Difficult questions of relativity theory

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The following concepts are discussed: covariance, invariance, the general, special, and kinematic principles of relativity, coordinate systems and frames of reference, and the energy–momentum tensor of the gravitational field. The relationships between the three canonical theories of gravitation are considered. The theory of the affine connection as applied to these questions is presented. Attention is drawn to inconsistency in terminology and the need for an explanatory dictionary for gravitational specialists. A contribution to the compilation of such a dictionary is made.

INTRODUCTION

In the theory of relativity, as in any developing theory, there are some difficult questions.

The first question arises in connection with the concept of relativity. Einstein wrote: "...the theory of relativity is like a house with two storeys: the special theory of relativity and the general theory of relativity." [1] It follows from Einstein's analogy that special relativity is neither a part of nor a special case of general relativity. We therefore speak of the special theory of relativity and not a partial theory of relativity. Now what is the relationship between special and general relativity?

The answer is complicated by the fact that one must consider not only the general and special principles of relativity but also a kinematic principle. We must therefore also consider the kinematic theory of relativity. Our question begins to divide itself—what is the relationship of the kinematic theory of relativity to special relativity and general relativity?

Further, the special and kinematic principles of relativity are intimately related to the concept of invariance, whereas the general principle of relativity has more to do with the concept of covariance than with invariance. The concept of covariance led to the modern theory of manifolds, but even today it has not yet achieved the degree of clarity that the concept of invariance achieved in the last century. This is the reason why there are today disputes about general relativity, whereas professional disagreements about special relativity died out half a century ago. So we have a second question: What is the relationship between the concepts of invariance and covariance?

Finally, the concept of covariance is intimately related to the concept of coordinate system, whereas the concept of invariance has more to do with the concept of a frame of reference than with the concept of a coordinate system. To explain why these two concepts must not be confused, we shall give some examples; as an abbreviation, we shall adopt cartographic terminology and, instead of "system of coordinates," use the word "map."

1. The transition from one inertial frame of reference to another in the old theory of relativity is given by a Galileo transformation $x_G = x - Vt$, $t_G = t$, and in the new theory of relativity by a Lorentz transformation $x_L = x\cosh\Psi - (ct)\sinh\Psi$, $ct_L = (ct)\cosh\Psi - x\sinh\Psi$, where V is the relative velocity of the frames of reference, $\tanh\Psi = Vc^{-1}$, and c is the velocity of light. They can equally well be regarded as coordinate transformations from the map (x,t) to the maps (x_G, t_G) and (x_L, t_L), respectively.

2. The d'Alembert transformation $u = x - ct$, $v = x + ct$, which is helpful when one is solving the wave equation $\varphi_{tt} = c^2\varphi_{xx}$, is not a transformation from any frame of reference to another in either the old or the new theory of relativity. However, in both cases it can be regarded as a coordinate transformation from the map (x,t) to the map (u,v).

3. The transition from the map (x, y, t) to the map (x', y', t'), where $x' = x\cos\Omega t - y\sin\Omega t$, $y' = x\sin\Omega t + y\cos\Omega t$, $t' = t$, can be regarded in the old theory of relativity as a transformation from an inertial frame of reference to a uniformly rotating frame but cannot be regarded in this way in the new theory of relativity.

And so a third question arises: What is the relationship between the concepts of a coordinate map and a frame of reference?

It can already be seen from these considerations that the terminology of relativity theory is very confused, and that the problems are by no means completely exhausted.

The theory of relativity took up non-Euclidean geometry as a tool. The creator of the new geometry himself, N. I. Lobachevskiĭ, was much concerned about its applications in physics, astronomy, and mechanics. The theory of relativity was a continuation of his work.[2]

The special theory of relativity was created at the beginning of this century on the basis of Maxwell's equations

$$\left.\begin{aligned} &\partial_a F_{mn} + \partial_n F_{am} + \partial_m F_{na} = 0; \quad F_{mn} + F_{nm} = 0; \\ &\sum_{m=1}^{4} \partial_m F^m_a = 0; \quad F^m_a = \sum_{n=1}^{4} h^{mn} F_{an}, \end{aligned}\right\} \tag{1}$$

where F_{mn} is the tensor of the electromagnetic field, and h^{mn} is the tensor that determines the cometric, which will be discussed below. In the new theory of relativity, Eqs. (1) play as characteristic a part as do Newton's equations

$$m_1 \frac{d^2\mathbf{r}_1}{dt^2} = \frac{\mathbf{r}}{r} F(r), \quad m_2 \frac{d^2\mathbf{r}_2}{dt^2} = -\frac{\mathbf{r}}{r} F(r) \tag{2}$$

for two bodies, where $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$, $r = |\mathbf{r}|$, in the old theory of relativity.

It is important that in the new theory of relativity the Euclidean geometry in the space of inertial systems is replaced by Lobachevskiĭ geometry.[3] We shall not attempt to consider why Einstein did not say anything about this question in print. With regard to Poincaré, he did not illuminate this question in his printed works, possibly, like Gauss, be-

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cause he did not wish to hear the "abuse of the Boeotians." However, the failure to mention Lobachevskiĭ's geometry does not need to be explained here at all. It is simply that in presenting his results, "in order to avoid every obscurity," he used a different terminology. For in a different case, when constructing the theory of automorphous functions, he made wide use of Lobachevskiĭ geometry, and in 1882 he wrote: "If these redesignations are accepted, then *Lobachevskiĭ's theorems are true*, i.e., to these new entities all the theorems of ordinary geometry apply except for those that are a consequence of Euclid's postulate. This terminology was a great assistance to me in my investigations, but, in order to avoid every obscurity, I shall not use this terminology here" (Ref. 4, p. 306 of the Russian translation).

Whatever the truth in this matter, the introduction of Lobachevskiĭ's geometry into the space of inertial systems made it necessary to achieve a reconciliation of the whole of physics, in particular, the following branches of it, which had been well developed by the end of the last century:

1. The mechanics of absolutely rigid bodies.
2. The mechanics of a single material point.
3. The mechanics of contact collisions.
4. Boltzmann's kinetic theory of gases.
5. Two-body mechanics.
6. Newton's law of universal gravitation.
7. The theory of the gravitational potential.

In all these branches, the basis at that time was provided by a Euclidean space of inertial systems, and nothing in them indicated the need for the new theory of relativity now called special relativity. What happened?

1. In special relativity, there is no rigid-body mechanics, since in special relativity the principle of kinematic relativity is violated.

2. The mechanics of a single material point in a field of external forces did not present fundamental difficulties in special relativity. It was formulated by Poincaré and Einstein. On the basis of their achievements, Minkowski developed the concept of the space–time world, in which special relativity is described by a geometry with metric

$$\sum_{a=1}^{4}\sum_{b=1}^{4}\theta_{ab}d^a\otimes d^b = -\frac{1}{c^2}(dx\otimes dx+dy\otimes dy+dz\otimes dz)+dt\otimes dt \quad (3)$$

and cometric

$$\sum_{a=1}^{4}\sum_{b=1}^{4}h^{ab}\partial_a\otimes\partial_b = \frac{\partial}{\partial x}\otimes\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\otimes\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\otimes\frac{\partial}{\partial z}-\frac{1}{c^2}\frac{\partial}{\partial t}\otimes\frac{\partial}{\partial t}, \quad (4)$$

where x, y, z are Cartesian coordinates and t is the time in one of the inertial frames of reference. The motion of a material point is represented in this world by a trajectory $x^a = x^a(\tau)$, on which the sum $\theta_{ab}dx^a dx^b$ is positive. The parameter τ is called the proper time of the material point and is a measure of its aging.

3. The mechanics of contact collisions of point particles also encountered no fundamental difficulties in the special theory of relativity. This was so because the mechanics of contact collisions can be reduced to statics in the space of inertial systems.

4. Boltzmann's kinetic theory can be well adapted to the conditions of special relativity. This is explained by the fact that all phenomena which unfold in a rarefied gas can be explained by contact collisions of particles and the motion of a single particle in an external force field in the intervals between collisions.

5. In contrast, Newtonian two-body mechanics cannot be adapted to the conditions of special relativity.

6. Newton's law of universal gravitation shares the fate of two-body mechanics.

7. For the theory of the gravitational potential too an adequate place in special relativity was not found, and this served as a stimulus to the creation of the general theory of relativity.

As a result of these discussions of questions in the theory of relativity we have found, as we see, a wide variety of terminology, and this suggests the need to create a special dictionary.[5] We are attempting to do what we can in this direction.[6,7]

Of course, some questions of the theory discussed here can be solved purely logically, and in this respect discussions are helpful. For example, they helped in establishing the theory of manifolds.[8] In turn, the theory of manifolds helps in a deeper penetration into the general theory of relativity.[9–12]

For the solution of other problems of general relativity there is, to put it mildly, a lack of experimental data. For example, one of the most important results of general relativity is the prediction of the existence of gravitational waves, and it long ago became necessary that experiments should be set up to discover gravitational waves under laboratory conditions.[13,14]

1. GROUPS, INVARIANCE, AND THE PRINCIPLE OF RELATIVITY

The main idea of the principle of relativity was formulated by Klein in 1872 in his Erlangen Program.[15] We give below a precise formulation of the concepts associated with it.

Group acting on a set

A group Γ (the concept of a group is assumed known) acts on a set P (the concept of a set is also assumed known) if the following conditions are satisfied.

1. To each element $\gamma\in\Gamma$ of Γ there corresponds a unique mapping $p\to P$: $\tilde{p} = \gamma\cdot p$, $p\in P$, $\tilde{p}\in P$.

2. For any two elements σ and γ of Γ the equation $(\sigma\cdot\gamma)\cdot p = \sigma\cdot(\gamma\cdot p)$ holds.

3. The identity e of the group Γ corresponds to the identity mapping $e\cdot p = p$.

COROLLARY. Besides the mapping $\tilde{p} = \gamma\cdot p$ there exists a unique inverse mapping $p = \gamma^{-1}\cdot\tilde{p}$. Indeed, $p = e\cdot p = (\gamma^{-1}\cdot\gamma)\cdot p\gamma^{-1}\cdot(\gamma\cdot p) = \gamma^{-1}\tilde{p}$. Thus, $\tilde{p} = \gamma\cdot p$ is a one-to-one mapping.

These concepts are discussed in Refs. 16–18.

