# The evolution of the Bogolyubov functional hypothesis and nonlocality of transport processes in hard-sphere hydrodynamics

N. G. Inozemtseva and B. I. Sadovnikov

*Moscow State University, Moscow*

Fiz. Elem. Chastits At. Yadra **18**, 878–903 (July–August 1987)

We discuss the Bogolyubov functional hypothesis and its fundamental role in the construction of the best possible description of multiparticle processes in the approach to equilibrium in macrosystems.

## INTRODUCTION

The study of transport phenomena occurring during the hydrodynamical stage of the approach of a macroscopic system to equilibrium is one of the most important applications of kinetic theory.

Throughout the long period of development of statistical mechanics it was quite naturally assumed that these phenomena can be described by local hydrodynamical equations which, in turn, could be obtained by constructing approximate solutions of the local kinetic equation.

Even in the very early stage of development of kinetic theory it was realized that the locality of the kinetic equations in statistical mechanics is a consequence of neglecting the finite size of the interaction region of the particles of the system and the duration of this interaction. However, it was assumed that both the nonlocal effects related to the finite extent of the binary interaction process and multiple-scattering processes could be taken into account as corrections to the local equation.

Prior to the development of rigorous mathematical methods of constructing the kinetic equations associated with the name of Bogolyubov[1] the attempts to study these effects were heuristic in nature (for example, in the Enskog equation for hard-sphere systems the Boltzmann collision integral was modified by taking into account the sizes of the colliding spheres).

Bogolyubov created a dynamical theory of kinetic phenomena and clearly laid down principles amenable to an exact mathematical formulation, making it possible to obtain local kinetic equations or the hydrodynamical equations directly, taking into account effects due to both multiple collisons and the spatial and temporal extent of the binary collison process.

Analysis of the features of approximate solutions of the exact dynamical Louiville equation permitted Bogolyubov to formulate his remarkable functional hypothesis, which provides the key to understanding the relation between the microscopic particle mechanics and the kinetic equations and serves as the basis for all the subsequent development of nonequilibrium statistical mechanics.

As is well known, the physical meaning of the functional hypothesis amounts to the introduction of the concept of different stages in the approach of a macroscopic system to equilibrium, with each stage characterized by a time scale and a method of reducing the description of the nonequilibrium states. Here the local Boltzmann equation and the corresponding local hydrodynamical equations arise as the approximations of lowest order in the dimensionless parameter $(nr_0^3)$, where $n$ is the number density of the particles of the system and $r_0$ is the characteristic size of the interaction region.

Until the 1970s, most efforts were directed to the microscopic justification of the phenomenological laws of Newton and Fourier and the calculation of the transport coefficients entering into them. These coefficients determine the relation between the stress tensor, the thermal-flux vector, the deformation-rate tensor, and the gradient of the local temperature:

$$P_{ik} = p\delta_{ik} - 2\eta S_{ik} - \varkappa\delta_{ik}\frac{\partial u_i}{\partial x_j}\,; \tag{1}$$

$$S_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) - \frac{1}{3}\delta_{ik}\frac{\partial u_i}{\partial x_i}\,; \tag{2}$$

$$q_j = -\frac{\lambda}{k_B}\frac{\partial \theta}{\partial x_i}\,.$$

Here $\lambda$, $\eta$, and $\varkappa$ are the coefficients of thermal conductivity and the shear and bulk viscosity, $\mathbf{u}$ and $\theta$ are the hydrodynamical velocity and local temperature, $k_B$ is the Boltzmann constant, and $p$ is the local pressure. If the particle interaction is short-range (which is certainly true in the case of hard spheres), bulk viscosity effects can be neglected for gases of moderate density ($nr_0^3 \sim 0.1$–$0.3$, where $n$ is the equilibrium particle number density and $r_0$ is the sphere diameter).

Use of (1) and (2) in the macroscopic conservation laws leads to the familiar Navier–Stokes equations:

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0;$$

$$\rho\frac{\partial u_i}{\partial t} + \rho u_k\frac{\partial u_i}{\partial x_k} = \frac{\partial}{\partial x_k}(-p\delta_{ik} + 2\eta S_{ik}); \tag{3}$$

$$\rho\frac{\partial\theta}{\partial t} + \rho u_i\frac{\partial\theta}{\partial x_i} = -\frac{2}{3}m\left(-\frac{\lambda}{k_B}\frac{\partial^2\theta}{\partial x_i^2} + p\frac{\partial u_i}{\partial x_i} - 2\eta S_{ij}\frac{\partial u_j}{\partial x_i}\right),$$

[$\rho(\mathbf{x},t) = mn(\mathbf{x},t)$ is the local equilibrium mass density].

In the theory dealing with the local kinetic equation for the one-particle distribution function $f_1(t,\mathbf{r},\mathbf{v})$ (Refs. 1–3) of the form

$$\frac{\partial f_1}{\partial t} + \mathbf{v}_1\frac{\partial f_1}{\partial \mathbf{r}_1} = \Phi\{f_1(t,\mathbf{r},\mathbf{v}),\mathbf{r}_1,\mathbf{v}_1\}, \tag{4}$$

the equations (3) are the result of the lowest-order approximation in the macroscopic gradients when the solutions of (4) are constructed by the Chapman–Enskog method. The structure of the functional $\Phi$ is determined by the dynamics of the particle interaction in the system.

Bogolyubov[1] gave a method of calculating the successive terms in the expansion of $\Phi$ in powers of the parameter $nr_0^3$, where the $k$-th term contains the contribution of $(k+1)$-particle collisions and also effects of the nonlocality of collisions of smaller numbers of particles. This sort of structure of the functional $\Phi$ naturally leads to a virial expansion for the transport coefficients. However, subsequent

 0090-4759/87/040376-11$04.40  

complicated analysis of the properties of the three- and four-particle collision operators revealed[2–4] that the higher terms in the virial expansion diverge, that is, the transport coefficients are not analytic functions of the parameter $nr_0^3$ in the neighborhood of the point $nr_0^3 = 0$. Cohen and Dorfman *et al.* (Refs. 2 and 3; see also Ref. 4 and references cited therein) determined the reason for the divergences in virial expansions in three-dimensional systems: they are due to correlated, successive double collisions in the four-particle system occurring at distances much greater than the mean free path. Since higher-order collisions necessarily occur at such distances, to eliminate the divergences it is necessary to successively take into account of arbitrary multiplicity.[3] In fact, divergences arise in dynamical processes in isolated small groups of particles. In a macroscopic system such groups are necessarily subjected to an external influence during a time $t_{\text{mfp}} \sim (nr_0^3)^{-1} t_{\text{coll}}$. This must lead to the cancellation of all divergences upon summation of the series determining the functional

$$\Phi\{f_1(t, \mathbf{r}, \mathbf{v}), \mathbf{r}_1, \mathbf{v}_1\} = \Phi^{(0)}\{f_1, \mathbf{r}_1, \mathbf{v}_1\} + nr_0^3\Phi^{(1)}\{f_1, \mathbf{r}_1, \mathbf{v}_1\} + (nr_0^3)^2\Phi^{(2)}\{f_1, \mathbf{r}_1, \mathbf{v}_1\}\ldots,$$

but this procedure leads to nonanalyticity of the local kinetic equation in the parameter $(nr_0^3)$ near the point $nr_0^3 = 0$. Physically, the appearance of divergences of this type indicates that there is no sharp boundary between the period of initial chaotization and the kinetic stage of the approach of the system to equilibrium. If we want to preserve the local structure of the kinetic equations, in order to derive them systematically and consistently we need to use a technique different from expansion of the functional $\Phi$ in a series in the parameter $nr_0^3$. Up to now no such technique has been found. It is not completely clear whether or not it is possible to preserve the local structure of the kinetic equation (4) in this case.

However, these difficulties with the local theory do not create any problems in the construction of higher-order approximations in the gradients, which lead to the Burnett equations, etc., instead of (3).

The evolution of the distribution functions is determined by the Bogolyubov–Born–Green–Kirkwood–Yvon (BBGKY) hierarchy, which for hard-sphere systems has the form[5]

$$\frac{\partial f_1(t, 1)}{\partial t} + L_0(1) f_1(t, 1) = n \int dx_2 \overline{T}(1, 2) f_2(t, 1, 2);$$
$$\frac{\partial f_2(t, 1, 2)}{\partial t} + (L_0(1, 2) - \overline{T}(1, 2)) f_2(t, 1, 2) = n \int dx_3 [\overline{T}(1, 3) + \overline{T}(2, 3)] f_3(t, 1, 2, 3), \qquad (5)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

where $f_1(t,1) = f_1(t,\mathbf{r}_1,\mathbf{v}_1)$, $x_i = (\mathbf{r}_i,\mathbf{v}_i)$, $L_0(1) = \mathbf{v}_1(\partial/\partial \mathbf{r}_1)$, $L_0(1,2) = \sum_{i=1}^{2} L_0(i)$, $\overline{T}(i,j)$ is the two-particle collision operator,

$$\overline{T}(i, j)\psi(\mathbf{v}_i, \mathbf{v}_j) = r_0^2 \int_{\mathbf{v}_{ij}\sigma \geqslant 0} d\sigma(\mathbf{v}_{ij}\sigma) \times [\delta(\mathbf{r}_{ij} - \sigma r_0)\psi(\mathbf{v}_i^*, \mathbf{v}_j^*) - \delta(\mathbf{r}_{ij} + \sigma r_0)\psi(\mathbf{v}_i, \mathbf{v}_j)], \quad (6)$$
$$\mathbf{v}_{ij} = \mathbf{v}_i - \mathbf{v}_j, \quad \mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j, \quad \mathbf{v}_{i,j}^* = \mathbf{v}_{ij} + \sigma(\mathbf{v}_{ij}\sigma),$$

and $\sigma$ is the unit vector. The solutions of the sytem (5) must satisfy the boundary condition corresponding to damping of the correlations:

$$f_s(t, 1, 2, \ldots, i, \ldots, j, \ldots, s) \to f_1(t, i) f_1(t, j) f_{s-2}(t, 1, \ldots, i-1, i+1, \ldots, j-1, j+1, \ldots, s-2)$$

for $|\mathbf{r}_{ij}| \to \infty$.

The assumption of locality of the kinetic equation for $f_1(t,\mathbf{r},\mathbf{v})$ follows from the more general hypothesis that $f_s(t,1,\ldots,s)$ is represented in the form of functionals $\tilde{f}_s[x_1,\ldots,x_s | f_1(t,x_1),\ldots,f_1(t,x_s)]$. As already noted above, here the formal evaluation of the hierarchy (5) neglecting higher-order correlations leads to divergences for the transport coefficients. In the experiments of Refs. 6 and 7 on machine modeling of the evolution of systems of hard disks and spheres evidence was obtained which favors the existence of long-range correlations due to particle interactions with collective excitations in the system—hydrodynamical modes which are weakly damped in time.

In particular, for systems of hard disks these long-range correlations are incompatible with the description of the hydrodynamical stage of relaxation to equilibrium by the Navier–Stokes equations, since the integrals of the time correlation functions of the microscopic currents, which determine the coefficients $\lambda$ and $\eta$, diverge for $t \to \infty$ (Ref. 4). For three-dimensional systems such divergences appear at the stage of the calculation of the Burnett coefficients.[8–10] The experimentally observed asymptotic dependence of the correlation functions $\langle J(0)J(t)\rangle$ on the time is of the form $\langle J(0)J(t)\rangle \sim t^{-d/2}$, where $d$ is the dimensionality of space; for $d = 2$ the divergence of the integrals of the correlation functions is logarithmic. We note that in the linear approximation the Boltzmann local kinetic equation leads to exponential falloff in time of the correlations of any microscopic quantities.

