The *N*-body quantum problem in configuration space

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The scattering problem for systems of N particles is formulated in the configuration representation. The cases of short-range two-body potentials, potentials with Coulomb long-range interaction, and also the boundary-condition model are considered. On the basis of the differential form of Faddeev's equations a numerical analysis is made of Nd scattering processes and various bound three-particle systems: the ^{3}He and ^{3}H nuclei, the positronium ion, and some baryons in the nonrelativistic quark model.

INTRODUCTION

One of the fundamental problems in the quantum theory of systems consisting of several particles is the development of model-independent methods for calculating the physical characteristics of such systems. With the growth of computing possibilities, it is becoming more and more obvious that the effectiveness of any particular method is determined by the extent to which it is mathematically sound—only mathematically correct methods in conjunction with a sufficiently powerful computer make it possible to calculate physically interesting quantities with controlled accuracy.

One of the most powerful and promising methods is based on the differential formulation of the scattering problem in the configuration space. In this approach, the integral equations for the components of the wave function, which are the mathematical basis of the problem, are used only to study the general properties of the solution—smoothness, asymptotic behavior, identification of the principal singularities, etc. In contrast, numerical calculations are made on the basis of the differential form of these equations, this form being obtained by inverting in some manner the integral operators.

What is attractive about this method in the first place is its universality. In an approach with a unified mathematical structure, one can consider very different physical systems with any type of interaction—systems of several nucleons, mesic atoms, and quark systems. From the point of view of the interaction, one can consider short-range, Coulomb, and rising potentials and the boundary-condition model.

A review of the methods based on the differential approach and their realization in specific problems is the subject of the present paper.

We review briefly the development of the quantum scattering theory for systems of several particles. Investigations and numerical calculations of the properties of such systems on the basis of correct mathematical methods became possible only once Faddeev had created a rigorous theory of scattering in three-body systems,[1] and Yakubovskiĭ[2] had generalized it for systems with an arbitrary number of particles. The Faddeev and Yakubovskiĭ equations became the basis for the creation of new computational methods in nuclear and atomic physics. Such methods were most strongly developed in nuclear physics, mainly to describe three-nucleon systems. It soon became clear that the direct solution of the Faddeev equations in their original integral formulation for realistic nuclear potentials requires huge computational resources. To simplify the procedure for solving the integral equations, it was proposed that separable approximations[3–5] and the quasiparticle method[6] should be used. However, these methods did not permit study of realistic NN interactions.

For numerical calculations with local internucleon potentials the differential form of the equations for the components proved to be more convenient. The differential Faddeev equations, proposed in Ref. 7, were successfully used for the first time to calculate the binding energies of the ^{3}H and ^{3}He nuclei.[8] After the scattering problem had been formulated in the coordinate representation,[9] these equations were also used to calculate the results of nd collisions with a realistic NN interaction.[10]

In the next stage, the Faddeev equations were generalized to systems of charged particles. At energies below the disintegration threshold, a modification of the integral equations was proposed in Ref. 11. In Ref. 12, the theory of scattering of three charged particles in the coordinate space was developed. The modified differential Faddeev equations proposed in these studies made it possible to take into account correctly the Coulomb interaction in the pd scattering problem.[13]

Later, the differential formalism was generalized to systems with arbitrarily many particles,[14,15] and also to three-body systems in which the interaction is specified by boundary conditions on the wave function.[16] The differential equations of Ref. 16 were used to calculate rigorously nd scattering processes in the boundary-condition model.[17]

We note also that the differential Faddeev equations have been used not only in the traditional group of problems in nuclear physics but have also been successfully employed to calculate the static characteristics of baryons in the nonrelativistic quark model.[18,19]

The present paper consists of three sections. In Sec. 1, we describe the general formalism of the differential approach. We consider in detail systems of three and four bodies, merely formulating the final results in the general case of the N-body problem. At the end of Sec. 1, we describe the differential formulation of the N-body problem in the boundary-condition model.

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The remainder of the paper is devoted to the application of the differential approach to calculate the properties of specific physical systems. In Sec. 2, we consider the problem of the scattering of nucleons by the deuteron. In Sec. 3, the differential Faddeev equations are used to analyze the characteristics of various bound three-particle systems: the ^{3}H and ^{3}He nuclei, the positronium ion, and light baryons in the nonrelativistic quark model.

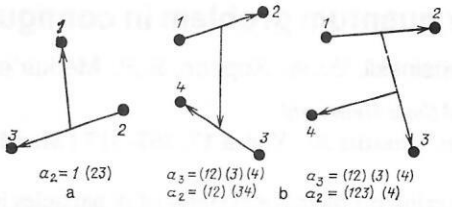

FIG. 1. Relative coordinates in systems of three (a) and four (b) bodies.

1. BASIC EQUATIONS

Differential formulation of the scattering problem

We describe the general differential formalism in the N-body problem. For simplicity, we shall assume that the particles are spinless and interact through two-body forces.

Notation. We first introduce some notation employed throughout the entire paper.

By a partition a_k we shall mean a way of dividing the system into k subsystems α_l of l particles. We shall say that a_k is a cluster partition if bound states exist for every subsystem α_l in a_k. Accordingly, a cluster will mean a bound state $\psi^j_{\alpha_l}$ of the subsystem and $-\varepsilon^j_{\alpha_l}$ will denote the binding energies of the clusters. By Ψ_{Ja_k} we denote the product of the cluster eigenfunctions describing the internal state of the partition a_k with energy $-\varepsilon_{Ja_k}$, which is equal to the sum of the cluster eigenvalues.

The index Ja_k, which denotes the set of numbers of the bound states of the subsystems $\omega_{l_1}, \omega_{l_2},..., \omega_{l_k}$, $J_{\alpha k} = \{i_{\omega_{l_1}}, i_{\omega_{l_2}},..., i_{\omega l_k}\}$, is too cumbersome. Therefore, in what follows we replace it by a single letter, for example. A, B, etc., associating the index $A = 0$ with the N free particles.

If partition a_k is obtained from a_i by division of subsystems into parts, we shall say that a_k follows a_i and write $a_k \subset a_i$ $(a_i \supset a_k)$, $i<k$. Further, we shall use identical letters to denote only those partitions connected by the relation $\subset$, i.e., i.e., $a_{N-1} \subset a_{N-2} \subset ... \subset a_2$. We shall call a sequence of such partitions, beginning with a certain k, $2 \leqslant k \leqslant N-1$, a chain of partitions and denote it by A_k, $A_k = \{a_k, a_{k+1},..., a_{N-1}\}$. The final partition in the chain is always a_{N-1}.

Note that every partition a_{N-1} is determined by a pair of particles α in a unique nontrivial subsystem. For this reason, we shall frequently identify the index a_{N-1} with the symbol of the subsystem α corresponding to it, $a_{N-1} \leftrightarrow \alpha$.

We shall also introduce relative coordinates, used to describe the system of N particles. Let r_i be the radius vector of particle i of mass m_i and ρ_{ω_k} be the coordinates of the center of mass of subsystem ω_k of mass m_{ω_k}. By $x_{i\omega_k}$ we denote the reduced relative coordinate of particle i and subsystem ω_k:

$$x_{i\omega_k} = \left(\frac{2m_i m_{\omega_k}}{m_i + m_{\omega_k}}\right)^{1/2} (r_i - \rho_{\omega_k}).$$

Consider the sequence of subsystems $\omega_2, \omega_3,...$, obtained by successive addition of the particles $i_2, i_3,..., i_N$ to particle i_1. With this sequence we associate the $N-1$ relative coordinates $x_{i_1 i_2}, x_{i_3 \omega_2},..., x_{i_N \omega_{N-1}}$. The set of these coordinates determines a point X in the configuration space R^{3N-3}. Transition from one set of relative coordinates to another corresponds to notation of the coordinate system in R^{3N-3}.

Let the partition a_k consist of the subsystems ω_{l_1}, $\omega_{l_2},..., \omega_{l_k}$. We denote by $x_{\omega_{l_i}}$ $(x_{\omega_{l_i}} \in R^{3l_i-3})$ the set of reduced relative coordinates of the subsystem ω_{l_i}. The set of vectors $\{x_{\omega_{l_1}}, x_{\omega_{l_2}},..., x_{\omega_{l_k}}\}$ is called the coordinate internal with respect to the partition a_k and is denoted by x_{a_k}. We denote the set of reduced relative coordinates $\rho_{\omega_{l_i}}$, regarded as the coordinates of simple particles, by y_{a_k}, $y_{a_k} \in R^{3k-3}$. We have the representation $X = \{x_{a_k}, y_{a_k}\}$, which determines an orthogonal coordinate system in R^{3N-3} corresponding to the partition a_k.

In the case of a three-particle system, there is only one type of relative coordinates $\{x_a, y_a\}$ $(\alpha = 1,2,3)$. It is convenient to represent these coordinates in the form of the transparent scheme in Fig. 1a.

In a four-body system, there are two types, shown in Fig. 1b.

These coordinates correspond to two types of partitions in the four-body system: $3+1$ and $2+2$. The possible relative coordinates in systems of five, six, etc., bodies are so numerous that it is already difficult to list them.

We denote by k_{a_i}, p_{a_i} the relative momenta conjugate to the coordinates x_{a_i}, y_{a_i}. The symbols $\hat{x}$, $\hat{k}$, etc., will denote unit vectors along the directions of the vectors x, k.

With this notation, the energy operator of the N-body system has the form

$$H = H_0 + \sum_{a_{N-1}} V_{a_{N-1}} \equiv -\Delta_X + \sum_{a_{N-1}} V_{a_{N-1}}(x_{a_{N-1}}), \quad (1)$$

where the summation is over all partitions a_{N-1} (i.e., over all particle pairs α). We denote by Δ_X the Laplacian in the $(3N-3)$-dimensional space, and by $V_{a_{N-1}}$ $(x_{a_{N-1}})$ the two-body interaction potentials. It is assumed that the center-of-mass motion is separated out.

Three-particle system. We describe first the differential formulation of the scattering problem for the example of a three-body system. We shall assume that the interaction potentials are rapidly decreasing functions.

Let Ψ be a solution of the Schrödinger equation

$$(H_0 + \sum_{\alpha} V_\alpha - E)\,\Psi = 0.$$

Here, in contrast to the general case (1), it is convenient to use the symbols α of the subsystems instead of the partitions a_2 corresponding to them. In the three-body problem, the wave function decomposes into three components,

$$\Psi_\alpha = -R_0(E+i0)V_\alpha\Psi, \quad R_0(z)=(H_0-z)^{-1}, \tag{2}$$

which are called Faddeev components. Summing over all α, we express Ψ in terms of these components:

$$\Psi = \sum_\alpha \Psi_\alpha.$$

Substituting this equation in the right-hand side of (2) instead of Ψ, we obtain integral equations for the components:

$$\Psi_\alpha = -R_0(E+i0)V_\alpha \sum_\beta \Psi_\beta.$$

The differential Faddeev equations are obtained from here by inverting the operator R_0 and transferring the term with $\beta = \alpha$ to the left-hand side:

$$(H_0+V_\alpha-E)\Psi_\alpha = -V_\alpha \sum_{\beta\neq\alpha}\Psi_\beta. \tag{3}$$

The integral Faddeev equations correspond to the inversion of the operator $H_0 + V_\alpha - E$ in this relation:

$$\Psi_\alpha = -R_\alpha(E+i0)V_\alpha \sum_{\beta\neq\alpha}\Psi_\beta, \quad R_\alpha(z)=(H_0+V_\alpha-z)^{-1}. \tag{4}$$

To determine a unique solution to the system of equations (3) identical to a definite solution of the integral equations (4), it is necessary to specify asymptotic boundary conditions. In the case of bound states, we have the condition of decrease at infinity (the eigenfunction must be square-integrable). In the case of scattering states, the conditions are much more complicated.

