Analytic description of the penetration of fast charged particles in matter

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The results of the analytic theory of the penetration of a beam of fast heavy charged particles (muons, protons, etc.) through matter are presented. Allowance is made simultaneously for the systematic energy losses as well as the fluctuations in the losses due to the probabilistic nature of energy transfer in ionization collisions, resulting in energy straggling, and the deflection of the particle paths from straight lines as a result of multiple Coulomb scattering. A study is made in detail of the fluctuations of the energy losses and the particle ranges in thick layers of matter, for which the influence of elastic Coulomb scattering on the spread in the particle energies is most strongly manifested. The distribution of stopped particles with respect to the penetration depth in matter and the transverse displacement from the beam axis is analyzed. Expressions are also given for the mean ranges and the mean penetration depth in matter as functions of the direction of motion of the particles and their displacement relative to the beam axis.

INTRODUCTION

The number of scientific and practical problems in which it is necessary to describe the passage of fast charged particles through matter is extremely large—in the analysis of experiments associated with measuring the energy-loss distributions and ranges of charged particles,[1] in particular, the determination from experiments of the basic parameters of the interaction of particles with matter (for example, determination of the mean ionization potential[2–9]); investigation of the effect of ionizing radiations in radiation physics and radiobiology[10]; the design of charged-particle detectors[11–14]; the penetration of cosmic rays through the atmosphere and below the earth[15]; and the calculation of the efficiencies of targets in nuclear-physics experiments.[16]

Each of the listed problems is characterized by specific features associated with either the geometry of the experiment, the particular range of particle energies, or the nature of the investigated processes (particle spectra, energy release, etc.). However, all these problems possess general features too. In them, as a rule, one requires information about the energy, angular, and spatial distribution of beam particles passing through matter.

We shall consider fast heavy charged particles ($v \gg v_{at}$, $m \gg m_e$, where v is the velocity of the incident particles, v_{at} is the characteristic velocity of the atomic electrons, m is the particle mass, and m_e is the electron mass) at energies that are not superhigh: $T \ll (m/m_e) mc^2$ (T is the kinetic energy of a particle measured in MeV, and c is the velocity of light; besides the usual units, MeV, we shall below also use as units mc^2, $E = T/mc^2$). By "heavy" we mean all charged particles except electrons and positrons, i.e., protons, muons, pions, etc.

In the studied range of energies, the penetration of particles through matter is accompanied by energy losses due to ionization and by multiple Coulomb scattering through small angles.[15,17–19] In addition, there may also be decay of the original particles, the production of new particles, and nuclear interactions (elastic and inelastic) in the collisions of the particles with the nuclei of the atoms in the matter.[15,19] In the considered range of energies, the losses due to bremsstrahlung, pair production, etc., are insignificant.[15,19]

If the nuclear interaction is entirely ignored, the distribution of the particles is determined solely by the multiple Coulomb scattering and the ionization losses (Refs. 17–19).[1]

Because the multiple elastic scattering associated with the penetration of fast heavy particles in matter is strongly anisotropic, and the fluctuations of the energy losses in the inelastic collisions with atoms are small, the slowing down of the particles can be described in a first approximation in the "continuous-slowing-down" model[17] in which the motion of the particles is treated as rectilinear and only the systematic losses are taken into account. For more detailed information about the spatial, angular, and energy distribution of the particles it is necessary to take into account the fluctuations of the energy losses in the inelastic collisions and the multiple elastic scattering.

Figure 1 shows schematically the trajectory of a particle in matter.

The fluctuations in the energy of a particle at the point **r** are due to two factors. First, an uncertainty in the energy arises from the probabilistic nature of the energy loss in the inelastic collisions and exists even in the case of a fixed length S of the trajectory. Second, the particle can reach the point **r** by following different spatial trajectories, traversing paths of different lengths and, thus, losing different energies in the process. In this case, an energy spread occurs even in the continuous-slowing-down approximation and is due to the fluctuations in the lengths of the particle paths due to multiple elastic scattering. In general, the two processes are equally important and must be taken into account simultaneously.

However, despite the fact that the penetration of fast charged particles has been investigated for more than one decade, until recently there has been no consistent analytic theory that includes a description of the energy distribution of the particles with simultaneous allowance for the factors

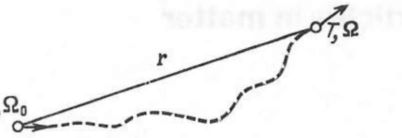

FIG. 1. Path of a particle in matter.

listed above.[6,17,21] The main difficulty in the analytic calculation of the energy spectrum was to take into account the influence of the multiple elastic scattering on the energy-loss distribution, i.e., to determine the spread of the particles with respect to the traversed paths.

At the beginning of the sixties numerical Monte Carlo calculations with various schemes of grouping of the collisions (Refs. 5–8 and 22–24) began to be employed in order to solve this problem. Despite its advantages, the Monte Carlo method is very laborious and requires much computing time. This is especially true of calculations with a satisfactory statistical accuracy of the differential distributions, for example, the energy spectrum or distribution of stopped particles. Therefore, whenever possible, preference was given to analytic results (Refs. 4, 6, 9, 24, and 25).

Hitherto, in the majority of studies on the penetration of heavy charged particles in matter (for example, Refs. 9, 24, and 25) the achievements of the analytic theory summarized in Ref. 26 were used. However, since the publication of the review of Ref. 26 there have been obtained various results[27–32] that greatly extend the possibilities of analytic description of the stopping of particles in matter. In particular, it became possible to solve analytically the problem of the influence of multiple elastic scattering on the energy–loss distribution of the particles.[30–32]

We present below an analytic theory of the penetration of a beam of fast heavy charged particles in matter with simultaneous allowance for the systematic slowing down, the fluctuations of the energy losses associated with inelastic collisions of the particles with atoms, and the bending of the trajectory due to multiple Coulomb scattering. We examine in particular the fluctuations of the energy losses and the ranges of the particles in thick layers of matter, when the influence of elastic scattering on the energy-loss distribution cannot be ignored.

1. SPATIAL, ANGULAR, AND ENERGY SPECTRUM OF A BEAM OF FAST CHARGED PARTICLES PENETRATING MATTER

To determine the spatial, angular, and energy distribution of the particles in the beam it is necessary to solve the Boltzmann kinetic equation (transport equation) for the flux density $N(\mathbf{r}, \mathbf{\Omega}, T)$ of particles at the point r moving in the direction $\mathbf{\Omega}$ and having energy T,[17]

$$\mathbf{\Omega}\frac{\partial}{\partial \mathbf{r}} N(\mathbf{r}, \mathbf{\Omega}, T) = \hat{I}_{\rm el} + \hat{I}_{\rm inel} \tag{1}$$

where $\hat{I}_{\rm el}$ and $\hat{I}_{\rm inel}$ are the elastic and inelastic collision integrals, respectively[17]:

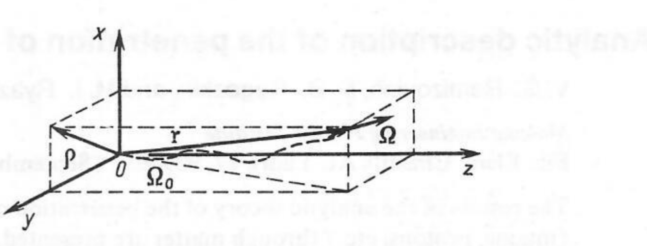

FIG. 2. Schematic representation of an infinitely narrow particle beam incident along the normal to the surface of matter.

$$\hat{I}_{\rm el} = - W_{\rm el}(T)\, N(\mathbf{r}, \mathbf{\Omega}, T) + \int_{4\pi} d\Omega'\, W_{\rm el}(T|\mathbf{\Omega}' \to \mathbf{\Omega})\, N(\mathbf{r}, \mathbf{\Omega}', T); \tag{2}$$

$$\hat{I}_{\rm inel} = - W_{\rm inel}(T)\, N(\mathbf{r}, \mathbf{\Omega}, T) + \int_{T}^{\infty} dT'\, W_{\rm inel}(T' \to T)\, N(\mathbf{r}, \mathbf{\Omega}, T'); \tag{3}$$

here, $W_{\rm el}$ and $W_{\rm inel}$ are, respectively, the probabilities of elastic and inelastic collisions per unit path length.

If an infinitely narrow monodirectional and monoenergetic particle beam is incident along the normal to the surface of the medium (Fig. 2) and reflection is ignored, then the boundary condition for Eq. (1) has the form[17]

$$N(z=0;\, \boldsymbol{\rho},\, \mathbf{\Omega},\, T) = N_0 \delta(\boldsymbol{\rho})\, \delta(\mathbf{\Omega} - \mathbf{\Omega}_0)\, \delta(T - T_0), \tag{4}$$

where N_0 is the flux of the incident particles, T_0 is their kinetic energy, $\mathbf{\Omega}_0$ is the unit vector along the velocity of the incident particles, the z axis is along $\mathbf{\Omega}_0$, the XY plane coincides with the surface of the medium, the origin is placed at the point of entry of the beam into the medium, and $\boldsymbol{\rho} = (x, y)$.

The solution of Eq. (1) that satisfies the boundary condition (4) determines the Green's function of the transport equation and makes it possible to calculate the spatial, angular, and energy distribution of particles incident on the surface of matter in a beam with arbitrary transverse dimensions, angular spectrum, and energy distribution. If reflection is to be ignored, the characteristic angles of incidence of the particles on the surface must not be near $\pi/2$ (the case of grazing incidence is ruled out).[33]

In general form it is not possible to solve Eq. (1) [with the definitions (2) and (3)] analytically without making simplifying assumptions. The possibility of solving this equation is based on the use of a number of physically justified assumptions that take into account the features of elastic and inelastic interactions of fast heavy charged particles with matter.

Energy distribution of particles in the case of ionization slowing down in matter without allowance for elastic scattering

If elastic scattering is completely ignored (the "straightforward" approximation[17]), the transport equation (1)–(4) takes the form

$$\frac{\partial N(z,T)}{\partial z} = \hat{I}_{\text{inel}}\ ;\ N(z=0;\ T) = N_0\delta(T-T_0). \qquad (5)$$

The calculation of the particle energy distribution in the approximation (5) is important not only as a preparatory stage to the solution of the transport equation with simultaneous allowance for elastic and inelastic collisions but is also of independent interest. The point is that $N(z, T)$ directly describes the energy spectrum of the fast charged particles in situations in which the bending of the particle trajectory due to elastic scattering is slight—in not too thick layers of matter $[z \ll R_0\,(m/m_e)^{1/3}Z^{-2/3}$, where R_0 is the total range of the particles,[17] and Z is the atomic number of the atoms of the matter$]$ and in light media ($Z \ll \sqrt{m/m_e}$) quite generally at all depths.

We review below the main results obtained by solving Eq. (5), establish the correspondence between them, and determine the region of applicability of each of them.

To transform the inelastic collision integral, we take into account the fact that in the case of ionization slowing down of fast heavy charged particles in matter the maximal energy $\varepsilon_{\max}(T)$ that can be transferred by a particle with energy T to an atomic electron in a collision process is small,

$$\varepsilon_{\max}(T) \simeq 2\left(\frac{m_e}{m}\right)\frac{T}{mc^2}(2mc^2+T) \ll T, \qquad (6)$$

and we represent $\hat{I}_{\text{inel}}$ in the form of a series in the energy transfer:

$$I_{\text{inel}} = \sum_{k=0}^{\infty}\frac{1}{k!}\int_0^{\varepsilon_{\max}(T)}\varepsilon^k\,d\varepsilon\,\frac{\partial^k}{\partial T^k}\{W_{\text{inel}}(T|\varepsilon)\,N(\mathbf{r},\Omega,T)\}, \qquad (7)$$

where $W_{\text{inel}}(T\varepsilon)$ is the probability that a particle with energy T loses energy ε per unit path length.

If in the expansion (7) we restrict ourselves to only the first term, then we obtain the inelastic collision integral in the continuous-slowing-down approximation.[17] In this approximation, we take into account only systematic energy losses, which are characterized by the stopping power $\bar{\varepsilon}$ of the medium:

$$\bar{\varepsilon}(T) = \int_0^{\varepsilon_{\max}(T)}\varepsilon\,d\varepsilon\ W_{\text{inel}}(T|\varepsilon). \qquad (8)$$

The value of $\bar{\varepsilon}$ (in units of mc^2) for fast charged particles is determined by the Bethe–Bloch formula (Refs. 15, 17, 19, and 34):

$$\bar{\varepsilon}(E) = 4\pi n_0 Zz^2 r_e^2\,\frac{m_e}{m}\,\frac{(1+E)^2}{E\,(2+E)}\,L_{\text{ion}}(E), \qquad (9)$$

where $r_e = e^2/m_e c^2$ is the classical electron radius, ze is the charge of the particle, n_0 is the number of atoms of the matter per unit volume, L_{ion} is the ionization logarithm,

$$L_{\text{ион}}(E) = \ln\frac{2E\,(2+E)\,m_e c^2}{I\,(Z)} - \frac{E\,(2+E)}{(1+E)^2} - \frac{1}{Z}\sum_{K,L\ldots}C_i - \delta, \qquad (10)$$

$I(Z)$ is the mean ionization potential of the atoms of the medium, C_i are corrections that take into account the coupling of the electrons in the $K, L,\ldots$ shells of the atoms, and δ is a correction for the density effect.

As a rule, it is not possible to calculate the value of the ionization potential from first principles, and $I(Z)$ is determined experimentally. The ionization potentials of different substances are given in Ref. 34. The corrections for the coupling of the electrons in the atom were calculated in Refs. 35 and 36; they are important in the nonrelativistic range of energies. The contribution of the density effect to the stopping power of the medium was studied in detail by Sternheimer *et al.*[37–40]

To determine $\bar{\varepsilon}$ in different energy ranges of the particles there exist detailed tables and approximation relations.[34] In the given range of energies one frequently employs for L_{ion} the approximation

$$L_{\text{ion}} \approx \ln\frac{2E\,(2+E)\,m_e c^2}{I_{\text{adj}}(Z)}, \qquad (11)$$

where $I_{\text{adj}}(Z)$ is the adjustable value of the ionization potential at which the best agreement with experiment is achieved.[41] The values of $I_{\text{adj}}(Z)$ for a number of substances and energy ranges are given in Ref. 41.

In the continuous-slowing-down approximation, the transport equation has the form[17]

$$\frac{\partial N}{\partial z} = \frac{\partial}{\partial T}\{\bar{\varepsilon}(T)\,N(z,\ T)\}. \qquad (12)$$

Its solution is determined by the expression

$$N(z,\ T) = \frac{N_0}{\bar{\varepsilon}(T)}\,\delta[R_0 - R(T) - z]. \qquad (13)$$

Here, $R(T)$ and R_0 are, respectively, the residual and total range of the particles in the matter:

$$R(T) = \int_0^T dT'/\bar{\varepsilon}(T');\ \ R_0 = R(T = T_0). \qquad (14)$$

The range–energy relation is one of the most important of the quantities measured and employed in practice to characterize the penetration of charged particles in matter.[1] For it, on the basis of calculations and generalization of experimental data, detailed tables have been compiled[19,34] and various approximate expressions have been proposed.[15,17,34] Assuming L_{ion} to be a smooth function of the energy, we can, to an accuracy of a few percent, readily obtain for the range–energy relation

$$R(E) = [4\pi n_0 Zz^2 r_e^2 L_{\text{ion}}\,(E_*)]^{-1}\,\frac{m}{m_e}\,\frac{E^2}{1+E}, \qquad (15)$$

where $E_* = [(1+E)/E][(2+E)/2 - (1/E)\ln(1+E)] - 1$.

It follows from (15) that for nonrelativistic particles $(E \ll 1)$

$$R(E) \simeq [4\pi n_0 Zz^2 r_e^2 L_{\text{ion}}]^{-1}\,\frac{m}{m_e}\,E^2, \qquad (16)$$

while for ultrarelativistic energies $(E \gg 1)$

$$R(E) \approx [4\pi n_0 Zz^2 r_e^2 L_{\text{ion}}]^{-1}\,\frac{m}{m_e}\,E. \qquad (17)$$

To include in the treatment the fluctuations of the energy losses due to the probabilistic nature of the process of inelastic collisions of the particles with the atoms of the matter, it is necessary to retain in the expansion at least one more term. This corresponds to the Fokker–Planck approximation or the diffusion approximation with respect to the energy.[17] In thick layers of matter, the inelastic collision integral, written down in the diffusion approximation, describes the penetration of fast charged particles with fairly high accuracy,[6,27] and in the majority of cases a restriction to precisely this approximation is made. However, in thin layers, it is necessary to take into account many terms[17] in the expansion (7) when calculating the energy spectrum, and the Fokker–Planck approximation is inadequate.