These facts, which are indicative of the inconsistency of the description of the hydrodynamical regime on the basis of the local kinetic equation, led to the formulation of a new approximation in the BBGKY hierarchy. This is associated with the names of Bogolyubov,[11] Ernst, and Dorfman.[12]

The approach of Ernst and Dorman was originally directed to the microscopic justification of the phenomenological theory of interacting modes,[13] which has been used successfully to describe the asymptotic behavor of time autocorrelation functions and kinetic phenomena near the critical point in liquids, and was formal in nature.

Bogolyubov established a paradoxical analogy between the new approximation chain and approximations in the microscopic theory of a nonequilibrium plasma known since the mid-fifties. He also made a fundamental contribution to the mathematical formulation of the latter approximations. In fact, collective phenomena due to long-range correlations play the dominant role in plasma dynamics. The idea proposed by Bogolyubov is that the approximation methods developed in plasma theory can be used to describe similar effects in a hard-sphere system (variations of the distribution function over distances and time intervals much greater than the mean free path and time). Therefore, in contrast to the traditional scheme,[1,14,15] where the functions $f_s$ are approximated by series in powers of the density,

$$f_s(t, 1, \ldots, s) = f_s^{(0)}(1, \ldots, s) f_1(t, 1)$$

$$\ldots f_1(t, s)$$

$$+] n f_s^{(1)} (1, \ldots, s/f_1(t, 1), \ldots, f_1(t, s)) + \ldots$$

and the expressions for $f_s^{(\alpha)}$ are found from (5) taking into account the damping of the correlations, in the new approximation[11,12] it is assumed that particle interactions with collective excitations can be described in lowest order in the density neglecting the three-particle correlation function $g_3(t,1,2,3)$ in (5):

$$g_3(t, 1, 2, 3) = f_3(t, 1, 2, 3) - \prod_{i=1}^{3} f_1(t, i) - \sum_{\substack{i=1 \\ k,\, l \neq i,\, k>l}}^{3} f_1(t, i)\, g_2(t, k, l) = 0, \quad (7)$$

where

$$g_2(t, 1, 2) = f_2(t, 1, 2) - f_1(t, 1) f_1(t, 2).$$

Here there are no *a priori* assumptions about the time dependence of $g_2(t,1,2)$. This dependence must be determined directly from the equations (5), which under the condition (7) form a closed system:

$$\frac{\partial f_1(t, 1)}{\partial t} + L_0(1) f_1(t, 1) = n \int dx_2 \overline{T}(1, 2) \{f_1(t, 1) f_1(t, 2) + g_2(t, 1, 2)\}; \quad (8a)$$

$$\frac{\partial g_2(t, 1, 2)}{\partial t} + \{\mathscr{L}(t, 1) + \mathscr{L}(t, 2) - \overline{T}(1, 2)\} g_2(t, 1, 2) = \overline{T}(1, 2) f_1(t, 1) f_1(t, 2). \quad (8b)$$

For the ordering of the notation here we have introduced the operators $\mathscr{L}(t,i)$ (Refs. 11 and 12), the action of which on a function of argument $(i)$ is defined as

$$\mathscr{L}(t, i)\chi(i) = L_0(i)\chi(i) - n \int dx_3 \overline{T}(i, 3) [f_1(t, i)\chi(3) + f_1(t, 3)\chi(i)]. \quad (9)$$

The factor $\overline{T}(1,2)$ on the left-hand side of (8b) is usually dropped; in the linearized variant it leads to corrections of higher order in the density. However, as noted in Ref. 11, the neglect of this factor can significantly change the behavior of $g_2(t,1,2)$ in the region $|\mathbf{r}_1 - \mathbf{r}_2| \sim r_0$. If, however, we are interested in correlations at distances comparable with the mean free path, the neglect of this factor is fully justified.

The solutions of (8a) and (8b) can be determined completely by specifying the initial conditions for $f_1(t,1)$ and $g_2(t,1,2)$ at some value $t_0$. Neglecting the contribution of the initial correlations[12] (which lead to terms of higher order in the density than deviations of $f_1(t_0,1)$ from the equilibrium value), we can write the solution of (8b) in the form[10]

$$g_2(t, 1, 2) = \int_0^t Q(t) Q^{-1}(\tau) \overline{T}(1, 2) f_1(\tau, 1) f_1(\tau, 2)\, d\tau, \quad (10)$$

where

$$Q(t) = T \exp\left(\int_0^t (\mathscr{L}(\tau, 1) + \mathscr{L}(\tau, 2))\, d\tau\right), \quad (11)$$

in which $T$ is the time-orderig operation, which is needed here because the operators $\mathscr{L}(\tau,i)$ taken at different instants of time do not commute. In (10) and (11) for convenience we have set $t_0 = 0$.

Substitution of (10) and into (8a) leads to a nonlocal kinetic equation for the one-particle distribution function $f_1$, and it is to be expected that it also contains new nonlinear effects arising from the functional dependence of the operators $\mathscr{L}(\tau,i)$ on $f_1$ (Ref. 10).

## 1. CONSTRUCTION OF THE HYDRODYNAMICAL APPROXIMATION

The asymptotic behavior of the solutions of (8a) and (8b) for arbitrary initial conditions is unknown at present, because even a formulation of the statement analogous to Boltzmann's $H$ theorem is lacking. We note that a similar situation was encountered in the case of the (generalized) Enskog equation before 1978, when a proof of the $H$ theorem in this important case was found by Resibois.[16] It is natural to assume that also in the system described by (8a) and (8b), during a time interval not exceeding the time for the mean free path $t_{\rm mfp} \sim (n r_0^2 (\theta/m)^{1/2})^{-1}$ a state is reached which is close to the "quasi-equilibrium" state described by Maxwellian distribution functions

$$f_1(t, 1) \sim f_M(1); \qquad g_2(t, 1, 2) \sim f_M(1) f_M(2) g(\mathbf{r}_1 - \mathbf{r}_2), \quad (12)$$

where

$$f_M(1) = \rho(\mathbf{r}, t) \left(\frac{m}{2\pi\theta(\mathbf{r}_1, t)}\right)^{3/2} \exp\left\{-\frac{m(\mathbf{v}_1 - \mathbf{u}(\mathbf{r}_1, t))}{2\theta(\mathbf{r}_1, t)}\right\}. \quad (13)$$

We note that the qualitatively new features of the solutions of the system (8a) and (8b) mentioned above are simpler to study if we neglect effects of the finite size of the collision region in the operators $\overline{T}_{ij}$. Henceforth, instead of $\overline{T}_{ij}$ we shall use the operators $\widetilde{T}_{ij}$, the action of which on functions of the form $\chi(i,j)$ is determined by the formula

$$T(i, j)\chi(i, j) = r_0^2 \int_{\mathbf{v}_{ij}\sigma \geqslant 0} d\sigma(\mathbf{v}_{ij}\sigma) \{\chi(\mathbf{r}_i, \mathbf{v}_i^*, \mathbf{r}_j, \mathbf{v}_j^*) - \chi(\mathbf{r}_i, \mathbf{v}_i, \mathbf{r}_j, \mathbf{v}_j)\} \delta(\mathbf{r}_i - \mathbf{r}_j). \quad (14)$$

To ensure that this approximation is self-consistent we must set $g(\mathbf{r}_1 - \mathbf{r}_2) = 0$ in (12).

The conservation laws for binary collisions contained in (8a) and (8b) admit the standard formulation in terms of the macroscopic parameters $\rho$, $\mathbf{u}$, and $\theta$ determining the quasi-equilibrium distribution function (13):

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \mathbf{r}}(\rho \mathbf{u}) = 0;$$

$$\rho\left(\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial r_k}\right) = -\frac{\partial P_{ik}}{\partial r_k}; \quad (15)$$

$$\frac{3}{2}\rho\left(\frac{\partial}{\partial t} + u_k \frac{\partial}{\partial r_k}\right)\theta = -\left(P_{ik} S_{ik} + \frac{\partial q_k}{\partial r_k}\right),$$

where

$$\left.\begin{aligned} P_{ik} &= \int (v_i - u_i)(v_k - u_k) f_1(t, x_1)\, d\mathbf{v}_1; \\ q_k &= \frac{1}{2}\int (v_k - u_k)(v_i - u_i)^2 f_1(t, x_1)\, d\mathbf{v}_1. \end{aligned}\right\} \quad (16)$$

In accordance with the structure of the collision operator (14), in (16) we omit the "potential" parts of the stress tensor and the thermal-flux vector, the inclusion of which leads only to corrections to the Boltzmann values of the transport coefficients.

Therefore, the problem of constructing the hydrodynamical equations amounts to calculating the integrals (16) with approximate solutions of the system (8a) and (8b) of a special form: these solutions must be determined by the gradients of the macroscopic parameters entering into (13). The method of constructing approximate solutions of this type actually originated with Chapman and Enskog. In the current formulation given by Bogolyubov in Ref. 17 it amounts to introducing a formal uniformity parameter $\mu$ into those terms of (8a) and (8b) which contain time and space gradients. This must be done in such a manner that the presence of the factor $\mu$ corresponds to the assumption of a small change of the properties of the system under spatial translations of the sytem as a whole[17] (at the end of the calculations we take $\mu = 1$). Then the solutions of the equations modified in this manner are usually written as series in the parameter $\mu$. For the system (8a) and (8b) these solutions look like the following[8]:

$$\left.\begin{aligned} f_1(t,1) &= f_M(1)\,(1+\mu\varphi_1(t,1)+\mu^2\varphi_2(t,1)+\ldots),\\ g_2(t,1,2) &= f_M(1)\,f_M(2)\,(\mu\psi_1(t,1,2)\\ &\quad +\mu^2\psi_2(t,1,2)+\ldots). \end{aligned}\right\} \quad (17)$$

However, detailed analysis of the resulting system of equations for $\varphi_\alpha$ and $\gamma_\alpha$ shows that already the third term of the first of the series (17) contains divergences[9] due to the singularities of the so-called ring operators, the properties of which will be discussed below. This indicates that the modified system (8a) and (8b)

$$\mu\left(\frac{\partial f_1(t,1)}{\partial t}+\mathbf{v}_1\frac{\partial}{\partial \mathbf{r}_1}f_1(t,1)\right) = n\int dx_2 T(1,2)\,\{f_1(t,1)f_1(t,2)+g_2(t,1,2)\}; \quad (18a)$$

$$\mu\left(\frac{\partial g_2(t,1,2)}{\partial t}+\frac{\mathbf{v}_1+\mathbf{v}_2}{2}\frac{\partial g_2(t,1,2)}{\partial \mathbf{R}}\right)+(\mathbf{v}_1-\mathbf{v}_2)\frac{\partial g_2(t,1,2)}{\partial \boldsymbol{\rho}} = n\sum_{i=1}^{2}\int dx_3 T(i,3)\,[f_1(t,i)\,g_2(t,3-i,3)+f_1(t,3)\,g_2(t,1,2)] + T(1,2)\,f_1(t,1)\,f_1(t,2); \quad (18b)$$

$$\mathbf{R}=\frac{\mathbf{r}_1+\mathbf{r}_2}{2},\quad \boldsymbol{\rho}=\mathbf{r}_1-\mathbf{r}_2$$

has no solutions $\{f_1,g_2\}$ clsoe to $\{f_M(1),0\}$ and analytic in $\mu$ near the point $\mu = 0$.

