Quantum method of Bogolyubov generating functionals in statistical physics: Lie algebra of the currents, its representations, and functional equations

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The review is devoted to a systematic exposition of a new approach to the investigation of the quantum method of Bogolyubov generating functionals in statistical physics on the basis of an analysis of the representations of the Lie algebra of the currents in nonrelativistic quantum mechanics and the functional equations corresponding to them. The quantum method of Bogolyubov generating functionals developed by the authors is shown to be effective for the important problem of calculating the correlation functions of many-particle systems. Explicit functional—operator expressions are obtained for the generating functional of the distribution functions in classical statistical mechanics in both the equilibrium and nonequilibrium cases. An analysis of the Wigner representation for the Bogolyubov generating functional is the basis of a proof for the first time of the Hamiltonicity of the nonequilibrium functional equation in the grand canonical ensemble of the classical statistical mechanics of many-particle systems with respect to a special Lie–Poisson–Vlasov symplectic structure on the orbits of the coadjoint representation of the semiclassical Lie algebra of observables. Examples of interacting one-dimensional Bose and Fermi gases of particles on an axis are considered; for them the calculations can be performed in closed form.

INTRODUCTION

In his fundamental study of Ref. 1, Bogolyubov proposed in the case of classical statistical mechanics the method of generating functionals for the study of the n-particle, $n \in \mathbf{Z}_+$, distribution functions of a many-particle dynamical system; this method proved[1,2] to be very effective when applied to definite physical problems. It is sufficient to point out that well-established results in classical statistical mechanics, such as the Mayer–Ursell expansion,[3–8] the representation of "collective" variables,[4,5] and others, became special cases of the general method of Bogolyubov generating functionals. Particularly effective was its quantum interpretation,[6–9] which in the limit of a Planck constant $\hbar$ tending to zero made it possible to obtain for the generating functional of the distribution functions of a classical dynamical system explicit functional–operator[6] expressions in not only the equilibrium but also the nonequilibrium case. Here, a very important part is played in the analysis of the properties of the generating functionals by the corresponding Bogolyubov functional equations,[1] whose solutions they are. Analyzing the quantum approach to the study of the solutions of the functional equations proposed in Ref. 6, Bogolyubov suggested the idea of its natural development to quantum dynamical systems of statistical physics, in which the generating-functional method and the Bogolyubov functional equations corresponding to them would provide the foundation. Developing this idea using the formalism of representations of the Lie algebra of the currents,[7,10–12] which is natural in this case, we give in the present paper a complete functional description of the initial quantum dynamical system, this generalizing the corresponding description of Bogolyubov in Refs. 1 and 13. Going to the classical limit on the basis of the Wigner representation for the Lie algebra of the observable operators, we prove the Hamiltonicity of the nonequilibrium Bogolyubov functional equation on a special infinite-dimensional symplectic manifold with Lie–Poisson–Vlasov bracket on the orbits of the coadjoint representation of the Lie algebra mentioned above.

It should be noted that in the framework of the alternative algebraic approach to the quantum theory of many-particle systems based on the methods of Hamiltonian algebra, generating functionals of Bogolyubov type were considered for the first time in Ref. 35. For these generating functionals there was obtained an associated equation in functional derivatives, this corresponding to the hierarchy of Bogolyubov–Gurov equations for the n-particle kernels of the operators of the distribution, from which the quantum Vlasov equation was derived in the limit of a weak interaction and a large mean number of particles. The same study also introduced the algebra of second-quantized observables, described by generating functionals on the basis of the method of canonical quantization (uniformization) of the classical Hamiltonian algebra of these functionals with a pointwise product and a bracket of Lie–Poisson type. Problems relating to the asymptotic determination of generating functionals of Bogolyubov type on arbitrary **C*** algebras were considered in Ref. 36, which contained a proof of proved a reconstruction theorem for a second-quantized system based on the above-mentioned properties of these functionals (a theorem of Gel'fand–Naĭmark–Segal–Araki type), and quantum kinetic equations of Boltzmann type corresponding to a Markovian dynamics of the second-quantized system were obtained.

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1. FORMALISM OF REPRESENTATIONS OF THE LIE ALGEBRA OF THE CURRENTS

Suppose we are given the Hilbert state space Φ in nonrelativistic quantum mechanics. There are defined on it canonical field operators of creation, $\psi^+(x)$, and annihilation, $\psi(y)$, with $x, y \in \mathbf{R}^3$, which satisfy commutation relations of Bose or Fermi type:

$$\left.\begin{aligned}&[\psi(x), \psi(y)]_\pm = [\psi^+(x), \psi^+(y)]_\pm = 0;\\ &[\psi(x), \psi^+(y)]_\pm = \delta(x-y).\end{aligned}\right\} \quad (1)$$

As the algebra of observable operators that describes interacting quantum particles, we choose the Lie algebra of the currents.[7] In order to construct it, we introduce the following fundamental operator quantities:

$$\rho(x) = \psi^+(x)\psi(x), \quad (2)$$

which is the particle-density operator, and

$$J(x) = \frac{1}{2i}[\psi^+(x)\nabla_x\psi(x) - \nabla_x\psi^+(x)\psi(x)], \quad (3)$$

which is the operator of the particle flux density at the point $x \in \mathbf{R}^3$. Then by direct calculations we can readily show that the operators

$$\rho(f) = \int_{\mathbf{R}^3} d^3x\, f(x)\,\rho(x) \text{ and } J(g) = \int_{\mathbf{R}^3} d^3x\, g(x)\, J(x),$$

where $f \in \mathscr{S}(\mathbf{R}^3; \mathbf{R}^1)$, $g \in \mathscr{S}(\mathbf{R}^3; \mathbf{R}^3)$ are rapidly decreasing Schwartz functions, satisfy the commutation relations

$$\left.\begin{aligned}&[\rho(f_1), \rho(f_2)] = 0;\\ &[\rho(f), J(g)] = i\rho(g\nabla f);\\ &[J(g_1), J(g_2)] = iJ(g_2\nabla g_1 - g_1\nabla g_2)\end{aligned}\right\} \quad (4)$$

for all $f, f_1, f_2 \in \mathscr{S}(\mathbf{R}^3; \mathbf{R}^1)$, $g, g_1, g_2 \in \mathscr{S}(\mathbf{R}^3; \mathbf{R}^3)$. The Lie algebra (4) of the currents will be the fundamental entity studied in our approach to the quantum method of Bogolyubov generating functionals. The current Lie algebra (4) corresponds to physically observable quantities, and therefore, naturally, the operators $\rho(f)$ and $J(g)$ must be self-adjoint, i.e., $\rho^+(f) = \rho(f)$, $J^+(g) = J(g)$ for all $f \in \mathscr{S}(\mathbf{R}^3; \mathbf{R}^1)$, $g \in \mathscr{S}(\mathbf{R}^3; \mathbf{R}^3)$. But, as is frequently the case in quantum theory, they may be unbounded operators on the Hilbert state space Φ. It is therefore natural and convenient to work with unitary operators $U(f)$ and $V(\varphi_t^g)$, which are determined by the formulas

$$U(f) = \exp[i\rho(f)]; \quad V(\varphi_t^g) = \exp[itJ(g)], \quad (5)$$

where

$$t \in \mathbf{R}^1; \ \frac{d}{dt}\varphi_t^g(x) = g \circ \varphi_t^g(x);$$

$$\varphi_0^g(x) = x \in \mathbf{R}^3; \quad g \circ \varphi_t^g(x)$$

$$= g(\varphi_t^g(x)) \text{ and } f \in \mathscr{S}(\mathbf{R}^3; \mathbf{R}^1);\ g \in \mathscr{S}(\mathbf{R}^3; \mathbf{R}^3).$$

$= g(\varphi_t^g(x))$ and $f \in \mathscr{S}(\mathbf{R}^3; \mathbf{R}^1)$; $g \in \mathscr{S}(\mathbf{R}^3; \mathbf{R}^3)$.

The exponential currents (5) form a group G with the composition law[10]

$$\left.\begin{aligned}&U(f_1)U(f_2) = U(f_1+f_2);\\ &V(\varphi)U(f) = U(f\circ\varphi)V(\varphi);\\ &V(\varphi_1)V(\varphi_2) = V(\varphi_2\circ\varphi_1).\end{aligned}\right\} \quad (6)$$

The group G (6) is the semidirect product $\mathscr{S} \wedge \mathrm{Diff}(\mathbf{R}^3)$, where $\mathscr{S} = \mathscr{S}(\mathbf{R}^3; \mathbf{R}^1)$, Diff $(\mathbf{R}^3)$ is the group of diffeomorphisms[14,15] of the space $\mathbf{R}^3$. It follows from (6) that for $f_1, f_2 \in \mathscr{S}$ and $\varphi_1, \varphi_2 \in \mathrm{Diff}(\mathbf{R}^3)$ the group law in $\mathscr{S} \wedge \mathrm{Diff}(\mathbf{R}^3)$ has the form

$$(f_1, \varphi_1) \circ (f_2, \varphi_2) = (f_1 + f_2\circ\varphi_1, \varphi_2\circ\varphi_1). \quad (7)$$

It is obvious that the algebra (4) is the Lie algebra $\mathfrak{G}$ of the group $G = \mathscr{S} \wedge \mathrm{Diff}(\mathbf{R}^3)$. To define the group $G = \mathscr{S} \wedge \mathrm{Diff}(\mathbf{R}^3)$, we introduce the group $\mathrm{Diff}_0(\mathbf{R}^3)$ of smooth diffeomorphisms of the space $\mathbf{R}^3$ with compact supports and with the standard group operation by means of the composition of mappings. The group $\mathrm{Diff}_0(\mathbf{R}^3)$ is topological; its topology is given by a countable system of metrics of the form

$$\langle\langle\varphi_1, \varphi_2\rangle\rangle_n = \max_{|m|=\overline{0,n}} \sup(1+|x|^2)^n\,|\varphi_1^{(m)}(x) - \varphi_2^{(m)}(x)|$$

for all $n \in \mathbf{Z}_+$, $\varphi_1, \varphi_2 \in \mathrm{Diff}(\mathbf{R}^3)$, where $(m) = (m_1, m_2, m_3)$ is a multiple index in the space $\mathbf{Z}_+^3$, $|m| = \Sigma_{j=1}^3 m_j$, $\varphi_k^{(m)}(x) = (\partial^{|m|}\varphi_k(x)/\partial x^{(m)})$, $k = 1, 2$. As a topological space, the group $\mathrm{Diff}(\mathbf{R}^3)$ can be defined by means of the operation of completion of the space $\mathrm{Diff}_0(\mathbf{R}^3)$ with the topology introduced above. Thus, the group $\mathrm{Diff}(\mathbf{R}^3)$ is topological and metrizable with a countable basis of the topology (neighborhoods) at each of its points. The group $\mathrm{Diff}(\mathbf{R}^3)$ contains diffeomorphisms with noncompact supports, but in the limit $|x| \to \infty$, $x \in \mathbf{R}^3$, they can be approximated by the identity mapping in $\mathrm{Diff}(\mathbf{R}^3)$. In particular, from the condition $g \in \mathscr{S}(\mathbf{R}^3; \mathbf{R}^3)$ the element $\varphi_t^g(x) \in \mathrm{Diff}(\mathbf{R}^3)$ for all $t \in \mathbf{R}^1$, $x \in \mathbf{R}^3$, and the mapping $\mathscr{S}(\mathbf{R}^3; \mathbf{R}^3) \ni g: \to \varphi_t^g \in \mathrm{Diff}(\mathbf{R}^3)$ is continuous. In addition, the group $\mathrm{Diff}(\mathbf{R}^3)$ is locally linearly connected, but it is not locally compact in the topology given above.

Let $G = \mathscr{S} \wedge \mathrm{Diff}(\mathbf{R}^3)$ be the semidirect product of the Abelian topological group $\mathscr{S}$ with the standard topology[16] and the topological group $\mathrm{Diff}(\mathbf{R}^3)$ described above, the action in which is defined by the expression (7). Different unitary representations of the group of currents $G = \mathscr{S} \wedge \mathrm{Diff}(\mathbf{R}^3)$ describe different physical systems. For example, a system of $N \in \mathbf{Z}_+$ identical Bose particles and a system of $N \in \mathbf{Z}_+$ identical Fermi particles correspond to two unitarily inequivalent representations of this group. Since the group $\mathscr{S} \wedge \mathrm{Diff}(\mathbf{R}^3)$ is an infinite-parameter group, it appears to be possible to describe a very broad spectrum of physical situations by means of its various unitary representations.[15]

The Hilbert space Φ for every irreducible representation of the group of currents $\mathscr{S} \wedge \mathrm{Diff}(\mathbf{R}^3)$ is unitarily equivalent[11] to the Hilbert space

$$\Phi = \oplus\int_{\mathscr{S}'} d\mu(F)\,\Phi_F, \quad (8)$$

where μ is a cylindrical measure on $\mathscr{S}'$, the space of continuous real and linear functionals on $\mathscr{S}$, and Φ_F, which are labeled by the index $F \in \mathscr{S}'$, are complex linear spaces. In physics applications,[11] it is necessary to choose $\dim \Phi_F = 1$; in this case $\Phi \approx L_2^{(\mu)}(\mathscr{S}'; \mathbf{C}^1)$ is the space of functions square-integrable with respect to the measure μ on $\mathscr{S}'$.

Now suppose the element $\omega(F) \in \Phi$ is arbitrary; then for the action of the group $\mathscr{S} \wedge \mathrm{Diff}(\mathbf{R}^3)$ on this element we have the representation

$$\left.\begin{aligned} &U(f)\,\omega(F) = \exp[\mathrm{i}(F, f)]\,\omega(F);\\ &V(\varphi)\,\omega(F) = \chi_\varphi(F)\,\omega(\varphi^* F)\left[\frac{d\mu(\varphi^* F)}{d\mu(F)}\right]^{1/2},\end{aligned}\right\} \tag{9}$$

where $(\varphi^* F, f) = (F, f \circ \varphi)$; $f \in \mathscr{S}$; $\varphi \in \mathrm{Diff}(\mathbf{R}^3)$; $F \in \mathscr{S}'$; $d\mu(\varphi^* F)/d\mu(F)$ is the Radon–Nikodým derivative[16] of the measure $\mu(\varphi^* F)$ with respect to the measure $\mu(F)$, and $\chi_\varphi(F)$ is a complex-valued factor of unit norm that satisfies the condition

$$\chi_{\varphi_2}(F)\,\chi_{\varphi_1}(F) = \chi_{\varphi_1 \circ \varphi_2}(F) \tag{10}$$

for all $\varphi_1, \varphi_2 \in \mathrm{Diff}(\mathbf{R}^3)$. If the Radon–Nikodým derivative in (9) is to exist, the measure μ on $\mathscr{S}'$ must be quasi-invariant with respect to the group $\mathrm{Diff}(\mathbf{R}^3)$, i.e., for any measurable set $Q \subset \mathscr{S}'$ and any $\varphi \in \mathrm{Diff}(\mathbf{R}^3)$ the measure $\mu(Q) = 0$ if and only if $\mu(\varphi^* Q) = 0$.

The representation (9) corresponding to a quantum-mechanical system of $N \in \mathbf{Z}_+$ identical particles has measure μ concentrated on Dirac delta functions, i.e., on functionals of the form[11,12]

$$= \sum_{j=1}^{N} \delta(x - x_j) \tag{11}$$

with measure of the form

$$d\mu(F) = \Omega^* \Omega \prod_{j=1}^{N} d^3 x_j\, \delta\Big(F - \sum_{j=1}^{N} \delta(x - x_j)\Big),$$

where $x, x_j \in \mathbf{R}^3$, $j = \overline{1, N}$ (i.e., $1, \ldots, n$), and $\Omega \in L_2^{(\pm)}(\mathbf{R}^{3N}; \mathbf{C}^1)$ is the symmetric or antisymmetric ground-state wave function of the N-particle dynamical system. The following formulas also hold: $\Omega(F) = 1$; $\omega \in L_2^{(\mu)}(\mathscr{S}'; \mathbf{C}^1)$;

$$\left.\begin{aligned} &\rho(x)\,\omega(F) = \sum_{j=1}^{N} \delta(x - x_j)\,\omega(F);\\ &J(x)\,\omega(F) = \frac{1}{2i} \sum_{j=1}^{N} [2\delta(x - x_j)\nabla_j - (\nabla\delta)(x - x_j)]\,\omega(F),\end{aligned}\right\} \tag{12}$$

where $\omega(F) = \omega(x_1, \ldots, x_N) \in \Phi \approx L_2^{(\pm)}(\mathbf{R}^{3N}; \mathbf{C}^1)$; x, $x_j \in \mathbf{R}^3$ $j = \overline{1, N}$. Consequences of (11) and (12) will be formulas for the representation of the group $\mathscr{S} \wedge \mathrm{Diff}(\mathbf{R}^3)$ on $\Phi \approx L_2^{(\pm)}(\mathbf{R}^{3N}; \mathbf{C}^1)$:

$$U(f)\,\omega(F) = \exp\Big[\mathrm{i} \sum_{j=1}^{N} f(x_j)\Big]\,\omega(F);$$

$$V(\varphi)\,\omega(F) = \omega(\varphi^* F)\left[\det\left\| \frac{\partial \varphi(x)}{\partial x} \right\|\right]^{1/2},$$

where we have set $\chi_\varphi(F) = 1$ for all $\varphi \in \mathrm{Diff}(\mathbf{R}^3)$, $\omega(\varphi^* F) = \omega(\varphi x_1, \ldots, \varphi x_N)$ in the case of Bose statistics.

In view of the particular effectiveness of the analysis of properties of representations of the group $G = \mathscr{S} \wedge \mathrm{Diff}(\mathbf{R}^3)$ by means of the generating-functional method,[11,17,18] which generalizes the approach of Refs. 1 and 6, we introduce some definitions in connection with this method.

