Self-consistent model of vibrational–rotational quadrupole–octupole excitations of deformed nuclei

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For the unified description of collective states of both positive and negative parity a model is developed in which not only quadrupole but also octupole degrees of freedom are taken into account. The analysis is based on a self-consistent treatment of the interaction of the vibrational degrees of freedom through rotation of the nucleus. The model can be formulated in such a way that there is no expansion with respect to the ratio of the vibration amplitude to the static deformation, and therefore deformed as well as transitional nuclei, in which the static deformation may be small, can be studied in a unified manner. The moments of inertia and the vibrational rigidities are determined in a self-consistent manner and depend essentially on the angular momentum of the level. This has the consequence that with increasing angular momentum there is a rearrangement of the spectrum from a rotational to an equidistant spectrum. The possibility of an additional branch of the spectrum beginning at a certain value of the angular momentum is discussed. The corresponding collective states are treated as dynamical nonaxial isomers.

INTRODUCTION

The last decade has been marked in nuclear theory by the rapid development of microscopic methods for describing nuclear structure (for more details, see Refs. 1–3). However, even in the simplest cases direct microscopic calculations are still very complicated and cumbersome, and their results can be evaluated and interpreted mainly only in the framework of model ideas (see, for example, Ref. 4). It is therefore sensible, using ideas advanced by Belyaev,[5] to modify somewhat the aims of microscopic theory, namely, to direct the main efforts not so much at obtaining exact numerical solutions to the equations of microscopic theory but rather to concentrate them on justifying the phenomenological models that have successfully proved themselves.

These models must, of course, at the same time be improved to take into account not only the accumulated experience in the description of the given nuclear phenomena but also the achievements of the microscopic theory.

The present paper is devoted to a review of the development of one such model. It is well known that in the excitation spectra of nuclei of the transition region and the actinides many levels are observed at energies of order 1–2 MeV that have a collective nature and are associated with excitations of the nuclear surface.

The phenomenological theory of vibrational–rotational spectra of deformed nuclei has been developed by many authors and was begun by the classical studies of Bohr and Mottelson.[6,7] The Hamiltonian constructed by Bohr was subsequently widely used in the work of Davydov and his collaborators[8–10] to study nonaxial nuclei (Davydov–Filippov model[8]), and also the coupling of β vibrations with rotations in the so-called soft nuclei, whose deformation changes appreciably with increasing angular momentum of the nucleus (Davydov–Chaban model[9]).

Faessler and Greiner[11,12] took into account systematically not only rotational but also vibrational (but only quadrupole) degrees of freedom. In their approach, which has become known as the "vibrational–rotational model" (or the Faessler–Greiner model), two circumstances are essentially used. First, the consideration of only quadrupole excitations has the consequence that the moment of inertia $\mathscr{J}_3$, associated with rotation around the symmetry axis depends only on a single vibrational variable. Second, the assumption of a strong static deformation means a weak dependence on the vibrational variables of the moments of inertia $\mathscr{J}_1$ and $\mathscr{J}_2$ associated with rotations about the transverse axes; in other words, the theory contains small parameters,

$$|(a_{\lambda\mu} - \bar{a}_{\lambda\mu})/\beta_0| \ll 1 \qquad (1)$$

($a_{\lambda\mu}$ are the vibrational variables, $\bar{a}_{\lambda\mu}$ are their mean values, and β_0 is the deformation parameter).

To describe the observed states with negative parity, it is necessary to include in the treatment octupole degrees of freedom. But then the moment of inertia $\mathscr{J}_3$ becomes a function of several vibrational variables $a_{\lambda\mu}$ with $\mu \neq 0$, which characterize the deformation of the shape of the nucleus from axial symmetry (when allowance is made for only the quadrupole degrees of freedom,[1)] the moment of inertia is determined by the single amplitude a_{22}), and as a result the vibrations are coupled to one another. Therefore, in the early studies[14–18] it was necessary to make additional approximations to describe the octupole vibrations—to assume an additional symmetry of the deformed potential or to ignore the quadrupole degrees of freedom altogether. In Refs. 19 and 20, not only octupole but also quadrupole variables were taken into account, but the method of approximate separation of the variables used in these studies cannot be regarded as sufficiently justified and is therefore suitable only for a preliminary description. The problem of the interaction of the quadrupole and octupole degrees of freedom is also the subject of Ref. 21. However, the wave functions are constructed in Ref. 21 without allowance for this interaction

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and therefore can be used only for subsequent diagonalization.

To describe the quadrupole–octupole excitations of the nuclear surface with allowance for their interaction through rotations about the symmetry axis a method was proposed in Refs. 22–25 on the basis of a self-consistent description of the interacting vibrations. This approach is based on the fact that the interaction potential is determined by the moments of inertia, while these themselves are determined by a whole set of vibrational amplitudes. It is therefore reasonable to assume that the potential is a fairly smooth function of each of the vibrational variables that describe the interacting degrees of freedom. In the studied problem, this is indeed the case, since already in the study of the quadrupole and octupole degrees of freedom four of them contribute to the moment $\mathscr{J}_3$ and, therefore, to the interaction potential of the vibrations. In principle, higher multipoles can also make a contribution; in this case, the validity of the approximation of self-consistent vibrations will be even better. We note that the summation over the multipoles is restricted in accordance with Ref. 26 by the inequality $\lambda \lesssim A^{1/3}$.

Separation of the vibrational variables by means of the variational principle makes it possible to obtain the best nuclear wave function (as a function of the collective variables) among all multiplicative functions that describe the independent or quasi-independent vibrations.

The fact that the number of vibration amplitudes that break the axial symmetry and therefore contribute to the moments of inertia $\mathscr{J}_3$ is fairly large also makes it possible to simplify considerably the structure of the Hartree equations for the individual vibrations. Thus, under the assumption that the amplitudes are fairly small the interaction of the vibrations leads only to a change in the rigidities and, therefore, frequencies. This renormalization of the frequencies depends on the rms values of the amplitudes and therefore for each band $\{K,\{n_{\lambda\mu}\}\}\equiv p$ is different. The energy of the system of interacting vibrations (phonons) is not equal to the sum of their energies but is determined by a more complicated expression.

For detailed comparison of the calculated and observed spectra, it is of course necessary to take into account the difference of the moments of inertia $\mathscr{J}_1$ and $\mathscr{J}_2$ from their static value $\mathscr{J}_0$, and also the nondiagonal elements of the inertia tensor that arise when the octupole degrees of freedom are taken into account. The corresponding part of the collective Hamiltonian, which is usually denoted by $\hat{H}_{\text{vib rot}}$, being an operator with respect to the vibrational as well as the rotational variables, leads to mixing of the rotational bands—the quantities K and $\{n_{\lambda\mu}\}$ cease to be exact quantum numbers, though it is convenient to use them to label the collective states.

The proposed method, which can be called the Hartree method for phonons or the method of self-consistent phonons) has made it possible to give a reasonable description of the vibrational–rotational spectra of deformed nuclei.

However, only the so-called rigid nuclei have been considered, and the conditions (1) have been used. In the present paper, the method of self-consistent phonons is presented in such a way that it becomes possible to treat not only rigid but also soft nuclei in which the dynamical deformation is comparable with the static deformation and the conditions (1) are not satisfied. The equilibrium deformation is then determined from the condition of a minimum of the total vibrational–rotational energy. Such an approach is due to the work of Davydov and Chaban in Ref. 9.

The essential dependence of all moments of inertia on the vibrational amplitudes has the consequence that the interaction between the vibrations is realized through rotations about all axes. As a result, the structure of the interaction becomes more complicated and it encompasses all vibration modes and not only the transverse modes ($\mu\neq 0$), as was the case for rigid nuclei. Such a treatment has the consequence that on the transition from one level to another in a given rotational band the moments of inertia can change appreciably; this leads to a significant rearrangement of the rotational spectra. The values of the moments of inertia are determined as functions of the angular momentum in a self-consistent manner.

1. VIBRATIONAL–ROTATIONAL HAMILTONIAN FOR EXCITATIONS OF ARBITRARY MULTIPOLARITY

The vibrational–rotational model is based on Bohr's classical paper of Ref. 6, in which he introduced an intrinsic coordinate system rigidly attached to the nucleus. Then the variables $\alpha_{\lambda\mu}$ that describe the shape of the nuclear surface in the laboratory system,

$$R(\theta,\ \varphi)=R_0\left(1+\sum_{\lambda\mu}\alpha_{\lambda\mu}Y^*_{\lambda\mu}(\theta,\ \varphi)\right), \tag{2}$$

are expressed in terms of the variables $a_{\lambda\mu}$ that describe the deformation of the nuclear surface in the intrinsic coordinate system,

$$R(\theta',\ \varphi')=R_0\left(1+\sum_{\lambda\mu}a_{\lambda\mu}Y^*_{\lambda\mu}(\theta',\ \varphi')\right), \tag{3}$$

and in terms of the Eulerian angles $\{\theta_1,\theta_2,\theta_3\}$, which specify the orientation of the nucleus as a whole:

$$\alpha_{\lambda\mu}=\sum_{\nu}D^{\lambda*}_{\mu\nu}(\theta_1,\ \theta_2,\ \theta_3)\,a_{\lambda\nu}. \tag{4}$$

Restricting the treatment to quadrupole deformations, Bohr fixed the intrinsic frame of reference by the conditions

$$a_{21}=a_{2-1}=0,\quad a_{2-2}=a_{22}, \tag{5}$$

which mean that the axes of this system are the principal axes of the ellipsoid (quadrupoloid) of the deformations.

Going over to deformations of higher multipolarity, we shall assume that they, like the quadrupole deformations, preserve the symmetry of the nucleus with respect to the plane ZOX, which passes through the symmetry axis of the nucleus. This assumption,

$$a_{\lambda-\mu}=(-1)^{\mu}a_{\lambda\mu}, \tag{6}$$

in conjunction with the requirement of reality of the nuclear radius (3) leads to reality of the amplitudes $a_{\lambda\mu}$:

$$a^*_{\lambda\mu}=a_{\lambda\mu}. \tag{7}$$

The same conclusion can be reached in a different way. As is well known, coordinates and velocities behave differently under time reversal: The coordinates do not change but the velocities are reversed. Accordingly, regarding the quantities $a_{\lambda\mu}$ directly as collective coordinates, it is natural to require their invariance under time reversal, and this reduces to the condition (7).

Since the vibration amplitudes $a_{\lambda\mu}$ and $a_{\lambda-\mu}$ are related to each other, we can regard as independent only those for which $\mu \geqslant 0$. Therefore, the quadrupole vibrations are described by two variables; the octupole, by four.

In this paper, we restrict the treatment to nuclei that are axisymmetric in the ground state; for these the minimum of the potential energy $V(\{a_{\lambda\mu}\})$ corresponds to values of the variables $a_{\lambda\mu}$ given by

$$a_{\lambda\mu} = \beta_\lambda \delta_{\mu 0}, \tag{8}$$

where β_λ is the parameter of the λ-complete deformation.

Assuming that the multipole deformations are of higher order than the quadrupole deformations and represent small corrections, we shall assume that the static deformation is purely quadrupole:

$$\beta_2 \equiv \beta_0, \quad \beta_3 = \beta_4 = \ldots = 0. \tag{9}$$

The vibrational–rotational model[6,10] is based on the classical energy of surface excitations:

$$E = \frac{1}{2} \sum_{\lambda\mu} B_\lambda |\dot{\alpha}_{\lambda\mu}|^2 + V(\{\alpha_{\lambda\mu}\}), \tag{10}$$

where B_λ are mass parameters. Going over by means of (4) from the collective variables $\alpha_{\lambda\mu}$ in the laboratory system to the vibrational variables $a_{\lambda\mu}$ in the intrinsic system and the Eulerian angles $\{\theta_1, \theta_2, \theta_3\}$, which characterize the orientation of the nucleus, we obtain

$$E = \frac{1}{2} \sum_{\lambda\mu} B_\lambda |\dot{a}_{\lambda\mu}|^2 + \sum_{i,\lambda\mu} \omega_i' a_{\lambda\mu} F_{i,\lambda\mu} + \frac{1}{2} \sum_{i,j} \mathcal{J}_{ij} \omega_i' \omega_j' + V(\{a_{\lambda\mu}\}), \tag{11}$$

where $V(\{a_{\lambda\mu}\})$ is the deformation potential, ω_i' are the rotational velocities around the axes of the intrinsic system, and $\omega_i' \equiv \dot{\beta}_i$; $\mathcal{J}_{ij}$ is the interia tensor,

$$\mathcal{J}_{ij} \equiv \sum_{\lambda\mu\nu\sigma} B_\lambda a_{\lambda\nu} a_{\lambda\mu} \langle \lambda\mu | \hat{L}_i | \lambda\sigma \rangle \langle \lambda\sigma | \hat{L}_j | \lambda\nu \rangle, \tag{12}$$

whose diagonal elements are the moments of inertia; the coefficients

$$F_{i,\lambda\mu} \equiv -\sum_{\nu} B_\lambda \operatorname{Im} \langle \lambda\nu | \hat{L}_i | \lambda\mu \rangle a_{\lambda\nu} \tag{13}$$

describe the "classical" interaction of the vibrations with the rotations.

By virtue of the definition of the intrinsic coordinate system (5), the quadrupole terms contribute only to the diagonal elements of the tensor $\mathcal{J}_{ij}$. In contrast, the nondiagonal elements of the tensor $\mathcal{J}_{ij}$ and the coefficients $F_{i,\lambda\mu}$ are determined by the octupole (and higher multipolarity) terms.

In what follows, we shall restrict ourselves (to be specific) to octupole deformations:

$$\left.\begin{aligned}
&\mathcal{J}_{11} = (\sqrt{3}\, a_{20} + \sqrt{2}\, a_{22})^2 B_2 + (6a_{30}^2 + 5a_{31}^2 + 8a_{32}^2 \\
&+ 3a_{33}^2 + 2\sqrt{30}\, a_{30}a_{32} + 2\sqrt{15}\, a_{31}a_{33}) B_3; \\
&\mathcal{J}_{22} = (\sqrt{3}\, a_{20} - \sqrt{2}\, a_{22})^2 B_2 + (6a_{30}^2 + 17a_{31}^2 + 8a_{32}^2 \\
&+ 3a_{33}^2 - 2\sqrt{30}\, a_{30}a_{32} - 2\sqrt{15}\, a_{31}a_{33}) B_3; \\
&\mathcal{J}_{33} = 8a_{22}^2 B_2 + (2a_{31}^2 + 8a_{32}^2 + 18a_{33}^2) B_3; \\
&\mathcal{J}_{12} = \mathcal{J}_{21} = \mathcal{J}_{23} = \mathcal{J}_{32} = 0; \\
&\mathcal{J}_{13} = \mathcal{J}_{31} = (2\sqrt{3}\, a_{30}a_{31} + 3\sqrt{10}\, a_{31}a_{32} + 5\sqrt{6}\, a_{32}a_{33}) B_3
\end{aligned}\right\} \tag{14}$$

and

$$\left.\begin{aligned}
&F_{1,\lambda\mu} = F_{3,\lambda\mu} = 0; \quad F_{i,2\mu} = 0; \\
&F_{2,30} = 2\sqrt{3}\, a_{31} B_3; \quad F_{2,31} = (\sqrt{10}\, a_{32} - 2\sqrt{3}\, a_{30}) B_3; \\
&F_{2,32} = (\sqrt{6}\, a_{33} - \sqrt{10}\, a_{31}) B_3; \quad F_{2,33} = -\sqrt{6}\, a_{32} B_3.
\end{aligned}\right\} \tag{15}$$

Then the kinetic energy of the nucleus can be represented in the form

$$T = \frac{1}{2} \sum_{\varkappa_1\varkappa_2} g_{\varkappa_1\varkappa_2} \dot{q}_{\varkappa_1} \dot{q}_{\varkappa_2}, \tag{16}$$

where $q_\varkappa$ is the set of collective variables

$$q_\varkappa = \{\beta_1, \beta_2, \beta_3, a_{20}, a_{22}, a_{30}, a_{31}, a_{32}, a_{33}\}, \tag{17}$$

and the matrix $g_{\varkappa_1\varkappa_2}$ can be expressed in terms of the mass parameters B_λ and the quantities $\mathcal{J}_{ij}$ and $F_{i,\lambda\mu}$:

$$g_{\varkappa_1\varkappa_2} = \begin{pmatrix}
\mathcal{J}_{11} & 0 & \mathcal{J}_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathcal{J}_{22} & 0 & 0 & 0 & F_{2,30} & F_{2,31} & F_{2,32} & F_{2,33} \\
\mathcal{J}_{13} & 0 & \mathcal{J}_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & B_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2B_2 & 0 & 0 & 0 & 0 \\
0 & F_{2,30} & 0 & 0 & 0 & B_3 & 0 & 0 & 0 \\
0 & F_{2,31} & 0 & 0 & 0 & 0 & 2B_3 & 0 & 0 \\
0 & F_{2,32} & 0 & 0 & 0 & 0 & 0 & 2B_3 & 0 \\
0 & F_{2,33} & 0 & 0 & 0 & 0 & 0 & 0 & 2B_3
\end{pmatrix}. \tag{18}$$