Let us explain the above using a simple example.[9] If the system is so close to equilibrium that the difference of $n(\mathbf{r},t)$, $\mathbf{u}(\mathbf{r},t)$ and $\theta(\mathbf{r},t)$ from their equilibrium values can be neglected on the right-hand sides of (18a) and (18b), then from (17), (18a), and (18b) it follows that

$$\psi_\alpha(t,x_1,x_2)=\left[(\mathbf{v}_1-\mathbf{v}_2)\frac{\partial}{\partial\boldsymbol{\rho}}-n\,(\Lambda_0(1)+\Lambda_0(2))\right]^{-1} \times T(1,2)\,(1+P_{1,2})\,\varphi_2(t,2)-\left(\frac{\partial}{\partial t}+\frac{\mathbf{v}_1+\mathbf{v}_2}{2}\frac{\partial}{\partial\mathbf{R}}\right) \times\left[\frac{\mathbf{v}_1-\mathbf{v}_2}{2}\frac{\partial}{\partial\boldsymbol{\rho}}-n\,(\Lambda_0(1)+\Lambda_0(2))\right]^{-1} \times T(1,2)\,(P_{1,2}+1)\frac{1}{n\Lambda_0(1)+\hat{z}}\,\pi(t,1), \quad (19)$$

where $P_{1,2}$ is an operator which interchanges the arguments of functions $P_{1,2}\psi(1)=\psi(2)$, $\Lambda_0(i)$ is the linearized Boltzmann operator

$$\begin{aligned} \Lambda_0(i)\,\psi(i,j) &= r_0^2\int dx_k f_{eq}(x_k)\,T(ik)\,(\psi(i,j)+\psi(k,j));\\ \pi(t,1) &= [f_M(1)]^{-1}\left(\frac{\partial}{\partial t}+\mathbf{v}_1\frac{\partial}{\partial\mathbf{r}_1}\right)f_M(1); \end{aligned} \quad (20)$$

and $\hat{z}$ is a ring operator, whose action on the function $\pi(t,1)$ is given by

$$\hat{z}\pi(t,1)=\int dx_2 T(1,2)\left[(\mathbf{v}_1-\mathbf{v}_2)\frac{\partial}{\partial\mathbf{r}}-n\,(\Lambda_0(1)+\Lambda_0(2))\right]^{-1} \times T(1,2)\,(1+P_{1,2})\,\pi(t,1).$$

The contribution to $\varphi_2(t,1)$ corresponding to (19) contains components of the form

$$\int\frac{d\mathbf{q}}{(2\pi)^3}\,\{i\mathbf{q}\,(\mathbf{v}_1-\mathbf{v}_2)-n\,(\Lambda_0(1)+\Lambda_0(2))\}^{-2} \frac{1}{n\,(\Lambda_0(1)+z)}\frac{\partial\pi(t,\mathbf{R},\mathbf{v}_1)}{\partial t}. \quad (21)$$

The expression $\{i_{\mathbf{q}}(\mathbf{v}_1-\mathbf{v}_2)-n(\Lambda_0(1)+\Lambda_0(2)\}^{-2}\times x(\mathbf{v}_1,\mathbf{v}_2)$ has a singularity at small $\mathbf{q}$ which comes from those terms of the expansion of $\chi(\mathbf{v}_1,\mathbf{v}_2)$ in a double series in eigenfunctions of the operators $\pm i\mathbf{q}\mathbf{v}_i-n\Lambda_0(i)$ for which the eigenvalues of these operators vanish for $q\to 0$. In fact, the operator

$$S(\mathbf{q},\mathbf{v}_1)=i\mathbf{q}\mathbf{v}_1-n\Lambda_0(1) \quad (22a)$$

has five such eigenfunctions corresponding to hydrodynamical modes. In zeroth-order perturbation theory in the small parameter $i\mathbf{q}\mathbf{v}_1$ these functions have the following form:

1. Functions corresponding to sound modes with eigenvalues $\{\pm i|\mathbf{q}|(5\theta/3m)^{-1/2}\Gamma q^2\}$:

$$\psi^{(\mathbf{q})}_{1,2}(\mathbf{v})=\frac{mv^2}{\theta\sqrt{30}}\pm\frac{\mathbf{q}\mathbf{v}}{|\mathbf{q}|}\left(\frac{m}{2\theta}\right)^{1/2} \quad (22b)$$

2. A function corresponding to a heat mode with eigenvalue $-D_Tq^2$:

$$\psi^{(\mathbf{q})}_{3}(\mathbf{v})=\frac{1}{\sqrt{10}}\left(\frac{mv^2}{\theta}-5\right). \quad (22c)$$

3. Functions corresponding to shear viscosity modes with eigenvalue $-D_\eta\mathbf{q}^2$:

$$\psi^{(\mathbf{q})}_{4,5}(\mathbf{v})=\left(\frac{m}{\theta}\right)^{1/2}\mathbf{e}_i\mathbf{v},\quad \mathbf{e}_i\mathbf{q}=0,\quad \mathbf{e}_i^2=1. \quad (22d)$$

Here $D_\eta$ and $D_T$ are the coefficient of kinematical viscosity and thermal diffusion, $\Gamma = 2/3(D_T + 2D_\eta)$, and all the eigenvalues have been calculated in the first two orders of perturbation theory in the small parameter $|\mathbf{q}|$.

Comparing the above eigenvalues of $S(\mathbf{q},\mathbf{v}_1)$ with (21), it is easily seen that the integrand in (21) can have a $1/\mathbf{q}^4$ type of singularity at small $\mathbf{q}$. This causes the integral determining the function of the second-order approximation of the Chapman–Enskog method $\varphi_2(1)$ to diverge.[9]

Therefore, the expansion (17) does not lead us to the desired singularity-free hydrodynamical equations, so it is necessary to seek a different method of constructing approximate solutions of (18a) and (18b).

Such a method was porposed in Ref. 10 and consists of choosing the following form of the solutions:

$$f_1(t, 1) = f_M(1)(1 + \mu\varphi_1(t, 1 \mid \mu)), \quad (23)$$

$$g_2(t, 1, 2) = f_M(\xi_1, \mathbf{v}_1) f_M(\xi_2, \mathbf{v}_2)(\mu\psi_1(t, 1, 2 \mid \mu)),$$

$$\xi_{1,2} = \mathbf{R} \pm \frac{\rho\mu}{2},$$

where the order of $\varphi_1(t,1|\mu)$, $\gamma_1(t,1,2|\mu)$ must be found from the system (18a) and (18b).

Since the conservation laws permit us to move products $f_M(i)f_M(j)$ to the left of operators $T(i,j)$, the equations (18a) taking into account (23) acquire the form

$$[f_M(1)]^{-1}\left[\frac{\partial}{\partial t} + \mathbf{v}_1 \frac{\partial}{\partial \mathbf{r}_1}\right] f_M(1) = n\Lambda(1, \mathbf{r}_1)\varphi_1(t, 1)$$

$$+ n \int d\mathbf{v}_2 f_M(\mathbf{r}_1, \mathbf{v}_2) \widetilde{T}(1, 2)\psi_1(t, \mathbf{r}_1, 0; \mathbf{v}_1, \mathbf{v}_2), \quad (24a)$$

where we have introduced the notation

$$\psi_1(t, 1, 2 \mid \mu) = \psi_1(t, \mathbf{R}, \boldsymbol{\rho}; \mathbf{v}_1, \mathbf{v}_2);$$

$$\left(\mathbf{R} = \frac{\mathbf{r}_1 + \mathbf{r}_2}{2}, \ \boldsymbol{\rho} = \mathbf{r}_1 - \mathbf{r}_2\right);$$

$$T(1, 2) = \widetilde{T}(1, 2)\,\delta(\mathbf{r}_1 - \mathbf{r}_2);$$

$$\Lambda(i, \xi)\chi(\mathbf{v}_i) = \int d\mathbf{v}_3 f_M(\xi, \mathbf{v}_3)\widetilde{T}(i, 3)$$

$$[\chi(\mathbf{v}_i) + \chi(\mathbf{v}_3)], \ i = 1, 2.$$

We note that because of (23), Eq. (24a) does not explicitly contain the uniformity parameter.

The second equation of the system (18a) and (18b) acquires the form

$$\left[\mu\left(\frac{\partial}{\partial t} + \frac{\mathbf{v}_1 + \mathbf{v}_2}{2}\frac{\partial}{\partial \mathbf{R}}\right) + (\mathbf{v}_1 - \mathbf{v}_2)\frac{\partial}{\partial \boldsymbol{\rho}}\right.$$

$$\left. + \mu\mathscr{L}(t, \mathbf{R}, \mathbf{v}_1, \mathbf{v}_2) - n(\Lambda(1, \xi_1) + \Lambda(2, \xi_2))\right]$$

$$\psi_1(t, \mathbf{R}, \boldsymbol{\rho}, \mathbf{v}_1, \mathbf{v}_2)$$

$$= \delta(\boldsymbol{\rho})\widetilde{T}(1, 2)[\varphi_1(t, \mathbf{R}, \mathbf{v}_1) + \varphi_2(t, \mathbf{R}, \mathbf{v}_2)]. \quad (24b)$$

Here we have introduced the notation

$$\mathscr{L}(t, \mathbf{R}, \boldsymbol{\rho}; \mathbf{v}_1, \mathbf{v}_2)$$

$$= \left[\frac{\partial}{\partial t} + \frac{\mathbf{v}_1 + \mathbf{v}_2}{2}\frac{\partial}{\partial \mathbf{R}} + \frac{1}{\mu}(\mathbf{v}_1 - \mathbf{v}_2)\frac{\partial}{\partial \boldsymbol{\rho}}\right]$$

$$\times \ln f_M(\xi_1, \mathbf{v}_1) f_M(\xi_2, \mathbf{v}_2).$$

Isolating in (24b) the terms of zeroth and first order in $\mu$, we find

$$\left\{(\mathbf{v}_1 - \mathbf{v}_2)\frac{\partial}{\partial \boldsymbol{\rho}} - n(\hat{\Lambda}_0(1, \mathbf{R}) + \hat{\Lambda}_0(2, \mathbf{R}))\right.$$

$$+ \mu\left[\frac{\partial}{\partial t} + \frac{1}{3}(\mathbf{v}_1 + \mathbf{v}_2)\frac{\partial}{\partial \mathbf{R}}\right.$$

$$\left.\left. - \frac{n\rho}{2}\left(\frac{\partial\hat{\Lambda}_0(1, \mathbf{R})}{\partial \mathbf{R}} - \frac{\partial\hat{\Lambda}_0(2, \mathbf{R})}{\partial \mathbf{R}}\right) + \widetilde{\mathscr{L}}_0\right]\right\}\psi_1$$

$$(t, \mathbf{R}, \boldsymbol{\rho}; \mathbf{v}_1, \mathbf{v}_2)$$

$$= \delta(\boldsymbol{\rho})\hat{T}(1, 2)[\varphi_1(t, \mathbf{R}, \mathbf{v}_1) + \varphi_2(t, \mathbf{R}, \mathbf{v}_2)]; \quad (24c)$$

$$\widetilde{\mathscr{L}}_0 = \sum_{\alpha=1}^{2}\{f_M(\mathbf{R}, \mathbf{v}_\alpha)\}^{-1}\left[\frac{\partial}{\partial t} + \mathbf{v}_\alpha\frac{\partial}{\partial \mathbf{R}}\right] f_M(\mathbf{R}, \mathbf{v}_\alpha),$$