DEFINITION 1.1. A generating functional on the group $G = \mathscr{S} \wedge \mathrm{Diff}(\mathbf{R}^3)$ is a complex-valued function E on G with the following conditions:

1) $E(1) = 1$, $1 \in G$;

2) $E[a_1 \exp(t, A) a_2]$ is a continuous function of the parameter $t \in \mathbf{R}^1$ for all $A \in \mathfrak{G}$ and $a_1, a_2 \in G$;

3) the matrix $\|E(a_i^{-1} a_j)\|$, $j = \overline{1, N}$, is positive definite for all $a_j \in G$, $j = \overline{1, N}$, and $N \in \mathbf{Z}_+$. The following theorem holds.[18]

THEOREM 1.2. The function E is a generating functional on G if and only if there exists a continuous unitary representation π: $G \to \mathrm{Aut}(\Phi)$ on Φ with a cyclic vector $\Omega \in \Phi$ such that

$$E(a) = (\Omega, \pi(a)\,\Omega). \tag{13}$$

The vector $\Omega \in \Phi$ is said to be cyclic with respect to the representation π if the set $\{\pi(a)\Omega \colon a \in G\}$ is complete in Φ, i.e., is dense in Φ if taken together with its linear combinations over $\mathbf{C}^1$; $(\cdot, \cdot)$ is the scalar product in Φ.

The significance of this theorem is that one can implicitly construct unitary representations of the group of currents $G = \mathscr{S} \wedge \mathrm{Diff}(\mathbf{R}^3)$ and, thus, the Lie algebra $\mathfrak{G}$ (4) of the currents by defining a generating functional on G appropriately. This is very important, since frequently the latter problem is much simpler than the initial problem.

We now consider the representation π: $G \to \mathrm{Aut}(\Phi)$ in Theorem 1.2 restricted to the Abelian subgroup $\mathscr{S}$, in the group $G = \mathscr{S} \wedge \mathrm{Diff}(\mathbf{R}^3)$ and its corresponding generating functional $\mathscr{S}(f)$, $f \in \mathscr{S}$, in the form

$$\mathscr{L}(f) = (\Omega, \exp[\mathrm{i}\rho(f)]\,\Omega) = \int_{\mathscr{S}'} d\mu(F) \exp[\mathrm{i}(F, f)], \tag{14}$$

where the cyclic vector $\Omega \in \Phi$ is normalized to unity: $(\Omega, \Omega) = 1$. In accordance with the principles of nonrelativistic quantum statistical mechanics,[8] the functional (14) can be rewritten[6] in the equivalent trace representation:

$$\mathscr{L}(f) = \mathrm{tr}\,(\mathscr{P} \exp[\mathrm{i}\rho(f)]), \tag{15}$$

where $\mathfrak{P}$: $\Phi \to \Phi$ is the corresponding statistical operator,[8] satisfying the condition $\mathrm{tr}\,\mathfrak{P} = 1$. As is demonstrated in Refs. 6 and 2, in many cases the representation (15) is in practice more convenient for investigation, since it reveals explicitly the operator structure of the cyclic vector $\Omega \in \Phi$.

Here, we shall, when necessary, use both representations (14) and (15).

The generating functional (14) possesses all the following necessary properties: 1) $\mathscr{L}(f) = \mathscr{L}^*(-f)$ for all $f \in \mathscr{S}$; 2) $\mathscr{L}(0) = 1$; 3) $|\mathscr{L}(f)| \leqslant 1$, $f \in \mathscr{S}$; 4) $\mathscr{L}(f)$ is a positive-definite functional, which means that for arbitrary $c_j \in \mathbf{C}^1$, $f_j \in \mathscr{S}$, $j = \overline{1, N}$, $N \in \mathbf{Z}_+$, the inequality $\Sigma_{j,k=1}^{N} c_j^* \mathscr{L}(f_k - f_j) \geqslant 0$ holds. As one can show, a functional $\mathscr{L}(f)$ with the properties 1–4 always defines a measure μ on $\mathscr{S}'$ for the representation of the Abelian subgroup $\mathscr{S}$ of the group of currents $\mathscr{S}\Lambda \mathrm{Diff}(\mathbf{R}^3)$. If the measure μ is in addition quasi-invariant and the factors $\chi_\varphi(F)$ in (9) are known for all $\varphi \in \mathrm{Diff}(\mathbf{R}^3)$, $F \in \mathscr{S}'$, then the representation (9) of the group of currents $\mathscr{S}\Lambda \mathrm{Diff}(\mathbf{R}^3)$ is thereby completely determined. In fact, we shall be interested in only irreducible representations of $\mathscr{S}\Lambda \mathrm{Diff}(\mathbf{R}^3)$. According to Ref. 11, reducible representations of $\mathscr{S}\Lambda \mathrm{Diff}(\mathbf{R}^3)$ correspond to systems of particles with different masses or with internal degrees of freedom. In the latter case, it is necessary to add to the Lie algebra of the currents (4) additional local currents of the observables in order to obtain a complete system of observables.[19] In this connection, we mention that a representation of $\mathscr{S}\Lambda \mathrm{Diff}(\mathbf{R}^3)$ will be irreducible if and only if the measure μ on $\mathscr{S}'$ is ergodic for $\mathrm{Diff}(\mathbf{R}^3)$, i.e., when for any measurable invariant set $Q \subset \mathscr{S}'$ either $\mu(Q)=0$ or $\mu(\mathscr{S}'\setminus Q)=0$.[20] For $F \in \mathscr{S}'$, we set $\mathrm{Or}(F)=\{\varphi * F\colon \varphi \in \mathrm{Diff}(\mathbf{R}^3)\}$, where the set $\mathrm{Or}(F)$ is called the orbit of the element $F\in\mathscr{S}'$ under the action of the group $\mathrm{Diff}(\mathbf{R}^3)$. An arbitrary invariant set is in the general case a nondenumerable union of a family of mutually nonintersecting orbits. Assuming that the orbits containing an invariant set $Q\subset\mathscr{S}'$ are measurable, we find only two possibilities for ergodicity of the cylindrical measure μ on $\mathscr{S}'$. Either it is concentrated on one orbit, or each orbit has zero measure. As is shown in Ref. 15, both situations occur for the group of currents $\mathscr{S}\Lambda \mathrm{Diff}(\mathbf{R}^3)$. In particular, the case when the measure μ is concentrated on functionals of the form (11), which was analyzed above, leads to irreducibility in $\Phi \approx L_2^{(\pm)}(\mathbf{R}^{3N};\mathbf{C}^1)$ of the representation (13) of $\mathscr{S}\Lambda \mathrm{Diff}(\mathbf{R}^3)$. Other irreducible representations of $\mathscr{S}\Lambda \mathrm{Diff}(\mathbf{R}^3)$ which are interesting from the physical point of view can be obtained[15] by means of the method of induced representations from the subgroups of the group $\mathrm{Diff}(\mathbf{R}^3)$. We hope to dwell on their application in the theory of the quantum method of Bogolyubov generating functionals in a following paper.

2. LIE ALGEBRA OF THE CURRENTS, THE HAMILTONIAN OPERATOR, AND THE BOGOLYUBOV FUNCTIONAL EQUATIONS

Suppose we are given a quantum system of particles with mean density $\bar{\rho}\in\mathbf{R}^1_+$ that is in an equilibrium ground state and a Hamiltonian $\mathbf{H}\colon\Phi\to\Phi$ that in the second-quantization representation[8] has the form

$$\mathbf{H}=\frac{\hbar^2}{2m}\int_{\mathbf{R}^3} d^3x\nabla_x\psi^+(x)\nabla_x\psi(x)+\frac{1}{2}\int_{\mathbf{R}^3} d^3x\int_{\mathbf{R}^3} d^3yV(x-y)\psi^+(x)\psi^+(y)\psi(y)\psi(x), \quad (16)$$

where $V(x-y)$ is the scalar two-particle interaction potential in the system, and $m\in\mathbf{R}^1_+$ is the mass of one particle. In terms of the current operators (2) and (3), the Hamiltonian operator (16) has[11] the representation

$$\mathbf{H}=\frac{\hbar^2}{8m}\int_{\mathbf{R}^3} d^3xK^+(x)\rho^{-1}(x)K(x)+\frac{1}{2}\int_{\mathbf{R}^3} d^3x\int_{\mathbf{R}^3} d^3yV(x-y):\rho(x)\rho(y):. \quad (17)$$

Here, we have the operator $K(x)=\nabla_x\rho(x)+2iJ(x)=2\psi^+(x)\nabla_x\psi(x)$, $x\in\mathbf{R}^3$, while the symbol :: denotes the standard operation of normal ordering[21] of creation and annihilation operators. This operation has an equivalent representation in terms of the density operators $\rho(x)$, $x\in\mathbf{R}^3$:

$$:\rho(x_1)\ldots\rho(x_n):=\prod_{j=1}^{n}\Big(\rho(x_j)-\sum_{k=1}^{j-1}\delta(x_j-x_k)\Big), \quad (18)$$

where $n\in\mathbf{Z}_+$ is arbitrary. The result (18) can be readily extracted from the N-particle representation (12) of the density operator in the Hilbert space Φ:

$$\left.\begin{aligned}&\rho(x)=\sum_{j=1}^{N}\delta(x-x_j);\\&\prod_{j=1}^{n}\rho(y_j)=\sum_{j_1=1}^{N}\ldots\sum_{j_n=1}^{N}\delta(y_1-x_{j_1})\ldots\delta(y_j-x_{j_n}),\end{aligned}\right\} \quad (19)$$

where $y_j\in\Lambda\subset\mathbf{R}^3$, $j=\overline{1,n}$ and $N/\Lambda=\bar{\rho}\in\mathbf{R}^1_+$ is fixed. From (19) the operation :: of normal ordering in the N-particle representation can be defined:

$$:\prod_{j=1}^{n}\rho(y_j):=\sum_{\{j_1,\ldots,j_n\}}\delta(y_1-x_{j_1})\ldots\delta(y_n-x_{j_n}), \quad (20)$$

where the summation in (20) is over only those sets of indices $\{j_1,\ldots,j_n\}$ such that $1\leqslant j_k\leqslant N$ and $j_k\neq j_p$ if $p\neq k$, $k,p=\overline{1,N}$. It is readily verified that the expression (20) is equivalent to the expression (18) with allowance for transition to the Fock representation. In particular, we obtain from (18) for $n=\overline{1,3}$

$$\left.\begin{aligned}&:\rho(x_1):=\rho(x_1);\\&:\rho(x_1)\rho(x_2):=\rho(x_1)[\rho(x_2)-\delta(x_1-x_2)];\\&:\rho(x_1)\rho(x_2)\rho(x_3):=\rho(x_1)\rho(x_2)\rho(x_3)\\&-\rho(x_1)\rho(x_2)\delta(x_1-x_3)-\rho(x_1)\rho(x_3)\delta(x_2-x_3)\\&-\rho(x_1)\rho(x_3)\delta(x_2-x_3)-\rho(x_2)\rho(x_3)\delta(x_1-x_2)\\&+2\rho(x_1)\delta(x_1-x_2)\delta(x_2-x_3).\end{aligned}\right\} \quad (21)$$

We consider in more detail the expression (17) for the Hamiltonian operator $\mathbf{H}$ in the Hilbert space Φ. In the case of a finite density $\bar{\rho}\in\mathbf{R}^1_+$ of our quantum system, the operator (17) in the grand canonical ensemble is an inadequately defined unbounded operator. To give a meaning to the operator (17), we consider[11] the following linear subset D in the Hilbert space Φ:

$$D=\mathrm{span}\,\{\exp\,[i\rho\,(f)]\,\Omega\} \quad (22)$$

for all $f\in\mathscr{S}$, where $\Omega\in\Phi$ is a cyclic vector, i.e., an irreducible ground-state vector. Since $\overline{D}=\Phi$, where $\overline{D}$ is the closure in Φ of the subset D, for any two vectors $\omega_1,\omega_2\in D$ the "matrix" element

$$(\omega_1,\mathbf{H}_0\omega_2)=\frac{\hbar^2}{8m}\int_{\mathbf{R}^3} d^3x\,(K(x)\,\omega_1,\rho^{-1}(x)\,K(x)\,\omega_2), \quad (23)$$

where $\mathbf{H}_0=(\hbar^2/8m)\int_{\mathbf{R}^3}d^3xK^+(x)\rho^{-1}(x)K(x)$, is a well-defined Hermitian form on $D\times D$. To generalize this construction to the total Hamiltonian $\mathbf{H}$ (17), we assume that the following conditions hold: i) there exists a normalized state $\Omega \in \Phi$ of the quantum system (here, at zero temperature) with lowest energy that is a ground state for $\mathbf{H}$, and $\mathbf{H} \geqslant 0$ and $\mathbf{H}\Omega = 0$; ii) the subset of vectors $D = \mathrm{span}\{\exp[i\rho(f)]\,\Omega : f \in \mathscr{S}\}$ is dense in Φ, $\overline{D} = \Phi$, and $D = \mathrm{dom}\,\mathbf{H}$; iii) the continuity equation holds for the currents: $i/\hbar[\mathbf{H}, \rho(f)] = (\hbar/m) J(\nabla f)$; iv) there exists an antiunitary operator of time reversal T which acts in such a way that $T\rho(f)T^{-1} = \rho(f)$, $TJ(g)T^{-1} = -J(g)$, and $T\Omega = \Omega$. For what follows, we shall find helpful Proposition 2.1.[11,12]

PROPOSITION 2.1. Let $|f\rangle = \exp[i\rho(f)]\,\Omega \in \Phi$, $f \in \mathscr{S}$ and suppose the representation $U(f)$, $V(\varphi)$ of the group $\mathscr{S} \wedge \mathrm{Diff}(\mathbf{R}^3)$ satisfies conditions (i)–(iv) with the cyclic vector $\Omega \in \Phi$. Then

$$\left.\begin{aligned} &\langle f_1 \mid J(g) \mid f_2\rangle = \frac{\hbar}{2m} \langle f_1 \mid \rho(g\cdot\nabla(f_1+f_2)) \mid f_2\rangle; \\ &\langle f_1 \mid \mathbf{H} \mid f_2\rangle = \frac{\hbar^2}{2m} \langle f_1 \mid \rho(\nabla f_1 \nabla f_2) \mid f_2\rangle. \end{aligned}\right\} \tag{24}$$

The proof of (24) is based on the identities

$$\left.\begin{aligned} &[\exp[i\rho(f)],\ J(g)] = -\frac{1}{2}[\exp[i\rho(f)],\ K(g)] \\ &\qquad = -\rho(g\cdot\nabla f)\exp[i\rho(f)]; \\ &\frac{i}{\hbar}[\exp[i\rho(f)],\ \mathbf{H}] \\ &\qquad = \frac{\hbar}{m}\Big[-J(\nabla f) + \frac{1}{2}\rho(\nabla f\cdot\nabla f)\Big]\exp[i\rho(f)], \end{aligned}\right\} \tag{25}$$

which readily follow from the relations (4) and the Campbell–Hausdorff formula:

$$e^A B e^{-A} = \sum_{n\in\mathbf{Z}_+} (n!)^{-1} (\mathrm{Ad}\,A)^n B, \tag{26}$$

where $(\mathrm{Ad}\,A)^0 B = B$; $(\mathrm{Ad}\,A)B = [A, B]$; $(\mathrm{Ad}\,A)^n B = [A, (\mathrm{Ad}\,A)^{n-1}B]$, $n \in \mathbf{Z}_+$. Using (25) and (26), we find

$$\left.\begin{aligned} &\langle f_1 \mid J(g) \mid f_2\rangle = (TJ(g)\exp[i\rho(f_2)]\,\Omega; \\ &T\exp[i\rho(f_1)]\,\Omega) = -(\Omega,\ \exp[i\rho(f_2)]\,J(g)\exp[-i\rho(f_2)]\,\Omega) \\ &= -(\Omega,\ \exp[-i\rho(f_1)]\,[J(g) \\ &\quad -\rho(g\cdot\nabla(f_1+f_2))]\exp[i\rho(f_2)]\,\Omega); \\ &\langle f_1 \mid \mathbf{H} \mid f_2\rangle = (\exp[i\rho(f_1)]\,\Omega,\ [\mathbf{H},\ \exp[i\rho(f_2)]]\,\Omega) \\ &= \Big(\exp[i\rho(f_1)]\,\Omega, \Big[J\Big(\nabla f_2 - \frac{1}{2}\rho(\nabla f_2\cdot\nabla f_2)\Big]\exp[i\rho(f_2)]\,\Omega\Big). \end{aligned}\right\} \tag{27}$$

From (27), the expression (24) already readily follows.

