Mathematical problems of the calibration of automated measuring systems for optical track detectors in high-energy physics

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Mathematical problems that arise in the calibration of automated systems for measuring data of optical track detectors in high-energy physics are reviewed. The calibration problem is formulated and the mathematical models generated by it are discussed. The stages of calibration and application of their results to the determination of the accuracy and stability of systems are described. Brief details are given of the algorithms and calibration programs developed at the Joint Institute for Nuclear Research.

INTRODUCTION

One of the topical directions in the automation of scientific investigations is the use of automated measuring devices for feeding into computers large amounts of experimental information.

The main source of experimental data in high-energy physics is the various detectors of elementary particles, among which optical track detectors such as bubble, spark, and streamer chambers are particularly common. By virtue of the visualization of the events that take place in these chambers, stereoprojections of the events can be recorded on photographic film, as is done in the most common film method, or sent directly to a computer memory after digitization and coding of the image in the case of filmless television recording.

As a long-life carrier of information, photographic film gives the most complete picture of an event, making it possible to reconstruct very accurately its topology and to determine the kinematic parameters from the curvature in a magnetic field and the ionization density of the tracks.

The great number of stereographic photographs, which in some experiments may reach 10^5–10^6 photographs per year, made the film method into a field of intensive penetration of automation, first in the analysis of the data and then in the very process of photograph measurement. This led to the development of a number of automatic measuring devices used, under the control of a minicomputer, to scan chamber photographs. According to the scanning method, i.e., the method used to scan the photograph with a light beam, these devices can be classified into several groups.

In the first of these we have devices with global examination of the entire scanning field; they are based on the principle of mechanical scanning and differ only in the scanning trajectory and shape of the light spot. The HPD device (Ref. 1) or analogous Soviet devices[2] employ parallel scanning by a spot of circular shape. The spiral reader[3,4] scans the image along the spiral of a radially oriented narrow slit from the event vertex, measured by the operator, for which purpose the construction of the spiral reader has not only the main polar coordinate system but also a rectangular system for manual measurements.

Another group of devices employs electron-beam scanning, the flexibility of which makes it possible to combine scanning with the search for vertices and tracks; this permits raster scanning of individual parts of the photographs. These include the devices PEPR, POLLY, ERASME, MELAS, and AÉLT-2, descriptions of which can be found in the reviews of Refs. 1 and 5 and the conference proceedings of Refs. 6–8. We mention also the devices of the SWEEPNIK type,[6] which realize instrumental following of the track by means of a rotating slit guided onto the track by the operator. The system HEVAS (see Ref. 8, p. 326) works similarly.

To reduce the stream of information received from the automatic scanners (particularly those of the first group), there is usually data selection in the scanning process by means of targetings obtained earlier by preliminary examination of the photographs and the measurement of some points on them. This is done on special scanning and measuring tables, which are operated in a semiautomatic regime (manual measurements with automatic digitization, verification, and data collection on magnetic tape). The problem of rapid reduction of the information stream arises even more acutely in the systems of filmless television data acquisition from spark and streamer chambers,[9,10] in which the stream can, given the high operating frequencies of the chamber, reach 10^6–10^7 bits/sec. The filmless method makes it possible to raise the event detection rate, eliminate the laborious and expensive stage of measurement of photographs, and control the course of the experiment and the quality of the detected events. The problem of reducing the data in the case when they are acquired by the filmless method is posed in a quite different way than in the case of the film method. In the television systems, the reduction must in principle be done in real time, i.e., in the time of the television scanning, and, remeasurement being impossible, there

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must be no information loss. To overcome the conflict of these requirements, fast reduction algorithms, designed for realization in the form of a special data-reduction apparatus, were proposed in Refs. 11 and 12. This reduces the digitized image of one of the projections of an event, consisting of 1–2 thousand points, to 1–2 hundred linear track elements.

Not only the automatic scanners and filmless acquisition systems but also the scanning and measuring tables produce data on the measurements in the form of ordered pairs of numbers, which represent the measured points of the photograph in a coordinate system specified by reference devices in the system.

Since the results of measurements can be fed into analysis programs only in a unified system of Cartesian coordinates (which is usually taken to be a rectangular coordinate system associated with the image), it is necessary to transform the digitization data produced by the device as it makes the measurement to this unified coordinate system. The coordinate systems in which the reference devices of the system operate depend on its construction and may be rectangular (such are the majority), polar (as for the spiral reader), and even bipolar (as, for example, in the CERN tables of the MYLADY system). The present paper is devoted to mathematical aspects of the problem of establishing the direct and inverse dependence between the coordinate systems of the device itself and the unified coordinates associated with the image.

After the formulation of the problem and the introduction of the necessary mathematical apparatus, the subsequent exposition will be made in accordance with the main stages in the analysis of calibration measurements.

Section 3 presents methods of determining the coordinates of the centers of crosses using scanning data of a calibration grid. To this end, it is first necessary that each cross be recognized among the scanning data either as a complete figure or (most frequently) as two intervals of the straight lines that form it. Since the approximate position of each cross and its shape (i.e., the angle between the arms, usually equal to 90°) are known in advance, the recognition problem is simplified. However, in the more general formulation, which takes into account possible significant distortions of the optical and electronic systems of the scanner and nonorthogonality of the scanning, it is necessary to use more complicated methods, which will be described in detail in the third section of the review.

After the arrays of points that form the arms of the cross have been found, they are approximated by straight lines or, if necessary, by polynomials of a suitable degree by means of least-squares fitting with the rejection of distant points or by more reliable, so-called robust fitting methods.

By simultaneous solution of the approximate equations of the arms, the coordinates of the centers of the crosses of the calibration grid and the rms errors of these coordinates are determined. On the basis of these data, in the next stage of the analysis, the direct and inverse calibration transformations are constructed, and the accuracy characteristics of the device are also determined. The main mathematical tool for this stage is provided by systems of two-dimensional orthogonal polynomials.

The problem of constructing and orthogonalizing a system of two-dimensional polynomials that are convenient from the computational point of view has passed through an evolution in their applications to direct and inverse calibration transformations.

In one of the early studies of PEPR calibration,[13] the idea of constructing a system of two-dimensional polynomials by multiplying one-dimensional Chebyshev polynomials was used.

A more detailed investigation of systems of two-dimensional polynomials specifically for the purpose of constructing the direct and inverse calibration transformations and also choosing the degree of such polynomials that is optimal from the point of view of the accuracy that can be achieved by the approximation was made in Refs. 14 and 15. Systems of polynomials more effective from the point of view of the speed of calculations were proposed in recent studies devoted to two-dimensional[16] and three-dimensional[17] polynomials. These results will be summarized in Sec. 4. Methods of raising the accuracy of the calibration transformations in the regions between the points of the calibration grid are described in Sec. 5, where we also give methods of investigating the stability of operation of measuring devices. We describe a method of using devices that have a drift of the calibration parameters in cases when this drift has a regular nature, as, for example, when due to heating of the device during several hours after it has been switched on.

Section 6 of the present paper is devoted to problems of program implementation of calibration algorithms and to a review of the existing standard calibration programs.

1. FORMULATION OF THE PROBLEM

We represent the image in the form of a two-dimensional set of points that are to be measured. Then with every point C in the plane of the image we can associate two pairs of numbers: its Cartesian coordinates (x, y) in this plane and a pair of integers (u, v) obtained from the reference devices of the system as a result of the measurement of the point C.

A calibration problem arises: to establish the form of the calibration transformation that permits a correct interpretation of the measurement data. It is also necessary to determine the extent to which one can trust the data, i.e., it is necessary to find the accuracy characteristics of the device at all points of the field of measurement and show that they are stable in time.

Forms of calibration devices

The main calibration method is to measure a special reference image, which consists of a set of objects suitable for automatic measurements distributed uniformly over the complete scanning field with values of the coordinates known to a high degree of accuracy ($\sim 1\,\mu$m); this makes it possible to identify them with "ideal" coordinates (x, y).

This could be a figure in the form of a cross with known coordinates of the center, i.e., of the point of intersection of the arms of the cross. If the ratio of the length of the arms to the scanning step is correctly chosen, 15–20 readings will be obtained from each arm, and this is sufficient to estimate the

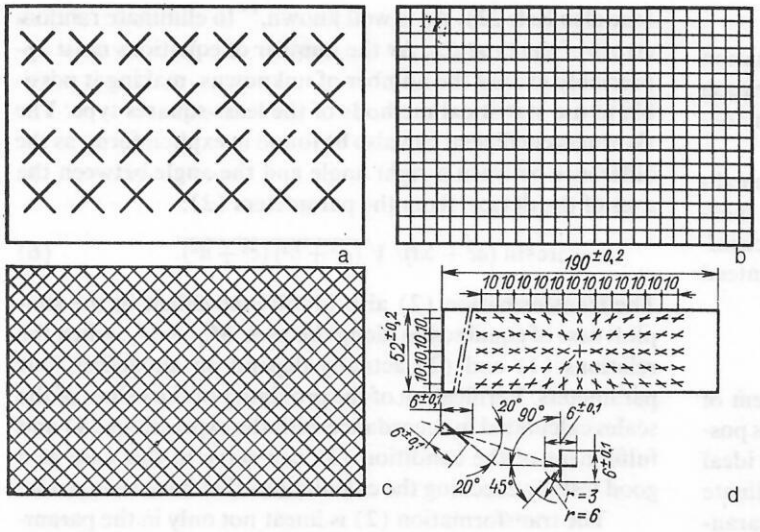

FIG. 1. Calibration grids of various types.

coordinates of the center of the cross. As is shown in the review of Ref. 5, to ensure the same spread of readings from each arm the cross must be oriented in such a way that the scanning line intersects the two arms at the same angle. From the point of view of obtaining equal errors of the center along the axes, the angle of the cross becomes optimal at 90°. Placing of the centers of the crosses at an equal step over the complete scanning field gives a calibration grid, whose crosses can either be separated from one another [Fig. 1(a)] or form a mesh of straight lines [Fig. 1(b)]. For manual measurements, and also for electron-beam scanning in automatic devices of the PEPR type with microraster of arbitrary orientation and a light spot in the form of a dash, it is more convenient to use a grid of horizontal and vertical lines—a so-called Gautier [Fig. 1(c)]. In the case of devices like the spiral reader, which combine polar and Cartesian coordinate systems, the construction of the calibration grid must permit measurements on it in both coordinate systems. To ensure that the measuring slit of the polar coordinate system of the spiral reader give identical signals from the two arms of each cross, all the crosses, apart from the central one, must be arranged symmetrically with respect to the polar radii described to their centers; moreover, to ensure that these signals can be detected, the angle between each of the arms and the slit must not exceed 20°. The form of the corresponding calibration plate used at the present time to calibrate the JINR spiral reader is shown in Fig. 1(d). We note that the opening of 40° gives unfavorable conditions for the measurements, but it cannot be reduced, since otherwise there is a sharp reduction in the radial accuracy with which the center of the cross is determined. The plate of the Stanford spiral reader,[18] for example, is free of this shortcoming, but in this case the determination of the centers of the crosses requires highly accurate digitization of the amplitudes of the signals received from the measuring slit (we note that, for example, this is not foreseen in the construction of the JINR spiral reader).

Stages in the analysis

After the calibration measurements have been made, they are analyzed mathematically in order to solve the calibration problem. Academician A. N. Tikhonov identifies the following main stages in the mathematical analysis of the results of measurements with a computer.[19]

1. Preliminary analysis, including normalization of the data and filtering to obtain the output results of the experiment.

2. Analysis of the devices, i.e., construction of an operator F corresponding to the model of the device. The analysis is made by: (a) special alignment experiments; (b) mathematical (numerical) modeling.

3. Interpretation of the results.

In the overall sequence of analysis of the data of the automatic measurements, the calibration embodies the second stage. The determination of the calibration transformation on the basis of the alignment measurements amounts to the construction of an operator that characterizes the influence of the measuring device in mathematical terms.

However, one can regard the calibration measurement itself as a certain experiment. In such an approach, Tikhonov's scheme determines the stages in the analysis of the calibration experiment.

First stage

Recognition and filtering of crosses and determination of their centers.

Second stage

Construction of the calibration transformation. We mention here the use of imitation modeling for analysis of the device and verification of the algorithms for analyzing the calibration data. We shall discuss in more detail below (in Sec. 2) the method of such modeling.

Third stage

The interpretation of the results, i.e., the determination of the reliability of the model used to derive the calibration transformation and the accuracy characteristics of the device.

Experience shows[20,21] that to test the quality of the calibration coefficients it is helpful to make an additional measurement of a simple but very accurately made reference image (usually, this is a straight line described with guaranteed absence of deviations greater than $\pm 1\,\mu$m).

Simplest models

In order to present the formulation of the problem of determining the calibration transformation as clearly as possible, we begin with a very simple case. We consider an ideal device without distortions with a rectangular coordinate system (u, v) for which orthogonality of the axes is guaranteed. Even in this simple case the transformation $(u, v) \xrightarrow{F} (x, y)$ requires the specification of five parameters: two scales along the axes, M_x and M_y, two numbers x_0 and y_0, which specify the initial positions of the reference systems along the axes, and the angle φ between the corresponding axes in the two coordinate systems. To determine these parameters, it is not sufficient simply to measure five points with known ideal coordinates, since every real measurement is made with an unknown random error, the projections of which onto the axes we shall denote by Δ_x, Δ_y. We obtain the simplest model for describing the relation $(u, v) \xrightarrow{F} (x, y)$:

$$\left.\begin{aligned} x &= x_0 + uM_x \cos\varphi + vM_y \sin\varphi + \Delta_x; \\ y &= y_0 - uM_x \sin\varphi + vM_y \cos\varphi + \Delta_y. \end{aligned}\right\} \quad (1)$$

We mention right away two circumstances that complicate the use of the relation (1): the nonlinearity of the model with respect to the parameters and the existence of a small nonorthogonality of the axes in practically all real measuring systems.