In Ref. 29, a method of approximate solution of the transport equation was developed, this making it possible to generalize the well-known results for thin[42–44] and thick [27,28,45–47] layers of matter. The method of Ref. 29 exploited the circumstance that in the overwhelming range of depths the energy spectrum of heavy particles has a well-defined (most probable) maximum at $T = T_{\rm m.p}(z)$,[2)] while $\varepsilon_{\max}(T)$ and $W_{\rm inel}(T|\varepsilon)$ are smooth functions of the energy, and approximately

$$\varepsilon_{\max}(T) \simeq \varepsilon_{\max}(T_{\rm m.p}(z)); \tag{18}$$

$$W_{\rm inel}(T \mid \varepsilon) \simeq W_{\rm inel}(T_{\rm m.p}(z) \mid \varepsilon). \tag{19}$$

The assumption (18)–(19) makes it possible to represent the inelastic collision integral (7) in the form

$$\hat{I}_{\rm inel} \simeq \sum_{k=1}^{\infty} \frac{1}{k!} \frac{\overline{\varepsilon^k(z)}}{[\bar{\varepsilon}(z)]^k} \frac{\partial^k}{\partial R^k} N, \tag{20}$$

where

$$\overline{\varepsilon^k(z)} = \int\limits_0^{\varepsilon_{\max}[T_{\rm m.p}(z)]} \varepsilon^k W_{\rm inel}(T_{\rm m.p}(z) \mid \varepsilon)\, d\varepsilon. \tag{21}$$

After this, an analytic solution of the transport equation (5) can be readily found:

$$N(z, T) = \frac{N_0}{\bar{\varepsilon}(T)} \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} d\omega \exp[-i\omega(R(T) - R_0 + z)]$$
$$\times \exp\left\{-\int\limits_{T_{\rm m.p}(z)}^{T_0} \frac{dT'}{\bar{\varepsilon}(T')} \int\limits_0^{\varepsilon_{\max}(T')} d\varepsilon\, W_{\rm inel}(T' \mid \varepsilon) \right.$$
$$\left. \times \left[1 - e^{-i\omega\frac{\varepsilon}{\bar{\varepsilon}(T')}} - i\omega\frac{\varepsilon}{\bar{\varepsilon}(T')}\right]\right\}. \tag{22}$$

The expression (22) describes the energy spectrum of the particles in matter without allowance for elastic scattering.

At small depths

$$z \lesssim \overline{\varepsilon^2}_{\max}/\overline{\varepsilon^2} \tag{23}$$

the width of the energy spectrum is comparable with the maximal energy lost by a particle in one collision. In this case, the integral (22) cannot be calculated analytically, and it is necessary to use numerical integration on a computer.

The mean-square energy (in units of mc^2) lost by the particle per unit path length that appears in the inequality (23) can be determined in the case of ionization slowing down in a medium by the relation (Refs. 6, 17, 19, 43, 44, and 48)

$$\overline{\varepsilon^2}(E) = 4\pi n_0 Z z^2 r_e^2 \left(\frac{m_e}{m}\right)^2 (1+E)^2 K(E). \tag{24}$$

The coefficient $K(E)$ in (24) takes into account the effects of binding of the electrons in an atom; if the electrons are regarded as free, $K(E) = 1$. In the region of nonrelativistic particle energies, the effects of the binding cannot, in general, be ignored, and an appropriate quantum-mechanical calculation[19,43,44,48] is needed to calculate $K(E)$.

Using the expressions (9), (15), and (24), we can rewrite the inequality (23) in the form

$$z \lesssim \left[4\, \frac{m_e}{m} \frac{(2+E_0)^2}{1+E_0} \frac{L_{\rm ion}}{K}\right] R_0. \tag{25}$$

It follows from this, in particular, that in the region of nonrelativistic energies ($E_0 \simeq 0.1$) for muons $z \lesssim 0.6R_0$ in Al and $z \lesssim 0.4R_0$ in Pb; for protons, $z \lesssim 0.07R_0$ in Al, and $z \lesssim 0.05R_0$ in Pb.

In the limit of small energy losses

$$z \ll R_0, \tag{26}$$

and setting in the expression (22)

$$R_0 - R(T) \simeq \frac{T_0 - T}{\bar{\varepsilon}(T_0)}, \tag{27}$$

we arrive at the well-known result of Vavilov[42]:

$$N(z, T) = \frac{N_0}{2\pi} \int\limits_{-\infty}^{\infty} d\omega$$
$$\times \exp\left\{i\omega(T_0 - T) - z \int\limits_0^{\varepsilon_{\max}(T_0)} d\varepsilon\, W_{\rm inel}(T_0 \mid \varepsilon)(1 - e^{-i\omega\varepsilon})\right\}. \tag{28}$$

The distribution (28) for different approximate values of $W_{\rm inel}(T_0|\varepsilon)$ was considered in Refs. 42–44. In the model of free electrons, the spectrum (28) was tabulated in Ref. 49.

In sufficiently thick layers of matter,

$$z \gg \varepsilon^2_{\max}/\overline{\varepsilon^2}; \tag{29}$$

the effective values of ω in (22) are small, and the exponential factor $\exp[-i\omega\varepsilon/\bar{\varepsilon}(T)]$ in the integrand can be expanded in a series with only the first few terms retained. This permits an analytic inverse Fourier transformation in (22) and a representation of the spectrum $N(z, T)$ in a form convenient for practical calculations.

If the exponential is expanded to the term linear in ω, we obtain the result (13) of the continuous-slowing-down model. This approximation gives a qualitatively correct characterization of the stopping of particles at large depths when the mean energy losses appreciably exceed the fluctuations of the losses.[27]

To take into account the fluctuations of the energy losses, it is necessary to retain at least one more term in the expansion of $\exp[-i\omega\varepsilon/\bar{\varepsilon}(T)]$ in a series in ω. We then obtain

$$N(z, T) = \frac{N_0}{\bar{\varepsilon}(T)\sqrt{2\pi\sigma^2(z)}} \exp\left\{-\frac{[R_0 - R(T) - z]^2}{2\sigma^2(z)}\right\}, \tag{30}$$

where

$$\sigma^2(z)=\int\limits_0^z dz'\,\frac{\overline{\varepsilon^2(z')}}{[\bar{\varepsilon}(z')]^2}\simeq\int\limits_{T_{\rm m.p}(z)}^{T_0} dT\,\frac{\overline{\varepsilon^2(T)}}{[\bar{\varepsilon}(T)]^3} \qquad (31)$$

is the variance of the range distribution. Using the expressions (9) and (24), we obtain for $\sigma^2(z)$

$$\sigma^2(z)=(4\pi n_0 Z z^2 r_e^2)^{-2}\int\limits_{E_{\rm m.p}(z)}^{E_0} dE\,\frac{[(1+E)^2-1]^3}{(1+E)^4 L^3_{\rm ion}(E)}K(E). \qquad (32)$$

If we ignore the dependence of the ionization logarithm on the energy and set $K(E)=1$, then from (32) we obtain[17]

$$\sigma^2(z)=R_0^2\sigma^2(\xi);$$
$$\sigma^2(\xi)=\begin{cases}[1-(1-\xi)^2]/2\nu, & E_0\ll 1,\\ [1-(1-\xi)^3]/3\nu, & E_0\gg 1,\end{cases} \qquad (33)$$

where the depth z is measured in units of the total range R_0 ($\xi=z/R_0$) and we have introduced the dimensionless parameter

$$\nu=\frac{[\bar{\varepsilon}(E_0)]^2}{\overline{\varepsilon^2}(E_0)}R_0=\frac{m}{m_e}\,\frac{1+E_0}{(2+E_0)^2}\,L_{\rm ion} \qquad (34)$$

which characterizes the fluctuations of the energy losses of the particles due to the probabilistic nature of the inelastic collisions [$\nu\sim R_0/\sqrt{\langle(\delta R_0)^2\rangle}$; $\sqrt{\langle(\delta R_0)^2\rangle}$ is the rms fluctuation of the range]. The larger ν, the smaller the fluctuations and the narrower the distribution function $N(z, T)$. The continuous-slowing-down approximation corresponds to the limit $\nu\to\infty$. For fast heavy particles, the parameter ν is large. For example, for protons with energy 10–100 MeV ($E_0\simeq 0.01$–0.1) the values of ν lie in the interval $2300\leqslant\nu\leqslant 3400$ for Al and $1400\leqslant\nu\leqslant 2500$ for Pb. For muons and pions, the corresponding values of ν are approximately an order of magnitude smaller.

The spectrum (30)–(31) corresponds to a "self-consistent" Gaussian approximation in the solution of the transport equation with an inelastic collision integral in the Fokker–Planck approximation.[17] The expression (30) was apparently proposed for the first time by Berger and Seltzer[6] as a generalization of the result of Bohr obtained for the distribution of stopped particles.[50] The energy spectrum (30) was found more rigorously, by solution of the transport equation, in Ref. 27.

To estimate the accuracy of the Fokker–Planck approximation and to make a more detailed calculation of the energy spectrum of the particles, it is necessary to retain one further term[29,33] in the expansion of the exponential in powers of ω. We then obtain[27,29,33]

$$N(z, T)=\frac{N_0}{\bar{\varepsilon}(T)}\left(\frac{2}{\zeta^3(z)}\right)^{1/3}\mathrm{Ai}\left[-\left(\frac{2}{\zeta^3(z)}\right)^{1/3}\right.$$
$$\left.\times\left(R_0-R(T)-z+\frac{1}{2}\,\frac{\sigma^4(z)}{\zeta^3(z)}\right)\right]$$
$$\times\exp\left\{\frac{\sigma^2(z)}{\zeta^3(z)}\left[R_0-R(T)-z+\frac{1}{3}\,\frac{\sigma^4(z)}{\zeta^3(z)}\right]\right\}, \qquad (35)$$

where

$$\zeta^3(z)=\int\limits_0^z dz'\,\frac{\overline{\varepsilon^3(z')}}{[\bar{\varepsilon}(z')]^3}\simeq\int\limits_{T_{\rm m.p}(z)}^{T_0} dT\,\frac{\overline{\varepsilon^3(T)}}{[\bar{\varepsilon}(T)]^4}, \qquad (36)$$

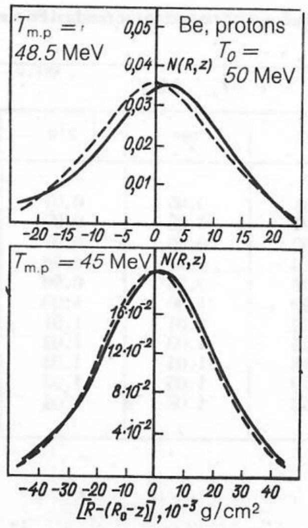

FIG. 3. Variation of the energy spectrum with increasing thickness of the matter layer.[27] The continuous curve represents the calculation on the basis of Eq. (35); the broken curve is for the Gaussian distribution (30).

and Ai(x) is the Airy function.[51]

The energy spectrum (35) was calculated for the first time by Payne,[27] who solved the transport equation (5) by the moment method.

The distribution (35) in the range variable differs from the Gaussian distribution, being somewhat asymmetric—the most probable range is not equal to the mean range (Fig. 3). However, with increasing depth z these differences rapidly decrease (Fig. 3).

In thick layers of matter ($z>\varepsilon^2_{\max}/\overline{\varepsilon^2}$), the corrections to the distribution (30) are small (Table I) and in practical calculations are, as a rule, ignored.[27]

We now turn to the Fokker–Planck approximation for the inelastic collision integral and consider in somewhat more detail the results obtained in this approximation. As quite a number have been obtained (Refs. 27, 28, 45–47, and 52), we give the correspondence between these results and the region of applicability of each of them.

1. Bohr's distribution[52]

$$N(z, T)=\frac{N_0}{\sqrt{2\pi\sigma_B^2(z)}}$$
$$\times\exp\left\{-\frac{[T-T_{\rm m.p}(z)]^2}{2\sigma_B^2(z)}\right\}, \qquad (37)$$

where

$$\sigma_B^2(z)=\int\limits_0^z dz'\,\overline{\varepsilon^2}(z')\simeq\int\limits_{T_{\rm m.p}(z)}^{T_0} dT\,\frac{\overline{\varepsilon^2}(T)}{\bar{\varepsilon}(T)}\quad, \qquad (38)$$

is physically justified in two cases: first, for particles of ultrarelativistic energies, when $\bar{\varepsilon}(T)\simeq$const; second, when the inequalities (26) and (29) are satisfied simultaneously, i.e., when, on the one hand, the energy losses are small and, on the other, the width of the spectrum exceeds the maximal energy lost by a particle in one collision.

2. Symon's result[45]

$$N(z, T)=\frac{N_0}{\sqrt{2\pi\sigma_S^2(z)}}\exp\left\{-\frac{[T-T_{\rm m.p}(z)]^2}{2\sigma_S^2(z)}\right\}, \qquad (39)$$

TABLE I. Corrections to the Gaussian distribution with respect to the residual Pb ranges[6] for protons with $T_0 = 340$ MeV.

$\frac{R-(R_0-z)}{\sqrt{\sigma^2(z)}}$	$T_{m.p}$, MeV				
	338	332	308	276	1
−1	0.83	0.91	0.95	0.97	0.98
−0.8	0.83	0.91	0.95	0.97	0.98
−0.6	0.84	0.92	0.96	0.97	0.98
−0.4	0.86	0.94	0.97	0.98	0.99
−0.2	0.90	0.96	0.98	0.99	0.99
0	0.95	0.99	1.00	1.00	1.00
0.2	1.01	1.02	1.01	1.01	1.01
0.4	1.09	1.05	1.03	1.02	1.01
0.6	1.17	1.08	1.04	1.03	1.02
0.8	1.25	1.10	1.05	1.03	1.02
1.0	1.31	1.13	1.06	1.04	1.02

where

$$\sigma_S^2(z) = [\bar{\varepsilon}(z)]^2 \int_0^z dz' \frac{\overline{\varepsilon^2}(z')}{[\bar{\varepsilon}(z')]^2} \simeq [\bar{\varepsilon}T_{m.p}(z)]^2 \int_{T_{m.p}(z)}^{T_0} dT \frac{\overline{\varepsilon^2}(T)}{[\bar{\varepsilon}(T)]^3} \quad (40)$$

follows directly from the relation (30) if T does not differ too strongly from $T_{m.p}(z)$ and one can set

$$R_0 - R(T) - z \simeq \frac{1}{\bar{\varepsilon}[T_{m.p}(z)]}[T - T_{m.p}(z)]. \quad (41)$$

3. In Refs. 46 and 47, Eq. (5) with a collision integral in the Fokker–Planck approximation was solved by direct numerical integration. As comparison shows,[53] the data of the numerical calculation hardly differ from the results of calculations in accordance with (30) and thus establish the validity of the procedure (18)–(19) (at least for the solution of the transport equation in the diffusion approximation).

4. In the region of nonrelativistic energies $\overline{\varepsilon^2} \simeq$ const, and if, in addition, we ignore the energy dependence of the ionization logarithm, the transport equation, written down in the Fokker–Planck approximation, can be solved analytically and exactly[28]:

$$N(z, T) = N_0 \frac{4\nu}{T_0} \frac{R_0}{z} \left(\frac{T}{T_0}\right)^{-\left(2\nu - \frac{1}{2}\right)} I_{2\nu+\frac{1}{2}}\left(4\nu \frac{R_0}{z} \frac{T}{T_0}\right) \times \exp\left[-2\nu \frac{R_0}{z}\left(1 + \frac{T^2}{T_0^2}\right)\right]. \quad (42)$$

Here, $I_{2\nu+1/2}$ is a modified Bessel function[51]; the dimensionless parameter ν is determined by Eq. (34).[c)]

At depths that do not yet exceed the stopping of particles, $[0 \leqslant z \lesssim R_0(1 - 1/\sqrt{\nu})]$, the distribution (42) agrees very well with the result (39) [if, of course, in (39) we set $\overline{\varepsilon^2}$ = const and $L_{\rm ion} \simeq$ const]. Compared with (39), the relation (42) has the advantage of making it possible to follow the variation of the energy spectrum as well in the region of depths at which the stopping of the particles occurs, where (39) is no longer valid.

In light media ($Z < \sqrt{m/m_e}$), the influence of elastic scattering on the energy spread of the particles is small, and the results obtained above are ready for direct comparison with experiment.

In Table II we give the data of an experiment[55] and of a calculation in accordance with Eq. (39) of the width of the energy spectrum of protons with initial energy 5.3 MeV in air.[27] The small differences between the experimental and theoretical results are within the limits of the accuracy of the relation (39).

Figure 4 shows the results of the experiment of Ref. 53 and of the calculation of the energy spectrum on the basis of the relations (30), (37), and (39), respectively. As follows from the figure, in the complete range of energy losses investigated in Ref. 53 the experimental data are excellently described by the distribution (30). At small $T - T_{m.p}(z)$, the expression (39) also gives good agreement with the experiment.

TABLE II. Widths of the energy spectrum of α particles ($T_0 = 5.3$ MeV) in air at different depths in accordance with the experimental data on Ref. 55 and the calculation[27] in accordance with Eq. 39.

z, cm	$\delta T_{\rm exp}$, keV	$\delta T_{\rm theor} = \sqrt{2\sigma_S^2}$, keV
0.833	32	31.0
1.388	43	41.4
1.943	54	51.9
2.554	70	68.6
2.998	92	89.0
3.276	109	113.0
3.526	122	127

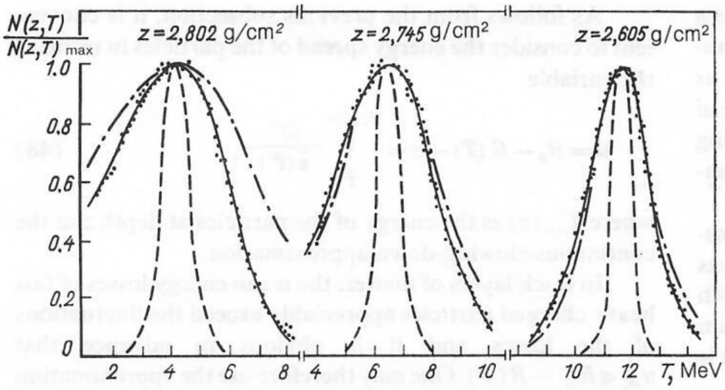

FIG. 4. Energy spectrum of protons with initial energy $T_0 = 49.1$ MeV in Al in accordance with experimental data and calculations in accordance with the expressions (30) (continuous curve), (37) (broken curve), and (39) (chain curve).[53]

Energy spectrum for penetration of a particle beam through thick layers of matter with simultaneous allowance for the probabilistic nature of inelastic collisions and the bending of the trajectory due to elastic scattering

As was pointed out in the Introduction, it is in general necessary, when determining the energy spread of the particles, to take into account not only the fluctuations of the energy losses in inelastic collisions but also the bending of the trajectory due to multiple Coulomb scattering.

