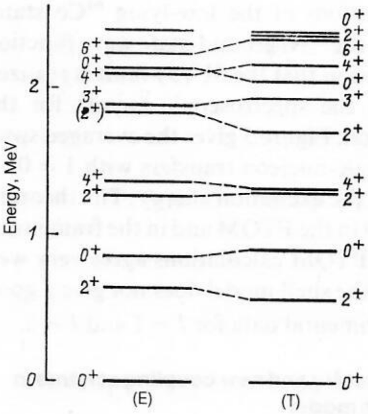


FIG. 1. Spectrum of excited states of ^{74}Se. The experimental data (E) are given on the left, and the results of the calculation based on the TQM on the right (T).

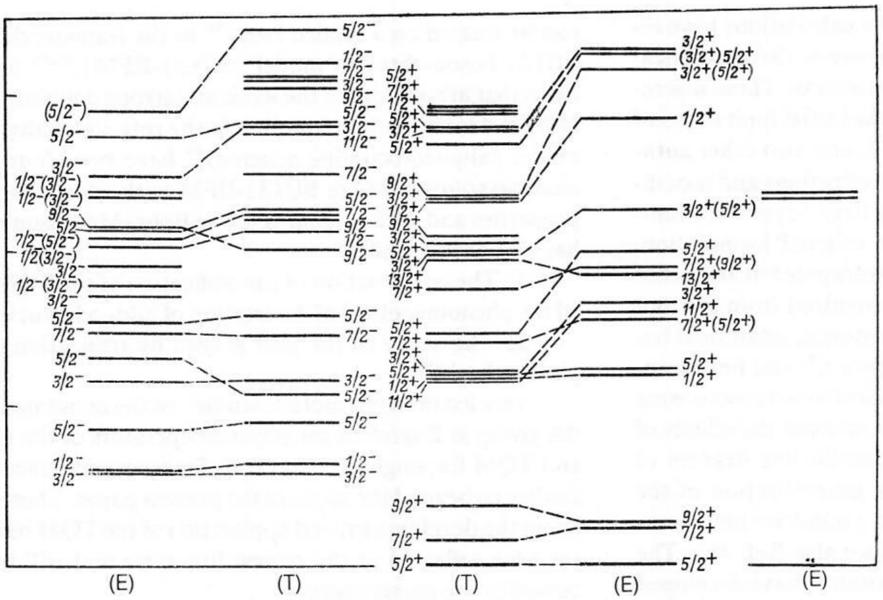


FIG. 2. Theoretical (PTQM) and experimental values of the energies of ^{75}Se excited states.

isfactory agreement with experiment was achieved.

The wave functions obtained by diagonalizing the Hamiltonian were used to calculate the probabilities of E2, Ml, and El transitions between the low-lying ^{75}Se states.

It can be seen from Table I that the agreement between the experimental data and the theoretical results is entirely satisfactory. The free parameters were fixed in such a way as to reproduce the experimental value of I_γ for the transition $(5/2^-)_1 \rightarrow (3/2^-)_1$. The probabilities of the E1 transitions $(5/2^-)_1 \rightarrow (5/2^+)_1$ and $(5/2^-)_1 \rightarrow (7/2^+)_1$ were calculated without free parameters and are in good agreement with experiment, although the $B(\mathrm{E}1)_{\mathrm{exp}}$ values for these transitions are very small. The calculations were made both with the renormalization (2) and without the renormalization of the effective charge (1).

In the framework of the PTQM, the spectroscopic factors for the ^{62}Ni$(d, {}^3\mathrm{He})$ ^{61}Co reaction were calculated. The ^{61}Co energy spectrum was calculated under the assumption that a proton quasiparticle is bound to the even–even ^{62}Ni core, which was described under the assumption that the SU(5) limit is realized. In Figs. 3 and 4, the results of the calculation are compared with the experimental level scheme. Figure 3 also gives the results of shell-model calculations. The parameters used in the calculations are given in Ref. 85. The wave functions of the low-lying ^{61}Co states found in the PTQM and the ^{62}Ni ground-state wave function found under the assumption that the SU(5) limit is realized were used to calculate the spectroscopic factors for the ^{62}Ni$(d, {}^3\mathrm{He})^{61}$Co reaction. Figure 5 gives the averaged spectroscopic factors for single-nucleon transfers with $1 = 0$, 1, 2, and 3 as a function of the excitation energy. The theoretical results were obtained in the PTQM and in the framework of the shell model. The PTQM calculations agree very well with experiment, while the shell model does not give a good description of the experimental data for $l = 1$ and $l = 3$.

Approximate supersymmetry and new coupling scheme in the SU(3) boson–fermion model

The SU(3)-BFM, which was proposed and investigated in Refs. 86 and 87, has been discussed in most detail in Ref. 93. As is well known,[7,10,29] the description of the rotational properties of even–even nuclei in the framework of the IBM is associated with the so-called SU(3) limit of SU(6) dynamical symmetry [chain (II)]. The SU(3)-BFM was developed to describe the rotational properties of odd nuclei and is in essence the SU(3) limit of the PTQM. This simple model contains the well-known weak, intermediate, and strong coupling schemes of the Bohr–Mottelson rotational model, and also a new analytic solution with supersymmetric properties.

The Hamiltonian describing a particle with angular momentum j coupled to an even–even core whose properties are determined by the SU(3) limit of the TQM can be represented in the form

$$h_{\mathrm{PTQM}}(SU(3)) = h_{\mathrm{TQM}}(SU(3)) + \Gamma (G_2^B G_2^F)_{(00)} + \text{exchange term}, \quad (69)$$

where

$$h_{\mathrm{TQM}}(SU(3)) = -\alpha (G_2^B G_2^B)_{(00)} + \delta' (I^B I^B)_{00} = -\frac{\alpha}{2} C_2^B + \delta (I^B I^B)_{(00)}, \quad (70)$$

and, since we consider the SU(3) limit of the TQM, the boson HP realization for the operators G_2^{B} and I_1^{B} can be used:

$$G_{2\mu}^B = b_\mu^+ \Big(N - \sum_\nu b_\nu^+ b_\nu\Big)^{1/2} + \Big(N - \sum_\nu b_\nu^+ b_\nu\Big)^{1/2} (-1)^\mu b_{-\mu} \pm \frac{\sqrt{7}}{2} (b_2^+ \tilde{b}_2)_{(2\mu)}; \quad (71)$$

$$I_{1m}^B = \sqrt{10}\, (b^+ \tilde{b})_{(1m)}. \quad (72)$$

It is known[6,7] that the five components of the mass quadrupole operator $\{G_{2\mu}\}$ (Ref. 94) together with the three components of the angular-momentum operator $\{I_{1m}\}$ close the SU(3) algebra. The positive and negative signs in (71) correspond to the oblate and prolate shapes of the even–even core, respectively.

The fermion operator corresponding to the particle–core interaction has the form

TABLE I. Comparison of theoretical and experimental values of the probabilities of E2, M1, and E1 transitions between low-lying ^{75}Se states.