$$\frac{\partial\hat{\Lambda}_0(1, \mathbf{R})}{\partial \mathbf{R}}\chi(\mathbf{v}_1) = \int d\mathbf{v}_3 \frac{\partial f_M(\mathbf{R}, \mathbf{v}_3)}{\partial \mathbf{R}}\widetilde{T}(1, 3)[\chi(\mathbf{v}_1) + \chi(\mathbf{v}_3)],$$

where $\hat{\Lambda}_0(i,\mathbf{R})$ are linearized operators of the type (20) with $f_{eq}(x)$ replaced by the quasi-equilibrium Maxwell function $f_M(\mathbf{R},\mathbf{v}_3)$. Expressing the time derivatives $\partial f_M/\partial t$ in (24b) in terms of the zeroth-order hydrodynamical equations (15) for $P_{ik} = (\theta/m)\delta_{ik}$, $\mathbf{q} = 0$, we obtain

$$n\hat{\Lambda}_0(1, 2)\varphi_1(t, \mathbf{r}_1, \mathbf{v}_1) + n\int d\mathbf{v}_2 f_M(\mathbf{r}, \mathbf{v}_2)$$

$$\widetilde{T}(1, 2)\psi_1(t, \mathbf{r}, 0; \mathbf{v}_1, \mathbf{v}_2)$$

$$= \frac{1}{2\theta}\frac{\partial\theta}{\partial \mathbf{r}}\mathbf{A}(\mathbf{V}_1) + \frac{m}{\theta}S_{ik}B_{ik}(\mathbf{V}_1); \quad (25a)$$

$$\left[\mu\left(\frac{\partial}{\partial t} + \frac{(\mathbf{v}_1 + \mathbf{v}_2)}{2}\frac{\partial}{\partial \mathbf{R}} - \frac{n\rho}{2}\left(\frac{\partial\hat{\Lambda}_0(1)}{\partial \mathbf{R}} - \frac{\partial\hat{\Lambda}_0(2)}{\partial \mathbf{R}}\right) + \mathscr{L}_0\right)\right.$$

$$\left. + (\mathbf{v}_1 - \mathbf{v}_2)\frac{\partial}{\partial \boldsymbol{\rho}} - n(\hat{\Lambda}_0(1, \mathbf{R}) + \hat{\Lambda}_0(2, \mathbf{R}))\right]$$

$$\psi_1(t, \mathbf{R}, \mathbf{r}; \mathbf{v}_1, \mathbf{v}_2)$$

$$= \delta(\boldsymbol{\rho})\hat{T}(1, 2)[\varphi_1(t, \mathbf{R}, \mathbf{v}_1) + \varphi_2(t, \mathbf{R}, \mathbf{v}_2)], \quad (25b)$$

where

$$\mathscr{L}_0 = \frac{1}{2\theta}\frac{\partial\theta}{\partial \mathbf{r}}\sum_{\alpha=1}^{2}\mathbf{A}(\mathbf{V}_\alpha) + \frac{m}{\theta}S_{ik}\sum_{\alpha=1}^{2}B_{ik}(\mathbf{V}_\alpha);$$

$$\mathbf{A}(\mathbf{V}_\alpha) = \mathbf{V}_\alpha\left(\frac{m\mathbf{V}_\alpha}{\theta} - 5\right); \ B_{ik}(\mathbf{V}_\alpha) = \mathbf{V}_{i\alpha}\mathbf{V}_{k\alpha} - \frac{1}{3}\delta_{ik}\mathbf{V}_\alpha^2;$$

$$\mathbf{V} = \mathbf{v} - \mathbf{u}(\mathbf{r}, t).$$

We note that the system (25a) and (25b) is linear. When the term linear in $\mu$ is discarded in (25b), the solution of this equation contains divergences. Keeping this term, we can write the solution of (25b) in the formal form

$$\psi_1(t, \mathbf{R}, \mathbf{r}; \mathbf{v}_1, \mathbf{v}_2) = \frac{1}{\mu}\int_0^t \hat{Q}(t, \mu)\hat{Q}^{-1}(\tau, \mu)\delta(\boldsymbol{\rho})\widetilde{T}(1, 2)$$

$$\times(\varphi_1(t, \mathbf{R}, \mathbf{v}_1) + \varphi_1(t, \mathbf{R}, \mathbf{v}_2))\,d\tau,$$

where

$$\hat{Q}(t, \mu) = T\exp\int_0^t$$

$$\left\{-\frac{(\mathbf{v}_1 + \mathbf{v}_2)}{2}\frac{\partial}{\partial \mathbf{R}} - \mathscr{L}_0 - \frac{n\rho}{2}\left(\frac{\partial\hat{\Lambda}_0(1)}{\partial \mathbf{R}} - \frac{\partial\hat{\Lambda}_0(2)}{\partial \mathbf{R}}\right)\right.$$

$$\left. + \frac{1}{\mu}\left((\mathbf{v}_1 - \mathbf{v}_2)\frac{\partial}{\partial \boldsymbol{\rho}} - n(\hat{\Lambda}_0(1, \mathbf{R}) + \hat{\Lambda}_0(2, \mathbf{R}))\right)\right\}d\tau, \quad (26)$$

where in the integrand of (26) all the operators $\hat{\Lambda}_0$ and also the term $\mathscr{L}_0$ must be taken at the time $t = \tau$. Using (26), it is not difficult to find the solution of (25a) which, according to (16) and (23), gives the desired hydrodynamical equations:

$$\varphi_1(t, \mathbf{r}, \mathbf{v}_1) = \frac{1}{n(\Lambda_0(1) + \hat{z})}\left\{\frac{1}{2\theta}\frac{\partial\theta}{\partial \mathbf{r}}\mathbf{A}(\mathbf{v}_1) + \frac{m}{\theta}S_{ik}B_{ik}(\mathbf{v}_1)\right\},$$

where

$$\hat{z}\chi(t, \mathbf{r}, \mathbf{v}_1) = \frac{1}{\mu}\int d\mathbf{v}_2 f_M(\mathbf{r}, \mathbf{v}_2)\widetilde{T}(1, 2)$$

$$\times\int_0^t \hat{Q}(t, \mu)\hat{Q}^{-1}(\tau, \mu)\delta(\boldsymbol{\rho})\widetilde{T}(1, 2)(\chi(\tau, \mathbf{r}, \mathbf{v}_1)$$

$$+ \chi(t, \mathbf{r}, \mathbf{v}_2))\,d\tau|_{\rho=0}. \quad (27)$$

Here at the final stage, after isolating the leading sing[ular terms] in $\mu$ for $\mu \to 0$, we must set $\mu = 1$ in the final expres[sions].

Naturally, this requires the explicit calculation of all the integrals entering into (23), (26), and (27) and cannot be done in practice. In the following sections we shall see how we can construct solutions of such hydrodynamical equations in simple cases.

## 2. THE LINEARIZED EQUATIONS AND DEPENDENCE OF THE FREQUENCIES OF THE HYDRODYNAMICAL MODES ON THE WAVE VECTOR

The method of constructing the hydrodynamical equations using the kinetic equations (8a) and (8b) which was described in Sec. 1 is simplified considerably when the system is in a state so close to equilibrium that all but the first power of $|n-\rho(\mathbf{r},t)|$, $|\mathbf{u}(\mathbf{r},t)|$, and $|\theta_0-\theta(\mathbf{r},t)|$ can be neglected in (23)–(27). In this case it is much easier to start directly from the kinetic equations (8a) and (8b) (Ref. 12), assuming that

$$f(t,1)=f_{eq}(1)(1+\delta f(t,1));$$

$$g(t,1,2)=f_{eq}(1)f_{eq}(2)(g_{eq}(\mathbf{r}_1-\mathbf{r}_2)+\delta g(t,1,2)),$$

and neglecting all but the first power of $\delta f(t,1)$ and $\delta g(t,1,2)$. The resulting system of linear kinetic equations is much simpler than the original one; in particular, the operators $\mathscr{L}(t,i)$ (9) become ordinary linear operators which commute at different times, so that a relation between $\delta g(t,1,2)$ and $\delta f(t,1)$ of the form (10) does not contain the $T$-ordering operation. The investigation of this system by Ernst and Dorfman led to the discovery of a new effect arising from the nonlocality of the kinetic equations—a nonanalytic dependence of the frequencies of the hydrodynamical excitations on the wave vector. For example, for sound modes it was found that

$$\omega(k)=ck+\frac{i}{2}\left(\Gamma k^2+\sum_{n=1}^{\infty}k^{3-2^{-n}}C_n+d_1k^2\ln k+\ldots\right),$$

where $c$ and $\Gamma$ are the speed of sound and the damping coefficient obtained from the linearized Boltzmann equation. Analytic expansions were obtained for the shear viscosity and heat-conduction modes which contain both an infinite power series in $k$ with exponents between 5/2 and 3 and logarithmic terms. The coefficients are functions of the density and temperature.

It was later shown that expansions similar to these also occur in the phenomenological theory of interacting hydrodynamical modes.[13] Therefore, the frequencies as functions of $k$ have a branch point at $k=0$, and the Riemann surface of the function $\omega(k)$ contains an infinite number of sheets.

By means of the Laplace and Fourier transformations

$$\delta\tilde{f}(z,1)=\int_0^{\infty}dt\,e^{-zt}\delta f(t,1);$$
$$\delta\tilde{\tilde{f}}(z,\mathbf{k},\mathbf{v})=\int d\mathbf{r}_1e^{-i\mathbf{k}\mathbf{r}_1}\delta\tilde{f}(z,1) \tag{28}$$

from (8a) and (8b) we can obtain a linear integral equation for $\delta\tilde{f}$ (Ref. 10):

$$[z+i\mathbf{k}\mathbf{v}-n\hat{\Lambda}_0(\mathbf{v})-n\hat{R}(\mathbf{k},\mathbf{v},z)]$$
$$\delta\tilde{\tilde{f}}(z,\mathbf{k},\mathbf{v})=\delta\tilde{f}(0,\mathbf{k},\mathbf{v}), \tag{29}$$

where $\delta\tilde{f}(0,\mathbf{k},\mathbf{v})$ is the value of the Fourier transform of $\delta f(t,1)$ at the time $t=0$ and $\hat{R}(\mathbf{k},z,\mathbf{v})$ is a ring operator:

$$\hat{R}(\mathbf{k},z,\mathbf{v}_1)\chi(\mathbf{v}_1)=\int d\mathbf{v}_2f_{eq}(\mathbf{v}_2)$$
$$\times\int\frac{d\mathbf{q}}{(2\pi)^3}\tilde{T}(1,2)(z+i\mathbf{q}\mathbf{v}_1+i(\mathbf{k}-\mathbf{q})\mathbf{v}_2$$
$$-n\hat{\Lambda}_0(\mathbf{v}_1)-n\hat{\Lambda}_0(\mathbf{v}_2))^{-1}\tilde{T}(1,2)(1+P_{1,2})\chi(\mathbf{v}_1). \tag{30}$$