We now define the following operator $A(x;\rho): \Phi \to \Phi$, which will be important in what follows and has the property

$$K(g)\,\Omega = A(g;\ \rho)\,\Omega \tag{28}$$

for all $g \in \mathscr{S}(\mathbf{R}^3; \mathbf{R}^3)$. For the representation $\pi: G \to \mathrm{Aut}\,L_2^{(\mu)}(\mathscr{S}'; \mathbf{C}^1)$ of the group $G = \mathscr{S} \wedge \mathrm{Diff}(\mathbf{R}^3)$ with the cyclic vector $\Omega(F) = 1$ we denote by $A(g)$ the operator of multiplication by the function $[K(g)\Omega](F)$ defined by the equation $[A(g)\omega](F) = [K(g)\Omega](F)\omega(F)$ for any $\omega(F) \in L_2^{(\mu)}(\mathscr{S}'; \mathbf{C}^1)$, $F \in \mathscr{S}'$. The domain of definition of the operator $A(g)$ has the form $\mathrm{dom}\,A(g) = \{\omega(F) \in L_2^{(\mu)}(\mathscr{S}';\mathbf{C}^1) : \int_{\mathscr{S}'} d\mu(F)\,|(A(g)\omega)\ (F)|^2 < \infty\}$. Since the operator $\exp[i\rho(f)]$ in $L_2^{(\mu)}(\mathscr{S}'; \mathbf{C}^1)$ is the operator of multiplication by the function $\exp[i(F, f)]$, we find that $[A(g), \exp[i\rho(f)]] = 0$ for all $f \in \mathscr{S}(\mathbf{R}^3; \mathbf{R}^1)$ and $g \in \mathscr{S}(\mathbf{R}^3; \mathbf{R}^3)$. In addition, $A(g)\exp[i\rho(f)]\Omega = \exp[i\rho(f)]\,K(g)\Omega$, i.e., $\mathrm{dom}\,A(g) \supset D$. From property (iv) above it follows that $[K(g)\Omega]^*(F) = [K(g)\Omega](F)$, i.e., our operator $A(g)$ is Hermitian and has domain of definition $\mathrm{dom}\,A(g) \supset D$ that is dense in $L_2^{(\mu)}(\mathscr{S}'; \mathbf{C}^1)$. Moreover, the operator $A(g): L_2^{(\mu)}(\mathscr{S}'; \mathbf{C}^1) \to L_2^{(\mu)}(\mathscr{S}'; \mathbf{C}^1)$ is self-adjoint. To prove this, it is sufficient to show that $\{\mathrm{range}\,(A(g) \pm i)\}^{\perp} = 0$.[16] We have therefore: If $\overline{\omega}(F) \in \{\mathrm{range}\,(A(g) \pm i)\}^{\perp}$, then $\int_{\mathscr{S}'} d\mu(F)\overline{\omega}(F)(A(g) \pm i)\omega(F) = 0$ for all $\omega(F) \in \mathrm{dom}\,A(g)$. Taking $\omega(F) = \chi_Q(F)$, the characteristic function of an arbitrary measurable set $Q \subset \mathscr{S}'$, we find $\int_Q d\mu(F)\overline{\omega}(F)[(K(g)\Omega)\ (F) \pm i] = 0$. Because the set $Q \subset \mathscr{S}'$ is arbitrary, we conclude that $\overline{\omega}(F)[(K(g)\Omega)(F) \pm i] = 0$, or $\overline{\omega}(F) = 0$, since $[K(g)\Omega](F) \pm i \neq 0$. We have thus shown that the operator $A(g)$, $g \in \mathscr{S}(\mathbf{R}^3; \mathbf{R}^3)$ is self-adjoint.

For effective study of representations of $\mathscr{S} \wedge \mathrm{Diff}(\mathbf{R}^3)$ with the conditions (i)–(iv) stated above in the construction of the cyclic vector $\Omega \in \Phi$, it is necessary to have an expression for the operator $A(g)$ as a function of the density operator ρ. Such a representation is possible, since $\rho: L_2^{(\mu)}(\mathscr{S}'; \mathbf{C}^1) \to L_2^{(\mu)}(\mathscr{S}'; \mathbf{C}^1)$ is an operator of multiplication and the set of polynomials in ρ is, in accordance with condition (ii), after application of them to a vector $\Omega \in \Phi$, dense in the space of all operators equivalent in $L_2^{(\mu)}(\mathscr{S}'; \mathbf{C}^1)$ to operators of multiplication by a function.

We now define the operator $\widetilde{K}(x) = K(x) - A(x;\rho)$, $x \in \mathbf{R}^3$; by virtue of the definition,

$$\widetilde{K}(x)\,\Omega = 0. \tag{29}$$

By means of the operator $\widetilde{K}(x): \Phi \to \Phi$ we construct the expression

$$\widetilde{\mathbf{H}} = \frac{\hbar^2}{8m} \int_{\mathbf{R}^3} d^3x \widetilde{K}^+(x)\,\rho^{-1}(x)\,\widetilde{K}(x). \tag{30}$$

The following theorem[11] is important.

THEOREM 2.2. The operator $\widetilde{\mathbf{H}}: \Phi \to \Phi$ is a well-defined Hermitian form with dense domain of definition $D \subset \Phi$. In addition, for all $\omega_1, \omega_2 \in D$

$$(\omega_1,\ \widetilde{\mathbf{H}}\omega_2) = (\omega_1,\ \mathbf{H}\omega_2). \tag{31}$$

Indeed, for all $|f_1\rangle$, $|f_2\rangle \in D$, where $|f_j\rangle = \exp[i\rho(f_j)]\Omega$, $j = 1, 2$,

$$\begin{aligned} \langle f_1 \mid \widetilde{\mathbf{H}} \mid f_2\rangle &= \frac{\hbar^2}{8m}\int_{\mathbf{R}^3} d^3x\,\langle \widetilde{K}(x)\exp[i\rho(f_1)]\,\Omega,\ \rho^{-1}(x) \\ &\quad \times \widetilde{K}(x)\exp[i\rho(f_2)]\,\Omega\rangle = \frac{\hbar^2}{8m}\int_{\mathbf{R}^3} d^3x \\ &\quad \times\langle -2i\nabla f_1(x)\,\rho(x)\exp[i\rho(f_1)]\,\Omega,\ \rho^{-1}(x)(-2i)\nabla f_2(x) \\ &\quad \times\rho(x)\exp[i\rho(f_2)]\,\Omega\rangle = \frac{\hbar^2}{8m}(\exp[i\rho(f_1)]\,\Omega,\ \rho(\nabla f_1\cdot\nabla f_2) \\ &\quad \times\exp[i\rho(f_2)]\,\Omega) = \langle f_1 \mid \mathbf{H} \mid f_2\rangle. \end{aligned} \tag{32}$$

In deriving (32), we have used the relation $[\exp[i\rho(f)], \widetilde{K}(x)] = -2i\nabla f(x)\rho(x)\exp[i\rho(f)]$, which holds for all $f \in \mathscr{S}(\mathbf{R}^3;\mathbf{R}^1)$. Extending now by linearity the relation (32) to the complete set $D \subset \Phi$, we obtain the result (31).

Thus, Theorem 2.2 asserts that the operator (30) as an operator in Φ with domain of definition $D \subset \Phi$ is identical to the original Hamiltonian operator $\mathbf{H}: \Phi \to \Phi$ with the same domain of definition, i.e., $\widetilde{\mathbf{H}} = \mathbf{H}$ on $D \subset \Phi$. In particular, the spectra of the operators $\widetilde{\mathbf{H}}$ and $\mathbf{H}: \Phi \to \Phi$ are identical. By direct calculations we also find that

$$\frac{i}{\hbar}[\widetilde{\mathbf{H}}, \rho(f)] = \frac{\hbar i}{8m}\int_{\mathbf{R}^3} d^3x\,(\widetilde{K}(x)^+\rho^{-1}(x)[\widetilde{K}(x), \rho(f)] + [\widetilde{K}(x)^+, \rho(f)]\rho^{-1}(x)\widetilde{K}(x)) = \frac{\hbar i}{4m}\int_{\mathbf{R}^3} d^3x\,(\widetilde{K}(x)^+ \times \rho^{-1}(x)\rho(x)\nabla f(x) - \nabla f(x)\rho(x)\rho^{-1}(x)\widetilde{K}(x)) = \frac{\hbar i}{4m}\int_{\mathbf{R}^3} d^3x\,\nabla f(x)[\widetilde{K}(x)^+ - \widetilde{K}(x)] = \frac{\hbar}{m}J(\nabla f). \quad (33)$$

We have here taken into account the equality $A(x;\rho)^+ = A(x;\rho)$ and also for all $\omega_1, \omega_2 \in D \subset \Phi$ the relation $(\omega_1, \widetilde{H}\rho(f)\omega_2) - (\rho(f)\omega_1, \widetilde{H}\omega_2) = -(i\hbar/m)(\omega_1, J(\nabla f)\omega_2)$, which makes it possible to equate on $D \subset \Phi$ the matrix elements of the operators $[\widetilde{\mathbf{H}}, \rho(f)]$ and $[\mathbf{H}, \rho(f)]$ for all $f \in \mathscr{S}(\mathbf{R}^3; \mathbf{R}^1)$. Thereby, Eq. (33) asserts the preservation on $D \subset \Phi$ of the continuity equation for the current operators under the substitution $\mathbf{H} \to \widetilde{\mathbf{H}}$ as the observable Hamiltonian operator in the Hilbert space Φ with cyclic vector $\Omega \in \Phi$ satisfying the conditions (i)–(iv). A consequence of Theorem 2.2 is the following proposition.[11,12]

PROPOSITION 2.3. The operator $\widetilde{\mathbf{H}}: \Phi \to \Phi$ (30) determines in $D \times D$ a positive-definite Hermitian form.

Suppose $\omega = \Sigma_{j=1}^n c_j U(f_j)\,\Omega \in D \subset \Phi$, then

$$(\omega, \widetilde{\mathbf{H}}\omega) = \frac{\hbar^2}{2m}\sum_{j,k=1}^{n} c_j^* c_k \langle f_j \mid \rho(\nabla f_j \cdot \nabla f_k) \mid f_k\rangle = \frac{\hbar^2}{2m}\int_{\mathscr{S}'} d\mu(F)\left(F, \left|\sum_{j=1}^{n} c_j \nabla f_j \exp[i(F, f_j)]\right|^2\right) \geqslant 0, \quad (34)$$

where we have used the isomorphism between Φ and $L_2^{(\mu)}(\mathscr{S}'; \mathbf{C}^1)$, and also the formula

$$(\omega, \rho(f)\,\omega) = \int_{\mathscr{S}'} d\mu(F)\,(F, f) \mid \omega(F)\mid^2. \quad (35)$$

The inequality in (34) follows from the natural physical assumption that for any $\omega \in D \subset \Phi$ the quantity $[\omega, \rho(f)\omega]$ for any function $f \in \mathscr{S}(\mathbf{R}^3; \mathbf{R}^1_+)$, as the mean value of the density operator $\rho(f)$ in the state $\omega \in D$, must be positive. By Friedrichs's theorem,[11,16] the operator expression (30) defines in Φ a positive self-adjoint operator $\widetilde{\mathbf{H}} = \mathbf{H}: \Phi \to \Phi$, which plays in our analysis the role of the observable Hamiltonian operator, expressed completely in terms of the current operators (2) and (3). This means that we have carried out the Araki[22] reconstruction procedure for the Hamiltonian operator $\mathbf{H}: \Phi \to \Phi$ in accordance with the irreducible unitary representation of the current group $G = \mathscr{S} \wedge \mathrm{Diff}(\mathbf{R}^3)$ given by the relations (6) with cyclic vector of the representation $\Omega \in \Phi$ satisfying the conditions (i)–(iv).

We now consider the generating functional $\mathscr{L}(f)$ (14) of a unitary representation of the Abelian subgroup $\mathscr{L}(\mathbf{R}^3; \mathbf{R}^1)$ of the group $G = \mathscr{S} \wedge \mathrm{Diff}(\mathbf{R}^3)$. In accordance with the relation (29), it satisfies the equation

$$0 = (\Omega, \exp[i\rho(f)]\,\widetilde{K}(x)\,\Omega) = (\Omega, \exp[i\rho(f)][\nabla\rho(x) - i\nabla f(x)\rho(x)]\,\Omega) - (\Omega, \exp[i\rho(f)]A(x;\rho)\,\Omega). \quad (36)$$

Bearing in mind that $(1/i)(\delta/\delta f(x))\mathscr{L}(f) = (\Omega, \rho(x)\exp[i\rho(f)]\Omega)$ for all $x \in \mathbf{R}^3$, $f \in \mathscr{S}(\mathbf{R}^3;\mathbf{R}^1)$, from (36) we directly obtain the first functional equation of Bogolyubov type:

$$[\nabla_x - i\nabla f(x)]\frac{1}{i}\frac{\delta\mathscr{L}(f)}{\delta f(x)} = A(x;\delta)\,\mathscr{L}(f), \quad (37)$$

where $A(x;\delta) = A(x;\rho)|_{\rho=(1/i)(\delta/\delta f)}$. Bearing in mind our further desire to work with the generating functional $\mathscr{L}(f)$ in the form (15) in the case of the grand canonical ensemble for a quantum system of particles with mean density $\bar{\rho} \in \mathbf{R}^1_+$, we augment conditions 1–4 of Sec. 1 by the following. 5) $\mathscr{L}(f)$ is an extremal solution of Eq. (37), i.e., it cannot be represented in the form of a convex linear combination of other solutions of the equation, i.e., the representation of the group $\mathscr{S} \wedge \mathrm{Diff}(\mathbf{R}^3)$ in Φ with cyclic vector $\Omega \in \Phi$ is irreducible; 6) $(1/i)(\delta\mathscr{L}/\delta f(x))|_{f=0} = (\Omega, \rho(x)\Omega) = \bar{\rho}$ is the mean particle density of the system; 7) $\mathscr{L}(f) = \mathscr{L}(f_a)$, where $f_a(x) = f(x-a)$, $x, a \in \mathbf{R}^3$, this being the condition of translational invariance; 8) $\lim_{|a|\to\infty} \mathscr{L}(f+h_a) = \mathscr{L}(f)\mathscr{L}(h)$, where $h_a(x) = h(x-a)$, this being the cluster condition, or Bogolyubov's principle of correlation weakening[1]; $f, h \in \mathscr{S}(\mathbf{R}^3; \mathbf{R}^1)$.

DEFINITION 2.4. We shall say that a representation of the group of currents $G = \mathscr{S} \wedge \mathrm{Diff}(\mathbf{R}^3)$ for which the generating functional $\mathscr{S}(f) f \in \mathscr{S}$, satisfies the conditions 1–8 is canonical.

By virtue of Theorem 2.2, we have the following proposition.

PROPOSITION 2.5. The expression $\widetilde{\mathbf{H}}: \Phi \to \Phi$ (30) defines in the case of a canonical representation of $G = \mathscr{S} \wedge \mathrm{Diff}(\mathbf{R}^3)$ a well-defined positive self-adjoint Hamiltonian operator of the initial quantum system of particles in the grand canonical ensemble at zero temperature.

We now consider the following n-particle distribution functions of the original quantum system of particles with the Hamiltonian (16) at temperature $\beta \in \mathbf{R}^1_+$ (introduced by Bogolyubov in Refs. 1 and 8):

$$F_n(x_1, \ldots, x_n) = \mathrm{tr}\,(\mathfrak{P}\colon \rho(x_1)\ldots\rho(x_n)\colon), \quad (38)$$

where $x_j \in \mathbf{R}^3$, $j = \overline{1,n}$, $n \in \mathbf{Z}_+$; $\mathfrak{P} = \exp(-\beta\bar{\mathbf{H}})[\mathrm{tr}\exp(-\beta\bar{\mathbf{H}})]^{-1}$ is the equilibrium statistical operator, $\bar{\mathbf{H}} = \mathbf{H} - \mu N$, $N = \int_{\mathbf{R}^3} d^3x\rho(x)$ is the particle-number operator, and $\mu \in \mathbf{R}^1$ is the chemical potential.[8] By virtue of the expression (15), we can write for the generating functional $\mathscr{L}(f), f \in \mathscr{S}$,

$$F_n(x_1, \ldots, x_n) = \colon\frac{1}{i}\frac{\delta}{\delta f(x_1)}\ldots\frac{1}{i}\frac{\delta}{\delta f(x_n)}\colon\mathscr{L}(f)\Big|_{f=0}, \quad (39)$$

where for all $n \in \mathbf{Z}_+$ the operator $(1/i)(\delta/\delta f(x_1))\ldots(1/$

$i)$ $(\delta/\delta f(x_n))$: is calculated in accordance with the rule

$$:\frac{1}{i}\frac{\delta}{\delta f(x_1)}\cdots\frac{1}{i}\frac{\delta}{\delta f(x_n)}: =\prod_{j=1}^{n}\left(\frac{1}{i}\frac{\delta}{\delta f(x_j)}-\sum_{h=1}^{j-1}\delta(x_j-x_h)\right). \quad (40)$$

Thus, the functional equation (37) is equivalent to Bogolyubov's infinite hierarchy of equations[1] for the distribution functions (38), generalizing to the quantum case the equations of Ref. 1. At the same time, it is readily seen that the Bogolyubov generating functional $\mathscr{L}_B(u)$, $u\in\mathscr{S}(\mathbf{R}^3;\mathbf{R}^1)$, of Ref. 1 is given by

$$\mathscr{L}_B(u)=\mathrm{tr}(\mathscr{P}:\exp[\rho(u)]:). \quad (41)$$

Expanding the expressions (15) and (41) under the trace operation in a series, we obtain the following important formal identity[6]:

$$\mathscr{L}_B(u)=\mathscr{L}(f), \quad (42)$$

where formally $u(x)=\exp[if(x)]-1$, $x\in\mathbf{R}^3$.