Theory of relativity

The theory of relativity (Γ,P) is concerned not with all properties of constructs belonging to the set P but only with those that are not changed under mappings $\tilde{p} = \gamma\cdot p$.

Invariant and invariance

Constructs belonging to the set P that do not change under the mappings $\tilde{p}=\gamma\cdot p$ are said to be invariant with respect to the group Γ. They are also called invariants of the group Γ.

If in a theory constructed in some manner one discovers invariance with respect to a group Γ acting on the set P, then this is the theory of relativity (Γ, P).

The term "invariant" was introduced into geometry by Sylvester.[19]

The principle of relativity

The principle of relativity is exhausted by the desire, need, or necessity to consider the theory of relativity (Γ,P).

Example. If we wish to study all properties of the set P, then the final result will be the theory of relativity (E,P), where E is the group consisting of the single element e.

Example. The number of elements of the set P (if P is a finite set) or the cardinality of the set P (if P is an infinite set) is an invariant of the group of all one-to-one mappings $P\to P$.

Remark. If Γ_0 is a subgroup of Γ, then the theory of relativity (Γ_0,P) is richer in content than the theory of relativity (Γ,P), since the invariants of the group Γ are fewer than those of the group Γ_0. It is obvious that the set of invariants of a group is a subset of the set of invariants of a subgroup of it. However, on the transition from the group Γ to its subgroup Γ_0 the heuristic role of the principle of relativity is reduced. In the case $\Gamma_0=E$, the principle of relativity plays no part at all.

Arithmetic space

The most important example of a set P is the arithmetic space A_N. The elements of this space are ordered sets $(x^1,\ldots,x^N)$ of all real numbers. They are called points of the arithmetic space. The symbol N denotes a positive integer. It is called the dimension of the arithmetic space. The geometry of the arithmetic space is the theory of relativity (E,A_N). The reader can find out about this concept in Ref. 20.

The affine group acting on the set A_N

The affine group Φ that acts on the set A_N is represented by the linear inhomogeneous mappings

$$\tilde{x}^a=\sum_{b=1}^{N}\Phi_b^a x^b+\Phi^a,\ a\in\{1,\ldots,N\},\tag{5}$$

with nonvanishing determinant, $\det(\Phi_b^a)\neq 0$. The theory of relativity (Φ,A_N) is called affine geometry, and the set A_N with the group Φ is called the affine space.

Invariant metric in the affine space

An invariant metric

$$\sum_{a=1}^{N}\sum_{b=1}^{N}\theta_{ab}d^a\otimes d^b\tag{6}$$

in the affine space is specified by a matrix (θ_{ab}) of numbers that transform under the mapping (5) in accordance with the tensor rule

$$\tilde{\theta}_{ab}=\sum_{p=1}^{N}\sum_{q=1}^{N}\theta_{pq}\Phi_a^p\Phi_b^q.\tag{7}$$

The condition of invariance $\tilde{\theta}_{ab}=\theta_{ab}$ determines in the group Φ the subgroup $\theta\Phi$ of mappings orthogonal with respect to the metric (6). The metric (6) is invariant in the theory of relativity $(\theta\Phi,A_N)$.

Example. The metric (3) is a special case of the metric (6).

Invariant cometric in the affine space

An invariant cometric

$$\sum_{a=1}^{N}\sum_{b=1}^{N}h^{ab}\partial_a\otimes\partial_b\tag{8}$$

is defined in the affine space by a matrix (h^{ab}) of numbers that transform under the mapping (5) in accordance with the tensor rule

$$\tilde{h}^{ab}=\sum_{p=1}^{N}\sum_{q=1}^{N}h^{pq}\Phi_p^a\Phi_q^b.\tag{9}$$

The condition of invariance $\tilde{h}^{ab}=h^{ab}$ determines in the group Φ its subgroup Φh of mappings orthogonal with respect to the cometric (8). The cometric (8) is invariant in the theory of relativity $(\Phi h,A_N)$.

Example. The cometric (4) is a special case of the cometric (8).

Nondegenerate metric and associated cometric

The metric (6) is said to be nondegenerate if it has determinant $\theta=\det(\theta_{ab})\neq 0$. The condition

$$\sum_{p=1}^{N}\theta_{ap}h^{pb}=-\frac{1}{c^2}\delta_a^b,\tag{10}$$

where δ_a^b is the Kronecker delta symbol and c is a real or imaginary but nonvanishing number, determines the cometric (8) associated with (6).

Example. The cometric (4) is associated with the metric (3).

Nondegenerate cometric and associated metric

The cometric (8) is said to be nondegenerate if it has determinant $h=\det(h^{ab})\neq 0$. The condition (10) determines the metric (6) associated with (8).

Remark. The metric (6) and cometric (8) are nondegenerate and associated with each other if they satisfy the condition (10). In such a case, the conditions of invariance

$$\left.\begin{aligned}\theta_{ab}&=\sum_{p=1}^{N}\sum_{q=1}^{N}\theta_{pq}\Phi_a^p\Phi_b^q;\\ h^{ab}&=\sum_{p=1}^{N}\sum_{q=1}^{N}h^{pq}\Phi_p^a\Phi_q^b\end{aligned}\right\}\tag{11}$$

are equivalent, so that the orthogonal subgroups $\theta\Phi$ and Φh are identical.

Metric which factorizes

The metric (6) is said to factorize if it decomposes into the product

$$\sum_{a=1}^{N}\sum_{b=1}^{N}\theta_{ab}d^a\otimes d^b=\theta\otimes\theta,\tag{12}$$

where

$$\theta=\sum_{a=1}^{N}\theta_a d^a,\tag{13}$$

and the numbers θ_a transform under the mapping (5) in accordance with the tensor rule

$$\widetilde{\theta}_a = \sum_{p=1}^{N} \theta_p \Phi_a^p. \quad (14)$$

Degenerate cometric of maximal rank

If the rank of the matrix (h^{ab}) is $N-1$, then (8) is called a degenerate cometric of maximal rank.

The Galileo group

The Galileo group is a subgroup of the affine group which is orthogonal to the metric (12) (which factorizes) and with respect to a degenerate cometric (8) of maximal rank. In addition, it is assumed that the metric (12) and cometric (8) are connected by the condition

$$\sum_{p=1}^{N} \theta_{ap} h^{pb} = 0, \quad (15)$$

that the quadratic form

$$\sum_{a=1}^{N} \sum_{b=1}^{N} h^{ab} p_a p_b \quad (16)$$

does not take negative values, and that $N = 4$.

The old special theory of relativity is the theory of relativity (Galileo group, A_4).

Usually, one sets

$$x^1 = x,\ x^2 = y,\ x^3 = z,\ x^4 = t, \quad (17)$$

where x, y, z are Cartesian coordinates, and t is the time in one of the inertial frames of reference. Then

$$(\theta_{ab}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (h^{ab}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (18)$$

$$(\theta_a) = (0\ 0\ 0\ 1),$$

so that

$$\sum_{p=1}^{N} \sum_{q=1}^{N} \theta_{ab} d^a \otimes d^b = dt \otimes dt, \quad (19)$$

$$\sum_{p=1}^{N} \sum_{q=1}^{N} h^{ab} \partial_a \otimes \partial_b = \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial z}, \quad (20)$$

$$\theta = \sum_{a=1}^{N} \theta_a d^a = dt. \quad (21)$$

In such a case, the Galileo group is formed by the mappings

$$\left.\begin{aligned} \widetilde{x}^a &= \sum_{b=1}^{3} \Gamma_b^a x^b + V^a t + \Gamma^a,\ a \in \{1,\ 2,\ 3\}, \\ \widetilde{t} &= \pm t + \Gamma^4, \end{aligned}\right\} \quad (22)$$

where Γ^a, Γ^4, and V^a are arbitrary constants, and Γ_b^a are constants that satisfy the conditions of orthogonality

$$\sum_{m=1}^{3} \sum_{n=1}^{3} h^{mn} \Gamma_m^a \Gamma_n^b = h^{ab}, \quad a,\ b \in \{1,\ 2,\ 3\}. \quad (23)$$

Equations (2) are invariant with respect to the mappings (22).

Orthochronous Galileo group

The orthochronous Galileo group is obtained by replacing in (22) the expression $\widetilde{t} = \pm t + \Gamma^4$ by $\widetilde{t} = t + \Gamma^4$. The linear metric (22) is invariant with respect to this group.

The Poincaré group

The Poincaré group is an orthogonal group with respect to the metric (3) or, equivalently, orthogonal with respect to the cometric (4), where c is the velocity of light.

The new special theory of relativity is the theory of relativity (Poincaré group, A_4). Equations (1) are invariant with respect to the Poincaré group.

A theory invariant with respect to the Poincaré group is said to be relativistic, and a theory invariant with respect to the Galileo group is said to be nonrelativistic. This is not logical, because we have a relativity principle in both cases. The word "nonrelativistic" must here be taken in quotation marks. We shall proceed in this manner always. For example, the metric (3) and the cometric (4) in the "nonrelativistic" limit $c \to \infty$ go over into (19) and (20).

Orthochronous Poincaré group

The orthochronous Poincaré group is characterized by the condition $\Phi_4^4 > 0$ in the representation (3).

The Lorentz group

The Lorentz group is the subgroup of the Poincaré group determined by the condition $\Phi^a = 0$ in the affine mapping (5).

Inertial motion of a particle

Inertial motion of a particle is represented in the four-dimensional affine space by the straight line determined by the equations

$$x^a = p^a \tau + x_0^a,\ a \in \{1,\ 2,\ 3,\ 4\}, \quad (24)$$

where τ is a parameter which takes all possible real values, and p^a and x_0^a are given.

The square of the rest mass of the particle is

$$m^2 = \sum_{a=1}^{4} \sum_{b=1}^{4} \theta_{ab} p^a p^b. \quad (25)$$

In the new theory of relativity, $m^2 = 0$ for a light ray and $m^2 < 0$ for a tachyon. In the old theory of relativity, $m^2 \geqslant 0$ and particles with $m = 0$ transmit interaction instantaneously between particles with $m > 0$.

Inertial system

An inertial system consists of particles with positive rest mass. All the particles in it move inertially with the same velocity. Thus, an inertial system is represented by a set of mutually parallel straight lines.