The influence of elastic scattering on the fluctuations of the energy losses was considered for the first time by Pomeranchuk.[56] He determined the mean path and the variance of the path of particles in a layer of matter of given thickness. For the mean path he obtained, for example, in Ref. 56 the relation

$$\langle S\rangle_z = z + \frac{1}{2}\int_0^z \langle\theta^2\rangle_{z'}\, dz', \qquad (43)$$

where $\langle\theta^2\rangle_z$ is the mean-square scattering angle of a particle in a layer of matter of thickness z.

Using the relation (43), for the energy spread due to elastic scattering we can readily obtain the estimate

$$(\delta T)_{\text{el}} \sim \bar{\varepsilon}\,\langle\theta_s^2\rangle\, z^2,$$

where $\langle\theta_s^2\rangle$ is the mean square of the scattering angle of a particle per unit path length.

The fluctuations of the energy losses in inelastic collisions lead, in their turn, to a spread $(\delta T)_{\text{inel}} \sim \sqrt{\overline{\varepsilon^2} z}$. Bearing in mind that for fast heavy charged particles $\overline{\varepsilon^2}/(\bar{\varepsilon})^2 \sim m_e/eR_0$, while $\langle\theta_s^2\rangle R_0 \sim m_e/mZ$, we obtain

$$\frac{(\delta T)_{\text{el}}}{(\delta T)_{\text{inel}}} \sim \frac{\bar{\varepsilon}\langle\theta_s^2\rangle z^2}{\sqrt{\overline{\varepsilon^2}\, z}} \sim Z\sqrt{\frac{m_e}{m}}\left(\frac{z}{R_0}\right)^{3/2}. \qquad (44)$$

Thus, the importance of multiple elastic scattering in forming the energy spectrum of the particles increases with increasing depths and is particularly great for the penetration of not too heavy particles (muons and pions) through thick layers ($z/R_0 \gtrsim Z^{-2/3}(m/m_e)^{1/3}$) of matter with large Z. In particular, elastic scattering must be taken into account in calculations of the energy-loss distribution and the energy release (Bragg curve) (Refs. 2, 3, 6–10, 24, and 25) and the spread of stopped particles.[5,57,58] It should also be noted that multiple elastic scattering is the reason for the dependence of the energy losses on the scattering angle of the particles[59] and their displacement relative to the beam axis.[10]

The problem of calculating not only the mean path and the variance of the path but also the actual distribution of the particles with respect to the traversed paths was first posed by Yang,[60] who considered the determination of the distribution of particles with respect to the path in the case of penetration of a narrow beam through a flat layer of matter. He assumed that the energy losses do not affect the elastic scattering (taking the mean square of the scattering angle per unit length to be constant, $\langle\theta_s^2\rangle = \text{const}$). In Ref. 60, Yang gave a formal solution of the corresponding transport equation in the form of a product of series in Hermite polynomials. The cumbersome nature of his solution made its practical use very complicated, and therefore Yang gave fairly simple final expressions for the path distributions only for the two simplest cases relating to wide-beam geometry—for particles moving in the original direction and without reference to the direction of motion of the particles. The initial assumption $\langle\theta_s^2\rangle = \text{const}$ of Ref. 60 restricted the applicability of the Yang distribution to the region of thin matter layers, in which, as was shown above, the part played by multiple elastic scattering is slight.

Yang's result was generalized to take into account the dependence of $\langle\theta_s^2\rangle$ on the particle energy by Spencer and Coyne[61] and Berger and Seltzer.[6]

Spencer and Coyne[61] found the path distribution of the particles for the case of a wide beam. However, the absence in Ref. 61 of analytic relations sufficiently convenient for calculations had the consequence that their solution was not subsequently used for practical calculations.

In contrast, the result of Berger and Seltzer,[6] though not rigorously obtained, became widely known and until recently was, in fact, the only analytic result used in practical calculations[6,9,24,25] to take into account the influence of multiple elastic scattering on the energy-loss distribution of particles in thick layers of matter. In Ref. 6, Yang's distribution[60] was renormalized on the basis of Eq. (43) for the mean path, $\langle\theta^2\rangle_z$ being calculated with allowance for the dependence of the elastic scattering cross section on the particle energy. Such a procedure is simple and logical but does not guarantee a correct description of the distribution as a whole. In particular, as is shown by comparison with numerical Monte Carlo calculations,[6,22] Yang's formula as modified by Berger and Seltzer overestimates the variance of the path distribution. This can also be readily seen by using the expression for the path variance obtained by Pomeranchuk.[56]

It should be noted that a method of solving the transport equation analogous to the one developed in Ref. 61 was used in Ref. 62 to calculate the particle distribution with respect to the traversed paths in an inhomogeneous medium (it was assumed that the dependence of $\langle\theta_s^2\rangle$ on the depth is due to the inhomogeneity of the medium).

In recent years, the authors of the present review have succeeded in solving completely Yang's problem, i.e., in finding the distribution of the particles with respect to the traversed paths in the case of penetration of a narrow beam through matter with allowance for the energy dependence of $\langle\theta_s^2\rangle$. This, in its turn, has made it possible to develop a consistent analytic theory of the penetration of a particle beam in thick layers of matter with allowance simultaneously for the fluctuations of the energy losses due to the probabilistic nature of the inelastic collisions and multiple elastic scattering.[30-32]

We present below the main results of this theory.

A feature of the Coulomb scattering of fast heavy charged particles by atoms of matter is that at energies $T \gtrsim 4\cdot 10^4 (m_e/m)^2 A^{-2/3} mc^2$ (A is the mass number of the matter) the particle wavelength λ becomes less than the characteristic scale $r_{\rm nuc}$ of the atomic nucleus, and at deflection angles $\theta \gtrsim \lambda/r_{\rm nuc}$ ($r_{\rm nuc} \simeq 0.45 A^{1/3} r_e$, where r_e is the classical electron radius[15]) it is necessary to take into account the nuclear form factor[15] in the elastic scattering cross section. This form factor describes a suppression of scattering at angles $\theta \gtrsim \lambda/r_{\rm nuc}$.

This makes it possible to use for the elastic collision integral the diffusion approximation[17]:

$$\hat{I}_{\rm el} = \frac{1}{4}\langle\theta_s^2(T)\rangle \Delta_{\Omega} N(\mathbf{r}, \Omega, T), \tag{45}$$

where Δ_Ω is the angular part of the Laplacian in spherical coordinates:

$$\Delta_\Omega = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}; \tag{46}$$

$\Omega = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, $\langle\theta_s^2(T)\rangle$ is the mean square of the scattering angle of a particle with energy T per unit path length,[17]

$$\langle\theta_s^2(E)\rangle = 16\pi n_0 Z(Z+1) z^2 r_e^2 \frac{m_e}{m}\frac{(1+E)^2}{E^2(2+E)^2} L_k, \tag{47}$$

Z is the atomic number of the atoms of the matter, ze is the charge of a particle, n_0 is the number of atoms of the medium per unit volume, and L_k is the Coulomb logarithm[15]: $L_k = \ln[210(AZ)^{-1/6}]$.

Overall, the approximation (45) describes correctly the angular spectrum of fast heavy charged particles in matter and is particularly fruitful for solving the problem of the influence of elastic scattering on the energy-loss distribution (see Refs. 17, 30–32, and 60–62).

As follows from the previous subsection, it is convenient to consider the energy spread of the particles in terms of the variable

$$u = R_0 - R(T) - z = \int_T^{T_{\rm c.s}(z)} \frac{dT'}{\bar{\varepsilon}(T')}, \tag{48}$$

where $T_{\rm c.s}(z)$ is the energy of the particles at depth z in the continuous-slowing-down approximation.

In thick layers of matter, the mean energy losses of fast heavy charged particles appreciably exceed the fluctuations of the losses and it is obvious in advance that $u_{\rm eff} \ll R_0 - R(T)$. One may therefore use the approximation (20)–(21) for the inelastic collision integral, and also, as in the case of (18) and (19), set

$$\langle\theta_s^2(T)\rangle \simeq \langle\theta_s^2[T_{\rm m.p}(z)]\rangle \equiv \langle\theta_s^2(z)\rangle. \tag{49}$$

After these manipulations, Eq. (1) and the boundary condition (4) take the form

$$\Omega\frac{\partial N}{\partial \mathbf{r}} + (1-\Omega_z)\frac{\partial N}{\partial u} = \frac{\langle\theta_s^2(z)\rangle}{4}\Delta_\Omega N + \sum_{k=2}^{\infty}\frac{(-1)^k}{k!}\frac{\overline{\varepsilon^k}(z)}{[\bar{\varepsilon}(z)]^k}\frac{\partial^k N}{\partial u^k}, \tag{50}$$

$$N(z=0, \boldsymbol{\rho}, \Omega, u) = N_0\delta(\boldsymbol{\rho})\,\delta(\Omega-\Omega_0)\,\delta(u). \tag{51}$$

The flux density $N(\mathbf{r}, \Omega, u)$ is related to $N(\mathbf{r}, \Omega, T)$ by

$$N(\mathbf{r}, \Omega, u) = \bar{\varepsilon}(T)\, N(\mathbf{r}, \Omega, T). \tag{52}$$

Since in the case of the scattering of fast heavy charged particles in matter the angle of deflection from the direction of the original motion is small over the complete path (at least where the particles can be assumed to be fast), $\langle\theta_s^2\rangle R_0 \ll 1$, for the subsequent transformation of Eq. (50) we can make a small-angle expansion of its coefficients:

$$\left.\begin{aligned} &\Omega_x = \sin\theta\cos\varphi \simeq \theta\cos\varphi = \theta_x;\\ &\Omega_y = \sin\theta\sin\varphi \simeq \theta\sin\varphi = \theta_y;\\ &\Omega_z = \cos\theta \simeq 1-\theta^2/2 = 1-(\theta_x^2+\theta_y^2)/2;\\ &\Delta_\Omega \simeq \frac{1}{\theta}\frac{\partial}{\partial\theta}\theta\frac{\partial}{\partial\theta} + \frac{1}{\theta^2}\frac{\partial^2}{\partial\varphi^2} = \frac{\partial^2}{\partial\theta_x^2} + \frac{\partial^2}{\partial\theta_y^2}, \end{aligned}\right\} \tag{53}$$

where θ_x and θ_y are the angles between the vector Ω (the direction of the motion of the particle) and the YZ and XZ planes, respectively (Fig. 5).

Retaining in the coefficients of Eq. (50) only the first nonvanishing terms, we reduce the transport equation (50) and the boundary condition (51) to the form

$$\frac{\partial N}{\partial z} + \theta_x\frac{\partial N}{\partial x} + \theta_y\frac{\partial N}{\partial y} + \frac{\theta_x^2+\theta_y^2}{2}\frac{\partial N}{\partial u} = \frac{\langle\theta_s^2(z)\rangle}{4}\left(\frac{\partial^2 N}{\partial\theta_x^2} + \frac{\partial^2 N}{\partial\theta_y^2}\right) + \sum_{k=2}^{\infty}\frac{(-1)^k}{k!}\frac{\overline{\varepsilon^k}(z)}{[\bar{\varepsilon}(z)]^k}\frac{\partial^k N}{\partial u^k}; \tag{54}$$

$$N(z=0\; x,\, y,\, \theta_x,\, \theta_y,\, u) = N_0\delta(x)\,\delta(y)\,\delta(\theta_x)\,\delta(\theta_y)\,\delta(u). \tag{55}$$

From the transport equation written in the form (50) we can see clearly where the incorrectness of the standard small-

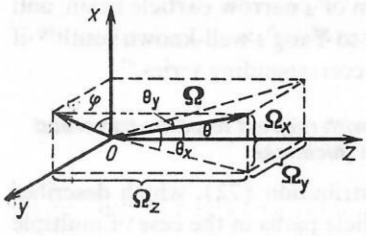

FIG. 5. Explanation of the derivation of the relations (53).

angle approximation[17,18] resides. Even in the case of small-angle scattering of the particles, it is possible to replace $\cos\theta$ by unity only in the first term on the left-hand side of Eq. (50). With regard to the second term, it is necessary to retain two terms in the expansion of $\cos\theta$ in a series in θ, since the value of the derivative $\partial N/\partial u$ is large. This term takes into account the influence of the bending of the trajectories of the particles as a result of multiple elastic scattering on their energy distribution.

The transport equation (54) with the boundary condition (55) describes the penetration of a beam of fast heavy charged particles in matter with allowance for the systematic slowing down, the fluctuations of the particle paths due to multiple elastic scattering and the probabilistic nature of the energy losses in inelastic collisions. This equation can be solved analytically in general form without making any new additional approximations.

To solve the transport equation (54) with the boundary condition (55), we can use a Fourier transformation with respect to the variable u:

$$N_\omega(z, x, y, \theta_x, \theta_y) = \int_{-\infty}^{\infty} du\, e^{-i\omega u} N(z, x, y, \theta_x, \theta_y, u).$$

As a result of this transformation, we obtain

$$\frac{\partial N_\omega}{\partial z} + \frac{\theta_x^2+\theta_y^2}{2} i\omega N_\omega + \theta_x \frac{\partial N_\omega}{\partial x} + \theta_y \frac{\partial N_\omega}{\partial y} = \frac{\langle\theta_s^2(z)\rangle}{4}\left(\frac{\partial^2}{\partial\theta_x^2}+\frac{\partial^2}{\partial\theta_y^2}\right) N_\omega + \sum_{k=2}^{\infty}\frac{(-i\omega)^k}{k!}\frac{\overline{\varepsilon^k}(z)}{[\bar\varepsilon(z)]^k} N_\omega; \quad (56)$$

$$N_\omega(z=0, x, y, \theta_x, \theta_y) = N_0\delta(x)\,\delta(y)\,\delta(\theta_x)\,\delta(\theta_y). \quad (57)$$

The solution $N_\omega(z, x, y, \theta_x, \theta_y)$ has the form

$$N_\omega(z, x, y, \theta_x, \theta_y) = N_0 F_\omega(z, x, y, \theta_x, \theta_y)\exp[-\tau(\omega, z)], \quad (58)$$

where

$$\tau(\omega, z) = -\sum_{k=2}^{\infty}\frac{(-i\omega)^k}{k!}\int_0^z \frac{\overline{\varepsilon^k}(z')}{[\bar\varepsilon(z')]^k}dz' \simeq \int_{T_{\mathrm{m.p}}(z)}^{T_0}\frac{dT'}{\bar\varepsilon(T')}\int_0^{\varepsilon_{\max}(T')} d\varepsilon\, W_{\mathrm{inel}}(T'|\varepsilon)\left[1-\frac{i\omega\varepsilon}{\bar\varepsilon(T')}-e^{-\frac{i\omega\varepsilon}{\bar\varepsilon(T')}}\right]. \quad (59)$$

Here, the function $F_\omega(z, x, y, \theta_x, \theta_y)$ satisfies the equation

$$\frac{\partial F_\omega}{\partial z} + \frac{\theta_x^2+\theta_y^2}{2} i\omega F_\omega + \theta_x\frac{\partial F_\omega}{\partial x} + \theta_y\frac{\partial F_\omega}{\partial y} = \frac{1}{4}\langle\theta_s^2(z)\rangle\left(\frac{\partial^2}{\partial\theta_x^2}+\frac{\partial^2}{\partial\theta_y^2}\right)F_\omega \quad (60)$$

with the boundary condition

$$F_\omega(z=0, x, y, \theta_x, \theta_y) = \delta(x)\cdot\delta(y)\,\delta(\theta_x)\,\delta(\theta_y). \quad (61)$$

Equation (60) becomes identical to the Fourier transform of the original transport equation (54) if we ignore in the latter the probabilistic nature of the inelastic collisions and describe the process of stopping of the particles in the medium with allowance for only the systematic losses and the path fluctuations due to the multiple elastic scattering.

From the physical point of view, the reason why the solution has the form (58) is to be found in the circumstance that the factors which determine the energy spread of the particles—the probabilistic nature of the energy transfer in inelastic collisions of particles with the atoms of the matter and the fluctuations of the path length due to multiple elastic scattering (i.e., the bending of the trajectories)—are quasi-independent. This last fact is a consequence of the relative smallness of the fluctuations due to each of these processes.