Consider, for example, the process that is the most important from the point of view of applications, scattering of a particle by a bound pair β. We shall append to the components Ψ_α an additional index B, this corresponding to the number of the bound state of the pair β, $B = \{i, \beta\}$. The following representation reflects the asymptotic behavior of the wave function:

$$\Psi_{\alpha B}(X, p_\beta) = \chi_B(X, p_\beta)\delta_{\alpha\beta} + \sum_A \psi_A(x_\alpha) U_{AB}(y_\alpha, p_\beta) + U_{0B}^{(\alpha)}(X, p_\beta). \tag{5}$$

Here, the first term χ_B describes the initial state,

$$\chi_B(X, p_\beta) = \psi_B(x_\beta)\exp\{i(p_\beta, y_\beta)\},$$

the term U_{AB} in the second group describes the processes of elastic scattering with possible excitation ($\alpha = \beta$) or rearrangement ($\alpha \neq \beta$), and the third term $U_{0B}^{(\alpha)}$ describes the processes of disintegration of the system into three free particles. The functions U_{AB} and $U_{0B}^{(\alpha)}$ go over asymptotically as $|y_\alpha| \to \infty$ and $|X| \to \infty$ into spherical waves with smooth bounded amplitudes:

$$\left.\begin{aligned} &U_{AB}(y_\alpha, p_\beta) \sim f_{AB}(\hat{y}_\alpha, p_\beta)|y_\alpha|^{-1}\exp\{i\sqrt{E+\varepsilon_A}\,|y_\alpha|\};\\ &U_{0B}^{(\alpha)}(X, p_\beta) \sim f_{0B}^{(\alpha)}(\hat{X}, p_\beta)|X|^{-5/2}\exp\{i\sqrt{E}\,|X|\},\\ &E = p_\beta^2 - \varepsilon_B. \end{aligned}\right\} \tag{6}$$

The cross sections of the corresponding processes are proportional to the squares of the moduli of the amplitudes[20]:

$$\left.\begin{aligned} &\frac{d\sigma}{d\hat{p}_\alpha} = \frac{2\pi|p_\alpha|}{|p_\beta|}\mu_\alpha\mu_B|f_{AB}(\hat{p}_\alpha, p_\beta)|^2;\\ &\frac{d\sigma}{d\hat{k}_\alpha\, d\hat{p}_\alpha\, dE_\alpha} = 16\pi\mu_\beta\left(\frac{m_1m_2m_3}{m_1+m_2+m_3}\right)^{3/2}|p_\alpha||k_\alpha||p_\beta|^{-1}\sum_\gamma |f_{0B}^{(\gamma)}|^2, \end{aligned}\right\} \tag{7}$$

$$\mu_\alpha = 2\Big(\sum_{i\subset\alpha} m_i^{-1}\Big)^{-1}.$$

Note that the proof of the asymptotic expressions (5) and (6) is essentially based on the integral equations (4) for the components, or rather, being more precise, the analogous inhomogeneous equations obtained after separation of the initial state χ_B are used for this purpose. Corresponding discussions can be found in Ref. 9.

Thus, we have formulated boundary-value problems on the basis of the differential Faddeev equations, which uniquely determine the components of the wave functions. Analogous boundary-value problems can also be obtained for the components of the Yakubovskiĭ wave functions. We begin by deriving the differential equations for such components for the example of the four-body system.

N-body systems. The Yakubovskiĭ components are labeled by chains of partitions A_2. In the case of four particles, the chain contains two partitions: a_2, a_3. The first of these partitions indicates the manner in which the four particles are partitioned into two subsystems, and the second specifies the manner in which a_2 is partitioned into three subsystems. There are altogether seven partitions a_2 [four $3+1$ partitions, i.e., partitions for which one of the subsystems contains three particles, and three $2+2$ partitions, when each of the subsystems contains two particles] and six partitions a_3. From these partitions, it is possible to form 18 different chains $A_2 = \{a_2, a_3\}$. Thus, for $N = 4$ it is necessary to introduce 18 components of the wave functions.

In the first stage, we introduce six Faddeev components by means of the standard definitions

$$\Psi_{a_3} = -R_0V_{a_3}\Psi.$$

Exactly as in the three-particle case, the components Ψ_{a_3} satisfy the Faddeev equations

$$(H_0+V_{a_3}-E)\Psi_{a_3} = -V_{a_3}\sum_{b_3\neq a_3}\Psi_{b_3}$$

or, in integral form

$$\Psi_{a_3} = -R_{a_3}(E+i0)V_{a_3}\sum_{b_3\neq a_3}\Psi_{b_3}, \quad R_{a_3}(z)=(H_0+V_{a_3}-z)^{-1}. \tag{8}$$

The components of the Yakubovskiĭ wave function, classified by the chains of partitions, are introduced by the equations

$$\Psi_{a_2a_3} = -R_{a_3}(E+i0)V_{a_3}\sum_{\substack{b_3\neq a_3\\ b_3\subset a_2}}\Psi_{b_3}, \tag{9}$$

i.e., on the right-hand side of (8) we retain only the terms for which $b_3 \subset a_2$. The Faddeev components can be expressed in terms of the Yakubovskiĭ components by means of a sum rule, this having for an arbitrary number of particles the form

$$\sum_{a_{k-1}} \sum_{\substack{b_k \neq a_k \\ b_k \subset a_{k-1}}} = \sum_{b_k \neq a_k}.$$

For $N = 4$, we obtain the representation

$$\Psi_{a_3} = \sum_{a_2} \Psi_{a_2 a_3}. \tag{10}$$

We now replace the components Ψ_{b_3} on the right-hand side of Eq. (9) by the corresponding expressions in terms of the Yakubovskiĭ components (10). As a result, we obtain the equations

$$\Psi_{a_2 a_3} = -R_{a_3}(E+\mathrm{i}0)\, V_{a_3} \sum_{\substack{b_3 \neq a_3 \\ b_3 \subset a_2}} \sum_{b_2} \Psi_{b_2 b_3}. \tag{11}$$

Applying to (11) the operator $H_1 + V_{a_3} - E$ and transferring to the left-hand side the term with $b_2 = a_2$, we transform the system (11) to the form

$$(H_0 + V_{a_3} - E)\, \Psi_{a_2 a_3} + V_{a_3} \sum_{\substack{c_3 \neq a_3 \\ c_3 \subset a_2}} \Psi_{a_2 c_3} = -V_{a_3} \sum_{b_2 \neq a_2} \sum_{\substack{b_3 \neq a_3 \\ b_3 \subset a_2}} \Psi_{b_2 b_3}. \tag{12}$$

This system is the required generalization of the differential equations for the components (3) to the case of the four-particle problem.

As in the three-body problem, there is an intimate connection between these equations and the compact integral equations, namely, the square-integrable solutions of the system of equations (12) are identical to the solutions of the homogeneous integral Yakubovskiĭ equations. But in the case of scattering states it is necessary to specify suitable asymptotic boundary conditions analogous to (5). Then the solutions of the differential equations will be identical to the corresponding solutions of the integral equations. In this sense, one can say that the differential equations (12) are equivalent to the integral Yakubovskiĭ equations.

We shall describe these asymptotic conditions below, after we have investigated the formal relationship between the differential and the integral equations.

Thus, we go over from the system (12) to the integral Yakubovskiĭ equations. To this end, we invert the matrix differential operator generated by the left-hand side of (12). We note first that this operator has a block structure. The matrix elements of the operator needed to invert one block satisfy the equation

$$(H_0 + V_{a_3} - z)\, R^{a_2}_{a_3 b_3}(z) + V_{a_3} \sum_{\substack{c_3 \neq a_3 \\ c_3 \subset a_2}} R^{a_2}_{c_3 b_3}(z) = \delta_{a_3 b_3} I, \tag{13}$$

where I is the identity operator, and $\delta_{a_3 b_3}$ is the Kronecker delta. We rewrite (13) in the integral form

$$R^{a_2}_{a_3 b_3}(z) = R_{a_3}(z)\, \delta_{a_3 b_3} - R_{a_3}(z)\, V_{a_3} \sum_{\substack{c_3 \neq a_3 \\ c_3 \subset a_2}} R^{a_2}_{c_3 b_3}. \tag{14}$$

Equations (14) are identical to the integral Faddeev equations for the components of the resolvent of the four-particle system in which the only nonvanishing interaction potentials are those between the particles in the subsystems of the partition a_2.[2] We denote by H_{a_2} the energy operator of this problem:

$$H_{a_2} = H_0 + \sum_{a_3 \subset a_2} V_{a_3}.$$

In this problem, the variables separate, and we have the representation

$$H_{a_3} = h_{a_2} \otimes 1 - 1 \otimes \Delta_{y_{a_2}},$$

where the energy operator h_{a_2} acts only on the variables x_{a_2}. This operator describes the three-particle subsystem in the case of a partition a_2 of $3+1$ type or a system of two pairs with no mutual interaction in the case of a partition a_2 of $2+2$ type.

The solution of (14) being unique, the operators $R^{a_2}_{a_3 b_3}$ are identical to the components of the Green's function[2]:

$$R^{a_2}_{a_3 b_3}(z) = R_0(z)\, \delta_{a_3 b_3} - R_0(z)\, V_{a_3} R_{a_2}(z), \qquad R_{a_2}(z) = (H_{a_2} - z)^{-1}.$$

Thus, the inversion of each block on the left-hand side of (12) can be implemented in terms of the operators $R^{a_2}_{a_3 b_3}$. As a result of this inversion, the system (12) can be transformed into

$$\Psi_{a_2 a_3} = -\sum_{b_2 \neq a_2} \sum_{\substack{c_3 \neq b_3 \\ c_3,\, b_3 \subset a_2}} R^{a_2}_{a_3 c_3} V_{c_3} \Psi_{b_2 b_3}. \tag{15}$$

The system (15) is identical to the homogeneous system of integral Yakubovskiĭ equations for the four-particle problem.[2] Thus, we have shown that (12) is the differential analog of the integral Yakubovskiĭ equations.

We now turn to the description of the asymptotic boundary conditions for (12) mentioned above. Note that these conditions can be obtained by means of the integral Yakubovskiĭ equations (15). We consider only the case of binary collisions. We shall endow the components of the wave functions with an additional index B, this describing the quantum numbers of the initial state of the system, which is specified by the wave function $\chi_B(X, p_{b_2}) = \psi_B(x_{b_2}) \times e^{i(p_{b_2} y_{b_2})}$.

The wave-function components $\Psi_{A_2, B}$ are represented in the form

$$\Psi_{A_2, B}(X, p_{b_2}) = \chi_{a_3 A}(X, p_{a_3})\, \delta_{AB} + \sum_A \psi_{a_3 A}(x_{a_3})\, U_{AB}(y_{a_3}, p_{b_2}) + \sum_A \psi_A(x_{a_3})\, U^{a_3}_{AB}(y_{a_3}, p_{b_2}) + U^{A_2}_{0B}(X, p_{b_2}). \tag{16}$$

The terms in this equation have the following meaning.