Kotel'nikov space or the space of inertial systems

A point of Kotel'nikov space is an inertial system. Kotel'nikov space is three-dimensional. In the old theory of relativity, Euclidean geometry holds in this space; in the new theory of relativity, we have Lobachevskiĭ's geometry. The distance s between two inertial systems, called the rapidity, is equal in the old theory of relativity to their relative velocity v, while in the new theory of relativity it is related to the relative velocity v by the formula $v/c = \tanh s/c$, where c is the velocity of light.

The transition from Euclidean geometry to Lobachevskiĭ's geometry in Kotel'nikov space amounts to the transition from the old to the new theory of relativity.

Thus, the creation of the new theory of relativity was predetermined by the creation of Lobachevskiĭ's geometry.

Lobachevskiĭ's geometry is becoming a tool that is ever more widely used by physicists studying high and superhigh energies.[21,22]

The group of isometries of Kotel'nikov space

The group of isometric mappings of the space of inertial systems is the homogeneous orthochronous Galileo group in the old theory of relativity and the orthochronous Lorentz group in the new theory. It is the group of isometries of Euclidean space in the first case and Lobachevskiĭ space in the second.

Note. This section has been set out after the manner of Ref. 23.

2. KINEMATIC PRINCIPLE OF RELATIVITY

The mappings $A_4 \to A_4$ with respect to which the metric (3) or, equivalently, the cometric (4), is invariant are necessarily linear. Such mappings form the Poincaré group. Since in the limit $c \to \infty$ the Poincaré group goes over into the Galileo group, one could suppose that the mappings $A_4 \to A_4$ with respect to which the metric (19) and the cometric (20) are invariant must also necessarily be linear. However, this is not correct. Such mappings form the Newton group,[24] and not the Galileo group. They have the form

$$\left.\begin{aligned} \tilde{x}^a &= \sum_{b=1}^{3} H_b^a(t)\, x^b + H^a(t), \quad a \in \{1, 2, 3\}, \\ \tilde{t} &= \pm t + H^4, \end{aligned}\right\} \tag{26}$$

where $H^4 = \Gamma^4$ is an arbitrary constant, $H^a(t)$ are arbitrary functions of the time t, and $H_b^a(t)$ are functions that at each instant of time t satisfy the conditions of orthogonality

$$\sum_{m=1}^{3} \sum_{n=1}^{3} h^{mn} H_m^a(t)\, H_n^b(t) = h^{ab}, \quad a,\ b \in \{1, 2, 3\}, \tag{27}$$

but are otherwise arbitrary.[1)]

The kinematic principle of relativity consists of the desire to study the invariants of the Newton group. The kinematic theory of relativity is the theory of the invariants of the Newton group.

The Galileo group is a subgroup of the Newton group. Situated between them is the group of mappings

$$\left.\begin{aligned} \tilde{x}^a &= \sum_{b=1}^{3} \Gamma_b^a x^b + H^a(t), \quad a \in \{1, 2, 3\}, \\ \tilde{t} &= \pm\, t + \Gamma^4 \end{aligned}\right\} \tag{28}$$

$A_4 \to A_4$. We call it the Galileo–Newton group.

The orthochronous mappings (26), (28), and (22) are different forms of motion of a rigid body in the old special theory of relativity.

In the new special theory of relativity, uniform and rectilinear motion of a rigid body without rotation is represented by an orthochronous Poincaré mapping. A rigid body cannot move differently in this theory, in which there is no place for the principle of kinematic relativity.

The Galileo–Newton group contains the subgroup of mappings of the form

$$\tilde{x} = x + H(t), \quad \tilde{t} = t, \tag{29}$$

and this explains the situation described in the popular literature by the image of a falling elevator. The relative motion of a closed system of material points is not changed if one applies to it forces $F_i = -m_i h(t)$, where i is the number of the point and m_i is its mass. The mapping (29), where $H(t)$ is the antiderivative of the function $h(t)$, eliminates the effect of such forces.

3. INTRODUCTION OF LOBACHEVSKIĬ'S GEOMETRY INTO MECHANICS

Lobachevskiĭ left a legacy: "It would remain to investigate the change that would result from the introduction of the conceived Geometry into Mechanics..." (Ref. 25, p. 49). He believed that the Conceived Geometry was admissible "either beyond the bounds of the visible world or in the close sphere of molecular attraction" (Ref. 26, p. 65).

In accordance with quantum mechanics, short distances mean large rapidities, so that allowance for Lobachevskiĭ's geometry in the close sphere of molecular attractions can be seen as the replacement of Euclidean geometry in Kotel'nikov space of inertial systems by Lobachevskiĭ's geometry. As we have seen, this replacement amounts to the transition from the old to the new special theory of relativity. And allowance for Lobachevskiĭ's geometry beyond the bounds of the visible world can be seen as the replacement of the Euclidean geometry in Newton's absolute space by Lobachevskiĭ's geometry. Under such a replacement, the principle of kinematic invariance is not in essence violated, since Lobachevskiĭ's space possesses the same mobility as Euclidean space. But the special principle of relativity loses its force. This is clear: If (in the best case) one of the points of a relative space moves uniformly along some straight line L and this space is not rotated around the axis L, then every point that does not lie on L moves uniformly along an equidistant path, an "equidistant," with base L. Since in Lobachevskiĭ's geometry an equidistant is not a straight line, every point that does not lie on L moves with acceleration. Therefore, the relative space is distinguished dynamically from the absolute space.

Newton has been criticized more than once for having introduced absolute space into his mechanics. But who knows whether Newton allowed the possibility of a non-Euclidean solution to the problem of parallels?

In this connection, it is interesting to note that the metric[2)]

$$\mathrm{ch}^2 \frac{r}{k}\, dt^2 - \frac{1}{c^2}\left[dr^2 + k^2\, \mathrm{sh}^2 \frac{r}{k}\, (d\theta^2 + \sin^2\theta\, d\varphi^2)\right], \tag{30}$$

which can be realized locally on the "sphere" $x_1^2 + x_2^2 - y_1^2 - y_2^2 - y_3^2 = k^2$ if the coordinates

$$x_1 = k\, \mathrm{ch} \frac{r}{k} \cos \frac{ct}{k}; \quad y_1 = k\, \mathrm{sh} \frac{r}{k} \sin\theta \cos\varphi;$$

$$x_2 = k\, \mathrm{ch} \frac{r}{k} \sin \frac{ct}{k}; \quad y_2 = k\, \mathrm{sh} \frac{r}{k} \sin\theta \sin\varphi;$$

$$y_3 = k\, \mathrm{sh} \frac{r}{k} \cos\theta$$

are chosen, introduces Lobachevskiĭ's geometry both into the base space–time manifold at the instant of time t and into the fiber—the space of velocities of a material point.

We have here approached the very important concept of the space of states of a material point. It is a seven-dimensional pencil B of timelike straight lines tangent to the four-dimensional world X. More precisely, B is the skew product $B = \bigcup_{x\in X} V(x)$ with base X, with fiber $V(x)$ that is the three-dimensional Lobachevskiĭ's velocity space, and with product group that is the orthochronous Lorentz group (Ref. 27, p. 89). This concept was of great assistance to us in the construction of the kinetic theory of gases in general relativity. The single-particle distribution function, for which the Boltzmann equation is derived, is a scalar function in the pencil B. The "relativistic" collision integral is calculated along the fiber $V(x)$.

4. THREE THEORIES OF GRAVITATION

We know three canonical theories of gravitation. One is based on Kepler's laws and Poisson's equations, and another is based on Newton's laws. The third theory is the general theory of relativity. It is constructed on the basis of Riemannian geometry in the space–time world and Lobachevskiĭ geometry in the velocity space.

First canonical theory of gravitation

The laws of motion of the planets were deduced by Kepler from the observations of Tycho Brahe.[28] These laws are as follows (Ref. 29, p. 89):

1. Each planet moves in an ellipse having the Sun at one of the foci.

2. The sectorial velocity of each planet with respect to the Sun is constant.

3. The ratio of the squares of the periods of revolution of the planets to the cubes of the semimajor axes of their orbits is the same for all planets.

Newton coded these laws in the form of the differential equation

$$\frac{d^2\mathbf{r}}{dt^2} = -\gamma \frac{M\mathbf{r}}{r^3}, \tag{31}$$

where M is the mass of the Sun, and γ is Newton's constant.

The motion of a test body in the gravitational field $\varphi(\mathbf{r},t)$ of the remaining masses placed in absolute space with density $\rho(\mathbf{r}, t)$ is determined by Newton's equation

$$d^2\mathbf{r}/dt^2 = -\nabla\varphi \tag{32}$$

and Poisson's equation

$$\Delta\varphi = 4\pi\gamma\rho. \tag{33}$$

If a test body is attracted by a single point body with mass M placed at the origin, then

$$\rho = M\delta(\mathbf{r}). \tag{34}$$

In such a case, it follows from Eq. (33) that

$$\varphi = -\gamma M/r. \tag{35}$$

Substituting (35) in (32), we obtain (31).

We emphasize that the first canonical theory of gravitation is valid only for test bodies and that the mass of the test body does not occur in its equations. It would be stupid to apply this theory to the motion of double stars.

Second canonical theory of gravitation

On the basis of Kepler's laws, Newton established that the gravitational potential is

$$U(r) = -\gamma \frac{m_1 m_2}{r}. \tag{36}$$

The system of equations (2) with such a potential, i.e., the system of equations

$$\frac{d^2\mathbf{r}_1}{dt^2} = \gamma \frac{m_2}{r^3}(\mathbf{r}_2 - \mathbf{r}_1); \quad \frac{d^2\mathbf{r}_2}{dt^2} = \gamma \frac{m_1}{r^3}(\mathbf{r}_1 - \mathbf{r}_2), \tag{37}$$

is called Newton's law of universal gravitation.

The transition from Eq. (31) to (37) is not continuous. For the difference $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ there follows from (37) the equation (31) with $M = m_1 + m_2$, whereas according to Kepler one should have $M = m_1$; moreover, $m_2 > 0$. By virtue of this, Kepler's third law is true only approximately.[28,29] This is due to the fact that the gravitational potential is proportional to the mass product $m_1 m_2$. Kepler's law is valid to a high degree of accuracy because the planets are test bodies relative to the Sun.