The small skewness of the axes can be taken into account by a dependence more general than (1), namely, an affine dependence; this is simultaneously more convenient from the computational point of view on account of the linearity with respect to the parameters:

$$\left.\begin{aligned} x &= x_0 + au + bv + \Delta_x; \\ y &= y_0 + cu + dv + \Delta_y. \end{aligned}\right\} \quad (2)$$

Therefore, the relation (2) is still preferable even if it is found after estimation of the six parameters that the orthogonality condition

$$ac + bd = 0 \quad (3)$$

of the axes is satisfied to a high degree of accuracy. Parameters with a more transparent meaning can be readily calculated from the relation (1) if we are given a, b, c, d:

$$M_x = \sqrt{a^2 + c^2}; \quad M_y = \sqrt{b^2 + d^2}; \quad (4)$$

$$\varphi = \operatorname{arctg}(ab - cd)/2ad \quad (5)$$

(see also Ref. 22). As is well known,[23] to eliminate random errors of the type Δx, Δy the number of equations must appreciably exceed the number of unknowns, making it possible to use statistical methods of the least-squares type. The skewness coefficient can also be found in explicit form, as the difference between a right angle and the angle between the axes of the device, from the parameters (2):

$$\varkappa = \arcsin(ac + bd)/\sqrt{(a^2 + b^2)(c^2 + d^2)}. \quad (6)$$

The transformation (2) also is still convenient in the simplest case of equal scales along the axes, $M_x = M_y$, when the relations (1) and (2) actually depend on altogether four parameters. Verification of the proximity of the values of the scales calculated in accordance with the expressions (4) and fulfillment of the condition (3) can serve in this case as a good test for checking the calculations that have been made.

The transformation (2) is linear not only in the parameters but also in the variables u and v that occur in it. This last circumstance is a consequence of the assumption made at the very beginning of the absence of distortions and requires verification even for such simple nonautomated devices for manual measurements such as the UIM microscope or the scanning and measuring table BPS-2.[22] Real automatic measuring devices are complicated systems consisting of optical and illuminating elements and mechanical or electronic devices that move the scanning spot. Each of these elements possesses definite distortions and even with the most careful alignment can introduce appreciable nonlinear distortions in the measured coordinates. These nonlinearities can be taken into account in different ways in the calibration transformation. One of the approaches was already outlined in the transformation (1), the coefficients of which were shown to be the parameters of the reference systems of the device. This way of introducing, into the nonlinear calibration transformation, coefficients that explicitly take into account the distortions provides an excellent possibility for rapid diagnosis of the sources of these distortions through the exceeding by one of the coefficients of a limit specified for it. The realization of such an approach in the calibration of spiral readers will be discussed below. However, despite the attractiveness of this approach for engineers who develop and use the devices, it cannot be regarded as a general approach, since it requires for each measuring device a strictly individual approach based on a detailed analysis of the optical, kinematic, and electronic circuits and, moreover, leads to computationally complicated problems of minimizing nonlinear functionals for the determination of the unknown parameters.

Linear approximation models

More universal is the approach using approximating functions, linear in the parameters, of the form

$$F(u, v, A) = \sum_{j=1}^{M} a_j \varphi_j(u, v), \quad (7)$$

where $\{\varphi_j(u, v)\}$ is a system of linearly independent continuous functions of two variables, and a_j are the components of an unknown vector A of parameters, which need not be

related to the distortions of specific elements of the instrument (although, when necessary, such a connection can be established with certain combinations of these parameters). Such an approach makes it possible to apply the full power of the well-developed linear regression methods of mathematical statistics to obtain optimal estimates of the vectors A and B that determine the transformations

$$\hat{x} = F(u, v, A), \quad \hat{y} = F(u, v, B) \tag{8}$$

with F of the form (7).

The problem of obtaining the calibration transformation that best corresponds to the calibration measurements which are made can be posed as follows.

Suppose a calibration grid consists of N crosses, for which the ideal coordinates of the centers are $\{x_k, y_k\}_{k=1}^N$. The analysis of the data of the automatic scanning of this grid give the measured values $\{u_k, v_k\}_{k=1}^N$ of these coordinates with an error matrix S_{2N}, which in the general case can take into account the correlations between the measurements of both coordinates at different points, i.e., $\mathrm{cov}(u_k, v_j) \neq 0$, with $j, k = 1,...,N$. Given these data of the calibration measurements, the problem is to find the best estimates A_N and B_N of the parameters that determine the transformation (8).

The quality of the transformation is judged by the proximity of the points (x, y) and $(\hat{x}, \hat{y})$; this proximity must be equally valid at all points of the measured field. Having N measured points uniformly distributed on the field, we can estimate the measure of proximity from the smallness of the absolute values of the residuals of the transformations (8):

$$\begin{aligned} dx_k &= x_k - F(u_k, v_k, A_N); \\ dy_k &= y_k - F(u_k, v_k, B_N), \end{aligned} \tag{9}$$

which form the vector $D_{2N} = \mathrm{col}(dx_1,...,dx_N, dy_1,..., dy_N)$.

For a sufficiently dense arrangement of crosses (i.e., sufficiently large N) the choice of the dimension M of the transformation and of the parameters A_N, B_N that ensure smallness of the absolute values of all the residues in fact means that in the transformations (8) we have succeeded in taking into account all the systematic deviations generated by the distortions of the device. Therefore, the so-called residual chart (Fig. 2), formed by the two-dimensional vectors $\{dx_k, dy_k\}$ emanating from the points $\{x_k, y_k\}$, multiplied by a factor 10^3–10^4, is a very helpful and transparent means of estimating the correctness of the choice of the parameters.

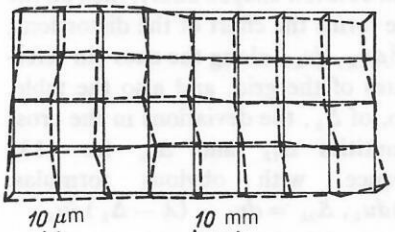

FIG. 2. Example of a chart of residuals.

As a measure of the proximity of the points (x, y) and $(\hat{x}, \hat{y})$ it is natural to take the rms value of the residuals averaged with allowance for their errors, i.e.,

$$L_2 = \sqrt{D_{2N}^T W_{2N} D_{2N}},$$

where T denotes transposition, and $W_{2N} = S_{2N}^{-1}$.

As is well known,[23] the optimal estimates of the parameters A_N and B_N can, under the assumption that the vector of residuals D_{2N} has a normal distribution, be obtained by minimizing the functional L_2. If the additional assumption of independence of the residuals is valid, then S_{2N} becomes diagonal, and the problem actually decomposes into the two independent problems of calculating the estimates of A_N and B_N by the minimization of functionals that, with allowance for (7)–(9), have the form

$$\left.\begin{aligned} \Phi_x^2 &= \sum_{k=1}^N w_{xk} [x_k - \sum_{j=1}^M a_j \varphi_j(u_k, v_k)]^2; \\ \Phi_y^2 &= \sum_{k=1}^N w_{yk} [y_k - \sum_{j=1}^M b_j \varphi_j(u_k, v_k)]^2 \end{aligned}\right\} \tag{10}$$

with weights

$$w_{xk} = \sigma_{uk}^{-2}, \quad w_{yk} = \sigma_{vk}^{-2}; \quad k = 1,...,N.$$

It is obviously important that there should be a correct parametrization of the calibration problem, i.e., a felicitous choice of the system of two-dimensional functions $\{\varphi_j\}$, which, on the one hand, must describe sufficiently well the types of distortions encountered in practice and, on the other, ensure computational convenience in the solution of the system of normal equations (rapid calculation on a computer, preclusion of situations with ill conditioning, etc.).

There is one further aspect of the parametrization problem. The transformations (8) give us the direct transition $(u, v) \xrightarrow{F} (x, y)$ from the measured to the ideal coordinates. However, in measurement practice the need arises of the inverse transformation $(x, y) \xrightarrow{F^{-1}} (u, v)$, which ensures, in particular, that some previously manually measured points can be tied to the data of the automatic measurements, this being necessary for the organization of rapid selection of data in the scanning process.

Unfortunately, the nonlinearity of the system $\{\varphi_j(u, v)\}_{j=1}^M$ hinders the direct inversion of (8), and therefore to obtain the function $F^{-1}(x, y, A')$ of the inverse transformation it is also expedient to construct it in the form of a linear combination of two-dimensional functions forming the system $\{\psi_j(x, y)\}_{j=1}^M$:

$$F^{-1}(x, y, A') = \sum_{j=1}^M a_j' \psi_j(x, y). \tag{11}$$

The coefficients of (11) can also be determined by the least-squares method using data of the measurements of $\{u_k, v_k\}$ and the ideal coordinates of the grid $\{x_k, y_k\}_{k=1}^N$; however, in contrast to the direct transformation, the minimized functional in the case of the inverse transformation is constructed on a fixed set of points $\{x_k, y_k\}_{k=1}^N$, and this

significantly simplifies the computational aspect of the problem.

Going over to the problem of constructing the systems of functions $\{\varphi_j\}$ and $\{\psi_j\}$, we outline one of the possible approaches to the construction, which is given below in the section on calibration for a device with a polar coordinate system. The functions $\{\varphi_j\}$ there are not orthogonal, since they are constructed by linearizing with respect to the parameters relations that take into account instrumental features and the distortions of the scanner. Although such experience has a particular nature, it is quite instructive, since it shows how one can establish relations for the functions $\{\varphi_j\}$ that permit diagnosis of the device.

However, systems of two-dimensional orthogonal polynomials are most frequently used for $\{\varphi_j\}$ and $\{\psi_j\}$. They are particularly convenient for the calibration of devices of the most widely used type with a rectangular coordinate system, since they simplify the calculations and make it possible to take into account well the most varied distortions, including appreciable cushion of barrel-shaped distortions inherent in ordinary devices with electron-beam scanning and also filmless systems. A more detailed description of different systems of polynomials is given in the following section with the other mathematical information needed to realize the stages of the analysis.

2. MATHEMATICAL APPARATUS

Imitation modeling

One of the most fruitful ways of testing the correctness of the actual mathematical model of a phenomenon, which depends on random parameters as well as on our ability to interpret sensibly the results of this phenomenon, is through imitation modeling. The Monte Carlo method makes it possible to feed into the model a stream of data obtained from the automatic scanning of the calibration grid, random errors that appear in the process of scanning, and systematic deviations due to distortions of the device that have been taken into account. The statistical analysis of such a model by means of a system of programs that implements certain principles and algorithms makes it possible to: a) verify the correctness of the algorithms and their ability to detect systematic errors; b) determine the accuracy of the calculated parameters; c) establish the characteristic rates and reliabilities and verify the complete system of programs. To ensure a high likelihood of the statistical conclusions, the modeling and subsequent analysis must be repeated a large number (10^3–10^4) of times.

As an example, we consider a model of a symmetric calibration grid consisting of N individually placed crosses with mutually perpendicular arms and ideal coordinates $\{x_k, y_k\}_{k=1}^N$ of the centers [see Fig. 1(a)].

As the origin of the ideal coordinate system, it is convenient to take the symmetry center of the grid, and since the origin of the reference systems of the device must be outside the scanning field, the arithmetic means of the coordinates of all the points obtained by scanning the grid can serve as estimates of the coefficients of transport $\hat{x}_0$, $\hat{y}_0$ between these systems. The construction of the device, and also the data of preceding or control measurements enable us to make initial estimates of the scales along the axes: M_{x0}, M_{y0}.

These *a priori* data are sufficient for the approximate determination of the boundaries of the square region containing the k-th cross. Its center has the coordinates

$$U_{k0} = x_k M_{x0} + \hat{x}_0,\ V_{k0} = y_k M_{y0} + \hat{y}_0;\ k = 1,...,N, \quad (12)$$

with side equal to $L/\sqrt{2}$, where L is the length of each of the arms of the cross.

This determines the usual procedure of the first stage of the analysis—the scanning data for the complete calibration grid are divided into N subsets formed by the points belonging to the regions corresponding to each cross, and these subsets are analyzed separately. In accordance with such an analysis scheme, one can implement the following simple imitation of the process of scanning of the calibration grid.

We transport the region associated with the k-th cross to the origin. We write the equations of the arms of the cross in the form

$$\left.\begin{aligned} v &= (\lambda + \Delta_k)\, u + \Delta_{1k}, \\ v &= -(\lambda - \Delta_k)\, u + \Delta_{2k} \end{aligned}\right\} \quad (13)$$

subject to the restriction $|u| \leqslant L/\sqrt{1+\lambda^2}$ (the arms being mutually perpendicular corresponds to $\lambda = 1$). Here, Δ_{1k} and Δ_{2k} are the segments cut off by the arms on the v axis, and Δ_k is the possible displacement of the direction of the arms. The appearance of these three parameters is due to the inaccuracy of the *a priori* information and the possible distortions of the device, which are assumed to be small in order to justify preserving rectilinearity of the arms of the cross. Specifying the scanning step h, we discretize the u axis: $u_i = ih$; $i = 0, \pm 1, ..., \pm n$; $n = [L/\sqrt{1+\lambda^2}]$, and from (13), for each u_i we obtain two values of v corresponding to the two arms of the cross.

To imitate the errors of the reference systems, it is necessary to add to each of the coordinates a normally distributed variable with zero mean values and standard deviations σ_u, σ_v. To take into account the possibility of failures of the reference systems, each of the points can, when the set of data is being accumulated, be omitted (not entered) with a small probability $P_{\text{om}} \sim 1/20$. In addition, noise readings are added to the set of cross scanning data. This is done as follows. For each u_i, with probability P_{noise}, one further value of v, in addition to the two values of v obtained from (13), is sampled from a uniform distribution in the interval $(-L\lambda/\sqrt{1+\lambda^2}, +L\lambda/\sqrt{1+\lambda^2})$.