Transition	$B(E2)$		$B(M1)$		$B(E1)$			I_γ		
	Experiment	Theory	Experiment	Theory	Experiment	Theory		Experiment	Theory	
						(1)	(2)		(1)	(2)
$\left(\frac{7^+}{2}\right)_1\left(\frac{5^+}{2}\right)_1$	0.40±0.08	0.400	0.032±0.005	0.124	—	—	—	—	—	—
$\left(\frac{9^+}{2}\right)_1\left(\frac{5^+}{2}\right)_1$	0.075	0.076	—	—	—	—	—	—	1.0	—
$\left(\frac{9^+}{2}\right)_1\left(\frac{7^+}{2}\right)_1$	—	0.340	0.084	0.085	—	—	—	—	0.3	—
$\left(\frac{3^-}{2}\right)_1\left(\frac{5^+}{2}\right)_1$	—	—	—	—	$1.4\cdot10^{-7}$	$1.4\cdot10^{-7}$	$1.4\cdot10^{-7}$	—	—	—
$\left(\frac{1^-}{2}\right)_1\left(\frac{3^-}{2}\right)_1$	—	0,0001	0.18	0.238	—	—	—	—	—	—
$\left(\frac{5^-}{2}\right)_1\left(\frac{5^+}{2}\right)_1$	—	—	—	—	$1\cdot10^{-7}$	$4.4\cdot10^{-8}$	$3.6\cdot10^{-8}$	2.9	1.2	(0.9)
$\left(\frac{5^-}{2}\right)_1\left(\frac{7^+}{2}\right)_1$	—	—	—	—	$5\cdot10^{-8}$	$5.1\cdot10^{-8}$	$4.8\cdot10^{-8}$	0.5	0.5	(0.5)
$\left(\frac{5^-}{2}\right)_1\left(\frac{3^-}{2}\right)_1$	—	0.371	—	0.038	—	—	—	4.5	4.5	—
$\left(\frac{3^-}{2}\right)_2\left(\frac{5^+}{2}\right)_1$	—	—	—	—	$5\cdot10^{-8}$	$6.5\cdot10^{-9}$	$7.8\cdot10^{-9}$	0.4	0.06	(0.07)
$\left(\frac{3^-}{2}\right)_2\left(\frac{3^-}{2}\right)_1$	—	0.028	—	0.034	—	—	—	0.7	4.7	—
$\left(\frac{3^-}{2}\right)_2\left(\frac{1^-}{2}\right)_1$	—	0.168	—	0.039	—	—	—	6.0	6.0	—
$\left(\frac{1^+}{2}\right)_1\left(\frac{5^+}{2}\right)_1$	—	0.014	—	—	—	—	—	6,0	6,0	—
$\left(\frac{1^+}{2}\right)_1\left(\frac{3^-}{2}\right)_1$	—	—	—	—	$9\cdot10^{-9}$	$1.7\cdot10^{-9}$	$1.6\cdot10^{-9}$	0,02	0.004	(0.004)
$\left(\frac{1^+}{2}\right)_1\left(\frac{1^-}{2}\right)_1$	—	—	—	—	$3\cdot10^{-7}$	$0.8\cdot10^{-8}$	$0.7\cdot10^{-8}$	0.6	0.02	(0.02)
$\left(\frac{5^+}{2}\right)_2\left(\frac{5^+}{2}\right)_1$	—	0.006	—	0.007	—	—	—	0.6	1.1	—
$\left(\frac{5^+}{2}\right)_2\left(\frac{7^+}{2}\right)_1$	—	0.012	—	0.039	—	—	—	2.8	2.8	—
$\left(\frac{5^+}{2}\right)_2\left(\frac{9^+}{2}\right)_1$	—	0.001	—	—	—	—	—	0.2	0.01	—
$\left(\frac{5^+}{2}\right)_2\left(\frac{3^-}{2}\right)_1$	—	—	—	—	$5\cdot10^{-7}$	$5.6\cdot10^{-9}$	$1.5\cdot10^{-9}$	0.09	0.001	(0.0003)
$\left(\frac{5^+}{2}\right)_2\left(\frac{5^-}{2}\right)_1$	—	—	—	—	—	$2.6\cdot10^{-10}$	$2.9\cdot10^{-11}$	—	0,000	(0.000)
$\left(\frac{5^-}{2}\right)_2\left(\frac{5^+}{2}\right)_1$	—	—	—	—	$0.8\cdot10^{-9}$	$1.0\cdot10^{-8}$	$1.3\cdot10^{-8}$	0.03	0.40	(0.51)
$\left(\frac{5^-}{2}\right)_2\left(\frac{7^+}{2}\right)_1$	—	—	—	—	$9\cdot10^{-9}$	$2.8\cdot10^{-9}$	$2.5\cdot10^{-9}$	0.2	0.06	(0.06)
$\left(\frac{5^-}{2}\right)_2\left(\frac{3^-}{2}\right)_1$	—	0.011	—	0.002	—	—	—	2.5	2.5	—
$\left(\frac{5^-}{2}\right)_2\left(\frac{1^-}{2}\right)_1$	—	0.0001	—	—	—	—	—	0.01	0.01	—
$\left(\frac{5^-}{2}\right)_2\left(\frac{5^-}{2}\right)_1$	—	0.020	—	0.001	—	—	—	0.5	0.35	—
$\left(\frac{7^-}{2}\right)_1\left(\frac{5^+}{2}\right)_1$	—	—	—	—	$6\cdot10^{-8}$	$1.1\cdot10^{-8}$	$1.0\cdot10^{-8}$	0.06	0.01	(0.01)

Continuation of Table I

Transition	$B\ (E2)$		$B\ (M1)$		$B(E1)$				I_γ	
	Experiment	Theory	Experiment	Theory	Experiment	Theory		Experiment	Theory	
						(1)	(2)		(1)	(2)
$\left(\frac{7^-}{2}\right)_1 \left(\frac{7^+}{2}\right)_1$	–	–	–	–	$2\cdot10^{-7}$	$5.4\cdot10^{-8}$	$4.9\cdot10^{-8}$	0.10	0.01	(0.01)
$\left(\frac{7^-}{2}\right)_1 \left(\frac{9^+}{2}\right)_1$	–	–	–	–	–	$3.6\cdot10^{-8}$	$3.1\cdot10^{-8}$	–	0.02	(0.02)
$\left(\frac{7^-}{2}\right)_1 \left(\frac{3^-}{2}\right)_1$	–	0.141	–	–	–	–	–	0.21	0.57	–
$\left(\frac{7^-}{2}\right)_1 \left(\frac{5^-}{2}\right)_1$	–	0,250	–	0.041	–	–	–	0.54	0.54	–
$\left(\frac{7^-}{2}\right)_1 \left(\frac{3^-}{2}\right)_2$	–	0.031	–	–	–	–	–	–	0.0007	–
$\left(\frac{7^-}{2}\right)_1 \left(\frac{5^+}{2}\right)_2$	–	–	–	–	–	$3.3\cdot10^{-9}$	$1.6-10^{-12}$	–	0.000	(0.000)
$\left(\frac{7^-}{2}\right)_1 \left(\frac{5^-}{2}\right)_2$	–	0.001	–	0.008	–	–	–	0.01	0.001	–
$\left(\frac{5^-}{2}\right)_3 \left(\frac{5^+}{2}\right)_1$	–	–	–	–	$5\cdot10^{-8}$	$0.9\cdot10^{-10}$	$6.6\cdot10^{-10}$	0.05	0.0001	(0.0007)
$\left(\frac{5^-}{2}\right)_3 \left(\frac{7^+}{2}\right)_1$	–	–	–	–	–	$1.2\cdot10^{-11}$	$2.3\cdot10^{-11}$	–	0.000	(0.000)
$\left(\frac{5^-}{2}\right)_3 \left(\frac{3^-}{2}\right)_1$	–	0.0005	–	0.006	–	–	–	0.76	0.18	–
$\left(\frac{5^-}{2}\right)_3 \left(\frac{1^-}{2}\right)_1$	–	0.179	–	–	–	–	–	0.58	0.84	–
$\left(\frac{5^-}{2}\right)_3 \left(\frac{5^-}{2}\right)_1$	–	0.0002	–	0.069	–	–	–	0.75	0,75	–
$\left(\frac{5^-}{2}\right)_3 \left(\frac{3^-}{2}\right)_2$	–	0.048	–	0.062	–	–	–	0.21	0.11	–
$\left(\frac{5^-}{2}\right)_3 \left(\frac{5^+}{2}\right)_2$	–	–	–	–	–	$1.5\cdot10^{-9}$	$4.5\cdot10^{-12}$	–	0.000	(0.000)
$\left(\frac{5^-}{2}\right)_3 \left(\frac{5^-}{2}\right)_2$	–	0.077	–	0.002	–	–	–	0.03	0.001	–
$\left(\frac{3^+}{2}\right)_1 \left(\frac{5^+}{2}\right)_1$	–	0.0001	–	0.058	–	–	–	3.7	3.7	–
$\left(\frac{3^+}{2}\right)_1 \left(\frac{7^+}{2}\right)_1$	–	0.008	–	–	–	–	–	0.1	0.12	–
$\left(\frac{3^+}{2}\right)_1 \left(\frac{3^-}{2}\right)_1$	–	–	–	–	–	$4.4\cdot10^{-11}$	$5.5\cdot10^{-11}$	–	0.000	(0.000)
$\left(\frac{3^+}{2}\right)_1 \left(\frac{1^-}{2}\right)_1$	–	–	–	–	–	$7.5\cdot10^{-10}$	$1.2\cdot10^{-9}$	–	0.0001	(0.0002)
$\left(\frac{3^+}{2}\right)_1 \left(\frac{5^-}{2}\right)_1$	–	–	–	–	–	$3.9\cdot10^{-10}$	$2.9\cdot10^{-10}$	–	0.000	(0.000)
$\left(\frac{3^+}{2}\right)_1 \left(\frac{3^-}{2}\right)_2$	–	–	–	–	–	$5.0\cdot10^{-8}$	$4.7\cdot10^{-8}$	–	0.0008	(0.0007)
$\left(\frac{3^+}{2}\right)_1 \left(\frac{1^+}{2}\right)_1$	–	0.071	–	0.284	–	–	–	0.4	0.37	–
$\left(\frac{3^+}{2}\right)_1 \left(\frac{5^+}{2}\right)_2$	–	0.152	–	0.124	–	–	–	0.8	0.13	–
$\left(\frac{3^+}{2}\right)_1 \left(\frac{5^-}{2}\right)_2$	–	–	–	–	–	$8.2\cdot10^{-11}$	$8.3\cdot10^{-11}$	–	0.000	(0.000)

$$G^{F}_{2m_j} = (c_j^+ \tilde{c}_j)_{(2m_j)}, \qquad (73)$$

where c_2 is the second-order Casimir operator of the SU(3) algebra. The exchange term was not taken into account in Refs. 86, 87, and 93, so that the results should be compared with the results of the particle–rotator model.[1] The authors of Refs. 86, 87, and 93 concentrated their attention on the states of the odd system associated with the rotational band of the core ground state. These states can be obtained by projecting $|jk\rangle|c\rangle$ onto the state with angular momentum $\mathscr{T}$ and its projections M and k (Refs. 84 and 85):

$$|k\mathscr{T}M\rangle_{\rm ss} = \sum_{I=0}^{2N} B_I^{-1} \langle jkI0|\mathscr{T}k\rangle\, |(jI)\,\mathscr{T}M\rangle; \qquad (74)$$

$$|c\rangle = \exp\left\{\beta b_{20}^{+} \left(N - \sum_{\nu} b_{2\nu}^{+} b_{2\nu}\right)^{1/2}\right\} |0\rangle$$

is the coherent state of the boson SU(3) group with $\beta = \sqrt{2}(-\sqrt{2})$ for the prolate (oblate) shape of the core, and $|jk\rangle$ is an adiabatic state of the odd system (see Ref. 96). The states

$$|(jI)\,\mathscr{T}M\rangle = \sum_{m_1 m_2} \langle jm_1 I m_2|\mathscr{T}M\rangle\, |jm_1\rangle\, |Im_2\rangle, \qquad (75)$$