Third canonical theory of gravitation

Through the work of Lobachevskiĭ, who discovered non-Euclidean geometry, Gauss, who created the intrinsic geometry of surfaces, and Riemann, who generalized Gauss's results to the multidimensional case, the third canonical theory of gravitation became possible. We consider it to the extent that it is analogous to the first theory. A "relativistic" analog of the second theory of gravitation is unknown.

In the third theory, one introduces in the space–time world M a Riemannian metric

$$ds^2 = \sum_{a=1}^{4} \sum_{b=1}^{4} g_{ab}(x)\, d^a x \otimes dx^b$$

of normal hyperbolic type. This means that at each point $x \in M$ in the pencil of straight lines tangent to M Lobachevskiĭ's geometry with characteristic constant equal to the velocity of light c holds.

Equation (32) is replaced by the geodesic equation

$$\frac{d^2x^a}{d\tau^2} + \left\{ {a \atop mn} \right\} \frac{dx^m}{d\tau} \frac{dx^n}{d\tau} = 0, \tag{38}$$

where

$$\left\{ {a \atop mn} \right\} = \frac{1}{2} g^{ak} (\partial_m g_{kn} + \partial_n g_{km} - \partial_k g_{mn}) \tag{39}$$

are the Christoffel symbols. Poisson's equation (33) is replaced by the equation

$$R_{ab} - \frac{1}{2} R g_{ab} = \frac{8\pi\gamma}{c^4} T_{ab}, \tag{40}$$

where R_{ab} is the Ricci tensor, and T_{ab} is the matter energy–momentum tensor.

The solution of Eq. (40) in the case analogous to (34) was found by Schwarzschild:

$$ds^2 = c^2V^2\, dt^2 - V^{-2}\, d\rho^2 - \rho^2\, (d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{41}$$

where

$$V^2 = 1 - 2\alpha/\rho; \quad \alpha = \gamma M/c^2; \quad \rho > 2\alpha.$$

5. NECESSARY INFORMATION ABOUT AFFINE CONNECTIONS

We give here the information about affine connections that is needed to understand the energy–momentum pseudotensor. Additional information can be found in Refs. 20 and

30. All geometrical objects in this paper relate to the coordinate basis $d^a = dx^a$ and the dual basis $\partial_a = \partial/\partial x^a$.

Affine connection

By a tensor field T of type $\binom{A}{B}$ we mean a tensor field with components $T^{a_1\cdots a_A}_{b_1\cdots b_B}$.

An affine connection Γ is introduced on a manifold in order to ensure that the covariant derivative ∇T of any tensor field T of type $\binom{A}{B}$ is a tensor field of type $\binom{A}{B+1}$.

For a covector field, one sets

$$\nabla_m T_n = \partial_m T_n - \Gamma^a_{mn} T_a. \tag{42}$$

It follows from this that the components Γ^a_{mn} of the connection on the transition from one coordinate system K to another system $\widetilde{K}$ transform in accordance with the rule

$$\widetilde{\Gamma}^a_{mn} = \left(\Gamma^i_{pq}\frac{\partial x^p}{\partial \widetilde{x}^m}\frac{\partial x^q}{\partial \widetilde{x}^n} + \frac{\partial^2 x^i}{\partial \widetilde{x}^m\,\partial \widetilde{x}^n}\right)\frac{\partial \widetilde{x}^a}{\partial x^i}. \tag{43}$$

It follows from (43) that for any vector field T the combinations

$$\nabla_m T^a = \partial_m T^a + \Gamma^a_{mn} T^n \tag{44}$$

are the components of a tensor field ΔT of type $\binom{1}{1}$.

The covariant derivative of any tensor field T of type $\binom{A}{B}$ is formed in accordance with the rule for differentiating a product. For example,

$$\left.\begin{aligned}
\nabla_k T_{mn} &= \partial_k T_{mn} - \Gamma^s_{km} T_{sn} - \Gamma^s_{kn} T_{ms};\\
\nabla_k T^a_m &= \partial_k T^a_m - \Gamma^s_{km} T^a_s + \Gamma^a_{ks} T^s_m;\\
\nabla_k T^{ab} &= \partial_k T^{ab} + \Gamma^a_{ks} T^{sb} + \Gamma^b_{ks} T^{as};\\
\nabla_k T^a_{mn} &= \partial_k T^a_{mn} - \Gamma^s_{km} T^a_{sn} -\\
&\quad - \Gamma^s_{kn} T^a_{ms} + \Gamma^a_{ks} T^s_{mn}.
\end{aligned}\right\} \tag{45}$$

For a scalar function T, one sets $\nabla_k T = \partial_k T$.

As follows from (43), the difference

$$S^a_{mn} = \Gamma^a_{mn} - \Gamma^a_{nm} \tag{46}$$

is a tensor. This tensor is called the torsion tensor. Sometimes half the difference (46) is called the torsion tensor.

We consider the operator

$$\nabla_{kl} = \nabla_k \nabla_l - \nabla_l \nabla_k + S^p_{kl} \nabla_p. \tag{47}$$

Applied to a scalar function T, it gives zero: $\nabla_{kl} T = 0$. Applied to a covector field T_n, it gives the result

$$\nabla_{kl} T_n = -R^a_{kln} T_a, \tag{48}$$

where

$$R^a_{kln} = \partial_k \Gamma^a_{ln} - \partial_l \Gamma^a_{kn} + \sum_{s=1}^{N} (\Gamma^a_{ks}\Gamma^s_{ln} - \Gamma^a_{ls}\Gamma^s_{kn}). \tag{49}$$

It follows from (48) that the combination (49) is a tensor. This tensor is called the curvature tensor or the Riemann–Christoffel tensor.

The contraction

$$R_{ln} = R^a_{aln} \tag{50}$$

is called the Ricci tensor.

Applied to the vector field T^a, the operator (47) gives

$$\nabla_{kl} T^a = R^a_{kln} T^n. \tag{51}$$

It is easy to see the analogy between Eqs. (42) and (44) and Eqs. (48) and (51). It extends to all tensor fields. For example,

$$\left.\begin{aligned}
\nabla_{kl} T_{mn} &= -R^s_{klm} T_{sn} - R^s_{kln} T_{ms};\\
\nabla_{kl} T^a_m &= -R^s_{klm} T^a_s + R^a_{kls} T^s_m;\\
\nabla_{kl} T^{ab} &= R^a_{kls} T^{sb} + R^b_{kls} T^{as};\\
\nabla_{kl} T^a_{mn} &= -R^s_{klm} T^a_{sn} - R^s_{kln} T^a_{ms} + R^a_{kls} T^s_{mn}.
\end{aligned}\right\} \tag{52}$$

Equations (52) can be obtained from Eqs. (45) by replacing $\nabla_k - \partial_k$ by ∇_{kl} and Γ^a_{km} by $R^{\ a}_{klm}$.

Finally, we note the obvious identity

$$R^a_{kln} + R^a_{lkn} = 0. \tag{53}$$

Contracted connection

The contracted (affine) connection is defined by the contraction

$$\Gamma_m = \Gamma^a_{ma}. \tag{54}$$

In accordance with (43), its components transform as follows:

$$\widetilde{\Gamma}_m = \frac{\partial x_p}{\partial \widetilde{x}^m}\,\Gamma_p + \frac{\partial}{\partial x^i}\,\frac{\partial x^i}{\partial \widetilde{x}^m}.$$

This formula can be reduced to the form

$$\widetilde{\Gamma}_m = \frac{\partial x^p}{\partial \widetilde{x}^m}\left(\Gamma_p + \frac{\partial}{\partial x^p}\ln J\right), \tag{55}$$

where J is the Jacobian of the transformation

$$J = \frac{\partial(x^1, \ldots, x^N)}{\partial(\widetilde{x}^1, \ldots, \widetilde{x}^N)}, \tag{56}$$

since the differential of the determinant φ of any matrix $(\varphi^{\ i}_k)$ is $d\varphi = \overline{\varphi}^{\ k}_i d\varphi^{\ i}_k$, where $\overline{\varphi}^{\ k}_i$ is the cofactor of the element $\varphi^{\ i}_k$.

The covariant derivative of a tensor of type $\binom{0}{N}$ is

$$\nabla_m \varepsilon_{k_1\ldots k_N} = \partial_m \varepsilon_{k_1\ldots k_N} - \Gamma^a_{mk_1}\varepsilon_{ak_2\ldots k_N} - \ldots - \Gamma^a_{mk_N}\varepsilon_{k_1\ldots a}.$$

If this tensor is antisymmetric with respect to any pair of symbols $k_1,\ldots,k_N$, then the combination

$$\Gamma^b_{ma}\varepsilon_{k_1\ldots k_N} - \Gamma^b_{mk_1}\varepsilon_{ak_2\ldots k_N} - \ \ldots - \Gamma^b_{mk_N}\varepsilon_{k_1\ldots k_{N-1}a}$$

is antisymmetric with respect to any pair of symbols a, $k_1,\ldots,k_N$. Therefore, it is equal to zero, and

$$\nabla_m \varepsilon_{k_1\ldots k_N} = (\partial_m - \Gamma_m)\,\varepsilon_{k_1\ldots k_N}. \tag{57}$$

It can be shown similarly that

$$\nabla_{mn}\varepsilon_{k_1\ldots k_N} = -R^a_{mna}\varepsilon_{k_1\ldots k_N}. \tag{58}$$

From (49), we find that

$$\Omega_{mn} = R^a_{mna} = \partial_m \Gamma_n - \partial_n \Gamma_m. \tag{59}$$

We call this tensor the curvature of the contracted connection.

On an orientable manifold there exists a nowhere vanishing N form. Choosing an atlas such that the Jacobians (56) are greater than zero, we can set $\varepsilon = \varepsilon_{1\ldots N} > 0$ for all maps forming the atlas. Indeed, if $\varepsilon > 0$, then also $\widetilde{\varepsilon} = \widetilde{\varepsilon}_{1\ldots N} = J\varepsilon > 0$. Assuming that the covariant derivative of the tensor $\varepsilon_{k_1\cdots k_N}$ is zero, we find from (57)

$$\Gamma_m = \partial_m \ln \varepsilon. \quad (60)$$

In such a case,

$$\Omega_{mn} = 0. \quad (61)$$

Essentially, we have been dealing with "extended" differentiation and an "extended" derivative.