If it is now required to model a particular picture of distortions (for example, cushion-shaped ones), then on the basis of this picture one forms the chart of the distortions, i.e., the displacements $\{du_k, dv_k\}$ along the axes (in reference units) at all N sites of the grid, and also the table, which depends on them, of Δ_k, the deviations in the cross orientations. The quantities Δ_{1k} and Δ_{2k} for (13) are calculated in accordance with obvious formulas: $\Delta_{1k} = dv_k + (\lambda + \Delta_k)du_k$, $\Delta_{2k} = dv_k - (\lambda - \Delta_k)du_k$.

If we now require a model of the complete set of data obtained by scanning the complete calibration grid, we must

add to the coordinates of all points of each of the subsets corresponding to the individual crosses the approximate coordinates of the centers of these crosses from (12), merge all these subsets into a single set, and arrange them for each u in ascending order of v.

If necessary, one can model similarly the set of connected readings that are obtained in the presence of scratches, grains, etc., on the calibration grid.

The model is much more complicated in the case of spiral scanning, since in that case the intersections of the straight lines corresponding to the arms of the crosses and the spiral lead to transcendental equations. In addition, the sampling of the coordinates of the noise points, which must be sampled uniformly over the length of the spiral arc, is much more complicated. Realization of this model, described in Refs. 24 and 25, made it possible to debug the calibration program already in the stage of development of a spiral reader, to propose and debug new algorithms for measuring reference crosses, and to verify the correctness of the program.

Robust estimates in regression problems

We consider the regression dependence $y = \Sigma_j x_j b_j + d$, where x_j are factors determined by the position at which the measurement is made, y is the response (the measured quantity), d is the error of the measurement, which has zero mean and variance σ^2, and b_j are unknown regression coefficients, $j = 1,...,m$. The measurements are made independently at n points, giving

$$X = \begin{pmatrix} x_{11} & \dots & x_{1m} \\ \vdots & & \vdots \\ x_{n1} & \dots & x_{nm} \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}.$$

Then $Y = XB + D$, where $B = \mathrm{row}(b_1,...,b_m)^{\mathrm{T}}$; $D = \mathrm{row}(d_1,...,d_m)^{\mathrm{T}}$ are unknown vectors (T denotes transposition). The random vector D has a scalar covariance matrix cov D.

It is well known[23] that when the variable d is distributed in accordance with a normal law the optimal estimate of the vector of parameters is the estimate obtained by the least-squares method:

$$\Phi(b_1, \dots, b_m) = \sum_i w_i \left(y_i - \sum_j x_{ij} b_j\right)^2 \to \min. \quad (14)$$

The least-squares estimate of B can be expressed linearly in terms of Y:

$$\hat{B} = (X^T W X)^{-1} X^T W Y. \quad (15)$$

Here, W is the matrix of weights, which for independent observations has a diagonal form, $w_{ii} = \sigma_i^{-2} = w_i$ (σ_i^2 is the variance of the i-th observation).

For clarity, we write out more fully the matrix expression (15) for the case $m = 2$ (a straight line: $y = ax + b$); this will be used frequently in what follows.

To determine the two parameters a and b, we minimize the functional

$$\Phi(a, b) = \sum_{i=1}^{n} w_i (y_i - a x_i - b)^2 = \sum_{i=1}^{n} w_i d_i^2. \quad (16)$$

To this end, we equate to zero the derivatives of $\Phi(a, b)$ and obtain a system of normal equations, which is usually written down by means of the so-called Gaussian bracket $[wx^p y^q] = \sum_{i=1}^{n} \omega_i x_i^p y_i^q$:

$$\left.\begin{aligned} a[wx^2] + b[w\ x] &= [wxy]; \\ a]wx] + b[w] &= [wy]. \end{aligned}\right\} \quad (17)$$

The determinant of the system (17) is det $= [wx^2][w] - [wx]^2$; the estimates of the parameters (i.e., the components of the vector $\hat{B}$ in the two-dimensional case),

$$\left.\begin{aligned} \hat{a} &= [wxy][w] - [wy][wx]/\det; \\ \hat{b} &= [wx^2][wy] - [wxy][wx]/\det, \end{aligned}\right\} \quad (18)$$

are random variables with means a and b and covariance matrix

$$\mathrm{cov}\,\hat{B} = \sigma^2 (X^T W X)^{-1} = \sigma^2 \frac{1}{\det} \begin{pmatrix} [w] & -[wx] \\ -[wx] & [wx^2] \end{pmatrix},$$

so that

$$\sigma_{\hat{a}} = \sigma \sqrt{\frac{[w]}{\det}}, \quad \sigma_{\hat{b}} = \sigma \sqrt{\frac{[wx^2]}{\det}},$$

$$\rho_{\hat{a},\hat{b}} = -\frac{[wx]}{\sqrt{[w][wx^2]}}. \quad (19)$$

The estimate of σ can be obtained as

$$\hat{\sigma} = \sqrt{\frac{\Phi_{\min}(a, b)}{n-2}} = \sqrt{\frac{\sum_i w_i \hat{d}_i^2}{n-2}},$$

where the residual at the i-th point is denoted by $\hat{d}_i = y_i - \hat{y}_i = y_i - a\hat{x}_i - \hat{b}$.

For problems involving the identification of images from automatic scanning data, the presence of readings generated by noise formations or failures of the device is characteristic. These junk readings are added to the data used to estimate the unknown parameters, and this nature of the inhomogeneous set complicates the use of the regression methods of statistics described above, which are based on the assumption that the distribution of all the data is normal. Therefore, to eliminate this junk various methods of so-called rejection are employed. They almost all reduce to a least-squares estimate of the regression parameters with respect to all the data with testing of the criterion $\hat{\sigma} < \sigma_{\lim}$ ($\sigma_{\lim}$ is a specified constant, usually $\sigma_{\lim} \sim 10$–$20\,\mu$m). If the criterion is violated, the corresponding data are rejected, i.e., the points are eliminated for which $|\hat{d}_i| > 3\hat{\sigma}$, or, in a more reliable variant,[26] the $\max_i |d_i|$ are eliminated. After this, the estimates $\hat{a}$, $\hat{b}$, $\hat{\sigma}$ are again found for the reduced set. The criterion for termination of the rejection process is smallness of $\hat{\sigma}$.

The cumbersome nature and inadequate reliability of such procedures led to wider use of so-called robust methods of estimating the parameters of inhomogeneous sets.

The essence of these methods consists of replacing the quadratic functional (14), (16) by a functional with a more

general contribution function $\psi(t)$:

$$\Phi_\psi(b_1, \ldots, b_m) = \sum_i \psi\left(\frac{\hat{d}_i}{s}\right), \quad (20)$$

where s, the estimate of the scale parameter, is equal to σ if σ is known.

The choice of the contribution function is determined by the degree of contamination with junk and the method used in the calculations. It is aimed at reducing the influence of the extraneous (junk) points, i.e., reducing the contribution to the functional of the points whose deviations $|\hat{d}_i|$ are large.

Detailed reviews of robust methods can be found in Refs. 27 and 28; briefer details with examples of applications to the analysis of track data are given in Ref. 29.

We here restrict ourselves to mentioning two types of contribution function—the convex one proposed by Huber,[28]

$$\psi_1(t) = \begin{cases} t^2/2, & |t| < c; \\ c|t| - t^2/2, & |t| \geqslant c, \end{cases}$$

and Tukey's nonconvex one[30]:

$$\psi_2(t) = \begin{cases} \frac{c^2}{6} + \frac{(t^2-c^2)^3}{6c^4}, & |t| < c. \\ c^2/6, & |t| \geqslant c. \end{cases} \quad (21)$$

Huber's method is correct (the functional $\Phi_{\psi 1}$ has a unique minimum) and is optimal in the minimax sense.[27] However, it gives satisfactory estimates only when there are not very heavy "tails" of the probability density of the deviation from the mean (excess coefficient of not more than 5–6). For cases with greater contamination of the data, the best results are given by methods with a nonconvex function of the form $\psi_2(t)$, which more actively suppresses the influence of points with large $|\hat{d}_i|$. The price to be paid for this is the appearance of spurious local minima of the functional (20).

These methods are implemented in the form of iterative procedures in which at each step the least-squares method is employed with weights $w_i = \psi'(\hat{d}_i/s)/\hat{d}$ calculated from the estimates of the parameters in the previous iterations. As initial approximations of the parameters one chooses either the least-squares estimates with $w_i = 1$ or *a priori* values of $\hat{a}_0$ and $\hat{b}_0$ if they are known, as, for example, in the case of the crosses of a calibration grid. As an estimate of the scale parameter s one uses $\hat{\sigma} = \text{med}|\hat{d}_i|/0.6745$, which is the normalized median (the central term of the variation series of the moduli of the residuals). The normalization coefficient 0.6745 is chosen to ensure that in the case of a Gaussian distribution $\hat{\sigma}$ is an unbiased estimate. If there is greater contamination, this coefficient must be increased. The constant c in Eq. (21) is usually taken to be equal to 5.[30] The investigation of Ref. 29 showed that the reliability of robust estimates depends strongly on the correct choice of the initial values of the parameters. In some perfectly real situations in which a clearly maverick point, an "outsider," appears after a large break in a set of points for which parameters are being estimated (Fig. 3), the use of the least-squares estimates with weights $w_i = 1$ gives values of the initial approximations $\hat{a}_0$, $\hat{b}_0$ near a spurious local minimum of the functional, into which the estimates of the parameters in the subsequent iterations then "roll" (see the dashed line in Fig. 3). As is shown in Ref. 29, this danger can be avoided by normalizing each residual by its own rms deviation δ_i, which is calculated for $m = 2$ as a polynomial of the second degree in x_i with coefficients found from the data of the previous iteration. This in fact amounts to a more complicated formula for Tukey's "biweights":

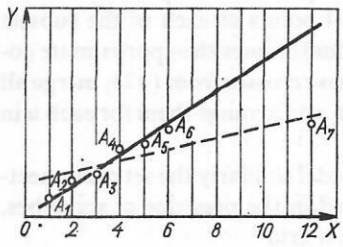

FIG. 3. Example of a regression with a pronounced outlier.

$$w_i = [1 - (\hat{d}_i/4\hat{\sigma}\delta_i)^2]^2. \quad (22)$$

Systems of polynomials and methods of orthogonalizing them

In this section, we shall consider some properties of linear metric spaces that relate to the numerical construction of calibration transformations.

Let Q_M be a linear real space of M dimensions. Suppose that for all $f, g \in Q_M$ a scalar product (f, g) satisfying the well-known axioms[31] is defined. Then Q_M is a metric space, and its basis $\{b_i\}_{i=1}^M$ is said to be orthogonal if

$$(b_i, b_j) = h_i^2 \delta_{ij} \quad (23)$$

(δ_{ij} is the Kronecker delta), and orthonormal if all $h_i^2 \equiv 1$.

For the reasons noted in the Introduction, the spaces Q_M of main interest to us for the approximation of calibration dependences are those whose elements are polynomials in the unknown variables.

We consider polynomials in n variables $x_1, x_2, \ldots, x_n$ (in physical space $n \leqslant 3$) as the elements of a real linear metric function space Q_M. We begin our consideration in the "natural" polynomial basis consisting of monomials of degrees of the coordinates of the form $x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}$, the total degree being given by the sum of the exponents $i_1 + i_2 + \ldots + i_n$. When M is finite, there is no doubt that the elements of the basis are countable. For $n = 1$, the elements of the basis can be ordered in a natural manner in ascending degrees. However, if $n > 1$, there exist several linearly independent monomials of the same total degree (for example, for $n = 2$ we have the second-degree monomial x_1^2, $x_1 x_2$, and x_2^2), the degree by itself is insufficient for the introduction of order. Combinatorial considerations lead to the conclusion[32,33] that there exist altogether

$$B_n^k = \frac{(n+k)!}{n!k!}$$

linearly independent polynomials in n variables of degree

not higher than k. In this case, one most frequently uses the so-called lexicographic ordering, which is introduced as follows. Of two monomials, the higher number is ascribed to the one that has the higher total degree or, in the case of equality of the degrees, the part corresponding to the variables with the higher numbers.

The existence of an ordered basis in Q_M makes it possible to consider the problem of finding the explicit form of the calibration transformation as a variant of the problem of decomposing a given vector $\mathbf{F}$ with respect to a basis,

$$\mathbf{F} = \sum_{i=1}^{J} a_i \mathbf{b}_i, \tag{24}$$

where $J \leqslant M$, $\{\mathbf{b}_i\}$ are the elements of the basis in the lexicographic order, and $\{a_i\}$ are the corresponding (unknown) coordinates of the vector. The specific methods of solving this problem depend to a considerable degree on the choice of the basis, the definition of its metric (the scalar product), and the criteria of optimality, on the basis of which the length J of the decomposition (24) is determined.

The natural basis introduced is to a fair degree elementary and transparent, and calculations in it are not too laborious if one uses the Horner rule[34] or its multidimensional generalizations. However, the condition numbers[a)] of the matrices that arise in this basis are so high that its practical applications are restricted to low degrees of the independent variables (for example, not more than 5–6 for $n = 1$) even in arithmetic of higher accuracy. This is clearly insufficient for the majority of calibration problems, and this justifies efforts made to improve the basis. From the theoretical point of view, the best basis would be one that ensures that the calibration problem has unit condition. The existence of such a basis was proved long ago,[35,36] and the constructive nature of the proof is obvious from frequent program implementations of the method.[37–39] We shall consider briefly different approaches to improving the properties of the basis by its orthogonalization and orthonormalization.