To calculate the energy density $\varepsilon(x)\in\mathbf{R}^1_+$, $x\in\mathbf{R}^3$, of the ground state of the Hamiltonian operator $\widetilde{\mathbf{H}}:\Phi\to\Phi$ (30) at zero temperature in the case of a canonical representation of the group of currents $\mathscr{S}\wedge\mathrm{Diff}(\mathbf{R}^3)$ in the Hilbert space Φ, we need to obtain a further functional equation,[11,7] which takes into account the second condition on the cyclic vector $\Omega\in\Phi$ of the canonical representation of the group of currents: the condition $\widetilde{\mathbf{H}}\Omega=0$. To write it in a convenient form in terms of the generating functional $\mathscr{L}(f)$ and its functional derivatives, we study first in more detail the structure of the operator $A(x;\rho)$, proceeding from its definition (28). In accordance with the properties of the time-reversal operator T when applied to the Lie algebra of currents $\mathfrak{G}$ (4), we conclude that the cyclic vector $\Omega\in\Phi$ is real, i.e., $\Omega^*=\Omega$.[7] Going over in accordance with Eqs. (11)–(13) to the N-particle representation of the current Lie algebra $\mathfrak{G}$ (4), $N\in\mathbf{Z}_+$, we readily find from (28)

$$A(x;\rho)=\sum_{j=1}^{N}\delta(x-x_j)(\nabla_{x_j}\ln\Omega^2)(x_1,\ldots,x_N). \quad (43)$$

Further, going over in accordance with Eq. (18) to the current representation, we obtain from (43)

$$A(x;\rho)=\int_{\mathbf{R}^3}d^3x_2\ldots\int_{\mathbf{R}^3}d^3x_N:\rho(x)\rho(x_2)\ldots\rho(x_N):\nabla_x\ln\Omega^2. \quad (44)$$

It is here appropriate to note that the expression (44) can also be readily obtained from (43) by going over to the second-quantized[8] representation with allowance for the expressions (2) and (18). For the case of the grand canonical ensemble, the expression (44) is inconvenient because it does not have a definite limit as $N\to\infty$. Therefore, the operator $A(x;\rho)$ must be represented in the form of a polynomial series in the operators $\rho(x)$, $x\in\mathbf{R}^3$,[7]

$$A(x;\rho)=\sum_{n\in\mathbf{Z}_+}(n!)^{-1}\int_{\mathbf{R}^3}d^3y_1\ldots\int_{\mathbf{R}^3}d^3y_n:\rho(x)\rho(y_1)\ldots\rho(y_n):\nabla_x\mathscr{A}_{n+1}(x,y_1,\ldots,y_n), \quad (45)$$

where the functions $\mathscr{A}_{n+1}(x,y_1,\ldots,y_n)$, $n\in\mathbf{Z}_+$, can be determined as follows. Consider the symmetric function $\ln\Omega^2(x_1,\ldots,x_N)$ in the form of the expansion

$$\ln\Omega^2(x_1,\ldots,x_N)=\sum_{j=1}^{N}\mathscr{A}(x_j)+\sum_{i<j}^{N}\mathscr{A}_2(x_i,x_j)+\sum_{i<j<h}^{N}\mathscr{A}_3(x_i,x_j,x_h)+\ldots+\mathscr{A}_N(x_1,\ldots,x_N), \quad (46)$$

where $x_j\in\Lambda\subset\mathbf{R}^3$, $j=\overline{1,N}$, and $\overline{\rho}=N/\Lambda$. To determine the terms of the series (46), we use the following proposition.[7]

PROPOSITION 2.6. (Campbell's lemma). Let $F(x_1,\ldots,x_N)$, $x_j\in\Lambda\subset\mathbf{R}^3$, $j=\overline{1,N}$, be a symmetric function of all its variables; then

$$F(x_1,\ldots,x_N)=\sum_{n=0}^{N}\sum_{\{1\leqslant j_1<j_2<\ldots<j_N\leqslant N\}}F_n(x_{j_1},\ldots,x_{j_n}), \quad (47)$$

where

$$F_n(x_1,\ldots,x_N)=\sum_{m=0}^{n}(-1)^{m+n}\sum_{\{j_1<j_2<\ldots<j_m\}}C_m^{(N)}(x_{j_1},\ldots,x_{j_m}); \quad (48)$$

$$C_m^{(N)}(x_1,\ldots,x_m)=\Lambda^{m-N}\int_\Lambda d^3x_{m+1}\ldots\int_\Lambda d^3x_NF(x_1,\ldots,x_N).$$

Campbell's lemma is proved[7] by the Fourier-transformation method, on which we shall not dwell here. Going in (46) to the limit $N\to\infty$, $\Lambda\nearrow\mathbf{R}^3$, $N/\Lambda=\overline{\rho}$, and also going over to the current representation, we obtain the required representation (45) for the operator $A(x;\rho)$ in the grand canonical ensemble, the coefficients in (45) being determined uniquely in accordance with the algorithm of Proposition 2.6.

We now consider the equation

$$(\Omega,\ \exp[i\rho(f)]\,\mathbf{H}\Omega)=0, \quad (49)$$

where the operator $\mathbf{H}$: $\Phi\to\Phi$ is given by (17). Using the expression (29), we obtain successively from (49)

$$\begin{aligned}(\Omega,\exp[i\rho(f)]\mathbf{H}\Omega)=0&=\frac{\hbar^2}{8m}\int_{\mathbf{R}^3}d^3x(\Omega,\exp[i\rho(f)]K^+(x)\\&\times\rho^{-1}(x)K(x)\Omega)+\frac{1}{2}\int_{\mathbf{R}^3}d^3x\int_{\mathbf{R}^3}d^3y\,V(x-y)(\Omega,\exp[i\rho(f)]\\&\times:\rho(x)\rho(y):\Omega)-\int_{\mathbf{R}^3}d^3x\,\varepsilon(x)(\Omega,\exp[i\rho(f)]\rho(x)\Omega)\\&=\frac{\hbar^2}{8m}\int_{\mathbf{R}^3}d^3x(\Omega,\exp[i\rho(f)]2\nabla(x)\cdot\rho^{-1}(x)A(x;\rho)\Omega)\\&-\frac{\hbar^2}{8m}\int_{\mathbf{R}^3}d^3x(\Omega,\exp[i\rho(f)][K(x),\rho^{-1}(x)A(x;\rho)]\Omega)\\&-\frac{\hbar^2}{8m}\int_{\mathbf{R}^3}d^3x(\Omega,\exp[i\rho(f)]A(x;\rho)\cdot\rho^{-1}(x)A(x;\rho)\Omega)\\&+\frac{1}{2}\int_{\mathbf{R}^3}d^3x\int_{\mathbf{R}^3}d^3y\,V(x-y):\frac{1}{i}\frac{\delta}{\delta f(x)}\frac{1}{i}\frac{\delta}{\delta f(y)}:\mathscr{L}(f)\\&-\int_{\mathbf{R}^3}d^3x\,\varepsilon(x)\frac{1}{i}\frac{\delta\mathscr{L}(f)}{\delta f(x)}\\&=\frac{\hbar^2}{8m}\int_{\mathbf{R}^3}d^3x(\Omega,\exp[i\rho(f)]i\nabla f(x)\cdot A(x;\rho)\Omega)\end{aligned}$$

$$-\frac{\hbar^2}{8m}\int\limits_{\mathbf{R}^3} d^3x\,(\Omega,\ \exp[i\rho(f)]\,\rho(x)\,\nabla\cdot(\rho^{-1}A(x;\rho))\,\Omega)$$

$$-\frac{\hbar^2}{8m}\int\limits_{\mathbf{R}^3} d^3x\,(\Omega,\ \exp[i\rho(f)]\,\tfrac{1}{2}[K(x),\ (\rho^{-1}(x)\,A(x;\rho))]\,\Omega)$$

$$+\frac{1}{2}\int\limits_{\mathbf{R}^3} d^3x\int\limits_{\mathbf{R}^3} d^3y\,V(x-y):\frac{1}{i}\frac{\delta}{\delta f(x)}\frac{1}{i}\frac{\delta}{\delta f(y)}:\mathcal{L}(f)$$

$$-\int\limits_{\mathbf{R}^3} d^3x\,\varepsilon(x)\frac{1}{i}\frac{\delta\mathcal{L}(f)}{\delta f(x)}, \qquad (50)$$

where $\varepsilon(x)\in\mathbf{R}^1_+$, $x\in\mathbf{R}^3$, is the energy density of the ground state of our system. In the derivation of Eq. (50), we have used the relations

$$\left.\begin{aligned}&(\Omega,\ \exp[i\rho(f)]\,K^+(x)\,\rho^{-1}(x)\,K(x)\,\Omega)\\&=\int\limits_{\mathbf{R}^3} d^3x\,(\Omega,\ \exp[i\rho(f)]\,A(x;\rho)\cdot\rho^{-1}(x)\,A(x;\rho)\,\Omega)\\&+\int\limits_{\mathbf{R}^3} d^3x\,(\Omega,\ \exp[i\rho(f)]\,2i\,\nabla f(x)\cdot A(x;\rho)\,\Omega);\\&\qquad 2\nabla\rho(x)=K^+(x)+K(x),\end{aligned}\right\} \qquad (51)$$

which follow directly from the definition of the operator $K(x)$: $\Phi\to\Phi$ and the property (28). Taking into account, besides (50), Eq. (37) and the relation

$$[K(x),(\rho^{-1}(x)\,A(x;\rho))]=2\rho(x)\left[\nabla\frac{\delta}{\delta\rho(x)}\right](\rho^{-1}(x)\,A(x;\rho)), \qquad (52)$$

we can finally write[7]

$$-\frac{\hbar^2}{8m}\int\limits_{\mathbf{R}^3} d^3x\,i\,\nabla f(x)\cdot[\nabla_x-i\,\nabla f(x)]\frac{1}{i}\frac{\delta\mathcal{L}(f)}{\delta f(x)}$$
$$+\frac{\hbar^2}{8m}\int\limits_{\mathbf{R}^3} d^3x\,B(x;\delta)\,\mathcal{L}(f)=\frac{1}{2}\int\limits_{\mathbf{R}^3} d^3x\int\limits_{\mathbf{R}^3} d^3y$$
$$\times V(x-y):\frac{1}{i}\frac{\delta}{\delta f(x)}\frac{1}{i}\frac{\delta}{\delta f(y)}:\mathcal{L}(f)-\int\limits_{\mathbf{R}^3} d^3x\,\varepsilon(x)\frac{1}{i}\frac{\delta\mathcal{L}(f)}{\delta f(x)}. \qquad (53)$$

Here, the operator $B(x;\rho)$ is given by the expression

$$B(x;\rho)=\rho(x)\,\nabla\cdot(\rho^{-1}(x;\rho))+\rho(x)\left[\nabla\frac{\delta}{\delta\rho(x)}\right](\rho^{-1}A(x;\rho)), \qquad (54)$$

and the symbol $[\nabla(\delta/\delta\rho(x))]$ denotes the operator of the gradient of the variational derivative $\delta/\delta\rho(x)$, and not the product of the operators of the gradient and the variational derivative; it is easy to see that this is important.

From the representation (45) for the operator $A(x;\rho)$: $\Phi\to\Phi$ and the expression (54) we can readily find by direct substitution that

$$B(x;\rho)=\sum_{m\in Z_+}(m!)^{-1}\int\limits_{\mathbf{R}^3} d^3y_1\ldots\int\limits_{\mathbf{R}^3} d^3y_m\,\{:\rho(x)\,\rho(y_1)\ldots$$
$$\ldots\rho(y_m):\nabla_x^2\mathcal{A}_{m+1}(x,y_1,\ldots,y_m)$$
$$+\rho(x)\left[\left(\nabla_x+\left[\nabla\frac{\delta}{\delta\rho(x)}\right]\right)\rho^{-1}(x):\rho(x)\,\rho(y_1)\ldots\right.$$
$$\left.\ldots\rho(y_m):\right]\nabla_x\mathcal{A}_{m+1}(x,y_1,\ldots,y_m)\}. \qquad (55)$$

It is easy to show that the second term in (55) is annihilated identically in accordance with (18) and the identity

$$\left(\nabla_{x_1}+\left[\nabla\frac{\delta}{\delta\rho(x_1)}\right]\right)\left(\rho(x_j)-\sum_{k=1}^{j-1}\delta(x_j-x_k)\right)=0_s \qquad (56)$$

which holds for all $j\in\mathbf{Z}_+$. Thus, from (53)–(56) we obtain

$$-\frac{\hbar^2}{8m}\int\limits_{\mathbf{R}^3} d^3x\,i\nabla f(x)\cdot[\nabla_x-i\,\nabla f(x)]\frac{1}{i}\frac{\delta\mathcal{L}(f)}{\delta f(x)}$$
$$+\int\limits_{\mathbf{R}^3} d^3x\,\varepsilon(x)\frac{1}{i}\frac{\delta\mathcal{L}(f)}{\delta f(x)}$$
$$+\frac{\hbar^2}{8m}\int\limits_{\mathbf{R}^3} d^3x\sum_{m\in Z_+}(m!)^{-1}\int\limits_{\mathbf{R}^3} d^3y_1\ldots\int\limits_{\mathbf{R}^3} d^3y_m:\frac{1}{i}\frac{\delta}{\delta f(x)}\frac{1}{i}\frac{\delta}{\delta f(y_1)}$$
$$\ldots\frac{1}{i}\frac{\delta}{\delta f(y_m)}:\mathcal{L}(f)\nabla_x^2\mathcal{A}_{m+1}(x,y_1,\ldots,y_m)$$
$$=\frac{1}{2}\int\limits_{\mathbf{R}^3} d^3x\int\limits_{\mathbf{R}^3} d^3y\,V(x-y):\frac{1}{i}\frac{\delta}{\delta f(x)}\frac{1}{i}\frac{\delta}{\delta f(y)}:\mathcal{L}(f). \qquad (57)$$

Equation (57) in conjunction with Eq. (37) is a system of Bogolyubov-type functional equations for the quantum generating functional $\mathcal{L}(f)$, $f\in\mathcal{S}$ (14). In the case when the expression $[\nabla_x-i\nabla f(x)]\,(\Omega,\exp[i\rho(f)]\,\rho(x)\Omega)$ tends to zero faster than $|x|^{-2}$ for $\mathbf{R}\ni x$ and $|x|\to\infty$, the first term in (57) can be rewritten in the equivalent form

$$\frac{\hbar^2}{8m}\int\limits_{\mathbf{R}^3} d^3x\,[\nabla_x-i\,\nabla f(x)]^2\frac{1}{i}\frac{\delta\mathcal{L}(f)}{\delta f(x)}=-\frac{\hbar^2}{8m}\int\limits_{\mathbf{R}^3} d^3x\,i\,\nabla f(x)$$
$$\times[\nabla_x-i\,\nabla f(x)]\frac{1}{i}\frac{\delta\mathcal{L}(f)}{\delta f(x)}, \qquad (58)$$

which somewhat simplifies the expression of Eq. (57). In terms of the Bogolyubov distribution functions (38), Eq. (58) is equivalent[1,7] to a condition of correlation weakening in the form

$$|F_n(x,y_1,\ldots,y_{n-1})-\bar{\rho}F_{n-1}(y_1,\ldots,y_{n-1})|\to 0 \qquad (59)$$

faster than $|x|^{-1}$ as $|x|\to\infty$ for all $y_j\in\mathbf{R}^3$, $j=\overline{1,n-1}$, $n\in\mathbf{Z}_+$. The condition (59) naturally depends on the interparticle interaction potential in the Hamiltonian operator (16), and if it is a short-range potential,[7] the condition must be satisfied. Bearing in mind further that for the functional $\mathcal{L}(f)$, $f\in\mathcal{S}$ (14) we have the expansion[6,7]

$$\mathcal{L}(f)=\sum_{n\in Z_+}(n!)^{-1}\int\limits_{\mathbf{R}^3} d^3x_1\ldots\int\limits_{\mathbf{R}^3} d^3x_n\prod_{j=1}^{n}\{\exp[if(x_j)]-1\}$$
$$\times F_n(x_1,\ldots,x_n), \qquad (60)$$

we readily obtain from (57), (37), and (60) the following hierarchy of Bogolyubov-type equations[1] for the distribution functions $F_n(x_1,\ldots,x_n)$, $n\in\mathbf{Z}_+$ (38) at zero temperature:

$$\frac{\hbar^2}{8m}\sum_{j=1}^{n}\nabla_{x_j}^2F_n(x_1,\ldots,x_n)=-\frac{\hbar^2}{8m}\int\limits_{\mathbf{R}^1} d^3x_{n+1}$$
$$\times\nabla_{x_{n+1}}^2[F_{n+1}(x_1,\ldots,x_{n+1})-F_1(x_{n+1})\,F_n(x_1,\ldots,x_n)]$$
$$+\sum_{m\in Z_+}\Bigg\{\sum_{(r=\max(m-n,\,0))}\left(\frac{1}{r!\,(m-r)!}\int\limits_{\mathbf{R}^3} d^3x_{n+1}\ldots\right.$$
$$\ldots\int\limits_{\mathbf{R}^3} d^3x_{n+r}\sum_{(j_1\neq j_2\neq\ldots\neq j_{m-r})}V_m(x_{j_1},\ldots,x_{j_{m-r}},x_{n+1},\ldots$$
$$\ldots,x_{n+r})\,F_{n+r}(x_1,\ldots,x_{n+r}))-(m!)^{-1}\int\limits_{\mathbf{R}^3} d^3x_{n+1}\ldots$$

$$\ldots \int_{\mathbf{R}^3} d^3x_{n+m} V(x_{n+1}, \ldots, x_{n+m}) F_m(x_{n+1}, \ldots$$
$$\ldots, x_{n+m}) F_n(x_1, \ldots, x_n) \Big\} ;$$

$$\nabla_{x_1} F_n(x_1, \ldots, x_n) = \sum_{m \in Z_+} \sum_{(r=\max(0,\, m+1-n))} \frac{1}{r!\,(m-r)!}$$
$$\times \sum_{(j_1 \neq j_2 \neq \ldots \neq j_{m-r})}^{n} \int_{\mathbf{R}^3} d^3x_{n+1} \ldots$$
$$\ldots \int_{\mathbf{R}^3} d^3x_{n+r} \nabla_{x_1} \mathcal{A}_{m+1}(x_1, x_{j_1}, \ldots$$
$$\ldots, x_{j_{m-r}}, x_{n+1}, \ldots, x_{n+r}) F_{n+r}(x_1, x_2, \ldots, x_{n+r}). \quad (61)$$

Here, we have introduced the following notation: for $n = 2$

$$V_2(x, y) = -\frac{\hbar^2}{8m}[\nabla_x^2 \mathcal{A}_2(x, y) + \nabla_y^2 \mathcal{A}(y, x)] + V(x-y) \quad (62)$$

and for $n \neq 2$

$$V_n(x_1, \ldots, x_n) = -\frac{\hbar^2}{8m}\frac{1}{(n-1)!}\sum_{\sigma} \nabla_{x_{\sigma(1)}} \mathcal{A}_n(x_{\sigma(1)}, \ldots, x_{\sigma(n)}),$$

where $\sigma \in S_n$ is an element of the group of permutations S_n. In the case $n = 0$, we also obtain an equation that determines the density $\varepsilon(x) \in \mathbf{R}^1$ of the ground-state energy of the Hamiltonian operator (16):

$$F_1(x)\,\varepsilon(x) = -\frac{\hbar^2}{8m}\nabla_x^2 F_1(x) + \sum_{n \in Z_+}(n!)^{-1}\int_{\mathbf{R}^3} d^3x_2 \ldots \int_{\mathbf{R}^3} d^3x_n$$
$$\times V_n(x, \ldots, x_n) F_n(x, x_2, \ldots, x_n). \quad (63)$$

In the case of a canonical representation of the group of currents $\mathscr{S} \wedge \mathrm{Diff}(\mathbf{R}^3)$ we readily find from the translational invariance of the generating functional $\mathscr{L}(f), f \in \mathscr{S}$, that $F_1(x) = \bar{\rho} \in \mathbf{R}^1_+$, which is the mean density of the system of quantum particles, and that $\nabla_x \mathscr{A}_1(x) = 0$ for all $x \in \mathbf{R}^3$. We note here that the expression (63) in the Fourier representation is analogous to Eq. (28) of Ref. 23, which develops the method of collective variables.[24] We note also an important property of (61), namely, depending on the choice of the representation of $\mathscr{S} \wedge \mathrm{Diff}(\mathbf{R}^3)$ in the Hilbert space Φ, these equations are suitable for studying the properties of both Bose and Fermi systems of identical quantum particles at zero temperature in the ground state.