Transformation of connections

Suppose now that on a manifold we are given two connections Γ and $\breve{\Gamma}$ with coefficients Γ^a_{mn} and $\breve{\Gamma}^a_{mn}$, respectively. The transition from one of them to the other is called a connection transformation.[30] It follows from (43) that the difference

$$P^a_{mn} = \breve{\Gamma}^a_{mn} - \Gamma^a_{mn} \quad (62)$$

is a tensor. This tensor is called the affine deformation tensor. Substituting the expression $\breve{\Gamma}^a_{mn}$ in (49), we obtain the law of variation of the curvature tensor under the connection transformation:

$$\breve{R}^a_{kln} = R^a_{kln} + S^m_{kl} P^a_{mn} + \nabla_k P^a_{ln} - \nabla_l P^a_{kn} + \sum_{s=1}^{N} (P^a_{ks} P^s_{ln} - P^a_{ls} P^s_{kn}). \quad (63)$$

The difference

$$P_m = \breve{\Gamma}_m - \Gamma_m = P^a_{ma} \quad (64)$$

is the covector of the affine deformation. From (59), we obtain the law of variation of the curvature tensor of the contracted connection:

$$\breve{\Omega}_{kl} = \Omega_{kl} + \partial_k P_l - \partial_l P_k, \quad (65)$$

and this agrees with (63).

Symmetric connection. Equiaffine connection

In what follows, we shall consider only symmetric connections, i.e., we shall set

$$\Gamma^a_{mn} = \Gamma^a_{nm}, \quad (66)$$

so that the torsion tensor will be zero.

In this case,

$$R^a_{kln} + R^a_{nkl} + R^a_{lnk} = 0. \quad (67)$$

As a result of contraction, we then find

$$\Omega_{kl} + R_{kl} - R_{lk} = 0. \quad (68)$$

The symmetric connection whose contraction is equal to (60) is called the equiaffine connection. In this case, the Ricci tensor is symmetric.

Besides the algebraic identities (53) and (67), there exists one further very important differential identity,

$$\nabla_i R^a_{kln} + \nabla_l R^a_{ikn} + \nabla_k R^a_{lin} = 0, \quad (69)$$

which is called the Bianchi–Padova identity (Ref. 30, p. 131).

For symmetric connections, the affine deformation tensor is also symmetric, i.e.,

$$P^a_{mn} = P^a_{nm}. \quad (70)$$

For symmetric connections, we write the law (63) in the two equivalent forms

$$\breve{R}^a_{kln} = R^a_{kln} + \nabla_k P^a_{ln} - \nabla_l P^a_{kn} + \sum_{s=1}^{N} (P^a_{ks} P^s_{ln} - P^a_{ls} P^s_{kn}); \quad (71)$$

$$R^a_{kln} = \breve{R}^a_{kln} + \breve{\nabla}_l P^a_{kn} - \breve{\nabla}_k P^a_{ln} + \sum_{s=1}^{N} (P^a_{ks} P^s_{ln} - P^a_{ls} P^s_{kn}). \quad (72)$$

For the equiaffine connections, we obtain the consequences

$$\breve{R}_{ln} = R_{ln} + \nabla_a P^a_{ln} - \nabla_l P_n + P_s P^s_{ln} - P^a_{ls} P^s_{an}; \quad (73)$$

$$R_{ln} = \breve{R}_{ln} + \breve{\nabla}_l P_n - \breve{\nabla}_a P^a_{ln} + P_s P^s_{ln} - P^a_{ls} P^s_{an}, \quad (74)$$

where

$$P_n = \partial_n \ln \frac{\breve{\varepsilon}}{\varepsilon}. \quad (75)$$

Christoffel connection

On the manifold we define a nondegenerate symmetric tensor g_{mn} and require that its covariant derivative with respect to the symmetric connection Γ^a_{mn} be zero, i.e., we set

$$\left. \begin{aligned} & g = \det(g_{mn}) \neq 0, \quad g_{mn} = g_{nm}, \\ & S^a_{mn} = 0, \quad \nabla_a g_{mn} = 0. \end{aligned} \right\} \quad (76)$$

It is easy to show that in such a case the connection Γ^a_{mn} is equal to the Christoffel symbol (39), where $g^{ma} g_{an} = \delta^m_n$.

From the equation $\nabla_a g_{mn} = 0$ there follows $\nabla_a g^{mn}$, and, hence, the operations of lowering and raising the tensor indices by means of g_{mn} and g^{mn} commute with the operation of covariant differentiation with respect to the Christoffel connection (39).

The connection contraction (39) is

$$\Gamma_m = \left\{ {a \atop ma} \right\} = \frac{1}{2} g^{ab} \partial_m g_{ab} = \frac{1}{2g} \partial_m g, \quad (77)$$

i.e., can be represented in the form (60), where

$$\varepsilon = \sqrt{|g|}. \quad (78)$$

Therefore, the Christoffel connection (39) is equiaffine.

Setting $T_{mn} = g_{mn}$ in the first equation of (52), we find that in the case of the Christoffel connection

$$R^s_{klm} g_{sn} + R^s_{kln} g_{sm} = 0. \quad (79)$$

In accordance with (53), (79), and (67), the tensor

$$R_{klmn} = R^a_{klm} g_{an} \quad (80)$$

is such that

$$R_{klmn} + R_{lkmn} = 0, \quad R_{klmn} + R_{klnm} = 0, \quad (81)$$

$$R_{klmn} + R_{mkln} + R_{lmkn} = 0. \quad (82)$$

We shall show that from Eqs. (81) and (82) there follows

$$R_{klmn} = R_{mnkl}. \quad (83)$$

For this, we make in (82) all cyclic permutations of the indices:

$$R_{klmn} + R_{mkln} + R_{lmkn} = 0;$$
$$R_{mnkl} + R_{kmnl} + R_{nkml} = 0;$$
$$R_{nklm} + R_{lnkm} + R_{klnm} = 0;$$
$$R_{lmnk} + R_{nlmk} + R_{mnlk} = 0.$$

Subtracting from the upper equations the lower equations, we obtain

$$R_{hlmn}-R_{hlnm}=R_{nhlm}+R_{lnhm}-R_{mhln}-R_{lmhn};$$
$$R_{mnhl}-R_{mnlh}=R_{lmnh}+R_{nlmh}-R_{hmnl}-R_{nhml}.$$

Hence and from (81) we obtain (83).

In the case of the Christoffel connection, we consider besides the Ricci tensor (50) the curvature scalar

$$R=g^{mn}R_{mn} \tag{84}$$

and the Einstein tensor

$$G_{ab}=R_{ab}-\frac{1}{2}Rg_{ab}, \tag{85}$$

which occurs on the left-hand side of (40).

Contracting the Bianchi–Padova identities (69) with the tensor $\delta_a^k g^{in}$, we find

$$2\nabla^n R_{nl}=\nabla_l R. \tag{86}$$

Therefore, the covariant divergence of the Einstein tensor (85) is zero.

Coordinate connection

We call the connection for which all the components vanish in a certain coordinate map $\widetilde{K}$ the coordinate connection generated by the map $\widetilde{K}$. Any coordinate map related linearly to $\widetilde{K}$ generates the same connection. According to (43), in a map related arbitrarily to $\widetilde{K}$ the coefficients of the coordinate connection are

$$\Gamma^a_{mn}=\sum_{i=1}^{N}\frac{\partial x^a}{\partial \widetilde{x}^i}\frac{\partial^2\widetilde{x}^i}{\partial x^m\,\partial x^n}. \tag{87}$$

The torsion tensor and the curvature tensor of the coordinate connection (87) vanish:

$$S^a_{mn}=0,\quad R^a_{hln}=0.$$

The contraction of the coordinate connection (87) is

$$\Gamma_m=\Gamma^a_{ma}=\frac{\partial}{\partial x^m}\ln\frac{\partial(\widetilde{x}^1,\ldots,\widetilde{x}^N)}{\partial(x^1,\ldots,x^N)},$$

so that the coordinate connection can be assumed to be equiaffine if we set $\varepsilon=\varepsilon_0 J^{-1}$, where $\varepsilon_0=\text{const}$, and J has the form (56).

If in Eqs. (71)–(74) we assume that Γ^a_{mn} is the coordinate connection, then they take the form

$$R^a_{hln}=-\nabla_h P^a_{ln}+\nabla_l P^a_{hn}-\sum_{s=1}^{N}(P^a_{hs}P^s_{ln}-P^a_{ls}P^s_{hn}); \tag{88}$$

$$\check{R}^a_{hln}=-\check{\nabla}_h P^a_{ln}+\check{\nabla}_l P^a_{hn}+\sum_{s=1}^{N}(P^a_{hs}P^s_{ln}-P^a_{ls}P^s_{hn}); \tag{89}$$

$$R_{ln}=-\nabla_a P^a_{ln}+\nabla_l P_n-(P_sP^s_{ln}-P^a_{ls}P^s_{an}); \tag{90}$$

$$\check{R}_{ln}=-\check{\nabla}_a P^a_{ln}+\check{\nabla}_l P_n+(P_sP^s_{ln}-P^a_{ls}P^s_{an}). \tag{91}$$

If both connections are coordinate connections, then the affine deformation tensor satisfies the following condition, which we express in two equivalent forms:

$$\nabla_h P^a_{ln}-\nabla_l P^a_{hn}=-\sum_{s=1}^{N}(P^a_{hs}P^s_{ln}-P^a_{ls}P^s_{hn});$$

$$\check{\nabla}_h P^a_{ln}-\check{\nabla}_l P^a_{hn}=+\sum_{s=1}^{N}(P^a_{hs}P^s_{ln}-P^a_{ls}P^s_{hn}).$$

If desired, a coordinate connection can be represented in the form of the Christoffel symbol (39) and is even nonunique. Indeed, take any nonsingular symmetric matrix (M_{ab}), where M_{ab} are numbers and not functions, and in the map K set

$$g_{ab}=\sum_{p=1}^{N}\sum_{q=1}^{N}M_{pq}\frac{\partial\widetilde{x}^p}{\partial x^a}\frac{\partial\widetilde{x}^q}{\partial x^b}. \tag{92}$$

For such a metric tensor

$$\left\{{a \atop mn}\right\}=\sum_{i=1}^{N}\frac{\partial x^a}{\partial\widetilde{x}^i}\frac{\partial^2\widetilde{x}^i}{\partial x^m\,\partial x^n}, \tag{93}$$

as one can readily show. Generally speaking, for any (not necessarily nonsingular) symmetric matrix of numbers (M_{ab}) the covariant derivative of the tensor (92) with respect to the connection (87) vanishes.