The set of the scalar products of both sides of Eq. (24) by the basis vectors $\{b_i\}$ can be written in matrix form as

$$G_J \mathbf{a}_J = \mathbf{f}_J, \tag{25}$$

where $\mathbf{a}_J = \mathrm{col}(a_1, a_2, \ldots, a_J)$, $\mathbf{f}_J = \mathrm{col}[(F, b_1), (F, b_2), \ldots, (F, b_J)]$, and G_J is the Gram matrix[40] of order J constructed from the basis vectors. Then the unknown coefficients $\{a_i\}$ can be expressed in the form

$$\mathbf{a}_J = G_J^{-1} \mathbf{f}_J. \tag{26}$$

The last two equations, (25) and (26), render the direct influence of the chosen basis on the conditioning of the calibration problem an obvious fact.

The matrix equation (25) leads to the idea of direct orthogonalization, namely, one can attempt to find numerically a matrix A that diagonalizes G_J:

$$AG_JA^{-1} = \mathrm{diag}\,(g_1, g_2, \ldots, g_J),$$

with all $g_i > 0$ in a nondegenerate basis. Then

$$AG_JA^{-1}A\mathbf{a}_J = A\mathbf{f}_J$$

and the components of the vector Aa_J can be readily expressed:

$$(A\mathbf{a}_J)_i = (A\mathbf{f}_J)_0/g_i, \quad i = i = 1, \ldots, J.$$

This method is laborious but attractive through the availability of the rich and well-developed arsenal of matrix calculations. It was used successfully in Ref. 41 to obtain more accurate calibration transformations.

A different and more direct route to basis orthogonalization before forming (25) consists of using the Gram–Schmidt orthogonalization procedure[40]: If $\{b_i\}$ is the original basis and $\{e_i\}$ is the orthogonal basis, then $e_1 = b_1$ and

$$e_i = b_i - \sum_{j=1}^{i-1} \frac{(e_i, b_i)}{e_j^2} e_j, \quad i = 2, \ldots, J.$$

It is easy to show that this procedure is universal, i.e., it ensures orthogonality of the set $\{e_i\}$ not only for polynomials but also in the case of all basis functions $\{b_i\}$. The price that must be paid for this useful quality is that the procedure is cumbersome, since Gram–Schmidt orthogonalization in conjunction with normalization of the new basis requires calculation of $J(J-1)/2$ scalar products and J scalar squares for dimension J, and the number of necessary scalar products does not depend on the dimension of the independent variable.

A third—and in our view, the most effective—method of orthogonalizing the polynomial basis makes use of recursions of the type of the Forsythe three-term relation.[35] This connects the values of the orthogonal polynomials of three successive orders in one variable and can be written in the form

$$P_{i+1}(x) = c_{i+1}[(x - \alpha_{i+1}) P_i(x) - \beta_i P_{i-1}(x)], \tag{27}$$

where x is an arbitrary but fixed value of the independent variable, and $\{\alpha_i\}$ and $\{\beta_i\}$ are recursion coefficients that are expressed in terms of scalar products and do not depend on the argument x. Strictly, Forsythe's relation (27) concerns the orthogonalization of the basis but not its normalization. The generalizations of this relation obtained in Refs. 36–39 extended it in such a way that:

(a) the connection between the normalizing coefficient c_i and the recursion coefficient β_i was established, and this made it possible to normalize the basis at the same time as the orthogonalization;

(b) the relation was augmented by a fourth term in order to extend it to all nontrivial derivatives and indefinite integrals of orthonormalized polynomials $\{P_i\}$ (Ref. 38);

(c) the analogous relation for polynomials in several variables was determined.

The recursion relations made possible the numerical or analytic construction of systems of orthonormalized polynomials for any definition of the scalar product and any number of independent variables. The general idea of the orthonormalizing algorithms of this class is as follows:

1. Preliminary calculation of the recursion coefficients for the complete range of required degrees.

2. Calculation of the values of the polynomials of the basis for an arbitrary fixed set of values of x (here, x denotes

TABLE I. Comparison of the numbers of scalar products whose calculation is needed for the orthogonalization of polynomials in one, two, and three variables by the Gram–Schmidt and Forsythe–Weisfeld methods.

Maximal degree of system of polynomials	Dimension 1			Dimension 2			Dimension 3		
	Number of linearly independent polynomials in system	Method		Number of linearly independent polynomials in system	Method		Number of linearly independent polynomials in system	Method	
		Gram-Schmidt	Forsythe–Weisfeld recursions		Gram-Schmidt	Forsythe–Weisfeld recursions		Gram-Schmidt	Forsythe–Weisfeld recursions
1	2	3	3	3	6	6	4	10	10
2	3	6	5	6	21	19	10	55	52
3	4	10	7	10	55	46	20	210	191
4	5	15	9	15	120	92	35	630	541
5	6	21	11	21	231	162	56	1596	1282
6	7	28	13	28	406	261	84	3570	2675

the independent variables, not necessarily in one dimension) by means of the appropriate recursion relation before the actual polynomials in the coefficients are written down.[42]

We do not believe that it is sensible to give here the explicit form of the complete set of computational formulas for the recursion coefficients, which can be found in the cited literature. We merely mention that there is a one-to-one correspondence between the polynomials of the natural basis and the orthonormalized basis, and this makes it possible to follow the same lexicographic order in the numbering of the elements of the orthonormal basis. We believe it is also important to describe the principle of the approach and persuade the reader of its greater efficiency, which is, moreover, many sided:

1. The recursive orthonormalization requires calculation of a smaller number of scalar products than the Gram–Schmidt method. This means that this approach is more economic from the point of view of preparatory calculations.

2. The recursive orthonormalization requires the retention of the calculated scalar products for calculating the values of the polynomials, whereas in the Gram–Schmidt method it is necessary to keep the values of the coefficients, the number of which is equal to the number of scalar products for the same Gram–Schmidt method. This means that the recursive approach is more economic from the point of view of the use of memory.

3. In the recursive approach, the method of calculating the values of the basis polynomials is simple and standard, whereas in the Gram–Schmidt method it is necessary to construct complicated multidimensional analogs of the Horner algorithm. In addition, in the one-dimensional case telescoping of the orthonormalized approximating series is possible.[38,43,48] We assume that telescoping is in principle also possible for multidimensional series, but we do not know of corresponding program implementations. This means that the recursive approach is also more economic from the point of view of the volume of basic calculations.

Quantitative data for comparison for degrees 1–6 for one, two, and three independent variables are given in Table I.

The comparison has a fundamental nature and is based on the intrinsic properties of two orthogonalization methods. In this connection, we should note the very interesting realization[17] of the Gram–Schmidt method that uses a simple multiplicative connection between the polynomials of the natural basis for the case of three variables; for it, the most advantageous order of calculating the scalar products was also chosen. On the basis of the results, Gorchakov[17] concluded that the Gram–Schmidt method has advantages over the recursive methods of orthogonalization. We do not share these views, since we believe that one should not confuse comparison of the methods with comparison of their actual realizations. The class of fast algorithms of the type of fast Fourier transformation is, in principle, also applicable to the recursive methods of orthogonalization. As can be seen from Table I, the relative difference between the volumes of the necessary calculations decreases with increasing number of independent variables. Nevertheless, because of the large absolute volume of the calculations it is necessary to use all available means to improve the speed of the programs, including a more economic method in conjunction with a felicitous algorithm.

An additional advantage of orthogonal expansions (independent of the method of orthogonalization and in addition to the improvement of the conditioning) is the mutual independence of the coefficients of the orthogonal series, the simplicity of calculating the total errors of the coefficients and the series themselves, and the facility of comparing series of different lengths J.

Summarizing the above, we also mention the three main advantages of Forsythe–Weisfeld orthonormalization, which completely compensate for the additional calculations in its realization: (1) the rapid and simple algorithm; (2) the normalization, which ensures a small rounding error (see also Ref. 39); (3) the simplification of the calculations when the necessary degree J of the polynomials is chosen, since the coefficients of the previously calculated polynomials are used unchanged (Refs. 16 and 37; see also Ref. 39).

Generally speaking, a scalar product is a transformation of convolution type and in most general form can be written as a Lebesgue integral[40]:

$$(f, g) = \int_{\Omega} f(x)\, g(x)\, d\mu, \tag{28}$$

where f and g are integrable on a set Ω of non-negative measure μ. In the given case, the independent variable x is necessarily one-dimensional. Since in our case x represents a coordinate continuum, without loss of generality we can write

$$d\mu = w(x)\, dx,$$

where $w(x)$ is a non-negative weight function. Then the different scalar products will differ in the domain of integration Ω and in the weight function $w(x)$. It can be seen that there is a wide choice of admissible scalar products. Some of them have been well studied and systematized; for example, for a one-dimensional argument and smooth weight functions there exist just five basically different families of orthogonal polynomials[43,44]: Legendre, Chebyshev, Hermite, Laguerre, and Jacobi. These are the so-called classical orthogonal polynomials.

If the reference points are more or less uniformly distributed over the sensitivity field Ω of the calibrated device and the accuracy of determining the reference coordinates is approximately the same, it is most natural to set

$$w(x) \equiv 1,$$

and then, by an elementary linear transformation, reduce Ω to the unit interval (or square for $n = 2$ and cube for $n = 3$) and use a scalar product of the form

$$(f, g) = \int_{\Omega} f(x)\, g(x)\, dx. \tag{29}$$

Orthonormalization of the basis by means of (29) does not reduce the matrix G in (25) to diagonal form but nevertheless makes it "quasidiagonal," i.e., with a sharply predominant principal diagonal. At the same time, the condition of G_J is significantly improved compared with the "natural"

basis, and the numerical solution of the system (25) is much more stable. This approach was used in Ref. 41, in which a two-dimensional orthogonalization was implemented analytically by means of the system REDUCE[45] on the basis of the Gram–Schmidt method. This technique was used to express the direct calibration transformations (i.e., to obtain the expected positions of the reference frames on the basis of their known actual positions), since in this case both the orthogonalization and the inversion of the Gram matrix are done only once and then used repeatedly. In this sense, one can assume that the labor in the preparatory calculations—the orthogonalization and inversion—is of secondary importance.

The accuracy in the determination of the reference positions $p(x)$ is, of course, never the same over the complete field Ω and in many cases gets worse at the edges. It is therefore helpful to use a weight function of the form

$$w(x) = 1/p^2(x), \tag{30}$$

if one can propose a reasonable analytic model for the accuracy of $p(x)$. The weights (30) have more physical significance and correspond better to the characteristics of the calibrated device. An example of their practical use in the case of a spiral reader will be given below.

For direct calibrations made periodically, it may happen that the set of measured coordinates of the reference frames is not constant (not all the frames can be seen in every picture), and the coordinates themselves and the accuracies of their determination are subject to drift; it would then be desirable to use a rapid and effective method of orthonormalization leading to a unit matrix G_J for the concrete set of coordinates and their errors. For these purposes, it is more convenient to return to the general definition (28) and introduce the measure in the form

$$d\mu = \sum_{i=1}^{N} w_i \delta(x - x_i)\, dx, \tag{31}$$

where the weight w_i is expressed in terms of the accuracy ΔF_i of determination of the i-th reference mark:

$$w_i = 1/(\Delta F_i)^2,$$

where $\delta(x - x_i)$ is the Dirac delta function, x_i are the observed coordinates of the i-th reference mark, and N is the total number of reference marks. Substitution of (31) in (28) leads to an expression for the scalar product in terms of a finite sum:

$$(f, g) = \sum_{i=1}^{N} f(x_i)\, w_i g(x_i). \tag{32}$$

A basis obtained by means of a scalar product of this type is said to be orthonormalized on a discrete point set. Since this set was chosen to coincide with the set of observed coordinates of the reference marks, and the weights were determined in accordance with the accuracy in the determination of the coordinates, it is easy to show that in this basis

$$G_J = I \tag{33}$$

and Eq. (26) gives directly $\mathbf{a}_J = \mathbf{f}_J$, i.e., the coefficients in the expansion of the required calibration transformation are the scalar products of the observed vector $\mathbf{F}$ and the corresponding basis vectors. It can also be seen from (33) that the condition of the problem (25) that we must solve is unity (i.e., optimal).

It is known that for a scalar product of the form (29) the number of elements of the generated orthonormalized basis is not bounded. However, since the actual number N of reference marks is bounded, we can in principle use only the first N polynomials—and in practice many fewer—in order to avoid transforming the calibration problem into an interpolation problem. In Ref. 16, a method is proposed for estimating the scalar square of the successive polynomials of the orthonormalized basis that does not required calculation of the values of the polynomial and is based on the values of the recursion coefficients. If this square is negative or smaller in modulus than the relative accuracy of the computer, then this is an indication that the dimension Q_M of the space has been reached and that further construction of polynomials higher in the lexicographic order is impossible.

In some practical cases, very simple geometrical or physical considerations lead to the conclusion that it is necessary to restrict the degrees with respect to one (or several) of the coordinates. This is done by imposing additional restrictions on the maximally admissible degrees with respect to some variables, as indicated by the broken line in Fig. 4. Another example of construction of an "incomplete" system of orthonormalized polynomials in three variables with restrictions on the degrees $i_x \leqslant 3$, $i_y \leqslant 2$, and $i_z \leqslant 6$ is shown in Fig. 5, which gives the complete hierarchy of the system to degree 6 and in which the elements of the basis omitted for the foregoing restrictions are hatched. We have no experience of the use of incomplete orthonormalized bases apart from the obvious one-dimensional cases of the expansion of even functions with respect to even polynomials and odd ones with respect to odd polynomials. We know that the

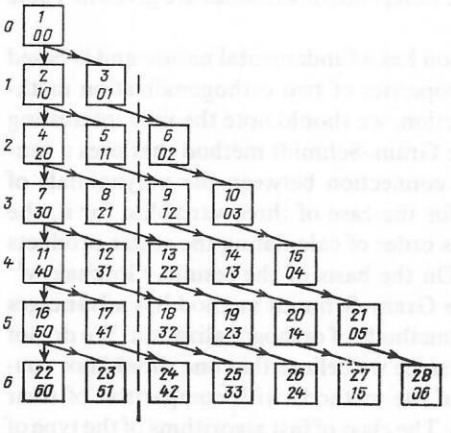

FIG. 4. Hierarchy and lexicographic order of polynomials in two variables of degrees 0–6. The total degree is indicated on the right. The number at the top of each cell is the serial number of the polynomial; the two numbers below it give the distribution of the degrees. If the condition of linearity along the J axis is imposed, it is necessary to omit (or regard as identical to zero) all terms of the family to the right of the vertical broken line.