The solution of Eq. (60) with the subsidiary condition (61) has the form

$$F_\omega(z, \rho, \theta) = \frac{1}{\pi^2 A_0(\omega, z) D^2(\omega, z)} \times \exp\left[-\frac{\rho^2 A_1(\omega, z) - 2\rho\theta A_2(\omega, z) + \theta^2 A_3(\omega, z)}{D^2(\omega, z)}\right], \quad (62)$$

where

$$A_1(\omega, z) = \frac{2}{i\omega}\frac{d}{dz}\ln A_0(\omega, z); \quad (63)$$

$$A_2(\omega, z) = \frac{2}{i\omega}\left[1-\frac{1}{A_0(\omega, z)}\right]; \quad (64)$$

$$A_3(\omega, z) = \frac{2}{i\omega}\int_0^z dz'\left[1-\frac{1}{A_0^2(\omega, z')}\right]; \quad (65)$$

$$D^2(\omega, z) = A_1(\omega, z)\,A_3(\omega, z) - A_2^2(\omega, z); \quad (66)$$

$$\rho = (x, y);\ \theta = (\theta_x, \theta_y). \quad (67)$$

The function $A_0(\omega, z)$, in terms of which $F_\omega(z, \rho, \theta)$ is expressed, can be determined from the equation

$$\left.\begin{aligned} &\frac{d^2}{dz^2}A_0 - \frac{1}{2} i\omega\langle\theta_s^2(z)\rangle A_0 = 0,\\ &A_0(z=0) = 1,\quad A_0'(z=0) = 0.\end{aligned}\right\} \quad (68)$$

Thus, the problem of calculating the spatial, angular, and energy distribution of the particles in a narrow beam, i.e., the function $N(z, \rho, \theta, u)$ of six variables, reduces to the much simpler problem of determining the function $A_0(\omega, z)$ of just one variable z (ω is a parameter) from Eq. (68). This equation has the same form as a one-dimensional time-independent Schrödinger equation. The solution of Eq. (68) is discussed below [see subsection (a)].

If the function $A_0(\omega, z)$ has been determined, then, substituting (62) in (58) and making the inverse Fourier transformation, we obtain the final expression for the particle flux density:

$$N(z, \rho, \theta, u) = \frac{N_0}{2\pi^3}\int_{-\infty}^{\infty} d\omega \frac{1}{A_0(\omega, z) D^2(\omega, z)} \times \exp\left[i\omega u - \tau(\omega, z) - \frac{\rho^2 A_1(\omega, z) - 2\rho\theta A_2(\omega, z) + \theta^2 A_3(\omega, z)}{D^2(\omega, z)}\right]; \quad (69)$$

$$u = \int_{T}^{T_{c.s}(z)} dT'/\bar{\varepsilon}(T') \simeq [4\pi n_0 Z z^2 r_e^2 L_{ion}(E_{c.s})]^{-1} \times \frac{m}{m_e}\left[\frac{E_{c.s}^2(z)}{1+E_{c.s}(z)} - \frac{E^2}{1+E}\right]. \quad (70)$$

The relation (69) can also be represented in the form

$$N(z, \rho, \theta, u) = N_0 \int_0^\infty du' F(z, \rho, \theta, u') H(z, u-u'), \quad (71)$$

where

$$F(z, \rho, \theta, u) = \frac{1}{2\pi^3}\int_{-\infty}^{\infty} d\omega \frac{1}{A_0(\omega, z) D^2(\omega, z)} \times \exp\left[i\omega u - \frac{\rho^2 A_1(\omega, z) - 2\rho\theta A_2(\omega, z) + \theta^2 A_3(\omega, z)}{D^2(\omega, z)}\right] \quad (72)$$

is the distribution of the particles with respect to the traversed paths, the angles, and the spatial coordinates with allowance for the bending of the trajectories due to multiple elastic scattering, and

$$H(z, u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \exp[i\omega u - \tau(\omega, z)] \quad (73)$$

is the distribution of the particles with respect to the ranges (energies) due to the fluctuations of the energy losses in the inelastic collisions.

If we ignore the fluctuations of the energy losses in inelastic collisions [$\tau(\omega,z)=0; H(z,u)=\delta(u)$], the particle spectrum (71) reduces to the distribution (72). In this approximation $u = S - z$, where $S = R_0 - R(T)$ is the path of a particle in matter traversed when it is decelerated to energy T.

The expression (69) [or (71)–(73)] determines the spatial, angular, and energy distribution of fast charged particles in a narrow beam with simultaneous allowance for the systematic slowing down and the probabilistic nature of the elastic and inelastic collisions of particles with the atoms of the medium.[4]

The distribution (69) is the most general of the previously obtained relations (Refs. 17, 27, 29, and 60–63) that describe the slowing down of fast heavy charged particles in matter and includes the results mentioned above as special cases. For example, if we restrict ourselves to the case of purely elastic scattering and do not take into account bending of the trajectory as a result of multiple elastic scattering, then the Fermi distribution[17] follows from (69). If in addition we take into account the systematic energy losses, we arrive at the result of Ref. 63. In addition, integrating (69) over the transverse coordinates, we obtain all previously known results relating to the case of a wide beam in which allowance is made for either only the fluctuations of the losses in inelastic collisions,[17,27,29] or only the bending of the trajectory as a result of multiple elastic scattering,[61,62] or both these processes at once.[17] If we make the same approximations as in Ref. 60, i.e., we do not take into account the probabilistic nature of the inelastic collisions and set $\langle\theta_s^2\rangle$ = const, then from the relation (69) we obtain a simple expression for the spectrum of a narrow particle beam, and this is completely identical to Yang's well-known result[60] if in his solution we sum the corresponding series.[e)]

a) Distribution of particles with respect to paths traversed in a layer of matter of given thickness

To investigate the distribution (72), which describes the fluctuations of the particle paths in the case of multiple elastic scattering in matter, it is convenient to use, alongside the usual variables, the dimensionless variables

$$\xi = z/R_0; \quad s = S/R_0; \quad (74)$$

$$\xi_\perp = \rho/R_0\sqrt{\gamma}; \quad \psi = \theta/\sqrt{\gamma}. \quad (75)$$

In the new variables, the expression for $F(z, \rho, \theta, S-z)$ takes the form

$$F(z, \rho, \theta, S-z)\, d^2\rho\, d^2\theta\, d(S-z) = F(\xi, \xi_\perp, \psi, s-\xi)\, d^2\xi_\perp\, d^2\psi\, d(s-\xi);$$

$$F(\xi, \xi_\perp, \psi, s-\xi) = \frac{1}{2\pi^3\gamma}\int_{-\infty}^{\infty} d\omega \frac{1}{A_0(\omega, \xi) D^2(\omega, \xi)} \times \exp\left[i\omega\frac{s-\xi}{\gamma} - \frac{\xi_\perp^2 A_1(\omega, \xi) - 2\xi_\perp\psi A_2(\omega, \xi) + \psi^2 A_3(\omega, \xi)}{D^2(\omega, \xi)}\right]. \quad (76)$$

The dimensionless parameter γ in (75) and (76) is equal to the ratio of the total range of the particles to the elastic-scattering transport length:

$$\gamma = \frac{1}{2} R_0 \langle\theta_s^2(T_0)\rangle = \frac{R_0}{l_{tr}(T_0)} \quad (77)$$

and it characterizes the influence of the bending of the trajectory on the distribution of the particles in the matter. Using (15) and (47), for the parameter γ we can obtain

$$\gamma = 2\frac{m_e}{m}(Z+1)\frac{L_k}{L_{ion}}\frac{1+E_0}{(2+E_0)^2}. \quad (78)$$

As can be seen from this relation, γ increases with decreasing mass of the particles and with increasing atomic number of the matter of the scatterer. For example, for 100-MeV protons ($E_0 \simeq 0.1$), $\gamma \simeq 2\times10^{-3}$ in Al and $\gamma \simeq 1.6\times10^{-2}$ in Pb. For muons of the same energy ($E_0 \simeq 1$), $\gamma = 1.3\times10^{-2}$ and $\gamma \simeq 0.8\times10^{-1}$, respectively.

In the new variables, Eq. (68) for the function A_0 takes the form

$$\left.\begin{aligned} &\frac{d^2}{d\xi^2}A_0 - i\omega\Lambda^2(\xi)A_0 = 0; \\ &A_0(\xi=0) = 1; \quad A_0'(\xi=0) = 0, \end{aligned}\right\} \quad (79)$$

where

$$\Lambda^2(\xi) = \frac{\langle\theta_s^2[T_{m.p}(z)]\rangle}{\langle\theta_s^2(T_0)\rangle}. \quad (80)$$

Equation (79) is a differential equation of second order, and an exact solution of it cannot be obtained analytically for arbitrary form of the function $\Lambda^2(\xi)$. However, if we ignore the dependence of the ionization logarithm on the energy, Eq. (79) can be solved exactly.

Instead of the dimensionless depth ξ, we introduce the new variable t, which is related to ξ by the relation (Refs. 30

and 31)[f)]

$$t(\xi) = \text{ch} \ln [1 + E_{c.s}(\xi)]. \quad (81)$$

If we now ignore the dependence of the ionization logarithm on the energy, then with allowance for (9) and (47) Eq. (79) can be written in the form

$$\left.\begin{aligned} &(t^2-1)\frac{d^2}{dt^2}A_0 - i\omega b^2 A_0 = 0; \\ &A_0(t=t_0)=1; \quad A_0'(t=t_0)=0, \end{aligned}\right\} \quad (82)$$

where

$$t_0 = t(\xi = 0) = \text{ch} \ln (1 + E_0); \; b = 1 + 2/E_0. \quad (83)$$

The solution of Eq. (82) is determined by the expression[30,31]

$$A_0(t, t_0, \omega) = \sqrt{t^2-1}\,[P_\mu^1(t)\,Q_\mu(t_0) - P_\mu(t_0)\,Q_\mu^1(t)], \quad (84)$$

where $P_\mu^\alpha(t)$ and $Q_\mu^\alpha(t)$ are associated Legendre functions of the first and second kind $(t \geqslant 1)$.[51,64] The degree μ is

$$\mu(\omega) = \frac{1}{2}\,[\sqrt{1 + 4i\omega b^2} - 1]. \quad (85)$$

In (81), $E_{c.s}(\xi)$ in the continuous-slowing-down approximation with L_{ion} = const has the form

$$E_{c.s}(\xi) = E_0 \frac{\sqrt{1-\xi}}{1+b}(\sqrt{1-\xi} + \sqrt{b^2 - \xi}). \quad (86)$$

If we do not ignore the dependence of the ionization logarithm on the energy, then Eq. (68) for the function A_0 takes the form

$$(t^2-1)\,L_{ion}(t)\frac{d}{dt}\,L_{ion}(t)\frac{d}{dt}A_0 - i\omega\frac{(Z+1)\,L_k}{2\pi n_0 Z z^2 r_e^2}A_0 = 0. \quad (87)$$

This equation cannot be solved exactly analytically, but to investigate it one can use the semiclassical approximation.[65] In it, we obtain[g)] for the function A_0

$$\begin{aligned} A_0(t, t_0, \omega) \simeq \sqrt[4]{\frac{t^2-1}{t_0^2-1}}\,[i\omega\varphi(t)\,\varphi(t_0)]^{1/2} \\ \times \{I_{-\frac{n+1}{n+2}}[\sqrt{i\omega}\,\varphi(t_0)]\,K_{\frac{1}{n+2}}[\sqrt{i\omega}\,\varphi(t)] \\ + K_{-\frac{n+1}{n+2}}[\sqrt{i\omega}\,\varphi(t_0)]\,I_{\frac{1}{n+2}}[\sqrt{i\omega}\,\varphi(t)]\}, \end{aligned} \quad (88)$$

where

$$\varphi(t) = \left[\frac{(Z+1)\,L_k}{2\pi n_0 Z z^2 r_e^2}\right]^{1/2} \int_1^t \frac{dt'}{\sqrt{t'^2-1}\,L_{ion}(t')}; \quad (89)$$

here $I_\alpha(x)$ and $K_\alpha(x)$ are a modified Bessel function and a Macdonald function, respectively.[51,64]

In the following discussions, to avoid encumbering the exposition with inessential details, we shall omit the dependence of the ionization logarithm on the energy. In the final expressions, on the basis of the solutions (88) and (89), we shall introduce corresponding changes, which make it possible to take into account the energy dependence of L_{ion}.

The expression (84) for the function A_0 makes it possible, using Eqs. (63)–(66), to determine the coefficients A_n $(n = 1, 2, 3)$ and D^2 and then, using the relations (69) or (71)–(73), to obtain the complete distribution of the particles in a narrow beam with respect to the angles, with respect to the transverse displacement from the beam axis, and with respect to the energies in the complete range of depths and energies in which the slowing down is due to ionization losses.

We consider first the spread of the particles with respect to the traversed paths, independently of the direction of their motion and deflection from the beam axis.

Integrating the relation (76) over the angles and the transverse coordinates (over the distance from the beam axis), and taking into account (84)–(86), we obtain

$$F(\xi, s-\xi) = \frac{1}{4\pi\gamma}\frac{b^2-1}{\sqrt{(b^2-\xi)(1-\xi)}} \times \int_{-\infty}^{\infty} d\omega \frac{\exp\left[\frac{i\omega}{\gamma}(s-\xi)\right]}{P_{\mu(\omega)}^1(t)\,Q_{\mu(\omega)}(t_0) - P_{\mu(\omega)}(t_0)\,Q_{\mu(\omega)}^1(t)}. \quad (90)$$

This expression is valid for a very wide range of energies of the incident particles, encompassing nonrelativistic $(E_0 \ll 1)$, relativistic $(E_0 \sim 1)$, and ultrarelativistic $(1 \ll E_0 \ll m/m_e)$ energies.

Note that the function $F(\xi, s-\xi)$ satisfies the equation

$$\int ds F(\xi, s-\xi) = 1, \quad (91)$$

which reflects the conservation of the total flux in the region of depth in which stopping of the particles has not yet occurred.

In general form, the integral (90) cannot be calculated exactly. However, in a number of limiting cases the expression (90) can be simplified, and the distribution $F(\xi, s-\xi)$ can be represented in the form of rapidly converging series.[30] Besides this, for the function $F(\xi, s-\xi)$ it is possible to obtain a very simple approximate formula valid for all initial energies of the incident particles.[30]

1. Yang's formula (the distribution of the particles with respect to the paths in the case $\langle\theta_s^2\rangle$ = const)[17,60]

We consider first the simplest case when it is possible to ignore the dependence of $\langle\theta_s^2\rangle$ on the particle energy [in Eq. (79), $\Lambda^2 = 1$].[17,60] This is always justified in the region of small depths $(\xi \ll 1)$. In addition in this simple example one can readily establish the main properties of the distribution of the particles with respect to the traversed paths that are also observed in the general case when the energy dependence of $\langle\theta_s^2\rangle$ is taken into account.

Setting $\Lambda^2 = 1$ in Eq. (79), we find

$$A_0(\omega, \xi) = \text{ch}(\xi\sqrt{i\omega}). \quad (92)$$

It follows from this that

$$F(\xi, s-\xi) = \frac{1}{\gamma\xi^2}\Phi\left(\frac{s-\xi}{\gamma\xi^2}\right), \quad (93)$$

where

$$\Phi(\lambda) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp\frac{e^{p\lambda}}{\text{ch}\sqrt{p}}; \quad \int_0^\infty d\lambda\Phi(\lambda) = 1. \quad (94)$$

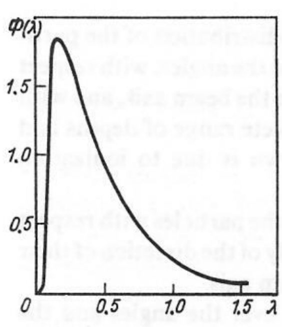

FIG. 6. Graph of the function $\Phi(\lambda)$.

The function Φ has a well-defined maximum at $\lambda_0 \simeq 1/6$ and is strongly asymmetric—for $\lambda < \lambda_0$, it decreases very rapidly, whereas in the region of large λ it has a long extended "tail" (Fig. 6).

Calculating the integral (94), we readily obtain two representations of the function $\Phi(\lambda)$ in the form of series:

$$\Phi(\lambda)=\frac{1}{\sqrt{\pi}\,\lambda^{3/2}}\sum_{k=0}^{\infty}(-1)^k(2k+1)\exp\left[-\frac{(2k+1)^2}{4\lambda}\right]; \quad (95)$$

$$\Phi(\lambda)=\pi\sum_{k=0}^{\infty}(-1)^k(2k+1)\exp\left[-\frac{\pi^2}{4}(2k+1)^2\lambda\right]. \quad (96)$$

The first term of the series (95) describes exactly the behavior of the function Φ in the region of comparatively small λ ($\lambda \lesssim 0.3$–0.4). It is important that the value λ_0, at which the function has a maximum, occurs in this region. The representation of $\Phi(\lambda)$ in the form (96) is distinguished by the fact that already the first term of this series describes well the behavior of the function Φ at relatively large λ ($\lambda \gtrsim 0.3$–0.4). In the complete interval of λ values, the function $\Phi(\lambda)$ can be represented with an error of not more than 1% in the form

$$\Phi(\lambda)=\begin{cases}\dfrac{1}{\sqrt{\pi}\,\lambda^{3/2}}\exp\left(-\dfrac{1}{4\lambda}\right), & 0\leqslant\lambda\leqslant 0.3;\\[2ex] \pi\exp\left(-\dfrac{\pi^2}{4}\lambda\right), & \lambda\geqslant 0.3.\end{cases} \quad (97)$$

We note that $\langle\lambda\rangle = 1/2$, $\langle\lambda^2\rangle = 5/12$.