Special form of the affine deformation tensor

We have

$$\left.\begin{aligned}\check{\nabla}_h g^{ab}&=\partial_h g^{ab}+\check{\Gamma}^a_{hs}g^{sb}+\check{\Gamma}^b_{hs}g^{as};\\ \partial_h g^{ab}&=-\left\{{a \atop hs}\right\}g^{sb}-\left\{{b \atop hs}\right\}g^{as}.\end{aligned}\right\} \tag{94}$$

Therefore, if in the difference the subtrahend is equal to (39), then

$$\check{\nabla}_h g^{ab}=P^a_{hs}g^{sb}+P^b_{hs}g^{as}. \tag{95}$$

Similarly,

$$\left.\begin{aligned}\check{\nabla}_h g_{mn}&=\partial_h g_{mn}-\check{\Gamma}^s_{hm}g_{sn}-\check{\Gamma}^s_{hn}g_{ms};\\ \partial_h g_{mn}&=\left\{{s \atop hm}\right\}g_{sn}+\left\{{s \atop hn}\right\}g_{ms}.\end{aligned}\right\} \tag{96}$$

Therefore,

$$\check{\nabla}_h g_{mn}=-P^s_{hm}g_{sn}-P^s_{hn}g_{ms}. \tag{97}$$

Since we have agreed to consider only symmetric connections,

$$P^a_{mn}=-\frac{1}{2}g^{ah}(\check{\nabla}_m g_{hn}+\check{\nabla}_n g_{hm}-\check{\nabla}_h g_{mn}). \tag{98}$$

We define the vector

$$\Phi^a=g^{mn}P^a_{mn}. \tag{99}$$

From (95) we obtain

$$\Phi^a=(\check{\nabla}_b-P_b)g^{ab}. \tag{100}$$

If the connection $\check{\Gamma}^a_{mn}$ is equiaffine, then

$$\Phi^a=\frac{\check{\varepsilon}}{\varepsilon}\check{\nabla}_b\left(\frac{\varepsilon}{\check{\varepsilon}}g^{ab}\right). \tag{101}$$

In what follows, we shall assume that the connection Γ^a_{mn} has the form (39), and the connection $\check{\Gamma}^a_{mn}$ has the form (87). The expression $\check{\varepsilon}^{-1}=J$ will have the form (56).

Transformation of the curvature scalar

It follows from (90) that

$$R=g^{mn}(P^a_{mb}P^b_{an}-P_sP^s_{mn})+\nabla_a\Omega^a, \tag{102}$$

where

$$\Omega^a=P_mg^{ma}-P^a_{mn}g^{mn}. \tag{103}$$

The Hilbert functional and its variation

The Hilbert functional of the metric tensor g^{mn} is

$$H = \int R\varepsilon\, dx^1 \ldots dx^N. \tag{104}$$

The law (73) of variation of the Ricci tensor immediately gives the variation δH; for, contracting (73) with the tensor $\breve{g}^{ln} = g^{ln} + \delta g^{ln}$ and ignoring the quadratic corrections, we find

$$\delta R = R_{ab}\delta g^{ab} - \nabla_a \omega^a, \tag{105}$$

where the vector ω^a is analogous to the vector (103):

$$\omega^a = g^{ma}\delta\Gamma^n_{mn} - g^{mn}\delta\Gamma^a_{mn}. \tag{106}$$

It follows from (78) and (77) that

$$\delta\varepsilon = -\frac{1}{2}\varepsilon g_{ab}\delta g^{ab}. \tag{107}$$

Thus,

$$\delta H = \int (G_{ab}\delta g^{ab} - \nabla_a\omega^a)\,\varepsilon\, dx^1 \ldots dx^N. \tag{108}$$

Transformation of the Einstein tensor

We consider the tensor

$$U^{ambn} = (\varepsilon J)^2 (g^{ab}g^{mn} - g^{an}g^{mb}), \tag{109}$$

where ε has the form (78), and $J = \breve{\varepsilon}^{-1}$ has the form (56). We have

$$\left.\begin{aligned} U^{ajkb}R_{jk} &= (\varepsilon J)^2 (R^{ab} - Rg^{ab}); \\ U^{akij}R^b_{ijk} &= 2(\varepsilon J)^2 R^{ab}. \end{aligned}\right\} \tag{110}$$

By means of these formulas and (89) and (91) we transform the tensor (85) as follows:

$$\begin{aligned} 2(\varepsilon J)^2 G^{ab} &= U^{akij}(\check{\nabla}_j P^b_{ik} + P^b_{is}P^s_{jk}) \\ &+ U^{ajkb}(\check{\nabla}_j P_k - \check{\nabla}_l P^l_{jk} + P_s P^s_{jk} - P^l_{js}P^s_{lk}). \end{aligned} \tag{111}$$

On the other hand, on the basis of (75) and (95), we find

$$\begin{aligned} (\check{\nabla}_s + 2P_s)U^{ambn} &= P^a_{sk}U^{kmbn} + P^m_{sk}U^{akbn} \\ &+ P^b_{sk}U^{amkn} + P^n_{sk}U^{ambk}. \end{aligned} \tag{112}$$

Further, we write

$$U^{abn} = \check{\nabla}_m U^{ambn}, \quad U^{ab} = \check{\nabla}_n U^{abn}. \tag{113}$$

It follows from (112) that

$$U^{abn} = P^b_{mk}U^{amkn} + P^n_{mk}U^{ambk} - P_m U^{ambn}. \tag{114}$$

We now find

$$2(\varepsilon J)^2 G^{ab} = U^{ab} + V^{ab}, \tag{115}$$

where

$$\begin{aligned} V^{ab} &= U^{akij}P^b_{is}P^s_{jk} + U^{ajkb}(P_s P^s_{jk} - P^l_{js}P^s_{lk}) \\ &- P^b_{mk}\check{\nabla}_n U^{amkn} - P^n_{mk}\check{\nabla}_n U^{ambk} + P_m\check{\nabla}_n U^{ambn}. \end{aligned} \tag{116}$$

In the last formula, using (112), we can eliminate the derivatives $\check{\nabla}_n$. We obtain

$$V^{ab} = P^a_{km}P^b_{ln}U^{klmn} + A_{mn}U^{ambn} + B^a_{kmn}U^{bkmn} + B^b_{kmn}U^{akmn}, \tag{117}$$

where

$$\left.\begin{aligned} A_{mn} &= 2P_s P^s_{mn} - P^l_{mk}P^k_{ln} - P_m P_n; \\ B^a_{kmn} &= P^a_{km}P_n + P^s_{km}P^a_{sn}. \end{aligned}\right\} \tag{118}$$

Setting here $\check{\Gamma}^a_{ma} = 0$, we obtain Synge's formula (Ref. 31, p. 220 of the Russian translation).

Energy–momentum pseudotensor of the gravitational field

Like the tensor field of the affine deformation

$$P^a_{mn} = \frac{\partial x^a}{\partial \tilde{x}^s}\frac{\partial^2 \tilde{x}^s}{\partial x^m \partial x^n} - \frac{1}{2}g^{ak}\left(\frac{\partial g_{kn}}{\partial x^m} + \frac{\partial g_{km}}{\partial x^n} - \frac{\partial g_{mn}}{\partial x^k}\right), \tag{119}$$

the tensor field

$$t^{ab} = -\frac{c^4}{16\pi\gamma}(\varepsilon J)^{-2}V^{ab} \tag{120}$$

(where c is the velocity of light and γ is Newton's constant) is a functional of the map $\tilde{K}$, in which the connection components $\check{\Gamma}^a_{mn}$ are zero. This functional is called the energy–momentum pseudotensor of the gravitational field. Note that the values of such functionals are not changed when the map $\tilde{K}$ in their arguments is replaced by any map $\tilde{\tilde{K}}$ linearly related to $\tilde{K}$.

6. SIMPLE CARTOGRAPHY

Affine maps

By means of the functions (5), we have specified a mapping $A_N \to A_N$: the point $(x^1,..., x^N)$ was associated with the point $(\tilde{x}^1,..., \tilde{x}^N)$. But the same functions can be used to define an affine map of the arithmetic space by assuming that in the affine map $\tilde{K}$ the point $(x^1,..., x^N)$ of the arithmetic space A_N has the coordinates $\tilde{x}^1,...,\tilde{x}^N$ [see (5)].

In particular, if $\Phi^a_b = \delta^a_b$ and $\Phi^a = 0$, then the affine map is a principal map K of the arithmetic space. However, from the point of view of affine geometry there is no difference between the principal map K and the affine map $\tilde{K}$.

Function of the class v

A single-valued real function $\tilde{x} = f(x^1,..., x^N)$ is said to be a function of the class v if it and its derivatives of all orders $u \leqslant v$ exist and are continuous at every point $(x^1,..., x^N)$ of the arithmetic space A_N. Here, v can be any positive integer.

Regular mapping of the class v

A one-to-one mapping $A_N \to A_N$ is said to be a regular mapping of the class v if it is defined by a set

$$\tilde{x}^a = f^a(x^1, \ldots, x^N), \quad a \in \{1, \ldots, N\}, \tag{121}$$

of functions of the class v with Jacobian

$$J = \frac{\partial(f^1, \ldots, f^N)}{\partial(x^1, \ldots, x^N)} = \det\left(\frac{\partial f^a}{\partial x^b}\right) \tag{122}$$

that is nonvanishing everywhere on the set A_N. In particular, in the case (5)

$$J = \det(\Phi^a_b). \tag{123}$$

Simple manifold of the class v

A set A_N with group F_v of regular mappings of the class v is said to be a simple manifold of the class v. The theory of such a manifold is the theory of relativity (F_v, A_N).

The group F_v includes all linear mappings (5) and many others, for example,

$$\tilde{x}^a = x^a + \frac{1}{2}\sin x^a, \quad a \in \{1, \ldots, N\}. \tag{124}$$

Global maps of the class v

The functions (121) can be used to determine a global map of the class v by assuming that in a global map the point $(x^1,...,x^N)$ of the arithmetic space A_N has the coordinates $\tilde{x}^1,...,\tilde{x}^N$ given by (121).