which form a convenient basis for diagonalizing the Hamil-

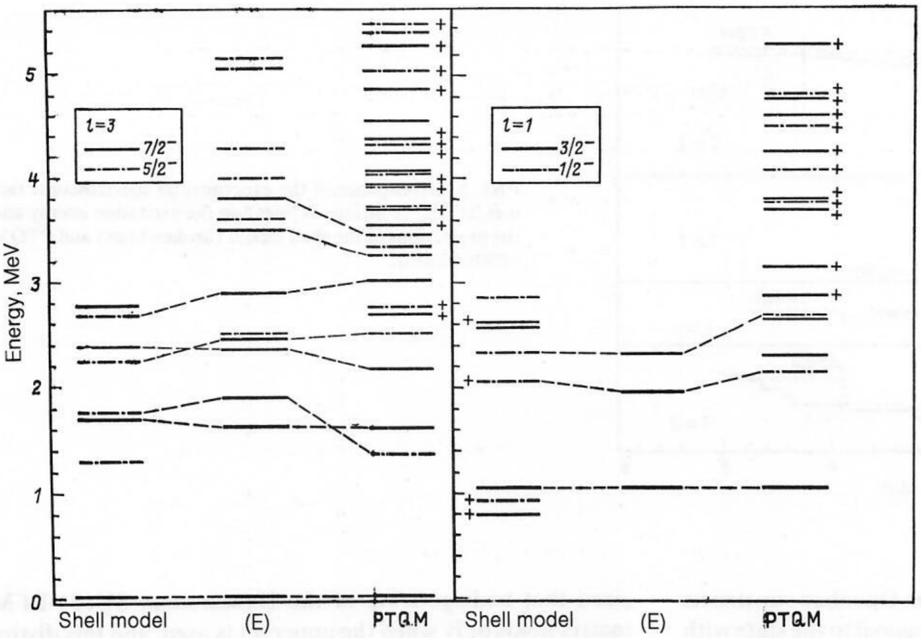


FIG. 3. Energy spectra of ^{61}Co: calculated in accordance with the shell model, the experimental spectrum, and the PTQM calculation. The theoretical levels labeled (+) have spectroscopic factors $(G) \leqslant 0.005$. The identification of experimental levels with theoretical levels is tentative.

tonian (69) for small coupling constants, are analogs of the basis states of the collective rotational model in the case of weak coupling, but with the difference that the angular momentum I is bounded by the value $I_{\max} = 2N$ [see (74)]. It will be seen from what follows that the basis given by (74) possesses the properties characteristic of the cases when dynamical supersymmetry is realized (hence the index ss).

The states $|I\rangle$ can be expanded with respect to a basis constructed from the quadrupole bosons as follows:

$$|I\rangle = B_I \sum_{n\varkappa} A_{n\varkappa I} |n\varkappa I\rangle.$$

Explicit expressions for B_I and $A_{n\varkappa I}$ are given in Ref. 84 [see Eqs. (50)–(52)].

If we limit the treatment to the states (74), the operator $C_2^{\rm B}$ gives only a constant energy shift, and therefore the corresponding part of the Hamiltonian (69) (with the exchange term ignored) can be written in the form

$$\delta h_{\rm BFM}(SU(3)) = \delta (I^B I^B)_{(00)} + \Gamma (G_2^B G_2^F)_{(00)}, \qquad (76)$$

and the wave functions will depend only on the ratio Γ/δ.

The states (75) are eigenstates of the Hamiltonian $h_{\rm BFM}(\mathrm{SU}(3))$ for $\Gamma/\delta = 0$. As will be shown below, the states $|k = j\mathscr{T} M\rangle_{\rm ss}$ become eigenstates of $h_{\rm BFM}(\mathrm{SU}(3))$ for a definite value of Γ/δ, which we shall denote by $(\Gamma/\delta)_{\rm ss}$

The states $|k\mathscr{T} M\rangle_{\rm ss}$ with different k are nonorthogonal, and they are therefore orthogonalized by means of the Gram–Schmidt procedure.[97] Two orthogonal sets of states $\{|k\mathscr{T} M\rangle_{\rm ss}\}$ are of physical interest.

a) The upper orthogonal set. For its construction for

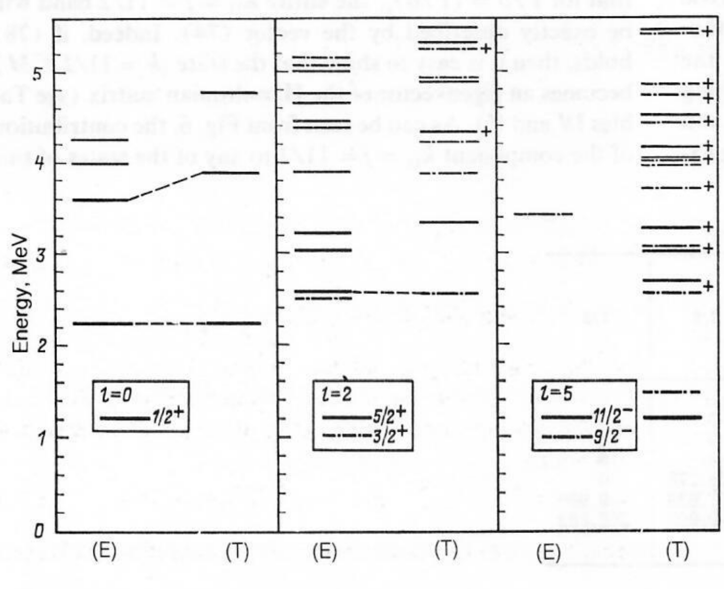


FIG. 4. Energy spectra of ^{61}Co states with quantum numbers $1/2^+$, $3/2^+$, $5/2^+$, $11/2^-$, and $9/2^-$: calculated in accordance with the PTQM and the experimental spectra. The theoretical levels labeled with the + have spectroscopic factor $(G) \leqslant 0.005$. The identification of the experimental levels with the theoretical levels is tentative.

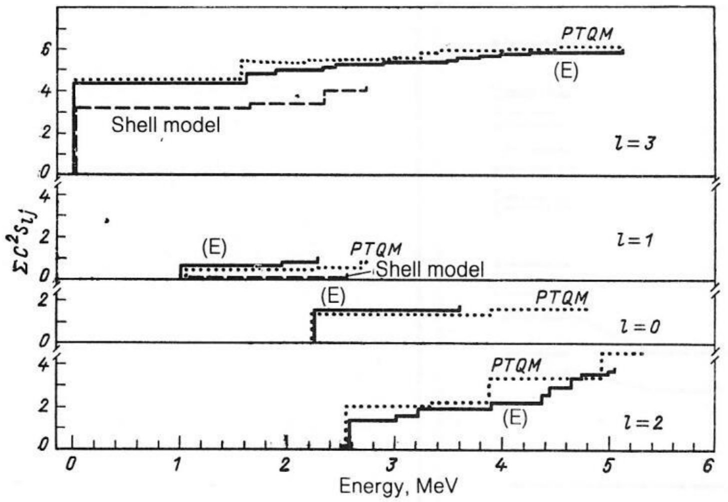


FIG. 5. Dependence of the experimental spectroscopic factors $\Sigma C^2 S_{ij}$ (continuous lines) on the excitation energy and the predictions of the shell model (broken lines) and PTQM (dotted lines).

fixed $\mathscr{T}$ one begins with the $k=j$ state. One then constructs the state with $k=j-1$, which is orthogonal to the state with $k=j$, and so forth, the process terminating with orthogonalization of the state with $k=\frac{1}{2}$. The quantum numbers k obtained as a result are denoted by $k_{\rm U}$, and the corresponding states by $|k_{\rm U}\mathscr{T}M\rangle$. By construction, the state $|k_{\rm U}=j\mathscr{T}M\rangle$ is identical to $|k=j\mathscr{T}M\rangle_{\rm ss}$.

b) The lower orthogonal set. For its construction for fixed $\mathscr{T}$ one begins (states with $k=\frac{1}{2}$) and completes the process of orthogonalization by the state with $k=j$. The corresponding quantum numbers k are denoted by $k_{\rm L}$, and the basis states by $|k_{\rm L}\mathscr{T}M\rangle$. By construction, the state $|k_{\rm L}=\frac{1}{2}\mathscr{T}M\rangle$ is identical to the state $|k=\frac{1}{2}\mathscr{T}M\rangle_{\rm ss}$. The matrix elements of $h_{\rm BFM}\,(\mathrm{SU}(3))$ in the basis $|k\mathscr{T}M\rangle_{\rm ss}$ can be calculated explicitly by using (74), (75), the Wigner–Eckart theorem, and the Wigner coefficients for the SU(3) group[98] and by noting that the generators $\{I_1^{\rm B}, \sqrt{8/3}\,G_{2\mu}^{\rm B}\}$ transform in accordance with the octet representation (1, 1).[99] Then, knowing the coefficients, which are obtained by Gram–Schmidt orthogonalization, we can also calculate the matrix elements

$$\langle k_b\mathscr{T}M|\, h_{\rm BFM}\,(SU\,(3))|k'_{\rm U}\mathscr{T}M\rangle. \tag{77}$$

It is difficult to obtain the matrix elements (77) in analytic form because of the complicated form of B_I and the orthogonalization coefficients. The calculations showed that the matrix (77) is always tridiagonal, i.e., it has nonvanishing elements only along the principal diagonal ($k_{\rm U}=k'_{\rm U}$) and for $|k_{\rm U}-k'_{\rm U}=1$ (Tables II and III). It should be emphasized that tridiagonality of the Hamiltonian SU(3)-BFM matrix holds only when the upper set is used, and this distinguishes this set from the others.