The functions $\chi_{a_3 A}$ are the components of the initial state of the system and are expressed in terms of the wave function $\chi_A(X, p_{a_2})$ of the initial state by

$$\chi_{a_3 A} = -R_0(E+\mathrm{i}0)\, V_{a_3} \chi_A.$$

The right-hand side here can be rewritten in the form of the product $\chi_{a_3 A} = \psi_{a_3 A}(x_{a_2}) \exp\{i(p_{a_2}, y_{a_2})\}$, where $\Psi_{a_3 A}$ are the Faddeev components (2) of the wave function of the three-particle bound state [if a_2 is a partition of $3+1$ type] or the Faddeev components of the wave function in the degenerate problem of two noninteracting pairs (in the case of a $2+2$ partition). In the latter case, the wave functions

$\psi_A(x_{a_2})$ are products of two-particle eigenfunctions.

In the second group of terms, the summation is over all cluster states of the partition a_2. For $a_2 = b_2$, these terms describe elastic processes with possible excitation; for $a_2 \neq b_2$, they describe rearrangement processes. In the third group, the summation is over the eigenvalues of the two-particle subsystems in the partition a_3. These terms correspond to breakup of the initial state into three clusters (2→3 process). Finally, the last term $U_{0B}^{A_2}$ describes complete disintegration into four free particles (2→4).

The functions U_{0B} and $U_{AB}^{a_2}$ in (16) are asymptotically equal to spherical waves with smooth bounded amplitudes:

$$U_{AB}(y_{a_2}, p_{b_2}) \sim f_{AB}(\hat{y}_{a_2}, p_{b_2})|y_{a_2}|^{-1}\exp\{i\sqrt{E+\varepsilon_A}\,|y_{a_2}|\};$$

$$U_{AB}^{a_2}(y_{a_3}, p_{b_2}) \sim f_{AB}^{a_2}(\hat{y}_{a_3}, p_{b_2})|y_{a_3}|^{-5/2}\exp\{i\sqrt{E+\varepsilon_A}\,|y_{a_3}|\};$$

$$U_{0B}^{A_2}(X, p_{b_2}) \sim f_{0B}^{A_2}(\hat{X}, p_{b_2})|X|^{-4}\exp\{i\sqrt{E}\,|X|\},$$

where $E = p_{b_2}^2 - \varepsilon_B$. The physical amplitudes of the 2→2, 2→3, and 2→4 processes are related to the amplitudes of the spherical waves by

$$f_{AB} \underset{2\to 2}{=} f_{AB}, \quad f_{AB} = \sum_{\substack{a_2 \\ 2\to 3}} f_{AB}^{a_2}, \quad f_{0B} = \sum_{\substack{A_2 \\ 2\to 4}} f_{0B}^{A_2}.$$

The effective cross sections of the 2→2, 2→3, and 2→4 processes can then be expressed in terms of these amplitudes by the ordinary formulas.[20]

With this, we conclude the description of the common aspects of the differential formalism in the four-body problem. We consider briefly the general case of N particles.

The differential analog of the integral Yakubovskiĭ equations for N particles was obtained in Refs. 14 and 15. The corresponding equations for the components Ψ_{A_2}, called the differential Yakubovskiĭ equations, have the form

$$(H_0 + V_{a_{N-1}} - E)\Psi_{A_2} + V_{a_{N-1}} \sum_{i=3}^{N-1} \sum_{\substack{c_i \neq A_i \\ c_i \subset a_{i-1}}}{}' \Psi_{a_2 a_3 \ldots a_{i-1} c_i} = -V_{a_{N-1}} \sum_{b_2 \neq a_2} \sum_{\substack{B_3 \neq A_3 \\ b_3 \subset a_2}}{}' \Psi_{B_2}. \quad (17)$$

The prime on the summation sign means that the summation is only over chains B_3 that are "linked" to A_3—for which the following conditions are satisfied:

$$b_{N-1} \neq a_{N-1}, \ldots, b_3 \neq a_3; \quad b_{N-1} \subset a_{N-2}, \ldots, b_4 \subset a_3.$$

The total wave function is related to the components Ψ_{A_2} by

$$\Psi = \sum_{A_2} \Psi_{A_2},$$

where the summation is over all chains A_2.

Note that the integral Yakubovskiĭ equations are obtained from (17) by inverting the matrix differential operator on the left-hand side.

The asymptotic boundary conditions for (17) can be obtained by means of the integral Yakubovskiĭ equations.[14] They have a form analogous to (16). In particular, the components of the wave functions corresponding to the two-cluster initial states in the configuration space are represented in the form

$$\Psi_{A_2B}(X, p_{b_2}) = \chi_{B_3B}(\chi, p_{b_2})\,\delta_{AB} + \sum_{i=2}^{N-1} \sum_{A} \Psi_{A_{i+1},A}(x_{a_i}) U_{AB}^{a_2 \ldots a_{i-1}}(y_{a_i}, p_{b_2}) + U_{0B}^{A_2}(X, p_{b_2}), \quad (18)$$

where χ_{B_3B} describes the initial state of the system:

$$\chi_{B_3B}(X, p_{b_2}) = \Psi_{B_3B}(x_{b_2}) \exp\{i(p_{b_2}, y_{b_2})\},$$

and $\Psi_{A_{i+1},A}$ are the Yakubovskiĭ components of the wave function of the i-cluster partition a_i. The function $U_{AB}^{a_2 \ldots a_{i-1}} \times (y_{a_i}, p_{b_2})$ goes over asymptotically in the limit $|y_{a_i}| \to \infty$ into a spherical wave in R^{3i-3}:

$$U_{AB}^{a_2 \ldots a_{i-1}}(y_{a_i}, p_{b_2}) \sim f_{AB}^{a_2 \ldots a_{i-1}}(\hat{y}_{a_i}, p_{b_2})\,|y_{a_i}|^{\frac{4-3i}{2}} \exp\{i\sqrt{E+\varepsilon_A}\,|y_{a_i}|\}. \quad (19)$$

The function $U_{0B}^{A_2}(X, p_{b_2})$ goes over in the limit $|X| \to \infty$ into a spherical wave in R^{3N-3} and correspond to disintegration into N particles:

$$U_{0B}^{A_2}(X, p_{b_2}) \sim f_{0B}^{A_2}(\hat{X}, p_{b_2})\,|X|^{\frac{4-3N}{2}} \exp\{i\sqrt{E}\,|X|\}. \quad (20)$$

The physical amplitudes of the $B \to A$ processes are related to the amplitudes of the spherical waves (19) by

$$f_{AB} = \sum_{a_2 \ldots a_{i-1}} f_{AB}^{a_2 \ldots a_{i-1}}.$$

Finally, the amplitude for disintegration of the two-cluster state into N free particles is given by the equation

$$f_{0B} = \sum_{A_2} f_{0B}^{A_2}.$$

Thus, we have shown how the wave functions can be determined by means of the differential equations for the components. We have hitherto assumed that the particles are neutral. Below, we generalize the differential formalism to the case of charged particles.

Systems of charged particles

In the case of systems of charged particles, the two-body potentials are sums of Coulomb and short-range parts:

$$V_{a_{N-1}}(x) = n_{a_{N-1}}|x|^{-1} + V_{a_{N-1}}^{(s)}(x).$$

As a result, the integral Faddeev and Yakubovskiĭ equations become noncompact because of the long-range nature of the Coulomb interaction. The problem of modifying these equations has been considered in many studies (Refs. 11–14 and 21–26). We shall consider here only the approach based on the differential formalism.

This approach is evidently the one best suited to formulate the scattering problem for systems of charged particles. This circumstance is reflected, in particular, in the fact that the necessary modifications reduce merely to corrections in the asymptotic behavior of the wave functions. Moreover, all such corrections can be interpreted in the framework of semiclassical approximations—eikonal approximations constructed on the basis of the trajectories of classical neutral

particles.[12] We shall not describe here the formalism needed to derive such asymptotic corrections but will restrict ourselves to the formulation of the final results.

We consider directly the general case of a system of N particles. We begin with the formulation of the modified differential equations for the components.

Let $\Omega_{a_{N-1}}(v)$ be the region of the configuration space in which the condition $|x_{a_{N-1}}| \leqslant C(1+|y_{a_{N-1}}|)^v$, $0<v<1/2$ is satisfied. We introduce a smooth function $\chi_{a_{N-1}}(X)$ of compact support, equal to 1 in $\Omega_{a_{N-1}}(v)$ and zero outside $\Omega_{a_{N-1}}(v_1)$, $v<v_1<1/2$. We split the interaction potential $V_{a_{N-1}}$ into short- and long-range parts by means of the function $\chi_{a_{N-1}}$:

$$V_{a_{N-1}}(x_{a_{N-1}}) = \hat{V}_{a_{N-1}}(X) + V^{(0)}_{a_{N-1}}(X), \tag{21}$$

where

$$\hat{V}_{a_{N-1}}(P) = V^{(s)}_{a_{N-1}} + \frac{n_{a_{N-1}}}{|x_{a_{N-1}}|}\chi_{a_{N-1}};$$

$$V^{(0)}_{a_{N-1}} := (1-\chi_{a_{N-1}})\frac{n_{a_{N-1}}}{|x_{a_{N-1}}|}.$$

In accordance with the division (21), we represent the energy operator of the system of N charged particles in the form

$$H = H_{\text{as}} + \sum_{a_{N-1}} \hat{V}_{a_{N-1}}. \tag{22}$$

Here, the asymptotic Hamiltonian H_{as} describes the motion of the system of N free particles on the background of the Coulomb "tails" $V^{(0)}_{a_{N-1}}$. It is given by

$$H_{\text{as}} = H_0 + \sum_{a_{N-1}} V^{(0)}_{a_{N-1}}.$$

Thus, we have included all the long-range parts of the potentials in the new "unperturbed" energy operator H_{as}. We can then literally repeat all the arguments that led us to the differential equations (3), (12), or (17) for the components. All the changes reduce to the replacement of the operator H_0 by H_{as}, and the interaction operator $V_{a_{N-1}}$ by its "short-range" part $\hat{V}_{a_{N-1}}$. The upshot is that we obtain the modified equations

$$(H_{\text{as}} + \hat{V}_{a_{N-1}} - E)\Psi_{A_2} + \hat{V}_{a_{N-1}} \sum_{i=3}^{N-1} \sum_{\substack{c_i \neq A_i \\ c_i \supset a_{i-1}}}{}' \Psi_{a_2 a_3 \ldots a_{i-1} c_i} = -\hat{V}_{a_{N-1}} \sum_{b_2 \neq a_2} \sum_{\substack{B_3 \neq A_3 \\ b_3 \supset a_2}}{}' \Psi_{B_2}. \tag{23}$$

As above, the wave function is equal to the sum of the components over all chains:

$$\Psi := \sum_{A_2} \Psi_{A_2}. \tag{24}$$

If we invert the matrix integral operator on the left-hand side of these equations, we can obtain the modified integral Faddeev ($N=3$) or Yakubovskiĭ ($N>3$) equations. For the three-body system, it has been shown that such equations are compact.[25] It may be expected that this property of the integral equations will still hold in the general case.