In the case of nonzero temperature, the arguments given above require a certain modification. Namely, the condition $\mathbf{H}\Omega = 0$ on the cyclic ground-state vector $\Omega \in \Phi$ is no longer valid, and therefore we must study in more detail the structure of the generating functional (15). Following Ref. 6, we obtain

$$\left.\begin{aligned}
&\mathscr{L}(f) = W(f)/W(0);\\
&W(f) = \mathrm{tr}(\exp(-\beta\overline{\mathbf{H}})\exp[i\rho(f)])\\
&= \mathrm{tr}(\exp(-\beta\overline{\mathbf{H}}_0)\,\mathbf{C}\exp(-\beta V(\rho))\exp[i\rho(f)])\\
&= (\Omega_0, \mathbf{C}\exp[-\beta V(\delta)]\exp[i\rho(f)]\,\Omega_0)\\
&= \exp[-\beta V(\delta)]C(\delta)\,\mathscr{L}_0(f),
\end{aligned}\right\} \quad (64)$$

where the operators $\mathbf{C}\colon \Phi \to \Phi$ and $C(\delta)$ are defined as follows:

$$\left.\begin{aligned}
&\mathbf{C} = \exp(\beta\overline{\mathbf{H}}_0)\exp(-\beta\overline{\mathbf{H}})\exp[\beta V(\rho)];\\
&\mathbf{C}^+\Omega_0 = C^+(\rho)\,\Omega_0, \quad C(\delta) = C(\rho)\big|_{\rho = \frac{1}{i}\frac{\delta}{\delta f}};\\
&V(\rho) = \frac{1}{2}\int_{\mathbf{R}^3} d^3x \int_{\mathbf{R}^3} d^3y V(x, y) : \rho(x)\rho(y) :
\end{aligned}\right\} \quad (65)$$

where $\Omega_0 \in \Phi$ is the ground state of the system of noninteracting particles at temperature $\beta \in \mathbf{R}^1_+$. The functional $\mathscr{L}_0(f)$ is the generating functional of the free system of noninteracting particles at temperature $\beta \in \mathbf{R}^1_+$; it satisfies a functional equation of the form (32), and

$$\mathscr{L}_0(f) = (\Omega_0, \exp[i\rho(f)]\,\Omega_0)$$
$$= \mathrm{tr}(\exp(-\beta\overline{\mathbf{H}}_0)\exp[i\rho(f)])\,[\mathrm{tr}\exp(-\beta\overline{\mathbf{H}}_0)]^{-1}. \quad (66)$$

Bearing in mind that the ground state $\Omega_0 \in \Phi$ of a noninteracting system of $N \in \mathbf{Z}_+$ particles is known,[8] we can also regard as known the operator $A_0(x;\rho)$ of the type (45) that occurs in the equation of the form (37):

$$[\nabla_x - i\nabla f(x)]\frac{1}{i}\frac{\delta \mathscr{L}_0(f)}{\delta f(x)} = A_0(x;\delta)\,\mathscr{L}_0(f). \quad (67)$$

By direct calculations one can readily show that $A_0(x;\rho) = 0, x \in \mathbf{R}^3$, in the case of a Bose system:

$$[\nabla_x - i\nabla f(x)]\frac{1}{i}\frac{\delta \mathscr{L}_0(f)}{\delta f(x)} = 0 \quad (68)$$

as $\hbar, \beta \to 0$. It is readily verified that Eq. (68) is solved by the functional[7,6]

$$\mathscr{L}_0(f) = \exp\Big(z\int_{\mathbf{R}^3} d^3x\{\exp[if(x)] - 1\}\Big). \quad (69)$$

We consider the case of the generating functional $\mathscr{L}_0(f)$ (66) for a Fermi system in the Appendix.

We now consider in more detail the structure of the operator $\mathbf{C}\colon \Phi \to \Phi$ (65). In accordance with the definition (65), the equation for the operator $\mathbf{C}\colon \Phi \to \Phi$ with respect to the variable $\beta \in \mathbf{R}^1_+$ has the form

$$\frac{\partial \mathbf{C}}{\partial \beta} = \mathbf{C}V(\rho) - V_\beta(\rho)\,\mathbf{C}, \quad (70)$$

where

$$V_\beta(\rho) = \exp(\beta\mathbf{H}_0)V(\rho)\exp(-\beta\mathbf{H}_0)$$
$$= V(\rho) + \beta[\mathbf{H}_0, V(\rho)] + \frac{\beta^2}{2!}[\mathbf{H}_0, [\mathbf{H}_0, V(\rho)]] + \ldots \quad (71)$$

Expanding the operator $\mathbf{C}\colon \Phi \to \Phi$ in a series in the parameter $\beta \in \mathbf{R}^1_+$ (high-temperature expansion), we find from (70)

$$\left.\begin{aligned}
&\mathbf{C} = 1 + \beta\mathbf{C}_1 + \frac{\beta^2}{2!}\mathbf{C}_2 + \frac{\beta^3}{3!}\mathbf{C}_3 + \ldots;\\
&\mathbf{C}_1 = 0, \quad \mathbf{C}_2 = -[\mathbf{H}_0, V(\rho)];\\
&\mathbf{C}_3 = [[\mathbf{H}_0, V(\rho)], \mathbf{H}_0 - V(\rho)];\\
&\mathbf{C}_4 = [\mathbf{C}_3, V(\rho)] + 3[\mathbf{H}_0, V(\rho)]^2 - \mathrm{Ad}^3(\mathbf{H}_0)V(\rho)\ldots
\end{aligned}\right\} \quad (72)$$

etc., where all the commutators in (72) are calculated by means of the relations (14) for the Lie algebra of the currents. In particular, from (33) we obtain

$$[\mathbf{H}_0, V(\rho)] = \Big[\mathbf{H}_0, \frac{1}{2}\int_{\mathbf{R}^3} d^3x \int_{\mathbf{R}^3} d^3y V(x, y) : \rho(x)\rho(y) :\Big]$$

$$= \frac{1}{2}\int_{\mathbf{R}^3} d^3x \int_{\mathbf{R}^3} d^3y V(x, y)[\mathbf{H}_0, \rho(x)]\rho(y)$$

$$+ \frac{1}{2}\int_{\mathbf{R}^3} d^3x \int_{\mathbf{R}^3} d^3y V(x, y)\rho(x)[\mathbf{H}_0, \rho(y)]$$

$$+ \frac{1}{2}\int_{\mathbf{R}^3} d^3x \int_{\mathbf{R}^3} d^3y V(x, y)\delta(x-y)[\mathbf{H}_0, \rho(x)]$$

$$= -\frac{\hbar^2}{2im}\int_{\mathbf{R}^3} d^3x \int_{\mathbf{R}^3} d^3y [\rho(x) J(y)\nabla_y V(x, y)$$

$$+ J(x)\nabla_x V(x, y)\rho(y)]$$

$$-\frac{1}{2}\int_{\mathbf{R}^3} d^3x \int_{\mathbf{R}^3} d^3y V(x, y)\delta(x-y)\frac{\hbar^2}{im}\nabla_x J(x). \qquad (73)$$

Since the formula

$$(\exp[i\rho(f_1)]\Omega_0, \quad J(g)\exp[i\rho(f)]\Omega_0)$$
$$= \frac{1}{2}(\exp[i\rho(f_1)]\Omega_0, \quad \rho(g\nabla(f_1+f))\exp[i\rho(f)]\Omega_0) \qquad (74)$$

is valid for all $f_1, f \in \mathcal{S}(\mathbf{R}^3;\mathbf{R}^1)$, $g \in \mathcal{S}(\mathbf{R}^3;\mathbf{R}^3)$, we find from (73) and (74)

$$(\Omega_0, [\mathbf{H}_0, V(\rho)]\exp[i\rho(f)]\Omega_0)$$
$$= -\frac{\hbar^2}{4im}\int_{\mathbf{R}^3} d^3x \int_{\mathbf{R}^3} d^3y (\Omega_0, \rho(x)\rho(\nabla_y V(x, y)\nabla_y f)$$
$$\exp[i\rho(f)]\Omega_0)$$

$$-\frac{\hbar^2}{4m}\int_{\mathbf{R}^3} d^3x \int_{\mathbf{R}^3} d^3y (\Omega_0, \nabla\rho(y)\nabla_y V(x, y)\exp[i\rho(f)]\Omega_0)$$

$$-\frac{\hbar^2}{4im}\int_{\mathbf{R}^3} d^3x \int_{\mathbf{R}^3} d^3y (\Omega_0, \rho(\nabla_x V(x, y)\nabla_x f)\rho(y)\exp[i\rho(f)]\Omega_0)$$

$$-\frac{\hbar^2}{4m}\int_{\mathbf{R}^3} d^3x \int_{\mathbf{R}^3} d^3y (\Omega_0, \nabla\rho(x)\nabla_x V(x, y)\exp[i\rho(f)]\Omega_0)$$

$$+\frac{\hbar^2}{2im}\int_{\mathbf{R}^3} d^3x \int_{\mathbf{R}^3} d^3y (\Omega_0, \nabla_x[V(x, y)\delta(x-y)]$$
$$J(x)\exp[i\rho(f)]\Omega_0)$$

$$= -\frac{\hbar^2}{4im}\int_{\mathbf{R}^3} d^3x \int_{\mathbf{R}^3} d^3y [\nabla_y V(x, y)\nabla_y f$$
$$+ \nabla_x V(x, y)\nabla_x f]\frac{1}{i}\frac{\delta}{\delta f(x)}$$
$$\times \frac{1}{i}\frac{\delta}{\delta f(y)}\mathcal{L}_0(f) - \frac{\hbar^2}{4m}\int_{\mathbf{R}^3} d^3x \int_{\mathbf{R}^3} d^3y \Big\{\nabla_x V(x, y)\Big[\nabla\frac{1}{i}\frac{\delta}{\delta f(x)}\Big]$$
$$+ \nabla_y V(x, y)\Big[\frac{1}{i}\frac{\delta}{\delta f(y)}\Big]\Big\}\mathcal{L}_0(f)$$
$$+\frac{\hbar^2}{4im}\int_{\mathbf{R}^3} d^3x \int_{\mathbf{R}^3} d^3y \{\nabla_x[V(x, y)\delta(x, y)]\nabla_x f\}$$
$$\frac{1}{i}\frac{\delta\mathcal{L}_0(f)}{\delta f(x)} = C_2(\delta)\mathcal{L}_0(f).$$

The representations of all the remaining terms in the expansion (72) for the operator $\mathbf{C}: \Phi \to \Phi$ can be calculated similarly.

We consider some obvious consequences of the results obtained above in the case of high temperatures $\mathbf{R}^1_+ \ni \beta \to 0$ or as $\hbar \to 0$, when the quantum expressions go over into the corresponding expressions of classical statistics.[6,8] From (64) in the limit $\hbar \to 0$ we obtain

$$\left.\begin{aligned} &\mathcal{L}(f) = W(f)/W(0);\\ &W(f) = \exp[-\beta V(\delta)]\mathcal{L}_0(f), \end{aligned}\right\} \qquad (75)$$

where the functional $\mathcal{L}_0(f)$ satisfies Eq. (67) in the case of a Fermi system and Eq. (69) in the case of a Bose system.

For the energy density $\varepsilon(x) \in \mathbf{R}^1$ of the ground state we have the integral expression

$$\int_{\mathbf{R}^3} d^3x\varepsilon(x)F_1(x) = \frac{\hbar^2}{8m}\Big(\Omega, \int_{\mathbf{R}^3} d^3x K^+(x)\rho^{-1}(x)K(x)\Omega\Big)$$
$$+\frac{1}{2}\int_{\mathbf{R}^3} d^3x \int_{\mathbf{R}^3} d^3y V(x, y)(\Omega, :\rho(x)\rho(y):\Omega). \quad (76)$$

To calculate it, we must use the relation (57) in the form (63), where the distribution functions $F_n(x_1,\ldots, x_n)$, $n \in \mathbf{Z}_+$, are found from the relation (39) by means of the generating functional (64).

If we again set $\hbar \to 0$, then from (67) and (75) we readily obtain a functional equation of Bogolyubov type[1] for the functional $\mathcal{L}(f), f \in \mathcal{S}(\mathbf{R}^3;\mathbf{R}^1)$,[37]

$$[\nabla_x - i\nabla f(x)]\frac{1}{i}\frac{\delta\mathcal{L}(f)}{\delta f(x)} = A_0(x;\delta)\mathcal{L}(f)$$
$$-\beta\int_{\mathbf{R}^3} d^3y\nabla_x V(x, y) : \frac{1}{i}\frac{\delta}{\delta f(x)}\frac{1}{i}\frac{\delta}{\delta f(y)} : \mathcal{L}(f). \qquad (77)$$

Equation (77) is valid for a system of particles satisfying either Fermi or Bose statistics (at high temperatures), and in the second case, as pointed out earlier, $A_0(x;\delta) = 0$ for all $x \in \mathbf{R}^3$. Thus, solving the functional equation (77), from Eq. (56) we find an explicit functional–operator expression for the generating functional $\mathcal{L}(f)$ of the distribution functions (38). In view of the great success[25–29] in the investigation of exactly solvable nonlinear models of nonrelativistic field theory, there is undoubted interest in comparing the results and combining the methods of investigation from Refs. 25–29 and 34 with the corresponding methods of the present paper. We hope to return to these and related questions in a further investigation.

3. QUANTUM METHOD OF BOGOLYUBOV GENERATING FUNCTIONALS IN NONEQUILIBRIUM STATISTICAL MECHANICS: LIE ALGEBRA OF THE CURRENTS, WIGNER REPRESENTATION, AND HAMILTONIAN STRUCTURE OF THE FUNCTIONAL EQUATIONS

We assume that for the group $\mathcal{S}\wedge\mathrm{Diff}(\mathbf{R}^3)$ of the Lie algebra of the currents (4) there exists a representation (6) with a so-called nonequilibrium cyclic vector $\Omega \in \Phi$ in the Hilbert space Φ for which the set $D = \mathrm{span}\{\exp[i\rho(f)]\Omega: f \in \mathcal{S}\}$ is dense in Φ, $\overline{D} = \Phi$, and $D \supset \mathrm{dom}\,\mathbf{H}$. For the density operator $\rho(f), f \in \mathcal{S}(\mathbf{R}^3;\mathbf{R}^1)$, the continuity equation holds:

$$\frac{\partial\rho(f)}{\partial t} = \frac{i}{\hbar}[\mathbf{H}, \rho(f)] = \frac{\hbar}{m}J(\nabla f), \qquad (78)$$

where $t \in \mathbf{R}^1$ is the evolution parameter of our quantum system. The time-reversal operator $T: \mathbf{R}^1 \ni t \to -t \in \mathbf{R}^1$ acts as follows:

$$T\rho(f)T^{-1}=\rho(f);\quad T\Omega=\Omega^*;\quad TJ(g)T^{-1}=-J(g);\quad THT^{-1}=\mathbf{H} \tag{79}$$

for all $f\in\mathscr{S}(\mathbf{R}^3;\mathbf{R}^1)$, $g\in\mathscr{S}(\mathbf{R}^3;\mathbf{R}^3)$. In accordance with the definition (15) for the generating functional $\mathscr{L}(f)f\in\mathscr{S}$, we can write down the following functional equation:

$$\frac{\partial\mathscr{L}(f)}{\partial t}=\operatorname{tr}\left(\mathscr{P}\frac{i}{\hbar}[\mathbf{H},\exp[i\rho(f)]]\right), \tag{80}$$

where we have used for the statistical operator $\mathfrak{P}:\Phi\to\Phi$ the Liouville dynamical equation[1,8]:

$$\frac{\partial\mathscr{P}}{\partial t}=\frac{i}{\hbar}[\mathscr{P},\mathbf{H}],\quad \operatorname{tr}\mathscr{P}=1. \tag{81}$$

Rewriting Eq. (80) in the equivalent form

$$\frac{\partial\mathscr{L}(f)}{\partial t}=\left(\Omega,\frac{i}{\hbar}[\mathbf{H},\exp[i\rho(f)]]\Omega\right), \tag{82}$$

we obtain from (82) and (25)

$$\frac{\partial\mathscr{L}(f)}{\partial t}=\left(\Omega,\frac{\hbar}{m}\left[J(\nabla f)-\frac{1}{2}\rho(\nabla f\nabla f)\right]\exp[i\rho(f)]\Omega\right). \tag{83}$$