The general quantization procedure (for more details, see Ref. 12) leads to a Hamiltonian of the form

$$\hat{H} = \frac{1}{2} \sum_{\varkappa_1\varkappa_2} \hat{p}_{\varkappa_1} g^{-1}_{\varkappa_1\varkappa_2} \hat{p}_{\varkappa_2} + V_{\text{add}} + V. \tag{19}$$

Here, $\hat{p}_\varkappa$ are the collective momenta corresponding to the collective coordinates (17):

$$\hat{p}_\varkappa = \left\{\hat{M}_1', \hat{M}_2', \hat{M}_3', -i\frac{\partial}{\partial a_{20}}, -i\frac{\partial}{\partial a_{22}}, -i\frac{\partial}{\partial a_{30}}, -i\frac{\partial}{\partial a_{31}}, -i\frac{\partial}{\partial a_{32}}, -i\frac{\partial}{\partial a_{33}}\right\}; \tag{20}$$

$\hat{M}_i'$ are the angular-momentum operators in the intrinsic system; V_{add} is the additional potential associated with the dependence of the mass coefficients $g_{\varkappa_1\varkappa_2}$ on the dynamical variables $a_{\lambda\mu}$:

$$V_{\text{add}}=\frac{1}{8D}\sum_{\lambda\mu}\frac{1+\delta_{\mu 0}}{B_\lambda}\left(\frac{\partial^2 D}{\partial a_{\lambda\mu}^2}-\frac{1}{2D}\left(\frac{\partial D}{\partial a_{\lambda\mu}}\right)^2\right); \quad (21)$$

here

$$D=\sqrt{\det g_{\varkappa_1\varkappa_2}}. \quad (22)$$

2. INTERACTION OF VIBRATIONS AND APPROXIMATION OF SELF-CONSISTENT PHONONS IN RIGID NUCLEI

When allowance is made for the octupole degrees of freedom, the matrix $g^{-1}_{\varkappa_1\varkappa_2}$ that determines the collective vibrational–rotational Hamiltonian has a rather complicated structure. In the vibrational–rotational model it is usually assumed that the vibrations take place with respect to a given deformed shape. In other words, when studying small vibrations about the equilibrium, it is convenient to separate from the variables $a_{\lambda\mu}$ their equilibrium values,

$$a_{\lambda\mu}=\beta_0\delta_{\lambda 2}\delta_{\mu 0}+a'_{\lambda\mu}, \quad (23)$$

and assume that the amplitudes $a'_{\lambda\mu}$ are small,

$$|a'_{\lambda\mu}/\beta_0|\ll 1, \quad (24)$$

and that a restriction can be made to the leading terms in expansions with respect to them.

Initially, we consider vibrational–rotational excitations satisfying the condition (24), having in mind applications to the so-called rigid nuclei, and then (beginning in Sec. 9) we generalize the treatment so that we can dispense with the assumption (24) and we therefore make the theory itself apply to the description of not only rigid but also soft nuclei, in which the static deformation can also be small.

We draw attention to the fact that the moments of inertia $\mathcal{J}_{11}$ and $\mathcal{J}_{22}$ associated with rotations around the axes 1 and 2 perpendicular to the symmetry axis contain the static component $\mathcal{J}_0$:

$$\mathcal{J}_0=\frac{1}{2}\sum_\lambda B_\lambda\lambda(\lambda+1)\beta_\lambda^2=3B_2\beta_0^2. \quad (25)$$

The remaining elements of the tensor $\mathcal{J}_{ij}$ are quadratic in the small amplitudes $a'_{\lambda\mu}$, while $F_{i,\lambda\mu}$ are linear. Therefore, the determinant of the matrix $g_{\varkappa_1\varkappa_2}$ reduces to the product of its diagonal elements:

$$\det g_{\varkappa_1\varkappa_2}\simeq \mathcal{J}_0^2\mathcal{J}_{33}2B_2^2 8B_3^4. \quad (26)$$

The leading terms in the expansion of the inverse matrix $g^{-1}_{\varkappa_1\varkappa_2}$ determine the first two terms in the vibrational–rotational Hamiltonian,

$$\hat H=\hat H_{\text{rot}}+\hat H_{\text{vib}}+\hat H_{\text{vibrot}}, \quad (27)$$

the rotational part $\hat H_{\text{rot}}$ including the terms

$$\hat H_{\text{rot}}\equiv\frac{\hat M_1'^2+\hat M_2'^2}{2\mathcal{J}_0}+\frac{\hat M_3'^2}{2\mathcal{J}_{33}}. \quad (28)$$

The vibrational part H_{vib} contains besides the deformation energy V,

$$V=\frac{1}{2}\sum_{\lambda\mu\geq 0}c_{\lambda\mu}a'^2_{\lambda\mu}, \quad (29)$$

the potential V_{add}, which in accordance with (21) is equal to

$$V_{\text{add}}\simeq\frac{1}{\mathcal{J}_{33}}\left(S-\frac{3}{4\mathcal{J}_{33}}\sum_{\lambda\mu}B_\lambda\mu^2(\mu^2-4)a'^2_{\lambda\mu}\right), \quad (30)$$

where

$$S\equiv\frac{1}{4}\sum_{\lambda\mu\geq 0}\mu^2-\frac{3}{2}. \quad (31)$$

Introducing the more compact notation

$$q_r\equiv a'_{\lambda\mu},\ B_r\equiv B_\lambda\frac{2}{1+\delta_{\mu 0}}, \quad (32)$$

where

$$r=\{20,\ 22,\ 30,\ 31,\ 32,\ 33\}, \quad (33)$$

we can represent the vibrational Hamiltonian as

$$\hat H_{\text{vib}}=\sum_r\left\{-\frac{1}{2B_r}\frac{\partial^2}{\partial q_r^2}+\frac{1}{2}\left(c_r-\frac{3B_r\mu^2(\mu^2-4)}{4\mathcal{J}_{33}^2}\right)q_r^2\right\}+\frac{S}{\mathcal{J}_{33}}. \quad (34)$$

With regard to the vibrational–rotational part $\hat H_{\text{vib rot}}$, it is determined by the following terms in the expansion of the inverse matrix $g^{-1}_{\varkappa_1\varkappa_2}$ in the small amplitudes of the vibrations and has a fairly complicated structure. The explicit form of $\hat H_{\text{vib rot}}$ will be given below in Sec. 5.

For the vibrational–rotation model it is traditional to adopt the approach in which one first constructs wave functions for the Hamiltonian

$$\hat H_0\equiv\hat H_{\text{vib}}+\hat H_{\text{rot}}, \quad (35)$$

and then, using already a diagonalization on these functions of the total collective Hamiltonian $\hat H$, one obtains the final solutions. Representing the wave function of the Hamiltonian $\hat H_0$ as a product of a vibrational and a rotational function,

$$\Phi(\{q_r\},\ \theta_1,\ \theta_2,\ \theta_3)=\chi_{IK}(\{q_r\})D^{I*}_{MK}(\theta_1,\ \theta_2,\ \theta_3), \quad (36)$$

where I is the total angular momentum of the nucleus and M and K are its projections onto the laboratory axis and the symmetry axis of the nucleus, we see that the vibrational function χ_{IK} satisfies the Schrödinger equation

$$\hat H^{\text{eff}}_{\text{vib}}\chi_{IK}(\{q_r\})=E_{IK}\chi_{IK}(\{q_r\}) \quad (37)$$

with effective vibrational Hamiltonian

$$\hat H^{\text{eff}}_{\text{vib}}=\hat H_{\text{vib}}+\frac{K^2}{2\mathcal{J}_{33}}+\frac{I(I+1)-K^2}{2\mathcal{J}_0}. \quad (38)$$

The Hamiltonian (38) describes vibrations of the surface of the rotating nucleus (with angular momentum I and projection K of the angular momentum onto the symmetry axis) and can be divided in accordance with its structure into three parts:

$$\hat H^{\text{eff}}_{\text{vib}}=\frac{I(I+1)-K^2}{2\mathcal{J}_0}+\sum_r\hat H_r^{(0)}+V_{\text{int}}. \quad (39)$$

The first of them is associated with rotation about the perpendicular (with respect to the symmetry axis) axis and is not an operator with respect to the vibrational variables. The

second part is the Hamiltonian of the system of noninteracting phonons:

$$\hat{H}_r^{(0)} = -\frac{1}{2B_r}\frac{\partial^2}{\partial q_r^2} + \frac{1}{2}c_r q_r^2. \tag{40}$$

Finally, the third part is formed by the terms in $\hat{H}_{\text{vib}}^{\text{eff}}$ that contain in the denominator the moment of inertia $\mathcal{J}_{33}$ (we recall that the axis 3 of the intrinsic system is the symmetry axis of the nucleus),

$$V_{\text{int}} = \frac{\delta_K}{\mathcal{J}_{33}} + \frac{\sum_r \gamma_r q_r^2}{\mathcal{J}_{33}^2}, \tag{41}$$

where

$$\delta_K = K^2/2 + S; \quad \gamma_r = -(3/8)B_r\mu^2(\mu^2 - 4). \tag{42}$$

In the Faessler–Greiner model,[11] in which one considers only quadrupole degrees of freedom, the moment of inertia $\mathcal{J}_{33}^{(2)}$ is determined solely by the variable a_{22}, and the part $V_{\text{int}}^{(2)}$ reduces simply to the centrifugal correction

$$V_{\text{int}}^{(2)} = -\frac{1-K^2}{16Ba_{22}^2} \tag{43}$$

and does not prevent separation of the vibrational variables.

In the general case, the moment of inertia $\mathcal{J}_{33}$ depends on all the variables $a_{\lambda\mu}$ with nonvanishing projections μ that correspond to vibrations that break the axial symmetry. Therefore, the part V_{int} describes the interaction between these degrees of freedom. The fact that when degrees of freedom with $\lambda > 2$ are included in the treatment the vibrational variables do not separate leads to certain difficulties in constructing the theory of low-lying collective states of axial nuclei. Despite great efforts to overcome these difficulties (see, for example, Refs. 19 and 21), this problem can hardly be regarded as solved.

It appears reasonable[22] to seek the collective wave function $\chi_{IK}(\{q_r\})$, which describes the quadrupole as well as the octupole (and, in principle, the higher multipolarity) vibrations, in the form

$$\chi_{IK}(\{q_r\}) = \prod_r \chi_r(q_r), \tag{44}$$

where the functions $\chi_r(q_r)$ are determined by means of the variational principle

$$\frac{\delta}{\delta\chi_r}\left\{\langle\chi_{IK}|\hat{H}_{\text{vib}}^{\text{eff}}|\chi_{IK}\rangle - \sum_{r'}\varepsilon_{r'}\langle\chi_{r'}|\chi_{r'}\rangle\right\} = 0 \tag{45}$$

(ε_r are Lagrangian multipliers). The relation (44) in conjunction with (45) is the Hartree approximation for the collective (vibrational) degrees of freedom and can be called the approximation of "self-consistent phonons."

The assumption of a self-consistent nature of the phonon interaction is based on the following considerations. The point is that already when the octupole degrees of freedom are taken into account four terms contribute to the moment of inertia $\mathcal{J}_{33}$, and therefore the part played by each of them separately is relatively small.

Since all these terms are of the same sign (positive), there can be no compensation effects, and the moment of inertia $\mathcal{J}_{33}$, and, therefore, the interaction potential V_{int}, is a fairly smooth function of each of the variables q_r for a certain (average) value of the remaining variables. With regard to the singularity of the potential at $q_{r_1} = q_{r_2} = \ldots = q_{r_N} = 0$ (N is the number of interacting degrees of freedom), its importance decreases with increasing dimension of the space, and already when $N = 4$ (quadrupole + octupole) this singularity is unimportant.

3. APPROXIMATION OF A LARGE NUMBER OF INTERACTING VIBRATIONAL DEGREES OF FREEDOM

The Hartree equations for the phonon wave functions $\chi_r(q_r)$ can be obtained and solved without additional approximations. However, these equations take a particularly simple form and the picture becomes rather clear if we exploit the fact that the number N of interacting degrees of freedom is rather large and we expand the interaction potential in powers of $1/N$.

Considering from this point of view the averaging with respect to a certain variable q_r of the quantity $1/\mathcal{J}_{33}$, which occurs in the interaction potential V_{int}, we can readily establish[22]

$$\left\langle\frac{1}{\mathcal{J}_{33}}\right\rangle_{q_r} \simeq \frac{1}{\langle\mathcal{J}_{33}\rangle_{q_r}}, \tag{46}$$

where terms of order $1/N^3$ and higher have been omitted. In principle, such terms can be retained and more accurate though cumbersome expressions can be obtained. We note that the third-order correction to (46) contains an additional small quantity, being proportional to the variance $\langle q_r^4\rangle - \langle q_r^2\rangle^2$ of the vibrations. Therefore, even with allowance for the accumulation of errors from the successive averaging over all the variables the approximation

$$\left\langle\frac{1}{\mathcal{J}_{33}}\right\rangle \simeq \frac{1}{\langle\mathcal{J}_{33}\rangle} \tag{47}$$

and the analogous approximation

$$\left\langle\frac{\sum_r \gamma_r q_r^2}{\mathcal{J}_{33}^2}\right\rangle \simeq \frac{\sum_r \gamma_r\langle q_r^2\rangle}{\langle\mathcal{J}_{33}\rangle^2} \tag{48}$$

are fairly good.

Using (47) and (48), we obtain for each function χ_r the Schrödinger equation

$$\hat{H}_r\chi_r = \varepsilon_r\chi_r(q_r) \tag{49}$$

with single-phonon oscillator Hamiltonian $\hat{H}_r$,

$$\hat{H}_r = -\frac{1}{2B_r}\frac{d^2}{dq_r^2} + \frac{1}{2}c_r^{(p)}q_r^2, \tag{50}$$

but rigidity $c_r^{(p)}$,

$$c_r^{(p)} = c_r - \frac{2(\delta_K\mu^2B_r - \gamma_r)}{\langle\mathcal{J}_{33}\rangle_{(p)}^2} - \frac{4\sum_{r'}\gamma_{r'}\langle q_{r'}^2\rangle_{(p)}}{\langle\mathcal{J}_{33}\rangle_{(p)}^3}, \tag{51}$$

which depends on the mean values of the complete set of vibrational coordinates. These, in their turn, are determined by the rigidities $c_r^{(p)}$,

$$\langle q_r^2\rangle_{(p)} = \frac{n_r + 1/2}{\sqrt{B_r c_r^{(p)}}}, \tag{52}$$

TABLE I. Rearrangement of nuclear characteristics.

Characteristic, keV	Initial value, keV	Band				
		g. b. (β)	$n_{22}=1$; $K=2$	$n_{21}=1$; $K=1$	$n_{32}=1$; $K=2$	$n_{33}=1$; $K=3$
$\omega_{22}^{(p)}$	965	832	842	814	841	823
$\omega_{31}^{(p)}$	806	790	780	780	780	774
$\omega_{32}^{(p)}$	977	847	857	830	856	838
$\omega_{33}^{(p)}$	2000	1715	1813	1736	1811	1797
$1/\overline{\mathscr{J}}_{33}$		124	79	106	80	77
$E^{(p)}$	($E_{g.s}=330$)		−104	−18	−102	−161

which depend, thus, on the state $p=\{K,\{n_r\}\}$ of the nucleus (we are of course speaking of the dependence on the set of numbers n_r with $\mu\neq 0$).