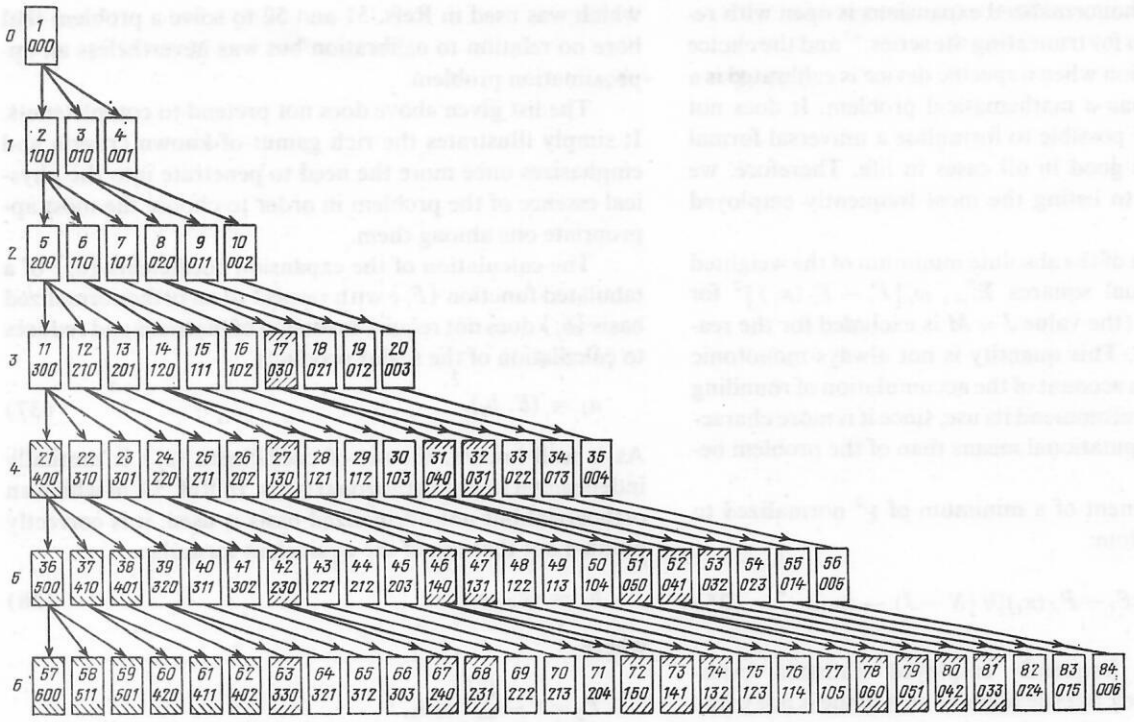

FIG. 5. Hierarchy and lexicographic order of polynomials in three variables of degrees 0–6. The total degree is given on the right. The number at the top of each cell is the serial number of the polynomial; the three numbers below it give the distribution of the degrees. The hatched members of the family must be omitted if the subsidiary restrictions $i_x \leqslant 3$ and $i_y \leqslant 2$ are imposed.

magnetic field of the RISK chamber[46] has been approximated in this way, but the results have not been published.

In the foregoing exposition, we did not impose any conditions on the spatial position of the reference points in the domain Ω. It is important to note that if Ω can be represented as a direct product of subspaces,

$$\Omega = \Omega_1 \times \Omega_2, \tag{34}$$

then separation of the variables is possible and the basis polynomials can be represented in Ω in the form of products of the elements of the bases in Ω_1 and Ω_2; moreover, the orthonormalization in Ω_1 is in no way related to the orthonormalization in Ω_2 and vice versa. Naturally, since the dimensions of the subspaces are lower than the dimension of Ω, such a representation is much more economic in the program realization—cf. the dimensions $n = 1$, $n = 2$, and $n = 3$ in Table I. Grids of reference marks for which (34) is possible can be called pseudoperiodic. Like periodic lattices, they consist of "unit" cells of similar shape but different sizes (Fig. 6).

And one further remark. Hitherto we have throughout assumed a rectangular Cartesian coordinate system, in which the polynomial representations are the most perspicuous. However, the recursive orthonormalization which we are considering does not necessarily require such a coordinate system. For example, there are no fundamental obstacles to carrying out the orthonormalization directly in polar coordinates, avoiding numerous laborious transitions to a Cartesian system, especially if we are concerned with images of localized objects (reference crosses). For work in a complete field, it is necessary to take into account the continuity and smoothness of the approximating function at $\theta = 0$ and $\theta = 2\pi$.

Now, assuming that the process of orthonormalization of the basis has been completed, we turn to the actual approximation of the calibration dependences by means of finite series of the form (24) for dimension M of the linear space Q_M, the relations $J \leqslant M \leqslant N$ holding. Note that for $J = M$ (N can be attained only if the reference points are not situated on an algebraic surface of lower order) this expression goes over into Lagrange's well-known interpolation formula[40] or its multidimensional analogs,[32] and this is undesirable for at least two reasons: (a) our reference marks are, of course, measured with finite accuracy, and there are no grounds for making the approximation exact at the corresponding points; (b) the Lagrange interpolation in high orders is extremely unstable in the interstices. Therefore, the first question when (24) is used concerns the determination of the optimal order J of the approximation. Generally speaking,

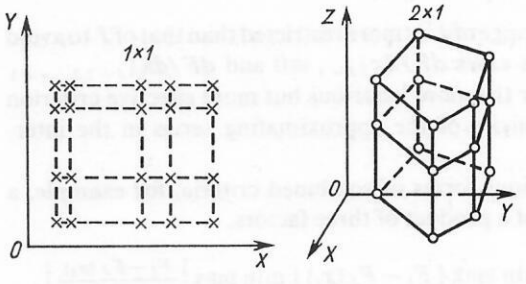

FIG. 6. Examples of grids that admit separation of the variables.

the method of orthonormalized expansions is open with respect to the criteria for truncating the series,[47] and the choice of a suitable criterion when a specific device is calibrated is a physical rather than a mathematical problem. It does not appear to us to be possible to formulate a universal formal criterion which is good in all cases in life. Therefore, we restrict ourselves to listing the most frequently employed criteria.

1. Attainment of the absolute minimum of the weighted sum of the residual squares $\Sigma_{i=1}^{N} \omega_i [F_i - F_J(x_i)]^2$ for $J = 1, 2, \ldots, M-1$ (the value $J = M$ is excluded for the reasons given above). This quantity is not always monotonic with respect to J on account of the accumulation of rounding errors. We do not recommend its use, since it is more characteristic of the computational means than of the problem being solved.

2. The attainment of a minimum of χ^2 normalized to one degree of freedom:

$$\chi^2_{N-J} = \sum_{i=1}^{N} w_i [F_i - F_J(x_i)]^2/(N-J) \to \min. \qquad (35)$$

This is the most frequently employed statistical criterion.[35,48] It is correct, but the reliance on it alone is not without danger.

3. Fisher's criterion,[49] i.e., the absence of a significant deviation from zero of the two higher expansion coefficients following J.

4. Fulfillment of the condition[21,38,48]

$$w_i [F_i - F_J(x_i)]^2 \leqslant 1, \quad i = 1, 2, \ldots, N. \qquad (36)$$

The geometrical meaning of this criterion is that the approximating curve (or surface) passes through the error corridors of all the reference points. The criterion is very reliable in the case of unbiased estimates of the positions of the reference marks and valid estimates of the errors, i.e., weights).

5. Attainment of the *a priori* absolute, $|F_i - F(x_i)| \leqslant \Delta F$, or relative, $|[F_i - F(x_i)]/F_i| \leqslant \Delta f$, accuracy at all the reference points. This criterion is suitable for a rapid current (verification) calibration if the device accuracy characteristics Δf or ΔF have been obtained earlier by different methods.

6. The minimax or uniform approximation criterion:

$$\min_J \max_i |F_i - F_J(x)|.$$

7. A smoothness criterion, for example,

$$\min_{J'} \max_i \left| \frac{dF_{J'}(x_i)}{dx} \right|,$$

where the range of J' is more restricted than that of J to avoid the obvious cases $dF/dx|_{J=1} \equiv 0$ and $dF/dx|_{J=2,3,\ldots,n+1} = \text{const}$, or the more laborious but more effective criterion of monotonicity of the approximating series in the interstices.[50]

8. Various forms of combined criteria, for example, a minimum of a product of three factors,

$$\chi^2_{N-J} \min_J \max_i |F_i - F_J(x_i)| \min_J \max_i \left| \frac{F_i - F_J(x_i)}{F_i} \right|,$$

which was used in Refs. 51 and 52 to solve a problem that bore no relation to calibration but was nevertheless an approximation problem.

The list given above does not pretend to completeness. It simply illustrates the rich gamut of known criteria and emphasizes once more the need to penetrate into the physical essence of the problem in order to choose the most appropriate one among them.

The calculation of the expansion coefficients $\{a_j\}$ of a tabulated function $\{F_i\}$ with respect to an orthonormalized basis $\{b_j\}$ does not require inversion of matrices and reduces to calculation of the scalar products

$$a_j = (F, b_j). \qquad (37)$$

As is well known, the set of coefficients $\{a_j\}$ is mutually independent in any orthogonal basis. In Ref. 53, in which an orthogonal but not normalized basis is used, it is correctly pointed out that use of the equivalent formula

$$a_j = (F_j, b_j), \qquad (38)$$

where

$$F_j = F - \sum_{k=1}^{j-1} a_k b_k,$$

leads to a smaller effect of the accumulation of rounding errors. This is indeed the case, although it is manifested less clearly when the basis is normalized. Equation (38) requires a greater volume of elementary calculations than (37) and, in addition, the mutual independence of the coefficients is lost when it is employed. One could resort to this method when it is known for certain that it is precisely the accumulation of rounding errors that is the cause of the unsatisfactory quality of the approximation. In our practical work in the calibration of various track measuring systems by means of orthonormalized polynomials, we have not encountered such situations. It was always possible to use the simpler formula (37) and preserve the independence of the expansion coefficients.

The total rms deviations $\{\Delta a_j\}$ do not depend on the index j and are numerically equal to[35,49]

$$\Delta a_j = \sqrt{\chi^2_{N-J}}, \quad j = \overline{1, J}. \qquad (39)$$

We mention in passing that χ^2_{N-J} can be expressed without recourse to summation of the series (24), but this property should not be used because of the danger of a loss of accuracy when nearly equal numbers are subtracted in a floating-point representation.

From (24) and (39) in the case of independent coefficients $\{a_j\}$ it follows that the accuracy of the approximating function ΔF at any point x can be expressed as

$$\Delta F(x) = [\chi^2_{N-J} \sum_{j=1}^{J} b_j^2(x)]^{1/2}$$

Once again, orthonormalization of the basis $\{b_j\}$ ensures a simple and reliable method of estimating $\Delta F(x)$.

To conclude this section, we may note that the advantages of orthonormalized polynomial bases are now in no

doubt. The formalism for them has been well developed theoretically, the technical details of the corresponding algorithmization have been clarified, and the program implementation does not present serious difficulties. The existing experience of the use of programs completely confirms these conclusions.

3. DETERMINATION OF THE CENTERS OF THE CROSSES

The first stage in the statistical analysis of the data of calibration measurements consists of recognition of the crosses and the determination of the centers and the errors in their values. This problem is important for the analysis of not only the calibration measurements, since a reference cross is one of the most common figures in photographs of track chambers. The centers of the reference marks are needed to reconstruct the spatial picture of the event and to take into account possible distortions of the device. Therefore, the centers must be measured with particular accuracy. In this connection, it is necessary to take into account the nonlinearity of the measuring devices, which is particularly important for those with electron-beam scanning and in filmless systems, in which appreciable distortions of the rectilinearity of the arms of some crosses and the angles between them are possible.[54] Nevertheless, in practically all cases the presence of distortions does not prevent the identification of rectangular regions of the approximate position of each of the crosses on the basis of *a priori* information about the actual calibration grid. Thus, the procedure of recognition and separation of the centers of the crosses is usually implemented in a cycle for all crosses of the grid, the following five operations being carried out for each cross: (a) separation of the region occupied by all the points of a cross and the transporting of this region to the origin in order to simplify the further calculations; (b) the division of all points of the region into classes corresponding to the arms of the cross and elimination of the noise points; (c) approximation of each of the arms of the cross; (d) determination of the coordinates of the center of the cross by simultaneous solution of the equations of the arms; (e) calculation of the errors of the coordinates of the centers.

This scheme is also basically retained in the case of a spiral reader, although in that case there are some specific features dictated by the existence of the two coordinate systems.

Cross recognition

The operation of identifying the region including all points of the k-th cross for a symmetric grid has effectively already been described at the beginning of Sec. 2. The *a priori* estimates of the scales along the axes, M_{x0} and M_{y0}, and also the coefficients of transport $\hat{X}_0$, $\hat{Y}_0$ between the coordinate systems of the device and the ideal system associated with the image make it possible to determine the transformation that transports the center of this region to the origin:

$$u'_{ik} = u_{ik} - U_{k0};\ v'_{ik} = v_{ik} - V_{k0};\ i = 1,...,N.$$

where U_{k0} and V_{k0} are determined in (12) and i is the index of the current point.