The hydrodynamical equations can be constructed from the solutions of (29) in the following manner. Macroscopic variables are introduced as velocity averages of functions corresponding to the hydrodynamical modes (22b) and (22c):

$$a_k^{(i)}(\mathbf{r},t)=\int f_{eq}(\mathbf{v})\psi_i^{(\mathbf{k})}(\mathbf{v})\,\delta f(t,\mathbf{r},\mathbf{v})\,d\mathbf{v}$$
$$=\langle\psi_i^{(\mathbf{k})}(\mathbf{v}),\ \delta f(t,\mathbf{r},\mathbf{v})\rangle, \tag{31}$$

where $\langle\ldots\rangle$ stands for the scalar product with weight $f_{eq}(\mathbf{v})$. In the Hilbert space of the functions $\{f\}$ defined by this scalar product one can introduce operators which project onto the hydrodynamical space and its orthogonal extension via the expressions[18]

$$Pf(\mathbf{v})=\sum_i\psi_i^{(\mathbf{k})}(\mathbf{v})\langle\psi_i^{(\mathbf{k})}(\mathbf{v}),\ f(\mathbf{v})\rangle,$$
$$P_\perp f(\mathbf{v})=(1-P)f(\mathbf{v}). \tag{32}$$

We note that these operators are Hermitian. By projecting (29) on both of these spaces we can find the desired linearized hydrodynamical equations for the quantitites $\tilde{\tilde{a}}^{(i)}(k,z)$:

$$\sum_{j=1}^{5}\hat{G}_{ij}\tilde{\tilde{a}}^{(j)}(\mathbf{k},z)=\tilde{\tilde{a}}^{(i)}(k,0)$$
$$+\langle\psi_i^{(\mathbf{k})},\ \mathscr{L}P_\perp(z-P_\perp\mathscr{L}P_\perp)^{-1}P_\perp\delta\tilde{f}(0,\mathbf{k},\mathbf{v})\rangle, \tag{33}$$

where

$$\hat{G}_{ij}=z\delta_{ij}-\langle\psi_i^{(\mathbf{k})},\ \mathscr{L}\psi_i^{(\mathbf{k})}\rangle$$
$$-\langle\psi_i^{(\mathbf{k})},\ \mathscr{L}P_\perp(z-P_\perp\mathscr{L}P_\perp)^{-1}P_\perp\mathscr{L}\psi_j^{(k)}\rangle; \tag{34}$$
$$\mathscr{L}=-i\mathbf{k}\mathbf{v}+n\hat{\Lambda}_0(\mathbf{v})+n\hat{R}(\mathbf{k},z,\mathbf{v}).$$

Let us discuss the roles of the various terms on the right-hand side of (33). The first is the value of the Fourier transform of one of the macroscopic variables at the time $t=0$ and is of zeroth order in $|\mathbf{k}|$ and $z$. The second term can be rewritten as

$$\langle\mathscr{L}^+\psi_i^{(\mathbf{k})},\ P_\perp(z-P_\perp\mathscr{L}P_\perp)^{-1}P_\perp\delta\tilde{f}(0,\mathbf{k},\mathbf{v})\rangle.$$

Using the relations

$$\hat{\Lambda}_0\psi_i^{(\mathbf{k})}=0,\quad\hat{R}(\mathbf{k},z,\mathbf{v})\psi_i^{(\mathbf{k})}=0, \tag{35}$$

it can be shown that this term is of order $|\mathbf{k}|$ and can be dropped in the hydrodynamical limit. The physical reason for this is obvious: the initial values of the projection of the distribution function on the space orthogonal to the hydrodynamical space must not affect the evolution of the macroscopic variables according to the principle of reduction of the description. Therefore, the problem is to calculate the matrix elements $\hat{G}_{ij}=z\delta_{ij}+i\mathbf{k}\Omega_{ij}(\mathbf{k})+k^2u_{ij}(\mathbf{k},z)$, that is, scalar products of the form

$$ik\Omega_{ij}(k)=i\langle\varphi_i^{(\mathbf{k})},\ \mathbf{k}\mathbf{v}\psi_j^{(\mathbf{k})}\rangle; \tag{36}$$

$$k^2u_{ij}(k,z)=\langle\psi_i^{(\mathbf{k})},\ \mathbf{k}\mathbf{v}P_\perp(z-P_\perp\mathscr{L}P_\perp)^{-1}$$
$$P_\perp\mathbf{k}\ \mathbf{v}\psi_j^{(\mathbf{k})}\rangle. \tag{37}$$

We note that we have used the relation (35) to write the matrix elements (34) in the form (36) and (37). The integrals (36) are easily calculated taking into account the explicit form of the functions $\{\psi_i^{(\mathbf{k})}\}$. As a result, we find

$$\Omega_{11} = -\Omega_{22} = (5m/3\theta)^{1/2}. \quad (38)$$

All the other elements of the matrix $\Omega$ are equal to zero. It is also easy to find the functions of the form $P_\perp(\mathbf{k},\mathbf{v})\psi_j^{(\mathbf{k})}(\mathbf{v})$. Application of the formula (32) for the projection operator $P_\perp$ gives

$$u_{ij}(k, z) = \langle \hat{j}_i^{(\mathbf{k})}, (z - P_\perp \mathscr{L} P_\perp)^{-1} \hat{j}_j^{(\mathbf{k})} \rangle, \quad (39)$$

where

$$\hat{j}_{1,2}^{(\mathbf{k})}(\mathbf{v}) = \left(\frac{mv^2}{\theta} - 5\right) \frac{\mathbf{ev}}{\sqrt{30}} \pm \left(\frac{m}{2\theta}\right)^{1/2} \left((\mathbf{ev})^2 - \frac{1}{3}\mathbf{v}^2\right);$$

$$\hat{j}_3(\mathbf{v}) = \left(\frac{mv^2}{\theta} - 5\right) \frac{\mathbf{ev}}{\sqrt{10}}; \quad \hat{j}_{4,5}(\mathbf{v}) = \left(\frac{m}{\theta}\right)^{1/2} (\mathbf{ev})(\mathbf{e}_\alpha \mathbf{v});$$

$\mathbf{e} = \mathbf{k}/|\mathbf{k}|$, and $\mathbf{e}_1$ and $\mathbf{e}_2$ together with $\mathbf{e}$ form an orthonormal basis. We can immdeiately conclude that $u_{4j} = u_{j4} = \delta_{j4} u_{44}$ and $u_{5j} = u_{j5} = \delta_{j5} u_{55}$, i.e., the matrix $u_{ij}$ (39) splits into an upper $(3\times3)$ and a lower diagonal $(2\times2)$ block. Equation (39) can be simplified further using the fact that in the hydrodynamical region, where $z$ and $|\mathbf{k}|$ are small, the quantities $z$ and $P_\perp(\mathbf{kv})P_\perp$ in the expression $(z - P_\perp \mathscr{L} P_\perp)^{-1}$ appearing in (39) can be neglected. Using the relations $P_\perp \hat{j}_i^{(\mathbf{k})}$ and $P_\perp \Lambda_0 = \Lambda_0$, we can rewrite (39) in the form

$$u_{ij}(\mathbf{k}, z) = -\frac{1}{n} \langle \hat{j}_i^{(\mathbf{k})}, (\Lambda_0 + \hat{R}(\mathbf{k}, z, \mathbf{v}))^{-1} \hat{j}_j^{(\mathbf{k})} \rangle. \quad (40)$$

The action of the operator $\hat{R}(\mathbf{k},z,\mathbf{v})$ (30) on functions of the argument $\mathbf{v}_1$ can be rewritten in the form

$$\hat{R}(\mathbf{k}, z, \mathbf{v}_1)\chi(\mathbf{v}_1) = \int d\mathbf{v}_2 f_{\text{eq}}(2) \int' \frac{d\mathbf{q}}{(2\pi)^3} \tilde{T}(1, 2) \sum_{i,j}' \frac{1}{z - z_i(\mathbf{q}) - z_j(\mathbf{k}-\mathbf{q})}$$

$$\times \| \psi_i^{(\mathbf{q})}(\mathbf{v}_1) \psi_j^{(\mathbf{k}-\mathbf{q})}(\mathbf{v}_2) \rangle\rangle \langle\langle \psi_j^{(\mathbf{k}-\mathbf{q})}(\mathbf{v}_2) \psi_i^{(\mathbf{q})}(\mathbf{v}_1) \|$$

$$\tilde{T}(1, 2)(1 + P_{1,2})\chi(\mathbf{v}_1)$$

$$+ \tilde{R}_{\text{reg}}(\mathbf{k}, \mathbf{v}, z)\chi(\mathbf{v}_1)$$

$$= \hat{R}_s(\mathbf{k}, z, \mathbf{v}_1)\chi(\mathbf{v}_1) + \hat{R}_{\text{reg}}(\mathbf{k}, z, \mathbf{v}_1)\chi(\mathbf{v}_1). \quad (41)$$

The region of integration over $\mathbf{q}$ in the first term in (41) is bounded by a sphere of radius $q_0$, where $q_0^{-1}$ is of the order of the mean free path; the summation runs over all values of the indices $(i,j)$ for which the sum of the eigenvalues corresponding to the functions $\psi_i^{(\mathbf{q})}(\mathbf{v}_1)$, $\psi_j^{(\mathbf{k}-\mathbf{q})}(\mathbf{v}_2)$, and $z_i(\mathbf{q}) + z_j(\mathbf{k}-\mathbf{q})$ is of order $\mathbf{q}^2$, $\mathbf{k}^2$, and $\mathbf{kg}$ for $|\mathbf{k}|, |\mathbf{q}| \to 0$. It is precisely this term (the second is determined by the relation $\bar{R}_{\text{reg}} = \hat{R} - \hat{R}_s$) which has interesting singularities at small $k$ and $z$, whereas $\hat{R}_{\text{reg}}$ leads only to corrections of higher order in the density.

The transformations (41), which amount to computation of the action of the operator $\tilde{T}$ on functions $\psi_i^{(\mathbf{q})}(\mathbf{v}_1)\psi_j^{(\mathbf{k}-\mathbf{q})}(\mathbf{v}_2)$, lead to the following result:

$$\hat{R}_s = \Lambda_0(\mathbf{v}) S_{\mathbf{k},z} \Lambda_0(\mathbf{v}),$$

where

$$S_{\mathbf{k},z}(\mathbf{v})\chi(\mathbf{v}) = \frac{1}{2} \sum_{i,j}' \int' \frac{d\mathbf{q}}{(2\pi)^3}$$

$$\times \frac{\psi_i^{(\mathbf{q})}(\mathbf{v}) \psi_j^{(\mathbf{k}-\mathbf{q})}(\mathbf{v})}{z - z_i(\mathbf{q}) - z_j(\mathbf{k}-\mathbf{q})} \langle \psi_j^{(\mathbf{k}-\mathbf{q})}(\mathbf{v}) \psi_i^{(\mathbf{q})}(\mathbf{v}), \chi(\mathbf{v}) \rangle. \quad (42)$$

Making the replacement $\hat{R} \to \hat{R}_s$ in (40) and using the relation

$$(A + B)^{-1} = A^{-1} - A^{-1} B A^{-1} + \ldots,$$

we find an interesting singularity in the hydrodynamical matrix $u_{ij}(\mathbf{k},z)$:

$$u_{ij}(\mathbf{k}, z) \approx u_{ij}^{(0)} + \frac{1}{n} \langle \hat{j}_i^{(\mathbf{k})}, \hat{S}_{\mathbf{k},z} \hat{j}_j^{(\mathbf{k})} \rangle, \quad (43)$$

where

$$u_{ij}^{(0)} = -\frac{1}{n} \langle \hat{j}_i^{(\mathbf{k})}, \Lambda_0^{-1} \hat{j}_j^{(\mathbf{k})} \rangle.$$