In a number of problems (for example, when the particles carry identical charges and the Coulomb interaction is appreciably weaker than the short-range interaction), it is convenient to use in the numerical calculations the simplest type of modified equations, obtained by setting $\chi_{a_{N-1}} \equiv 0$. This means that in this case the entire Coulomb interaction is included in the "unperturbed" Hamiltonian H_{as}:

$$H_{\text{as}} := H_0 + \sum_{a_{N-1}} n_{a_{N-1}} |x_{a_{N-1}}|^{-1},$$

and the total energy operator is represented in the form

$$H = H_{\text{as}} + \sum_{a_{N-1}} V^{(s)}_{a_{N-1}}.$$

For example, in the three-body problem this procedure leads to the equations[26]

$$\left(H_0 + \sum_{\alpha} \frac{n_\alpha}{|x_\alpha|} + V^{(s)}_\alpha - E\right)\Psi_\alpha = -V^{(s)}_\alpha \sum_{\beta \neq \alpha} \Psi_\beta. \tag{25}$$

After the inversion of the operator on the left-hand side, we obtain from here the Noble-Faddeev equations, which were first proposed in Ref. 22 (see also Ref. 12).

We now formulate boundary conditions that uniquely determine a solution of the modified equations (23). As above, we consider only scattering with two clusters in the initial state.

For the sake of clarity, we consider first the following model scattering problem. Let the clusters ω_l and ω_k forming the partition b_2 have charges q_{ω_l} and q_{ω_k}, respectively. We regard these clusters as structureless point particles. Such effective particles interact only through the Coulomb potentials

$$V^c_{\omega_l\omega_k} = \frac{n_{\omega_l\omega_k}}{|y_{b_2}|}; \quad n_{\omega_l\omega_k} = (2\mu_{\omega_l\omega_k})^{1/2} q_{\omega_l} q_{\omega_k},$$

where $\mu_{\omega_l\omega_k}$ is the reduced mass of the clusters ω_l and ω_k, and y_{b_2} is their reduced relative coordinate.

The elastic scattering of these particles is described by the wave function

$$\chi_{b_2}(X) = F(\xi_{b_2}) \exp\{i(p_{l_2}, y_{b_2})\}, \tag{26}$$

where the function $F(\xi)$ can be expressed in terms of the confluent hypergeometric function:

$$F(\xi) = e^{-\pi\eta_{b_2}} \Gamma(1+i\eta_{b_2}) \Phi(-i\eta_{b_2}, 1, i|p_{b_2}|\xi),$$

where η_{b_2} is the characteristic Coulomb parameter

$$\eta_{b_2} = n_{\omega_l\omega_k}/2|p_{b_2}|,$$

and ξ_{b_2} is a parabolic coordinate, $\xi_{b_2} = |y_{b_2}| - (y_{b_2}, \hat{p}_{b_2})$. If the directions of the vectors y_{b_2} and p_{b_2} do not coincide (i.e. we do not consider the forward scattering direction), the function χ_{b_2} takes asymptotically the form of a sum of distorted plane and spherical waves:

$$\chi_{b_2}(y_{b_2}, p_{b_2}) \sim \exp\{i(p_{b_2}, y_{b_2}) + i\eta_{b_2} \ln|p_{b_2}|\xi_{b_2}\} + a^c_{b_2}(\hat{y}_{b_2}, p_{b_2})|y_{b_2}|^{-1} \exp\{i|p_{b_2}||y_{b_2}| - i\eta_{b_2} \ln 2|p_{b_2}||y_{b_2}|\}.$$

The amplitude of a spherical wave is given by

$$a^c_{b_2} = -\frac{\eta_{b_2}}{2\,|p_{b_2}|}\,\frac{\exp\left\{2i\left(-\eta_{b_2}\ln\sin\frac{\theta_{b_2}}{2}+\Delta^c_{b_2}\right)\right\}}{\sin^2\frac{\theta_{b_2}}{2}}, \quad (27)$$

where

$$\Delta^c_{b_2} = \arg\Gamma(1+i\eta_{b_2}); \quad \cos\theta_{b_2} = (\hat{y}_{b_2},\ \hat{p}_{b_2}).$$

We shall call this the purely Coulomb amplitude.

We consider further the physical wave function $\Psi_B(X, p_{b_2})$, which describes scattering with two clusters (ω_k and ω_1) in the initial state. As in a system of neutral particles, the index B denotes the internal state of the subsystems.

The initial state of the system in such a process is described by the function

$$\chi_B(X,\ p_{b_2}) = \chi_{b_2}(y_{b_2},\ p_{b_2})\,\psi_B(x_{b_2}), \quad (28)$$

where the model function χ_{b_2} was introduced above. The wave function is equal to the sum (24) of components, which can be represented in a form analogous to (18):

$$\Psi_{A_2B}(X,\ p_{b_2}) = \chi_{B_3B}(X,\ p_{b_2})\,\delta_{AB} + \sum_{i=2}^{N-1}\sum_{A}\psi_{A_{i+1},A}(x_{a_i})\,Q^{a_2\ldots a_{i-1}}_{AB}(y_{a_i},\ p_{b_2}) + Q^{A_2}_{0B}(X,\ p_{b_2}). \quad (29)$$

Here, the terms on the right-hand side are associated with the same physical processes as in systems of neutral particles and have a nontrivial difference from (18) in the asymptotic region.

The first term in (29), which corresponds to the initial state of the system, can be expressed in terms of the components of the eigenfunction $\psi_B(x_{b_2})$ and the function χ_{b_2}:

$$\chi_{B_3B}(X,\ p_{b_2}) = \psi_{B_3B}(x_{b_2})\,\chi_{b_2}(y_{b_2},\ p_{b_2}).$$

The asymptotic behavior of this term has been described above.

The functions $Q^{a_2\ldots a_{i-1}}_{AB}$ and $Q^{A_2}_{0B}$, which are associated with processes of rearrangement and disintegration of the clusters, are transformed asymptotically into distorted spherical waves with bounded smooth amplitudes:

$$Q^{a_2\ldots a_{i-1}}_{AB}(y_{a_i},\ p_{b_2}) \sim h^{a_2\ldots a_{i-1}}_{AB}(\hat{y}_{a_i},\ p_{b_2})\,|y_{a_i}|^{\frac{4-3i}{2}} \times \exp\{i\sqrt{E+\varepsilon_A}\,|y_{a_i}| + iW_{AB}(y_{a_i})\}; \quad (30)$$

$$Q^{A_2}_{0B}(X,\ p_{b_2}) \sim h^{A_2}_{0B}(\hat{X},\ p_{b_2})\,|X|^{\frac{4-3N}{2}} \exp\{i\sqrt{E}\,|X| + iW_{0B}(X)\}. \quad (31)$$

The Coulomb phase shifts W_{AB} and W_{0B}, which distort the spherical waves, are given by

$$W_{AB} = -\sum_{\substack{\omega_l,\ \omega_k \\ l\neq k}}\frac{n_{\omega_l\omega_k}}{2\sqrt{E+\varepsilon_A}}\,\frac{|y_{a_i}|}{|y_{\omega_l\omega_k}|}\ln 2\sqrt{E+\varepsilon_A}\,|y_{a_i}|; \quad (32)$$

$$W_{0B}(X) = -\sum_{i\neq j}\frac{\sqrt{2\mu_{ij}}\,q_iq_j}{2\sqrt{E}}\,\frac{|X|}{|y_{ij}|}\ln 2\sqrt{E}\,|X|. \quad (33)$$

The summation in (32) is over the different subsystems of the partition a_l; $y_{\omega_l\omega_k}$ is the reduced relative coordinate of the subsystems ω_1 and ω_k. In (33), the indices i and j label the different particles of the N-particle system; μ_{ij} and y_{ij} are the reduced mass and the reduced relative coordinate of particles i and j; and q_i is the charge of particle i.

As we noted, we can separate the plane and spherical waves in the limit $\xi_{b_2}\to\infty$ in the first term in (29). Therefore, the $B\to B$ elastic scattering amplitude contains two terms—the purely Coulomb amplitude $a^c_{b_2}$ and the amplitude h_{BB} from (30), which takes into account the effective short-range interaction between the clusters:

$$f_{BB} = a^c_{b_2} + h_{BB}.$$

The amplitudes f_{AB} for excitation and rearrangement of the clusters for $2\to 2$ processes are equal to the amplitudes H_{AB} ($A \equiv J_{a_2}$). But the amplitude of $2\to 1$, $1>2$, processes, when the formation of more than two clusters in the final state is possible, are obtained from $h^{a_2\ldots a_{l-1}}_{AB}$ by summing over the components:

$$f_{AB} = \sum_{a_2\ldots a_{l-1}} h^{a_2\ldots a_{l-1}}_{AB},\ A = J_{a_l}; \qquad f_{0B} = \sum_{A_2} h^{A_2}_{0B}.$$

The effective cross sections of these processes can be expressed in terms of the amplitudes by the same expressions as in the case of neutral particles.

It is necessary to mention particularly the nature of the asymptotic conditions in the case of equations of the type (25) that are obtained by taking $\chi_{a_{N-1}} = 0$. If all the particles have the same charge, the asymptotic behavior of the components has the same form as the asymptotic behavior of the analogous components in the case of (23) with $\chi_{a_{N-1}} \neq 0$. This asymptotic behavior is described by the expressions (29)–(33). But if the system contains particles with different charges, it is as yet impossible to say anything definite about the asymptotic behavior of the components (25). As examples show, the asymptotic behavior will depend on the number of differently charged pairs. At the same time, it is necessary to take into account the contribution of "unphysical" channels, which correspond to bound states for the "purely" Coulomb potentials. This asymptotic behavior appears more complicated than even the asymptotic behavior of the total wave function. Therefore, in the case of differently charged particles it is necessary to use only (23) for $\chi_{a_{N-1}} \neq 0$.

Boundary-condition model

We consider one further problem that can be solved in the configuration space—the boundary-condition model. In this model, the interaction between the particles is specified not only by potentials but also by boundary conditions on the wave function. Thus, this problem can be formulated in terms of the coordinate representation. We shall show that its solution can also be obtained in a natural manner in this representation. We begin with the three-body problem.

System of three hard spheres. We shall assume for simplicity that the two-body potentials are equal to zero, and

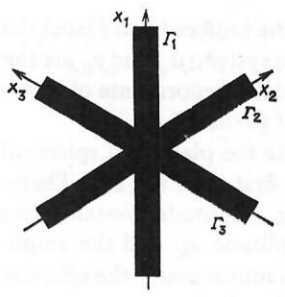

FIG. 2. Configuration space of three identical one-dimensional particles in the boundary-condition model.

consider first the case of zero-value boundary conditions (the model of three hard spheres[27,28]).

In the boundary-condition model, the configuration space Ω consists of points that satisfy the conditions

$$|x_\alpha| \geqslant C_\alpha \quad (\alpha = 1,\ 2,\ 3),$$

where the parameter C_α determines the minimal distance to which the hard spheres can approach each other. The energy operator is determined by the expression

$$Hf(X) = -\Delta f(X) \tag{34}$$

on functions f that satisfy the Dirichlet boundary conditions

$$f|_{\partial\Omega} = 0 \tag{35}$$

on the boundary of Ω. The example of the configuration space for three identical one-dimensional spheres is shown in Fig. 2.

From the point of view of local potentials, a hard core corresponds to an infinitely strong repulsion for $|x_\alpha| \leqslant C_\alpha$. This interpretation of the boundary conditions helps one to understand why the Faddeev equations (4), which were obtained under the assumption of sufficient smoothness of the potentials, cannot be used in this case to formulate the scattering problem. The question of the modification of these equations has been discussed in numerous studies (see, for example, Refs. 29–31).