It is not surprising that the matrix $I^{\rm B}I^{\rm B}$ was found to be tridiagonal, since $\mathbf{II}=(\vec{\mathscr{T}}-\mathbf{j})(\vec{\mathscr{T}}-\mathbf{j})$ contains the Coriolis interaction $\vec{\mathscr{T}}\cdot\mathbf{j}$, for which the selection rule $\Delta k=1$ holds.[1] What is surprising is the fact that the $\Delta k=1$ rule also holds for the $G_2^{\rm B}G_2^{\rm F}$ interaction matrix. One would expect the explicit presence of the fermion operator $G_2^{\rm F}$ to lead to violation of the selection rules that hold in the collective subspace.

The tridiagonality property of the matrix (77) can be used to construct a special solution having properties characteristic of Hamiltonians based on dynamical supersymmetries.

We fix Γ/δ to ensure that

$$k_{\rm U}=j\mathscr{T}M|(I^BI^B)_{(00)}+(\Gamma/\delta)_{\rm ss}\,(G_2^BG_2^F)_{(00)}|k_{\rm U}=j-1\mathscr{T}M\rangle=0. \tag{78}$$

For the special case $j=11/2$, we obtain $(\Gamma/\delta)_{\rm ss}=46.3$. In realistic cases, $\delta\approx 0.015$ MeV. It follows that $\Gamma_{\rm ss}\approx 1$ MeV, this corresponding to intermediate coupling. It was also found that the $(\Gamma/\delta)_{\rm ss}$ obtained from (78) depends neither on the maximal number N of bosons nor on $\mathscr{T}$. This means that for $\Gamma/\delta=(\Gamma/\delta)_{\rm ss}$ the entire $k_{\rm U}=f=11/2$ band will be exactly described by the vector (74). Indeed, if (78) holds, then it is easy to show that the state $|k=11/2\mathscr{T}M\rangle$ becomes an eigenvector of the Hamiltonian matrix (see Tables IV and V). As can be seen from Fig. 6, the contribution of the component $k_{\rm U}=j=11/2$ to any of the states of this

TABLE II. The matrix elements $\langle k_{\rm U}\mathscr{T}=17/2|I^BI^B|k'_{\rm U}\mathscr{T}=17/2\rangle$.

$k'_{\rm U}$ \ $k_{\rm U}$	11/2	9/2	7/2	5/2	3/2	1/2
11/2	40.350	−17.821	0	0	0	0
9/2	−17.821	56.265	−26.719	0	0	0
7/2	0	−26.719	70.913	−32.760	0	0
5/2	0	0	−32.760	92.203	−28.278	0
3/2	0	0	0	−28.278	141.906	−9.980
1/2	0	0	0	0	−9.980	208.362

TABLE III. The matrix elements $\langle k_{\mathrm{U}}\mathscr{T} = 17/2|G_2^{\mathrm{B}}\cdot G_2^{\mathrm{F}})_0|k'_{\mathrm{U}}\mathscr{T} = 17/2\rangle$.

k_{U} k'_{U}	11/2	9/2	7/2	5/2	3/2	1/2
11/2	−2.121	0.385	0	0	0	0
9/2	0.385	−0.844	0.462	0	0	0
7/2	0	0.462	0.076	0.425	0	0
5/2	0	0	0.425	0.618	0.245	0
3/2	0	0	0	0.245	0.872	0.043
1/2	0	0	0	0	0.043	1.399

band is exactly 100%. It should be noted that the $k_U = j = 11/2$ band is an ideal rotational band. For other bands, deviations from the $\mathscr{T}(\mathscr{T}+1)$ rule are observed, these being larger, the smaller k_{U}. One can also see signature effects, as in the Bohr–Mottelson model.[1] Thus, the energies of the states of the band with $k_{\mathrm{U}} = 11/2$ are given by

$$E_{\mathscr{T}}(k_{\mathrm{U}} = j = 11/2) = \delta\mathscr{T}(\mathscr{T}+1), \qquad (79)$$

where δ is the same parameter as in the expression $E_I - \delta I(I+1)$ for the energy of the states of the ground rotational band of the even–even core. It is for this reason that the basis (74) was said to be supersymmetric (the even–even and odd systems are described in a unified manner, and the moments of inertia are equal).

The properties of the Hamiltonian (76) for $\Gamma = -\Gamma_{\mathrm{ss}}$ were also investigated. (This corresponds to a particle bound to a core having a prolate shape, or a hole bound to an oblate core.) In this case the upper set was found to be inconvenient, and the lower set was used. It was found[93] that in this basis even the matrix $I^{\mathrm{B}}I^{\mathrm{B}}$ is not tridiagonal. The Hamiltonian (76) mixes the bands with different k_{L}. The k_{L} contribution is given as a percentage next to each state in Fig. 7. It can be seen from the figure that the band with $k_{\mathrm{L}} = 1/2$ is the lowest band and exhibits a strong signature effect. The band with $k_{\mathrm{L}} = 11/2$ is situated above all of the others in energy and has an unusual parabolic shape [in the $\mathscr{T}(\mathscr{T}+1)$ scale]. In this band, the state with $\mathscr{T} = 29/2$ has the greatest energy. The subsequent lowering of the states is evidently due to the truncation with respect to the phonon number N. We now consider the yrast states (Fig. 7); these are all states in the bands with $k_{\mathrm{L}} = 1/2$, then the states with $\mathscr{T} = 21/2$ and $\mathscr{T} = 23/2$ in the band with $k_{\mathrm{L}} = 3/2$, etc., up to the states with $\mathscr{T} = 37/2$ and $\mathscr{T} = 39/2$ in the $k_{\mathrm{L}} = 11/2$ band. The B(E2) values for transitions between these states are given in Fig. 8. It can be seen that the states are divided into two groups according to the dependence of B(E2) on $\mathscr{T}$ for the transitions with $\Delta\mathscr{T} = 2$. The E2 transitions $\mathscr{T}+1 \to \mathscr{T}$ between these groups are weaker than the E2 transitions $\mathscr{T}+2 \to \mathscr{T}$. This served as the basis for a new classification of the states, which is shown in Fig. 9. The states are distributed over rising bands, exhibiting a signature effect, this being the stronger, the lower they are in energy. These bands are identified by the quantum number k'_{L}. As can be seen from Fig. 9, the bands with $k'_{\mathrm{L}} = 1/2$, 3/2, and 5/2 have branches of different signature, connected by the broken line, and along these branches the B(E2) values are enhanced. The lower branch of the band with $k'_{\mathrm{L}} = 1/2$, which includes states with $\mathscr{T} = j, j+2, j+4, \ldots, 39/2$, recalls the decoupled band of the rotational-alignment model, for which the projection of the angular momentum onto the rotation axis is j.[88]

In this model, the following relation holds[88]:

$$E_{\mathscr{T}=I_{GSB}+j}(\alpha = j) = E_{I_{GSB}}\,(\mathrm{core}), \qquad (80)$$

where

$$E_{I_{GSB}}\,(\mathrm{core}) = \delta I_{\mathrm{GSB}}(I_{\mathrm{GSB}}+1). \qquad (81)$$

In the lower part of Fig. 9, the dotted line connects states whose energies were calculated by means of (80) and (81) and the value $\delta = 0.015$. It can be seen that these states are close in energy to the exactly calculated states of the lower branch of the $k'_{\mathrm{L}} = 1/2$ band.

The connection with the rotational-alignment model was also established by comparing the wave functions. Instead of the functions $d^j_{k_\alpha}(\pi/2)$ the authors of Ref. 86 used the Clebsch–Gordan coefficient $\langle 2N\,0jk\,|2N+\alpha k\rangle$, which in the asymptotic limit $2N \to \infty$ is equal to $d^j_{k_\alpha}(\pi/2)$. Thus, in the case $\Gamma = -\Gamma_{\mathrm{ss}}$ the $k'_{\mathrm{L}} = 1/2$ band is the analog of the

TABLE IV. Matrix elements of the Hamiltonian $\mathrm{H}'_{\mathrm{PTQM}} = I^{\mathrm{B}}I^{\mathrm{B}} + (\Gamma/\delta)_{\mathrm{ss}}(G_2^{\mathrm{B}}G_2^{\mathrm{F}})_0$ for $(\Gamma/\delta)_{\mathrm{ss}} = 46.3$.

k_{U} k'_{U}	11/2	9/2	7/2	5/2	3/2	1/2
11/2	−57.9	0	0	0	0	0
9/2	0	17.2	−5.3	0	0	0
7/2	0	−5.3	74.4	−13.1	0	0
5/2	0	0	−13.1	120.8	−17.0	0
3/2	0	0	0	−17.0	182.3	−8.0
1/2	0	0	0	0	−8.0	273.1

TABLE V. Eigenvalues and eigenfunctions of the Hamiltonian from Table IV.

n	$E_{17/2,\ n}$	$\eta_{\mathcal{T}}^{k_U} = 17, 2, n$					
		$k_U = 11/2$	$k_U = 9/2$	$k_U = 7/2$	$k_U = 5/2$	$k_U = 3/2$	$k_U = 1/2$
1	−57.9	1.000	0	0	0	0	0
2	16.7	0	0.995	0.094	0.012	0.001	0.000
3	71.3	0	−0.094	0.958	0.267	0.041	0.002
4	120.0	0	0.014	−0.268	0.929	0.254	0.013
5	186.1	0	−0.001	0.030	−0.256	0.962	0.088
6	273.9	0	0.000	−0.001	0.010	−0.089	0.996

decoupled band of the rotational-alignment model, but in a space with a restricted number of quadrupole phonons.