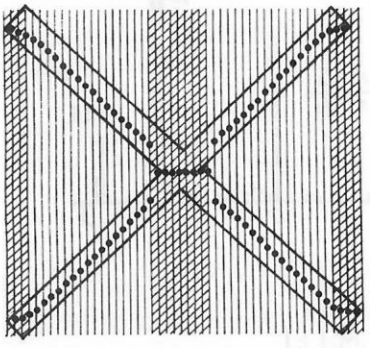

FIG. 7. Examples of distortions produced by digitization of a cross.

The transition to such a local system of Cartesian coordinates appears much more complicated when one is analyzing data of a spiral reader,[21] for which the boundaries of the region in the polar system of coordinates R, θ form a symmetric trapezium, and transition to a local Cartesian system, even approximately, requires estimates of not only the scales M_R and M_θ along the axes but also a preliminary estimate θ_0 of the initial angle of reference. The crosses of the calibration grid of a spiral reader in the local system are rotated in such a way that the OX' axis is approximately along the bisector of the smaller angle of the cross in order to ensure that its arms satisfy Eq. (13).

It should be pointed out immediately that in any system that scans globally the complete calibration grid the scanning lines pass at an angle to the arms of the crosses. The finite resolution of the scanning spot leads in this case to a displacement of the readings at the ends of the crosses and to a merging of them at the middle of the cross (Fig. 7). The points from these sections, which are hatched in Fig. 7, must be eliminated, i.e., simply not transferred to the local coordinate system.

The choice of the cross recognition method depends on the distortions introduced by the measuring system and on the noise contamination of the image.

When both are small, one can use one of the clustering methods (see, for example, Ref. 26), i.e., the combining of points in accordance with certain criteria of proximity. Allowance for proximity with respect to distance and direction [by verification of the proximity of the angle coefficients of the straight lines joining a chosen point to all remaining points; Fig. 8(a)] permits recognition of both arms of a cross even if their orientation is unknown. However, this "liberality" of the method makes it rather unrobust with respect to noise generally and especially contamination in the form of extraneous dashes, scratches, etc.

More robust and therefore more widely used is the method of histogramming points along given directions with choice of the interval that makes the main contribution to the histogram [Fig. 8(b)]. By combining intervals next to the maximal interval, this method makes it possible to find the arms of a cross even if the direction is specified approximately [Fig. 8(b)].

In the cases when the distortions of the measuring devices are so small that they do not change the angle between

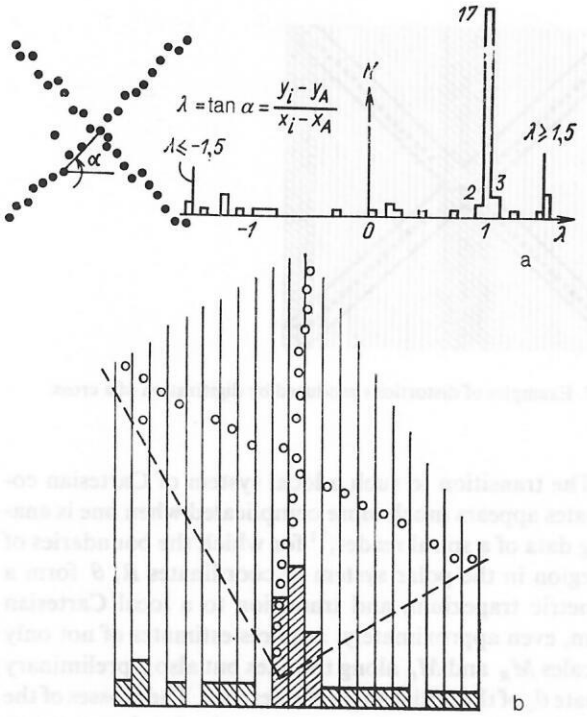

FIG. 8. Methods of recognizing a cross: (a) the clustering method; (b) the histogramming method.

the arms of the cross, it is most effective to use the method of determining the cross as a whole,[55] i.e., as a degenerate curve of second order.

Multiplying the equations of the arms (13), we eliminate the second-order terms containing the products of the quantities Δ_k, Δ_{1k}, Δ_{2k}, which are small by hypothesis, and introduce the new notation $a = \Delta_{1k} + \Delta_{2k}$, $b = -\lambda(\Delta_{2k} - \Delta_{1k})$, $c = 2\Delta_k$.

We obtain the functional

$$\Phi_k = \sum_{i=1}^{N} (v_{ik}^2 - \lambda^2 u_{ik}^2 - a v_{ik} - b u_{ik} - c u_{ik} v_{ik})^2,$$

whose normal system of equations is linear in the new variables. The functional being approximate, the exact solution requires several iterations. However, in Ref. 21 only the first iteration is given. The accuracy obtained in the determination of Δ_k, Δ_{1k}, Δ_{2k} from the values found for a, b, c is sufficient for classifying all points from the region of the cross into two groups according as they belong to the two arms of the cross.

The smallness assumptions for the distortions are by no means always satisfied. For example, preliminary experiments on the calibration of a filmless readout system showed that even in the regions of the individual crosses nonlinear distortions of the arms can be clearly observed (Fig. 9). The causes of the nonlinearity are here associated with the nonlinearity of the electron beam scanning, the distortion due to the amplifier of the brightness of the light (image tube), the instability of the electronic scanning circuits, and the aberrations of the optical systems.[10] Therefore, the arms of the reference crosses cannot in all cases be assumed to be intervals of straight lines,[54] or the angle between them to be equal to the corresponding angle for the calibration grid.

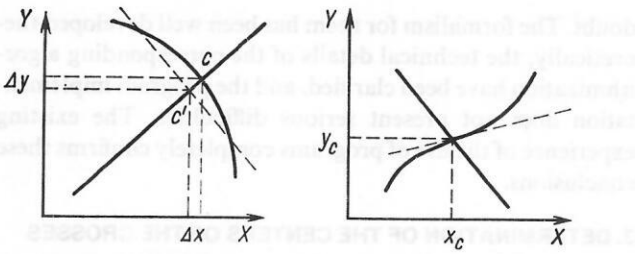

FIG. 9. Nature of the distortions produced by neglect of local nonlinearity: (a) shift of the center; (b) increasing of the errors.

In this connection, the method of simultaneous robust fitting of both arms of the cross by means of polynomials of suitable degrees is recommended as the most universal method not requiring verification of the smallness of the distortions.

Determination of cross centers

If we denote the coordinates of the point of intersection of the two arms by u_c, v_c, we can write the equations of the arms in the form of the overdetermined system

$$\left.\begin{aligned} f_1(u_{1i}, v_{1i}) &= f_1(u_c, v_c), \quad 1,\ldots,N_1, \\ f_2(u_{2i}, v_{2i}) &= f_2(u_c, v_c), \quad 1,\ldots,N_2 \end{aligned}\right\} \tag{40}$$

and seek the unknowns u_c, v_c as the coordinates of the points that minimize the sum of the residual squares of the equations, i.e., the functional

$$\chi^2 = \sum_{i=1}^{N_1} w_i [f_1(u_{1i}, v_{1i}) - f_1(u_c, v_c)]^2 + \sum_{i=1}^{N_2} w_i [f_2(u_{2i}, v_{2i}) - f_2(u_c, v_c)]^2. \tag{41}$$

The choice of weights ensuring stability of the solution with respect to contaminating data is important. The specific nature of the problem (the good initial approximation for the directions of the arms and the absence of situations of the type shown in Fig. 3) makes it possible to use instead of the weight function (22) Tukey's simpler biweight[30]

$$w_i = (1 - (\hat{d}_i / 5\hat{\sigma})^2)^2$$

with $\hat{\sigma}$ estimated from the median $|\hat{d}_i|$ for the initial values of the parameters.

Before we turn to the description of the method for solving this problem, we must consider the types of nonlinearities of the functions f_1 and f_2; this will enable us to sketch the qualitative way to its solution.

One observes the following types of nonlinearity (Fig. 9): (a) practically undistorted arms that can be approximated fairly accurately by straight lines; (b) convex or concave arms with curvature of constant sign, which is a form that can be represented with sufficient accuracy by a polynomial of second degree; (c) S-shaped arms, for which the curvature changes sign. For this case, a polynomial of third degree serves as a good approximation.

Then f_1 and f_2 can be represented in the form

$$f_j = \sum_{i=0}^{l_j} c_{ji} u_j^i - v_j, \quad j = 1, 2, \tag{42}$$

where $l_j = 1, 2, 3$ and each of them can be determined individually and independently for each arm. In addition, the distortions may change the angles between the arms of the cross, and we therefore give up the assumption of perpendicularity of the arms and assume only that the arms of the reference cross intersect at a definite but unknown angle. This assumption frees us from the need to go over to linear scales in an early stage of the calibration and, in addition, simplifies the algorithm [there is no need to construct the tangents to the curves f_1 and f_2 at the point (u_c, v_c)] and makes it more universal, since it permits the use of a skew reference mesh and different scales along the axes u and v.

The substitution of (42) in (40) and minimization of the functional (41) lead to a nonlinear system of normal equations, which can be decomposed into two independent linear systems of the type (17) for $\{c_{ji}\}$; a detailed description of the numerical–analytic algorithm for its solution can be found in Ref. 54.

In the case of the filmless method, it is necessary to take into account additional distortions that arise from information reduction, i.e., the replacement of the arms of the crosses by linear track elements.[11,56] In Ref. 54, a comparison is also made of the accuracy of the coordinates found for the centers of the crosses for different methods of preliminary analysis of the information (for example, with and without reduction of the data).

The most radical example of the use of robust polynomial fitting of the arms of crosses is the idea of using this method for the direct determination of the polar coordinates of the centers of the crosses in the case of spiral scanning, avoiding thereby the laborious operations of transporting all points to the local system of Cartesian coordinates and vice versa.

Calculation of the errors of the coordinates of the centers

When the approximating equations for the arms of the cross are solved simultaneously, the coordinates of the points of intersection of the arms, i.e., the center (u_c, v_c), are determined as functions of the coefficients of the approximating polynomials. For each of these polynomials, the covariance matrices of the coefficients are calculated in the course of the standard least-squares procedure, this making it possible to calculate the errors of the coordinates u_c, v_c as well. For this, it is necessary to substitute the values found for u_c, v_c in the functional (41) and calculate their errors, forming the Jacobian of the equations at the point of the solution.

As an example of a simpler way, we consider the case $l_1 = l_2 = 1$. In accordance with (18), we obtain estimates of the coefficients $\hat{a}_j, \hat{b}_j, j = 1, 2$, which determine the two straight lines $v = \hat{a}_j u + \hat{b}_j$, and we find the coordinates of their point of intersection:

$$\hat{u}_c = (\hat{b}_2 - \hat{b}_1)/(\hat{a}_1 - \hat{a}_2); \quad \hat{v}_c = (\hat{a}_1\hat{b}_2 - \hat{a}_2\hat{b}_1)/(\hat{a}_1 - \hat{a}_2). \tag{43}$$

Following Ref. 57, we expand $\hat{u}_c$ and $\hat{v}_c$ in Taylor series with respect to the unknown mean values of the parameters, i.e., $a_j, b_j, j = 1, 2$. Restricting ourselves to only the first term of the expansion and averaging the squares $(\hat{u}_c - u_c)^2$ and $(\hat{v}_c - v_c)^2$, we obtain approximate expressions for the variances:

$$\sigma^2_{\hat{u}_c} \approx [(\sigma^2_{\hat{a}_1} + \sigma^2_{\hat{a}_2})\,\hat{u}_c^2 + 2(\rho_{\hat{a}_1\hat{b}_1}\hat{} + \rho_{\hat{a}_2\hat{b}_2}\hat{})\frac{\hat{u}_c}{(\hat{a}_1 - \hat{a}_2)} + (\sigma^2_{\hat{b}_1} + \sigma^2_{\hat{b}_2})]/(\hat{a}_1 - \hat{a}_2)^2;$$

$$\sigma^2_{\hat{v}_c} \approx [(\hat{a}_2^2\sigma^2_{\hat{a}_1} + \hat{a}_1^2\sigma^2_{\hat{a}_2})\,\hat{u}_c^2 + 2(\hat{a}_2^2\rho_{\hat{a}_1\hat{b}_1}\hat{} + \hat{a}_1^2\rho_{\hat{a}_2\hat{b}_2}\hat{})\frac{\hat{u}_c}{(\hat{a}_1 - \hat{a}_2)} + (\hat{a}_2^2\sigma^2_{\hat{b}_1} + \hat{a}_1^2\sigma^2_{\hat{b}_2})]/(\hat{a}_1 - \hat{a}_2)^2,$$

where $\sigma^2_{\hat{a}_j}, \sigma^2_{\hat{b}_j}, \rho_{\hat{a}_j\hat{b}_j}; j = 1, 2$, are calculated in accordance with the expressions (19).

As already noted, in the case of spiral scanning when the data of each cross are transferred to the local coordinate system in order to obtain there the coordinates of the center (43) there is a rotation through an angle $\theta_k = \tan^{-1}(v_k/u_k)$, which leads to the appearance in the error matrix of nonvanishing correlation terms of the form $\sigma^2(\lambda^2 - 1)\cos\theta_k \sin\theta_k$.[21] This value is already, for example, when $\theta_k \sim \pi/4$, comparable with the value of the diagonal terms of the error matrix and must be taken into account in estimates of the calibration parameters.