We shall not dwell here on the analysis of the corresponding approaches. We merely mention that in their framework it has not proved possible to overcome completely the basic difficulties of the problem. In particular, the equations proposed in these studies cannot be used to investigate the singularities of the Green's function or, accordingly, to justify the scattering problem.

In Ref. 16, integral equations of Faddeev type that make it possible to solve these problems were proposed. We describe briefly the basic idea of that paper. In the traditional method of investigating boundary-value problems, the solution is constructed by the methods of potential theory.[32,33] In particular, for the Green's function of the energy operator (34) a representation in the form of the potential of a simple layer is employed. The surface $\partial\Omega$ on which the density of the potential is specified is unbounded. As a result, in contrast to problems with compact boundaries,[32,33] the integral equations of a simple layer are not Fredholm equations. To go over to equations of Fredholm type, one can use Faddeev's method, namely, explicit inversion of the diagonal singular parts of the integral operators. Such reconstructed equations can also be formulated in differential form. This last approach is the one best suited for numerical calculations.

We shall describe the integral Faddeev equations for a simple-layer potential. The Green's function of the energy operator (34) satisfies the equation

$$(-\Delta_X - z)\, R(X,\ X',\ z) = \delta(X - X') \tag{36}$$

for X, X' in Ω and the boundary conditions

$$R(X,\ X',\ z)|_{\partial\Omega} = 0 \tag{37}$$

with respect to each variable X, X'. By means of Green's formula, we can obtain the relation

$$R(X,\ X',\ z) = R_0(X,\ X',\ z) - \int_{\partial\Omega} dS R_0(X,\ S,\ z) \frac{\partial}{\partial n_e} R(S,\ X',\ z), \tag{38}$$

where $R_0(X, X', z)$ is the Green's function of the free Hamiltonian H_0. As follows from this representation, the Green's function $R(X, X', z)$ can be expressed explicitly in terms of the value of its normal derivative on the boundary. This derivative determines the density $\mu(S) = (\partial/\partial n_e) R(S, X', z)$ of a simple-layer potential:

$$U(X,\ z) = \int_{\partial\Omega} dS R_0(X,\ S,\ z)\, \mu(S).$$

We recall that this potential has continuous normal derivatives when the variable X takes a value on the boundary:

$$\lim_{X \to S \mp 0} \frac{\partial U}{\partial n_{i,\,e}} = \mp \frac{1}{2} \mu(S) + \int_{\partial\Omega} \frac{\partial}{\partial n} R_0(S,\ S',\ z)\, \mu(S')\, dS'. \tag{39}$$

Here, the signs $-$ and $+$ correspond to the interior (i) and exterior (e) limits, respectively.

Differentiating Eq. (38) and taking into account (39), we obtain the following integral equation for the density $\mu(S)$:

$$\frac{1}{2} \mu(S) + \int_{\partial\Omega} \frac{\partial}{\partial n} R_0(S,\ S',\ z)\, \mu(S')\, dS' = \frac{\partial}{\partial n} R_0(S,\ X',\ z). \tag{40}$$

As noted above, this is a non-Fredholm equation, the surface $\partial\Omega$ being unbounded. We now go over to equations of Faddeev type for the components of the density.

Faddeev equations for a simple-layer potential. We introduce first some new notation. We denote by Γ_α the hypercylinder $|x_\alpha| = C_\alpha$ in the space R^6. We denote the part of this cylinder that belongs to the boundary $\partial\Omega$ by Γ_α^e and write Γ_α^i for the complement of Γ_α^e in Γ_α. We shall call Γ_α^e and Γ_α^i, respectively, the exterior and interior parts of the cylinder (with respect to Ω). It is clear that the surface $\partial\Omega$ is the union of Γ_α^e, $\partial\Omega = \cup_\alpha \Gamma_\alpha^e$. We shall call the function $\mu_\alpha^e(S)$, defined on Γ_α^e, a component of the density. It is convenient to regard the components as functions defined on the complete hypercylinder. Then by μ_α^e and μ_α^i we denote, respectively, the nontrivial parts of $\mu_\alpha(S)$ defined on the exte-

rior and interior parts of Γ_α, and by P^e_α and P^i_α the operators of multiplication by the characteristic functions of Γ^e_α and Γ^i_α. Then, for example, $P^e_\alpha \mu_\alpha = \mu^e_\alpha$ on Γ^e_α and $P^e_\alpha \mu_\alpha = 0$ on Γ^i_α.

With this notation, Eq. (40) can be rewritten as a system of three equations for the components μ_α:

$$\frac{1}{2} P^e_\alpha \mu_\alpha = P^e_\alpha \vec{V}_\alpha R_0 - P^e_\alpha \vec{V}_\alpha R_0 \sum_{\beta \neq \alpha} P^e_\beta \mu_\beta. \tag{41}$$

We have here used the notation $\vec{V}_\alpha R_0$ for the integral operator with kernel equal to the normal derivative of the kernel R_0:

$$(\vec{V}_\alpha R_0)(S, X', z) = \frac{\partial}{\partial n} R_0(S, X', z), \quad S \in \Gamma_\alpha, \quad X' \in \Omega. \tag{42}$$

On the other hand, this equation can be used to determine the interior part of the component μ^i_α—the operators P^e_α must be replaced by the operators P^i_α. Then the total components will satisfy the system of equations

$$\frac{1}{2} \mu_\alpha = \vec{V}_\alpha R_0 - \vec{V}_\alpha R_0 \sum_{\beta \neq \alpha} P^e_\beta \mu_\beta. \tag{43}$$

Further, in accordance with Faddeev's method we must transfer the diagonal term with $\beta = \alpha$ to the left-hand side and invert the operator $1/2\, I + \vec{V}_\alpha R_0$ which then arises. As a result, we obtain the following Faddeev equations for the simple-layer potential[16]:

$$\mu_\alpha = \vec{V}_\alpha R_0 + \frac{1}{2} \vec{V}_\alpha \rho_\alpha \mu^i_\alpha - \vec{V}_\alpha R_\alpha \sum_{\beta \neq \alpha} \mu^e_\beta. \tag{44}$$

Here, R_α is the Green's function for the energy operator of a system of two spheres in the three-particle space. This operator is determined by the relations (36) and (37), Dirichlet boundary conditions being imposed on the hypercylinder Γ_α; ρ_α is a double-layer density that determines the Green's function $R_\alpha(X, X', z)$ in the form

$$R_\alpha(X, X', z) = R_0(X, X', z) - \int_{\Gamma_\alpha} dS \rho_\alpha(X, S, z) \frac{\partial}{\partial n} R_0(S, X', z). \tag{45}$$

At the same time, both the function ρ_α and the kernel R_α can be explicitly expressed in terms of the two-sphere problem.[16]

The resulting equations can be investigated by the ordinary methods of potential theory.[32,33] In particular, one can show that the matrix integral operator determined by the right-hand side of (44) can be represented as a sum of a completely continuous operator and an operator with norm less than unity.[16] The latter is generated by the neighborhoods of the angles formed by the intersection of the hypercylinders Γ_α. Thus, the Fredholm alternative applies of (44), and by means of these equations it is possible to investigate the properties of the density $\mu(S)$. Finally, knowing the behavior of $\mu(S)$, we can study the properties of the Green's function $R(X, X', z)$ by means of the representation (38). The subsequent procedure for constructing the wave functions and investigating their properties (completeness, orthogonality, asymptotic behavior, etc.) is completely analogous to the procedure developed for the three-particle problem with bounded smooth potentials.[25]

Thus, we have described the integral equations that can serve as a basis for the proper posing of the scattering problem in the system of three hard spheres. In the next stage, we go over to differential equations for the components, which are convenient to use for numerical calculations.

Formalism of generalized potentials. To obtain differential equations for the components, it is convenient to use the formalism of generalized potentials. We briefly describe this approach.

We start with heuristic arguments that provide the basis for the method of generalized potentials. We note first that Eq. (38) can be interpreted as a Lippmann-Schwinger equation with generalized potential $\vec{V}$ that acts on functions $f(X)$ in accordance with the equation

$$\vec{V} f = \delta(\partial\Omega) \frac{\partial}{\partial n_e} f. \tag{46}$$

Here, $\delta(\partial\Omega)\varphi$ is a generalized function, called a simple layer,[34] and $(\partial/\partial n_e)f$ are the limiting values from Ω on $\partial\Omega$ of the normal derivative of f. Note that the generalized function $\delta(\partial\Omega)\varphi$ acts in accordance with the equation

$$(g, \delta(\partial\Omega)\varphi) = \int_{\partial\Omega} dS g(S) \varphi(S).$$

Following this formal analogy, we consider alongside the integral equation a Schrödinger equation with generalized potentials. To make the formal analogy with the three-particle system complete, we shall, as in the derivation of (44), add to the potentials nontrivial terms localized on the surfaces Γ^i_α. As a result, we represent the generalized potential in the form of the sum $\Sigma_\alpha \vec{V}_\alpha$, where each term acts in accordance with (46) on the complete hypercylinder Γ_α. We determine the corresponding energy operator by the expression

$$Hf(X) = -\Delta f(X) + \sum_\alpha \vec{V}_\alpha f(X), \tag{47}$$

where the variable X ranges over the complete space R^6. Thus, we extend the domain of definition of the operator (34) to functions defined both without and within the surface $\partial\Omega$. We note further that in accordance with (39) these functions are to be regarded as continuous together with their derivatives up to the surfaces Γ_α and to admit the existence of discontinuities

$$f^e - f^i, \quad \frac{\partial}{\partial n_e} f - \frac{\partial}{\partial n_i} f$$

on the transition through these surfaces. We denote the set of such functions by $\mathscr{D}$.

On this set, the action of the Laplacian can be understood in the sense of generalized functions as follows[34]:

$$-\Delta f = -\Delta_X f - \sum_\alpha \delta(\Gamma_\alpha) \left(\frac{\partial f}{\partial n_e} - \frac{\partial f}{\partial n_i} \right) + \sum_\alpha \frac{\partial}{\partial n} \delta(\Gamma_\alpha)(f^i - f^e). \tag{48}$$

Here, Δ_X is the Laplacian understood in the usual sense in

the space R^6 with the surfaces Γ_α eliminated; $\delta(\Gamma_\alpha)\varphi$ is a simple layer on the surface Γ_α, and $(\partial/\partial n)(\delta(\Gamma_\alpha)\varphi)$ is a double layer on this surface.

Thus, we have arrived at the conclusion that it is expedient to consider the energy operator (47) defined on the set $\mathscr{D}$ by Eqs. (47) and (48). We now consider what boundary conditions are satisfied for the functions that satisfy the Schrödinger equation with this operator:

$$(-\Delta+\sum_\alpha \vec{V}_\alpha)\,\Psi=E\Psi. \tag{49}$$

For this, we expand the left-hand side of (49) by means of (46) and (48). We find that outside the surfaces Γ_α these functions satisfy the equation

$$(-\Delta - E)\,\Psi = 0. \tag{50}$$

Equating to zero the remaining terms, we arrive at two-sided boundary conditions that must be satisfied on Γ_α:

$$\frac{\partial}{\partial n_i}\Psi|_{\Gamma_\alpha}=0; \tag{51}$$

$$\Psi^i|_{\Gamma_\alpha}=\Psi^e|_{\Gamma_\alpha}. \tag{52}$$

Thus, within the hypercylinders Γ_α the Helmholtz equations (50) with the Neumann boundary conditions (51) must be satisfied. It is necessary to consider the set of such problems corresponding to the possible regions bounded by the intersections of the surfaces Γ_α (see Fig. 2). On the other hand, if the point E does not belong to the discrete spectrum of these interior problems, the corresponding solutions are identically equal to zero, $\Psi^i = 0$. Therefore, it follows from (52) that in the exterior region Ω Eq. (50) with the Dirichlet boundary condition (35) must be satisfied.