The first term in (43) corresponds to the Laplace and Fourier transformation of the linearized Navier–Stokes equations. The second can be written as

$$\tilde{u}_{ij}(\mathbf{k}, z) = \frac{1}{(2\pi)} \int' \frac{d\mathbf{q}}{(2\pi)^3} \sum_{\alpha,\beta}' \frac{w_{\alpha,\beta}^i(\mathbf{k}, \mathbf{q}) w_{\alpha,\beta}^j(\mathbf{k}, \mathbf{q})}{z - z_\alpha(\mathbf{q}) - z_\beta(\mathbf{k}-\mathbf{q})}, \quad (44)$$

where

$$w_{\alpha,\beta}^i(\mathbf{k}, \mathbf{q}) = \langle \hat{j}_i^{(\mathbf{k})}, \psi_\alpha^{(\mathbf{q})} \psi_\beta^{(\mathbf{k}-\mathbf{q})} \rangle.$$

The function $\tilde{u}_{ij}(\mathbf{k},z)$ at small $|\mathbf{k}|$ and $z$ has singularities related to the singularity of the integrand in (44). In particular, for $\mathbf{k} = 0$, $\tilde{u}_{ij} \sim \text{const}\sqrt{z}$, and for $z \sim k, k^2$, $\tilde{u}_{ij} \sim \text{const}\sqrt{k} + O(k \ln k, k^2, \ln k)$. These singularities are doubled in the solutions of the generalized hydrodynamical equations (33). First of all, the frequencies of the hydrodynamical modes determined by the condition

$$\det(z\delta_{ij} + ik\Omega_{ij} + k^2 u_{ij}(\mathbf{k}, z)) = 0, \quad (45)$$

are not analytic functions of the wave vector $|\mathbf{k}|$. Since at small $|\mathbf{k}|$, $z \sim k$, $k^2$ and $\tilde{u}_{ij} \sim \text{const}\sqrt{k}$, it is clear that the general form of the $k$ dependence of $z$ given by (45) is

$$z_{1,2} = \pm |\mathbf{k}| \, i \left(\frac{5m}{3\theta}\right)^{1/2} - \frac{1}{2}\Gamma |\mathbf{k}|^2 + |\mathbf{k}|^{5/2} \Delta_{1/2};$$

$$z_3 = -D_T k^2 + \Delta_3 |\mathbf{k}|^{5/2}; \quad (46)$$

$$z_{4,5} = -D_\eta k^2 + \Delta_{4,5} |\mathbf{k}|^{5/2}.$$

Explicit expressions for the $\Delta_i$ can be obtained by direct calculation of the coefficients of the leading singularities in the integrals (44) (Ref. 16):

$$\Delta_{1,2} = (1 \mp i)(4\pi)^{-1} \left(\frac{5\theta}{12m}\right)^{1/4} \frac{\theta}{mn} \left[\frac{14}{45(2D_\eta)^{3/2}} + \frac{2}{9(D_\eta + D_T)^{3/2}} + \frac{2^{19/2}}{(11)!!\,\Gamma^{3/2}}\right];$$

$$\Delta_3 = (21\pi)^{-1} \left(\frac{5\theta}{12m}\right)^{1/4} \frac{\theta}{mn} \Gamma^{-3/2};$$

$$\Delta_{4,5} = (77\pi)^{-1} \left(\frac{5\theta}{12m}\right)^{1/4} \frac{\theta}{mn} \Gamma^{-3/2},$$

where $\Gamma = \frac{2}{3}(D_T + 2D_\eta)$ and $D_T$ and $D_\eta$ are the coefficients of the kinematical viscosity and heat diffusion in the hard-sphere model determined from the relations

$$D_\eta = -\frac{m}{\theta n} \langle (\mathbf{e}_1\mathbf{v})(\mathbf{e}_2\mathbf{v}), \Lambda_0^{-1}(\mathbf{e}_1\mathbf{v})(\mathbf{e}_2\mathbf{v}) \rangle;$$

$$D_T = -\frac{1}{10n} \left\langle \left(\frac{mv^2}{\theta} - 5\right)(\mathbf{ev}), \Lambda_0^{-1} \left(\frac{mv^2}{\theta} - 5\right)(\mathbf{ev}) \right\rangle.$$

Detailed calculation of the roots of Eq. (45) based on the method of successive approximations shows that the expansion (46) also contains terms of the form $|\mathbf{k}|^{3-2^{-n}}, n = 2,\ldots$. This fact, which was first noticed in Ref. 13, indicates that logarithmic branch points of the functions $z_j(\mathbf{k})$ may be present. The relative magnitude of the terms in (46) which are nonanalytic in $|\mathbf{k}|$ is very small in gases of moderate density. This makes it extremely difficult to detect such terms in experimental studies of the propagation of hydrodynamical perturbations in gases and liquids. At the same time, the existence of such a nonanalyticity is in conflict with the description of the hard-sphere system using the linearized Burnett equations.

The existence of branch points of $u_{ij}(\mathbf{k},z)$ also leads to new singularities in the solutions of the hydrodynamical equations (33). In particular, the time dependence of these equations is determined by the inverse Laplace transform

$$\tilde{a}^{(i)}(\mathbf{k},\ t) = \int\limits_{\Gamma} \frac{dze^{zt}}{2\pi i}\,(z + ik\Omega(\mathbf{k}) + k^2 u(\mathbf{k},\ t))_{ij}^{-1}\,\tilde{a}^{(i)}(\mathbf{k},\ 0), \tag{47}$$

where the contour $\Gamma$ runs parallel to the imaginary axis to the right of all the singularities of the integrand. In addition to the poles determined by (45), (47) can also receive contributions from integrals along the branch cuts

$$-\infty \langle z + i\,|\mathbf{k}|\left(\frac{5\theta}{3m}\right)^{1/2} < 0, \quad -\infty < z \leqslant \frac{1}{2} D_{T,\,\eta} k^2,$$

which correspond to the branch points of $\tilde{u}_{ij}(\mathbf{k},z)$ in the complex plane.[19,20]

The asymptote of the hydrodynamical modes $\tilde{a}^{(i)}(\mathbf{k},t)$ for $t\to\infty$ determined by the pole contributions has the standard form:

$$\tilde{a}^{(1,\,2)}(\mathbf{k},\ t) \sim \exp\left\{\pm ik\left(\frac{5\theta}{3m}\right)^{1/2} - \frac{1}{2}\Gamma k^2 t\right\}\{1 + O(k)\},$$
$$\tilde{a}^{(3)}(\mathbf{k},\ t) \sim \exp\{-k^2 D_T t\}\{1 + O(\mathbf{k}^2)\},$$
$$\tilde{a}^{(4,5)}(\mathbf{k},\ t) \sim \exp\{-k^2 D_\eta t\}\{1 + O(k^2)\}.$$

At small $t/\tau_i$ ($\tau_i = \{2/\Gamma k^2, 1/D_T k^2, 1/D_\eta k^2\}$) the pole contribution dominates. The contribution from the cuts becomes important for $t/\tau_i \gg 1$, since the branch points of $u_{ij}(\mathbf{k},z)$ lie to the right of the corresponding poles. In particular, for the hydrodynamical viscosity and heat-conduction modes the asymptote for $t\to\infty$ due to the branch cuts has the form[20]

$$\left.\begin{aligned} \tilde{a}^{(3)}(\mathbf{k},\ t) &\sim \exp\left\{-\frac{k^2}{2} D_T t\right\} f_T(t);\\ \tilde{a}^{(4,\,5)}(\mathbf{k},\ t) &\sim \exp\left\{-\frac{k^2}{2} D_\eta t\right\} f_\eta(t), \end{aligned}\right\} \tag{48}$$

where $f_T(t)$ and $f_\eta(t)$ have a power-law falloff for $t\to\infty$. We also note that the $z$ dependence of the hydrodynamical matrix (47) does not permit us to write the hydrodynamical equations for $\tilde{a}^{(i)}(\mathbf{k},t)$ in a local form in the variable $t$.

## 3. DESCRIPTION OF NONLINEAR TRANSPORT PROCESSES IN A STATIONARY FLUX WITHOUT HEAT EXCHANGE: THE NONANALYTIC DEPENDENCE OF THE VISCOSITY ON THE VELOCITY GRADIENT

Here we shall study the simplest nonlinear phenomenon occurring in the generalized hydrodynamical theory for a hard-sphere system, which arises in the study of a stationary uniform flux with a given simple distribution of the local macroscopic velocity $\mathbf{u}(\mathbf{r})$:

$$\mathbf{u}(\mathbf{r}) = \mathbf{e}\,(u_0 + \varkappa r), \tag{49}$$

where the vectors $\mathbf{e}$ and $\boldsymbol{\varkappa}$ are perpendicular (for simplicity, we shall henceforth assume that $\mathbf{u}$ is directed along te $y$ axis and that the velocity gradient $\boldsymbol{\varkappa}$ lies along the $x$ axis). The value of $\boldsymbol{\varkappa}$ is assumed to be so small that effects $\sim|\boldsymbol{\varkappa}|^2$ (in particular, heat exchange) can be neglected. We can therefore consider the values of all the macroscopic variables as given $[\rho(\mathbf{r},t) = \theta(\mathbf{r},t) = \text{const}]$, and the probelm is to calculate the stress tensor (16) and, in particular, its nondiagonal component:

$$R_{xy} = \int f_1(t,\ \mathbf{r},\ \mathbf{v})\,d\mathbf{v}\,(v_x(v_y - u_0 - |\,\boldsymbol{\varkappa}\,|\,x)). \tag{50}$$

Standard calculations using the Boltzmann equation in the Navier–Stokes approximation show that $\lim\limits_{|\varkappa|\to 0}(P_{xy}/|\varkappa|) = -\eta$, where $\eta$ is the constant viscosity coefficient. The Burnett approximation must lead to an expansion for $P_{xy}/|\boldsymbol{\varkappa}|$ of the form

$$\frac{P_{xy}}{|\boldsymbol{\varkappa}|} = -\eta + \eta_B\,|\varkappa|, \tag{51}$$

that is, to a linear dependence on the velocity gradient. However, as was first shown by Kawasaki and Gunton,[21] when the effects of singularities in the ring operator are taken into account the dependence (51) does not occur, since the expression for $\eta_B$ contains divergent integrals. The authors of Refs. 21 and 22 used a phenomenological approach taking into account the interaction of the hydrodynamical modes and showed that the $|\varkappa|$ dependence of $P_{xy}$ is not analytic. We shall show how this result arises in the hydrodynamical theory based on the kinetic equations (8a) and (8b) (Ref. 23).