The wave functions (75) can be used as a basis for constructing analogs of Nilsson states in the framework of the SU(3)-BFM. In the general case, we can write

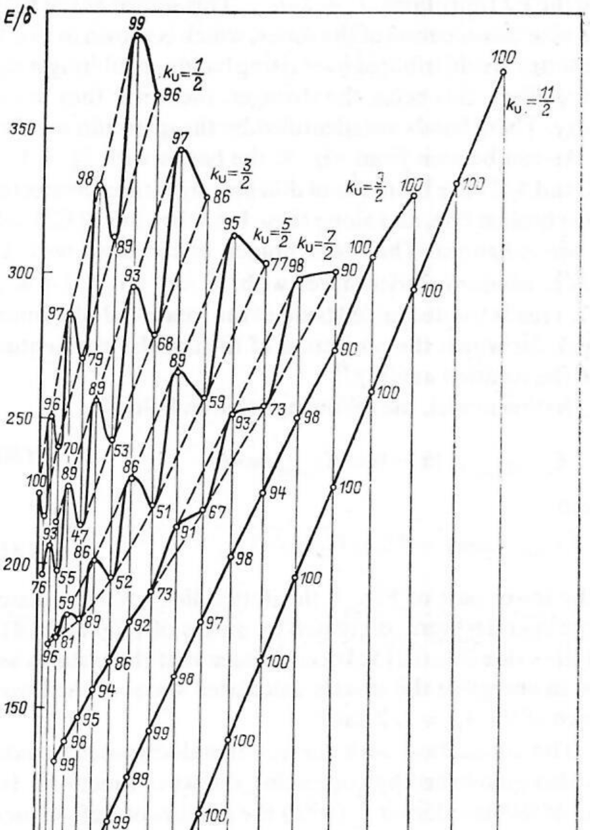


FIG. 6. Spectrum calculated in accordance with Eq. (76) for the case $j = 11/2$, $N = 7$, $\Gamma/\delta = (\Gamma/\delta)_{ss} = 46.3$. The Hamiltonian was diagonalized in the basis $\{|k_U \mathcal{T} M\rangle_{ss}\}$. For the angular momenta, the $\mathcal{T}(\mathcal{T}+1)$ scale was used. Next to each state we give the percentage of the dominant component k_U in the wave function of this state. The energy along the ordinate is given in units of δ.

$$|k\mathcal{T} M\rangle = \sum_I f_{N\mathcal{T}}(I) \langle jkI0|\mathcal{T} k\rangle | (jI_{0c}) \mathcal{T} M\rangle, \qquad (82)$$

where the nature of the quantum number k is determined by the form of the weight function $f_{N\mathcal{T}}(I)$. In the two cases considered above, the weight function

$$f_{N\mathcal{T}}(I) = B_I^{-1}$$

was used. To treat the limiting case of strong coupling in the

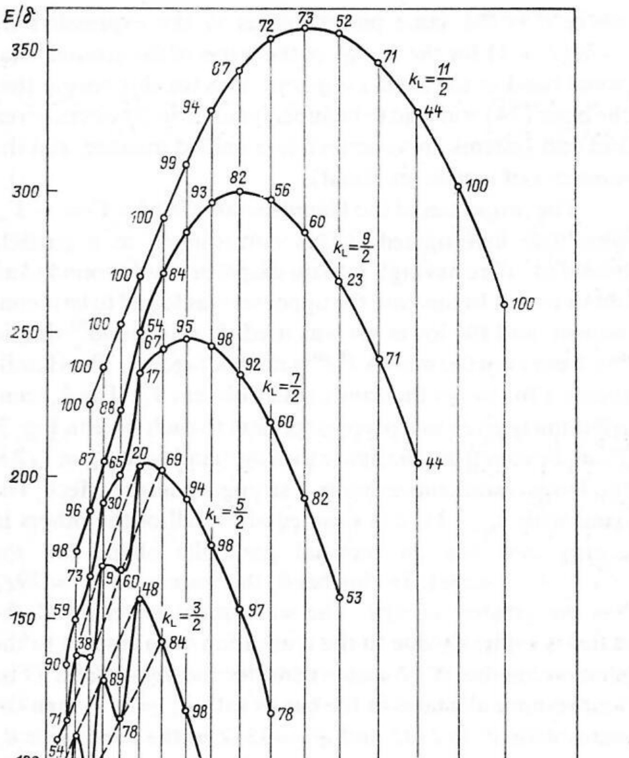


FIG. 7. The spectrum calculated in accordance with Eq. (76) with the same parameter values as in Fig. 6 but with the opposite sign of Γ, i.e., for $\Gamma/\delta = -(\Gamma/\delta)_{ss}$. In addition, the lower set $\{|k_L \mathcal{T} M\rangle_{ss}\}$ was used. For each state we give as a percentage the dominant component k_L in the wave function of this state. The energy along the ordinate is given in units of δ.

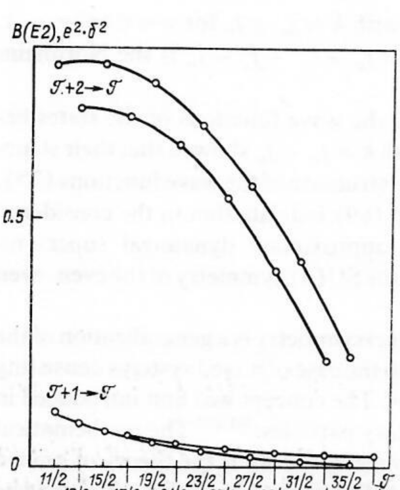


FIG. 8. Calculated values of $B(\mathrm{E2})$ transitions between states of the lower branch (corresponding to the $\alpha = j = 11/2$ decoupled band) between states of the upper branch of the band with $k'_\mathrm{L} = 1/2$ and between the states of the two branches. The wave functions correspond to the same parametrization as in Fig. 7, but the upper set $|k_\mathrm{U}\mathscr{T}M\rangle_\mathrm{ss}$ has been used. The effective charges are $e^{\text{sing}\cdot\text{part}} = 1.5$, $e^{\text{vibr}} = 2$, and the radial matrix element is $\langle r^2\rangle = 1.44A^{2/3}$, where $A = 100$.

framework of the SU(3)-BFM, the weight function must be taken by analogy with the rotational BFM in the form

$$f_{N,\mathscr{T}}(I) = \left(\frac{4I+2}{2\mathscr{T}+1}\right)^{1/2}. \tag{83}$$

This limiting case of the SU(3)-BFM was considered in detail in Ref. 87. The corresponding transformation has the form

$$|k\mathscr{T}M\rangle_\mathrm{s} \qquad \sum_{I=0}^{2N}\left(\frac{4I+2}{2\mathscr{T}+I)}\right)^{1/2}\langle jkI0|\mathscr{T}k\rangle|(jI_{0c})\mathscr{T}M\rangle. \tag{84}$$

The unitarity of the transformation is violated if the triangle rules admit values $I \geqslant 2N+2$. The maximal angular momentum $\mathscr{T}$ for which (84) is unitary is $\mathscr{T}_\mathrm{max} = 2N - j + 1$. In the general case, the maximal angular momentum in the k band is

$$\mathscr{T}^k_\mathrm{max} = 2N - j + 2k. \tag{85}$$

It follows that (84) is unitary for all states of the $k = 1/2$ band. In the other k bands, the transformation (84) is not unitary for the angular momenta that do not occur in the $k = 1/2$ band. For these states, the Gram–Schmidt procedure must be used. In the basis (84), the matrix $I^\mathrm{B}I^\mathrm{B}$ is always tridiagonal for $\mathscr{T} < \mathscr{T}_\mathrm{max}$. This property is also preserved when $\mathscr{T} > \mathscr{T}_\mathrm{max}$ if the upper set is used. However, the interaction matrix is never tridiagonal in the basis $|k\mathscr{T}M\rangle_s$. In this case, it behaves in exactly the same way as in the rotational model, for which the selection rule $\Delta k = 1$, which holds for $I^\mathrm{B}I^\mathrm{B}$, is violated for the term describing the particle–core interaction. Since the matrix $\mathrm{G}_2^\mathrm{B}G_2^\mathrm{F}$ cannot be reduced to a tridiagonal form in the basis (84), there also does not exist a finite coupling constant Γ for which the states (84) will be eigenstates of the Hamiltonian (76). [An exception is the trivial case of a 2×2 Hamiltonian matrix (76).] By construction, it is to be expected that the states (84) will give a good approximation to the eigenstates of the SU(3)-BFM Hamiltonian in the limit of large Γ and N. It was shown[87] that