Thus, in Ω the solution of the Schrödinger equation (49) with generalized potentials is identical to the solution of the Schrödinger equation for three hard spheres. It is on this result that the method of generalized potentials in the boundary-condition model is based.

We now show how the method of generalized potentials can be treated in the framework of integral equations. We introduce the generalized Green's function $G(X, X', z)$ by means of Eq. (49) with the δ function $\delta(X-X')$ on the right-hand side. As in the case of the function (36), one can show that $G(X, X', z)$ satisfies the Lippmann-Schwinger equation (38), where as the surface of integration in the given case it is necessary to take the set of surfaces Γ_α, i.e., $\partial\Omega\to\cup_\alpha \Gamma_\alpha$. In operator form, this equation is

$$G(z)=R_0(z)-R_0(z)\sum_\alpha \vec{V}_\alpha G(z). \tag{53}$$

As in the case of wave functions, one can show that $G(X, X', z)$ is equal to the Green's function (36) when the variables X, X' lie in Ω (in other cases, the function G is expressed by means of the Green's functions of the interior Neumann problems).

To study the properties of the function G, it is convenient to go over to the equations of potential theory. In this case, in contrast to (40), the density $\mu(S)$ is specified from the very beginning not only on Γ_α^e but also on Γ_α^i. For the components $\mu_\alpha = (\partial/\partial n_e)G|_{\Gamma_\alpha}$ of the density of the simple-layer potential we can obtain in the same way as in the case of the function $R(X, X', z)$ the Faddeev equations

$$\mu_\alpha=\vec{V}_\alpha G_\alpha-\vec{V}_\alpha G_\alpha\sum_{\beta\neq\alpha}\mu_\beta. \tag{54}$$

Here G_α is a generalized Green's function, defined as the solution of Eq. (49) containing the single generalized potential $\vec{V}_\alpha$.

Equations (54) are considered on functions defined on the surfaces Γ_α. They can be rewritten as equations in configuration space by going over to operators $M_\alpha = \vec{V}_\alpha G$, which can be interpreted as generalized components of a T matrix. They are related to the densities by the equations $M_\alpha = \delta(\Gamma_\alpha)\mu_\alpha$. The equations for M_α have the form

$$M_\alpha=\vec{V}_\alpha G_\alpha-\vec{V}_\alpha G_\alpha\sum_{\beta\neq\alpha}M_\alpha. \tag{55}$$

By definition, the solutions of (54) are identical to the components that satisfy Eq. (44) on the exterior part of the cylinders Γ_α. One can show that on the interior parts of Γ_α these solutions are equal to zero. It follows from this in particular that the representations (38) and (53) for the Green's function are equivalent.

Thus, we have described an alternative formalism in the problem of three hard spheres in which the boundary conditions are reproduced by means of generalized potentials. In the framework of this approach, it is easy to include as well an additional interaction specified by means of ordinary potentials. All that needs to be done is to replace in the relations (47)–(55) the generalized potentials $\vec{V}_\alpha$ by the sum $\vec{V}_\alpha + V_\alpha$ of generalized and ordinary potentials.

One can similarly treat the model in which use is made of arbitrary boundary conditions of the third kind:

$$\left(\frac{\partial f}{\partial n_e}+\tau f\right)\Big|_{\partial\Omega}=0. \tag{56}$$

The formal scheme is unchanged. In this case, the generalized potentials are determined by the equation

$$\vec{V}_\alpha f=-\delta(\Gamma_\alpha)\,\tau f^e+\frac{\partial}{\partial n_e}(\delta(\Gamma_\alpha)\,f^e). \tag{57}$$

We now turn to the formulation of the differential equations for the components.

Differential equations for the components. In the formalism of generalized potentials, the components and the differential equations for them are constructed in the same way as in the case of ordinary potentials. One can also consider a combination of interactions, when there are smooth potentials together with boundary conditions. For example, in the three-body problem the components of the wave function are introduced by the equation

$$\Psi_\alpha = -R_0(V_\alpha+\vec{V}_\alpha)\,\Psi.$$

The differential Faddeev equations for these components have the form

$$(-\Delta+V_\alpha+\vec{V}_\alpha-E)\,\Psi_\alpha=-(V_\alpha+\vec{V}_\alpha)\sum_{\beta\neq\alpha}\Psi_\beta. \tag{58}$$

The generalized potentials generate boundary conditions for the components. For example, in the hard-sphere model

these conditions are

$$\left(\frac{\partial\Psi_\alpha}{\partial n_i}+\sum_{\beta\neq\alpha}\frac{\partial\Psi_\beta}{\partial n_e}\right)\Big|_{\Gamma_\alpha}=0,\ \Psi^i_\alpha|_{\Gamma_\alpha}=\Psi^e_\alpha|_{\Gamma_\alpha}. \tag{59}$$

On the other hand, since Eq. (58) is equivalent to the Schrödinger equation (49) with the boundary conditions (51) and (52), these components will also satisfy the simpler conditions

$$\sum_\alpha \Psi_\alpha|_{\Gamma_\alpha}=0.$$

These conditions are sufficient to determine the component Ψ_α in the exterior part of the hypercylinder Γ_α.

To determine the wave functions uniquely, we must also augment Eq. (58) with asymptotic conditions. These have the same form as in the case (5), (6) of ordinary potentials.

Finally, long-range potentials can also be included in this above scheme. For this, it is necessary in the modified differential equations for the components to replace the short-range potentials by a sum of short-range and generalized potentials: $V^{(s)}_\alpha \to V^{(s)}_\alpha + \vec{V}_\alpha$.

N-body problem. We shall restrict ourselves to describing the formalism of generalized potentials and will not discuss questions of its justification. The boundary conditions on the wave function are imposed in this case the hypercylinders $\Gamma_{\alpha_{N-1}}$, which are specified by the condition $|x_{a_{N-1}}| = C_{a_{N-1}}$.

The corresponding boundary-value problem in the exterior of these hypercylinders can be solved by means of the Schrödinger equation with generalized potentials

$$(-\Delta_X+\sum_{a_{N-1}}(V_{a_{N-1}}+\vec{V}_{a_{N-1}})-E)\,\Psi=0.$$

We can go over from the Schrödinger equation to the equations for the components Ψ_{A_2} (17), in which it is necessary to take a sum of ordinary and generalized potentials: $V_{a_{N-1}} \to V_{a_{N-1}} + \vec{V}_{a_{N-1}}$. The generalized potentials generate boundary conditions. For example, in the model of N hard spheres

$$\Psi^i_{A_2}|_{\Gamma_{a_{N-1}}}=\Psi^e_{A_2}|_{\Gamma_{a_{N-1}}},\ \left(\frac{\partial\Psi_{A_2}}{\partial n_i}+\frac{\partial\widetilde{\Psi}_{A_2}}{\partial n_e}\right)\Big|_{\Gamma_{a_{N-1}}}=0,$$

where

$$\widetilde{\Psi}_{A_2}=\sum_{i=3}^{N-1}\sum_{\substack{c_i\neq A_i\\ c_i\subset a_{i-1}}}{}'\Psi_{a_2\ldots a_{i-1}c_i}+\sum_{b_2\neq a_2}\sum_{\substack{B_3\neq A_3\\ b_3\subset a_2}}{}'\Psi_{B_2}.$$

The asymptotic boundary conditions for the components of the wave functions remain the same as in the case (18)–(20) of ordinary potentials.

Finally, we note that the modified equations for the components in the case of long-range potentials can also be used after the formulation of the boundary conditions. It is merely necessary to replace the short-range potentials by the sum $V^{(s)}_{a_{N-1}} + \vec{V}_{a_{N-1}}$.

With this, we conclude the description of the differential formalism in the boundary-condition model. We see that the basic equations considered above retain their form if generalized potentials are used. We shall show below that this approach can be successfully used for numerical calculations.

2. NUCLEON-DEUTERON SCATTERING

Taking the example of the simplest three-particle systems of nuclear physics, *nnp* and *npp*, we shall describe here the application of the differential Faddeev equations to the numerical solution of the scattering problem. We make a partial-wave analysis of these equations and formulate asymptotic conditions for the partial-wave components corresponding to *Nd* scattering processes. We study the problem directly for the more general case of *pd* scattering, for which the system contains not only the nuclear but also the long-range Coulomb interaction.

The *S*-wave Faddeev equations

In the isospin formalism, all particles of the system (*npp*) are assumed to be identical. The wave function of the system is a spinor in the space $\mathscr{H}=\mathscr{H}_s\otimes\mathscr{H}_\tau$, where $\mathscr{H}_s$ and $(\mathscr{H}_\tau)$ are, respectively, the spaces of the spinors that characterize the spin and isospin state of the system. The Faddeev components Ψ_α are also spinors in $\mathscr{H}$. They satisfy the system of modified Faddeev equations (25). In the given case, these are equations in the space $L_2(R^6)\otimes\mathscr{H}$:

$$(-\Delta+V^c+V_\alpha-E)\,\Psi_\alpha=-V_\alpha\sum_{\beta\neq\alpha}\Psi_\beta, \tag{60}$$

where V^c and V_α are the Coulomb and nuclear potentials. They are operators in the space $\mathscr{H}$. For example, the Coulomb potential has the form

$$V^c=\sum_\alpha\frac{n}{|x_\alpha|}\prod_{i\subset\alpha}\frac{1}{2}(1+\tau^z_i),\ n=\frac{m_N e^2}{\hbar^2}, \tag{61}$$

where τ^z_i is the operator of projection of the isospin of particle i from the pair of particles α.

The identity of the particles leads to a simple relationship[9] between the Faddeev components and makes it possible to express the wave function Ψ in terms of one component with some fixed index α:

$$\Psi=(1+\mathscr{P}^+ +\mathscr{P}^-)\,\Psi_\alpha, \tag{62}$$

where $\mathscr{P}^\pm$ are the operators of cyclic permutation of the particles:

$$\mathscr{P}^+\ (123)=(312),\quad \mathscr{P}^-\ (123)=(231). \tag{63}$$

To be specific, we shall assume that pair α in (62) consists of particles 2 and 3. We shall denote all quantities relating to this partition by symbols without indices to label the partition ($\Psi=\Psi_\alpha$, $V=V_\alpha$, $x=x_\alpha$, etc.).

The representation (62) reduces the system (60) to a single equation for the component Ψ:

$$(-\Delta+V^c+V-E)\,\Psi=-V(\mathscr{P}^+ +\mathscr{P}^-)\,\Psi. \tag{64}$$

To reduce this equation to a form suitable for numerical calculations, it is necessary to separate in (64) the spin–isospin and angular variables. To this end, we introduce in

TABLE I. Elements of the spin basis.

Elements of basis	s_{23}	S
χ_0	0	1/2
χ_1	1	1/2
χ_2	1	3/2

the spin space $\mathscr{H}_s$ a basis whose elements are classified by the values of the total spin S of the system and the total spin s_{23} of pair (2, 3). The elements χ_i $(i=0, 1, 2)$ of this basis are characterized in Table I and are described in detail, for example, in Ref. 35. The basis in the isospin space $\mathscr{H}_\tau$ is formed by analogous spinors η_i $(i=0, 1, 2)$. Thus, in the space $\mathscr{H}$ the basis is formed by nine spinors $\chi_i \otimes \eta_k$.