If we introduce the frequencies ω_r and $\omega_r^{(p)}$,

$$\omega_r=\sqrt{c_r/B_r},\quad \omega_r^{(p)}=\sqrt{c_r^{(p)}/B_r}, \tag{53}$$

then we can cast the relations (51) into the form

$$\omega_r^{(p)2}=\omega_r^2-\frac{\mu^2}{\overline{\mathscr{J}}_{33}^2}\left\{K^2+\frac{3}{4}(\mu^2-4)+\varkappa_{(p)}+2S\right\}, \tag{54}$$

where the mean value of the moment of inertia in the given state is determined in accordance with

$$\overline{\mathscr{J}}_{33}=\sum_r \mu^2\frac{n_r+1/2}{\omega_r^{(p)}} \tag{55}$$

and we have introduced the notation

$$\varkappa_{(p)}=-\frac{3}{2\overline{\mathscr{J}}_{33}}\sum_r \mu^2(\mu^2-4)\frac{n_r+1/2}{\omega_r^{(p)}} \tag{56}$$

[we recall that the constant S is defined in accordance with (31)].

Equations (54) form in conjunction with (55) and (56) a closed system of coupled algebraic equations for the frequencies $\omega_r^{(p)}$ characterizing the state p. When the vibrational–rotational interaction $\hat{H}_{\text{vib rot}}$ is ignored, these frequencies are independent of the total angular momentum I and therefore characterize the entire rotational band based on the given level p.

Equations (54)–(56) reduce to a system of two equations with two unknowns, and, as analysis shows, this system has a unique solution.

In Table I, we give as an example the frequencies $\omega_r^{(p)}$ for the different bands p (the choice of the initial frequencies ω_r corresponds to the nucleus ^{238}U; see below). We see that already the zero-point vibrations in the ground state significantly disturb the axial symmetry of the nucleus, and this leads to a significant difference between the frequencies $\omega_r^{(g)}$ and the initial ω_r. Although the change in each of the rigidities as a result of excitation is not particularly large [the values of $\omega_r^{(p)}$ for different bands are similar], the total effect of the additional breaking of the axial symmetry is not so small—the moment of inertia $\overline{\mathscr{J}}_{33}$ varies significantly from band to band. On the other hand, even following the excitation of a vibration quantum, when the contribution of the corresponding degree of freedom to $\overline{\mathscr{J}}_{33}$ increases by three times, the value of $1/\overline{\mathscr{J}}_{33}$ changes by not more than 1.5 times. This confirms the assumption made above concerning the nature of the dependence of the part V_{int} on the vibrational variables.

4. SPECTRUM OF THE HAMILTONIAN $\hat{H}_0$

The frequencies $\omega_r^{(p)}$ directly determine the phonon oscillator functions $\chi_r(q_r)$, from which the total vibrational function χ_{IK} is constructed in accordance with (44), and the total collective function Φ_{IKM} in accordance with (36). However, it must be borne in mind that the choice of the axes of the intrinsic coordinate system is not unique in the sense that there exist different possibilities for choosing the indexing and directions of the coordinate axes (for a given triaxial system).[6] Since a unique set of collective variables $\alpha_{\lambda\mu}$ corresponds to a given shape of the nucleus in the laboratory system, the collective wave function, expressed in terms of the Eulerian angles and the vibration amplitudes in the intrinsic coordinate system, must be invariant with respect to the choice of the intrinsic system. For this, the wave function must be invariant with respect to the transformations determined by the operators $\hat{R}_1$ (change of the direction of reference with respect to the axes 2 and 3) and $\hat{R}_2$ (rotation around axis 3 through $\pi/2$). Acting on the function (36), we obtain[24] for the considered model

$$K=\sum_r \mu_r n_r. \tag{57}$$

The following function has the required property of invariance under the transformation $\hat{R}_1$:

$$\Psi_{I|K|M}=\frac{1}{\sqrt{2(1+\delta_{K0})}}\{\Phi_{IK}+(-1)^{I-K}\pi\Phi_{I,-K}\}, \tag{58}$$

where

$$\pi = (-1)^{\sum_r \lambda n_r} \tag{59}$$

is the parity of the state.

From (58) for $K = 0$ there follows the well-known selection rule

$$\pi = (-1)^I, \quad K = 0. \tag{60}$$

We note that the symmetrization (58) of the wave functions has an effect when they are used to calculate the nondiagonal matrix elements.

The energy of the system of interacting phonons is determined by the expectation value of the Hamiltonian $\hat{H}^{\text{eff}}_{\text{vib}}$ with respect to the wave functions χ_{IK}, and this leads to

$$E_{IK}(\{n_r\}) = \frac{I(I+1)-K^2}{2\mathscr{J}_0} + \sum_r \langle \chi_r | \hat{H}^{(0)}_r | \chi_r \rangle + \frac{\delta_K + \frac{\varkappa^{(p)}}{4}}{\overline{\mathscr{J}}_{33}}$$

$$= \frac{I(I+1)-K^2}{2\mathscr{J}_0} + \sum_r (n_r + 1/2)\left(\omega_r + \frac{(\omega_r^{(p)}-\omega_r)^2}{2\omega_r^{(p)}}\right) + \frac{K^2/2 + \varkappa_p/4 + S}{\overline{\mathscr{J}}_{33}}. \tag{61}$$

The excitation energy $E^*_{IK}(\{n_r\})$ is obviously the difference between the energies of the given state $\{IK,\{n_r\}\}$ and the ground state $\{00,\{n_r = 0\}\}$:

$$E_{IK}(\{n_r\}) = E_{IK}(\{n_r\}) - E_{00}(\{n_r = 0\}). \tag{62}$$

We consider three types of excited states.

1. Ground-state band, $p = g \equiv \{0,\{n_r = 0\}\}$.

The excitation energy is purely rotational (around an axis transverse to the symmetry axis):

$$E^*_{I0}(\{n_r = 0\}) = I(I+1)/2\mathscr{J}_0. \tag{63}$$

2. Excitations that do not disturb the axial symmetry of the nucleus, i.e., $K = 0$, and only some of the numbers n_r with $\mu = 0$ are nonzero. To these excitations there belong above all the β bands $n_{20} = 1,2,...$, and also the octupole excitations $n_{30} = 1,2,...$, etc. Excitations of this type do not change the moment of inertia $\overline{\mathscr{J}}_{33}$ and the frequencies $\omega_r^{(p)}$ from their values in the ground state:

$$\omega_r^{(p)} = \omega_r^{(g)}, \tag{64}$$

with

$$\omega_{\lambda 0}^{(p)} = \omega_{\lambda 0} \tag{65}$$

(for all states).

The excitation energy consists in this case of rotational energy and the energy of the excited phonons:

$$E^*_{I0}(\{n_{\lambda\mu} \sim \delta_{\mu 0}\}) = \frac{I(I+1)}{2\mathscr{J}_0} + \sum_\lambda \omega_{\lambda 0} n_\lambda. \tag{66}$$

3. Excitations that disturb the axial symmetry of the nucleus. The breaking of the symmetry (in addition to that produced in the ground state by the zero-point vibrations) leads to a change in the moment of inertia $\overline{\mathscr{J}}_{33}$ and the frequencies $\omega_r^{(p)}$:

$$\langle \mathscr{J}_{33} \rangle_{(p)} \neq \langle \mathscr{J}_{33} \rangle_{(g)}, \quad \omega_r^{(p)} \neq \omega_r^{(g)} \ (\mu \neq 0). \tag{67}$$

As a result, the excitation energy contains an additional part $E^{(p)}$ associated with the rearrangement,

$$E^{(p)} = \sum_r \frac{1}{2}\left(\frac{(\omega_r^{(p)}-\omega_r)^2}{2\omega_r^{(p)}} - \frac{(\omega_r^{(g)}-\omega_r)^2}{2\omega_r^{(g)}}\right) + \frac{\frac{\varkappa^{(p)}}{4}+S}{\langle \mathscr{J}_{33}\rangle_p} - \frac{\frac{\varkappa^{(g)}}{4}+S}{\langle \mathscr{J}_{33}\rangle_{(g)}} \tag{68}$$

(see Table I), and determined by the expression

$$E^*_{IK}(\{n_r\}) = \frac{I(I+1)-K^2}{2\mathscr{J}_0} + \sum_r \left(\omega_r + \frac{(\omega_r^{(p)}-\omega_r)^2}{2\omega_r^{(p)}}\right) n_r + \frac{K^2}{2\overline{\mathscr{J}}_{33}} + E^{(p)}. \tag{69}$$

5. ALLOWANCE FOR THE VIBRATIONAL–ROTATIONAL INTERACTION

The next step is to take into account[24,25] the part $\hat{H}_{\text{vib rot}}$ of the total collective Hamiltonian (27) responsible for the deviation of the energies in a rotational band from the law $E_I \sim I(I+1)$. The exact wave functions and energies are then obtained by diagonalizing the total Hamiltonian on the basis of the functions constructed above,

$$\Psi^{(p)}_{IM} = \sum_{p'} \Psi_{IM,p'} c^p_{p'}. \tag{70}$$

Note that wave functions $\Psi_{IM,p}$ relating to different states p_1 and p_2 are not, strictly speaking, orthogonal to each other. However, in reality the parameters $\omega_r^{(p)}$ of the "single-particle" functions $\chi_r(q_r)$ do not differ too strongly (see Table I), and therefore the effects of the nonorthogonality are negligibly small. (Even for $\Delta\omega/\omega = 0.1$, the overlap integral is 0.99943.)

The vibrational–rotational part $\hat{H}_{\text{vib rot}}$ can be represented as the sum

$$\hat{H}_{\text{vib rot}} = \hat{H}^{(2)}_{\text{vib rot}} + \hat{H}^{(3)}_{\text{vib rot}} \tag{71}$$

of the quadrupole operator $\hat{H}^{(2)}_{\text{vib rot}}$, which is identical to the operator $\hat{H}_{\text{vib rot}}$ in the Faessler–Greiner model (see Ref. 12),

$$\hat{H}^{(2)}_{\text{vib rot}} = \frac{1}{2\mathscr{J}_0}\left\{\hat{M}'^2_1\left(-2\frac{a'_{20}}{\beta_0} - 2\frac{\sqrt{6}}{3}\frac{a_{22}}{\beta_0} + 3\frac{a'^2_{20}}{\beta^2_0} + 2\frac{a^2_{22}}{\beta^2_0} + 2\sqrt{6}\frac{a'_{20}a_{22}}{\beta^2_0}\right) + \hat{M}'^2_2\left(-2\frac{a'_{20}}{\beta_0} + \frac{2\sqrt{6}}{3}\frac{a_{22}}{\beta_0} + 3\frac{a'^2_{20}}{\beta^2_0} + 2\frac{a^2_{22}}{\beta^2_0} - 2\sqrt{6}\frac{a'_{20}a_{22}}{\beta^2_0}\right)\right\}, \tag{72}$$

and the operator $\hat{H}^{(3)}_{\text{vib rot}}$, which arises because the octupole degrees of freedom are taken into account,

$$H^{(3)}_{\text{vib rot}} = \frac{1}{\mathscr{J}_0}\left\{\frac{1}{2}\sum_{\mu_1\mu_2}\hat{p}_{3\mu_1}f_{3\mu_1}f_{3\mu_2}\hat{p}_{3\mu_2} - \hat{M}'_2\sum_\mu f_{3\mu}\hat{p}_{3\mu} - \frac{\mathscr{J}_{13}}{2\mathscr{J}_{33}}(\hat{M}'_1\hat{M}'_3 + \hat{M}'_3\hat{M}'_1) + \frac{\mathscr{J}^2_{13}}{2\mathscr{J}_0\mathscr{J}_{33}}\hat{M}'^2_1 - \frac{\mathscr{J}^{(3)}_{11}}{2\mathscr{J}_0}\hat{M}'^2_1\right\}, \tag{73}$$

where $\mathscr{J}^{(3)}_{11}$ is the octupole part of the moment of inertia $\mathscr{J}_{11}$ [see (14)], and the amplitudes $f_{\lambda\mu} \equiv F_{2,\lambda\mu}/B_{\lambda\mu}$ are given by

TABLE II. Rearrangement of rotational energy for different bands.

Band	g. b	β	γ	Octupoles				β octupoles
$n_{\lambda\mu}$; K	$n_{\lambda\mu}=0$; $K=0$	$n_{20}=1$; $K=0$	$n_{22}=1$; $K=2$	$n_{30}=1$; $K=0$	$n_{31}=1$; $K=1$	$n_{32}=1$; $K=2$	$n_{33}=1$; $K=3$	$n_{20}=n_{30}=1$; $K=0$
$\varepsilon^{(p)}/\varepsilon_0$	1.08	1.20	1.15	1.02	1.07	1.05	1.07	1.15

$$\left.\begin{aligned} f_{30}&=2\sqrt{3}\,a_{31},\quad f_{31}=\sqrt{5/2}\,a_{32}-\sqrt{3}\,a_{30};\\ f_{32}&=\sqrt{3/2}\,a_{33}-\sqrt{5/2}\,a_{31},\quad f_{33}=-\sqrt{3/2}\,a_{32}. \end{aligned}\right\} \tag{74}$$

Each term in the operator $\hat{H}_{\text{vib rot}}$ is a product of angular-momentum operators, which act only on the rotational variables, and the "vibrational" operators $a_{\lambda\mu}$ and $\hat{p}_{\lambda\mu}$.

Omitting the cumbersome expressions for the matrix elements of the vibrational–rotational interaction (which can be calculated by the well-known method; see, for example, Ref. 12), we restrict ourselves to a discussion of their structure.

We note first of all that the diagonal part of $\hat{H}_{\text{vib rot}}$ already leads to a renormalization of the moments of inertia for the individual bands (Table II). This renormalization is characterized by the small parameter ε_0/ω [$\varepsilon_0 \equiv 1/(2\mathscr{J}_0)$ is the rotational energy, and ω is the energy of one of the vibration quanta $\omega_{\lambda\mu}$].

Further, since part of $\hat{H}_{\text{vib rot}}$ consists of terms linear or quadratic in the angular-momentum operators $\hat{M}_i$, the matrix elements with $|\Delta K| = 0,1,2$ are nonzero. For states with positive parity, only even K are admissible, and matrix elements with $|\Delta K| = 1$ do not arise. But for the negative-parity states it is precisely these matrix elements that play an important part. The Coriolis forces ($|\Delta K| = 1$) are due both to the direct interaction of the vibrations with the rotations [the terms with $F_{i,\lambda\mu}$ in the kinetic energy (11)] and to the nondiagonality of the moment of inertia tensor $\mathscr{J}_{ij}$; moreover, whereas the former are characterized by only rotational energy, the latter are also proportional to the ratio $1/(\mathscr{J}_{33}|\omega)$.

The matrix elements with $|\Delta K| = 0$ and 2 are appreciably smaller than the Coriolis elements, since in addition to ε_0 they also contain one or two factors $\sqrt{\varepsilon_0/\omega}$. As an illustration, Table III gives the numerical values of the different matrix elements from the calculation for the nucleus ^{238}U:

$$\begin{aligned} &\langle IK'\{n'_{\lambda\mu}\}|\hat{H}_{\text{vib rot}}|IK\{n_{\lambda\mu}\}\rangle\\ &\equiv \gamma_{pp'}\begin{cases} I(I+1)-K^2,\ \Delta K=0;\\ \sqrt{I(I+1)-K(K+1)},\ |\Delta K|=1;\\ \sqrt{I(I+1)-K(K+1)}\sqrt{I(I+1)-(K+1)(K+2)},\\ \quad |\Delta K|=2. \end{cases} \end{aligned} \tag{75}$$

It is of interest to follow the rearrangement of the states when the nondiagonal part of the vibrational–rotational interaction is "switched on" (by means of a coefficient that varies from zero to unity). Figure 1 shows the rearrangement of the energies of the 3^- states of the different octupole bands, the mean values of the corresponding number K,

$$\langle K\rangle_p = \sum_{p'} (c^p_{p'})^2 K, \tag{76}$$

and the reduced probabilities of $E3$ transitions from the ground state to the given state. The parameter ω_{33} for this calculation was chosen in such a way that the "initial" energies of the two 3^- levels from the bands with $K = 3$ and $K = 2$ were nearly equal. It is natural that precisely these states are the most strongly rearranged. However, two other states with $K = 1$ and $K = 0$ are also subject to a fairly strong influence of the vibrational–rotational interaction. This is reflected especially in the coefficients c^p_p and, therefore, in the probabilities of electromagnetic transitions between the collective states.

6. PARAMETERS AND THEIR CHOICE

The collective Hamiltonian used in the present approach includes as parameters the mass coefficients B_λ, the rigidities $c_{\lambda\mu}$, and the static-deformation parameters β_λ. The nucleus is characterized by nine quantities (assuming a

TABLE III. Nondiagonal matrix elements of the vibrational–rotational interaction $\gamma_{pp'}$, keV [see (75)].