2. Nonrelativistic particles ($E_0 \ll 1$)

In this case, $t \simeq \cosh E_{\rm c.s}(\xi)$, $t_0 \simeq \cosh E_0$, $E_{\rm c.s}(\xi) = \sqrt{1-\xi}$, and the parameter b is large: $b \simeq 2/E_0 \gg 1$. Since the distribution $F(\xi, s-\xi)$ is essentially nonzero when $s-\xi \lesssim \gamma$, the main contribution to (90) is made by the region $|\omega_{\rm eff}| \sim 1$, in which $\mu \gg 1$, since $b \gg 1$. Therefore, when calculating the integral (90) we can use the well-known asymptotic representation of the Legendre functions at large values of the degree.[51,64] After this, the integral (90) can be calculated exactly by means of residue theory, and the expression for $F(\xi, s-\xi)$ can be represented in the form a rapidly converging series.[30]

3. Ultrarelativistic particles ($1 \ll E_0 \ll m/m_e$)

In the complete range of depths for which the particles are still ultrarelativistic ($1-\xi \gg 1/E_0$), $t \simeq E_{\rm c.s}(\xi)/2$, $t_0 \simeq E_0/2$, $E_{\rm c.s}(\xi) = E_0(1-\xi)$, and the value of the parameter b is near unity: $b - 1 = 2/E_0 \ll 1$. The range of effective values of ω in (90), $|\omega_{\rm eff}| \sim 1$, is such that $\mu \sim 1$, whereas t, $t_0 \gg 1$. This makes it possible to use the asymptotic representation of the Legendre functions at large values of the argument.[51,64] After this, as in the case of particles of nonrelativistic energies, the integral (90) can be calculated by means of residue theory and the distribution with respect to the traversed paths can be represented in the form of a rapidly converging series.[30]

4. Calculation of the distribution with respect to paths in the general case

Whereas the distribution with respect to the traversed path can be calculated directly on the basis of Eq. (84) for the function $A_0(\omega,\xi)$ in the limiting cases of nonrelativistic and ultrarelativistic energies, in the region of relativistic energies ($E_0 \sim 1$) the integral (90) cannot be calculated exactly.

However, there are two methods by means of which it is possible to calculate with good accuracy the distribution of the particles with respect to the traversed paths at all energies of the incident particles.

The first method was proposed initially by Berger and Seltzer[6] and was subsequently used in many studies (Refs. 9, 17, 24, and 25). What it lacks in rigor it makes up in simplicity and clarity.

The gist of the method is as follows. The Yang distribution (93) is modified in such a way that when the mean path $\langle S\rangle_z$ of the particles in a layer of matter of thickness z is calculated the result obtained is equal to the expression (43). The Yang distribution corrected in this manner has the form

$$F(z, S-z)=\frac{1}{2\langle S-z\rangle_z}\Phi\left(\frac{S-z}{2\langle S-z\rangle_z}\right), \quad (98)$$

where the function $\Phi(\lambda)$ is determined by (94) and $\langle S-z\rangle_z \equiv \langle S\rangle_z - z$ by (43).

Comparison with the results of Ref. 30 shows that the distribution (98) overestimates the variance of the paths of particles in thick layers of matter [for nonrelativistic particles, for example, by 10%–30%].

The second method[30] is based on the following considerations.

In the case when the difference between the path traversed by the particle and its depth of penetration into the medium is small, the main contribution to the integral (90) is made by the region $|\omega_{\rm eff}| \gg 1$. Therefore, to calculate the distribution $F(\xi, s-\xi)$ it is sufficient to know the value of the function $A_0(\omega,\xi)$ only at large ω. To this end, we use the asymptotic representation of the Legendre functions at large values of the degree[51,64]:

$$P_\mu^\alpha[\operatorname{ch}\ln(1+E)] \simeq \frac{1}{\sqrt{2\pi\,\operatorname{sh}\ln(1+E)}}\times\left[\frac{\Gamma\left(\mu+\frac{1}{2}\right)}{\Gamma(1+\mu-\alpha)}e^{\left(\mu+\frac{1}{2}\right)\ln(1+E)}+\frac{\Gamma\left(-\mu-\frac{1}{2}\right)}{\Gamma(-\mu-\alpha)}e^{-\left(\mu+\frac{1}{2}\right)\ln(1+E)}\right]; \quad (99)$$

$$Q_\mu^\alpha [\mathrm{ch}\ln(1+E)]$$

$$\simeq e^{i\pi\alpha}\frac{\Gamma(1+\mu+\alpha)}{\Gamma(\mu+3/2)}\sqrt{\frac{\pi}{2\,\mathrm{sh}\ln(1+E)}}\,e^{-\left(\mu+\frac{1}{2}\right)\ln(1+E)}. \quad (100)$$

This is in fact equivalent to the semiclassical approximation (88) with $n = 0$. Substituting (99) and (100) in (90) and making some simple calculations, we obtain[30]

$$F(\xi,\ s-\xi)\simeq\frac{\sqrt[4]{\Lambda^2(\xi)}}{\gamma b^2\ln^2\dfrac{1+E_0}{1+E_{\mathrm{m.p}}(\xi)}}\,e^{-\frac{s-\xi}{4\gamma b^2}}\Phi(\lambda). \quad (101)$$

Here, $\Phi(\lambda)$ is determined by the expression (94) and

$$\lambda=\frac{s-\xi}{\gamma b^2\ln^2\dfrac{1+E_0}{1+E_{\mathrm{m.p}}(\xi)}}\equiv\frac{S-z}{\varkappa(z)}, \quad (102)$$

where we have introduced the notation

$$\varkappa(z)=\gamma R_0 b^2\ln^2\frac{1+E_0}{1+E_{\mathrm{m.p}}(z)}. \quad (103)$$

The relation (101) describes well the distribution with respect to the traversed paths in the region that includes the point of the maximum of the function F. Therefore, for the most probable path traversed by a particle in a layer of matter of thickness ξ we find[30]

$$S_{\mathrm{m.p}}(\xi)\simeq\xi+\frac{1}{6}\gamma b^2\ln^2\frac{1+E_0}{1+E_{\mathrm{m.p}}(\xi)}. \quad (104)$$

In the special cases of nonrelativistic and ultrarelativistic particles, and also at small depths ($\xi\ll1$), the well-known results[17,30] follow from the expression (104).

At large λ, the approximate expression (101) gives overestimated values. A consequence of this is a violation of the normalization condition (91).

On the basis of the result (101), it is of interest to obtain an expression for the distribution of the particles with respect to the traversed paths that not only correctly describes the behavior of the distribution function near the maximum but also does not violate the condition of conservation of the total particle flux (91).

Analysis shows that if these requirements are to be satisfied it is expedient to express $F(\xi, s-\xi)$ in the form

$$F(\xi,\ s-\xi)=\frac{\sqrt[4]{\Lambda^2(\xi)}}{\gamma b^2\ln^2\dfrac{1+E_0}{1+E_{\mathrm{m.p}}(\xi)}}\,e^{-\delta(\xi)\lambda}\Phi(\lambda), \quad (105)$$

i.e., to set

$$A_0(\omega,\ \xi)=\frac{1}{\sqrt[4]{\Lambda^2(\xi)}}\,\mathrm{ch}\sqrt{i\omega b^2\ln^2\frac{1+E_0}{1+E_{\mathrm{m.p}}(\xi)}+\delta(\xi)} \quad (106)$$

and to determine the unknown function $\delta(\xi)$ from the normalization condition (91) $[A_0(\omega=0,\xi)=1]$. Implementing this procedure, we obtain

$$\delta(\xi)=\ln^2\left[\sqrt[4]{\Lambda^2(\xi)}+\sqrt{\sqrt{\Lambda^2(\xi)}-1}\right]. \quad (107)$$

Thus, the approximate expression for the distribution of particles of arbitrary energies with respect to the paths has the form[30]

$$F(z,\ S-z)=\frac{\sqrt[4]{\Lambda^2(z)}}{\varkappa(z)}\Phi\left(\frac{S-z}{\varkappa(z)}\right)$$
$$\times\exp\left\{-(S-z)\frac{\ln^2\left[\sqrt[4]{\Lambda^2(z)}+\sqrt{\sqrt{\Lambda^2(z)}-1}\right]}{\varkappa(z)}\right\}. \quad (108)$$

Direct comparison of the results of calculations made for nonrelativistic and ultrarelativistic particles in accordance with the general formula (108) and on the basis of the exact expressions given in Ref. 30 shows that the accuracy of the relation (108) is fairly high. For example, the error in the calculation of $F(z, S-z)$ in accordance with (108) at depth $z/R_0 = 0.7$ does not exceed 2%. The error of the analogous calculation in accordance with (98) is significantly larger.

It is important to note that the method of obtaining the result (108) is not in general associated with neglect of the dependence of the ionization logarithm on the particle energy. Taking into account both (98) and the energy dependence of L_{ion}, we find

$$\varkappa(z)=\frac{(Z+1)L_k}{2\pi n_0 Z z^2 r_e^2}\left[\int_{E_{\mathrm{m.p}}(z)}^{E_0}\frac{dE}{(1+E)L_{\mathrm{ion}}(E)}\right]^2. \quad (109)$$

With increasing depth, the accuracy of the result (108) gradually decreases. Therefore, for detailed calculations when $1-\xi\ll1$ it is better to use the variant of the semiclassical solution with $n=-1$, which takes into account the singularity $\Lambda^2\sim1/(1-\xi)$. In this case, we can obtain for the path distribution

$$F(z,\ S-z)=\sqrt[4]{\frac{t_0^2-1}{t^2-1}}\,[\varphi(t)\varphi(t_0)]^{-1/2}e^{-(S-z)\delta}$$
$$\times\sum_{k=1}^{\infty}q_k\frac{J_0(q_k)J_1[q_k\varphi(t)/\varphi(t_0)]}{J_0^2(q_k)-J_1^2[q_k\varphi(t)/\varphi(t_0)]}\exp\left[-q_k^2\frac{S-z}{\varphi^2(t_0)}\right], \quad (110)$$

where q_k are the roots of the equation

$$J_0(q_k)N_1\left(q_k\frac{\varphi(t)}{\varphi(t_0)}\right)=J_1\left(q_k\frac{\varphi(t)}{\varphi(t_0)}\right)N_0(q_k); \quad (111)$$

$J_\alpha(x)$ and $N_\alpha(x)$ are Bessel and Neumann functions, respectively;[51,64] the value of the parameter δ is determined from the equation

$$\sqrt{\delta}\sqrt[4]{\frac{t^2-1}{t_0^2-1}}\,[\varphi(t)\varphi(t_0)]^{1/2}$$
$$\times\left[I_0(\sqrt{\delta}\varphi(t_0))K_1(\sqrt{\delta}\varphi(t))+K_0(\sqrt{\delta}\varphi(t_0))I_1(\sqrt{\delta}\varphi(t))\right]=1. \quad (112)$$

A procedure analogous to the one described above is also considered in Sec. 2 in the calculation of the distribution of stopped particles.

The method presented here for approximate calculation of the distribution of the particles with respect to the traversed paths is valid for any (sufficiently smooth) dependence of the mean square of the scattering angle on the depth and can also be used to describe the penetration of particles in inhomogeneous media.

(b) Energy spectrum in thick layers of matter without reference to the direction of motion of the particles and their displacement from the beam axis

The results of subsection (a) make it possible, on the basis of the general expression (69), to make specific calculations of the spatial, angular, and energy distribution of the particles in a beam. We consider first the energy spectrum of the particles without reference to the direction of their motion and distance from the beam axis.

Integrating the relation (69) over the angles and transverse coordinates, we obtain for the energy distribution of the particles the expression

$$N(z,\ u)=\frac{N_0}{2\pi}\int_{-\infty}^{\infty}d\omega\,\frac{e^{i\omega u-\tau(\omega,\ z)}}{A_0(\omega,\ z)},\tag{113}$$

where the functions $\tau(\omega,z)$ and $A_0(\omega,z)$ are determined by (59) and (84) [or (106)], respectively.

At small depths ($z\lesssim\varepsilon_{\max}^2/\overline{\varepsilon^2}$), the part played by the bending of the trajectory in forming the energy spectrum of fast heavy particles is always slight and the energy spread is completely determined by the probabilistic nature of the inelastic collisions (see Refs. 17, 27, 29, 42–44, and 49).

In contrast, in thick layers of matter the particle distribution is formed in general under the influence of two processes: the fluctuations of the energy losses in inelastic collisions with atoms of the medium and fluctuations of the particle paths due to bending of the trajectory as a result of multiple elastic scattering. Depending on the energy of the particles, their masses, and the atomic number Z of the atoms of the matter, the relative importance of each of these processes will vary. Therefore, at large depths it is in the general case necessary to take into account simultaneously the probabilistic nature of the energy transfer in inelastic collisions and the bending of the paths.

We shall consider the fluctuations of the energy in inelastic collisions on the basis of the self-consistent Gaussian approximation[17]: $\pi(\omega,z)=\frac{1}{2}\omega^2\sigma^2$, where σ^2 is determined by Eqs (31) and (32). At large depths in media with atomic number $Z\gtrsim(m/m_e)^{1/3}$ there is no point at all in attempting to take into account more accurately the probabilistic nature of the inelastic collisions [i.e., to retain in the expression for $\tau(\omega,z)$ the term proportional to ω^3]. This is so because in such media the fluctuations of the energy losses due to multiple elastic scattering are more important than the nondiffuseness (i.e., the nonvanishing of $\overline{\varepsilon^3}$) of the process of energy loss in inelastic collisions.

With allowance for what we have said, the relation (113) can be rewritten in the form

$$N(z,u)=\frac{N_0}{2\pi}\int_{-\infty}^{\infty}d\omega\,\frac{\exp\left[i\omega u-\frac{1}{2}\omega^2\sigma^2(z)\right]}{A_0(\omega,\ z)}.\tag{114}$$

If in (114) we ignore the probabilistic nature of the inelastic collisions, $\nu\to\infty$, we arrive at the expression (90). In this case, the spread of the particle energies is due solely to the bending of their paths by multiple elastic scattering.

In contrast, if we ignore the elastic scattering ($\gamma\to0$), then the result of the self-consistent Gaussian approximation (30) follows from (114).

In the general case (for an arbitrary ratio of γ and ν), combining the results of the previous section [(98) or (108)] with (30), for the energy spectrum of the particles we obtain the expression

$$N(z,\ u)=N_0\exp\left[-\frac{u^2}{2\sigma^2(z)}\right]\times\sum_{k=1}^{\infty}a_k\exp\left[\eta_k^2(z,\ u)\right]\{1-\mathrm{erf}\left[\eta_k(z,\ u)\right]\},\tag{115}$$

where erf(x) is the error function.[51]

If the contribution of the elastic scattering to the spread of the particles over the energies is taken into account on the basis of the result (108), then the values of a_k and $\eta_k(z,u)$ are determined by

$$a_k=\frac{\pi}{2}(-1)^{k-1}(2k-1)\frac{\sqrt[4]{\Lambda^2}}{\varkappa};\tag{116}$$

$$\eta_k=\sqrt{\frac{\sigma^2}{2}}\left[\frac{1}{\varkappa}\left(\frac{\pi^2}{4}(2k-1)^2+\delta\right)-\frac{u}{\sigma^2}\right],\tag{117}$$

and if we use the modified Yang distribution (98), then

$$a_k=\frac{\pi}{4\langle S-z\rangle_z}(-1)^{k-1}(2k-1);\tag{118}$$

$$\eta_k=\sqrt{\frac{\sigma^2}{2}}\left[\frac{\pi^2(2k-1)^2}{8\langle S-z\rangle_z}-\frac{u}{\sigma^2}\right].\tag{119}$$

The first term of the series (115) describes the low-energy part of the particle spectrum corresponding to relatively large energy losses. In this region of the spectrum we find mainly strongly scattered particles that have traversed a relatively long path and, as a result of this, have lost more energy than particles that have hardly been deflected from the direction of the original motion.

In the calculation of the energy spectrum in the region of relatively small energy losses, it is necessary to retain in the sum (115) an appreciable number of terms, and it is not particularly convenient to use the representation (115). In this case, using the asymptotic expression

$$\mathrm{erf}\,(x)\simeq1-\frac{1}{\sqrt{\pi}\,x}\exp\left(-x^2\right)$$

for the error function and taking into account the relation

$$\frac{1}{\mathrm{ch}\,x}=\pi\sum_{k=1}^{\infty}(-1)^{k-1}\frac{(2k-1)}{\frac{\pi^2}{4}(2k-1)^2+x^2},\tag{120}$$

we can represent the energy spectrum (115)–(117) in the form

$$N(z,\ u)=N_0\exp\left(-\frac{u^2}{2\sigma^2}\right)\left\{\frac{\sqrt[4]{\Lambda^2}}{\sqrt{2\pi\sigma^2}\,\mathrm{ch}\sqrt{\delta-\frac{\varkappa u}{\sigma^2}}}+\sum_{k=1}^{\infty}a_k\left[\exp(\eta_k^2)\left(1-\mathrm{erf}(\eta_k)\right)-\frac{1}{\sqrt{\pi}\,\eta_k}\right]\right\}.\tag{121}$$

An analogous relation can also be written down for the case (118), (119).

For $|u|\gg\sigma^2/\varkappa$ $(u<0)$ we obtain from the expression (121)

$$N(z,\ u)\simeq N_0\sqrt{\frac{2}{\pi\sigma^2}}\sqrt[4]{\Lambda^2}\exp\left(-\frac{u^2}{2\sigma^2}-\sqrt{\frac{\varkappa\,|u|}{\sigma^2}}\right).\tag{122}$$

Two or three terms of the representations (115) and (121) make it possible to calculate the spectrum $N(z,u)$ in the complete range of the variable u.

Thus, the energy spectrum of fast heavy charged particles deep in matter can be described by the fairly simple analytic expressions (115)–(119) and (121), which are valid for all energies of the particles incident on the surface of the matter. These expressions determine the energy distribution of the particles in a wide depth range, bounded, on the one hand, by the region of thin layers, in which the energy spread of the particles is comparable with the maximal energy transferred in one inelastic collision, and, on the other, by the region of depths in which the particles are brought to rest after their slowing down and the energy spectrum no longer has a well-defined maximum.

The conservation of the total particle flux in this interval of depths $[1-\xi \gtrsim \max(\gamma, 1/\sqrt{\nu})]$ is reflected in the normalization condition

$$\int_{-\infty}^{\infty} du N(z, u) = 1, \tag{123}$$

which is satisfied by the distributions (115) and (121).