Since the total spin of the system is conserved, the partial-wave analysis can be made independently for $S=1/2$ (doublet) and $S=3/2$ (quartet). For $S=3/2$, the Ψ component is a three-dimensional spinor in the space with basis ${}^4e_i = \chi_2 \otimes \eta_i, i=0, 1, 2$, and for $S=1/2$ it is a spinor in the six-dimensional space with basis ${}^2e_{ik} = \chi_i \otimes \eta_k$, $i=0, 1$, $k=0, 1, 2$.

We make the partial-wave analysis for the simpler quartet case. For this, we first describe the representations of the operators V^c, V, and $\mathscr{P}^{\pm}$ in the basis $\{{}^4e_i\}$.

A simple calculation shows that the representation of the permutation operators is determined by the matrix[35]

$$ {}^4P^{\pm} = \{\langle {}^4e_k \,|\mathscr{P}^{\pm}|\, {}^4e_i\rangle\} = \begin{pmatrix} -1/2 & \mp\sqrt{3}/2 & 0 \\ \pm\sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. $$

The representation of the operator of the Coulomb interaction (61) can also be readily calculated and is given by the matrix[35]

$$ W^c = \{\langle {}^4e_k \,|V^c|\, {}^4e_i\rangle\} = \begin{pmatrix} w_{11} & w_{12} & -\sqrt{2}\,w_{12} \\ w_{12} & w_{22} & w_{23} \\ -\sqrt{2}\,w_{12} & w_{23} & w_{33} \end{pmatrix} $$

with elements

$$ \left.\begin{aligned} &w_{11}=\frac{1}{2}(V_2^c+V_3^c),\ w_{12}=\frac{1}{2\sqrt{3}}(V_2^c-V_3^c); \\ &w_{22}=\frac{2}{3}\left(V_1^c+\frac{1}{2}\,w_{11}\right),\ w_{23}=\frac{\sqrt{2}}{3}(V_1^c-w_{11}); \\ &w_{33}=\frac{1}{3}\sum_{\alpha=1}^{3}V_\alpha^c, \end{aligned}\right\} \quad (65) $$

where V_α^c is the Coulomb potential in the particle pair α:

$$ V_1^c = n/|x|,\ V_2^c(x,\ y) = V_3^c(x,\ -y) = n\left|\frac{x}{2}+\frac{\sqrt{3}\,y}{2}\right|^{-1}. $$

(Here, 1/2 and $\sqrt{3}/2$ are the coefficients of the transition from one system of reduced Jacobi coordinates to another in the case of equal particle masses.)

The dependence of the nuclear interaction on only the total spin of the pair of particles leads to diagonality of the potential V in the basis $\{{}^4e_i\}$: $\langle {}^4e_i \,|V|\, {}^4e_k\rangle = \delta_{ik}\, V_k(x)$. At the same time, $V_k(x)=0$ for $k=1, 2$, since in the singlet–singlet $(s_{23}=\tau_{23}=0)$ and triplet–triplet $(s_{23}=\tau_{13}=1)$ states there is no nuclear interaction.

We now decompose the component Ψ with respect to the spin–isospin basis:

$$ \Psi = \sum_{i=0}^{2} \Psi_i(x,\ y)\, {}^4e_i $$

and project Eq. (64) onto the elements of this basis. As a result, we obtain a system of equations for the coefficients Ψ_i:

$$ \left.\begin{aligned} &(-\Delta+V_0+w_{11}-E)\,\Psi_0+w_{12}(\Psi_1-\sqrt{2}\,\Psi_2)=-V_0\widetilde{\Psi}_0; \\ &(-\Delta+w_{22}-E)\,\Psi_1+w_{12}\Psi_0+w_{23}\Psi_2=0; \\ &(-\Delta+w_{33}-E)\,\Psi_2-\sqrt{2}\,w_{12}\Psi_0+w_{23}\Psi_1=0, \end{aligned}\right\} \quad (66) $$

where

$$ \widetilde{\Psi}_0 = \sum_{i=0}^{2} ({}^4P_{0i}^{+}\mathscr{P}^{+}+{}^4P_{0i}^{-}\mathscr{P}^{-})\,\Psi_i. $$

The next step is the separation of the angular variables in (66). This is done conveniently in the bispherical basis

$$ |\lambda l L\rangle = \sum_{m+\mu=M} \langle lm\lambda\mu \mid LM\rangle\, Y_l^m(\hat{x})\, Y_\lambda^\mu(\hat{y}), \quad (67) $$

where L is the total orbital angular momentum of the system.

The representation of the permutation operators in the basis (67) was calculated in Ref. 9 and is given by

$$ \begin{aligned} &\langle \lambda' l' L' |\ \mathscr{P}^{+}U(|x|,\ |y|)\,|\lambda l L\rangle \\ &\qquad = (-1)^{\lambda+\lambda'}\langle \lambda' l' L' |\ \mathscr{P}^{-}U(|x|,\ |y|)\,|\lambda l L\rangle \\ &\qquad = \frac{1}{2}\,\delta_{LL'}\,(\hat{h}^L_{\lambda' l',\lambda l}U)(|x|,\ |y|), \end{aligned} $$

where $\hat{h}^L_{\lambda' l',\lambda l}$ is an integral operator that in the polar coordinates $\rho^2=x^2+y^2$, $\tan\theta=|y|/|x|$ has the form

$$ \left.\begin{aligned} &(\hat{h}^L_{\lambda' l',\lambda l}U)(\rho,\ \theta)=\int_{\theta^-}^{\theta^+} h^L_{\lambda' l',\lambda l}(\theta,\ \theta')\,U(\rho,\ \theta')\,d\theta', \\ &\theta^+=\frac{\pi}{2}-\left|\theta-\frac{\pi}{6}\right|,\ \theta^-=\left|\theta-\frac{\pi}{3}\right|. \end{aligned}\right\} \quad (68) $$

The kernel of the operator (68) is described in Ref. 9.

The nuclear interaction is diagonal in the bispherical basis:

$$ \langle \lambda l L\ |V_0|\ \lambda' l' L'\rangle = \delta_{LL'}\delta_{\lambda\lambda'}\delta_{ll'}V_0^l(x). $$

For the potentials that act only in the s state, only the matrix element V_0^0 is nonzero. Following the tradition, we denote it by V^t.

Expanding the functions Ψ_i in series in the bispherical harmonics,

$$ \Psi_i = \sum_{\lambda,\, l} \frac{\Psi_i^{\lambda l}(|x|,\ |y|)}{|x|\ |y|}\,|\lambda l L\rangle, \quad (69) $$

and then separating the angular variables in (66), we obtain an infinite system of coupled equations for the partial-wave components $\Psi_i^{\lambda l}$. These equations are coupled by the matrix elements

$$\langle \lambda l L \mid w_{11} \mid \lambda' l' L \rangle, \quad \langle \lambda l L \mid w_{12} \mid \lambda' l' L \rangle \tag{70}$$

of the Coulomb potentials in (65).

It is clear that for the numerical solution of the problem this infinite system of equations must be truncated, i.e., only a few partial-wave terms are taken into account in the expansions (69) of the components. Let us consider, for example, the equations that are obtained when we retain in (69) only the first term with $l = 0, \lambda = L$.

The matrix elements (70) between the functions $|L\,0L\rangle$ are given in polar coordinates by

$$\langle L0L \mid w_{11} \mid L0L \rangle = n \int_{-1}^{1} du\, [x^2 - 2\sqrt{3}\,|x|\,|y|\,u + 3y^2]^{-1/2} = \frac{n\mu(\theta)}{\rho},$$

$$\mu(\theta) = 2/\max(\sqrt{3}\sin\theta,\ \cos\theta);$$

$$\langle L0L \mid w_{12} \mid L0L \rangle = 0.$$

By virtue of the last equation, the system (66) when truncated in this manner decouples into a homogeneous system for Ψ_i^{L0}, $i = 1, 2$, and an independent inhomogeneous equation for Ψ_0^{L0}. Therefore, by the uniqueness theorem of Ref. 12, $\Psi_i^{L0} = 0$ for $i = 1, 2$. The remaining component $\Psi_0^{L0} \equiv \Psi^L$ satisfies the so-called s-wave Faddeev equation[13]:

$$\left(H_{L0} + V^t + \frac{n\mu(\theta)}{\rho} - E\right)\Psi^L = -\frac{1}{2}V^t \hat{h}^L_{L0,\,L0}\Psi^L, \tag{71}$$

where

$$H_{\lambda l} = -\partial^2_{|x|} - \partial^2_{|y|} + \lambda(\lambda+1)\,|y|^{-2} + l(l+1)\,|x|^{-2}. \tag{72}$$

The integral operator $\hat{h}^L_{L0,L0}$ in (71) is determined by Eq. (68). Its kernel can be expressed in terms of Legendre polynomials[9]:

$$h^L_{L0,\,L0}(\theta,\ \theta') = \frac{4L!}{\sqrt{3}}\left(\frac{-\sin\theta}{2\sin\theta'}\right)^L \sum_{k=0}^{L} \frac{P_k(u)\,(\sqrt{3}\cot\theta)^k}{k!\,(L-k)!}; \tag{73}$$

where $u = (\cos 2\theta + 2\cos 2\theta')/\sqrt{3}\sin\theta$.

Thus, as a result of the $l = 0, \lambda = L$ truncation the system of equations (66) is reduced to the single s-wave Faddeev equation (71). Note that in the (nnp) system (when there is no Coulomb potential) this equation is exact for nuclear potentials that act only in the s state. For the (npp) system, the s-wave equation is approximate, since it does not take into account the higher partial-wave components of the Coulomb interaction. However, as the results of calculations show, this approximation describes very accurately processes of pd scattering at not too low energies. According to our estimates, the contribution of the terms that are not taken into account in it does not exceed 1% at energies $E^p_{\text{lab}} \gtrsim 0.5$ MeV, and this figure is at the level of accuracy of the calculations. The reason for this is that at distances of the order of the range of the nuclear forces the Coulomb interaction is much weaker than the nuclear interaction (since the proton reduced charge is small, $n = 0.035\ \text{F}^{-1}$).

However, at very low energies it is necessary to take into account the Coulomb interaction in the higher partial waves.[36] The region of applicability of the s-wave approximation will be discussed in more detail below.

Thus, in the case of quartet scattering the modified Faddeev equation (64) is reduced to the integro-differential equation (71).