$\pi=+1$				$\pi=-1$				
Band	g. b	β	γ	Band	$n_{30}=1$; $K=0$	$n_{31}=1$; $K=1$	$n_{32}=1$; $K=2$	$n_{33}=1$; $K=3$
g. b	—	−2.0	−1.3	$n_{30}=1$; $K=0$	—	−25.4	−0.2	—
β	−2.0	—	+0.6	$n_{31}=1$; $K=1$	−25.4	—	−13.2	−0.04
γ	−1.3	+0.6	—	$n_{32}=1$; $K=2$	−0.2	−13.2	—	−11.5

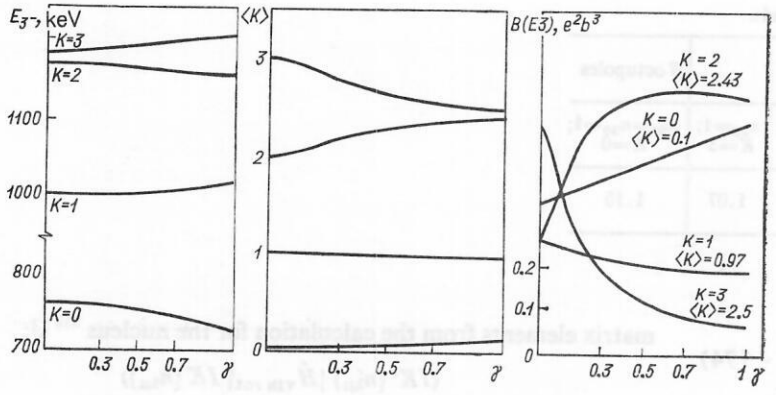

FIG. 1. Rearrangement of 3^- states when the nondiagonal part of the vibrational–rotational interaction with $\omega_{33} = 1010$ keV is included.

quadrupole nature of the static deformation).

$$\beta_2 \equiv \beta_0,\ B_2,\ B_3,\ c_{20},\ c_{22},\ c_{30},\ c_{31},\ c_{32},\ c_{33}. \qquad (77)$$

To determine the wave functions of the Hamiltonian H_0 it is sufficient to specify the six frequencies

$$\omega_{20},\ \omega_{22},\ \omega_{30},\ \omega_{31},\ \omega_{32},\ \omega_{33}. \qquad (78)$$

To find the energy levels of the total collective Hamiltonian, it is also necessary to know the rotational energy

$$\varepsilon_0 \equiv 1/(2\mathcal{I}_0) = 1/(6B_2\beta_0^2), \qquad (79)$$

which can be regarded as an independent parameter instead of the mass coefficient B_2 (if the values of β_0 are well known from experiment). These same quantities (78) and (79) in conjunction with β_0 also determine the probabilities of $E2$ transitions between the positive-parity states. But the probabilities of $E3$ transitions, in which octupole quanta are excited, are inversely proportional to the mass coefficient B_3.

In the spectra of many deformed nuclei (four among them have been chosen) one can observe not only the ground-state band but also bands identified as β ($n_{20} = 1$), $\gamma(n_{22} = 1)$, and octupole bands with $K = 0,1,2$ ($n_{3K} = 1$) excitations. In accordance with this, five parameters of the set (78) were chosen in such a way as to achieve fitting to one level (as a rule, the head line) of each of these bands. The rotational energy ε_0 was determined, as is usually done, from the 2^+ level of the ground-state band.

Unfortunately, reliable experimental data permitting such a determination of the parameter ω_{33} are not available—a band with $K^{(\pi)} = 3^-$ is not observed. Therefore, the dependence of the results of the theory on the choice of the parameter ω_{33} was investigated. Figure 2 shows the energies of the 3^- states, the mean values $\langle K \rangle$ of the projections of the angular momentum onto the symmetry axis, and the probabilities of transitions from the ground state to the given state as functions of the parameter ω_{33} in the "resonance" region (the remaining parameters were determined for each value of ω_{33} independently by the method described above).

The quantities given in Fig. 2 are very sensitive to the choice of the parameter ω_{33} when the energy of the 3^- ($K = 3$) level is close to the energy of any other 3^- level and the interaction between them is particularly important. The remaining quantities (the energies of the positive-parity levels and the probabilities of transitions between them) hardly change in a wide range of ω_{33} values—from 1000 keV to 100 MeV.

For the subsequent calculations, values of ω_{33} appreciably exceeding the "resonance" values were used, and therefore the results given below depend weakly on the choice of this parameter. Such an approach, in which the parameters of the collective Hamiltonian are determined from an appropriate number of experimental facts (level energies), is typical for the collective model. If in this approach one can successfully describe a much larger number of observed facts, this indicates that the chosen variant of the collective model correctly encompasses the nature of the investigated collective motion.

The Bohr Hamiltonian has a fairly clear physical meaning and the future microscopic theory of the nucleus will

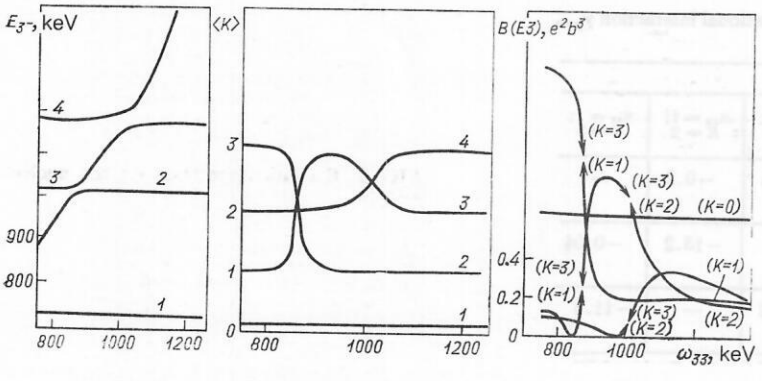

FIG. 2. Dependence of characteristics of 3^- states on the parameter ω_{33}.

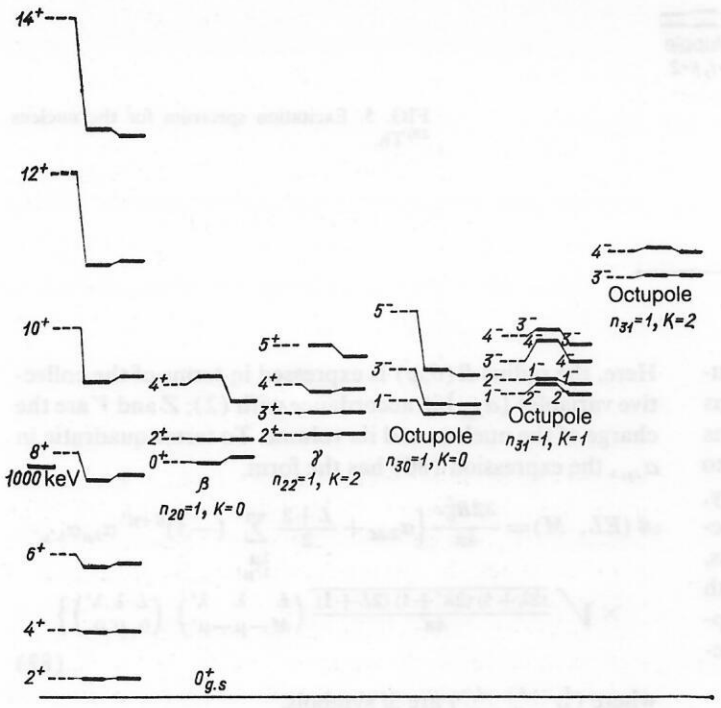

FIG. 3. Excitation spectrum for the nucleus ^{156}Gd.

very probably be able to calculate its parameters. It will then no longer be necessary to fit the parameters to several observed levels. This will make it possible to study nuclei for which experimental information is sparse or even completely absent. At the same time, the parameters of a Hamiltonian of a collective model that has successfully proved itself can also serve as a test for future microscopic calculations.

7. EXCITATION SPECTRA OF RIGID NUCLEI

Figures 3–6 show the observed and calculated[24,25] spectra of the collective excitations of the nuclei ^{156}Gd, ^{228}Th, ^{230}Th, and ^{238}U. The employed values of the parameters (78) are given in Table IV.

For each band the dotted lines on the left in Figs. 3–6 show the energies without allowance for the nondiagonal part of the vibrational–rotational interaction; the lines in the center show the results of diagonalizing the total collective Hamiltonian; and, finally, the lines on the right show the corresponding experimental values of the energies of the collective states. Of course, the vibrational–rotational interaction leads to a mixing of the states with respect to K and $n_{\lambda\mu}$, and therefore the (nonvanishing) values of these quantities given in the figures are used only to label the bands and correspond to the limit $\hat{H}_{\text{vib rot}} \to 0$. A clear picture of the mixing of the bands by the interaction $H_{\text{vib rot}}$ is given by the mean values $\langle K \rangle$, which can be found in Table V.

Comparison of the calculated and observed results indicates that the unified description proposed in this paper for the quadrupole and octupole degrees of freedom of deformed nuclei reflects their structure quite well. One can not only explain a large number of collective levels of both positive and negative parity but also reproduce the anomalous order $2^-,1^-,4^-,3^-$ of the levels in the octupole band $K^\pi = 1^-$ in ^{156}Gd. The following circumstance should be noted. Allowance for not only the quadrupole but also the octupole degrees of freedom also has a qualitative effect on the classification of the positive-parity states, namely, the γ excitations. In the Faessler–Greiner model, the energy of the rotation around the symmetry axis $K^2/(2\mathcal{J}^{(2)}_{33})$ is fairly large

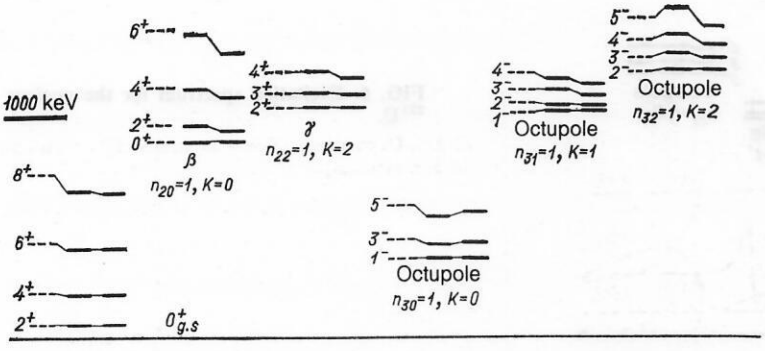

FIG. 4. Excitation spectrum for the nucleus ^{228}Th.

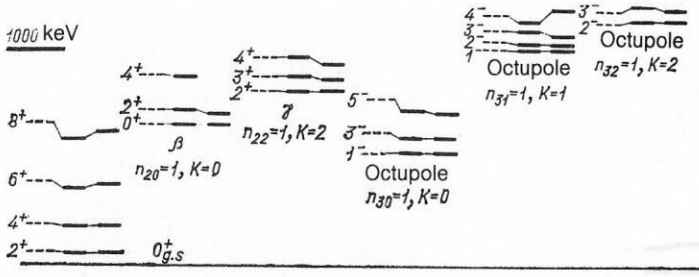

FIG. 5. Excitation spectrum for the nucleus ^{230}Th.

($\sim\omega$), since the moment $\mathcal{J}_{33}^{(2)}$ is determined by the quadrupole amplitude a_{22} alone. Therefore, the first excitations with $K=2$ must be interpreted as no-phonon excitations ($n_{22}=0$). At the same time, the octupole excitations lead to a much stronger dynamical breaking of the axial symmetry, and as a result $\overline{\mathcal{J}}_{33}$ is several times greater, and the corresponding rotational energy $K^2/(2\overline{\mathcal{J}}_{33})$ is several times less, than in the Faessler–Greiner model, which deals only with quadrupole excitations. Therefore, in the considered approach the γ excitations are naturally classified as single-phonon states with $n_{22}=1$ and $K=2$.

8. PROBABILITIES OF ELECTROMAGNETIC TRANSITIONS

The reduced transition probabilities give important information about the structure of the collective states. From this point of view, it seems to us that the most interesting probabilities are $B(E2)$ and $B(E3)$, which are measured in the case of Coulomb excitation of the levels 2^+ and 3^-. The matrix elements of such transitions can be determined by the standard method[10,12] by means of multipole operators in the collective space,

$$\mathcal{M}(EL,\ M)=e\int dr\, r^L Y_{LM}(\Omega)\,\rho_p(r,\ \{\alpha\}),\quad L\geqslant 2, \tag{80}$$

where $\rho_p(r,\{\alpha\})$ is the density of the protons for the given values of the collective variables $\{\alpha\}$, this density being assumed for simplicity to be constant within the nucleus:

$$\rho_p(r,\ \{\alpha\})=Z/V\Theta(R(\theta,\varphi)-r). \tag{81}$$

Here, the radius $R(\theta,\varphi)$ is expressed in terms of the collective variables $\{\alpha_{\lambda\mu}\}$ in accordance with (2); Z and V are the charge of the nucleus and its volume. To terms quadratic in $\alpha_{\lambda\mu}$, the expression (80) has the form

$$\mathcal{M}(EL,\ M)=\frac{3ZR_0^Le}{4\pi}\Big\{\alpha_{LM}+\frac{L+2}{2}\sum_{\substack{\lambda\mu\\ \lambda'\mu'}}(-1)^{\mu+\mu'}\alpha_{\lambda\mu}\alpha_{\lambda'\mu'}\times\sqrt{\frac{(2\lambda+1)(2\lambda'+1)(2L+1)}{4\pi}}\begin{pmatrix}L & \lambda & \lambda'\\ M & -\mu & -\mu'\end{pmatrix}\begin{pmatrix}L & \lambda & \lambda'\\ 0 & 0 & 0\end{pmatrix}\Big\}, \tag{82}$$

where $\begin{pmatrix}L & \lambda & \lambda'\\ M & -\mu & -\mu'\end{pmatrix}$ are $3j$ symbols.

To calculate the matrix elements between the states described by wave functions of the form (36), it is convenient to transform the expression for the multipole operators by going over from the variables $\alpha_{\lambda\mu}$ in the laboratory system to the vibrational variables $a_{\lambda\mu}$ in the intrinsic system and to the Eulerian angles that characterize the rotation of the nucleus. This is done in accordance with (4):

$$\mathcal{M}(EL,\ M)=\frac{3ZR_0^Le}{4\pi}\sum_m D^{L*}_{Mm}\Big\{a_{Lm}+\frac{L+2}{2}\times\sum_{\lambda\nu\lambda'\nu'}(-1)^{\nu+\nu'}a_{\lambda\nu}a_{\lambda'\nu'}\times\sqrt{\frac{(2\lambda+1)(2\lambda'+1)(2L+1)}{4\pi}}\begin{pmatrix}L & \lambda & \lambda'\\ M & -\nu & -\nu'\end{pmatrix}\begin{pmatrix}L & \lambda & \lambda'\\ 0 & 0 & 0\end{pmatrix}\Big\}. \tag{83}$$

We note that for states with $K\neq 0$ the wave functions

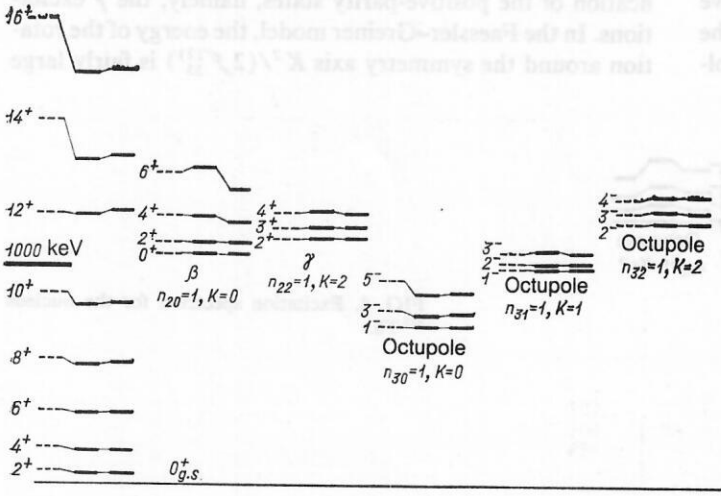

FIG. 6. Excitation spectrum for the nucleus ^{238}U.

TABLE IV. Values of the parameters of the theory (the last column gives for comparison the parameters of the Faessler–Greiner model[12]), keV.