To make the expression on the right-hand side of (43) effective, we note that for the current $J(x):\Phi\to\Phi$ there exists the representation[10,11]

$$J(x)=\rho(x)\left[\frac{1}{i}\nabla\frac{\delta}{\delta\rho(x)}\right]+J_0(x;\rho), \tag{84}$$

where $J_0(x;\rho)$, $x\in\mathbf{R}^3$, is an arbitrary operator function. Then from (43) and (44) we find

$$\frac{\partial\mathscr{L}(f)}{\partial t}=\frac{\hbar}{m}\left(\Omega,\int_{\mathbf{R}^3}d^3x\left(\rho(x)\left[\frac{1}{i}\nabla\frac{\delta}{\delta\rho(x)}\right]\nabla_x f\right)\exp[i\rho(f)]\Omega\right)$$

$$-\frac{\hbar}{2m}(\Omega,\rho(\nabla f\nabla f)\exp[i\rho(f)]\Omega)+(\Omega,J_0(\nabla f;\rho)\exp[i\rho(f)]\Omega)$$

$$=\left(\Omega,\frac{\hbar}{m}\rho(\nabla f\nabla f)\exp[i\rho(f)]\Omega\right)+(\Omega,J_0(\nabla f;\delta)\exp[i\rho(f)]\Omega)$$

$$-\frac{\hbar}{2m}(\Omega,\rho(\nabla f\nabla f)\exp[i\rho(f)]\Omega)$$

$$=\frac{\hbar}{2m}\int_{\mathbf{R}^3}d^3x(\nabla_x f\nabla_x f)\frac{1}{i}\frac{\delta\mathscr{L}(f)}{\delta f(x)}+J_0(\nabla f;\delta)\mathscr{L}(f), \tag{85}$$

where, as usual, we have denoted $J_0(x;\delta)=J_0(x;\rho)|_{\rho=(1/i)(\delta/\delta f)}$. In order to particularize the operator $J_0(x;\delta)$ in (85), we again consider an N-particle representation of the Lie algebra of the currents (4) in the Hilbert space $\Phi\approx L_2^{(\pm)}(\mathbf{R}^{3N};\mathbf{C}^1)$:

$$\left.\begin{aligned}&\rho(x)\omega(x_1,\ldots,x_N)=\sum_{j=1}^{N}\delta(x-x_j)\omega(x_1,\ldots,x_N);\\&J(x)\omega(x_1,\ldots,x_N)\\&=\frac{1}{2i}\sum_{j=1}^{N}[-\nabla_x\delta(x-x_j)+2\delta(x-x_j)\nabla_{x_j}]\omega(x_1,\ldots,x_N),\end{aligned}\right\} \tag{86}$$

where $N\in\mathbf{Z}_+$; $\omega\in L_2^{(\pm)}(\mathbf{R}^{3N};\mathbf{C}^1)$. The generating functional $\mathscr{L}(f,g)$, $f\in\mathscr{S}(\mathbf{R}^3;\mathbf{R}^1)$, $g\in\mathscr{S}(\mathbf{R}^3;\mathbf{R}^3)$, of the representation (86) is given[7] by

$$\mathscr{L}(f,g)=(\Omega,\exp[i\rho(f)]\exp[iJ(g)]\Omega)=\int_{\mathbf{R}^3}d^3x_1\ldots\int_{\mathbf{R}^3}d^3x_N\Omega^*(x_1,\ldots,x_N)$$

$$\times\prod_{j=1}^{N}\exp[if(x_j)]\exp[i\xi(x_j;g)]\Omega(x_1,\ldots,x_N), \tag{87}$$

where $\xi(x;g)=(1/2i)\quad[2g(x)\nabla_x+\Delta g(x)]$, $x\in\mathbf{R}^3$; $\Omega\in L_2^{(\pm)}(\mathbf{R}^{3N};\mathbf{C}^1)$ is the ground state. For any function $\omega\in L_2^{(\pm)}(\mathbf{R}^{3N};\mathbf{C}^1)$ the operator $\exp[i\xi(x;g)]$ acts in accordance with the rule

$$\exp[i\xi(x;g)]\omega(x_1,\ldots,x_N)=(\varphi^*\omega)(x,\ldots,x_N)\left[\det\left\|\frac{\partial\varphi(x)}{\partial x}\right\|\right]^{1/2}. \tag{88}$$

Here, $(\varphi^*\omega)(x,\ldots,x_N)=\omega(\varphi x,\ldots,\varphi x_N)$; $\varphi\in\mathrm{Diff}(\mathbf{R}^3)$ is the flux corresponding to the vector field $g\in\mathscr{S}(\mathbf{R}^3;\mathbf{R}^3)$, i.e., $\varphi(x)=\varphi_1^g(x)$, where $(d/dt)\varphi_t^g(x)=g(\varphi_t^g(x))$, and the relations (5) hold. From the generating functional (87) we can obtain

$$(\Omega,J(\Delta f)\exp[i\rho(f)]\Omega)=\int_{\mathbf{R}^3}d^3x\frac{1}{i}\frac{\delta\mathscr{L}(f,g)}{\delta g(x)}\bigg|_{g=0} \tag{89}$$

and thereby determine the operator $J_0(x;\rho)$ in (85). For $\mathbf{Z}_+\ni N\to\infty$, $\Lambda\nearrow\mathbf{R}^3$, $N/\Lambda=\bar\rho\in\mathbf{R}^1_+$, the expression (87) goes over[7] into

$$\mathscr{L}(f,g)=\sum_{n\in\mathbf{Z}_+}(n!)^{-1}\int_{\mathbf{R}^3}d^3x_1\ldots\int_{\mathbf{R}^3}d^3x_n\int_{\mathbf{R}^3}d^3y_1\ldots\int_{\mathbf{R}^3}d^3y_n$$

$$\times\prod_{j=1}^{n}[\delta(x_j-y_j)\{\exp[if(x_j)]\exp[i\xi(x_j;g)]-1\}$$

$$\times F_n(y_1,\ldots,y_n;x_1,\ldots,x_n), \tag{90}$$

where for all $n\in\mathbf{Z}_+$

$$F_n(y_1,\ldots,y_n;x_1,\ldots,x_n)=(\Omega,\psi^+(y_n)\ldots\psi^+(y_1)\psi(x_1)\ldots\psi(x_n)\Omega). \tag{91}$$

It is also obvious that $F_n(x_1,\ldots,x_n;x_1,\ldots,x_n)=F_n(x_1,\ldots,x_n)$, $n\in\mathbf{Z}_+$, where

$$F_n(x_1,\ldots,x_n)=(\Omega,:\rho(x_1)\ldots\rho(x_n):\Omega). \tag{92}$$

In the limit $\hbar\to0$, we now introduce the following[6,8,9] quantized Wigner operator $w(x,p):\Phi\to\Phi$:

$$w(x,p)=\frac{1}{(2\pi)^3}\int_{\mathbf{R}^3}d^3\alpha e^{i\alpha p}\psi^+\left(x+\frac{\hbar\alpha}{2}\right)\psi\left(x-\frac{\hbar\alpha}{2}\right), \tag{93}$$

where $(x,p)\in\mathbf{R}^3\times\mathbf{R}^3$. Making the transformation (93) in (91), we go over to the Wigner representation[9] for the classical nonequilibrium Bogolyubov distribution function $F_n(x_1,p_1;\ldots;x_n,p_n)$, $n\in\mathbf{Z}_+$,[1]

$$F_n(x_1,p_1;\ldots;x_n,p_n)=(\Omega,:w(x_1,p_1)\ldots w(x_n,p_n):\Omega), \tag{94}$$

where the pair (x_j,p_j), $j=\overline{1,n}$, are the coordinate and momentum of particle j. Going over similarly to the Wigner representation for the generating functional $\mathscr{L}(f,g)$ (90), we can obtain

$$\mathscr{L}(f, g) \to \mathscr{L}(f) = \sum_{n \in \mathbf{Z}_+} (n!)^{-1} \int_{\mathbf{R}^3 \times \mathbf{R}^3} d^3x_1 d^3p_1 \times \int_{\mathbf{R}^3 \times \mathbf{R}^3} d^3x_n d^3p_n \prod_{j=1}^{n} \{\exp[if(x_j, p_j)] - 1\} \times F_n(x_1, p_1; \ldots; x_n, p_n), \quad (95)$$

where, by definition, $f \in \mathscr{S}(\mathbf{R}^6; \mathbf{R}^1)$ and in the limit $\hbar \to 0$

$$\exp[if(x, p)] \simeq \hbar^3 \int_{\mathbf{R}^3} d^3\beta\delta(\hbar\beta) \left\{\exp\left[if\left(x - \frac{\hbar\beta}{2}\right)\right] \times \exp\left[i\xi\left(x - \frac{\hbar\beta}{2}; g\right)\right]\right\} \exp(i\beta p). \quad (96)$$

Thus, in accordance with (94)–(96) we have for the generating functional (90) its Wigner representation in the form

$$\mathscr{L}(f) = (\Omega, \exp[iw(f)]\Omega), \quad (97)$$

where $w(f) = \int_{\mathbf{R}^3 \times \mathbf{R}^3} d^3x\, d^3p\, f(x, p) w(x, p), f \in \mathscr{S}(\mathbf{R}^6; \mathbf{R}^1)$. In the Wigner representation (93), the Hamiltonian $\mathbf{H}$: $\Phi \to \Phi$ (17) is given by the expression[6,8]

$$\mathbf{H} = \int_{\mathbf{R}^3 \times \mathbf{R}^3} d^3x d^3p T(p) w(x, p) + \frac{1}{2} \int_{\mathbf{R}^3 \times \mathbf{R}^3} d^3x d^3p \int_{\mathbf{R}^3 \times \mathbf{R}^3} d^3y d^3\xi V(x, y) : w(x, p) w(y, \xi):, \quad (98)$$

where $T(p) = p^2/2m, p \in \mathbf{R}^3$.

In order to study the dynamics with respect to the evolution variable $t \in \mathbf{R}^1$ of the functional $\mathscr{L}(f)$ (97), we note the validity of the following formulas, which are all understood in the weak sense:

$$\left.\begin{aligned}
&[w(x, p), w(y, \xi)] \simeq 0, \quad \hbar \to 0; \\
&:w(x, p) w(y, \xi): = w(x, p)[w(y, \xi) - \delta(x - y)\delta(p - \xi)]; \\
&\frac{i}{\hbar}[\mathbf{H}_0, w(x, p)] \overset{\hbar \to 0}{\simeq} \left\{\frac{p^2}{2m}, w(x, p)\right\}^{(1)}; \\
&\frac{i}{\hbar}[\mathbf{H} - \mathbf{H}_0, w(x, p)] \overset{\hbar \to 0}{\simeq} \int_{\mathbf{R}^3 \times \mathbf{R}^3} d^3y d^3\xi \\
&\times \{V(x, y), :w(x, p), w(y, \xi):\}^{(2)},
\end{aligned}\right\} \quad (99)$$

where $\{\cdot, \cdot\}^{(j)}$ is the corresponding canonical classical Poisson bracket on the manifold of variables $\mathbf{R}^{3j} \times \mathbf{R}^{3j}, j \in \mathbf{Z}_+$. The (local) momentum operator in the Wigner representation has the form

$$J(g) \overset{\hbar \to 0}{\simeq} \int_{\mathbf{R}^3 \times \mathbf{R}^3} d^3x d^3p g(x, p) p w(x, p), \quad (100)$$

where $g \in \mathscr{S}(\mathbf{R}^6; \mathbf{R}^3)$. The properties (99) of the Wigner operators $w(x, p), (x, p) \in \mathbf{R}^3 \times \mathbf{R}^3$, make it possible to carry out a complete algebraization of all the basic relations of the dynamical theory as $\hbar \to 0$. For this purpose, we consider the algebra A of self-adjoint operators of the form

$$\mathbf{K}_n = \int_{\mathbf{R}} d^3q_1 \ldots \int_{\mathbf{R}^3} d^3q_n K_n(q_1, \ldots, q_n) : w(q_1) \ldots w(q_n):, \quad (101)$$

where $q_j = (x_j, p_j) \in \mathbf{R}^3 \times \mathbf{R}^3$, $j = \overline{1, n}$; $K_n \in \mathrm{sym}[\mathbf{R}^{3n} \times \mathbf{R}^{3n}; \mathbf{R}^1]$, the space of real symmetric functions, $n \in \mathbf{Z}_+$. The set A of operators of the form (101) and their linear combinations over $\mathbf{R}^1$ obviously forms a Lie algebra with respect to the ordinary commutator bracket $[\cdot, \cdot]$ $(i/\hbar)$, and it is easy to show that

$$\frac{i}{\hbar}[A_j, A_k] \subset \sum_{l = \max(j, k)}^{j+k-1} A_l, \quad (102)$$

where $A_j = \{K_j \in A\}, j \in \mathbf{Z}_+, A = \Sigma_{j \in \mathbf{Z}_+} A_j$. Using formulas of the form (99) and their generalizations, we can in the limit $\hbar \to 0$ obtain from $(i/\hbar)$ $[\cdot, \cdot]$ the new bracket $[\cdot, \cdot]_0$ by means of the rule

$$[\cdot, \cdot]_0 = \lim_{\hbar \to 0} \frac{i}{\hbar}[\cdot, \cdot], \quad (103)$$

where the limit in (103) is considered in the weak sense. In particular, for operators $K_j \in A_j, j \in \mathbf{Z}_+$,

$$[\mathbf{K}_j, \mathbf{K}_n]_0 = \sum_{l=1}^{\min(j, n)} \int_{\mathbf{R}^6} d^3q_1 \ldots \int_{\mathbf{R}^3} d^3q_l : w(q_1) \ldots w(q_l) \times \left\{\frac{\delta^l \mathbf{K}_j}{\delta w(q_1) \ldots \delta w(q_l)}, \frac{\delta^l \mathbf{K}_n}{\delta w(q_1) \ldots \delta w(q_l)}\right\}^{(l)} :. \quad (104)$$

The bracket (104) is a natural generalization of the well-known Poisson–Vlasov bracket[8,30] in the theory of kinetic equations. In view of its structure, we shall say that the Lie algebra $A = \Sigma_{j \in \mathbf{Z}_+} A_j$ is *hierarchical* if the Poisson bracket in it is given in accordance with (104). By virtue of (104), it is meaningful to consider on the set of operators A a new Lie algebra $\mathfrak{A}$ of the form

$$\mathfrak{A} = \bigoplus_{j \in \mathbf{Z}_+} A_j, \quad (105)$$

in which the structure of the Lie algebra is given by the bracket $[\![\cdot, \cdot]\!]$, for which we have

$$[\![\mathfrak{A}, \mathfrak{A}]\!] = \bigoplus_{l \in \mathbf{Z}_+} \sum_{j, k \in \mathbf{Z}_+} [A_j, A_k]_0^{(l)}, \quad (106)$$

where, by definition, $[A_j, A_k]_0^{(l)} \in A_l$, and, in accordance with (104), $[A_j, A_k]_0 = \Sigma_{l \in \mathbf{Z}_+} [A_j, A_k]_0^{(l)}, j, k \in \mathbf{Z}_+$.

We now consider the linear mapping $\alpha: \mathfrak{A} \to A$, where for any element $(\mathbf{K}_1, \ldots, \mathbf{K}_j, \ldots) \in \mathfrak{A}$

$$\alpha(\mathbf{K}_1, \ldots, \mathbf{K}_j \ldots) = \sum_{j \in \mathbf{Z}_+} \mathbf{K}_j \in A, \quad (107)$$

the sum in (107) being regarded as an operator in the initial Hilbert space Φ. By direct calculations we can show that the mapping $\alpha: \mathfrak{A} \to A$ is a homomorphism of the Lie algebra $\mathfrak{A}$ and A with brackets $[\![\cdot, \cdot]\!]$ and $[\cdot, \cdot]_0$, respectively. Besides the mapping $\alpha: \mathfrak{A} \to A$, we consider the dual mapping α^*: $A^* \to \mathfrak{A}^*$, where, by definition,

$$\mathfrak{A}^* = \bigoplus_{j \in \mathbf{Z}_+} A_j^*, \quad (108)$$

$$A^* = \{F \in \mathscr{D}(A) : F\mathbf{K} = \mathrm{tr}(\mathscr{P}\mathbf{K}), \ \mathbf{K} \in A\},$$

and $\mathfrak{P}$ is the statistical operator of our "semiclassical" system as $\hbar \to 0$; it satisfies the Liouville equation. It is readily verified that for an element $F \in A^*$ the expression

$$\alpha^*F=(F_1, \ldots, F_j, \ldots)=\mathscr{F}\in\mathfrak{A}^* \quad (109)$$

defines a mapping on the manifold $\mathfrak{A}^*$ of distribution functions $F_j=\operatorname{tr}(\mathfrak{P}: \omega(q_1),\ldots, \omega(q_j):)$, $j\in\mathbf{Z}_+$, and for any $k\in\mathfrak{A}$

$$\mathscr{F}\mathbf{k}=(F_1, F_2, \ldots, F_j, \ldots)\circ(K_1, K_2, \ldots, K_j, \ldots)=\sum_{j\in Z_+}\int_{\mathbf{R}^s} d^3q_1 \ldots \int_{\mathbf{R}^s} d^3q_j F_j(q_1, \ldots, q_j) K_j(q_1, \ldots, q_j). \quad (110)$$

Let $B(F)$, $C(F)\in\mathscr{D}(A^*)$, where A^* are functionals over the dual space A^*. Then on $\mathscr{D}(A^*)$ there is defined the Lie–Poisson bracket[30] $\{\cdot,\cdot\}_0$ in accordance with the rule

$$\{B(F), C(F)\}_0=F\cdot[\mathbf{B}, \mathbf{C}]_0, \quad (111)$$

where $\mathbf{B}, \mathbf{C}\in A$ are such that $F\cdot\mathbf{B}=B(F)$, $F\cdot\mathbf{C}=C(F)$. Similarly, on the set of functionals $\mathscr{D}(\mathfrak{A}^*)$ over the dual space $\mathfrak{A}^*$ (108) the dual Lie–Poisson bracket $\{\{\cdot,\cdot\}\}$ is defined in accordance with the rule

$$\{\{\mathscr{B}(\mathscr{F}), \mathscr{C}(\mathscr{F})\}\}=\mathscr{F}\cdot[[\mathbf{b}, \mathbf{c}]], \quad (112)$$

where $\mathbf{b}, \mathbf{c}\in\mathfrak{A}$ are such that $\mathscr{F}\cdot\mathbf{b}=\mathscr{B}(\mathscr{F})$, $\mathscr{F}\cdot\mathbf{c}=\mathscr{C}(\mathscr{F})$ for all $\mathscr{F}\in\mathfrak{A}^*$.