4. CONSTRUCTION OF CALIBRATION TRANSFORMATIONS

Proceeding to the second stage of the calibration, we have at our disposal sets of pairs of ideal, $\{x_i, y_i\}_{i=1}^{N}$, and measured, $\{u_{jk}, v_{jk}\}_{j=1\,k=1}^{N'_k\,L}$, coordinates, where $N'_k \leqslant N$ (for whatever reason, certain reference marks could be omitted in the k-th measurement), and L is the number of dimensions. Depending on the method used to obtain the set $\{u_{jk}, v_{jk}\}$ in a semiautomatic or automatic regime of scanning of the grid, the method may be either complete, as indicated above, or may provide values averaged over the measurements: $\{u_j, v_j\}_{i=1}^{N'}$. Generally speaking, the averaging may take place in parallel with the construction of the calibration transformations and need not precede it. In principle, averaging is admissible only for homogeneous data. If the condition of homogeneity is not satisfied (for example, if there is a shift of the origin or of the direction of the polar axis in a spiral reader), it is necessary to average after the inhomogeneities have been taken into account or compensated for. In all cases, an important aspect of the data is also the accuracy of the measurement, $\{\Delta u_{jk}, \Delta v_{jk}\}$ or $\{\Delta u_j, \Delta v_j\}$. In what follows, we shall use a notation with a single index, understanding both the situations.

Direct and inverse transformations

It is usual to call the operation $\{u_j, v_i\} \overset{F}{\to} \{x_i, y_j\}$, carrying the measured coordinates into the ideal coordinates, the direct calibration transformation and the operation $\{x_i, y_i\} \overset{F^{-1}}{\to} \{u_i, v_i\}$ the inverse transformation. Although

these expressions have become the standard jargon, they should not be ascribed mathematical significance, from the point of view of which each of the problems of finding the explicit form of both the direct and the inverse calibration transformations is a typical inverse problem with all the features associated with such problems, i.e., they may be improperly posed, require the use of *a priori* information, etc.[58]

The explicit forms of the operators F and F^{-1} are found approximately by using the linear approximation model (24). Four types of basis $\{\varphi_i\}$ have been used to do this[14,59]:

$\{\varphi_i^{(1)}\}$, the natural basis $1, x, y, x^2, xy, y^2, \ldots$ (see Sec. 2);

$\{\varphi_i^{(2)}\}$, the basis orthogonalized in accordance with Ref. 45 on the unit square $x, y \in [-1, 1]$ with a scalar product of the form (29);

$\{\varphi_i^{(3)}\}$, the basis, orthogonal on a discrete fixed set, obtained from a set of ideal coordinates by the method of (26);

$\{\varphi_i^{(4)}\}$, the basis orthonormalized on an arbitrary discrete set by the Forsythe–Weisfeld recursive method with arbitrary positive weights.

We have included $\{\varphi_i^{(1)}\}$ because of its perspicuity and for logical completeness, though its practical significance is restricted to cases in which there are no distortions. In $\{\varphi_i^{(2)}\}$ and $\{\varphi_i^{(3)}\}$, the condition is much better; these bases can be used, respectively, for the direct and inverse transformations. The basis $\{\varphi_i^{(4)}\}$ is the most universal one and can be used in both cases with optimal (unit) condition. A comparison of the sets $\{\varphi_i^{(1)}\}$–$\{\varphi_i^{(4)}\}$ can be found in Ref. 60.

In the calculation of the explicit form of F one usually employs the weights

$$w_i = 1/(\Delta u_i^2 + \Delta v_i^2),$$

the operator itself being regarded as a two-component vector (F_x, F_y). Both components are expanded with respect to the same basis but with different coefficients; see (8). Since the indicated weights express the degree of correspondence between the i-th measured pair and the ideal pair, they should also be used for the numerical construction of the basis of the inverse transformation F^{-1}. Of course, F being nonlinear with respect to the coordinates, and also approximate, direct inversion, $F \to F^{-1}$ is impossible.

The question of the optimal length J of the approximating series (24) is settled by means of an appropriate criterion (or combination of criteria; see Sec. 2) chosen with allowance for the specific features of the calibrated device. For example, $J = 3$ for the scanning and measuring table CAMET, but $J = 21$ for maximal degree 5 in the case of the automatic scanners ERASME and AELT-2/160. The corresponding investigation was made in Ref. 14.

The case of spiral readers

Since spiral readers employ two coordinate systems, the Cartesian $(X, Y)_{sr}$ and the polar $(R, \theta)_{sr}$, the calibration problem for them is more complicated. Besides the direct and inverse transformations $(X, Y)_{sr} \leftrightarrow (X, Y)_{cg}$ between the Cartesian coordinates of the spiral reader and the calibration grid, it is also necessary to establish the transformations $(X, Y)_{cg} \leftrightarrow (R, \theta)_{sr}$ and $(X, Y)_{sr} \leftrightarrow (R, \theta)_{sr}$.

Because the last transformation can be obtained by successive use of the first two, it need not be determined.

An investigation[61] of the accuracy of the system $(X, Y)_{sr}$ showed that one can restrict oneself to measuring only five crosses situated at the intersection and at the ends of the $(X, Y)_{cg}$ coordinate axes in order to obtain the six coefficients of the affine transformation (2), $(X, Y)_{sr} \leftrightarrow (X, Y)_{cg}$, and calculate the parameters (4) and (5). The skewness of the spiral reader system at the Joint Institute for Nuclear Research was found to be nonzero but small: $|\varkappa| \lesssim 0.0003$.[21]

Analysis of the $(R, \theta)_{sr}$ data revealed the existence of significant distortions, these being particularly clearly seen in the scanning data for a reference straight line, the distortions giving rise to an S-shaped curve with deflections up to 20μm. As was established in Ref. 61, most of these nonlinear distortions can be compensated by a simple linear correction of the form

$$\theta = \theta_0 + \alpha R. \tag{44}$$

However, the remaining distortions were significant (in particular, they contained a spurious curvature that introduced a systematic shift in the values of the particle momenta). Investigation of the form of the remaining distortions revealed[21] the presence of a discontinuity at the pole of the system and characteristic deflections in the region from -10 to $+10$ mm, i.e., precisely where the calibration grid [see Fig. 1(d)] does not contain sufficient information for the interpretation of these distortions. Therefore, the results obtained by using only data from the calibration grid could not, even with the most refined analysis (including the use of the polynomial systems $\{\varphi_i^{(3)}\}$), ensure elimination of the distortions. In this connection, it was proposed in Ref. 21 to carry out a simultaneous analysis of the scanning data for both the calibration grid and the reference straight line. Instead of a straight line, a more suitable object would be one in the form of a multiprong star of six–eight intersecting segments of straight lines. However, the isotropy of the dependence of (44) on α established in Ref. 61 made it possible to restrict the scanning to just one straight line.

The analysis of these data also represents the start of the determination of the nonlinear distortions through calculation of the coefficients of an angular correction of the form

$$\theta_{corr} = \theta_0 + \alpha R + \beta/R + \gamma f(R), \tag{45}$$

where $f(R)$ depends on the form of the residual distortions. In particular, for the JINR spiral reader $f(R) = R/[1 + c(R - R_M)^2]$.

This attempt at a more complicated correction by means of orthogonal polynomials[61] was in fact unsuccessful, since it required the use of polynomials of degree 13 (on the basis of the criterion 1; see Sec. 2), the coefficients of which varied strongly in the case of small random deviations of the data.

After the determination of the parameters (45), all the scanning data of the calibration grid were corrected by means of β and γ; this made it possible to take into account

TABLE II. Coefficients of calibration transformations.

j	a_j	φ_j	ψ_j
1	x_0	1	0
2	y_0	0	1
3	$M_R \cos\theta_0$	$R\cos\theta$	$R\sin\theta$
4	$M_R \sin\theta_0$	$R\sin\theta$	$-R\cos\theta$
5	$M_R R_0 \cos\theta_0$	$-\cos\theta$	$-\sin\theta$
6	$M_R R_0 \sin\theta_0$	$-\sin\theta$	$\cos\theta$
7	$\alpha M_R \cos\theta_0$	$R^2\sin\theta$	$R^2\cos\theta$
8	$\alpha M_R \sin\theta_0$	$-R^2\cos\theta$	$-R^2\sin\theta$

the residual distortions in the required transformation $(R, \theta)_{sr} \to (X, Y)_{cg}$ by means of a simple six-parameter expression:

$$\left.\begin{aligned}\hat{x}_{cg} &= x_0 + M_R (R - R_0)\cos(\theta - \theta_0 - \alpha R);\\ \hat{y}_{cg} &= y_0 + M_R (R - R_0)\sin(\theta - \theta_0 - \alpha R).\end{aligned}\right\} \quad (46)$$

The nonlinearity of (46) with respect to the parameters made it necessary, for minimization of the corresponding functional, to use the program FUMILI,[62] which implements a method of functional linearization but without allowance for the correlation of the original data, to which attention was drawn at the end of Sec. 3.

Instead of this, the dependence (46) itself can be linearized. We expand the expressions for the cos and sin of the triple angles and, using the fact that α and R_0 are small, make the substitutions $\cos\alpha R \sim 1$ and $\sin\alpha R \sim \alpha R$, omitting the terms with the product αR_0. We obtain the expressions

$$\hat{x}_{cg} = \sum_{j=1}^{8} a_j \varphi_j(R, \theta); \quad \hat{y}_{cg} = \sum_{j=1}^{8} a_j \psi_j(R, \theta), \quad (47)$$

where the connection between the new and the old parameters, and also the form of the functions φ_j and ψ_j are given in Table II.

The parameters of the linear dependence (47) and their errors are found by minimizing the functional

$$\chi^2 = \sum_{i=1}^{N} [w_{xi}(x_i - \hat{x}_i)^2 + 2w_{xyi}(x_i - \hat{x}_i)(y_i - \hat{y}_i) + w_{yi}(y_i - \hat{y}_i)^2], \quad (48)$$

in which (x_i, y_i) are the ideal coordinates of the center of the i-th cross of the calibration grid, $\hat{x}_i$ and $\hat{y}_i$ are obtained by substituting in (47) the measured values R_i and θ_i of these centers, and w_{xi}, w_{yi}, and w_{xyi} are obtained by inverting the complete error matrix of the measured centers.

The quantity (48) is minimized by means of any standard program of the weighted least-squares method.

The old parameters, containing information about the reference systems of the spiral reader, can be determined from the resulting estimates by means of simple expressions:

$$x_0 = \hat{a}_1;\ y_0 = \hat{a}_2;\ M_R = \sqrt{\hat{a}_3^2 + \hat{a}_4^2};\ R_0 = \sqrt{\hat{a}_5^2 + \hat{a}_6^2}/M_R;$$
$$\theta_0 = \operatorname{arctg}(\hat{a}_4/\hat{a}_3);\quad \alpha = \sqrt{\hat{a}_7^2 + \hat{a}_8^2}/M_R.$$

Apart from this, the recovery of the "old" parameters is also needed to solve the problem of the inverse calibration transformation, since (46), despite its nonlinearity, admits simple inversion:

$$R = R_0 + 1/M_R \sqrt{(x_{cg} - x_0)^2 + (y_{cg} - y_0)^2},$$
$$\theta = \theta_0 + \alpha R + \operatorname{arctg}[(y_{cg} - y_0)/(x_{cg} - x_0)].$$

Experience demonstrated[21] the importance of repeated use of the scanning data for the straight line. Application to them of the transformation (46) or (47) with coefficients calculated using the scanning data for the calibration grid is a very sensitive criterion for the correctness of the parametrization, the choice of the initial approximations in the minimization, and the correctness of the calculations.

Investigation and use of the chart of the residuals

The most complete, i.e., effective in the complete field of measurement, indicator of the quality of the calibration transformation is the chart of the residuals, i.e., the quantities $dx_i = x_i - \hat{x}_i$, $dy_i = y_i - \hat{y}_i$, $i = 1,...,N$ (Fig. 2). In this connection, we recommend careful study of the residual chart by its visualization, by the construction of histograms of the actual quantities dx_i, dy_i, and also the moduli $d_i = \sqrt{dx_i^2 + dy_i^2}$, their moments $\bar{d}$ and σ_d, and $\max_i d_i$ In the case of a large number of crosses, one uses decision functions, which give diagnostic information when any of the quantities d_i, $\bar{d}$, σ_d, and $\max_i d_i$ exceed given thresholds. The investigation into the choice of the best system of polynomials and the optimal degree of them made in Ref. 14 was based on study of the histograms d_i, and also $\max_i d_i$, $\bar{d}$, and σ_d calculated both at the points of the calibration grid as well as at the central points between them. As this investigation showed, the appearance of significant residuals at the interstices of the calibration grid may indicate that it is too coarse for describing the structure of the distortions. Increasing the degree of the approximating two-dimensional polynomial in such cases does not improve and may actually worsen the situation.

The way out of the difficulty in such situations is to use the information contained in the residual chart itself. If the stability of the device has been confirmed, making it possible to average the residual charts over several calibration cycles in order to eliminate random outliers, then the corrections dx and dy obtained from such an averaged residual chart by

two-dimensional interpolation can compensate the presence of even appreciable (up to 10–15 μm) systematic deviations. A possible method of such interpolation is proposed in Ref. 21.