Nucleus	ω_{20}	ω_{22}	ω_{30}	ω_{31}	ω_{32}	ω_{33}	ω_{22}^{FG}
^{156}Gd	1027	1028	1169	1111	1485	2000	1122
^{228}Th	831	852	256	804	929	2000	—
^{230}Th	635	710	448	827	922	2000	764
^{238}U	993	965	625	806	977	2000	1047

must be symmetrized in accordance with (58). The transition matrix element consists of two terms,

$$\langle \Psi_K \mid \mathscr{M} \mid \text{g.s}\rangle = \frac{1}{\sqrt{2}}\left(\langle \Phi_K \mid \mathscr{M} \mid \text{g.s}\rangle + \langle \Phi_{-K} \mid \mathscr{M} \mid \text{g.s}\rangle\right), \tag{84}$$

which are equal to each other, and therefore

$$\langle \Psi_K \mid \mathscr{M} \mid \text{g.s}\rangle = \sqrt{2}\,\langle \Phi_K \mid \mathscr{M} \mid \text{g.s}\rangle. \tag{85}$$

The considered transition corresponds to excitation of a phonon of multipolarity λ with projection K, and therefore the matrix element with respect to the vibrational variables contains $1/\sqrt{B_{\lambda K}}$, with $B_{\lambda K} = 2B_\lambda$ when $K \neq 0$. As a result, the additional factors $\sqrt{2}$ (compared with the case $K = 0$) in the numerator and denominator cancel. This cancellation is hardly fortuitous, since both factors arise for the same reason—the correlation of deformations with opposite values of the projections K.

For the transitions $0^+\text{g.s}\to 2^+(K,\{n_\nu\})$ the matrix elements are equal to

$$\left.\begin{aligned}
&\langle 2M0,\ \text{g.b} \mid \mathscr{M}(E2,\ M) \mid \text{g.s}\rangle \\
&\quad = \frac{3ZR_0^2 e}{4\pi}\,\frac{1}{\sqrt{5}}\,\beta_0\left(1+\frac{2}{7}\sqrt{\frac{5}{\pi}}\,\beta_0\right); \\
&\langle 2M0,\ n_{20}=1 \mid \mathscr{M}(E2,\ M) \mid \text{g.s}\rangle \\
&= \frac{3ZR_0^2 e}{4\pi}\,\frac{1}{\sqrt{5}}\,\frac{1}{\sqrt{2B_2\omega_{20}}}\left(1+\frac{4}{7}\sqrt{\frac{5}{\pi}}\,\beta_0\right); \\
&\langle 2M2,\ n_{22}=1 \mid \mathscr{M}(E2,\ M) \mid \text{g.s}\rangle \\
&= \frac{3ZR_0^2 e}{4\pi}\,\frac{1}{\sqrt{5}}\,\frac{1}{\sqrt{2B_2\omega_{22}}}\left(1-\frac{4}{7}\sqrt{\frac{5}{\pi}}\,\beta_0\right); \\
&\langle 2M2,\ n_{20}=n_{22}=1 \mid \mathscr{M}(E2,\ M) \mid \text{g.s}\rangle \\
&\quad = \frac{3ZR_0^2 e}{4\pi}\,\frac{-1}{\sqrt{5}}\,\frac{\frac{4}{7}\sqrt{\frac{5}{\pi}}}{\sqrt{2B_2\omega_{20}}\sqrt{2B_2\omega_{22}}}.
\end{aligned}\right\} \tag{86}$$

The quadrupole mass parameter B_2 that occurs here can be expressed in terms of the rotational energy in accordance with (79). The matrix elements (86) directly determine the reduced probabilities of the corresponding transitions:

$$B(EL,\ i\to f) = \frac{1}{2I_i+1}\sum_{M_iM_fM} \mid \langle I_fM_fK_f \mid \mathscr{M}(EL,\ M) \mid I_iM_iK_i\rangle \mid^2, \tag{87}$$

where $I_iM_iK_i$ and $I_fM_fK_f$ are the angular momenta and their projections in the initial and final states. These same matrix elements make it possible to calculate the probabilities of excitation of "mixed" states described by wave functions of the form (70):

$$\begin{aligned}
B(E2,\ \text{g.s}\to 2^+) = \frac{9Z^2}{16\pi^4}\,\sigma^2 e^2\Bigg\{ & c_g\beta_0\left(1+\frac{2}{7}\sqrt{\frac{5}{\pi}}\,\beta_0\right) \\
& + c_\beta\sqrt{\frac{3\varepsilon_0\beta_0^2}{\omega_{20}}}\left(1+\frac{4}{7}\sqrt{\frac{5}{\pi}}\,\beta_0\right) \\
& + c_\gamma\sqrt{\frac{3\varepsilon_0\beta_0^2}{\omega_{22}^{(p)}}}\left(1-\frac{4}{7}\sqrt{\frac{5}{\pi}}\,\beta_0\right) \\
& - c_{\beta\gamma}\frac{3\varepsilon_0\beta_0^2}{\sqrt{\omega_{20}\omega_{22}^{(p)}}}\,\frac{4}{7}\sqrt{\frac{5}{\pi}}\Bigg\}^2,
\end{aligned} \tag{88}$$

where $\sigma = \pi R_0^2$, and the coefficients $c_g, c_\beta c_\gamma$, and $c_{\beta\gamma}$ are the projections of the given 2^+ state onto the corresponding "pure" state.

The matrix elements of the transitions to the pure octupole states, $\text{g.s}\to 3^-(K,n_{3K}=1)$, are given by

$$\langle 3MK,\ n_{3K}=1 \mid \mathscr{M}(E3,\ M) \mid \text{g.s}\rangle = \frac{3ZR_0^3}{4\pi}\,e\,\frac{1+(-1)^K\beta_0\frac{4-K^2}{3}\sqrt{\frac{5}{4\pi}}}{\sqrt{2B_3\omega_{3K}^{(p)}}}\,\frac{1}{\sqrt{7}}, \tag{89}$$

TABLE V. Mean values $\langle K\rangle$ of the projection.

I	^{156}Gd			^{228}Th			^{238}U		
	$K=0$	$K=1$	$K=2$	$K=0$	$K=1$	$K=2$	$K=0$	$K=1$	$K=2$
1	0.396	0.604	—	0.006	0.994	—	0.021	0.979	—
2	—	1.016	—	—	1.075	1.925	—	1.033	1.967
3	0.482	0.585	1.942	0.032	1.150	1.819	0.100	1.000	1.901
4	—	1.065	1.959	—	1.207	1.795	—	—	1.886
5	0.520	—	1.765	0.070	—	1.647	0.186	—	—

TABLE VI. Excitation probabilities $B(E2,\text{g.s}\to 2^+)$ of 2^+ states measured in units of b^2e^2 (energy measured in keV).

Band	Energy probability	^{156}Gd theory	^{156}Gd experiment	^{156}Gd reference	^{230}Th theory	^{230}Th experiment	^{230}Th reference	^{238}U theory	^{238}U experiment	^{238}U reference
g. b.	E	88.0	89.0	[22]	52.6	53.2	[22]	44.7	44.9	[22]
	B	3.39	—	—	6,98	—	—	11.65	11.7—13.2	[24]
β	E	1151	1129.5	[22]	698.6	677.8	[22]	1043	1937	[22]
	B	0.01	0.07 0.013	[22]	0.18	0.046	[23]	0.20	0.63 0.080 0.164	[23] [24] [22]
γ	E	1164	1152.2	[22]	781.4	781.4	[22]	1060	1060	[22]
	B	0,16	0.06 0.12	[22]	0.176	0.123 0.14	[23] [22]	0,20 0.22	0.127 0.090	[23] [24]

and the reduced probabilities for excitation of the octupole states can be calculated in accordance with the formula

$$B(E3,\ \text{g.s}\to 3^-,\ K)=\left(\frac{3Ze}{4\pi^2}\right)^2\frac{1}{\pi}\sigma^3\vartheta\varepsilon_0\beta_0^2\frac{B_2}{B_3}$$
$$\times\left|\sum_{K'=0}^{3}c_{K'}^{K}\frac{1+(-1)^{K'}}{3}\frac{4-K'^2}{\sqrt{\omega_{3K'}^{p}}}\beta_0\sqrt{\frac{5}{4\pi}}\right|^2. \quad (90)$$

In Tables VI and VII, we give the results of calculations of the reduced probabilities for excitation of the 2^+ and 3^- states of the nuclei ^{156}Gd, ^{230}Th, and ^{238}U. Since the quantities B_3 are unknown, we give, to avoid introducing additional parameters, the calculated values of $B(E3)$ in units of $2B_2/(3B_3)$, and also the ratios of the excitation probabilities of the different 3^- states, which do not depend on the parameters B_3 at all.

9. METHOD OF SELF-CONSISTENT PHONONS IN THE GENERAL CASE

Above, when constructing the quadrupole–octupole model, we restricted ourselves, as is also done in the Faessler–Greiner model, by making the assumption that the static deformation significantly exceeds the vibrational amplitudes. However, the method of self-consistent phonons

TABLE VII. Excitation probabilities $B(E3,\text{g.s}\to 3^-)$ of 3^- states measured in units of $e^2b^3\times 2B_2/3B_3$ (energy measured in keV) and their ratios.

K	Energy, probability	^{156}Gd theory	^{156}Gd experiment	^{156}Gd reference	^{230}Th theory	^{230}Th experiment	^{230}Th reference	^{238}U theory	^{238}U experiment	^{238}U reference
0	E	1279	1276.2	[22]	571.0	571.8	[22]	727	731.3	[22]
	B	0.13	0.16	[22]	0.43	0.64	[23]	0.55	0.64	[23]
1	E	1611	1539,0	[22]	1034	1012.2	[22]	0.31 1018	0.59 998	[26] [22]
	B	0.0035	—	—	0.11	0.50	[23]	0.08 0.10	0.24 0.19	[23] [26]
2	E	1851	1852.1	[22]	1114	1127.9	[22]	1179	1169	[22]
	B	0.044	—	—	0.06	—	—	0.14 0.10	0.28	[23]
$0^+\to 3^-(1)$ / $0^+\to 3^-(0)$		0.03	—	—	0.26	—	—	0.26 0.32	0.38 0.32	[23] [26]
$0^+\to 3^-(2)$ / $0^+\to 3^-(0)$		0.35	—	—	0.14	—	—	0.33 0.32	0.44 0.25	[23] [26]

makes it possible to dispense with this restriction and study in a unified manner nuclei with either large or small static deformation, i.e., one can also study the so-called soft nuclei. The more general treatment also makes it possible to follow the rearrangement when rigid nuclei spin; in the limit $\omega \mathscr{J}_0 \gg I$ (ω is the energy of the vibrational quantum, $\mathscr{J}_0$ is the static moment of inertia, and I is the total angular momentum of the state) the generalized scheme simplifies appreciably and goes over into the one already described.

Turning now to the unified description of both rigid and soft nuclei, it is convenient from the point of view of the rotational variables to separate in the total Hamiltonian (19) the diagonal part $\hat{H}_0$, which consists of a rotational, $\hat{H}_{\text{rot}}$, and a vibrational, $\hat{H}_{\text{vib}}$, Hamiltonian:

$$\hat{H}_0 = \hat{H}_{\text{rot}} + \hat{H}_{\text{vib}}, \tag{91}$$

where we define the rotational Hamiltonian as

$$H_{\text{rot}} = \frac{\hat{M}_1'^2 + \hat{M}_2'^2}{4}\left(\frac{1}{\mathscr{J}_1} + \frac{1}{\mathscr{J}_2}\right) + \frac{\hat{M}_3'^2}{2\mathscr{J}_3}, \tag{92}$$

in which $\hat{M}_i'$ are the operators of rotations around the axes of the intrinsic system; $\mathscr{J}_i$ are the corresponding moments of inertia (for brevity, we identify the diagonal elements of the tensor by a single index), which depend essentially on the vibrational variables.

In the vibrational Hamiltonian, we include, as before, the so-called additional potential V_{add}, which is associated with the dependence of the inertia coefficients on the dynamical variables:

$$\hat{H}_{\text{vib}} = \sum_r \frac{\hat{p}_r^2}{2B_r} + V(\{a_r\}) + V_{\text{add}}. \tag{93}$$

We recall that the index r denotes the set of indices λ and μ, $B_r \equiv 2B_\lambda/(1+\delta_{\mu 0})$, B_λ are mass parameters, and $V(\{a_r\})$ is the deformation potential.

The remaining part $\hat{H}'$ is a nondiagonal (at least with respect to the rotational variables) operator which arises both because of the nondiagonality of the moment of inertia tensor and because $\mathscr{J}_1 \neq \mathscr{J}_2$. The solutions for the total Hamiltonian $\hat{H}$ are found in two stages. First, one constructs wave functions for the Hamiltonian $\hat{H}_0$, which are then used as a basis for diagonalizing H'.

The wave function of the Hamiltonian $\hat{H}_0$ can be sought in the form of the product (36) of a vibrational function and a rotational function; the vibrational wave function χ_{IK} does not depend on the projection M and satisfies the Schrödinger equation with the effective vibrational Hamiltonian

$$H_{\text{vib}}^{\text{eff}} = H_{\text{vib}} + \frac{I(I+1)-K^2}{4}\left(\frac{1}{\mathscr{J}_1} + \frac{1}{\mathscr{J}_2}\right) + \frac{K^2}{2\mathscr{J}_3}. \tag{94}$$

The terms in the effective Hamiltonian (94) associated with the rotation depend on an entire set of vibrational amplitudes a_r and are therefore equivalent to interaction between the different vibration modes,

$$V_{\text{int}} = \frac{I(I+1)-K^2}{4}\left(\frac{1}{\mathscr{J}_1} + \frac{1}{\mathscr{J}_2}\right) + \frac{K^2}{2\mathscr{J}_3} + V_{\text{add}}, \tag{95}$$

this interaction depending rather strongly on the angular momentum I of the nucleus. This leads to a basic difference of the situations for soft and rigid nuclei. Indeed, in the case of rigid nuclei the moments of inertia $\mathscr{J}_1$ and $\mathscr{J}_2$ were determined mainly by the static component, and therefore the main part of the rotational Hamiltonian did not depend on the vibrational variables and, thus, did not lead to interaction of the vibrational modes.

Assuming that the vibrations are small, we choose the deformation potential in the form

$$V = \sum_r \frac{1}{2} c_r (a_r - \beta_r)^2, \tag{96}$$

where c_r is the rigidity and β_r is the static deformation.

The additional potential that occurs in (95) can be expressed in accordance with (21) through the determinant of the matrix of inertia coefficients and can be represented as

$$V_{\text{add}} \simeq \frac{1}{8}\sum_r \frac{1}{B_r}\left\{\sum_i \left(\frac{1}{\mathscr{J}_i}\frac{\partial^2 \mathscr{J}_i}{\partial a_r^2} - \frac{1}{\mathscr{J}_i^2}\left(\frac{\partial \mathscr{J}_i}{\partial a_r}\right)^2\right) + \frac{1}{4}\left(\sum_i \frac{1}{\mathscr{J}_i}\frac{\partial \mathscr{J}_i}{\partial a_r}\right)^2\right\}. \tag{97}$$

We have ignored here the square of the nondiagonal element $\mathscr{J}_{13}$ of the moment of inertia tensor compared with the product $\mathscr{J}_1 \mathscr{J}_3$ of the diagonal elements.

We draw attention to the important part played by the classical interaction of the vibrations with the rotations. If the conditions (5) and (7) are satisfied, the term in the kinetic energy proportional to the rate of the octupole vibrations and the rotational velocity around axis 2 is nonzero. After quantization, the contribution of this term in the Hamiltonian completely compensates the influence of the octupole amplitudes on the rotation around axis 2—the expressions (95) and (97) contain the purely quadrupole part of the moment of inertia $\mathscr{J}_2$:

$$\mathscr{J}_2 = B_2(\sqrt{3}\,a_{20} - \sqrt{2}\,a_{22})^2. \tag{98}$$

At the same time, the moments $\mathscr{J}_1$ and $\mathscr{J}_2$ in these expressions are determined by both the quadrupole and the octupole parts.

Assuming that the interaction of the vibrations is realized through the average field produced by all the phonons and is determined in a self-consistent manner, we shall again seek the vibrational wave function in the form of the product (44), where the functions $\chi_r(q_r)$ satisfy the Hartree equations, which can be obtained from a variational principle analogous to (45). The single-phonon Hamiltonian can be represented in the form

$$\hat{H}_r = \frac{p_r^2}{2B_r} + V_r(a_r), \tag{99}$$

where

$$V_r(a_r) = \langle V^{\text{eff}} \rangle_{[r]} \equiv \int \{\prod_{r' \neq r} da_{r'}\, \chi_{r'}(a_{r'})\}\, V^{\text{eff}} \tag{100}$$

is the potential of the average field, and $V^{\text{eff}} = V + V^{\text{int}}$.