The contributions of the fluctuations of the energy losses in inelastic collisions and of the fluctuations of the losses due to the bending of the paths by multiple elastic scattering to the energy distribution of the particles are characterized by $\sqrt{\sigma^2(z)}$ and $\varkappa(z)$, respectively. Depending on the ratio of these quantities, the one or the other process will be dominant.

The importance of multiple elastic scattering in forming the energy spread of the particles increases with increasing depths: $\varkappa(z)/\sqrt{\sigma^2(z)} \gtrsim \gamma\sqrt{\nu}\,(z/R_0)^{3/2}$. In view of this, it is particularly important to take into account the bending of the paths when the distribution of the stopped particles is calculated.

(c) Dependence of the energy losses on the displacement of the particles relative to the beam axis and the direction of their motion

We considered above the pure energy spectrum of the particles without regard to their position in space and the direction of their motion. It is, however, of particular interest to investigate the energy spread of the particles as a function of their displacement relative to the beam axis and the scattering angle or, which is the same thing, the spatial and angular distributions of particles of a given energy.

The analysis of this question is all the more important in view of the fact that hitherto information about the distribution of particles in matter could be obtained only on the basis of laborious Monte Carlo calculations.[6,10] Analytic results were exhausted by investigation of the angular spectrum and were obtained either without allowance for the fluctuations of the energy losses in the inelastic collisions[61] or with neglect of the bending of the paths by multiple elastic scattering.[28] The spatial distribution of particles of a given energy has not hitherto been investigated at all.

Integrating the relation (69) over the transverse displacement relative to the beam axis or over the angles, we obtain, respectively, expressions for the angular and energy distributions and the spatial–energy spectrum of particles that have passed through a layer of matter:

$$N(z, \theta, u) = -\frac{N_0}{2\pi^2}\frac{\partial}{\partial(\theta^2)}\int_{-\infty}^{\infty} d\omega\, \frac{\exp\left[i\omega u - \frac{1}{2}\omega^2\sigma^2(z) - \theta^2/A_1(\omega_1 z)\right]}{A_0(\omega, z)}; \tag{124}$$

$$N(z, \rho, u) = -\frac{N_0}{2\pi^2}\frac{\partial}{\partial(\rho^2)}\int_{-\infty}^{\infty} d\omega\, \frac{\exp\left[i\omega u - \frac{1}{2}\omega^2\sigma^2(z) - \rho^2/A_3(\omega, z)\right]}{A_0(\omega, z)}, \tag{125}$$

where A_1 and A_3 are determined by (63) and (65), respectively.

In the region of relatively small energy losses, the integrals (124) and (125) reduce to simple analytic expressions.

Using the asymptotic representations of the functions A_0, A_1, and A_3 at large ω,

$$A_0(\omega, z) \simeq \frac{1}{2\sqrt[4]{\Lambda^2(z)}} \exp\sqrt{i\omega\varkappa(z)};$$
$$A_1(\omega, z) \simeq \frac{2}{\sqrt{i\omega}}\left(\sqrt{\frac{\langle\theta_s^2(z)\rangle}{2}} - \frac{1}{4\sqrt{i\omega}}\frac{d}{dz}\ln\langle\theta_s^2(z)\rangle\right);$$
$$A_3(\omega, z) \simeq \frac{2z}{i\omega} \tag{126}$$

and calculating the integrals (124) and (125) by the method of steepest descent, we find

$$N(z, \theta, u) \simeq \frac{N_0}{\pi\sqrt{2\langle\theta_s^2\rangle}}\left(\frac{2\sqrt{\Lambda^2}}{\pi\sigma^2}\right)^{1/2}\left(\frac{\sqrt{\varkappa}+\theta^2/\sqrt{2\langle\theta_s^2\rangle}}{4\sigma^2}\right)^{1/3}$$
$$\times\exp\left(\frac{\theta^2}{4\langle\theta_s^2\rangle}\frac{d}{dz}\ln\langle\theta_s^2\rangle\right) h\left\{\frac{u}{[(\sqrt{\varkappa}+\theta^2/\sqrt{2\langle\theta_s^2\rangle})^2\sigma^2]^{1/3}}\right\}$$
$$\times f\left\{\frac{(\sqrt{\varkappa}+\theta^2/\sqrt{2\langle\theta_s^2\rangle})^2}{\sqrt{\sigma^2}};\ \frac{u}{[(\sqrt{\varkappa}+\theta^2/\sqrt{2\langle\theta_s^2\rangle})^2\sigma^2]^{1/3}}\right\}; \tag{127}$$

$$N(z, \rho, u) \simeq \frac{N_0}{2\pi z}\left(\frac{2\sqrt{\Lambda^2}}{\pi\sigma^2}\right)^{1/2}\left(\frac{\varkappa}{16\sigma^4}\right)^{1/3} h^2\left[\frac{u-\rho^2/2z}{(\varkappa\sigma^2)^{1/3}}\right]$$
$$\times f\left[\frac{\varkappa}{\sqrt{\sigma^2}};\ \frac{u-\rho^2/2z}{(\varkappa\sigma^2)^{1/3}}\right]. \tag{128}$$

Like (127) and (128), we can readily obtain relations for the mean squares of the scattering angle and transverse displacement relative to the beam axis, respectively:

$$\langle\theta^2\rangle_{z,u} \simeq \sqrt{2\langle\theta_s^2(z)\rangle}\left(\frac{16\sigma^4}{\varkappa}\right)^{1/6} h^{-1}\left\{\frac{u}{[\varkappa(z)\,\sigma^2(z)]^{1/3}}\right\}; \tag{129}$$

$$\langle\rho^2\rangle_{z,u} \simeq 2z\left(\frac{16\sigma^4}{\varkappa}\right)^{1/3} h^{-2}\left\{\frac{u}{[\varkappa(z)\,\sigma^2(z)]^{1/3}}\right\}. \tag{130}$$

The functions $f(x; y)$ and $h(y)$ in (127)–(130) are determined by

$$f(x; y) = [1 + h^{-3}(y)]^{-1/2}\exp[-x^{2/3}g(y)]; \tag{131}$$

$$h(y) = \left[\left(1+\sqrt{1+\frac{16}{27}y^3}\right)^{1/3} + \left(1-\sqrt{1+\frac{16}{27}y^3}\right)^{1/3}\right]; \tag{132}$$

$$g(y) = 3\cdot 2^{-8/3}h(y) - 2^{-7/3}yh^2(y). \tag{133}$$

As follows from the expressions (129) and (130), the angular and spatial spread of the particles also decreases with decreasing energy that is lost:

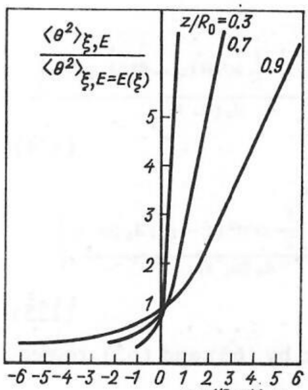

FIG. 7. Dependence of the angular spread of nonrelativistic particles ($E_0 \ll 1$) that have passed through a layer of matter of thickness z on their energy.

$$\langle\theta^2\rangle_{z,u} \simeq \sqrt{2\langle\theta_s^2(z)\rangle}\left(\frac{\sigma^2}{|u|}\right)^{1/2}, \quad \langle\rho^2\rangle_{z,u} \simeq 2z\left(\frac{\sigma^2}{|u|}\right)$$
$$(u<0,\ |u| \gg (\varkappa\sigma^2)^{1/3}).$$

This feature of the distribution of particles that have traversed a layer of matter (a kind of "focusing") is due to the fact that particles that have been strongly scattered and deflected from the beam axis traverse a greater distance in the matter and lose more energy than particles that move basically straight forward (i.e., are scattered at very small angles and are hardly deflected from the beam axis). Because of this, we find in the high-energy part of the spectrum mainly particles that have not been too strongly deflected from the direction of the initial motion and have been displaced little relative to the beam axis. Figure 7 shows the dependence of the mean square of the scattering angle of particles that have passed through a layer of matter of given thickness on their energy. Since with increasing layer thickness the energy spread of the particles increases, the dependence of $\langle\theta^2\rangle_{z,u}$ ... the energy becomes smoother and smoother. Precisely the same features of the angular spectrum are observed experimentally.[59]

Note that if in the relation (127) we ignore the probabilistic nature of the inelastic collisions ($\sigma^2 \to 0$), then we arrive at the result of Ref. 61.

The expressions (127) and (128) indicate that with increasing distance from the beam axis and increasing deflection angle of the particles from the direction of the original motion the energy spectrum is displaced into the region of greater energy losses and becomes broader.

This feature of the energy spectrum is also confirmed by the expressions for the mean ranges.

Proceeding directly from the general relations (124) and (125) and Eq. (68), we can obtain expressions for the mean residual ranges of particles that have traversed a layer of matter of thickness z and have been deflected through the distance ρ from the beam axis or move at angle θ relative to the direction of the initial motion. For this, there is no need to solve Eq. (68); it is sufficient to determine only the values of the derivatives $\partial A_0/\partial\omega|_{\omega=0}$ and $\partial^2 A_0/\partial\omega^2|_{\omega=0}$. After simple calculations, we obtain

$$\langle R\rangle_{z,\rho} = R_0 - z - \frac{1}{2}\int_0^z (z-z')\langle\theta_s^2(z')\rangle\,dz'$$
$$-\frac{1}{2}\,\frac{\int_0^z dz'\left[\int_0^{z'} dz''(z-z'')\langle\theta_s^2(z'')\rangle\right]^2}{\left[\int_0^z dz'(z-z')^2\langle\theta_s^2(z')\rangle\right]^2}\left[\rho^2 - \int_0^z dz'(z-z')^2\langle\theta_s^2(z')\rangle\right]; \quad (134)$$

$$\langle R\rangle_{z,\theta} = R_0 - z - \frac{1}{2}\int_0^z (z-z')\langle\theta_s^2(z')\rangle\,dz'$$
$$-\frac{1}{2}\,\frac{\int_0^z dz'\left[\int_0^{z'} dz''\langle\theta_s^2(z'')\rangle\right]^2}{\left[\int_0^z dz'\langle\theta_s^2(z')\rangle\right]^2}\left[\theta^2 - \int_0^z dz'\langle\theta_s^2(z')\rangle\right]. \quad (135)$$

It follows from (134) and (135) that with increasing ρ and θ the energy spectrum is displaced in the direction of greater energy losses (smaller residual ranges). At the same time, if

$$\rho^2 < \langle\rho^2\rangle_z = \int_0^z dz'(z-z')^2\langle\theta_s^2(z')\rangle,$$

where $\langle\rho^2\rangle_z$ is the mean square of the transverse displacement of the particles at depth z independently of their energy,[17] then the mean residual range $\langle R\rangle_{z,\rho}$ is greater than the mean range of the particles at depth z without regard to their position in space:

$$\langle R\rangle_z = R_0 - z - \frac{1}{2}\int_0^z (z-z')\langle\theta_s^2(z')\rangle\,dz'.$$

But if $\rho^2 > \langle\rho^2\rangle_z$, then $\langle R\rangle_{z,\rho} < \langle R\rangle_z$. The same can be said of the variation of the energy spectrum with increasing angle θ.

Note that for the variance of the ranges $\langle(R-\langle R\rangle_z)^2\rangle_z$ we obtain from (113) and (68) an expression identical to Pomeranchuk's well-known result[56]:

$$\langle(R-\langle R_z\rangle)^2\rangle_z = \frac{1}{2}\int_0^z (z-z')\left[\int_0^{z'} dz''\langle\theta_s^2(z'')\rangle\right]^2 dz' + \sigma^2(z). \quad (136)$$

For the complete spatial, angular, and energy distribution (69) we can without great difficulty obtain relations analogous to (127)–(130), and we can also calculate the mean range $\langle R\rangle_{z,\rho,\theta}$. In addition, following the procedure described above, we can readily determine the mean ranges of the particles and the variances of the ranges for different spatial and angular parameters of the beam and the detector. However, the corresponding expressions are rather cumbersome and are therefore omitted.

In conclusion, we emphasize that all the features of the spatial and angular distributions of the particles mentioned above are due to the combined effect of multiple elastic scattering and the energy losses, this being taken into account in the original equation (54) by the term $[(\theta_x^2 + \theta_y^2)/2](\partial N/\partial u)$. But if we ignore this term, then the process of energy loss is independent of the multiple elastic scattering. Then the energy and spatial–angular distributions effectively separate, and all the features formed in the angular spectrum of the particles are due solely to the relatively smooth dependence of $\langle\theta_s^2\rangle$ on the particle energy.[28]

2. DISTRIBUTION OF THE STOPPED PARTICLES

The expressions (69), (71)–(73), (113)–(115), (121), (124), and (125) describe the distribution of the particles with respect to the ranges (energies) in a wide range of depths z in which there is almost no stopping of the particles and the energy spectrum has a well-defined maximum. The stopping of the particles occurs mainly in the short final section of the path, whose length $\delta z \sim R_0 \max(\gamma, 1/\sqrt{\nu})$ is only a small fraction of the total range.[17] This circumstance makes it possible when calculating the distribution of the stopped particles to use the relation obtained above for the energy spectrum.

Since the values of γ and $1/\sqrt{\nu}$ are small, it is obvious in advance that the entire spread of the particle ranges is formed at depths $R_0 - z \gg R_0 \max(\gamma, 1/\sqrt{\nu})$ and at $z \sim R_0 - R_0 \max(\gamma, 1/\sqrt{\nu})$ is already so great that the fluctuations over the section $\delta z \sim R_0 \max(\gamma, 1/\sqrt{\nu})$ hardly change it. Therefore, the distribution of the stopped particles can be determined by extrapolating the expression (125) to $R(T) = 0$.

Following the method which we have described, we can readily obtain from (125) the complete spatial distribution $W(z,\rho,T_0)$ of the stopped particles[32] [$W(z,\rho,T_0)d^2\rho dz$ is the probability that a particle with initial energy T_0 "gets stuck" in the volume element $2\pi\rho d\rho dz$ at depth z at distance ρ from the beam axis]:

$$W(z, \rho, T_0) = -\frac{1}{2\pi^2}\frac{\partial}{\partial(\rho^2)}\int_{-\infty}^{\infty} d\omega \frac{\exp\left[i\omega(R_0 - z) - \frac{\rho^2}{A_3(\omega, R_0)} - \tau(\omega, R_0)\right]}{A_0(\omega, R_0)}. \quad (137)$$

The function $\tau(\omega, R_0)$ in (137) is determined by the expression

$$\tau(\omega, R_0) \equiv \tau(\omega, T_0) = -\sum_{k=2}^{\infty}\frac{(-i\omega)^k}{k!}\int_0^{T_0} dT \frac{\overline{\varepsilon^k}(T)}{[\bar{\varepsilon}(T)]^{k+1}}. \quad (138)$$

If we restrict ourselves to the Fokker–Planck approximation, then

$$\tau(\omega, T_0) = \frac{1}{2}\omega^2\sigma_0^2; \quad \sigma_0^2 = \int_0^{T_0} dT \frac{\overline{\varepsilon^2}(T)}{[\bar{\varepsilon}(T)]^3}. \quad (139)$$

The distribution (137) satisfies the normalization condition

$$\int 2\pi\rho \, d\rho \, dz W(z, \rho, T_0) = 1. \quad (140)$$

Integrating the relation (137) over the transverse displacement from the beam axis, we find the distribution of the stopped particles with respect to the penetration depths in matter[31]:

$$W(z, T_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{\exp[i\omega(R_0 - z) - \tau(\omega, R_0)]}{A_0(\omega, R_0)}$$

$$= \int_0^{R_0} dz' \, W_{\text{inel}}(z + R_0 - z', R_0) \, W_{\text{el}}(z', R_0), \quad (141)$$

where

$$W_{\text{inel}}(z, R_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \exp[i\omega(R_0 - z) - \tau(\omega, R_0)]; \quad (142)$$

$$W_{\text{el}}(z, R_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{\exp[i\omega(R_0 - z)]}{A_0(\omega, R_0)}. \quad (143)$$

Ignoring the dependence of L_{ion} on the energy, we can represent W_{el} in the form[31]

$$W_{\text{el}}(z, E_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{\exp[i\omega(R_0 - z)]}{P_{\mu(\omega\gamma R_0)}[\text{ch}\ln(1 + E_0)]}. \quad (144)$$

The value of the degree $\mu(\omega)$ is determined in accordance with (85).

The expressions (137) and (141) describe the distribution of the stopped particles for a very wide range of energies of the incident particles; the range is bounded below by the energies corresponding to the velocities of the atomic electrons and above by the region of superhigh energies ($E_0 \ll m/m_e$).

If multiple elastic scattering is ignored ($\gamma \to 0$), then the spread of the stopped particles with respect to the depth will be due solely to the probabilistic nature of the energy transfer in inelastic collisions, and W reduces to W_{inel}. For heavy charged particles, the approximation (139) works to a good accuracy, and this leads to Bohr's well-known formula[50]

$$W_{\text{inel}}(z) = \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{(R_0 - z)^2}{2\sigma_0^2}} \quad (145)$$

The corrections to the expression (145) can be readily estimated by using the distribution (35).[h)] In the nonrelativistic region of energies they are small; it is only sensible to take into account the corrections to (145) for relatively light particles (muons and pions) in light media. In the ultrarelativistic region, the accuracy of Bohr's formula decreases somewhat with increasing energy.[29] In the following calculations, we use for $\tau(\omega, T_0)$ the approximation (139).

We now analyze the distribution (141) for different initial energies of the incident particles.