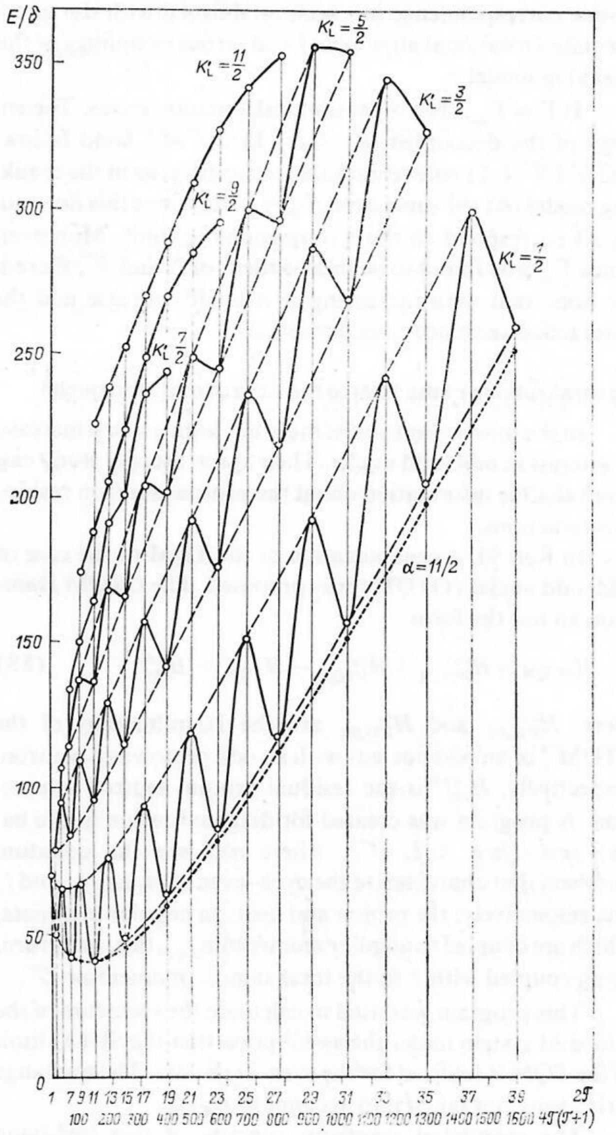


FIG. 9. The same spectrum as in Fig. 7, but the classification of the states in terms of k'_L has been used. The decoupled $\alpha = 11/2$ band in the model of rotational alignment, calculated in accordance with (80) and (81), is shown by the dotted curve.

$$\lim_{N\to\infty} {}_\mathrm{s}\langle k\mathscr{T}M|(G_2^BG_2^F)_{(00)}|k'\mathscr{T}M\rangle_\mathrm{s}$$
$$= -2\sqrt{2}N\left[\frac{(2j-2)!}{(2j+3)!}\right]^{1/2}[3k^2 - j(j+1)]\,\delta_{kk'}. \tag{86}$$

This means that at large N only the Coriolis interaction $I^\mathrm{B}I^\mathrm{B}$ can be the source of the nonvanishing nondiagonal matrix elements.

It was also shown that

$$\lim_{N\to\infty}\frac{\Gamma_\mathrm{s}}{\delta_\mathrm{s}}\,\frac{\langle G_2^BG_2^F\rangle_\mathrm{s}}{\langle I^BI^B\rangle_\mathrm{s}} = \frac{\Gamma N}{\delta}, \tag{87}$$

i.e., for $N \gg 1$ the term describing the interaction of the fermions with the core is dominant. In addition, it was found that signature effects are manifested only for the band with $k = 1/2$, in complete agreement with the strong-coupling limit in the Bohr–Mottelson cranking model.

Thus, for two of the limiting cases considered in the framework of the SU(3)-BFM, $\Gamma = -\Gamma_\mathrm{ss}$ and $\Gamma \gg 1$, a one-to-one correspondence has been established with the intermediate (rotational alignment) and strong couplings of the cranking model.

If $\Gamma = \Gamma_{ss}$, then a paradoxical situation arises. The energy of the decoupled $|k_U = j = 11/2\mathcal{T}M\rangle$ band follows the $\mathcal{T}(\mathcal{T}+1)$ rule remarkably accurately, as in the cranking model. At the same time, $\Gamma_{ss} \approx 1$ MeV, but this does not at all correspond to the strong-coupling limit. Moreover, since Γ_{ss} was found to be independent of N and $\mathcal{T}$, there is no hope that with increasing N it could increase and the interaction term becomes dominant.

Generalizations of the TQM to the case of odd–odd nuclei

In the most recent years, there has been a sharp increase of interest in odd–odd nuclei. Their spectroscopic study can give valuable information about the proton–neutron residual interactions.

In Ref. 91, a generalization of the TQM to the case of odd–odd nuclei (OTQM) was proposed. The OTQM Hamiltonian has the form

$$H_{\mathrm{OTQM}} = H^{(\pi)}_{\mathrm{PTQM}} + H^{(\nu)}_{\mathrm{PTQM}} - H_{\mathrm{TQM}} + H^{(\pi\nu)}_{\mathrm{res}} \tag{88}$$

Here, $H^{(\pi)}_{\mathrm{PTQM}}$ and $H^{(\nu)}_{\mathrm{PTQM}}$ are the Hamiltonians of the PTQM[74] of an odd nucleus with an odd proton and neutron, respectively; $H^{(\pi\nu)}_{\mathrm{res}}$ is the residual proton–neutron interaction. A program was created for diagonalization in the basis$|(j\pi j\nu)\ j\pi\nu,\ n\varkappa\mathrm{I};\ \mathcal{T}\rangle$, where $n\varkappa I$ are the quantum numbers that characterize the even–even core, and j_π and j_ν are, respectively, the proton and neutron angular momenta, which are coupled to angular momentum $j_{\pi\nu}$, this, in its turn, being coupled with I to the total angular momentum $\mathcal{T}$.

This program was used to calculate the spectrum of the odd–odd system under the assumption that the SU(3) limit of the TQM is realized for the even–even core. The exchange terms were not taken into account in H_{PTQM}.

The calculated spectrum consists of two low-lying bands based on states with angular momenta $\Gamma = j_\pi + j_\nu$ and $\mathcal{T} = |j_\pi - j_\nu|$ and bands that lie higher in energy.

It was shown that the wave functions of the state with angular momentum $\mathcal{T}$ belonging to the $(j_\pi + j_\nu)$ band can be represented in the same form as the wave functions (74) for the odd system, except that it is necessary to make the substitutions

$$|j\rangle \to |(j_\pi j_\nu)\, j_{\pi\nu}\rangle;$$
half-integral $k \to$ integral k;
half-integral $\mathcal{T}M \to$ integral $\mathcal{T}M$.

The energies of the states of this band are proportional to $\mathcal{T}(\mathcal{T}+1)$

$$E\,(k = j_\pi + j_\nu,\ \mathcal{T}) = \delta\mathcal{T}\,(\mathcal{T}+1), \tag{89}$$

where δ is the same parameter as in the case of the even–even and odd systems. It was established that these two lowest bands, i.e., $|k = j_\pi + j_\nu, \mathcal{T}\rangle$ and $|k = |j_\pi - j_\nu|, \mathcal{T}\rangle$, are analogs of the Gallagher–Mozkowski bands based on $(\Omega_\pi = j_\pi,\ \Omega_\nu = j_\nu)$. The results obtained in the framework of the OTQM indicate that the difference of energies of the bases of these bands is sensitive to the residual spin–spin interaction. As in the case of the parabolic rule,[100] the spin–spin interaction leads to a lowering of the bases of the bands with $k = |j_\pi - j_\nu|$ or with $k = j_\pi + j_\nu$ for $\nu = 0$ or $|\nu| = 1$, respectively, where $\nu = j_\pi - l_\pi + j_\nu - l_\nu$ is the Nordheim number.

Direct analysis of the wave functions of the states belonging to the band with $k = j_\pi + j_\nu$ showed that their structure is analogous to the structure of the wave functions (75). This fact, and also Eq. (89) indicate that in the considered odd–odd system the approximate dynamical supersymmetry associated with the SU(3) symmetry of the even–even core is realized.

The concept of supersymmetry is a generalization of the concept of symmetry to the case of mixed systems consisting of bosons and fermions. The concept was first introduced in the physics of elementary particles.[101,102] The mathematical formalism of supersymmetry theory is the theory of graded Lie algebras,[103] instead of ordinary Lie algebras. It should be noted that at the present time nuclear physics appears to be the only branch of physics in which one can test whether or not supersymmetries are realized in nature. The approach based on dynamical supersymmetries makes it possible to describe even–even, odd, and odd–odd nuclei on a unified basis. Initially, an approach was developed in the framework of the interacting-boson–fermion model to a unified description of even–even and odd nuclei using exact boson–fermion symmetries, the so-called spinor symmetries.[105–110] Somewhat later, in Ref. 89 and in the framework of the OTQM, this approach was generalized to the case of odd–odd nuclei. The basic idea of the generalization of the method to this type of nucleus is to construct the maximal group of transformations for the odd fermions (π and ν) and to use embedded chains that lead to the necessary spin composition of the considered nuclear system. Once this structure has been found, it is necessary to analyze all possible couplings of the two odd fermions to the core. As a result of this analysis, one obtains corresponding mass formulas for the spectrum of the odd–odd system. In Ref. 89, a study was made of the case when odd fermions in configurations with $j = 3/2$ are coupled to a collective core described under the assumption that the SO(6) limit is realized. The following chain of embedded boson–fermion groups was considered:

$$\begin{tikzcd}[column sep=small]
U^B(6) & \otimes & U^{F_\pi}(4) \arrow[d, phantom, "\cup"] & \otimes & U^{F_\nu}(4) \arrow[d, phantom, "\cup"] \\
\cup & & SU^{F_\pi}(4) \arrow[dr, dash] & \otimes & SU^{F_\nu}(4) \arrow[dl, dash] \\
SO^B(6) \arrow[dr, dash] & \otimes & & SU^{F_{\pi\nu}}(4) \arrow[dll, dash] & \\
& \mathrm{Spin}\ BF_{\pi\nu}(6) \arrow[d, phantom, "\cup"] & & & \\
& \mathrm{Spin}\,BF_{\pi\nu}(5) \arrow[d, phantom, "\cup"] & & & \\
& \mathrm{Spin}\,BE_{\pi\nu}(3) \arrow[d, phantom, "\cup"] & & & \\
& \mathrm{Spin}BF_{\pi\nu}(2). & & & (90)
\end{tikzcd}$$