For a calibration grid of $m \times n$ crosses separated by distance h_x along the horizontal and h_y along the vertical, the residual chart must be transformed into a two-dimensional array dx_{ij}, dy_{ij}, with $i = 1,...,m$ and $j = 1,...,n$. The corrections dx and dy at an arbitrary point (x, y) are calculated using the following algorithm: (1) the indices I and J of the bottom left-hand point of the cell containing the point (x, y) are calculated:

$$I = [(m + 1)/2 - x/h_x],\ J = [(n + 1)/2 - y/h_y]$$

($[\cdot]$ is the symbol for the integral part); (2) then follows a calculation of the quantities

$$X = |x - h_x((m + 1)/2 - I)|,$$
$$Y = |y - h_y((n + 1)/2 - J)|$$

and of the weight functions at the four grid points surrounding the point:

$$\left.\begin{aligned} w_1 &= (h_x - X)(h_x - Y)/h_x h_y; & w_2 &= X(h_y - Y)/h_x h_y; \\ w_3 &= XY/h_x h_y; & w_4 &= (h_x - X)N/h_x h_y; \end{aligned}\right\} \quad (49)$$

(3) finally, there is a calculation of the corrections

$$dx = w_1 dx_{IJ} + w_2 dx_{I+1,J} + w_3 dx_{I+1,J+1} + w_4 dx_{I,J+1};$$
$$dy = w_1 dy_{IJ} + w_2 dy_{I+1,J} + w_3 dy_{I+1,J+1} + w_4 dy_{I,J+1}.$$

If the indices I and J exceed the ranges of their variation, 1,...,m and 1,...,n, respectively, then $dx = dy = 0$. The system of weight functions (49) is convenient in that on the edges of the cells it is only the weights of the two grid points at the ends of the edge that are nonzero, the values $w_1 = 1$, $w_2 = w_3 = w_4 = 0$ being automatically obtained when the point (x, y) coincides with the grid point (I, J).

By means of such a procedure of two-dimensional interpolation of higher order one can compensate the growth of the residual chart due to insufficient degree of the approximating polynomials. The choice of a sensible compromise between these two methods of compensating the distortions must be determined in each particular case, the factors taken into account being the speed of the methods and the determination of which of them performs the compensation better.

5. ACCURACY AND STABILITY

If the residual chart is to characterize reliably the accuracy of the calibration transformation, it is necessary to take into account the statistical contribution of each cross to the minimized functional, i.e., to multiply the residuals by the values of their weights. Significant deviations of the weighted residuals indicate an incorrect parametrization, this meaning in the case of polynomial approximation that the system of polynomials is not suitable.

Thus, we arrive at Eq. (36), i.e., criterion 4 of Sec. 2. No less important is criterion 2, which gives a characterization of the accuracy of the measurements that is common to all the crosses, i.e., (35). In addition, $\min \chi^2_{N-J}$ estimates the variance σ^2 per unit value of the weight.

Since σ^2 determines the error matrix and its inverse, the matrix of weights W, which is used to calculate and minimize the functional (14) before the estimate (35) is known, one either specifies σ in advance at the level of the mean accuracy of the measurements (generally between 3 and 7 μm) or uses a certain "good" calibration to choose an estimate of σ that ensures $\chi^2_{N-J} \sim 1$. (In this way, criterion 4 of Sec. 2 is actually satisfied.)

We have already noted the necessity of making several calibration cycles in order to verify the stability of the calibration parameters and, if verified, to average them. As a simplest criterion of stability one can take the absence of drift (systematic shift) of the parameters in successive calibrations.

A more accurate criterion is based on the ratio of the q_k^2 and s_k^2 statistics[63] calculated using the data of the measurements of the coordinates of the k-th cross $\{u_{ki}, v_{ki}\}_{k=1}^{N}{}_{i=1}^{L}$. For brevity, we restrict ourselves to one of the coordinates $\{u_{ki}\}$. Then

$$q_k^2 = \frac{1}{2(L-1)} \sum_{i=1}^{L} (u_{k,i+1} - u_{ki})^2;$$
$$s_k^2 = \frac{1}{L-1} \sum_{i=1}^{L} (u_{ki} - \bar{u}_k)^2,$$

where $\bar{u}_k = (1/L) \sum_{i=1}^{L} u_{ki}$.

We determine the quotient $r_k = q_k^2/s_k^2$. If in the course of the measurements a systematic shift is present, one must expect $s_k^2 \gg q_k^2$. Using the tables of the quantiles r_{kp} of order p of the distribution of the random variable r_k (Ref. 63) (usually $p = 0.05$), we establish for $r_k < r_{kp}$, violation of the hypothesis of stability for the k-th cross.

Analogous statistical calculations must also be made for the coordinates $\{v_{ki}\}$ of all crosses. In the absence of cases when r_k goes beyond the boundary of the critical region for all crosses $k = 1,...,N$, it can be assumed that the results of the calibration measurements do not depend on the time.

The reasons for the loss of stability are mainly associated with refitting, seasonal variation of the temperature, and the temperature drift due to heating of the device after it has been switched on. This last factor, which sometimes exerts an influence over 6–10 h, seriously restricts the time of functioning of the device in work in which it is used intermittently.[64,65]

The investigations made in this connection (see, for example, Ref. 64) showed that it is necessary to identify calibration parameters determined by the drift of the measured coordinates and choose an interval of time (usually 1 h) after which the calibration measurements are made, linear interpolation being employed in the intervals to correct these important parameters.

6. PROBLEMS OF PROGRAM IMPLEMENTATION

In recent years, several descriptions of calibration programs for automated measuring devices and systems of film-

less readout have been published. We mention, for example, those in Refs. 4, 14, 15, 21, 22, and 66–69. Except for simple programs designed for semiautomatic analysis of measuring tables[22] and realized by a control minicomputer, the overwhelming majority of the calibration programs use large off-line computers. Although the various applications differ, we shall attempt to identify some general features characteristic of such programs.

Structure of the data

The initial data are the results of scanning of the calibration plate (between two and four thousand points), the ideal coordinates of the grid, and, in the case of two-coordinate systems of the spiral-reader type, manual measurements in the second coordinate system. The initial data are recorded on magnetic tape in the form of files, consisting of records of a standard length and equipped with passports containing information about the device, the operator who made the measurement, the date and time of measurement, etc.

The initial data usually require recoding from the format of the controlling minicomputer into the format of the large computer used for the analysis. If economy of the memory is necessary and the recoding is done rapidly, the original set of data is kept in the memory without recoding, which is done when it is necessary to extract a particular measurement using its pointer index in the set of data. In many programs one introduces a special internal floating-point format, into which the initial data are translated with a normalization to ensure that they are close to standard units of measurement (millimeters, radians, etc.).

The importance of control of the initial data must be mentioned. Systematic control in the input, recoding, etc., programs makes it possible to eliminate wastage of time on the analysis of false information, to avoid distortion of the output parameters, and to establish operatively the defect of the measuring system. There are various forms of such control: (a) geometrical control (based on monotonicity of one of the coordinates or on the given distances between the crosses); (b) bit-by-bit control; (c) control on the basis of the number of points found in a given region (in one cross, there must be not less than 20 points, on one arm not less than 10), etc.

The output data are the coefficients of the direct and inverse transformations (for example, for polynomials of fifth degree we obtain 42 numbers), the chart of the corrections (up to 600 numbers), and the values of the normalized χ^2_{N-J} and χ^2 for the individual crosses (a further 300 numbers).

Verification of the output data is also very important. In Secs. 4 and 5, we have noted the need to print out various histograms and the quantities $\bar{d}$, σ_d, and $\max_k d_k$ and to check that all these parameters lie within definite ranges. Combinations of the calibration parameters, for example, the scales (4) along the axes, the skewness parameter (6), etc., are also verified.

Architecture of the calibration program

This architecture is due to the stages of the analysis: obtaining the centers, obtaining the transformation coefficients, calculation of the accuracy characteristics, and the control analysis of an additional reference image.

All these stages are repeated several (10–15) times in order to eliminate random outliers and to estimate the error in the correction chart. The calibration program also includes numerous verification subprograms and service pro-

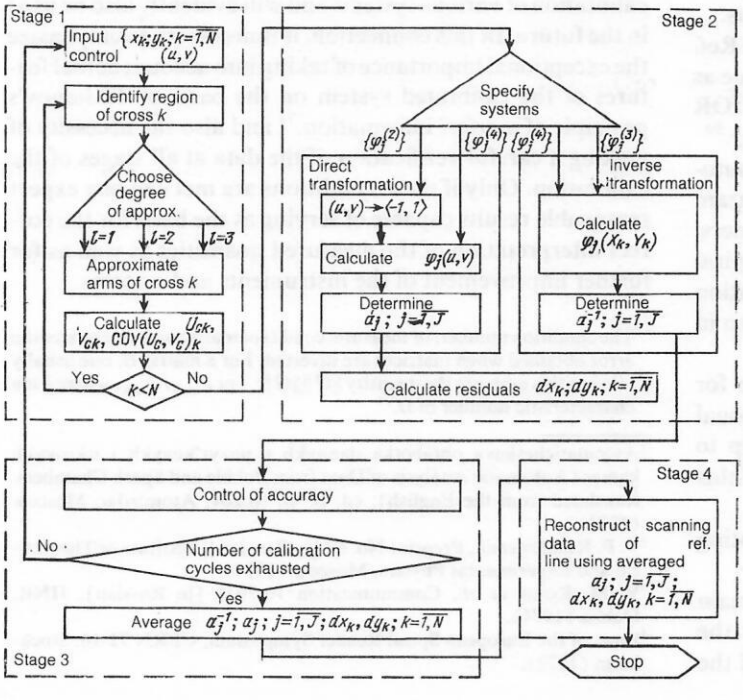

FIG. 10. Block diagram of calibration program.

grams for printing, histogramming, etc.

As an example, we give the block diagram shown in Fig. 10.

Examples of standard calibration programs

We give below information on some standard calibration programs developed with the participation of the authors and employed at the Laboratory of Computational Technology and Automation at the JINR. These programs realize the algorithms described in the previous sections for implementing all the stages of the calibration and are designed for devices that execute parallel scanning over the complete field of a grid of the type of those shown in Figs. 1(a) and 1(b).

The first stage is implemented by means of the FORTRAN program NIGEN,[15] which gives the set of centers of the crosses and their errors. In the second stage, this set is used to obtain the coefficients of the direct and inverse transformations. In one of the first standard calibration programs, this was done in the programs GENCAL (direct) and ORCAL (inverse transformation),[14] which use the systems of polynomials $\{\varphi_i^{(1)}\}$–$\{\varphi_i^{(3)}\}$. Later, these programs were improved and combined into the program CALORT, which uses the complete set of systems of polynomials $\{\varphi_i^{(1)}\}$–$\{\varphi_i^{(4)}\}$.[59]

In the meanwhile, the program NIGEN has been augmented by the program NORTCR, which realizes algorithms for finding the centers of the crosses on the basis of approximation by one-dimensional orthonormalized polynomials.[54]

The size of the operative memory of the computer and the computing time for the program NIGEN depend on the number of crosses and keys that activate the output printing. A typical calculation with 36 crosses and detailed printing requires 20 K words and 30 sec of computing with the CDC-6500 computer.

Similarly, for the programs GENCAL and ORCAL one requires 25 K words and 20 sec of CDC-6500 time to obtain the coefficients of the direct and inverse transformations.

A detailed description of these programs is given in Ref. 14. The program CALORT uses the same calling sequence as GENCAL, but the values of the parameters NPOL and IOR are augmented to call the system of polynomials $\{\varphi_i^{(4)}\}$.[59]

There exist two forms of programs for calibrating automatic devices of the spiral-reader type. The program SCALP[69] is designed for the SAAB family of spiral readers, which have constructive differences from the JINR spiral reader. For the calibration of this last type, the calibration program CALIBR, a detailed description of which is given in Ref. 21, has been developed.

The program REDUCE-2 in algorithmic language for symbolic operations for the generation of two-dimensional orthogonal polynomials of the system $\{\varphi_i^{(2)}\}$ with up to $J = 50$ coefficients, and also an example of the output of this program can be found in Refs. 41 and 45.

In conclusion, we can give some general figures relating to the average indices for calibration programs.

In the general body of the analysis programs, the main part (more than 50%) is made up of the programs for the geometrical reconstruction, the kinematic analysis, and the statistical programs, which operate using tapes of the end results. The calibration and filtering programs have approximately the same volume and make up the remaining 45%–50%.

The accuracy of measurement of the coordinates from the film depends on the construction of the automatic device and may be in the range 1.6–3 μm. Factors such as the accuracy of the spatial coordinates, "pulse" curvature, and the total speed of analysis depend on the remaining programs of analysis and on the title data determined by the experimental device.

CONCLUSIONS

The measuring systems that have undergone calibration have already passed through a long evolution based on the development of the methods of high-energy physics and also measuring and computational techniques. The construction and use of complex scanning systems, in particular systems with electronic and mechanical or purely electronic scanning have revealed—in addition to their high productivity—pronounced nonlinearity. The need to take into account nonlinearity under conditions of a relatively high noise level stimulated the development of a mathematical formalism that combines optimal condition of the systems of orthonormalized polynomials with robust methods of statistical rejection of noise points. This mathematical formalism is used in both stages of the calibration—in the determination of the coordinates of the reference crosses and the construction of the calibration transformations.[70] The algorithms and programs developed by means of this formalism are proof of its viability and practical utility. The introduction of rapid computational algorithms in the already existing programs has promising prospects.

At the Joint Institute for Nuclear Research the standard calibration program packages GENCAL, ORCAL, and CALORT have been developed. They have been used in the calibration of various systems and will evidently also be used in the future. In this connection, it is necessary to emphasize the exceptional importance of taking into account actual features of the calibrated system on the basis of Tikhonov's principle of *a priori* information,[71] and also the necessity of making a careful verification of the data at all stages of the calibration. Only if these conditions are met can one expect reasonable results capable of serving as the basis for the correct interpretation of the measured quantities as well as for further improvement of the instruments and systems.

[a] The condition number, or measure, cond G of matrices characterizes the error obtained when matrices are inverted. For a matrix G, one usually takes as this measure the quantity $\|G\|\|G\|^{-1}$ or $\lambda_{\max}/\lambda_{\min}$, where λ is a characteristic number of G.[31]

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