DEFINITION 3.1. We shall say that mapping of Lie algebras α: $\mathfrak{A}\to A$ is canonical (or Poisson[30,31]) if for all $B(F)$, $C(F)$

$$\alpha^*\{B(F), C(F)\}_0=\{\{\alpha^*B(\mathscr{F}), \alpha^*C(\mathscr{F})\}\}, \quad (113)$$

where $\mathscr{F}=\alpha^*F\in\mathfrak{A}^*$.

The following proposition[30] will be helpful.

PROPOSITION 3.2. Let A and $\mathfrak{A}$ be two arbitrary Lie algebras and α: $\mathfrak{A}\to A$ be a linear mapping. Then the dual α^*: $\mathscr{D}(A^*)\to\mathscr{D}(\mathfrak{A}^*)$ is canonical if and only if α: $\mathfrak{A}\to A$ is a homomorphism of the Lie algebras.

Since the mapping α: $\mathfrak{A}\to A$ is by virtue of its explicit construction a homomorphism of Lie algebras, we have in accordance with Proposition 3.2 established the following proposition:

PROPOSITION 3.3. The dual mapping α^*: $\mathscr{D}(A^*)\to\mathscr{D}(\mathfrak{A}^*)$ is canonical.

We now consider the generating functional $\mathscr{L}(f)$ in the form (97). To obtain for it an evolution Bogolyubov functional equation,[1] we apply the algebraic technique developed above to the calculation of the following quantity:

$$\frac{\partial\mathscr{L}(f)(\mathscr{F})}{\partial t}=\operatorname{tr}\left(\mathscr{P}[\mathbf{H}, \exp[iw(f)]]\frac{i}{\hbar}\right) \quad (114)$$

as $\hbar\to 0$, $t\in\mathbf{R}^1$.

In accordance with Eqs. (95) and (104), we find

$$\frac{\partial}{\partial t}\mathscr{L}(f)(\mathscr{F})=\operatorname{tr}(\mathscr{P}[\mathbf{H}, :\exp[w(e^{if}-1)]:]_0)$$
$$=\alpha^*\{H(\mathscr{F}), \mathscr{L}(f)(\mathscr{F})\}_0, \quad (115)$$

where we have used the fact that for the functional $\mathscr{L}(f)(\mathscr{F})$ the following identity holds:

$$\mathscr{L}(f)(\mathscr{F})=\mathscr{L}_B(u)(\mathscr{F}), \quad (116)$$

where formally $u=\exp(if)-1$ and

$$\mathscr{L}_B(u)=\operatorname{tr}(\mathscr{P}:\exp[w(u)]:) \quad (117)$$

is the Bogolyubov functional from Ref. 1 in the limit $\hbar\to 0$. In addition, for the functional $H(F)\in\mathscr{D}(A^*)$

$$H(F)=\operatorname{tr}(\mathscr{P}\mathbf{H})=\int_{\mathbf{R}^s} d^3qT(p)F_1(q)+\frac{1}{2}\int_{\mathbf{R}^s} d^3q_1\int_{\mathbf{R}^s} d^3q_2 V(x_1-x_2)F_2(q_1,q_2), \quad (118)$$

and obviously $\alpha^*\mathscr{L}(f)(F)=\mathscr{L}(f)(\mathscr{F})$, $\alpha^*H(F)=\mathscr{H}(\mathscr{F})\in\mathscr{D}(\mathfrak{A}^*)$. Taking into account in accordance with Proposition 3.3 the canonicity (113) of the mapping α: $\mathfrak{A}\to A$, we obtain an important Hamiltonicity theorem:

THEOREM 3.4. The Bogolyubov generating functional $\mathscr{L}(f)$ (97) satisfies on the phase space $\mathscr{D}(\mathfrak{A}^*)$ a Hamiltonian dynamical system of the form

$$\frac{\partial}{\partial t}\mathscr{L}(f)(\mathscr{F})=\{\{\mathscr{H}(\mathscr{F}), \mathscr{L}(f)(\mathscr{F})\}\} \quad (119)$$

with the Lie–Poisson bracket (112) and Hamiltonian function $\mathscr{H}(\mathscr{F})=H(F)$ (118).

Noting further that for the Lie–Poisson bracket (113) on the orbits of the coadjoint representation of the Lie algebra $\mathfrak{A}$

$$\begin{aligned}\{\{\mathscr{B}(\mathscr{F}), \mathscr{C}(\mathscr{F})\}\}=&\int_{\mathbf{R}^s} d^3q_1F_1(q_1)\left\{\frac{\delta\mathscr{B}(\mathscr{F})}{\delta F_1(q_1)}, \frac{\delta\mathscr{C}(\mathscr{F})}{\delta F_1(q_1)}\right\}^{(1)}\\&+\int_{\mathbf{R}^s} d^3q_1\int_{\mathbf{R}^s} d^3q_2F_2(q_1, q_2)\Big[2\left\{\frac{\delta\mathscr{B}(\mathscr{F})}{\delta F_1(q_1)}, \frac{\delta\mathscr{C}(\mathscr{F})}{\delta F_2(q_1, q_2)}\right\}^{(1)}\\&+2\left\{\frac{\delta\mathscr{B}(\mathscr{F})}{\delta F_2(q_1, q_2)}, \frac{\delta\mathscr{C}(\mathscr{F})}{\delta F_1(q_1)}\right\}^{(1)}+\left\{\frac{\delta\mathscr{B}(\mathscr{F})}{\delta F_2(q_1, q_2)}, \frac{\delta\mathscr{C}(\mathscr{F})}{\delta F_2(q_1, q_2)}\right\}^{(2)}\Big]\\&+\int_{\mathbf{R}^s} d^3q_1\ldots\int_{\mathbf{R}^s} d^3q_3F_3(q_1, \ldots, q_3)\\&+\Big[3\left\{\frac{\delta\mathscr{B}(\mathscr{F})}{\delta F_1(q_1)}, \frac{\delta\mathscr{C}(\mathscr{F})}{\delta F_3(q_1, q_2, q_3)}\right\}^{(1)}\\&+3\left\{\frac{\delta\mathscr{B}(\mathscr{F})}{\delta F_3(q_1, q_2, q_3)}, \frac{\delta\mathscr{C}(\mathscr{F})}{\delta F_1(q_1)}\right\}^{(1)}\\&+3\left\{\frac{\delta\mathscr{B}(\mathscr{F})}{\delta F_3(q_1, \ldots, q_3)}, \frac{\delta\mathscr{C}(\mathscr{F})}{\delta F_2(q_1, q_2)}\right\}^{(2)}\\&+\left\{\frac{\delta\mathscr{B}(\mathscr{F})}{\delta F_3(q_1, q_2, q_3)}, \frac{\delta\mathscr{C}(\mathscr{F})}{\delta F_3(q_1, q_2, q_3)}\right\}^{(3)}\\&+4\left\{\frac{\delta\mathscr{B}(\mathscr{F})}{\delta F_2(q_1, q_2)}, \frac{\delta\mathscr{C}(\mathscr{F})}{\delta F_2(q_1, q_2)}\right\}^{(1)}\Big]+\cdots,\end{aligned} \quad (120)$$

where $\mathscr{B}(\mathscr{F}), \mathscr{C}(\mathscr{F})\in\mathscr{D}(\mathfrak{A}^*)$, $\mathscr{F}\in\mathfrak{A}^*$, we obtain from (119) the following Bogolyubov functional equation[6]:

$$\frac{\partial\mathscr{L}(f)}{\partial t}=\int_{\mathbf{R}^s} d^3q\left\{T(p), \frac{1}{i}\frac{\delta\mathscr{L}(f)}{\delta f(q)}\right\}^{(1)}+\frac{1}{2}\int_{\mathbf{R}^s} d^3q_1\int_{\mathbf{R}^s} d^3q_2\left\{V(x_1-x_2), :\frac{1}{i}\frac{\delta}{\delta f(q_1)}\frac{1}{i}\frac{\delta}{\delta f(q_2)}:\right\}^{(2)}\mathscr{L}(f). \quad (121)$$

From (121), we readily find a functional equation for the functional (117):

$$\frac{\partial\mathscr{L}_B(u)}{\partial t}=\int_{\mathbf{R}^s} d^3q[1+u(q)]\left\{T(p), \frac{1}{i}\frac{\delta\mathscr{L}_B(u)}{\delta u(q)}\right\}^{(1)}+\frac{1}{2}\int_{\mathbf{R}^s} d^3q_1\times\int_{\mathbf{R}^3} d^3q_2[1+u(q_1)][1+u(q_2)]\left\{V(x_1-x_2), \frac{\delta^2\mathscr{L}_B(u)}{\delta u(q_1)\delta u(q_2)}\right\}^{(2)}, \quad (122)$$

which was obtained for the first time by Bogolyubov[1] in 1946.

We now consider the problem of integrating the Bogolyubov functional equations (121) and (122). From the representation (97) we readily find that[38]

$$\mathscr{L}(f)=\exp[(t-t_0)V\{\delta\}]\mathscr{L}_0(f). \qquad (123)$$

Here, we have used the explicit operator solution of the Liouville equation (81) in the form

$$\mathscr{P}(t)=\exp\left[\frac{\mathrm{i}}{\hbar}(t_0-t)\,\mathbf{H}\right]\overline{\mathscr{P}}\exp\left[\frac{\mathrm{i}}{\hbar}(t-t_0)\,\mathbf{H}\right],\quad \mathrm{tr}\,\overline{\mathscr{P}}=1, \qquad (124)$$

where $\overline{\mathfrak{P}}=\mathfrak{P}(t)|_{t=t_0}$ is the initial value of the statistical operator at the time $t_0\in\mathbf{R}^1$, and we have also introduced the notation

$$\mathscr{L}_0(f)=\mathrm{tr}\,(\mathscr{P}_0\exp[iw(f)]) \qquad (125)$$

for the generating functional of the noninteracting system of particles, and

$$V\{\delta\}=-\frac{1}{2}\int_{\mathbf{R}^6}d^3q_1\int_{\mathbf{R}^6}d^3q_2\left\{V(x_1-x_2),:\frac{1}{\mathrm{i}}\frac{\delta}{\delta f(q_1)}\frac{1}{\mathrm{i}}\frac{\delta}{\delta f(q_2)}:\right\}^{(2)} \qquad (126)$$

Equation (123) is very helpful for applications,[30-32] in particular, in the case when Bogolyubov's condition of correlation weakening[1] is satisfied:

$$|F_n(q_1,\ldots,q_n)-\prod_{j=1}^{n}F_1(q_j)|\to 0$$

for $t\to-\infty$, $n\in\mathbf{Z}_+$. From the explicit expression for $\mathscr{L}_0(f)$ of the form

$$\mathscr{L}_0(f)=\sum_{n\in\mathbf{Z}_+}(n!)^{-1}\int_{\mathbf{R}^6}d^3q_1\ldots\int_{\mathbf{R}^6}d^3q_n\prod_{j=1}^{n}\left\{\exp[if(q_j)]-1\right\}$$
$$\times\overline{F}_n\left(x_1-\frac{p_1}{m}(t-t_0),p_1;\ldots;x_n-\frac{p_n}{m}(t-t_0),p_n\right), \qquad (127)$$

where $\overline{F}_n(q_1,\ldots,q_n)=\mathrm{tr}\,(\overline{\mathscr{P}}:w(q_1)\ldots w(q_n):)$, $q_n=(x_n,p_n)\in\mathbf{R}^3\times\mathbf{R}^3$, $n\in\mathbf{Z}_+$, one can obtain the nonlinear equations of hydrodynamics[1,31] in any order of the "cluster" perturbation theory obtained from the expression (123) by expansion of the exponential in a series. We hope to dwell in more detail in another investigation on questions relating to further applications of the results which we have obtained.

APPENDIX

Suppose that the system which we study is a free Fermi gas at zero temperature. To obtain in explicit form a functional $\mathscr{L}_0(f)$, $f\in\mathscr{S}(\mathbf{R}^3;\mathbf{R}^1)$ and the functional equation (77), we first consider the N-particle ground-state function $\Omega_F^{(N)}\in\Phi$ of the system (in the equilibrium case), $N\in\mathbf{Z}_+$. We have[8,11]

$$\Omega_F^{(N)}(x_1,\ldots,x_N)=(N!)^{-1/2}\det\|f_n(x_m)\|_{n,m=1}^{N}, \qquad (A1)$$

where $f_n(x)=\Lambda^{-1/2}\exp(ik_nx)$, $x\in\Lambda\subset\mathbf{R}^3$; $N/\Lambda=\bar\rho$; $k_n=(2\pi/l)\in\mathbf{R}^3$ and the numbers $\alpha_n\in\mathbf{Z}^3$ are such that $|k_n|\leqslant k_F$, where

$$\left(\frac{1}{2\pi}k_F\right)^3\frac{4}{3}\pi=\rho,\quad N=\sum_{|k_n|\leqslant k_F}1; \qquad (A2)$$

$k_F\in\mathbf{R}^1_+$ is the so-called Fermi momentum[3,8] (for convenience, we do not distinguish the symbol for a volume $\Lambda\subset\mathbf{R}^3$ and its numerical value $|\Lambda|=l^3\in\mathbf{R}^1_+$; this will obviously not lead to confusion). Thus, by virtue of (109) and (110), we obtain from (A1)

$$F_n^{(N)}(x_1,\ldots,x_N)=\frac{(N!)}{(N-n)!}\int_{\mathbf{R}^3}d^3x_{1+n}\ldots\int_{\mathbf{R}^3}d^3x_N\,|\,\Omega_F^{(N)}\,|^2=\det\|K_N(x_j,x_i)\|_{i,j=1}^{N}, \qquad (A3)$$

where $K_N(x,y)=\Sigma_{j=1}^N f_j(x)f_j(y)$; $\int_{\mathbf{R}^3}d^3xf_j^*(x)f_i(x)=\delta_{ij}$, $i,j=\overline{1,N}$. From (A3), therefore, we can find that

$$F_n^{(N)}(x_1,\ldots,x_N)=\det\|G_N(x_j-x_i)\|_{i,j=1}^{n}, \qquad (A4)$$

where $G_N(x)=\Lambda^{-1}\Sigma_{|k_n|<k_F}\exp(ik_nx)$, $x\in\Lambda\subset\mathbf{R}^3$, $N\in\mathbf{Z}_+$. For $\lim_{N\to\infty,\,\Lambda\nearrow\mathbf{R}}N/\Lambda=\bar\rho$ $G_N(x)\to G(x)$, where

$$G(x)=(2\pi)^{-3}\int_{|k|<k_F}d^3k\exp(ikx)=3\bar\rho\,(\sin z-z\cos z)/z^3|_{z=k_F|x|}. \qquad (A5)$$

At the same time, $F_n^{(N)}(x_1,\ldots,x_n)\to F_n(x_1,\ldots,x_n)=\det\|G(x_i-x_j)\|_{i,j=1}^{n}$, $n\in\mathbf{Z}_+$. For the generating functional $\mathscr{L}(f)$ we find from (60)

$$\mathscr{L}(f)=\sum_{n\in\mathbf{Z}_+}(n!)^{-1}\int_{\mathbf{R}^3}d^3x_1\ldots\int_{\mathbf{R}^3}d^3x_n\det\|G(x_k-x_j)\|_{k,j=1}^{n}\times\prod_{j=1}^{n}\{\exp[if(x_j)]-1\}. \qquad (A6)$$

To obtain the cluster expansion for (A6) in the Mayer–Ursell form,[3,7] we note that the following identity holds:

$$\int_{\mathbf{R}^3}d^3x_1\ldots\int_{\mathbf{R}^3}d^3x_N(\det\|h_j^*(x_k)\|_{j,k=1}^{N})(\det\|g_j(x_k)\|_{j,k=1}^{N})=N!\left(\det\left\|\int_{\mathbf{R}^3}d^3xh_j^*(x)g_k(x)\right\|_{j,k=1}^{N}\right) \qquad (A7)$$

Then from (91), (A1), and (A7) we obtain

$$\mathscr{L}^{(N)}(f)=(N!)^{-1}\int_{\Lambda}(d^3x_1/\Lambda)\ldots\int_{\Lambda}(d^3x_N/)$$
$$\times(\det\|\exp(ik_nx_m)\|_{n,m=1}^{n})^*$$
$$\times\prod_{j=1}^{N}\exp[if(x_j)](\det\|\exp(ik_nx_m)\|_{n,m=1}^{N})$$
$$=\det\left(\left\|\int_{\Lambda}(d^3x/\Lambda)\exp[if(x)]\exp[i(k_n-k_m)x]\right\|_{n,m=1}^{N}\right)$$
$$=\det\left\|\delta_{mn}+\int_{\Lambda}(d^3x/\Lambda)\{\exp[if(x)]-1\}\exp[i(k_n-k_m)x]\right\|_{n,m=1}^{N}. \qquad (A8)$$

Using now an expansion of the form

$$\det(1+A)=\exp[\mathrm{tr}\ln(1+A)]=\exp\left[\sum_{n=1}^{\infty}n^{-1}(-1)^{n+1}\,\mathrm{tr}\,A^n\right], \qquad (A9)$$

where $A: \mathbf{C}^N \to \mathbf{C}^N$ is an arbitrary matrix, we find from (A9) and (A8)

$$\mathscr{L}^{(N)}(f) = \exp\Big[\sum_{n=1}^{\infty} \frac{1}{n!}(-1)^{n+1} \int_\Lambda d^3x_1 \times \dots$$

$$\dots \times \int_\Lambda d^3x_n \prod_{j=1}^{n} \{\exp[\mathrm{i}f(x_j)]-1\} T_n^{(N)}(x_1, \dots, x_n), \quad \text{(A10)}$$

where

$$T_n^{(N)}(x_1, \dots, x_n) = \frac{(n-1)!}{\Lambda} \sum_{k_1} \dots \sum_{k_n} \{\exp[\mathrm{i}(k_1-k_2)x_1] \times \exp[\mathrm{i}(k_2-k_3)x_2] \dots \exp[\mathrm{i}(k_n-k_1)x_n]\}$$
$$= (n-1)!\, G_N(x_1-x_2)\, G_N(x_2-x_3) \dots G_N(x_n-x_1). \quad \text{(A11)}$$

In the limit $N/\Lambda \to \bar{\rho}$, $N\to\infty$, $\Lambda \nearrow \mathbf{R}^3$, we obtain

$$\mathscr{L}^{(N)}(f) \to \mathscr{L}(f) = \exp\Big[\sum_{n=1}^{\infty} \frac{1}{n!}(-1)^{n+1}$$

$$\times \int_{\mathbf{R}^3} d^3x_1 \dots \int_{\mathbf{R}^3} d^3x_n \prod_{j=1}^{n} \{\exp[\mathrm{i}f(x_j)]-1\} T_n(x_1, \dots, x_n)\Big], \quad \text{(A12)}$$

where for all $n \in \mathbf{Z}_+$

$$T_n(x_1, \dots, x_n) = (n-1)!\, G(x_1-x_2)\, G(x_2-x_3) \dots G(x_n-x_1), \quad \text{(A13)}$$

and, obviously,

$$(n!)^{-1} \sum_\sigma T_n(x_{\sigma(1)}, \dots, x_{\sigma(n)}) \quad \text{(A14)}$$

is the Ursell cluster function[7] of the distribution functions $F_n(x_1, \dots, x_n)$, $n \in \mathbf{Z}_+$.