When the boson and fermion groups were combined into boson–fermion groups, essential use was made of the homeomorphism $\mathrm{SO}(6) \approx \mathrm{SU}(4)$. By analogy with the case of odd nuclei,[105,106] the generators of the spinor groups in the chain (90) were obtained by combining the generators of the boson and fermion groups in such a way that the algebra of the corresponding spinor groups closes. The basis states corresponding to the chain have the form

$$|\,[N],\ \{M_\pi=1\},\ \{M_\nu=1\}.\ \Sigma,\ (\xi_1,\ \xi_2,\ \xi_3),\ (\sigma_1,\ \sigma_2,\ \sigma_3),\\ (\mathscr{T}_1,\ \mathscr{T}_2)\,(n_\delta)\,\mathscr{T},\ M\rangle,$$

where the quantum numbers $N, M_\pi, M_\nu, \Sigma, (\xi_1, \xi_2, \xi_3), (\sigma_1, \sigma_2, \sigma_3), (\mathscr{T}_1, \mathscr{T}_2), (n_\delta), \mathscr{T}$, and M characterize irreducible representations of the groups that participate in the chain (90): $U^B(6)$, $U^{F_\pi}(4)$, $U^{F_\nu}(4)$, $SO^B(6)$, $SU^{F_{\pi\nu}}(4)$, $\mathrm{Spin}^{BF_{\pi\nu}}(6)$, $\mathrm{Spin}^{BF_{\pi\nu}}(5)$, $\mathrm{Spin}^{BF_{\pi\nu}}(3)$, and $\mathrm{Spin}^{BF_{\pi\nu}}(2)$, respectively. Using the group-theoretical methods developed in Refs. 44, 111, and 112, the authors of Ref. 89 constructed representations of each of the subgroups in (90) contained in a corresponding larger group.

In the case when the OTQM Hamiltonian (88) can be expressed in terms of the Casimir operators of the subgroups that occur in the chain (90), its eigenvalues, i.e., the spectrum of the odd–odd system, can be represented as follows:

$$\begin{aligned} E = E^{(0)}\ (N, M_\pi = 1), M_\nu = 1)\\ -\frac{1}{4}A_1'\,[\xi_1(\xi_1+4)+\xi_2(\xi_2+2)+\xi_3^2]\\ -\frac{1}{4}A_1\Sigma(\Sigma+4)-\frac{1}{4}A\,[\sigma_1(\sigma_1+4)+\sigma_2(\sigma_2+2)+\sigma_3^2]\\ +\frac{1}{6}B\,[\mathscr{T}_1(\mathscr{T}_1+3)+\mathscr{T}_2(\mathscr{T}_2+1)]+C\mathscr{T}(\mathscr{T}+1). \quad (91)\end{aligned}$$

The parameters A_1, A_2, B, C were introduced in such a way that the expression (91) resembles the form of the Spin(6) mass formula for odd nuclei.[106] The expression $E^{(0)}(N, M_\pi, M_\nu)$ contains terms linear and quadratic in N, M_π, M_ν that contribute only to the binding energy.

Figure 10 gives the spectrum of eigenvalues calculated in accordance with (91) with the parameters of Ref. 113 corresponding to the typical Spin(6) spectrum of odd nuclei.

Figure 11 shows the yrast bands: the 0_1 band, $3/2_1$ band, and 3_1 band in the even–even, odd, and odd–odd systems, respectively, calculated in accordance with the SO(6) mass formula,[30] the Spin(6) mass formula,[105] and in accordance with the formula (91), respectively. The total number of bosons was taken to be seven, and the remaining parameters were the same as in Fig. 10. Manifestation of the Spin(6) boson–fermion symmetry is to be expected in the Ir–Pt–Au region, where SO(6) boson symmetry is realized in the even–even nuclei and there are low-lying $\pi\tilde{d}_{3/2}$ and $\nu\tilde{p}_{3/2}$ single-quasiparticle states. If certain restrictions are imposed on the constants of the boson–boson, boson–fermion, and proton–neutron residual interactions, the OTQM Hamiltonian (88) for $j_\pi = j_\nu = 3/2$ can be rewritten in terms of the Casimir operators of the chain (90), and this leads to the spectrum given by Eq. (91). This means that in (91) allowance is made for the neutron–proton residual interaction but with a constant fixed in a definite manner.

In Ref. 89, it was assumed that both odd fermions are particles. New possibilities arise if one bears in mind that the odd fermion may be either a particle or a hole. In Ref. 114, it was shown that for the isotopes of Au ($Z = 79$) with $N = 117$ and $N = 119$ one can achieve a good description of the spectra of excited states by assuming that the odd neutron is a particle and the odd proton is a hole.

The Spin(6) symmetry considered in Ref. 89 can be extended by embedding the group $U^B(6) \otimes U^{F_\pi}(4)\,U^{F_\nu}(4)$, with which the chain (90) begins, in the graded Lie algebra $U(6/4\pi + 4\nu)$. This generalization of the idea proposed in Ref. 107 makes it possible to treat on a unified basis not only the even–even and odd nuclei but also odd–odd nuclei. In this case, the superalgebra has proton and neutron subsectors in its Fermi sector.

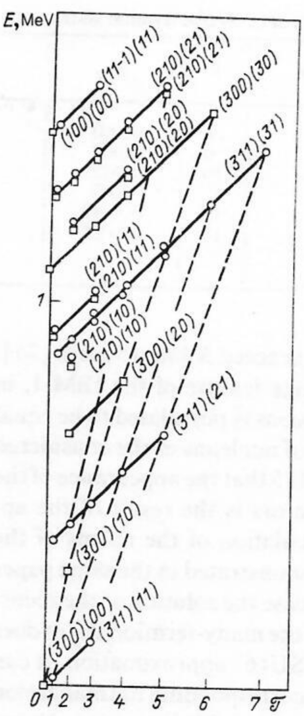


FIG. 10. Spectrum of excited states of the odd nucleus calculated in accordance with Eq. (91). The parameters are typical of odd nuclei described by means of the group Spin(6): $A_1 = 0$, $A/4 = 90$ keV, $B/6 = 60$ keV, $C = 10$ keV, $N = 2$, $A_1 = 0$. For $\Sigma = 2$, the states with $(\xi_1, \xi_2, \xi_3) = (1, 0, 0)$ and $(1, 1, 1)$ are shown by the squares and circles, respectively. Each group of states with the same values of (ξ_1, ξ_2, ξ_3), $(\sigma_1, \sigma_2, \sigma_3)$, and (τ_1, τ_2) is connected by a straight line. To each group the quantum numbers $(\sigma_1, \sigma_2, \sigma_3)$ and (τ_1, τ_2) have been ascribed. The states of the four bands nearest the yrast band are joined by the broken lines.

On the effect of truncation of the phonon space of the TQM

A characteristic feature of the TQM is that it operates with a finite-dimensional phonon space. The Hamiltonian and physical operators of the TQM are defined on a phonon

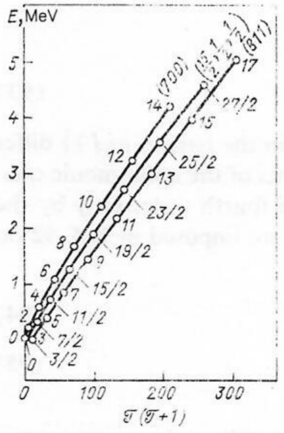


FIG. 11. Yrast bands constructed on the states 0_1, $(3/2)_1$, and 3_1 in even–even, odd, and odd–odd systems. The bands are identified by the quantum numbers $(\sigma_1, \sigma_2, \sigma_3) = (7, 0, 0)$, $(15/2, 1/2, 1/2)$, and $(8, 1, 1)$ for the even–even, odd, and odd–odd systems, respectively. For each level, its angular momentum is given.

TABLE VI. Normalization coefficients of collective fermion states.

N	$\overline{\mathcal{N}}_N$	$\overline{\mathcal{N}}_N^{SU(6)}$	N	$\overline{\mathcal{N}}_N$	$\overline{\mathcal{N}}_N^{SU(6)}$
1	1	1	8	0.060	0.018
2	0.90	0.90	9	0.028	0.004
3	0.73	0.72	10	0.012	0.0004
4	0.535	0.504	11	0.005	0
5	0.356	0.302	12	0.0019	—
6	0.215	0.151	13	0.0007	—
7	0.119	0.060			

space whose dimension does not exceed N [see (5) and (7)]. (Finiteness of N is also a specific feature of the IBM-1, in which the total number N of bosons is postulated to be equal to the number of valence pairs of nucleons of the considered nucleus.) It was shown in Ref. 115 that the appearance of the square root in the TQM operators is the result of the approximations made in the calculation of the norms of the many-fermion states. It was demonstrated in the same paper (Table VI) that in the general case the solution of the recursion equations for the norms of the many-fermion states does not agree with the result of the SU(6) approximation. It can be seen from Table VI that the corresponding normalization coefficients $\overline{\mathcal{N}}_N$ decrease smoothly with increasing N and do not vanish anywhere [in contrast to the $\overline{\mathcal{N}}_N$ found in the SU(6) approximation], this corresponding to an unrestricted space of collective states. Because of the restriction of the phonon space in the SU(3) limit of the TQM (IBM-1), rotational bands that terminate at $I = 2N$ arise.[7,10] This termination is not observed experimentally. The part played by the truncation effect was studied in Ref. 92, in which it was shown that the occurrence of the rotational bands is a property characteristic of a large class of quadrupole-phonon models independently of the SU(3) symmetry. To demonstrate this, the authors of Ref. 92 introduced a collective Hamiltonian containing cutoff factors in the form of an arbitrary function of the operator of the number of quadrupole bosons:

$$H^f = h_1\hat{N} + h_2\{(b^+b^+)_{(00)}f(\hat{N})f(\hat{N}+1) + \mathrm{h.c.}\}$$
$$+ h_3\{(b^+b^+\tilde{b})_{(00)}f(\hat{N}) + \mathrm{h.c.}\} + \sum_{L=0,\,2,\,4} h_{4L}\{(b^+b^+)_L(\tilde{b}\tilde{b})_L\}_{(00)}, \tag{92}$$

where $\hat{N} = \Sigma_\mu b_\mu^+ b_\mu$.