For finding the distribution function $f(t,\mathbf{r},\mathbf{v})$ determining the stress tensor it is possible, as in Sec. 1, to use the generalized Chapman–Enskog method, introducing expansions similar to (23):

$$\left.\begin{aligned} &f_1(t,\ \mathbf{r},\ \mathbf{v}) = f_M(1)\,(1 + |\boldsymbol{\varkappa}|\,h(\mathbf{r},\ \mathbf{v}) + \delta h(\mathbf{r},\ \mathbf{v};\ |\boldsymbol{\varkappa}|));\\ &g_2(\mathbf{r}_1,\ \mathbf{r}_2;\ \mathbf{v}_1,\ \mathbf{v}_2) =\\ &= f_M(1)\,f_M(2)\,\delta g\left(\frac{\mathbf{r}_1 - \mathbf{r}_2}{2},\ \mathbf{r}_1 - \mathbf{r}_2,\ \mathbf{v}_1,\ \mathbf{v}_2,\ \boldsymbol{\varkappa}\right), \end{aligned}\right\} \tag{52}$$

where the distribution function $f_M(1)$ is fixed:

$$f_M(1) = n\left(\frac{m}{2\pi\theta}\right)^{3/2} \exp\left[-\frac{m\,(\mathbf{v}_1 - \mathbf{u}(\mathbf{r}_1))}{2\theta}\right], \tag{53}$$

and the coordinate dependence of the macroscopic velocity $\mathbf{u}$ is given by (49). Substitution of the functions (52) into (8a) and separation of terms of different order in $|\boldsymbol{\varkappa}|$ leads to two equations, one which determines $h(\mathbf{r},\mathbf{v})$, while the other gives the relation between $\delta h(\mathbf{r},\mathbf{v};|\boldsymbol{\varkappa}|)$ and $\delta g(\mathbf{r}_1,\mathbf{r}_2,\mathbf{v}_1,\mathbf{v}_2;|\boldsymbol{\varkappa}|)$:

$$h(\mathbf{r},\ \mathbf{v}) = \frac{m}{\theta}\,\Lambda^{-1}(\mathbf{V})\,(\mathbf{V}_x\mathbf{V}_y),$$
$$\delta h(\mathbf{r},\ \mathbf{v};\ |\boldsymbol{\varkappa}|) = -n\Lambda^{-1}(\mathbf{V}) \int \frac{d\mathbf{k}}{(2\pi)^3}$$
$$\times \int d\mathbf{V}_2\Phi_M(\mathbf{V}_2)\,T_0(1,\ 2)\,\delta\tilde{g}(\mathbf{r},\ \mathbf{k};\ \mathbf{v}_1,\ \mathbf{v}_2). \tag{54}$$

Here we have introduced the notation

$$\Phi_M(\mathbf{V}_2)=n\left(\frac{m}{2\pi\theta}\right)^{3/2}\exp\left\{-\frac{m\mathbf{V}_2^2}{2\theta}\right\};$$

$$\delta\tilde{g}(\mathbf{r},\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)=\int d\xi\delta g(\mathbf{r},\xi,\mathbf{V}_1,\mathbf{V}_2;|\mathbf{x}|),\ \mathbf{V}=\mathbf{v}-\mathbf{u}(\mathbf{r}).$$

The equation for $\delta\tilde{g}$, which together with (54) can be used to find $\delta h(\mathbf{r},\mathbf{v};|\mathbf{x}|)$, should be obtained from Eq. (8b) of the kinetic system taking into account (52). The linear coordinate dependence of the macroscopic velocity (49) is extremely important for writing (8b) in a compact form. In fact, it follows from (49) that $\delta g$ depends only on the $x$ component of the vector $\mathbf{r}$. Therefore, instead of $x$, we can introduce into (8b) a new variable $u_y=u_0+x|\mathbf{x}|$ and write the differential operator

$$\hat{\Omega}=\mathbf{v}_1\frac{\partial}{\partial\mathbf{r}_1}+\mathbf{v}_2\frac{\partial}{\partial\mathbf{r}_2}=(\mathbf{v}_1-\mathbf{v}_2)\frac{\partial}{\partial\xi}+\frac{(\mathbf{v}_1+\mathbf{v}_2)}{2}\frac{\partial}{\partial\mathbf{r}}$$

in the form

$$\hat{\Omega}=(\mathbf{v}_1-\mathbf{v}_2)\frac{\partial}{\partial\xi}+\frac{|\mathbf{x}|}{2}(v_{1x}+v_{2x})\frac{\partial}{\partial u_y}. \quad (55)$$

Next, in the operators $\mathscr{L}_i$ (9) we can expand in powers of $|\mathbf{x}|$, using the $\delta$ functions in the operators $\overline{T}(1,2)$ and $\overline{T}(2,3)$ and the definition of $f_M$:

$$\overline{T}(1,3)f_1(3)=\delta(\mathbf{r}_1-\mathbf{r}_3)T(1,3)f_1(\mathbf{r}_1,\mathbf{v}_3)$$
$$=\delta(\mathbf{r}_1-\mathbf{r}_3)T(1,3)f_1(\mathbf{r}+\xi,\mathbf{v}_3)=\delta(\mathbf{r}_1-\mathbf{r}_3)T(1,3)'$$
$$=\Phi_M(\mathbf{V}_3)\left[1+\frac{1}{2}\xi_y|\mathbf{x}|\frac{m}{\theta}\mathbf{V}_{3y}+|\mathbf{x}|h(\mathbf{r},\mathbf{v}_3)\right]. \quad (56)$$

Calculation of the Fourier transform of both sides of (8b) with respect to the variable $\xi$ taking into account (55) and (56) allows us to write the desired equation for $\delta\tilde{g}(\mathbf{r},\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)$ in the form[23]

$$i\mathbf{k}(\mathbf{v}_1-\mathbf{v}_2)-\Lambda(\mathbf{v}_1)-\Lambda(\mathbf{v}_2)$$
$$+|\mathbf{x}|\hat{R}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)]\delta\tilde{g}(\mathbf{r},\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)$$
$$=|\mathbf{x}|T(1,2)(1+P_{1,2})h(\mathbf{v}_1), \quad (57)$$

where to simplify the notation we have introduced the notation $h(\mathbf{r}_1,\mathbf{v}_1)=h(\mathbf{v}_1)$,

$$\tilde{R}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)\chi(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)$$
$$=\left[-\frac{1}{2}(v_{1x}+v_{2x})\left(\frac{\partial}{\partial v_{1y}}+\frac{\partial}{\partial v_{2y}}\right)-\frac{i}{2}\right.$$
$$\frac{d}{dk_x}(1-P_{1,2})\hat{\Lambda}\left(\mathbf{v}_1\middle|\frac{m\mathbf{v}_{1y}}{2\theta}\right)$$
$$\left.-(1+P_{1,2})\hat{\Lambda}(\mathbf{v}_1|h(\mathbf{v}_1))\right]\chi(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2), \quad (58)$$

$$\tilde{\Lambda}(\mathbf{v}_1|\psi(\mathbf{v}_1))\chi(\mathbf{v}_1)$$
$$=n\int d\mathbf{v}_2T(1,2)(1+P_{12})\Phi_M(\mathbf{v}_2)\psi(\mathbf{v}_2)\chi(\mathbf{v}_1)$$

We note that it follows from (57) and (58) that $\delta\tilde{g}$ is independent of the first of these arguments. According to (52) and (54), the component of the stress tensor $P_{xy}$ (50) can be written as a sum of two terms, the first of which is linear in $|\mathbf{x}|$ and determined by the function $f_1(\mathbf{r},\mathbf{v})$ in the Navier–Stokes approximation. The second term is defined by the function $\delta\tilde{g}(\mathbf{r},\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)$:

$$\delta P_{xy}=-mn^2\int\frac{d\mathbf{k}}{(2\pi)^3}\int d\mathbf{v}_1d\mathbf{v}_2\Phi_M(\mathbf{v}_1)$$
$$\times\Phi_M(\mathbf{v}_2)v_{1x}v_{1y}\Lambda^{-1}(\mathbf{v}_1)T(1,2)\delta\tilde{g}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2). \quad (59)$$

It follows directly from (57) that $\delta P_{xy}$ contains a regular part, which is proportional to $|\mathbf{x}|$ and represents a higher-order density correction to the $(-\eta|\mathbf{x}|)$ Navier–Stokes approximation. Expansion of the solution of (57) in powers of $|\mathbf{x}|$ leads to divergence due to the mechanism discussed in the preceding sections.

In fact, writing $\delta\tilde{g}$ as

$$\delta\tilde{g}=|\mathbf{x}|\delta\tilde{g}^{(1)}+|\mathbf{x}|^2\delta\tilde{g}^{(2)}+\ldots,$$

we see that from (57) it follows that

$$\left.\begin{array}{l}\delta\tilde{g}^{(1)}=\hat{Q}T(1,2)(1+P_{1,2})h(\mathbf{v}_1),\\ \delta\tilde{g}^{(2)}=-\hat{Q}\hat{R}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)\hat{Q}T(1,2)(1+P_{1,2})h(\mathbf{v}_1),\end{array}\right\} \quad (60)$$

where $\hat{Q}=[i\mathbf{k}(\mathbf{v}_1-\mathbf{v}_2)-\hat{\Lambda}(\mathbf{v}_1)-\hat{\Lambda}(\mathbf{v}_2)]^{-1}$.

It follows from (60) that at small $|\mathbf{k}|$, $\delta\tilde{g}^{(2)}$ contains terms $\sim\text{const}/|\mathbf{k}|^4$ for which the integral (59) diverges. Therefore, the solution (57) is not an analytic function of $|\mathbf{x}|$ near the point $|\mathbf{x}|=0$. It is easy to study its leading singularity qualitatively: we write $\delta\tilde{g}$ as a double series in the eigenfunctions of the linearized Boltzmann operator $\psi_\lambda^{(\mathbf{k})}(\mathbf{v}_1)$, $\psi_\mu^{(-\mathbf{k})}(\mathbf{v}_2)$:

$$\delta\tilde{g}\sim x\sum_{\lambda,\mu}{}'\psi_\lambda^{(\mathbf{k})}(\mathbf{v}_1)\psi_\mu^{(-\mathbf{k})}(\mathbf{v}_2)B^{\lambda\mu}(\mathbf{k})+\delta\tilde{g}_{\text{reg}}. \quad (61)$$

The symbol $\Sigma'$ in (61) denotes summation over those eigenvalues of $\hat{\Lambda}_0$ for which the eigenfunctions (22a)–(22c) are also eigenfunctions of the zeroth approximation for the operators $i\mathbf{k}\mathbf{v}_1+\Lambda(\mathbf{v}_1)$ and $i\mathbf{k}\mathbf{v}_2+\Lambda(\mathbf{v}_2)$, where $\hat{Q}\psi_\lambda^{(\mathbf{k})}(\mathbf{v}_1)\psi^{(-\mathbf{k})}(\mathbf{v}_2)\sim(k^2)^{-1}\psi_\lambda^{(\mathbf{k})}(\mathbf{v}_1)\psi^{(-\mathbf{k})}(\mathbf{v}_2)$ at small $|\mathbf{k}|$. It follows from (57) that at small $|\mathbf{k}|$ the quantities $B^{\lambda\mu}(\mathbf{k})$ have corrections of order $(k^2+\text{const}\,x)^{-1}$. Substitution of (61) into (59) allows us to find the magnitude of the contribution of these quantities to $\delta P_{xy}(\mathbf{x})$:

$$\delta P_{xy}\sim|\mathbf{x}|\int_0^{k_0}\frac{d\mathbf{k}\,k^2}{k^2+\text{const}\,|\mathbf{x}|}\sim|\mathbf{x}|^{3/2}. \quad (62)$$

The result (62) is in qualitative agreement with the predictions of the phenomenological approach.[21,22] Therefore, the Burnett approximation leading to the expansion (51) for $P_{xy}$ is not valid in this theory: the dependence of $P_{xy}$ on the velocity gradient is not analytic, and it is possible to go beyond the Navier–Stokes approximation only by constructing the generalized hydrodynamical equations. The exact calculation of the coefficient of $|\mathbf{x}|^{3/2}$ in (62) goes as follows: Eq. (57) must be projected onto the subspace of products of the functions (22b)–(22d) corresponding to the hydrodynamical modes, and then the expansion (61) must be used, neglecting the term $\delta\tilde{g}_{\text{reg}}$. As a result, we obtain a closed system of equations for the functions $B^{\lambda\mu}(\mathbf{k})$ (Ref. 23):

$$\sum_{\{\nu,\rho\}}[\delta_{\lambda\nu}\delta_{\mu\rho}(D_\lambda+D_\mu)k^2+xR_{\nu\rho}^{\lambda\mu}(\mathbf{k})]B^{\nu\rho}(\mathbf{k})$$
$$=-\frac{nm}{\theta}\langle\psi_\lambda^{(\mathbf{k})}(\mathbf{v}_1)\psi_\mu^{(-\mathbf{k})}(\mathbf{v}_1)v_{1x}v_{1y}\rangle. \quad (63)$$