In the case of nonrelativistic energies ($E_0 \ll 1$), using the asymptotic representation of the Legendre functions of the first kind at large values of μ, we obtain

$$W(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{\exp\left[i\omega(R_0 - z) - \frac{\omega^2\sigma_0^2}{2}\right]}{I_0(2\sqrt{i\omega\gamma R_0})}. \quad (146)$$

The distribution (146) can be represented in the form of the series[31]

$$W(z) = \frac{1}{4\gamma R_0}\exp\left[-\frac{(R_0 - z)^2}{2\sigma_0^2}\right]\sum_{k=1}^{\infty}\frac{q_k}{J_1(q_k)}$$

$$\times \exp[\eta_k^2(z)]\{1 - \text{erf}[\eta_k(z)]\}, \quad (147)$$

where

$$\eta_k(z) = \sqrt{\frac{\sigma_0^2}{2}}\left(\frac{q_k^2}{4\gamma R_0} - \frac{R_0 - z}{\sigma_0^2}\right); \quad (148)$$

q_k are the zeros of the Bessel function of zeroth order.[51]

At large depths ($z - R_0 \gtrsim (R_0/\sqrt{\nu})(1 - 1/\gamma\sqrt{\nu})$) the convergence of the series (147) is not sufficiently rapid, and it is not very convenient to use the representation (147) directly. In this case, taking into account the asymptotic expression for the error function and the formula[64]

$$2\sum_{k=1}^{\infty} \frac{q_k}{(q_k^2 - x^2) J_1(q_k)} = \frac{1}{J_0(x)},$$

we can transform the relation (147) to the form

$$W(z) = \frac{1}{4\gamma R_0} \exp\left[-\frac{(R_0 - z)^2}{2\sigma_0^2}\right] \left\{ \frac{4\gamma R_0}{\sqrt{2\pi\sigma_0^2}\, I_0\left(\sqrt{\frac{4\gamma R_0 (R_0 - z)}{\sigma_0^2}}\right)} + \sum_{k=1}^{\infty} \frac{q_k}{J_1(q_k)} \left[\exp \eta_k^2 (1 - \mathrm{erf}\, \eta_k) - \frac{1}{\sqrt{\pi}\, \eta_k}\right] \right\}. \quad (149)$$

For fast heavy charged particles, the contribution to the distribution of the stopped particles from the fluctuations of the energy losses in inelastic collisions and from the bending of the particle paths by elastic scattering is characterized by

$$\gamma\sqrt{\nu} = 2\sqrt{\frac{m_e}{m}(Z+1)}\left[\frac{1+E_0}{(2+E_0)^2}\right]^{3/2}\sqrt{\frac{L_k^2}{L_{\rm ion}}}. \quad (150)$$

If $\gamma\sqrt{\nu} \gg 1$, then one can ignore the statistical nature of the inelastic collisions. Conversely, in the case $\gamma\sqrt{\nu} \ll 1$ one can ignore the bending of the particle paths. When $\gamma\sqrt{\nu} \sim 1$, the contribution of the two processes to the particle distribution are of the same order.

For nonrelativistic particles, the two probabilistic processes are equally important in forming the distribution of the stopped particles when

$$Z \sim 4\sqrt{\frac{m}{m_e}}\sqrt{\frac{L_{\rm ion}}{L_k^2}}. \quad (151)$$

Therefore, for the slowing down of nonrelativistic muons the influence of multiple elastic scattering on the spread of the particles with respect to the penetration depths in the matter is important for the majority of elements of the periodic table ($Z \gtrsim 10$–20). In substances with atomic number $Z \gtrsim 50$–60 the effect of the bending of the trajectory is dominant. For protons, allowance for the fluctuations of the paths of the particles due to multiple elastic scattering is important above all in heavy media ($Z \gtrsim 60$–70).

In the case of ultrarelativistic energies ($E_0 \gg 1$), the value of $\gamma\sqrt{\nu}$ is

$$\gamma\sqrt{\nu} \simeq 2\sqrt{\frac{m_e}{m}(Z+1)}\, E_0^{-3/2} \sqrt{\frac{L_k^2}{L_{\rm ion}}} \quad (152)$$

and, in contrast to the nonrelativistic case, depends strongly on the energy of the incident particles and decreases fairly rapidly with increasing E_0. Therefore, even for muons the bending of the trajectory need be taken into account only in

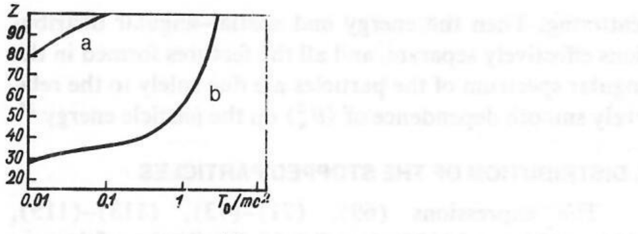

FIG. 8. Diagram of $\gamma\sqrt{\nu} = 1$: a) protons ($m/m_e \simeq 1836$); b) muons ($m/m_e \simeq 207$).

sufficiently heavy media ($Z \gtrsim 3E_0^{3/2}$) and at not too high energies ($E_0 \lesssim 10$).

In the region of ultrarelativistic energies, $\gamma\sqrt{\nu} \ll 1$ in media with any Z, and the main contribution to the integral (141) is made by relatively small ω such that $|\gamma R_0 \omega_{\rm eff}| \ll 1$. This circumstance makes it possible to represent μ in the form $\mu \simeq i\omega\gamma R_0 + \omega^2\gamma^2 R_0^2$ and, taking into account the asymptotic relation for a Legendre function of the first kind at large values of the argument,[64]

$$P_\mu(t) \simeq \frac{(2t)^\mu\, \Gamma(\mu + 1/2)}{\sqrt{\pi}\, \Gamma(\mu+1)} \left(\mathrm{Re}\, \mu > -\frac{1}{2}\right),$$

we find

$$W(z) \simeq \frac{1}{\sqrt{2\pi(\sigma_0^2 + 2\gamma^2 R_0^2 \ln E_0)}} \exp\left[-\frac{(R_0 - z - \gamma R_0 \ln E_0)^2}{2(\sigma_0^2 + 2\gamma^2 R_0^2 \ln E_0)}\right]. \quad (153)$$

Thus, in the region of ultrarelativistic energies allowance for multiple elastic scattering leads in the first place to a decrease in the mean depth of penetration of the particles into the medium. However the form of the actual distribution is Gaussian, this corresponding to a dominant role of the fluctuations of the energy losses in inelastic collisions in forming the spread of the stopped particles with respect to the depth.

Figure 8 is a diagram which shows which of the probabilistic processes will make the dominant contribution to the distribution of the stopped particles (protons and muons) with respect to the penetration depths in matter with atomic number Z as a function of the initial particle energy T_0. In regions lying above the curves, the influence of multiple elastic scattering on the spread of the particles is dominant.

In the region of intermediate energies ($E_0 \sim 1$), where the results (147)–(149) and (153) do not hold, to calculate the distribution $W(z)$ it is convenient to use the semiclassical solution (88)–(89) of Eq. (68). However, whereas in the calculation of the energy spectrum (88) it was sufficient to make a restriction to the case $n = 0$, it is now necessary to use the solution with $n = -1$. The point is that such a solution will describe more accurately the elastic scattering in the region of large energy losses, when $\Lambda^2(z) \sim (1 - z/R_0)^{-1}$. Using (88) with $n = -1$, for $A_0(\omega, R_0)$ we find

$$A_0(\omega, R_0) \simeq \sqrt{\frac{\ln(1+E_0)}{\mathrm{sh}\,\ln(1+E_0)}}\, I_0\left(\sqrt{i\omega\varkappa_0}\right), \quad (154)$$

where $\varkappa_0 \equiv \varkappa(z = R_0)$ is determined by the expression[i)]

$$\varkappa_0 = \gamma R_0 b^2 \ln^2(1 + E_0). \quad (155)$$

The semiclassical approximate solution (154) works well for sufficiently large ω, the asymptotic behaviors of (154) and the exact solution (84) (for $t=1$) being identical. In addition, the semiclassical solution does not differ at all from the exact solution for nonrelativistic energies ($E_0 \ll 1$). However, in the region of small ω corresponding to the tail of the distribution the expression (154) leads to overestimated values of $W(z)$ for $E_0 \gtrsim 1$. To eliminate this shortcoming, we can proceed in the same way as in the calculation of the distributions (108) and (110), i.e., one can represent the function $A_0(\omega, R_0)$ in the form

$$A_0(\omega,\ R_0)=\sqrt{\frac{\ln(1+E_0)}{\operatorname{sh}\ln(1+E_0)}}\, I_0\left(\sqrt{i\omega\varkappa_0+\delta_0}\right) \tag{156}$$

and determine the value $\delta_0=\delta(E_0)$ from the normalization condition (140). The result is

$$I_0\left(\sqrt{\delta_0}\right)=\left[\frac{\operatorname{sh}\ln(1+E_0)}{\ln(1+E_0)}\right]^{1/2}. \tag{157}$$

It follows from this equation that for nonrelativistic particles, naturally, $\delta_0=0$, and for ultrarelativistic particles

$$\delta_0 \simeq \left[\frac{1}{2}\ln(1+E_0)+\frac{1}{4}\right]^2.$$

In the limit $\gamma\sqrt{\nu}\ll 1$, (156) leads with allowance for this last relation to the expression (153).

Using the expression (156), we can write the distribution for the stopped particles with respect to the penetration depths in matter for arbitrary energy of the incident particles in the form

$$W(z)=\frac{1}{\varkappa_0}\sqrt{\frac{\operatorname{sh}\ln(1+E_0)}{\ln(1+E_0)}}\exp\left[-\frac{(R_0-z)^2}{2\sigma_0^2}\right] \times\sum_{k=1}^{\infty}\frac{q_k}{J_1(q_k)}\exp\left[\eta_k^2(z,\ R_0)\right]\{1-\operatorname{erf}[\eta_k(z,\ R_0)],\tag{158}$$

where q_k are the zeros of the Bessel function of zeroth order, and

$$\eta_k(z,\ R_0)=\sqrt{\frac{\sigma_0^2}{2}}\left(\frac{q_k^2+\delta_0}{\varkappa_0}-\frac{R_0-z}{\sigma_0^2}\right). \tag{159}$$

The expression which we have obtained describes the spread of the stopped particles with respect to the depth due to multiple Coulomb scattering and the fluctuations of the energy losses in inelastic collisions with atoms in the complete range of energies of the incident particles in which these processes are dominant. In the cases of nonrelativistic and ultrarelativistic energies the relations already obtained [(147) and (153), respectively] follow from (158).

At large depths $\{z-R_0 \gtrsim (R_0/\sqrt{\nu})\,[1-(1/\gamma\sqrt{\nu})]\}$, where the series (158) does not converge sufficiently rapidly, it can be readily transformed to a form analogous to (149).

In the region $R_0-z \lesssim \gamma R_0(1-1/\nu\gamma^2)$, using the asymptotic value of the function A_0 at large ω, we can obtain for the distribution $W(z)$

$$W(z)\simeq\left(\frac{\varkappa_0}{2\sigma_0^4}\right)^{1/3}\left(\frac{\operatorname{sh}\ln(1+E_0)}{\ln(1+E_0)}\right)^{1/2} \times\exp\left[-\frac{R_0-z-(\sigma_0^2/2\varkappa_0)\,\delta_0}{\varkappa_0}\delta_0\right] \times f\left[\frac{\varkappa_0}{\sqrt{\sigma_0^2}};\ \frac{R_0-z-(\sigma_0^2/\varkappa_0)\,\delta_0}{(\varkappa_0\sigma_0^2)^{1/3}}\right]h^{1/2}\left[\frac{R_0-z-(\sigma_0^2/\varkappa_0)\,\delta_0}{(\varkappa_0\sigma_0^2)^{1/3}}\right], \tag{160}$$

where the functions $f(x;y)$ and $h(y)$ are determined by the relations (131)–(133).

If in the expression (160) we set $\sigma_0^2=0$, i.e., we ignore the probabilistic nature of the inelastic collisions, then in the nonrelativistic case we arrive at the equation

$$W_{\text{el}}\ (z)\simeq\frac{1}{4}\left(\frac{\varkappa_0}{R_0-z}\right)^2 e^{-\frac{1}{4}\frac{\varkappa_0}{(R_0-z)}}, \tag{161}$$

which is a more accurate form of Spencer's well-known result[67] (in Ref. 67, the pre-exponential factor was incorrectly calculated).

The approximate formula (160) can be used in the case $\gamma\sqrt{\nu}\gtrsim 1$, i.e., when the part played by the fluctuations of the particle paths due to multiple scattering is important. Figure 9 shows the distribution of stopped muons with initial energy $E_0=1$ ($T_0=105.7$ MeV) in lead ($\gamma\sqrt{\nu}\simeq 2$). It can be seen that the relation (160) works well in the region of relatively large depths, including the most probable depth of penetration of the particles in the matter.

Until recently, the only serious analytic result permitting calculation of the contribution of the elastic scattering to the spread of the stopped particles with respect to the penetration depths in matter was Yang's distribution as modified by Berger and Seltzer[6] to take into account the energy losses (see also Refs. 17, 24, and 25). To obtain the distribution of the stopped particles on the basis of the modified Yang formula, it is necessary to substitute in (143) a function A_0 of the form

$$A_0(\omega,\ R_0)=\operatorname{ch}\sqrt{2i\omega\langle R_0-z\rangle}, \tag{162}$$

where $\langle R_0-z\rangle$ is determined from Pomeranchuk's formula[56]

$$\langle R_0-z\rangle=\frac{1}{2}\int_0^{R_0}(R_0-z)\,\langle\theta_s^2(z)\rangle\,dz \tag{163}$$

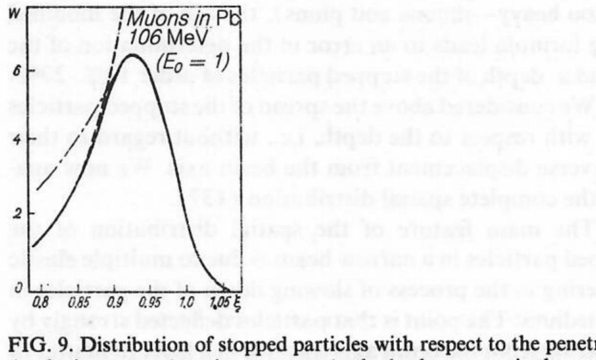

FIG. 9. Distribution of stopped particles with respect to the penetration depths in matter (continuous curve); the broken curve is the result of calculation on the basis of the first term of the series (158); the chain curve is the result of the calculation in accordance with (160).

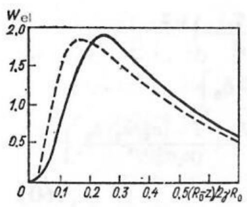

FIG. 10. The distribution $W_{el}(z)$ for nonrelativistic particles. The continuous curve is the result of calculation in accordance with (144), and the broken curve is calculated in accordance with the Yang distribution modified on the basis of Pomeranchuk's formula.

or the formula of Lewis (Refs. 6 and 68)[j]

$$\langle R_0 - z\rangle = \int_0^{R_0} dz\left[1-\exp\left(-\frac{1}{2}\int_0^z \langle\theta_s^2(z')\rangle\, dz'\right)\right]. \qquad (164)$$

In (98) one can also simply set $R(T)=0$ $(S=R_0)$.

Although the modified Yang spectrum correctly describes the first moment of the distribution, i.e., $\langle R_0 - z\rangle$ (which is as correct as the relations of Pomeranchuk and Lewis are exact), it reproduces the form of the actual distribution W_{el} only approximately (Fig. 10). Comparison shows that use of the function (162) overestimates the variance of the distribution $W_{el}(z)$. For example, in the region of nonrelativistic energies the expression (144) and Pomeranchuk's calculations[56] give $\langle (z-\langle z\rangle)^2\rangle = \frac{1}{2}\gamma^2 R_0^2$, whereas the Yang distribution modified on the basis of (162) and (163) gives $(z-\langle z\rangle^2) = 2\gamma^2 R_0^2/3$. Thus, the contribution of the elastic scattering to the variance of the distribution of the stopped particles with respect to the depths is overestimated by Berger and Seltzer's result by a factor 1.3. This is also confirmed by comparison with the data of numerical Monte Carlo calculations.

Figure 11 shows the particle depth distributions calculated without allowance for the fluctuations of the losses in the inelastic collisions on the basis of the relations (143) and (156) and on the basis of the modified Yang distribution (143), (162), (164) (Ref. 6), and the data of numerical computer simulation.[6] It can be seen that the result (143), (156) agrees with the calculations by the Monte Carlo method much better than the Yang distribution.

In the situation when the part played by elastic scattering is appreciable (media with large Z and particles that are not too heavy—muons and pions), the use of the modified Yang formula leads to an error in the determination of the spread in depth of the stopped particles of order 10%–20%.

We considered above the spread of the stopped particles only with respect to the depth, i.e., without regard to their transverse displacement from the beam axis. We now analyze the complete spatial distribution (137).