Since

$$f(\hat{N})\mid n\varkappa I\rangle = f(n)\mid n\varkappa I\rangle, \tag{93}$$

the matrix elements of $\hat{H}^f$ (92) in the basis $\{|n\varkappa I\rangle\}$ differ from the standard matrix elements of the anharmonic quadrupole–phonon Hamiltonian of fourth order only by the factors $f(n)$. Two restrictions were imposed in Ref. 92 on $f(\hat{N})$:

$$f(\hat{N})\mid n = N\varkappa I\rangle = 0; \tag{94}$$

$$\int_0^N f(n)\,dn = \frac{2}{3}N^{3/2}. \tag{95}$$

Note that the conditions (94) and (95) are in no way related to symmetry considerations. Three particular choices of $f(\hat{N})$ were considered:

$$1)\ f(\hat{N}) = (N - \hat{N})^{1/2}. \tag{96}$$

In this case $\hat{H}^f \equiv \hat{H}_{\mathrm{TQM}}$ [see (5)];

$$2)\ f(\hat{N}) = \frac{2}{3}\sqrt{N}\,\theta(N - \hat{N}), \tag{97}$$

where the operator step $Q(N - \hat{N})$ is defined as usual;

$$\theta(N - \hat{N})|n\varkappa I\rangle = \begin{cases} |n\varkappa I\rangle & \text{for } n < N; \\ 0 & \text{for } n \geqslant N. \end{cases} \tag{98}$$

As was noted in Ref. 74, $\hat{H}^f$ in this case corresponds exactly to an anharmonic quadrupole–phonon Hamiltonian of fourth order of special form (see Ref. 74 and the references given there). This Hamiltonian was diagonalized in an N-dimensional phonon space. The results of the diagonalization depend on N. This dependence was studied in Refs. 116 and 117 before the introduction of the TQM and the IBM-1;

$$3)\ f(\hat{N}) = \frac{4}{3}\sqrt{N}\left(1 - \frac{\hat{N}}{N}\right)\theta(N - \hat{N}). \tag{99}$$

This case was first considered in Ref. 92. Figure 12 gives the diagonal matrix elements $f(n) = \langle n\varkappa I\,|\,f(\hat{N})|n\varkappa I\rangle$ corresponding to the three forms of the cutoff factor: 1, 2, 3.

It is known[7,29] that in the case when the SU(3) limit is realized the parameters $\{h\} \equiv \{h_1, h_2, h_3, h_{40}, h_{42}, h_{44}\}$ can be expressed by means of N and two parameters: α and β. We call this parametrization the SU(3) parametrization of $\{h\}$ SU(3). [The explicit form of the $\{h\}_{\mathrm{SO}(3)} \leftrightarrow \{\alpha, \beta, N\}$ connection is given in Ref. 74; see Eq. (38) in Ref. 74.] The spectra of the Hamiltonians $\hat{H}^f$ were calculated for the three forms of $f(\hat{N})$ given by (96), (97), and (99); the same SU(3) parametrization was used in all three cases, and it was assumed that $N = 7$. The calculated spectra are shown in Figs. 13–15, respectively. The cases when $f(\hat{N})$ are determined by the relations (97) and (99) do not have any bearing on the SU(3) symmetry. However, it can be seen from Figs. 13–15 that the spectra corresponding to the three forms of the cutoff factor are remarkably similar. A similar result is obtained for the electromagnetic properties calculated in the three cases. On this basis, it was concluded in

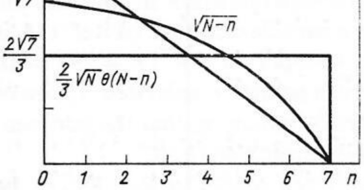
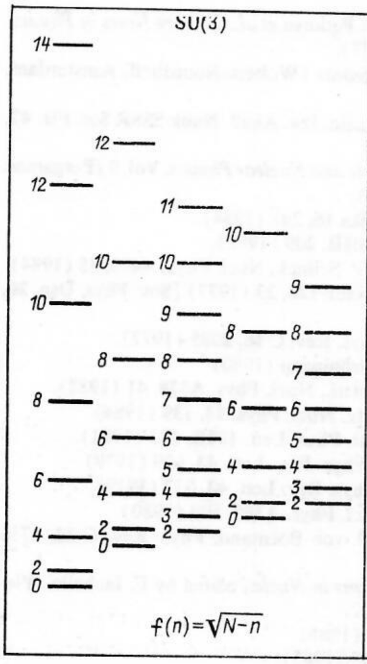
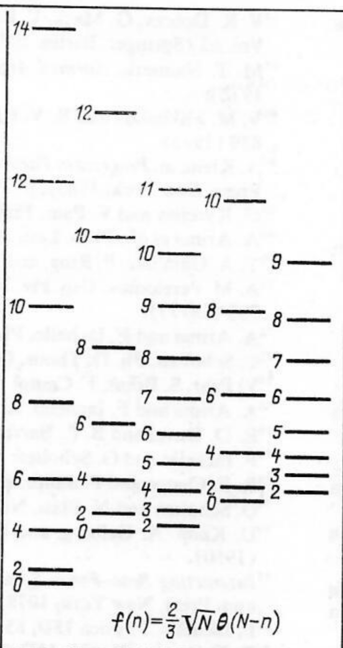


FIG. 12. Diagonal matrix elements $f(n) = \langle n\varkappa I\,|\,f(\hat{N})|n\varkappa I\rangle$ for three different forms of the cutoff factor for $N = 7$.





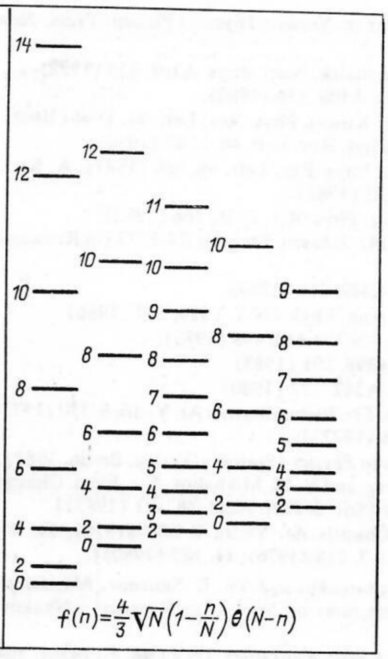


FIG. 13. The spectrum of $\hat{H}^f$ for $f(\hat{N})=(N-\hat{N})^{1/2}$. The following parametrization was used: $h_1=0.0225$, $h_2=-0.0224$, $h_3=-0.0592$, $h_{40}=-0.1175$, $h_{42}=-0.0811$, $h_{44}=0.285$, and $N=7$. This corresponds to the parameters $\alpha=0.1$ and $\beta=0.2$ for the SU(3) parametrization. The numbers next to the levels give the spin.

FIG. 14. The spectrum of $\hat{H}^f$ for $f(\hat{N})=(2/3)\sqrt{N}\theta(N-\hat{N})$. The parameters are the same as in Fig. 13.

FIG. 15. The spectrum of $\hat{H}^f$ for $f(\hat{N})=(4/3)\sqrt{N}(1-\hat{N}/N)\theta(N-\hat{N})$. The parameters are the same as in Fig. 13.

Ref. 92 that the occurrence of finite rotational bands (Figs. 13–15) is not due to realization of the SU(3) limit of the TQM but rather reflects definite relationships between the constants of the collective Hamiltonian acting on the restricted phonon space. The circumstance that the conditions of occurrence of rotational bands must be weaker than SU(3) symmetry requires was pointed out earlier by Bohr and Mottelson.[118]

5. CONCLUSIONS

Our review consists of two main parts. For those interested in the mathematical formalism of the theory, greater interest attaches to the first part (Secs. 1–3), where we have presented the results of a detailed mathematical investigation of the relationship between the two very popular collective models (TQM and IBM-1); this connection was a controversial one. We have established equivalence of the two models at the level of the matrix elements and operators. We have given a rigorous proof that in the physical subspace of the boson state space the TQM and IBM-1 are unitarily equivalent. The question of the relationship between the TQM and the IBM-1 can now be regarded as completely studied.

For the reader having more interest in the physical applications Sec. 4 will be more relevant; for here we give the results of the latest development and application of the TQM to particular physical systems.

We are very grateful to É. Nadzhakov for helpful discussions of a number of questions considered in the review, and to V. G. Solov'ev and I. N. Mikhaĭlov for their interest in this paper.

[1)] Although the TQM and IBM-1 are equivalent (see Sec. 2), each has its own specific features. When there is no need to distinguish the TQM and IBM-1, we shall use the general designation IBM.

[2)] It must be emphasized that this is a somewhat nominal definition. Usually, a dynamical symmetry is defined[39–41] in terms of concepts such as algebra, generating spectrum, enveloping algebra, irreducible representation, etc.

[3)] The Lie group SU(6) has not only algebraic but also geometrical structure[44]; for the points of the SU(6) factor space stand in a one-to-one correspondence with the SU(6) coherent state, which can be parametrized by means of variable forms and Eulerian angles.

[4)] An isomorphism is a generalization of unitary equivalence [see (43)].

[5)] A model of many interacting bosons was recently formulated.[104] As an example, a model of interacting *spdf* bosons was considered.

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