Similarly, for the functional $\mathscr{L}(f, g)$ (90) we can obtain[33]

$$\mathscr{L}(f, g) = \sum_{n \in \mathbf{Z}_+} (n!)^{-1} \int_{\mathbf{R}^3} d^3x_1 \dots \int_{\mathbf{R}^3} d^3x_n \int_{\mathbf{R}^3} d^3y_1 \dots$$

$$\dots \int_{\mathbf{R}^3} d^3y_n \prod_{j=1}^{n} \delta(x_j - y_j)\{\exp[\mathrm{i}f(x_j)]\exp[\mathrm{i}\xi(x_j; g)]-1\}$$
$$\times F_n(y_1, \dots, y_n; x_1, \dots, x_n), \quad \text{(A15)}$$

where $f \in \mathscr{S}(\mathbf{R}^3; \mathbf{R}^1)$, $g \in \mathscr{S}(\mathbf{R}^3; \mathbf{R}^3)$ and

$$F_n(y_1, \dots, y_n; x_1, \dots, x_n) = (\det \| G(x_m - y_h) \|_{h, m=1}^n), \quad \text{(A16)}$$

$$\xi(x; g) = \frac{1}{2\mathrm{i}}[2g(x)\nabla_x + (\nabla g)(x)].$$

In the Ursell cluster form, the functional (A15) has the expression.[33]

$$(f, g) = \exp \sum_{n=1}^{\infty} (n!)^{-1}(-1)^{n+1} \int_{\mathbf{R}^3} d^3x_1 \dots \int_{\mathbf{R}^3} d^3x_n$$

$$\times \int_{\mathbf{R}^3} d^3y_1 \dots \int_{\mathbf{R}^3} d^3y_n \prod_{j=1}^{n} \delta(x_j - y_j)\{\exp[\mathrm{i}f(x_j)]$$

$$\times \exp[\mathrm{i}\xi(x_j; g)]-1\} T_n(y_1, \dots, y_n; x_1, \dots, x_n)\Big], \quad \text{(A17)}$$

where

$$T_n(y_1, \dots, y_n; x_1, \dots, x_n) = (n-1)!\, G(x_1-y_2)\, G(x_2-y_3) \dots G(x_n-y_1). \quad \text{(A18)}$$

We now consider a special case of our system—a noninteracting Fermi gas in one spatial dimension. To obtain in explicit form the functional equation (77), we use the reduction of the ground state $\Omega_n^{(N)}$ (A1) for the one-dimensional case:

$$\Omega_F^{(N)} = (N!\, l^N)^{-1/2} \prod_{j>k=1}^{N} [2\sin(\pi(x_j - x_k)/l)]$$
$$= (N!\, l^N)^{-1/2} (\det \| \exp(\mathrm{i}k_j x_n) \|_{j, n=1}^N), \quad \text{(A19)}$$

where

$$k_j = 2\pi j/l,$$

$$j = -\frac{1}{2}(N-1), -\frac{1}{2}(N-3), \dots, \frac{1}{2}(N-1), \quad N \in \mathbf{Z}_+.$$

Using Eqs. (46)–(48), from (A19) for the coefficients $c_m \in \mathbf{R}^1$, $m = \overline{0, N}$, we successively find

$$\left.\begin{aligned}
&c_0^{(N)} = \frac{1}{2}N(N-1)c + \text{const}; \quad c = \frac{2}{l}\int_{-l/2}^{l/2} dx \ln|\sin(\pi x/l)|;\\
&c_1^{(N)} = \frac{1}{2}N(N-1)c; \quad c_n^{(N)} = 2\sum_{j>k\geqslant 1}^{N} \ln|\sin[\pi(x_j-x_k)/l]|\\
&+\frac{1}{2}(N-n)(N+n-1)c,
\end{aligned}\right\} \quad \text{(A20)}$$

whence for the operator $A_0(x; \delta)$ in (77) we have the expression (as $N\to\infty$, $l\to\infty$, $N/l\to\rho = \text{const}$)

$$A_0(x; \delta) = 2\nabla_x \int_{\mathbf{R}^1} dy \ln|x-y| : \frac{1}{\mathrm{i}}\frac{\delta}{\delta f(x)} \frac{1}{\mathrm{i}}\frac{\delta}{\delta f(y)} : . \quad \text{(A21)}$$

A similar expression[7] can also be obtained for a one-dimensional system of Bose particles on the axis $\mathbf{R}^1$ with Hamiltonian $\mathbf{H}: \Phi \to \Phi$ in an N-particle representation of the form

$$\mathbf{H} = -\frac{\hbar^2}{2m}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + \frac{\hbar^2\lambda(\lambda-1)}{m}\sum_{j<k}^{N}[(l/\pi)\sin(\pi x/l)]^{-2}, \quad \text{(A22)}$$

where $\lambda \in \mathbf{R}^1$ is an arbitrary numerical parameter. Since the ground state $\Omega_B^{(N)} \in \Phi$ for the Hamiltonian (A22) is given by the explicit expression

$$\Omega_B^{(N)} = \text{const} \prod_{j>k=1}^{N} |2\sin[\pi(x_j-x_k)/l]|^\lambda, \quad \text{(A23)}$$

for the numbers $c_j^{(N)} \in \mathbf{R}^1$, $j = \overline{0, N}$, we find accordingly[7] from (A23)

$$\left.\begin{aligned}
&c_0^{(N)} = \text{const} + \frac{1}{2}N(N-1)c(\lambda), \quad c_1^{(N)} = \frac{1}{2}N(N-1)c(\lambda);\\
&c_n^{(N)} = 2\lambda\sum_{j<k}^{N}\ln|\sin[\pi(x_j-x_k)/l]| + \frac{1}{2}(N-n)(N+n-1)c(\lambda);\\
&c(\lambda) = \frac{2\lambda}{l}\int_{-l/2}^{l/2} dx \ln|\sin(\pi x/l)|.
\end{aligned}\right\} \quad \text{(A24)}$$

Therefore, in accordance with Eqs. (45)–(48) we obtain for the operator $A(x; \delta)$ in the functional equation (37) an explicit expression in the limit $N\to\infty$, $l\to\infty$, $N/l\to\bar{\rho}$:

$$A(x; \delta) = 2\lambda \int_{\mathbf{R}^1} dy \nabla_x \ln|x-y| : \frac{1}{\mathrm{i}}\frac{\delta}{\delta f(x)} \frac{1}{\mathrm{i}}\frac{\delta}{\delta f(y)} : . \quad \text{(A25)}$$

As a consequence of the form of the expressions (A21) and (A25), we conclude that in the limit $\lambda \to 1$ the Bose system of particles (A22) goes over into a noninteracting Fermi system. The reason for this is to be sought in the fact that the corresponding unitary representations of the group of currents $\mathscr{S} \wedge \mathrm{Diff}(\mathbf{R}^1)$ are unitarily equivalent. In order that the ground state $\Omega_F^{(N)} \in \Phi$ for the Fermi system with Hamiltonian (A22) be determined in the N-particle representation, we make the following constructions. Let $\mathbf{R}^1[<] \subset \mathbf{R}^1$ be the region for which $x_1 < x_2 < \dots < x_N$. Also, let $\sigma \in S_N$ be a permutation such that $(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(N)}) \in \mathbf{R}^1[<]$. We now set $\Omega_F^{(N)} = (-1)^6\, \Omega_B^{(N)}$; it is readily verified that the state $\Omega_F^{(N)} \in \Phi$, $N \in \mathbf{Z}_+$, is antisymmetric, real, and satisfies the Schrödinger equation $\mathbf{H}\Omega_F^{(N)} = E_0 \Omega_F^{(N)}$, where $E_0 \in \mathbf{R}^1$ is the energy of the ground state. At the same time, the number $N \in \mathbf{Z}_+$ must be odd in order to ensure periodicity of the ground state. Thus, in both the Bose and Fermi cases of particles with the Hamiltonian (A22) the functional equation (37) has the form

$$[\nabla_x - i\nabla f(x)]\, \frac{1}{i}\, \frac{\delta \mathscr{L}(f)}{\delta f(x)} = 2\lambda \int_{\mathbf{R}^1} dy\, \nabla_x \ln|x-y| : \frac{1}{i}\frac{\delta}{\delta f(x)}\, \frac{1}{i}\frac{\delta}{\delta f(y)} : \mathscr{L}(f) \qquad \text{(A26)}$$

for all $\lambda \in \mathbf{R}^1$. By virtue of the results of Sec. 2, we obtain from Eq. (64) the following explicit functional–operator solution of Eq. (A26):

$$\left.\begin{aligned} &\mathscr{L}(f) = W(f)/W(0);\\ &W(f) = \exp\left[\lambda \int_{\mathbf{R}^1} dx \int_{\mathbf{R}^1} dy \ln|x-y| : \frac{1}{i}\frac{\delta}{\delta f(x)}\, \frac{1}{i}\frac{\delta}{\delta f(y)} : \right] \mathscr{L}_0(f);\\ &\mathscr{L}_0(f) = \exp\left(\alpha \int_{\mathbf{R}^1} dx\, \{\exp[if(x)] - 1\}\right) \end{aligned}\right\} \qquad \text{(A27)}$$

where the numerical parameter $\alpha \in \mathbf{R}^1_+$ is determined from the following boundary condition: for all $x \in \mathbf{R}^1$

$$\frac{1}{i}\, \frac{\delta \mathscr{L}(f)}{\delta f(x)}\bigg|_{f=0} = \bar{\rho} \in \mathbf{R}^1_+. \qquad \text{(A28)}$$

To make this effective, we note that the functional $\mathscr{L}(f)$ (A27) satisfies[6] the following equation of Kirkwood–Salsburg–Symanzik type:

$$\exp[if(x)]\, \frac{1}{i}\, \frac{\delta \mathscr{L}(f)}{\delta f(x)} = \alpha \mathscr{L}(f(\cdot) + i2\lambda \ln|\cdot - x|) \qquad \text{(A29)}$$

for all $x \in \mathbf{R}^1$. By the translational invariance of our system of particles, we can in (A29) set $f = 0$ and $x = 0$, from which we find

$$\bar{\rho} = \alpha \mathscr{L}(2i\lambda \ln|\cdot|). \qquad \text{(A30)}$$

In accordance with (A23) and the equation $\mathscr{L}(f) = \lim_{N\to\infty} \mathscr{L}^{(N)}(f)$, which follows from (87), where

$$\left.\begin{aligned} &\mathscr{L}^{(N)}(f) = \int_0^l dx_1 \dots \int_0^l dx_N\, |\Omega^{(N)}|^2 \prod_{j=1}^{N} \exp[if(x_j)],\\ &\Omega^{(N)} = \{\Gamma^N(\lambda)/\Gamma(\lambda N)\, l^N\}^{1/2} \prod_{j>k}^{N} |2\sin[\pi(x_j - x_k)/l]|^{\lambda} \end{aligned}\right\} \qquad \text{(A31)}$$

[$\Gamma(\cdot)$ is the gamma function], we readily obtain

$$\bar{\rho} = \alpha \lim_{N\to\infty} \{\Gamma^N(\lambda)/\Gamma(\lambda N)\, l^N\} \int_0^l dx_1 \dots \int_0^l dx_N \prod_{j>k}^{N} |2\sin[\pi(x_j - x_k)/l]|^{2\lambda} \prod_{j=1}^{N} |\sin(x_j \pi/l)|^{2\lambda}. \qquad \text{(A32)}$$

From (A32), calculating the limit $N \to \infty$, $l \to \infty$, $N/l = \bar{\rho}$, we determine the parameter $\alpha \in \mathbf{R}^1_+$ explicitly, on which we shall not dwell here.

To calculate the functional $\mathscr{L}(f)$ (A27), we use the fact that by virtue of (A31) it has the representation

$$\left.\begin{aligned} &\mathscr{L}^N(f) = \exp\left[\lambda \int_0^l dx \int_0^l dy \ln|\sin[\pi(x-y)/l]|\right.\\ &\qquad \left.\times : \frac{1}{i}\, \frac{\delta}{\delta f(x)}\, \frac{1}{i}\, \frac{\delta}{\delta f(y)} : \right] \mathscr{L}_0^{(N)}(f),\\ &\mathscr{L}_0^{(N)}(f) = \exp\left(\alpha \int_0^l dx\, \{\exp[if(x)] - 1\}\right). \end{aligned}\right\} \qquad \text{(A33)}$$

Expanding the exponential in the first equation of (A33) in a series in powers of the parameter $\lambda \in \mathbf{R}^1_+$ we readily[6,7] obtain for the functional $\mathscr{L}^{(N)}(f)$ the diagrammatic representation

$$\left.\begin{aligned} &\mathscr{L}^{(N)}(f) = W^{(N)}(f)/W^{(N)}(0);\\ &W^{(N)}(f) = \exp\left[\sum_{N=1}^{\infty} \frac{1}{N!} \sum_{G_N^{(c)}} w(G_N^{(c)})\right]. \end{aligned}\right\} \qquad \text{(A34)}$$

Here, $G_N^{(c)}$ is a connected simple graph consisting of $N \in \mathbf{Z}_+$ vertices $x_1, \dots, x_N \in [0, 1]$ and of directed lines that join them (without self-repetitions) arbitrarily, each vertex in the graph $G_N^{(c)}$ being associated with the factor $\int_0^l dx \exp[if(x)]$, and each line joining the vertices x_j and x_k being associated with the factor $\exp\{2\lambda \ln|\sin[\pi(x_j - x_k)/l]|\} - 1$, $j, k = \overline{1, N}$. The result of a calculation in accordance with this rule for the given diagram $G_N^{(c)}$ will be the functional $w(G_N^{(c)})$, and summation over all such connected simple graphs leads after division by the factor $N!$ to the functional $W^{(N)}(f)$, which specifies in accordance with (A34) the required functional $\mathscr{L}^{(N)}(f)$. Making the calculations in accordance with (A34) and then going to the limit $N \to \infty$, $l \to \infty$, $N/l = \bar{\rho}$, we can finally determine the generating functional $\mathscr{L}(f) = \lim_{N\to\infty} \mathscr{L}^{(N)}(f)$ in (A27) and, thus, the distribution functions for one-dimensional Bose as well as Fermi gases of particles in the ground state $\Omega \in \Phi$ (at zero temperature). In accordance with the results presented above, this same approach is also valid for the calculation of the generating functional $\mathscr{L}(f)$, $f \in \mathscr{S}(\mathbf{R}^3; \mathbf{R}^1)$ in the physical space $\mathbf{R}^3$ in the case of noninteracting Bose and Fermi gases of particles of fixed density in the ground state at arbitrary temperatures. We note in conclusion that the original approach to the calculation of the correlation functions of a one-dimensional Fermi gas was developed in Ref. 34 on the basis of an analysis of isomonodromic deformations of a special lattice approximation of the nonlinear completely integrable Schrödinger model. The combination of the ideas of this study with the approach of the present review as applied to a more general nonlinear model of Schrödinger type[25] is at present an interesting and topical problem.

There is also definite interest in studying the topological[39,40] structure of the set of solutions of the Bogolyubov functional equation (37) in the quantum case.

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ERRATA

Erratum: "Long-distance" neutrinos. Physical bases and geophysical applications [Sov. J. Part. Nucl. 17(3), 167–185 (May–June 1986)]

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Figure 1 should read:

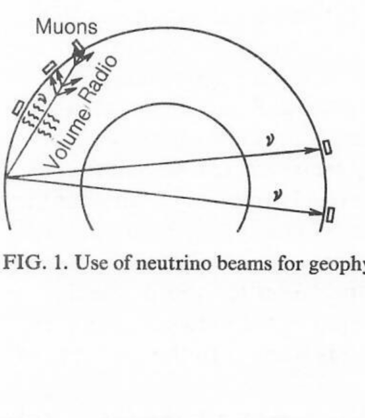

FIG. 1. Use of neutrino beams for geophysical studies.

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