The Hartree equations form a system of equations that describe these coupled vibrations. Assuming that the interaction, modifying these vibrations, leaves them small, we

represent the potentials V_r in the form

$$V_r(a_r) \simeq V_r(\tilde{\beta}_r) + \frac{1}{2} c_r^p (a_r - \tilde{\beta}_r)^2, \quad (101)$$

where the modified rigidities

$$c_r^{(p)} = \frac{\partial^2 V_r}{\partial a_r^2}\Big|_{a_r=\tilde{\beta}_r}, \quad (102)$$

and the positions $\tilde{\beta}_r$ of the displaced minima are determined from the conditions

$$\frac{\partial V_r}{\partial a_r}\Big|_{a_r=\tilde{\beta}_r} = 0. \quad (103)$$

If we now introduce the amplitudes of the vibrations with respect to the displaced equilibrium positions,

$$q_r = a_r - \tilde{\beta}_r, \quad (104)$$

then they will obviously possess the properties

$$\bar{q}_r = 0, \ \bar{q}_r^2 = \frac{n_r + 1/2}{\sqrt{B_r c_r^{(p)}}}, \quad (105)$$

where the bar denotes integration with the square of the corresponding wave function.

Equations (102)–(105) and (100), which form a closed system, can be significantly simplified, and they reduce to purely algebraic equations, if, separating from the moments their mean (and not their static!) values

$$\mathcal{I}_i = \bar{\mathcal{I}}_i + \delta\mathcal{I}_i, \quad (106)$$

we expand the inverse moments $1/\mathcal{I}_i$ in the potential V_r with respect to the ratio of the fluctuation $\delta\mathcal{I}_i$ of the moment to the mean value $\bar{\mathcal{I}}_i$:

$$\frac{1}{\mathcal{I}_i} = \frac{1}{\bar{\mathcal{I}}_i} - \frac{\delta\mathcal{I}_i}{\bar{\mathcal{I}}_i^2} + \frac{(\delta\mathcal{I}_i)^2}{\bar{\mathcal{I}}_i^3}. \quad (107)$$

This makes it possible to calculate the integrals in (100) and determine the potentials V_r in explicit form. Representing the interaction potential (95) as

$$V^{\text{int}} = \sum_i \left(\frac{\delta_i}{\mathcal{I}_i} - \frac{1}{2}\gamma_{ii}\frac{1}{\mathcal{I}_i^2}\right) + \frac{1}{8}\sum_{i,k}\gamma_{ik}\frac{1}{\mathcal{I}_i\mathcal{I}_k}, \quad (108)$$

where

$$\left.\begin{aligned} \gamma_{ik} &= \frac{1}{4}\sum_r \frac{1}{B_r}\mathcal{I}_{i,r}\mathcal{I}_{k,r}; \\ \delta_{1,2} &= \frac{I(I+1)-K^2}{4} + S_{1,2}; \quad \delta_3 = \frac{K^2}{2} + S_3; \\ S_i &= \frac{1}{8}\sum_r \frac{1}{B_r}\mathcal{I}_{i,rr}; \\ \mathcal{I}_{i,r} &\equiv \frac{\partial\mathcal{I}_i}{\partial a_r}; \quad \mathcal{I}_{i,rr} \equiv \frac{\partial^2\mathcal{I}_i}{\partial a_r^2}, \end{aligned}\right\} \quad (109)$$

we transform Eqs. (103) and (102) to the form

$$c_r(\tilde{\beta}_r - \beta_r) = \sum_i \left\{\frac{\delta_i}{\bar{\mathcal{I}}_i^2}\mathcal{I}_{i,r} + \frac{1}{2}\frac{1}{\bar{\mathcal{I}}_i^2}\gamma_{ii,r} - \bar{\gamma}_{ii}\frac{1}{\bar{\mathcal{I}}_i^3}\mathcal{I}_{i,r}\right\} - \frac{1}{8}\sum_{i,k}\frac{1}{\bar{\mathcal{I}}_i\bar{\mathcal{I}}_k}\left\{\gamma_{ik,r} - \bar{\gamma}_{ik}\left(\frac{1}{\bar{\mathcal{I}}_i}\mathcal{I}_{i,r} + \frac{1}{\bar{\mathcal{I}}_k}\mathcal{I}_{k,r}\right)\right\} \quad (110)$$

and

$$c_r^{(p)} = c_r + \sum_i \left\{\frac{\delta_i}{\bar{\mathcal{I}}_i^2}\left(\frac{2}{\bar{\mathcal{I}}_i}\mathcal{I}_{i,r}^2 - \mathcal{I}_{i,rr}\right) - \frac{1}{2\bar{\mathcal{I}}_i^2}\left(\gamma_{ii,rr} - \frac{4}{\bar{\mathcal{I}}_i}\mathcal{I}_{i,r}\gamma_{ii,r} - \frac{2}{\bar{\mathcal{I}}_i}\bar{\gamma}_{ii}\mathcal{I}_{i,rr} + \frac{6}{\bar{\mathcal{I}}_i^2}\bar{\gamma}_{ii}\mathcal{I}_{i,r}^2\right)\right\} + \frac{1}{8}\sum_{i,k}\frac{1}{\bar{\mathcal{I}}_i\bar{\mathcal{I}}_k}\left\{\gamma_{ik,rr} - 2\gamma_{ik,r}\left(\frac{1}{\bar{\mathcal{I}}_i}\mathcal{I}_{i,r} + \frac{1}{\bar{\mathcal{I}}_k}\mathcal{I}_{k,r}\right) + \bar{\gamma}_{ik}\left(\frac{2}{\bar{\mathcal{I}}_i^2}\mathcal{I}_{i,r}^2 + \frac{2}{\bar{\mathcal{I}}_k^2}\mathcal{I}_{k,r}^2 + \frac{2}{\bar{\mathcal{I}}_i\bar{\mathcal{I}}_k}\mathcal{I}_{i,r}\mathcal{I}_{k,r} - \frac{1}{\bar{\mathcal{I}}_i}\mathcal{I}_{i,rr} - \frac{1}{\bar{\mathcal{I}}_k}\mathcal{I}_{k,rr}\right)\right\}, \quad (111)$$

where the derivatives $\mathcal{I}_{i,r}$ and $\gamma_{ik,r}$ are taken at the point $a_r = \tilde{\beta}_r$. Equations (110) and (111) in conjunction with the expressions for the moments of inertia (14) and (98), the definitions (109), and the properties of the vibrational amplitudes (105) and (104) form a system of nonlinear algebraic equations for the rigidities $c_r^{(p)}$, which determine the phonon wave functions, and the equilibrium positions $\tilde{\beta}_r$. This system is a generalization of the system of equations (51) discussed above for rigid nuclei.

10. SOLUTION OF THE BASIC EQUATIONS. SELF-CONSISTENT MOMENTS OF INERTIA

We first of all analyze some properties that follow from the structure of Eqs. (110) and (111) for the basic characteristics of the collective vibrational–rotational states. We note first that since the expressions for the moments of inertia (14) and (98) do not contain terms that are cross products of a quadrupole amplitude and an octupole amplitude, such terms are also absent in the expressions for the quantities γ_{ik}, defined in accordance with (109), and therefore the system of equations (110) decomposes into two subsystems of linear equations for the quadrupole and octupole quantities $\tilde{\beta}$ separately (or rather quasilinear equations, since the quantities $\tilde{\beta}$ also occur in the expressions for the mean values $\bar{\mathcal{I}}_i$ and $\bar{\gamma}_{ik}$). It follows from this that the presence of the effective deformation $\tilde{\beta}_r$ is directly determined by the presence of the unrenormalized deformation of the same multipolarity in the potential (96). In other words, from the assumption of the absence of octupole deformation,

$$\beta_{3\mu} = 0 \quad (\mu = 0, 1, 2, 3), \quad (112)$$

it follows from (110) that

$$\tilde{\beta}_{3\nu} = 0 \quad (\nu = 0, 1, 2, 3). \quad (113)$$

With regard to the quadrupole deformation, the breaking of the symmetry between the axes 1 and 2 associated with the presence of the octupole degrees of freedom has the consequence that an effective nonaxial deformation $\tilde{\beta}_{22}$ arises even in the case when the deformation potential (96) presupposes axial symmetry:

$$\beta_{20} = \beta_0 \neq 0, \quad \beta_{22} = 0. \quad (114)$$

It is convenient to measure $\tilde{\beta}_{20}$ and $\tilde{\beta}_{22}$ in units of the unre-

normalized deformation β_0 and to characterize them by the parameters

$$x \equiv \tilde{\beta}_{20}/\beta_0,\ y \equiv \sqrt{2/3}\,\tilde{\beta}_{22}/\beta_0. \tag{115}$$

Further, the mean values $\overline{\mathscr{J}}_i$ and $\bar{\gamma}_{ik}$, which occur on the right-hand sides of Eqs. (110) and (111), are determined in accordance with (14), (98), (105) and (110), (111) by all the quantities $c_r^{(p)}$ and by the nonvanishing quantities $\tilde{\beta}_r$. Therefore, the equations for all these variables, which characterize the given collective states $|I,K\{n_r\}\rangle$, are coupled to each other.

Instead of the rigidities c_r and $c_r^{(p)}$ it is more convenient to consider the frequencies ω_r and $\omega_r^{(p)}$, defined in accordance with (53), and to write the equations in the form

$$x = 1 + f(x,\ y,\ \{\omega_r^{(p)}\}); \tag{116}$$

$$y = g(x,\ y,\ \{\omega_r^{(p)}\}); \tag{117}$$

$$\omega_r^{(p)2} = \omega_r^2 + F_r(x,\ y,\ \{\omega_r^{(p)}\}), \tag{118}$$

where the functions f, g, and F_r on the right-hand sides are related to the interaction of the collective modes through the rotation and depend on the total angular momentum I of the level, its projection K onto the symmetry axis, the set of oscillator numbers $\{n_r\}$, and all the required characteristics x, y, and $\{\omega_r^{(p)}\}$.

Equations (116)–(118) simplify in the limiting case of rigid nuclei, for which $|q_r| \ll \beta_0$ and the moments $\mathscr{J}_1$ and $\mathscr{J}_2$ contain the large static component $\mathscr{J}_0$,

$$\omega\mathscr{J}_0 \gg 1 \tag{119}$$

(ω is the excitation energy of the vibrational quantum), and in the functions F_r the dominant terms are the ones determined by the moment $\mathscr{J}_3$. At the same time, in the limit $\omega\mathscr{J}_0 \to \infty$ four equations take the form

$$x = 1,\ y = 0,\ \omega_{20}^{(p)} = \omega_{20},\ \omega_{30}^{(p)} = \omega_{30}, \tag{120}$$

and the four remaining equations go over into the equations (54) considered above for $\omega_{\lambda\mu}^{(p)}$ with $\mu \neq 0$, the solutions of which we denote by $(\omega_{\lambda\mu}^{(p)})_{\text{rig}}$:

$$\omega_{\lambda\mu}^{(p)} = (\omega_{\lambda\mu}^{(p)})_{\text{rig}} \quad (\mu \neq 0); \tag{121}$$

the quantities $(\omega_{\lambda\mu}^{(p)})_{\text{rig}}$ do not depend on the angular momentum I but are different for different bands $\{K,\{n_r\}\}$.

With increasing I, the corrections to the solutions (120)–(121) can become appreciable even for fairly rigid nuclei,

$$x = 1 + \frac{3}{(\omega_{20}\mathscr{J}_0)^2}\{I(I+1) - K^2 + 6\}; \tag{122}$$

$$y = \frac{2}{(\omega_{22}\mathscr{J})^2}\left\{\frac{7}{2} + \left(\frac{\mathscr{J}_0}{\mathscr{J}_1} - \frac{\mathscr{J}_0}{\mathscr{J}_2}\right)\left(I(I+1) - K^2 - \frac{\mathscr{J}_0}{2\mathscr{J}_3} - \frac{5}{2}\right)\right\}; \tag{123}$$

$$\omega_{20}^{(p)2} = \omega_{20}^2 + \frac{9}{\mathscr{J}_0^2}\{I(I+1) - K^2 + 6\}; \tag{124}$$

$$\omega_{30}^{(p)2} = \omega_{30}^2 - \frac{3}{\mathscr{J}_0^2}\left\{I(I+1) - K^2 + \frac{75}{4}\right\}; \tag{125}$$

$$\omega_r^{(p)2} = (\omega_r^{(p)})_{\text{rig}}^2 + \sum_{r'} \frac{\partial(\omega_r^{(p)})_{\text{rig}}^2}{\partial\omega_{r'}^2} k_{r'}\ \frac{1}{\mathscr{J}_0^2}\{I(I+1) - K^2\},$$
$$(r,\ r' = 22,31,\ 32,33), \tag{126}$$

where

$$k_{22} = 3,\ k_{31} = -5/2,\ k_{32} = -4,\ k_{33} = -3/2. \tag{127}$$

It can be seen from these expressions that the real small parameter that characterizes the deformability of the nucleus is $I/(\omega\mathscr{J}_0)$ and not $1/(\omega\mathscr{J}_0)$.

We now turn to the solution of Eqs. (116)–(118) for the basic characteristics in the general case of arbitrary relations between the excitation energies of the vibrational quanta ω_r and the static moment $\mathscr{J}_0$. It is rather natural and, at the same time, fairly effective to use the iterative method with initial values

$$x_0 = 1,\quad y_0 = 0,\quad (\omega_r^{(p)})_0 = \omega_r, \tag{128}$$

which correspond to the solution in the absence of interaction of the collective modes. For convergence of the iterative process, it is necessary to introduce mixing, i.e., to determine the values of the variables in the $(n+1)$th iteration as

$$z_{n+1} = z_n + S\delta z_n, \tag{129}$$

where z_n is any of the required quantities $\{x,y,\{\omega_r^{(p)}\}\}$ in the nth iteration; δz_n is the correction to it calculated in the nth iteration by comparing the left- and right-hand sides of the corresponding equation of the system (116)–(118). The parameter S, chosen in the interval $0 \leqslant S \leqslant 1$, characterizes the rate of the iterative process and influences its convergence. A felicitous choice is the value $S = 0.5$.

The dependence on the angular momentum I of the required characteristics of the levels of the ground-state band is given in Table VIII for the rigid nucleus ^{238}U and in Table IX for the soft nucleus ^{194}Pt. As can be seen from the tables, the quadrupole rigidities increase with increasing I, whereas the octupole rigidities decrease. This is due to the assumption of a quadrupole nature of the static deformation.

With regard to the effective deformation $\tilde{\beta}_r$, the longitudinal deformation $\tilde{\beta}_{20}$ increases with increasing angular momentum. At the same time, the effective nonaxial deformation $\tilde{\beta}_{22}$ initially decreases to zero, and then, changing sign (relative to the axial deformation), begins to increase. Such behavior is observed both for the rigid nucleus ^{238}U and for the soft nucleus ^{194}Pt, although, of course, the actual values of $\tilde{\beta}_{22}/\beta_0$ for the soft nucleus are much greater.

We draw attention to the different behavior of the moments of inertia $\overline{\mathscr{J}}_i$ as functions of I. Thus, for the rigid nucleus ^{238}U the moments $\overline{\mathscr{J}}_1$ and $\overline{\mathscr{J}}_2$ are not too far from the static value $\mathscr{J}_0$, increasing by $I = 24$ by about 20%. At the same time, the moment $\overline{\mathscr{J}}_3$ is approximately an order of magnitude less than $\mathscr{J}_0$ and almost independent of I. For the soft nucleus ^{194}Pt the moments $\overline{\mathscr{J}}_1$ and $\overline{\mathscr{J}}_2$ differ from $\mathscr{J}_0$ by two or three times and differ appreciably at small I. With increasing I, the moment of inertia $\overline{\mathscr{J}}_2$ increases much more rapidly, and therefore already when $I = 8$ we have $\overline{\mathscr{J}}_1(I) \approx \overline{\mathscr{J}}_2(I)$. The moment of inertia $\overline{\mathscr{J}}_3$, which is close to $\mathscr{J}_0$ for $I = 0$, has decreased by about 1.5 times by $I = 20$ and is, as also in the case of rigid nuclei, appreciably less than the transverse moments of inertia $\overline{\mathscr{J}}_1$ and $\overline{\mathscr{J}}_2$.

TABLE VIII. Dependence of the basic characteristics of collective states on the angular momentum for the ground-state band of the nucleus ^{238}U.