In particular, if

$$f^a(x^1, \ldots, x^N) = \sum_{b=1}^{N} \Phi_b^a x^b + \Phi^a, \quad a \in \{1, \ldots, N\}, \tag{125}$$

then the global map of the class v is affine. However, from the point of view of the theory of relativity (F_v, A_N) there is no difference between the affine map and the global map of class v.

Oriented affine space

The affine group Φ contains the subgroup Φ^+ of mappings with determinant (123) greater than zero. The set A_N with group Φ^+ is called an oriented affine space. The theory of such a space is the theory of relativity (Φ^+, A_N).

Oriented simple manifold of the class v

The group F_v of regular mappings of the class v contains the subgroup F_v^+ of mappings with the Jacobian (122) greater than zero. The set A_N with group F_v^+ is called an oriented simple manifold of the class v. The theory of this manifold is the theory of relativity (F_v^+, A_N).

Right and left maps

A map obtained from a principal map by a transformation (121) with positive Jacobian is called a right map. A map obtained from the principal map by a transformation (121) with negative Jacobian is called a left map.

From the point of view of the theory of relativity (Φ, A_N) there is no difference between left and right affine maps, and from the point of view of the theory of relativity (F_v, A_N) there is no difference between left and right global maps of the class v.

Orienting a simple manifold

The set of global maps of a simple manifold can be divided into two classes in accordance with the following criterion: If the Jacobian (122) of the transformation (121) from the map K to the map $\widetilde{K}$ is greater than zero, then the two maps belong to the same class; but if it is greater than zero, then they belong to opposite classes. Calling one of the two such classes the right class, and the other the left class, we orient the manifold.

Introduction on a simple manifold of the structure of an affine space

The set of global maps of a simple manifold can be divided into an infinite set of classes in accordance with the following criterion: If the transformation (121) from the map K to the map $\widetilde{K}$ is linear, then the two maps belong to the same class; but if it is nonlinear, then they belong to different classes. Calling one of these classes affine, we introduce on the manifold the structure of an affine space.

Example of a nonglobal map

In a number of problems, nonglobal maps of a simple manifold are convenient. For example, polar coordinates r and φ defined in accordance with the formulas

$$x^1 = r\cos\varphi; \; x^2 = r\sin\varphi \tag{126}$$

are convenient. They take values in the region $r>0$, $-\pi<\varphi<\pi$ and define a nonglobal map. In contrast to a global map, one such map does not form an atlas. But with two such maps, one can cover the complete manifold, and they form an atlas.

Example of a system of equations in the theory of relativity (F_3, A_N)

The system of equations

$$S_{mn}^a = 0, \quad R_{kln}^a = 0 \tag{127}$$

for the N^3 functions Γ_{mn}^a of $x^1,...,x^N$ is invariant in the theory of relativity (F_3, A_N).

The general solution of this system can be obtained from the particular solution

$$\Gamma_{mn}^a(x^1, \ldots, x^N) = 0 \tag{128}$$

by means of a regular mapping (121) of the class $v = 3$ and the law of transformation (43) of the components of the affine connection. The general solution satisfies the following system of linear algebraic equations:

$$\sum_{s=1}^{N} \frac{\partial f^a}{\partial x^s} \Gamma_{mn}^s = \frac{\partial^2 f^a}{\partial x^m \partial x^n}, \quad a \in \{1, \ldots, N\}. \tag{129}$$

Among these solutions of the system (127) is the solution (128) obtained from (129) in the case (125).

7. BIMETRIC FORMALISM

The bimetric formalism (which we abbreviate to bimetrism) is based on the formulas given above in the subsection "Transformation of connections." In this formalism, it is assumed that Γ_{mn}^a and $\check{\Gamma}_{mn}^a$ are the Christoffel connections for two preassigned metric tensors g_{mn} and $\check{g}_{mn}$. This explains the name—bimetric formalism.

Conformal correspondence

The metrics g_{mn} and $\check{g}_{mn}$ are in conformal correspondence if

$$\check{g}_{mn} = \Omega^2 g_{mn}, \tag{130}$$

where Ω is a scalar function. In such a case, the affine deformation tensor (62) is equal to (98) and is given by

$$P_{mn}^a = \Omega^{-1}(\delta_n^a \Omega_m + \delta_m^a \Omega_n - \Omega^a g_{mn}), \tag{131}$$

where $\Omega_n = \partial_n \Omega$, $\Omega^a = g^{as}\Omega_s$. The covector (64) of the affine deformation is

$$P_m = N\Omega^{-1}\Omega_m. \tag{132}$$

These formulas simplify if we express Ω in the form

$$\Omega = \Omega_0 e^{-\psi}, \tag{133}$$

where $\Omega_0 = \text{const}$, and ψ is a scalar function. In accordance with (63), (131), and (133)

$$\check{R}_{kln}^a = R_{kln}^a + \delta_k^a \psi_{ln} - \delta_l^a \psi_{kn} + \psi_k^a g_{ln} - \psi_l^a g_{kn}, \tag{134}$$

where

$$\left.\begin{array}{l}\psi_{hn}=\nabla_h\nabla_n\psi+\psi_h\psi_n-\frac{1}{2}(\psi,\psi)g_{hn};\\ \psi_n^a=g^{as}\psi_{sn};\ \psi_n=\partial_n\psi;\ \psi^a=g^{as}\psi_s;\\ (\psi,\psi)=g^{mn}\psi_m\psi_n.\end{array}\right\} \quad (135)$$

Hence we obtain

$$\check{R}_{ln}=R_{ln}+(N-2)\psi_{ln}+\psi_a^a g_{ln}; \quad (136)$$

$$\Omega^2\check{R}=R+2(N-1)\psi_a^a, \quad (137)$$

where

$$\psi_a^a=\Box\psi+\frac{2-N}{2}(\psi,\psi),\ \Box\psi=g^{mn}\nabla_m\nabla_n\psi. \quad (138)$$

Beltrami parameters

For a scalar function ψ, the square (ψ,ψ) of its gradient in (135) is called the differential Beltrami parameter of the first kind.

The differential Beltrami parameter of the second kind $\Box\psi$ for the scalar function ψ occurs in (138). It can be represented in the form

$$\Box\psi=\frac{1}{\varepsilon}\frac{\partial}{\partial x^a}\left(\varepsilon g^{ab}\frac{\partial\psi}{\partial x^b}\right), \quad (139)$$

where ε denotes (78).

Klein–Fock equation in a Riemannian world

If in the original Klein–Fock equation, we replace the partial derivatives by covariant derivatives, we obtain the equation

$$\Box\phi+(mc/\hbar)^2\phi=0. \quad (140)$$

Instead of (140), we must, however, consider the equation

$$\Box\phi-\frac{N-2}{4(N-1)}R\phi+\left(\frac{mc}{\hbar}\right)^2\phi=0, \quad (141)$$

since the last equation is conformally invariant for $m=0$.[32] Indeed, setting

$$\check{\phi}=\Omega^{(2-N)/2}\phi, \quad (142)$$

we obtain

$$\Omega^{(N+2)/2}\check{\Box}\check{\phi}=\Box\phi+\frac{N-2}{2}\left[\Box\psi-\frac{N-2}{2}(\psi,\psi)\right]\phi. \quad (143)$$

Comparing this with Eqs. (137) and (138), we find the identity

$$\left(\Box-\frac{N-2}{4(N-1)}R\right)\phi=\Omega^{\frac{N+2}{2}}\left(\check{\Box}-\frac{N-2}{4(N-1)}\check{R}\right)\Omega^{\frac{2-N}{2}}\phi, \quad (144)$$

which enables us to prove our assertion.

Equation (141) corresponds to the classical equation

$$g^{mn}P_mP_n=m^2c^2, \quad (145)$$

so that the operator of the square of the momentum is

$$-\hbar^2\left(\Box-\frac{N-2}{4(N-1)}R\right). \quad (146)$$

Hilbert's energy–momentum tensor

Hilbert defined the matter energy–momentum tensor[33] as the variational derivative of the action function with respect to the metric tensor:

$$\delta S=\frac{1}{2}\int T_{mn}\delta g^{mn}\,dv, \quad (147)$$

where $dv=\varepsilon dx^1\cdots dx^N$.

Let us consider this definition for the example of the scalar field ϕ. We write the field action in the form

$$S^{(A)}=\int\mathscr{L}^{(A)}dv, \quad (148)$$

where $A=1$ if the field satisfies Eq. (140), and $A=2$ if the field satisfies Eq. (141). At the same time,

$$\left.\begin{array}{l}\mathscr{L}^{(1)}=\frac{1}{2}\left[(\phi,\phi)-\left(\frac{mc}{\hbar}\right)^2\phi^2\right];\\ \mathscr{L}^{(2)}=\mathscr{L}^{(1)}+\frac{N-2}{8(N-1)}R\phi^2.\end{array}\right\} \quad (149)$$

The canonical energy–momentum tensor is

$$T_{mn}^{(A)}=\phi_m\phi_n-\mathscr{L}^{(A)}g_{mn}. \quad (150)$$

For $A=1$, it is identical to the Hilbert tensor, but for $A=2$ it is not identical to it. In the second case, the Hilbert energy–momentum tensor is[32]

$$T_{mn}=T_{mn}^{(2)}+\frac{N-2}{4(N-1)}\{R_{mn}-\nabla_m\nabla_n+g_{mn}\Box\}\phi^2. \quad (151)$$

This tensor has the properties

$$T_{mn}=T_{nm},\quad g^{mn}T_{mn}=\left(\frac{mc}{\hbar}\phi\right)^2,\quad \nabla_nT^{na}=0. \quad (152)$$

It is remarkable that also in the Minkowski world, when the curvature tensor is zero, this tensor differs from the canonical tensor, although in this case Eqs. (140) and (141) are obviously identical.

In this connection, suppose that someone wishes to continue working in the framework of the special theory of relativity and does not wish to know about general relativity and Riemannian geometry. Would general relativity give him anything useful? I answer: yes. For example, it gives him the necessary energy–momentum tensor; for to take the variational derivative with respect to the metric tensor, it is necessary to go beyond the framework of Euclidean geometry and, therefore, beyond the framework of the special theory of relativity. Having obtained what is required, one can then go back to the convenient framework.