The summation in (63) runs only over those values of the indices $\nu$ and $\rho$ for which the eigenvalues of the operator $i\mathbf{k}(\mathbf{v}_1-\mathbf{v}_2)-\Lambda(\mathbf{v}_1)-\Lambda(\mathbf{v}_2)$ corresponding to the functions $\psi_\nu^{(\mathbf{k})}(\mathbf{v}_1)\psi_\rho^{(-\mathbf{k})}(\mathbf{v}_2)$ are of order $k^2$ for $\lambda=1,2$.

$$D_\lambda=\begin{cases}\frac{\Gamma}{2},\ \lambda=1,2,\\ D_T,\ \lambda=3,\\ D_\eta,\ \lambda=4,5,\end{cases}$$

where $\Gamma$, $D_T$, and $D_\eta$ are the coefficients of the sound-mode damping, thermal diffusion, and kinematical viscosity in the Navier–Stokes approximation. The quantities $R^{\lambda\mu}_{\nu\rho}(\mathbf{k})$ are matrix elements of the operator $R(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)$:

$$R^{\lambda\mu}_{\nu\rho}(\mathbf{k}) = \int d\mathbf{v}_1\, d\mathbf{v}_2 \Phi_M(\mathbf{v}_1)\, \Phi_M(\mathbf{v}_2)\, \psi^{(\mathbf{k})}_\lambda (\mathbf{v}_1)\, \psi^{(-\mathbf{k})}_\mu (\mathbf{v}_2)\, \psi^{(\mathbf{k})}_\nu (\mathbf{v}_1)\, \psi^{(-\mathbf{k})}_\rho (\mathbf{v}_2).$$

Since $R(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)$ contains the derivative operator $d/dx_k$, (63) is a system of first-order linear differential equations. Its solution can be obtained as an expansion in powers of $|\mathbf{x}|$, which, however, is not applicable to the calculation of $\delta P_{xy}$ because each term of this expansion will give a contribution to the integrand of (59) which diverges for small $|\mathbf{k}|$. In the region $|\mathbf{k}| \sim k_0 \sim n r_0^{-2}$ it is natural to expect that the exact solution of (63) for $|\mathbf{x}| \ll k_0^2 D_\lambda$ will correspond to the first term of this expansion. This allows us to write down the boundary conditions for $B^{\lambda\mu}(\mathbf{k},|\mathbf{x}|)$, which together with (63) determine these functions, in the form

$$B^{\lambda\mu}(\mathbf{k}_0,\ 0) = -\frac{nm}{\theta}(D_\lambda + D_\mu)^{-1} k_0^{-2} \langle \psi^{(\mathbf{k}_0)}_\lambda (\mathbf{v}_1)\, \psi^{(-\mathbf{k})}_\mu (\mathbf{v}_1)\, v_{1x} v_{1y} \rangle.$$

Calculation of the integral (59) with the functions $B^{\lambda\mu}(\mathbf{k},|\mathbf{x}|)$ obtained in this manner leads to the following expression[23] for the coefficient of $|\mathbf{x}|^{3/2}$ in the asymptotic expansion of $\delta P_{xy}$ at small $|\mathbf{x}|$:

$$\delta P_{xy} = -\eta\,|\mathbf{x}| + |\mathbf{x}|^{3/2}\,\theta \left[ \frac{M_{\eta\eta}}{(2D_\eta)^{3/2}} + \frac{M_{+-}}{\Gamma^{3/2}} \right], \qquad (64)$$

where $M = -0.002\ 59$ and $M = -0.004\ 06$.

In experiments on computer modeling of multiparticle dynamical events[24,25] attempts have been made to establish a dependence of $\delta P_{xy}$ on the velocity gradient of the form (64). It turned out that such a dependence of $\delta P_{xy}$ does actually occur, but the coefficient of $|\mathbf{x}|^{3/2}$ is roughly two orders of magnitude greater than the theoretical value. One explanation[26] for this is that the experiments have studied very dense systems, in which in important role can be played by more complicated mechanisms giving rise to contributions to $P_{xy}$ which are nonanalytic in $|\mathbf{x}|$. In addition, relatively large values of $|\mathbf{x}|$ have been used in experiments.

For two-dimensional systems divergences arise in $P_{xy}$ in the application of the standard Chapman–Enskog method already at the stage where the Navier–Stokes approximation is constructed[22–28]: a modification of the method similar to that discussed above leads to a logarithmic dependence of the generalized viscosity coefficient on $|\mathbf{x}|$:

$$P_{xy}(\mathbf{x}) \sim \text{const} - \left(\frac{\theta}{32\pi}\right)(D_\eta^{-1} + \Gamma^{-1}) \ln \frac{D_\eta k_0^2}{|\mathbf{x}|}. \qquad (65)$$

The appearance of the logarithm in (65) is easily understood by estimating an integral analogous to (62) for the two-dimensional case:

$$\delta P_{xy} \sim |\mathbf{x}| \int_0^{k_0} \frac{k\,dk}{k^2 + \text{const}\,|\mathbf{x}|} \sim |\mathbf{x}| \ln \frac{1}{|\mathbf{x}|}.$$

We note that Eqs. (64) and (65) have been obtained in the simplest case where the macroscopic velocity depends linearly on one of the coordinates. Up to now it has not been possible to find any more complicated nonlinear problem for which it would be possible to obtain a compact expression for the stress tensor and the thermal-flux vector starting from the kinetic equatons (8a) and (8b).

## CONCLUSION

At present the problem of constructing the hydrodynamical equations on the basis of a kinetic theory which generalize the Navier–Stokes equations is far from a final resolution. As shown by experiments on the modeling of molecular dynamics, such equations should be nonlocal and should contain nonanalytic dependences of the characteristics of transport processes on the gradients of the macroscopic variables. These effects are of higher order in the particle density of the system than for the dynamics of binary collisions leading to the usual dissipative processes in the Navier–Stokes approximation. They are due to the slow damping of long-range correlations, which are extremely small for gases of moderate density and are present even in hard-sphere systems. However, the existence of such effects indicates that there is no sharp boundary between the kinetic and hydrodynamical stages of the approach of the system to equilibrium. This is manifested in the divergence of the formal expansions in the small parameter of the theory. Divergences are also characteristic of the kinetic state, where they arise from correlated successive collisions of duration much greater than $t_{\text{mfp}}$.

Therefore, study of the features of the Bogolyubov functional hypothesis with reference to the problem of constructing the equations of hydrodynamics has proved to be extremely fruitful and has led to an understanding of the important role played by kinetic phenomena in macroscopic time intervals for hard-sphere systems. Further study of this group of problems, first posed by Bogolyubov several decades ago, will in the near future probably result in the discovery of the most suitable approach to describing the multiparticle dynamical processes establishing equilibrium in macrosystems. This will apparently require significant changes in the existing mathematical technique based on expansion in the small parameters of the theory. One thing is certain: the further dynamical investigation of the problem will involve the development and justification of the functional hypothesis concerning the properties of solutions of the equations of the Bogolyubov chain.

The authors are deeply grateful to Academician N. N. Bogolyubov for his constant interest in the study and for valuable advice and remarks.


[1]N. N. Bogolyubov, "Problems of a dynamical theory in statistical physics," in: Studies in Statistical Mechanics, eds. J. de Boer and G. E. Uhlenbeck, Vol. 1, North-Holland, Amsterdam, 1961, pp. 5–515.

[2]E. G. D. Cohen, *Fundamental Problems in Statistical Mechanics* (North-Holland, Amsterdam, 1966).

[3]L. K. Heines, J. R. Dorfman, and M. H. Ernst, Phys. Rev. **144**, 207 (1966).

[4]S. G. Brush, *Kinetic Theory*, Vol. 3 (Pergamon, Oxford, 1972).

[5]M. H. Ernst, J. R. Dorfman, W. R. Hoegy, and J. M. J. van Leeuwen, Physica **45**, 127 (1969).

[6]B. J. Adler and T. E. Wainwright, Phys. Rev. A **1**, 18 (1970).

[7]B. J. Adler, D. M. Gass, and T. E. Wainwright, Phys. Rev. A **4**, 233 (1971).

[8]N. G. Inozemtseva and B. I. Sadovnikov, Teor. Mat. Fiz. **48**, 137 (1981); **51**, 142 (1982).

[9]B. I. Sadovnikov and N. G. Inozemtseva, Physica **94A**, 615 (1978); B. I.

Sadovnikov, N. G. Inozemtseva, and S. N. Bochkov, Physica **110A**, 329 (1982).
[10]N. G. Inozemtseva and B. I. Sadovnikov, Proc. of the Second Intern. Symposium on Selected Problems in Statistical Mechanics, Dubna, 1981, p. 351 [in Russian].
[11]N. N. Bogolyubov, Preprint E17-10514, JINR, Dubna, 1977.
[12]M. H. Ernst and J. R. Dorfman, Physica **61**, 157 (1972); J. Stat. Phys. **12**, 311 (1975).
[13]Y. Pomeau, Phys. Lett. **38A**, 245 (1972); Y. Pomeau and P. Resibois, Phys. Rep. **19C**, 63 (1975); P. Resibois and M. F. de Leener, Classical Kinetic Theory of Fluids, Wiley, New York, 1977.
[14]J. H. Ferziger and H. G. Kaper, *Mathematical Theory of Transport Processes in Gases* (North Holland, Amsterdam, 1972).
[15]G. E. Uhlenbeck and G. W. Ford, Lectures in Statistical Mechanics, American Mathematical Society, Providence, R. I., 1963.
[16]P. Resibois, Phys. Rev. Lett. **40**, 1409 (1978); J. Stat. Phys. **19**, 593 (1978).
[17]N. N. Bogolyubov, Collected Works on Statistical Physics, Nauka, Moscow, 1979, pp. 270–292 [in Russian].
[18]R. Zwanzig, Lectures in Theoretical Physics, Vol. III, Interscience, New York, 1961.
[19]J. W. Dufty, Phys. Rev. A **5**, 2247 (1972).
[20]Y. Pomeau, Phys. Rev. A **5**, 2569 (1972).
[21]K. Kawasaki and J. D. Gunton, Phys. Rev. A **8**, 2048 (1973).
[22]T. Yamada and K. Kawasaki, Prog. Theor. Phys. **53**, 111 (1975).
[23]M. H. Ernst, B. Cichocki, J. R. Dorfman *et al.*, J. Stat. Phys. **18**, 237 (1978).
[24]W. T. Ashurst and W. G. Hoover, Phys. Rev. A **11**, 658 (1975); Phys. Lett. **61A**, 175 (1977).
[25]J. R. Dorfman, Physica **106A**, 77 (1981).
[26]A.Onuki, Phys. Lett. **70A**, 31 (1979).
[27]D. J. Evans, Phys. Rev. A **22**, 290 (1980).
[28]N. G. Inozemtseva and B. I. Sadovnikov, Proc. of the Third Intern. Symposium on Selected Problems in Statistical Mechanics, Dubna, 1984, Vol. 1, p. 310 [in Russian].

Translated by Patricia Millard