I	$\omega_{20}^{(p)}y_0$	$\omega_{22}^{(p)}y_0$	$\omega_{30}^{(p)}y_0$	$\omega_{31}^{(p)}y_0$	$\omega_{32}^{(p)}y_0$	$\omega_{33}^{(p)}y_0$	x	y	$y_0/\bar{y}_1$	$y_0/\bar{y}_2$	$y_0/\bar{y}_3$
0	66.56	56.43	45.30	56.83	60.88	114.41	1.0032	0.0021	0.855	0.967	8.588
2	66.85	55.54	45.16	56.78	60.81	114.39	1.0066	0.0018	0.850	0.961	8.589
4	67.52	56.80	44.85	56.68	60.65	114.34	1.0142	0.0014	0.840	0.946	8.593
6	68.48	57.18	44.38	56.52	60.41	114.28	1.0256	0.0007	0.624	0.925	8.598
8	69.67	57.66	43.79	56.33	60.11	114.20	1.0401	0.0000	0.805	0.899	8.603
10	70.99	58.20	43.10	56.10	59.76	114.12	1.0572	—0.0009	0.783	0.870	8.609
12	73.40	58.79	42.33	55.86	59.39	114.04	1.0761	—0.0017	0.759	0.840	8.612
14	73.83	59.39	41.52	55.60	58.99	113.95	1.0964	—0.0025	0.735	0.809	8.614
16	75.25	60.00	40.67	55.33	58.59	113.87	1.1177	—0.0034	0.711	0.778	8.616
18	76.64	60.60	39.81	65.07	58.19	113.80	1.1396	—0.0042	0.687	0.749	8.616
20	77.98	61.19	38.94	54.81	57.78	113.73	1.1619	—0.0049	0.664	0.720	8.615
22	79.27	61.76	38.06	54.55	57.39	113.67	1.1844	—0.0056	0.641	0.693	8.613
24	80.50	62.31	37.20	54.29	57.01	113.61	1.2070	—0.0062	0.620	0.667	8.612

Note. $\omega_{20}y_0=66.30$, $\omega_{22}y_0=66.01$, $\omega_{30}y_0=45.80$, $\omega_{31}y_0=60.61$, $\omega_{32}y_0=67.34$, $\omega_{33}y_0=134.68$.

TABLE IX. Dependence of the basic characteristics of collective states on the angular momentum for the ground-state band of the nucleus ^{194}Pt.

I	$\omega_{20}^{(p)}y_0$	$\omega_{22}^{(p)}y_0$	$\omega_{30}^{(p)}y_0$	$\omega_{31}^{(p)}y_0$	$\omega_{32}^{(p)}y_0$	$\omega_{33}^{(p)}y_0$	x	y	$y_0/\bar{y}_1$	$y_0/\bar{y}_2$	$y_0/\bar{y}_3$
0	15.48	5.89	12.17	24.65	24.23	22.91	1.0059	0.2343	0.451	1.38	1.08
2	16.36	4.91	11.95	24.60	24.04	22.93	1.0568	0.1535	0.481	0.99	1.24
4	17.19	4.35	11.36	24.48	23.61	22.02	1.1231	0.0073	0.523	0.69	1.30
6	18.12	5.30	10.61	24.34	23.52	21.51	1.2031	—0.0615	0.517	0.56	1.41
8	19.00	6.32	9.88	24.20	23.26	21.13	1.2879	—0.0923	0.484	0.485	1.49
10	19.78	7.14	9.23	24.09	23.05	20.86	1.3722	—0.1056	0.444	0.429	1.54
12	20.46	7.78	8.66	24.00	22.88	20.66	1.4540	—0.1119	0.406	0.386	1.58
14	21.05	8.30	8.15	23.92	22.73	20.51	1.5326	—0.1152	0.372	0.351	1.61

Note. $\omega_{20}y_0=15.58$, $\omega_{22}y_0=5.84$, $\omega_{30}y_0=12.71$, $\omega_{31}y_0=\omega_{32}y_0=\omega_{33}y_0=24.77$.

TABLE X. Spectrum of vibrational–rotational excitations of the nucleus ^{150}Sm (energy measured in keV).

Band	I	Experiment[23]	T	$\tilde{T}$	RV 1 (Ref. 20)	RV 2 (Ref. 20)	RV 3 (Ref. 20)
g	0	333.4	334.5	333.3	268.0	225.0	356.0
	4	772.6	823.8	819.5	830.0	937.0	1044.0
	6	1277.8	1356.0	1347.0	1244.0	1074.0	2047.0
	8	1836.0	1914.0	1899.0	1860.0	558.0	3388.0
	10	2431.7	2492.4	2469.0	2538.0	—	5078.0
	12	3046.2	3085.4	3055.0	—	—	—
	(14)	3676.0	3691.0	3653.0	—	—	—
β	0	740.4	762.2	762.2	740.0	740.0	740.0
	2	1046.1	1046.3	1046.0	953.0	1916.0	1143.0
	4	1449.1	1573.3	1572.0	1362.0	3292.0	2005.0
	6	1926.0	2176.6	2175.0	—	—	—
γ	2	1165.5	1165.0	1166.0	1194.0	3453.0	1364.0
	3	1504.5	1463.4	1453.0	1335.0	—	1629.0
	4	1643.0	1748.0	1753.0	1713.0	9998.0	2985.0
	5	2020.4	2043.8	2044.0	—	—	—

11. SPECTRA OF VIBRATIONAL–ROTATIONAL EXCITATIONS

The energy of the collective state described by the wave function (44) is determined by the expectation value of the collective Hamiltonian (94) with respect to these functions and can be represented in the form

$$E = E_0 + E_1 + E_2 + E_3, \tag{130}$$

where the part E_0 is associated with the vibrations alone,

$$E = \left\langle \chi_{IK} \middle| \sum_r \left(\frac{\hat{p}_r^2}{2B_r} + \frac{1}{2} c_r q_r^2 \right) \middle| \chi_{IK} \right\rangle$$
$$= \sum_r \omega_r (n_r + 1/2) + \sum_r \frac{(\omega_r - \omega_r^{(p)})^2}{2\omega_r^{(p)}} (n + 1/2), \tag{131}$$

and the part E_1 is due to the fact that because of the interaction of the collective modes the effective deformation $\tilde{\beta}_r$ differs from the static value β_r, which minimized the deformation potential (96),

$$E_1 = \frac{1}{2} \sum_r c_r (\tilde{\beta}_r - \beta_r)^2. \tag{132}$$

The quantities E_2 and E_3 represent the main part and the correction to it from the rotational and additional energies:

$$E_2 = \sum_i \left(\frac{\delta_i}{\mathcal{I}_i} - \frac{1}{2} \frac{\bar{\gamma}_{ii}}{\bar{\mathcal{I}}_i^2} \right) + \frac{1}{8} \sum_{i,k} \frac{\bar{\gamma}_{ik}}{\bar{\mathcal{I}}_i \bar{\mathcal{I}}_k}; \tag{133}$$

$$E_3 = \frac{1}{2} \sum_r \left\{ c_r^p - c_r + \sum_i \left((\delta_i - \bar{\gamma}_{ii}/\bar{\mathcal{I}}_i) \mathcal{I}_{i,rr} + \frac{1}{2} \gamma_{ii,rr} \right) \frac{1}{\bar{\mathcal{I}}_i^2} \right.$$
$$\left. + \frac{1}{8} \sum_{i,k} \left(\bar{\gamma}_{ik} \left(\frac{\mathcal{I}_{i,rr}}{\bar{\mathcal{I}}_i} + \frac{\mathcal{I}_{k,rr}}{\bar{\mathcal{I}}_k} \right) - \gamma_{ik,rr} \right) \frac{1}{\bar{\mathcal{I}}_i \bar{\mathcal{I}}_k} \right\} \bar{q}_r^2. \tag{134}$$

The excitation energy of the collective state characterized by the angular momentum I, its projection K onto the intrinsic symmetry axis, and by the set of oscillator numbers $\{n_r\}$ is determined by the relation (62), in which $E_{IK}\{n_r\}$ and $E_{00}\{0\}$ are the energies of the excited and ground states of the nucleus calculated in accordance with Eqs. (130)–(134).

As an illustration, we give in Tables X–XIII the energies of the vibrational–rotational excitations calculated in this manner for the nuclei ^{150}Sm, ^{156}Dy, ^{194}Pt, and ^{238}U (T column) in comparison with the experimental data and other calculations.[13,32,33] The results obtained by diagonalizing the total Hamiltonian (19) are given in the column $\tilde{T}$. The parameters of the theory were chosen as follows: the frequencies ω_r from the positions of the 2^+ levels in the β and γ bands and the 3^- levels in the existing octupole bands; the rotational energy $\varepsilon_0 \equiv 1/(2\mathcal{I}_0)$, from the position of the 2^+ level of the ground-state band. The absence of data on a number of octupole bands makes the choice of the corresponding parameters $\omega_{3\mu}$ difficult. However, these quantities have (if the remaining parameters are appropriately chosen) little influence on the results presented in Tables X–XIII (for the corresponding bands, we have, as in Ref. 25, chosen $\omega_r = 2000$ keV, which appreciably exceeds the remaining ω_r).

For small angular momenta I, the results for the nucleus ^{238}U are close to those obtained above in Sec. 7. However, with increasing angular momentum appreciable differences appear. This reflects the fact that the nucleus becomes softer and we no longer have fulfillment of the approximations for which the more general scheme discussed here reduces to the simpler scheme considered earlier for the description of rigid nuclei.

We emphasize that the decomposition of the collective Hamiltonian into a rotational part and a vibrational–rotational part now differs from the decomposition that is traditional for the vibrational–rotational model—the rotational Hamiltonian includes in a natural manner an appreciable part of the interaction of the vibrational modes through the rotation (except for the nondiagonal part with respect to the rotational variables, in particular, the interaction of the negative-parity bands with $\Delta K = 1$). Therefore, in contrast to

TABLE XI. Spectrum of vibrational–rotational excitations of the nucleus ^{156}Dy (energy measured in keV).

Band	I	Experiment[23]	T	$\tilde{T}$	RV 1 (Ref. 20)	RV 2 (Ref. 20)
g	2	137.8	137.9	137.8	133.6	141.0
	4	404.1	410.0	408.2	408.0	435.0
	6	770.3	761.5	756.0	770.0	849.0
	8	1215.7	1162.6	1152.0	1201.0	1381.0
	10	1725.0	1597.9	1551.0	1708.0	2038.0
	12	2285.9	2058.9	2035.0	2301.0	2824.0
	14	2887.8	2540.1	2509.0	—	—
	16	3498.9	3037.9	2999.0	—	—
	18	4026.1	3549.5	3502.0	—	—
	20	4635.6	4073.1	4018.0	—	—
	(22)	5320.3	4607.1	4544.0	—	—
β	0	675.6	683.1	683.4	675.0	675.0
	2	828.7	829.3	829.0	802.0	855.0
	4	1988.3	1127.5	1125.0	1081.0	1200.0
	6	1437.1	1517.6	1511.0	1384.0	1727.0
	8	1658.6	1959.1	1949.0	1761.0	2444.0
	10	2315.6	2431.5	2417.0	2236.0	3353.0
	12	2706.9	2924.9	2908.0	2813.0	4452.0
	14	3065.9	3434.2	3415.0	—	—
	16	3523.3	3956.1	3936.0	—	—
	18	4178.2	4448.9	4469.0	—	—
	20	4859.0	5030.8	5011.0	—	—
γ	2	890.7	869.6	890.1	75.0	5.0
	3	1022.1	1026.8	1027.0	1046.0	1047.0
	4	1168.5	1190.7	1195.0	1216.0	1416.0
	5	1335.2	1373.9	1374.0	—	—
	6	1525.2	1571.7	1582.0	—	—
	7	1728.8	1780.7	1781.0	—	—
	8	1957.2	1998.6	2018.0	—	—
	9	2191.6	2224.1	2224.0	—	—
	10	2448.0	2455.7	2485.0	—	—
	11	2712.4	2693.0	2693.0	—	—
	12	2997.2	2935.1	2974.0	—	—

the studies based on the vibrational–rotational model of Ref. 11, the vibrational–rotational part of the total Hamiltonian is not, at least for the positive-parity states, particularly important.

The nuclei ^{150}Sm, ^{156}Dy, and ^{194}Pt were considered in Ref. 13, in which the authors proposed for the description of three bands (ground, β, and γ) a modified variant of the vibrational–rotational model with five parameters; namely, besides the two quadrupole rigidities and the mass parameter that determines the moment of inertia $\mathcal{J}_0$, they introduced additional mass parameters for the β and γ bands, thus making the moments of inertia of the β and γ bands independent adjustable parameters. We note in this connection that in our approach essentially only (the first) three

TABLE XII. Spectrum of vibrational–rotational excitations of the nucleus ^{194}Pt (energy measured in keV).

Band	I	Experiment[23]	T	$\tilde{T}$	RV 1 (Ref. 20)	RV 2 (Ref. 20)
g	2	328.5	330.8	328.4	331.0	345.0
	4	811.2	822.8	800.2	810.0	1046.0
	6	1481.6	1481.9	1427.0	1376.0	2129.0
	8	2099.2	2230.6	2144.0	2115.0	3567.0
	10	2438.2	3038.5	2920.0	3038.0	—
	12	2829.5	3893.1	3744.0	—	—
β	0	1479.3	1344.0	1344.0	1267.0	1267.0
	2	1622.3	1622.3	16622.0	1514.0	1780.0
	4	—	2132.2	2130.0	2274.0	3159.0
γ	2	622.1	620.0	622.0	637.0	641.0
	3	922.8	872.2	872.2	781.0	1252.0
	4	1229.5	1179.2	1201.0	1235.0	1782.0
	5	1498.7	1528.9	1529.0	—	—
	6	1925.7	1911.3	1963.0	—	—

TABLE XIII. Spectrum of vibrational–rotational excitations of the nucleus ^{238}U (energy measured in keV).

Band	I	Experiment[23]	T	$\tilde{T}$	*VMI* (Ref. 32)	*MVMI* (Ref. 33)
g	2	44.9	44.91	44.9	44.7	44.5
	4	148.4	148.3	148.2	148.2	148.3
	6	307.2	307.2	306.7	308.4	308.0
	8	518.8	517.2	515.9	522.5	519.0
	10	775.7	774.0	771.1	786.7	776.4
	12	1076.5	1072.7	1068	1076	1100
	14	1415.3	1409.3	1401	1458	1414
	16	1788.2	1779.7	1768	1856	1786
	18	2190.7	2180.5	2164	2191	2191
	20	2618.7	2608.9	2587	2626	2626
	22	3067.2	3062.0	3034	–	3088
	(24)	3534.5	3503.9	3503	–	3576
β	0	993	989.2	989.2	–	–
	2	1037.3	1037.4	1037	–	–
	4	1127	1148.3	1148	–	–
γ	2	1060.3	1060.4	1060	–	–
	3	1105.6	1106.6	1107	–	–
	4	1167.7	1167.7	1168	–	–
$K^\pi = 0^-$	1	679.8	686.2	681.1	–	–
	3	731.3	756.0	731.0	–	–
	5	326.4	381.9	830.0	–	–
	7	965.9	1058.3	982.5	–	–
	9	1150.3	1281.9	1188	–	–
	11	1378.4	1548.6	1443	–	–
	13	1648.9	1854.4	1743	–	–
	15	1958.6	2195.5	2083	–	–
	17	2305.0	2568.4	2459	–	–
$K^\pi = 1^-$	1	930.8	904.7	909.8	–	–
	2	950.0	934.1	934.6	–	–
	3	997.5	977.9	997.5	–	–
$K^\pi = 2^-$	2	1128.7	1120.1	1123	–	–
	3	1169.4	1163.7	1169	–	–
	4	1242.9	1221.3	1230	–	–

parameters are used to describe the same three bands.

In Table XIII, we give for comparison the level energies of the ground-state band of the nucleus ^{238}U calculated in the model of a variable moment of inertia[32] and in the quite recently proposed anharmonic vibration model.[33] Although these models were intended for phenomenological approximation of the observed levels of just the ground-state band by means of two and three parameters, respectively, the quality of the description of the spectrum in the present paper is at least not worse. We recall that the only parameter used to specify the ground-state band is chosen on the basis of the position of the 2^+ level.