8. HARMONIC COORDINATES

The question of harmonic coordinates has again become topical in connection with the appearance of the relativistic theory of gravitation.[34–37] Therefore, we shall attempt to present the theory of harmonic coordinates as simply as possible, without referring to the preceding material.

Harmonic coordinates were introduced by Einstein[38] in the case of weak gravitational fields, and then by De Donder[39] and Lanczos[40] in the general case. They were used with great success by Fock.[41] Also interesting in this connection are the papers of Rosen,[42] Papapetrou,[43] Gupta,[44] Asanov,[45] and others.

In the two-dimensional case, orthogonal harmonic coordinates are called isometric coordinates. They have been well studied.[46] Also of interest are coordinates in which the determinant of the metric tensor is constant. They were used by Einstein in the early stages in his development of general relativity and by Schwarzschild[47] in the derivation of the metric that carries his name. Similar coordinates can be introduced in the general case of spaces of equiaffine connection. We shall call them equiaffine coordinates.

The affine deformation tensor generated by a map

An affine connection is defined by components Γ^a_{mn} that transform in accordance with the rule

$$\Gamma^a_{mn} = \left(\tilde{\Gamma}^b_{pq} \frac{\partial \tilde{x}^p}{\partial x^m} \frac{\partial \tilde{x}^q}{\partial x^n} + \frac{\partial^2 \tilde{x}^b}{\partial x^m \partial x^n}\right) \frac{\partial x^a}{\partial \tilde{x}^b} \quad (153)$$

on the transition from one coordinate map to another. The simplest example is the coordinate connection

$$\check{\Gamma}^a_{mn} = \frac{\partial x^a}{\partial \tilde{x}^b} \frac{\partial^2 \tilde{x}^b}{\partial x^m \partial x^n} \quad (154)$$

generated by the map $\tilde{x}$, in which its components are equal to zero. The difference

$$P^a_{mn} = \check{\Gamma}^a_{mn} - \Gamma^a_{mn} = -\tilde{\Gamma}^b_{pq} \frac{\partial \tilde{x}^p}{\partial x^m} \frac{\partial \tilde{x}^q}{\partial x^n} \frac{\partial x^a}{\partial \tilde{x}^b} \quad (155)$$

is a tensor (special form of the affine deformation tensor; we call it the affine deformation tensor generated by the map $\tilde{x}$). For the coordinate $\tilde{x}^a$ (since any coordinate is a scalar function)

$$\nabla_m \nabla_n \tilde{x}^a = \frac{\partial \tilde{x}^a}{\partial x^s} P^s_{mn}, \quad P^a_{mn} = \frac{\partial x^a}{\partial \tilde{x}^b} \nabla_m \nabla_n \tilde{x}^b. \quad (156)$$

Another example is the Christoffel connection, which is expressed in terms of the metric tensor: $g_{ab} dx^a \otimes dx^b = \tilde{g}_{mn} d\tilde{x}^m \otimes d\tilde{x}^n$. In this case, the second differential Beltrami parameter is defined. For the coordinate $\tilde{x}^a$ it is

$$\Box \tilde{x}^a = \frac{1}{\varepsilon} \frac{\partial}{\partial x^m} \left(\varepsilon g^{mn} \frac{\partial \tilde{x}^a}{\partial x^n}\right) = g^{mn} \nabla_m \nabla_n \tilde{x}^a, \quad (157)$$

where $\varepsilon = \sqrt{|g|}$. We have the vector

$$\Phi^a = g^{mn} P^a_{mn} = \frac{1}{\tilde{\varepsilon}} \check{\nabla}_b (\varepsilon g^{ab}) = \frac{1}{\varepsilon} \check{\nabla}_b (\varepsilon g^{ab}), \quad (158)$$

since the covariant derivative $\check{\nabla}_b$ of the Jacobian $|\partial \tilde{x}^a / \partial x^b|$ is equal to zero. In the basis $\partial / \partial \tilde{x}^a$, its components are

$$\Box \tilde{x}^a = \frac{\partial \tilde{x}^a}{\partial x^s} \Phi^s = \frac{1}{\tilde{\varepsilon}} \frac{\partial}{\partial \tilde{x}^b} (\tilde{\varepsilon} \tilde{g}^{ab}) = -\tilde{g}^{mn} \tilde{\Gamma}^a_{mn}. \quad (159)$$

The coordinates $\tilde{x}^a$ are said to be harmonic if all $\Box \tilde{x}^a = 0$, and this is equivalent to a covariant condition: all $\Phi^a = 0$.

A third example is the equiaffine connection for which

$$\Gamma_m = \Gamma^a_{ma} = \partial_m \ln \varepsilon, \quad (160)$$

where ε varies in accordance with the rule

$$\varepsilon = \frac{\partial (\tilde{x}^1, \ldots, \tilde{x}^N)}{\partial (x^1, \ldots, x^N)} \tilde{\varepsilon} \quad (161)$$

when the map $\tilde{K}$ is replaced by the map K. In this case, the affine deformation covector is a gradient:

$$P_m = P^a_{ma} = \partial_m \ln \tilde{\varepsilon}. \quad (162)$$

In equiaffine coordinates, $\tilde{\varepsilon} = \text{const}$ and $P_m = 0$.

Einstein's harmonic map

In the case of a weak gravitational field, Einstein assumed that the components of g_{ab} are[38]

$$g_{ab} = \eta_{ab} + \Delta_{ab}, \quad (163)$$

where η_{ab} do not depend on the coordinates, and Δ_{ab} are very small quantities. Up to the first order,

$$g = \eta\,(1 + \Delta^s_s), \quad g^{ab} = \eta^{ab} - \Delta^{ab}, \quad (164)$$

where η is the determinant of the matrix (η_{ab}), the numbers η^{ab} are found from the condition $\eta^{ak}\eta_{kb} = \delta^a_b$, and

$$\Delta_{as}\eta^{sb} = \Delta^b_a = \eta_{as}\Delta^{sb}. \quad (165)$$

In accordance with (164),

$$\frac{1}{\varepsilon} \frac{\partial}{\partial x^b} (\varepsilon g^{ab}) = -\frac{\partial}{\partial x^b} \theta^{ab}, \quad (166)$$

where

$$\theta^{ab} = \Delta^{ab} - \frac{1}{2} \Delta^s_s \eta^{ab}. \quad (167)$$

In the given case, the conditions of harmonicity $\Phi^a = 0$ amount to precisely the conditions

$$\frac{\partial}{\partial x^b} \theta^{ab} = 0, \quad (168)$$

which Einstein imposed on the metric (163). They mean that the Cartesian coordinates for the metric

$$d\check{s}^2 = \eta_{ab}\, dx^a \otimes dx^b \quad (169)$$

are harmonic coordinates for the metric

$$ds^2 = g_{ab}\, dx^a \otimes dx^b. \quad (170)$$

Note that the Christoffel connection for the metric (169) is the coordinate connection.

Fock's harmonic map

In Fock's harmonic map, the Schwarzschild metric has the form

$$ds^2 = \frac{\sqrt{x^2+y^2+z^2} - \alpha}{\sqrt{x^2+y^2+z^2} + \alpha} c^2\, dt^2 - \frac{(\sqrt{x^2+y^2+z^2} + \alpha)^2}{x^2+y^2+z^2} (dx^2 + dy^2 + dz^2) - \alpha^2 \frac{\sqrt{x^2+y^2+z^2} + \alpha}{\sqrt{x^2+y^2+z^2} - \alpha} \frac{(x\,dx + y\,dy + z\,dz)^2}{(x^2+y^2+z^2)^2}. \quad (171)$$

In this connection, Fock wrote: "...we make a remark concerning the definition of a straight line in the theory of gravitation. How is one to define a straight line: As a ray of light or as a straight line in the Euclidean space in which the harmonic coordinates x, y, z are Cartesian coordinates? It seems to us that only the second definition is correct" (Ref. 10, p. 273 of the Russian original).

It can be seen that Fock gave a real meaning to the connection $\check{\Gamma}^a_{mn}$, whose components in his chosen map are equal to zero. Such a connection endows the Schwarzschild world with the structure of an affine space but does not give any information about the Lorentz group. However, Fock later wrote: "*In the case of an isolated system of masses, there exists a coordinate system, namely, a harmonic system, that, up to Lorentz transformations, is uniquely determined by the imposition of additional conditions*" (Ref. 10, p. 445 of the Russian original).

From this the conclusion may be drawn (which we did in Ref. 6) that besides the Schwarzschild metric (171) Fock considered the metric

$$d\check{s}^2 = c^2\, dt^2 - dx^2 - dy^2 - dz^2, \quad (172)$$

although he did not report this in his book.[10]

But Einstein too in the paper of Ref. 38 considered not only the de Sitter metric

$$ds^2 = \left(1 - \frac{2\alpha}{\sqrt{x^2+y^2+z^2}}\right) c^2\, dt^2 - \left(1 + \frac{2\alpha}{\sqrt{x^2+y^2+z^2}}\right) (dx^2 + dy^2 + dz^2) \quad (173)$$

but also the Minkowski metric (172), and at large values of the sum of the squares $x^2 + y^2 + z^2$ the Schwarzschild metric goes over into the de Sitter metric.

Thus, Einstein's approach to the problem of a massive source of the gravitational field differs from Fock's approach only insignificantly; it is merely that Einstein considered the field far from the source, while Fock considered it everywhere in the region $x^2 + y^2 + z^2 > \alpha^2$. But in essence Fock's approach to this problem is identical to Einstein's.

Concluding the review, we cannot but agree with the current opinion that the theory of relativity is confused:

"The D.A.R. [Daughters of the American Revolution] (reflected the cynic Doremus Jessup that evening) is a somewhat confusing organization—as confusing as Theosophy, Relativity, or the Hindu Vanishing Boy Trick, all three of which it resembles." [48]

Who would enjoy sorting out the confusion?

[1)] *Translator's Note.* The spatial part of the group called here the Newton group is identical to the spatial part of what Bertotti and I have called the *Leibniz group*; see J. B. Barbour and B. Bertotti, Nuovo Cimento **B38**, 1 (1977) and Proc. R. Soc. London Ser. A **246**, 326 (1982), where dynamical theories invariant with respect to the Leibniz group are constructed.

[2)] *Translator's Note.* The Russian notation for the trigonometric, inverse trigonometric, hyperbolic functions, etc., is retained here and throughout the article in the displayed equations.

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