The main feature of the spatial distribution of the stopped particles in a narrow beam is due to multiple elastic scattering in the process of slowing down of the particles in the medium. The point is that particles deflected strongly by scattering from the beam axis traverse in a layer of matter of given thickness a greater path than particles which move along the axis. Therefore, they lose more energy and are

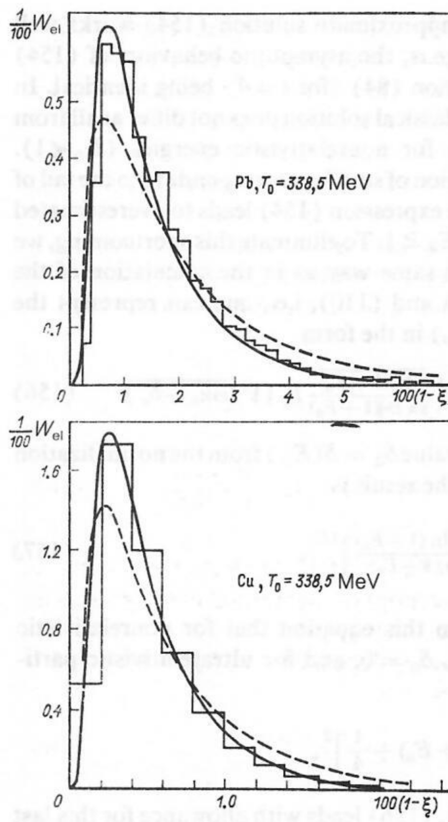

FIG. 11. Distribution $W_{el}(z)$ of particles with respect to the penetration depths based on numerical computer simulation[6] and calculation on the basis of the relations (143) and (156) (continuous curve) and (143), (162), and (164)[6] (broken curve).

stopped at shorter depths. As a result, it is the particles deflected little from the direction of the initial motion, i.e., the ones that move mainly "straight forward," that penetrate with large probability to appreciable depths. This leads to a decrease in the spread of the particles with respect to the transverse displacement relative to the beam axis with increasing depth or, and this is the same thing, to a decrease in the penetration depth of particles in the matter with increasing distance from the beam axis.

In the region of relatively large depths $R_0 - z - \rho^2/2R_0 \lesssim \gamma R_0(1-1/\gamma^2\nu)$, using for A_0 and A_3 the asymptotic representations at large ω, we find[32]

$$W(z,\rho)\simeq \frac{1}{\pi R_0}\left(\frac{\varkappa_0}{16\sigma_0^4}\right)^{2/3}\left(\frac{\mathrm{sh}\,\ln(1+E_0)}{\ln(1+E_0)}\right)^{1/2} \times h^{5/2}\left[\frac{R_0-z-\rho^2/2R_0}{(\varkappa_0\sigma_0^2)^{1/3}}\right] f\left[\frac{\varkappa_0}{\sqrt{\sigma_0^2}};\ \frac{R_0-z-\rho^2/2R_0}{(\varkappa_0\sigma_0^2)^{1/3}}\right]. \qquad (165)$$

The expression (165) makes it possible to estimate the spread of the stopped particles with respect to the transverse displacement from the beam axis[32]:

$$\langle\rho^2\rangle_z \simeq 2R_0\left(\frac{16\sigma_0^4}{\varkappa_0}\right)^{1/3} h^{-2}\left[\frac{R_0-z}{(\varkappa_0\sigma_0^2)^{1/3}}\right]. \qquad (166)$$

As follows from this relation, $\langle\rho^2\rangle_z$ decreases with increasing depth and for $z - R_0 \gg (\varkappa_0\sigma_0^2)^{1/3}$ is equal to

$$\langle\rho^2\rangle_z \simeq 2\sigma_0^2/(z-R_0).$$

We emphasize that allowance for the deviation from rectilinearity of the motion of the particles as they are decelerated in the matter leads to a dependence of the penetration depths of the particles on their transverse displacement from the beam axis, the strongly deflected particles being stopped at smaller depths. If the bending of the trajectory is not taken into account, there is no such dependence.[17]

The method of calculating the mean values presented in subsection (c) makes it possible to find the dependence of the mean penetration depths on the displacement of a particle from the beam axis:

$$\langle z\rangle_\rho = R_0 - \frac{1}{2}\int_0^{R_0} dz\,(R_0 - z)\,\langle\theta_s^2(z)\rangle - \frac{1}{2}\int_0^{R_0} dz\left[\int_0^z dz'\,(R_0 - z')\,\langle\theta_s^2(z')\rangle\right]^2 \frac{\rho^2 - \langle\rho^2\rangle}{\langle\rho^2\rangle^2}, \qquad (167)$$

where $\langle\rho^2\rangle = \int_0^{R_0} dz(R_0 - z)^2\langle\theta_s^2(z)\rangle$ is the mean square of the transverse displacement of the stopped particles without relation to their penetration depth.[17]

Ignoring the energy dependence of L_{ion} $[\Lambda^2(\xi) \simeq b^2/(1-\xi)(b^2-\xi)]$, we find

$$\langle z\rangle_\rho = R_0(1 - \gamma\alpha_1) - \alpha_2\rho^2/R_0, \qquad (168)$$

where

$$\left.\begin{aligned} \alpha_1 &= b^2\left[\ln\frac{b^2}{b^2-1}\left(1-(b^2-1)\ln\frac{b^2}{b^2-1}\right)^{-1} - 2\right];\\ \alpha_2 &= \frac{1}{2}\,\frac{2-(b^2-1)\ln\frac{b^2}{b^2-1}\left(2+\ln\frac{b^2}{b^2-1}\right)}{\left[1-(b^2-1)\ln\frac{b^2}{b^2-1}\right]^2}. \end{aligned}\right\} \qquad (169)$$

In the special case of nonrelativistic energies ($E_0 \ll 1$), we obtain from (168)

$$\langle z\rangle_\rho = R_0\left(1 - \frac{1}{3}\gamma\right) - \frac{2}{3}\,\frac{\rho^2}{R_0}. \qquad (170)$$

The relation (170) is actually the equation of the surface (a paraboloid of revolution) on which the density of the stopped particles of a narrow beam passing through matter is maximal.

3. INFLUENCE OF NUCLEAR INTERACTIONS ON THE PENETRATION OF FAST CHARGED PARTICLES IN MATTER

Above, when we calculated the particle distribution, we did not take into account the possibility of their decay or nuclear interaction in collisions with nuclei of the atoms in the medium. In the investigated range of energies, such a treatment is always justified for the description of the slowing down of muons in condensed media (in solids and liquids, the muon decays after it has been stopped[15]) and strongly interacting particles (protons, α particles, etc.) at not too high energies, for which the total range R_0 of the particles is appreciably less than the nuclear interaction length l_{nuc}.

With increasing energy of the incident particles, the range R_0 increases, while the length l_{nuc} changes weakly.[15,19] This has the consequence that, beginning at a certain energy (for protons, around 100 MeV), a large fraction of the particles undergoes a nuclear interaction in the process of slowing down. In general, a complicated internuclear cascade process develops in the matter.[20,69] However, as was shown in Refs. 7–9 and 20, even in such a case the distribution of the particles at $z \sim R_0$ is formed by the particles that have not undergone a nuclear interaction (for them, nuclear collisions are tantamount to absorption[7–9,19,25]).

The reason for this can be readily understood. In contrast to Coulomb scattering and ionization slowing down, nuclear interaction results in the particle losing energy in large portions and being scattered through relatively large angles.[15] Therefore, the initial particles that undergo a nuclear collision, and also the secondary charged particles (reaction products) reach a given depth with a large loss of energy and are decelerated in thinner ($z < R_0$) layers of matter (Refs. 7–9, 20, 25, and 70).

The quantities least sensitive to the presence of the nuclear interaction are the differential characteristics—the energy spectrum and the spatial distribution of particles stopped in the region $z \sim R_0$. At not too high energies (for protons up to several hundred mega-electron-volts) they are formed by the particles that do not undergo a nuclear interaction, the number of which decreases with the depth in proportion to $\exp[-\int_T^{T_0} dT w_{\text{nuc}}(T)/\bar{\varepsilon}(T)]$, where $w_{\text{nuc}}(T)$ is the probability of nuclear collisions per unit path length (Refs. 15, 19, and 70).[k)] The products of the nuclear interactions mainly influence the low-energy "tail" of the spectrum and, accordingly, the distribution of the stopped particles at $R_0 - z > R_0 \max(\gamma, 1/\sqrt{\nu})$. The integrated characteristics, in particular the Bragg curve, depend on the nuclear collisions to a greater degree.[7,20] The nuclear interaction has a particularly strong influence on the shape of the Bragg curve in the region $R_0 - z \gg R_0 \max(\gamma, 1/\sqrt{\nu})$, where it is evidently important to take into account the contribution of the products of the nuclear reactions.[7,20]

With increasing initial energy of the particles, the nuclear collisions acquire an essentially multiple nature, and the particles that undergo a strong interaction influence not only the low-energy tail of the spectrum but also the entire distribution of the particles. Then, to determine the energy spectrum and the spatial distribution of the stopped particles (to say nothing of the Bragg curve) in this region of energies (for protons $T_0 \gtrsim 0.5$–1 GeV), it is now necessary to calculate the entire internuclear cascade process.

We note finally that if on the right-hand side of Eq. (1) we introduce the term $w_{\text{nuc}}(T)N$, then the resulting equation will describe the motion of the particles up to a nuclear collision (or decay) or between successive nuclear interaction events. For the solution of this equation, the methods considered in Sec. 1 can be used. Knowing the solution of the equation and the corresponding differential cross sections, it is possible to determine the initial spectrum of particles that have undergone a single nuclear interaction, or the initial spectrum of the secondary particles. To describe the motion of these particles, it is again necessary to use the original equation, but now with the appropriate initial distribution.[l)] This procedure can, for example, be used to calculate the particle distribution for pions penetrating matter with allowance for their decay ($\pi^{\pm} \rightarrow \mu^{\pm} + \nu_{\mu}$) and nuclear interaction (as in Refs. 10 and 58).

4. CONCLUSIONS

In this paper, we have presented an analytic theory of the penetration of a beam of fast heavy charged particles through matter, taking into account simultaneously the systematic slowing down and the fluctuations of the energy loss due to the probabilistic nature of the inelastic collisions and multiple Coulomb scattering. This theory is the most general means for obtaining analytic results describing the penetration of fast heavy charged particles in matter, and includes the well-known results of Bohr,[50] Fermi,[17] Eyges,[63] Pomeranchuk,[56] Yang,[60] Vavilov,[42] Spencer and Coyne,[61] Berger and Seltzer,[6] and Payne[27] as appropriate limiting cases.

We have considered in detail the fluctuations of the energy losses and the ranges of particles in thick layers of matter, in which the influence of elastic scattering on the energy distribution of the particles is most clearly manifested.

We have found the distribution of the particles with respect to the lengths of the path traversed in a layer of matter. We have found that there is good agreement between the analytic results and the data of numerical Monte Carlo calculations. We have shown that Yang's distribution as modified by Berger and Seltzer overestimates the contribution of multiple Coulomb scattering to the variance of the distribution of the stopped particles with respect to the depth by not less than 30%.

We have calculated the energy spectrum of the fast heavy charged particles in matter with allowance for the ionization energy losses and the bending of the particle paths by multiple scattering. We have also obtained estimates of the spatial and angular spread of the particles that have traversed a layer of matter of given thickness as functions of the energy which they have lost.

We have given a general expression for the spatial distribution of the stopped particles. In this expression, we have taken into account not only the ionization losses but also multiple elastic scattering. We have calculated the distribution of the stopped particles with respect to the penetration depths in the matter. We have analyzed the relative importance of the fluctuations in the ranges of the particles due to the probabilistic nature of the ionization collisions and Coulomb elastic scattering as functions of the atomic number of the atoms of the medium and the mass and energy of the particles. We have considered the distributions of the particles with respect to the transverse displacement from the beam axis at large depths. We have shown that the simultaneous effect of the ionization slowing down and the multiple Coulomb scattering leads to a decrease in the spread of the particles with respect to the transverse displacement from the beam axis with increasing penetration depth.

On the basis of the general expression for the distribution of the particles with respect to the energies, coordinates, and angles we have proposed a simple method for calculating the moments of the particle energy spectrum and the spatial distribution of the stopped particles. We have obtained relations for the mean ranges and the mean penetration depths as functions of the direction of motion of the particles and their transverse displacement from the beam axis.

The results listed above are valid in the complete range of particle energies in which the main contribution to the experimentally measured characteristics of particle penetration in matter is made by particles that have not undergone nuclear collisions.

The analytic theory presented in the present paper can be used to analyze experiments that measure the energy spectra of particles, the distribution of the energy losses and the energy release (Bragg curve), the spatial distribution of the stopped particles, and the dependence of the integrated particle flux on the depth and displacement from the beam axis (transmission curve); in particular, it can be used to determine from an experiment the main parameters in the theory of the interaction of fast charged particles with matter [the stopping power $\bar{\varepsilon}$, the ionization potential $I(Z)$, the spread of the energy losses $\overline{\varepsilon^2}$ per unit path length, etc.].

When they were first made, the calculations of the decrease of the particle penetration depth due to multiple elastic scattering made it possible to determine with high accuracy from experiments the stopping power $\bar{\varepsilon}$ and one of the most important parameters of the interaction of charged particles and matter—the mean ionization potential $I(Z)$.[2–9] The logical continuation of these studies would be the experimental measurement of $\overline{\varepsilon^2}$, in particular, the acquisition of data on the influence on $\overline{\varepsilon^2}$ of the effect of the binding of the electrons in the atoms.[21]

However, until recently systematic calculations giving an idea of the contribution of multiple elastic scattering to the variance of the energy and range distributions did not exist.[19] This, in its turn, meant that it was not possible to extract from experiments the values of $\overline{\varepsilon^2}$, or at least not values sufficiently accurate to permit estimation of the importance of the corrections for the effect of binding of the electrons in the atoms. This is most important for media with large Z in which, on the one hand, elastic scattering is important and, on the other, the effect of the electron binding must be appreciable even at fairly high particle energies.[6,48]

The results considered above describe fairly accurately the contribution of multiple elastic scattering to the energy and range distributions of the particles and thus make it possible to extract from experiments data on the influence of the electron binding in the atoms on $\overline{\varepsilon^2}$. For this, it is more expedient to use experiments that measure the energy spectra and distributions of the stopped particles rather than integrated characteristics, i.e., the Bragg curves and transmission curves.

[a] For muons this is always valid. With regard to strongly interacting particles, at not too high energies, when the total range of the particles in matter does not greatly exceed the nuclear interaction length, many important experimentally measured quantities are determined by the particles that have not undergone nuclear collisions (Refs. 7–9, 19, and 20). For them, the influence of the strong interaction reduces to an exponential attenuation. For protons, for example, such a situation obtains up to several hundred mega-electron-volts (Refs. 7–9, 19, and 20).

[b] In a first approximation $T_{\mathrm{m.p}}(z) \simeq T_{\mathrm{c.s}}(z)$, where $T_{\mathrm{c.s}}(z)$ is the particle

energy at depth z in the continuous-slowing-down approximation:

$$R\,[T_{\text{н.з}}(z)] = R_0 - z;\quad z = \int\limits_{T_{\text{н.з}}(z)}^{T_0} \frac{dT'}{\bar{\varepsilon}\,(T')}.$$

c) In Ref. 28, an error made in Ref. 54 is corrected. The index of the Bessel function in the solution of Ref. 54 must be increased by unity.

d) The result (69) can be used to describe the penetration of particles in not only homogeneous but also inhomogeneous media. The corresponding modifications associated with the inhomogeneity of the matter affect only the expressions for $\langle \theta_s^2(z)\rangle$ and $\overline{\varepsilon k}\,(z)$.

e) Under the condition that the original equation in Yang's paper of Ref. 60 is correctly expressed. In Ref. 60 there is an error—instead of the term $[(\theta_x^2 + \theta_y^2)/2]\,(\partial N/\partial u)$ the term $[(\theta_x^2/2)\,(\partial/\partial u_x) + (\theta_y^2/2)\,(\partial/\partial u_y)]N$ is written.

f) *Translator's Note.* The Russian notation for the trigonometric, inverse trigonometric, hyperbolic functions, etc., is retained here and throughout the article in the displayed equations.

g) The solution (88) corresponds to the semiclassical approximation for an equation of the form $f'' + x^n \varphi(x) f = 0$, where $\varphi(x)$ is a smooth function of the argument x.[65]

h) The corrections to (145) were calculated for the first time for nonrelativistic particles by Lewis, who solved the transport equation by the moment method.[66]

i) With allowance for the energy dependence of the ionization logarithm

$$\varkappa_0 = \varphi^2\,(t_0) = \frac{(Z+1)\,L_h}{2\pi n_0 Z z^2 r_e^2}\left[\int\limits_0^{E_0} \frac{dE}{(1+E)\,L_{\text{ион}}\,(E)}\right]^2.$$

j) The calculations of Ref. 6 give overestimated values for $\langle R_0 - z\rangle$. This is due to the fact that the relation for $\langle \theta_s^2\rangle$ used in Ref. 6 was first obtained for electrons and not for heavy particles and does not take into account the finite size of the atomic nucleus.

k) Elastic nuclear scattering through small angles ($\theta < \sqrt{\langle\theta^2\rangle_z}$) can be taken into account by a renormalization of the mean square $\langle\theta_s^2\rangle$ of the scattering angle per unit path length. Then $w_{\text{nuc}}(T)$ in the argument of the exponential must be replaced by $\widetilde{\omega}_{\text{nuc}}(T) = w_{\text{nuc}}(T) - w_{\text{el}}^{\langle\theta^2\rangle_{z,p,T}}$, where $w_{\text{el}}^{\langle\theta^2\rangle_{z,p,T}}$ is the cross section of elastic nuclear scattering through angles $\theta < \sqrt{\langle\theta^2\rangle_{z,p,T}}$.

l) The motion of particles that undergo a nuclear interaction and also the motion of secondary particles can in general, because of the large spread in their initial distributions, be calculated already in the continuous-slowing-down approximation.

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Translated by Julian B. Barbour