12. ADDITIONAL SOLUTIONS AND THE POSSIBLE EXISTENCE OF DYNAMICAL SHAPE ISOMERS

The essential nonlinearity of Eqs. (110) and (111) for the new equilibrium positions $\tilde{\beta}_r$ and rigidities $c_r^{(p)}$ means that the question of the uniqueness (for a given set of quantum numbers $I, K, \{n_r\}$) of the solution found above in Sec. 10, which represents the "free" solution (128) modified by the interaction of the vibrations, is nontrivial. It can be shown that at not too small angular momenta I, beginning with $I \gtrsim \omega \mathcal{J}_0$ (ω is the minimal energy of a vibration quantum, and $\mathcal{J}_0$ is the static moment of inertia) the interaction of the collective modes becomes sufficiently strong for a qualitative rearrangement to occur, and there then arise additional solutions characterized by an appreciable nonaxial deformation, $y > x$.

Before we give the numerical results, let us follow the appearance of the additional solutions (we shall say that they are anomalous, in contrast to those, the normal solutions, investigated above) in the limiting case $I \to \infty$, when on the right-hand sides of Eqs. (110) and (111) it is sufficient to retain only the terms proportional to $I(I+1)$,

$$c_r(\tilde{\beta}_r - \beta_r) = \frac{I(I+1)}{4}\left(\frac{y_{1,r}}{\bar{y}_1^2} + \frac{y_{2,r}}{\bar{y}_2^2}\right); \quad (135)$$

$$c_r^{(p)} = c_r + \frac{I(I+1)}{4}\left(\frac{2y_{1,r}^2}{\bar{y}_1^3} + \frac{2y_{2,r}^2}{\bar{y}_2^3} - \frac{y_{1,rr}}{\bar{y}_1^2} - \frac{y_{2,rr}}{\bar{y}_2^2}\right). \quad (136)$$

Under the assumption (6) and (112) that we made above, Eqs. (136) for the octupole modes $r = 3\mu$ reduce to

$$c_r^{(p)} = c_r - \frac{I(I+1)}{4}\frac{y_{1,rr}}{\bar{y}_1^2} \quad (r = 3\mu), \quad (137)$$

from which it follows that the amplitudes of the octupole

vibrations increase with increasing I. This has the consequence that the moment of inertia $\overline{\mathscr{J}}_1$ containing the contribution of the octupole vibrations is greater than the moment of inertia $\overline{\mathscr{J}}_2$ that does not contain this contribution, and therefore for the analysis we can ignore the terms $1/\overline{\mathscr{J}}_1$ compared with the terms $1/\overline{\mathscr{J}}_2$ and reduce the equations for the quadrupole modes to the form

$$x-1=\frac{I(I+1)}{2}\frac{3}{(\omega_{20}\overline{\mathscr{J}}_0)^2}\frac{x-y}{j_2^2}; \quad (138)$$

$$y=-\frac{I(I+1)}{2}\frac{1}{(\omega_{22}\overline{\mathscr{J}}_0)^2}\frac{x-y}{j_2^2}; \quad (139)$$

$$(\omega_{20}^{(p)}\overline{\mathscr{J}}_0)^2=(\omega_{20}\overline{\mathscr{J}}_0)^2+3I(I+1)\left(\frac{2(x-y)^2}{j_2}-\frac{1}{2}\right)\frac{1}{J_2^2}; \quad (140)$$

$$(\omega_{22}^{(p)}\overline{\mathscr{J}}_0)^2=(\omega_{22}\overline{\mathscr{J}}_0)^2+I(I+1)\left(\frac{2(x-y)^2}{j_2}-\frac{1}{2}\right)\frac{1}{J_2^2}; \quad (141)$$

$$j_2\equiv\frac{\overline{\mathscr{J}}_2}{\overline{\mathscr{J}}_0}=(x-y)^2+\frac{3(n_{20}+1/2)}{\omega_{20}^{(p)}\overline{\mathscr{J}}_0}+\frac{n_{22}+1/2}{\omega_{22}^{(p)}\overline{\mathscr{J}}_0}. \quad (142)$$

For analytic investigation, it is convenient to subtract (139) from (138), after which the result can be represented as

$$1-\frac{I(I+1)}{2}\frac{1}{j_2^2}\left(\frac{3}{(\omega_{20}\overline{\mathscr{J}}_0)^2}+\frac{1}{(\omega_{22}\overline{\mathscr{J}}_0)^2}\right)=\frac{1}{x-y}. \quad (143)$$

Considering the cases $(x-y)^2>j_2/4$ and $(x-y)^2<j_2/2$, it is easy to show that the left-hand side of (143) has as a function of $x-y$ the form shown in Fig. 7, the minimum at $x-y=0$ sinking ($I_1>I_2$) with increasing angular momentum I. As can be seen from Fig. 7, for any value of I there is a point of intersection with the hyperbola $(x-y)^{-1}$ in the first quadrant; it is this point that corresponds to the normal

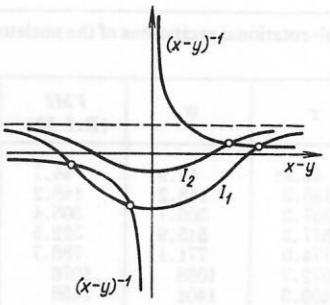

FIG. 7. Graphical analysis of Eq. (143).

solution discussed above, $x>y$. In addition, beginning at a certain value of I there appear two additional points of intersection in the third quadrant, the solutions corresponding to them corresponding to vibrations about an essentially nonaxial deformation ($y>x$). It can be seen from Eqs. (137) and (138) that solutions differing strongly from $x=1, y=0$ can occur only when $I\gtrsim\omega\overline{\mathscr{J}}_0$.

In the numerical search for anomalous solutions of Eqs. (116)–(118), we use the fact that the "main" nonlinearity is, as was shown in the investigation of the rigid limit [see (122)–(126)], contained in Eq. (117). Therefore, it is convenient, using (116) and (118) to express the remaining unknowns in terms of y, to reduce the problem to the solution of the nonlinear equation

$$y-g(y)=0. \quad (144)$$

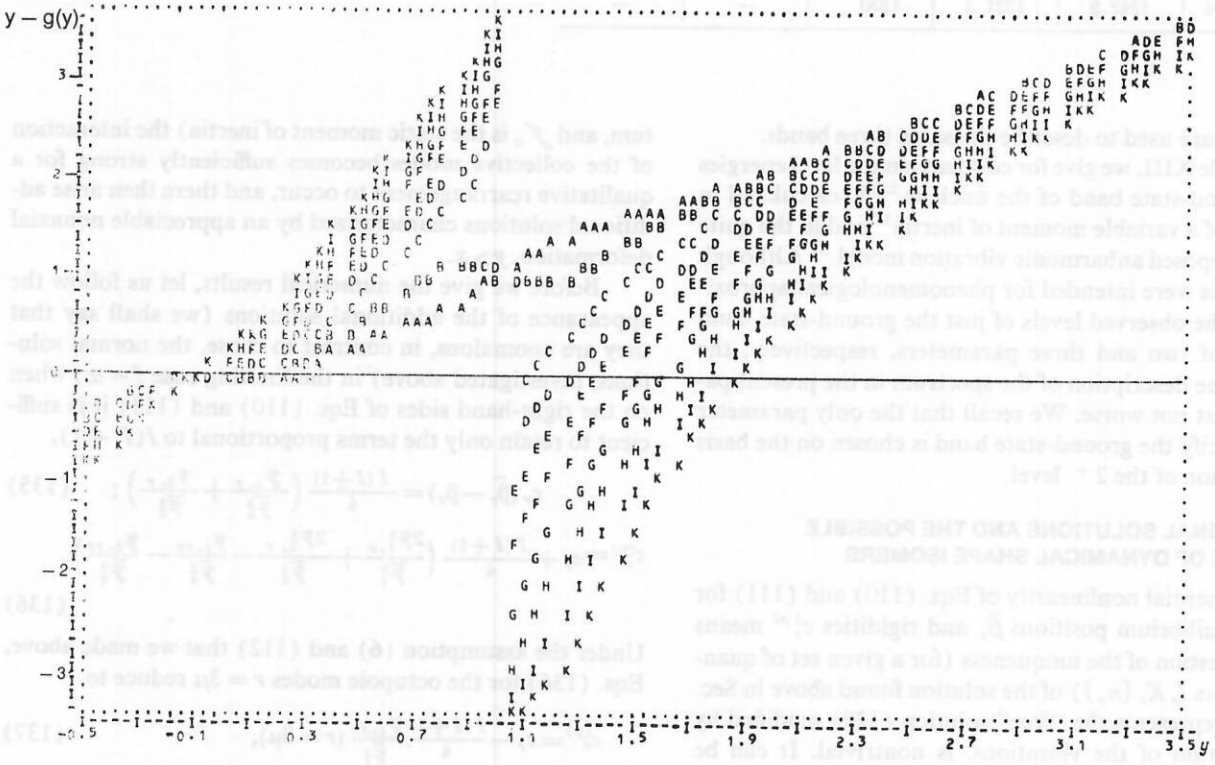

FIG. 8. Graphical analysis of Eq. (144) for the moments: A) $I=0$; B) $I=2$; C) $I=4$; D) $I=6$; E) $I=8$; F) $I=10$; G) $I=12$; I K) $I=18$.

TABLE XIV. Characteristics of normal (N) and anomalous (A) states of the ground-state band of the nucleus ^{194}Pt.

I	$\frac{N}{A}$	E	x	y	$\omega_{20}^{(p)}\gamma_0$	$\omega_{22}^{(p)}\gamma_0$	$\omega_{30}^{(p)}\gamma_0$
4+	N	823	1.1229	0.0007	17.19	4.347	11.36
6+	N	1482	1.2030	−0.0617	18.12	5.299	10.61
6+	A	4895	0.5564	1.267	25.40	13.04	12.24
8+	N	2230	1.2879	−0.0924	19.00	6.327	9.88
8+	A	5945	0.5308	1.394	25.18	12.87	12.12
10+	N	3038	1.3722	−0.1056	19.78	7.143	9.23
10+	A	6940	0.5129	1.509	24.97	12.72	11.99

Note. $\omega_{20}\gamma_0=15.58$, $\omega_{22}\gamma_0=5.839$, $\omega_{30}\gamma_0=12.71$.

The graphical analysis of this equation is shown in Fig. 8 for different angular momenta I (for the nucleus ^{194}Pt). Besides the root that lies near the point $y=0$, beginning at angular momentum $I=6$ there appear two additional roots in the region $y\gtrsim 1$. To these solutions there correspond a local minimum of the energy as a function of the unknowns $\tilde{\beta}_r, c_r^{(p)}$ (right-hand root) and a saddle point between this minimum and the main minimum. The characteristics of the corresponding states are given in Table XIV.

Since the dynamical-deformation parameters x and y differ appreciably for the anomalous and normal states, the transitions between them can be appreciably weakened. Thus, the reduced probability of an $E2$ transition from the anomalous state with angular momentum I to the normal state with angular momentum $I-2$ is determined by the expression

$$B(E2, I_A \to (I-2)_N) \simeq \left(\frac{3ZeR_0^2}{4\pi}\right)^2 \beta_0^2 \left(\frac{\omega_{20}^{(N)}x_N+\omega_{20}^{(A)}x_A}{\omega_{20}^{(N)}+\omega_{20}^{(A)}}\right)^2 \times \frac{3I(I-1)}{2(4I^2-1)} \exp\left\{-\frac{1}{3}\tilde{\omega}_{20}\gamma_0(x_A-x_N)^2-\tilde{\omega}_{22}\gamma_0(y_A-y_N)^2\right\}, \quad (145)$$

where

$$\tilde{\omega}_r = \omega_r^{(N)}\omega_r^{(A)}/(\omega_r^{(N)}+\omega_r^{A}). \quad (146)$$

In particular, for the most probable transition from the lowest anomalous state 6^+ to the 4^+ level of the ground-state band in ^{194}Pt we have

$$B(E2, 6_{(A)}^+ \to 4^+)/B(E2, 6_{(N)}^+ \to 4^+) \simeq 0.8\cdot 10^{-3}, \quad (147)$$

while the corresponding probabilities of transition per unit time, expressed in terms of the reduced probabilities in accordance with the well-known formulas (see, for example, Ref. 10), are

$$T(E2, 6_{(A)}^+ \to 4^+) \simeq 7.2T(E2, 6_{(N)}^+ \to 4^+) \approx 1.1\cdot 10^{-15}\,\text{sec}^{-1}. \quad (148)$$

Thus, the model discussed in this review predicts long-lived states with high angular momenta representing dynamical shape isomers. The experimental investigation of such states would be of interest in its own right as well as for the verification of the correctness of the model.

CONCLUSIONS

The vibrational–rotational properties of nuclei are investigated in the framework of the classical approach proposed by Bohr[6] and consisting of the introduction of collective variables and the construction of a corresponding Hamiltonian on the basis of clear physical (geometrical) ideas about the nuclear deformation and rotation. An alternative to such an approach is the interacting-boson approximation (IBA), which appeared subsequently[34,35] and is today widely used; in it, the collective variables (bosons) are constructed using the symmetry of the nuclear spectra.

The treatment given in the present review synthesizes much from the earlier treatments based on the Bohr Hamiltonian. Thus, as in the Faessler–Greiner model,[11] we separate in the collective Hamiltonian a main part (diagonal with respect to the rotational variables), on the eigenfunctions of which the total Hamiltonian is then diagonalized. On the other hand, as in the Davydov–Chaban model,[9] use is made of an expansion of the effective vibrational potential near the point of dynamical equilibrium. The important difference from the earlier studies is the picture of the interaction of the collective modes through a certain average field. Such a representation becomes possible when the number of interacting degrees of freedom is sufficiently large and one can meaningfully speak of an average field that depends weakly on each of them separately. Thus, the simultaneous treatment of both the quadrupole and octupole excitations makes possible a different approach to the problem of finding the collective wave function. Of course, the inclusion in the treatment of hexadecupole variables would only improve the validity of such an approach. (Unfortunately, experimental information about them is as yet very sparse and inadequate for an unambiguous choice of the corresponding parameters.)

The possibility of simplification of the problem by the

significant increase in the number of degrees of freedom treated explicitly may be unexpected but is not unique in its kind. Indeed, the situation greatly resembles the one encountered in the classical formulation of the theory of the nucleus when one attempts to solve the Schrödinger equation for A nucleons by passing successively from a nucleus consisting of two nucleons to one consisting of three, from one consisting of three to one consisting of four, etc. It is a different matter if one attempts to begin right at the start with a "fairly large" nucleus, $A \gg 1$. Then one can use many methods developed in the theory of nuclear matter, for example, the theory of finite Fermi systems and the Hartree–Fock method, to say nothing of the shell model.

We note in this connection that in recent years wide use has been made in quantum mechanics of the so-called $1/N$ expansion, which was proposed initially for the solution of special problems in nuclear physics,[36] the theory of critical phenomena, and quantum field theory,[37] and was later extended to many other problems (for more details, see Ref. 38).

The idea of a self-consistent interaction of the vibrational–rotational degrees of freedom makes it possible to construct a theory of such excitations that includes in a natural manner not only the quadrupole degrees of freedom but also the octupole degrees and, in principle, the degrees of higher multipolarity. The fact that it is then not necessary to make an expansion with respect to the ratio of the vibration amplitude to the static deformation makes it possible to treat in a unified manner not only deformed but also transitional nuclei, in which the static deformation may be small.

The resulting dependence of the moments of inertia on the angular momentum I, which is also taken into account in a self-consistent manner, leads to a significant rearrangement of the spectrum—from the rotational law $E_I \sim I(I+1)$ at small I to an equidistant law at large I, and this corresponds to the actually observed picture. Comparison of the results of the calculation with the experimental data indicates that the proposed description is adequate for both rigid and soft nuclei up to fairly high angular momenta.

The collective Bohr Hamiltonian contains several parameters that characterize the collective properties of the nucleus (mass coefficients, rigidities, deformation parameters) and are adjusted using several experimental facts. In this sense, the theory based on the Bohr Hamiltonian is phenomenological. Microscopic calculation of the parameters of the Hamiltonian would permit investigation of nuclei for which information about the spectra is sparse or entirely nonexistent. On the other hand, parameters found from comparison with experiment could be used to test future microscopic theories.

One such theory of collective motion is the method of generalized hyperspherical functions,[39] which has been further developed using the technique of coherent states.[40] In this approach, the problem of introducing collective variables describing quadrupole and monopole motions has been consistently solved, and a microscopic justification of the collective "quadrupole" Bohr Hamiltonian has also been obtained. Generalization of the theory to the case of octupole collective motions (and ones of higher multipolarity) could be the solution to the problem under discussion.

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