For $S = 1/2$, the partial-wave analysis is analogous and leads to the following system of s-wave Faddeev equations[13,35]:

$$\left(H_{L0} + \frac{n\mu(\theta)}{\rho} - E\right)\Psi_0^L = -V^t\tilde{\Psi}_0^L; \tag{74}$$

$$\left(H_{L0} + \frac{2n}{3\,|x|} + \frac{n\mu(\theta)}{\rho} - E\right)\Psi_1^L + \frac{\sqrt{2}\,n}{3}\left(\frac{1}{|x|} - \frac{\mu(\theta)}{\rho}\right)\Psi_2^L = -V^s\tilde{\Psi}_1^L;$$

$$\left(H_{L0} + \frac{n}{2\,|x|} + \frac{2n\mu(\theta)}{3\rho} - E\right)\Psi_2^L + \frac{\sqrt{2}\,n}{3}\left(\frac{1}{|x|} - \frac{\mu(\theta)}{\rho}\right)\Psi_1^L = -V^s\tilde{\Psi}_2^L,$$

the inhomogeneous terms of which are

$$\tilde{\Psi}_i^L = \Psi_i^L + \sum_{k=0}^{2} \hat{h}^L_{L0,\,L0} G_{ik}\Psi_k^L,$$

where the matrix G is

$$G = \begin{pmatrix} 1/4 & -3/4 & 0 \\ -3/4 & 1/4 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}. \tag{75}$$

The components Ψ_i^L correspond to the different spin–isospin states of the system. The first two of them correspond to a doublet with respect to the total isospin, while the component Ψ_2^L reflects the contribution of the quartet component with $T = 3/2$. Note that for the (nnp) system the total isospin is conserved and is equal to 1/2. Accordingly, for $n = 0$ the third equation in (74) "decouples" from the first two, and therefore has the trivial solution $\Psi_2^L = 0$. The nontriviality of the component Ψ_2^L for the (npp) system expresses the violation of the isospin conservation law due to the presence of the electromagnetic interaction.

Asymptotic behavior of partial-wave components

For a unique solution of Eqs. (71) and (74), boundary conditions for the components Ψ_i^L must be specified. The first of them follows from the regularity of the wave function at the origin:

$$\Psi_i^L\big|_{|x|=0} = \Psi_i^L\big|_{|y|=0} = 0. \tag{76}$$

In addition, it is necessary to impose asymptotic conditions at large distances. They are obtained by partial-wave expansion of the asymptotic behaviors (29)–(33) of the Faddeev components in R [6]. For example, in the case of quartet Nd scattering the asymptotic behavior of Ψ^L is

$$\Psi^L \underset{\rho\to\infty}{\sim} \chi_L + {}^4a_L\,\varphi_d(|x|)Q^L(|y|, k) + {}^4A_L(\theta)\,Q_0(\rho,\ \theta) + O(\rho^{-3/2}). \tag{77}$$

Here, φ_d is the deuteron wave function with binding energy $-\varepsilon$, $k^2 = E + \varepsilon$, and χ_L is the partial-wave component of the initial-state wave function:

$$\chi_L = \varphi_d(|x|)\,e^{\,i\Delta^c_L}F_L(\eta,\ k\,|\,y\,|), \quad \Delta^c_L = \arg\Gamma(L+1+i\eta), \tag{78}$$

where $\eta = n/\sqrt{3}k$, and $F_L(\eta, \rho)$ is the Coulomb wave function.[37]

The coefficients Q^L and Q_0 in (77) are outgoing plane and spherical waves,

$$Q^L = \exp\{i k |y| + i W - i\pi L/2\};$$
$$Q_0 = \rho^{-1/2} \exp\{i\sqrt{E}\rho + iW_0\},$$

distorted by the Coulomb phase shifts:

$$W = -\eta \ln 2k|y|, \quad W_0 = -\frac{n}{2\sqrt{E}}\mu(\theta)\ln 2\sqrt{E}\rho.$$

In the case of doublet scattering, to describe the asymptotic behaviors of the components it is convenient to go over in (74) to the representation in which the Coulomb interaction is diagonal.[13,35] This transition is realized by a unitary transformation $\hat{\Psi}^L = U\Psi^L$ with matrix

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & \frac{-1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \end{pmatrix}. \tag{79}$$

The asymptotic behavior of the transformed components is

$$\left.\begin{aligned} &\hat{\Psi}^L_0 \sim \chi_L + {}^2a_L\varphi_d Q^L + \hat{A}^L_0(\theta)\,Q_0; \\ &\hat{\Psi}^L_1 \sim \hat{A}^L_1(\theta)\exp\left\{i\sqrt{E}\rho + \frac{in\sec\theta}{2\sqrt{E}}\ln 2\sqrt{E}\rho\right\}\rho^{-1/2}; \\ &\hat{\Psi}^L_2 \sim \hat{A}^L_2(\theta)\,Q_0. \end{aligned}\right\} \tag{80}$$

All physical characteristics of *Nd* scattering are determined by the coefficients in the asymptotic behaviors (77) and (80). For example, the elastic-scattering phase shifts ${}^{2S+1}\Delta_L = \Delta^c_L + {}^{2S+1}\delta_L$ and the inelasticity coefficients ${}^{2S+1}\eta_L$ are related to ${}^{2S+1}a_L$ by

$${}^{2S+1}a_L = e^{2i\Delta^c_L}\{{}^{2S+1}\eta_L \exp(2i\,{}^{2S+1}\delta_L) - 1\}/2i,$$

where ${}^{2S+1}\delta_L$ is the contribution to the nuclear-interaction phase shift.

The elastic scattering amplitude is given by the partial-wave series

$${}^{2S+1}f(k, \theta) = a^c + k^{-1}\sum_{L=0}^{\infty}(2L+1)^{2S+1}a_L P_L(\cos\theta).$$

Here, a^c is the Coulomb scattering amplitude (27).

By means of the doublet, ${}^2f$, and quartet, ${}^4f$, amplitudes we can obtain the differential cross section for elastic scattering of an unpolarized beam of nucleons by the deuteron:

$$\frac{d\sigma}{d\Omega} = \frac{1}{3}|{}^2f|^2 + \frac{2}{3}|{}^4f|^2. \tag{81}$$

The partial-wave terms ${}^{2S+1}\mathscr{A}^L$ of the disintegration amplitude can be expressed in terms of the amplitudes of the spherical waves in (77) and (80). For $S = 3/2$, they have the form

$${}^4\mathscr{A}^L = \left(1 - \frac{1}{2}\hat{h}^L_{L0,\,L0}\right){}^4A_L.$$

In the doublet case, the amplitude ${}^2\mathscr{A}^L$ is a vector with three components ${}^2\mathscr{A}^L_i$, $i = 0, 1, 2$, these corresponding to the different spin–isospin quantum numbers of the system in the final state. This vector is related to the vector $\hat{A}^L = (\hat{A}^L_0, \hat{A}^L_1, \hat{A}^L_2)$ by

$${}^2\mathscr{A}^L = (1 + \hat{h}^L_{L0,\,L0}G)\,U^{-1}\hat{A}^L, \tag{82}$$

where U and G are the matrices in (75) and (79). The total amplitude of disintegration can be expressed in terms of its components ${}^2\mathscr{A}^L_i$ and ${}^4\mathscr{A}^L$ by a partial-wave series.[9]

Nucleon–deuteron scattering in the boundary-condition model

We now briefly describe the method of solving the *Nd* scattering problem in the boundary-condition model. In this case the Faddeev equations (58) are equivalent to a single equation analogous to (64):

$$(-\Delta + V^c + V + \vec{V} - E)\Psi = -(V + \vec{V})(\mathscr{P}^+ + \mathscr{P}^-)\Psi, \tag{83}$$

where $\vec{V}$ is a generalized potential. Depending on the type of boundary conditions, $\vec{V}$ has the form (46) or (57). This generalized potential generates boundary conditions of the type (59) on the cylinder $|x| = C$, where C is the diameter of the nucleon core.

By a partial-wave analysis, Eq. (83) for $|x| \neq C$ is reduced to ordinary s-wave equations for the partial-wave components (71) and (74). These equations are augmented by additional two-sided boundary conditions on the line $|x| = C$, the conditions being obtained by separating the angular variables in (59).

For example, in the case of a hard core [i.e., in the case of the Dirichlet conditions (35)], the boundary conditions for $S = 3/2$ are[17]:

$$\left.\begin{aligned} &\frac{\partial}{\partial|x|}U_L(|x|, |y|)\,|_{|x|=C-0} = 0; \\ &U_L(C+0, |y|) = U_L(C-0, |y|), \end{aligned}\right\} \tag{84}$$

where

$$U_L = \Psi^L - \frac{1}{2}\hat{h}^L_{L0,\,L0}\Psi^L.$$

We recall that $\hat{h}^L_{L0,L0}$ is the integral operator (68) with the kernel (73). Note that (84) is equivalent to the simpler condition $U_L(C, |y|) = 0$, which it is convenient to use in numerical calculations.

If the two-body interactions are specified by conditions of the third kind, the boundary conditions for the partial-wave components have the form

$$\left[\frac{\partial}{\partial|x|} + \tau^t\right]U_L(|x|, |y|)\,|_{|x|=C+0} = 0, \tag{85}$$
$$U_L(C-0, |y|) = 0,$$

where τ^t and τ^s are, respectively, the parameters of the boundary conditions in the triplet and singlet spin states of the particle pair (2, 3).

Thus, the *Nd* scattering problem in the boundary-condition model is described by the s-wave equations (71) and (74), augmented by the boundary conditions (84) or (85) and the asymptotic behaviors (77) and (80) of the components at large distances.

Numerical method

The algorithm for solving the boundary-value problems formulated above is based on finite-difference approximation of Eqs. (71) and (74) in polar coordinates. Such a method was developed for the first time in Ref. 10 to calculate *nd* scattering in a potential model. In Ref. 13, it was then generalized to the *pd* system, and in Ref. 17 to *nd* scattering in the boundary-condition model. We shall describe the main aspects of the implementation of this method for the example of quartet *Nd* scattering [Eq. (71)] in the potential model.

First of all, in Eq. (71) we must explicitly separate from the component Ψ^L the initial-state wave function χ_L [see (78)]. As a result, (71) is reduced to an inhomogeneous equation for the remainder $\Psi_L = \Psi^L - \chi_L$, which differs from (71) only by the substitution $\Psi^L \to \Psi_L$ and the addition to the right-hand side of an inhomogeneous term:

$$\Phi_L = \frac{n}{\rho}\left(\frac{2}{\sqrt{3}\sin\theta} - \mu(\theta)\right)\chi_L - \frac{1}{2}V^t \hat{h}^L_{L0,\,L0}\chi_L.$$

For the difference approximation of the equation for the function Ψ_L in the quadrant $|x| \geqslant 0$, $|y| \geqslant 0$ we take a grid having N_θ points $\{\theta_i\}$ on the arc $\rho = \text{const}$ and $N_\rho + 1$ points $\{\rho_i\}$ on the ray $\theta = \text{const}$. On a fixed arc $\rho = \rho_i$, the values of the functions Ψ_L and Φ_L at the grid points form vectors $U^{(i)}, I^{(i)} \in R^{N_\theta}$ with components $U^{(i)}_k = \Psi_L(\rho_i, \theta_k)$ and $I^{(i)}_k = \Phi_L(\rho_i, \theta_k)$. The set of N_ρ vectors $U^{(i)}$ and $I^{(i)}$ specifies the values of the functions Ψ_L and Φ_L at the grid points on the first N_ρ arcs, i.e., determines the vectors U, $I \in R^{N_m}$, $N_m = N_\rho N_\theta$:

$$U = \sum_{i=1}^{N_\rho} \oplus U^{(i)}; \quad I = \sum_{i=1}^{N_\rho} \oplus I^{(i)}. \tag{86}$$

In such a representation, the finite-difference form of Eq. (71) is the system of equations

$$\left.\begin{aligned} &L_i U^{(i-1)} + (M_i - E)\,U^{(i)} + R_i U^{(i+1)} = I^{(i)}, \quad i = 1, \ldots, N_\rho, \\ &U^{(0)} = 0. \end{aligned}\right\} \tag{87}$$

Here, L_i, M_i, R_i are matrices of rank N_θ. The matrices L_i and R_i are generated by the radial part of the Laplacian in (71) and are therefore diagonal. The nondiagonal matrix M_i describes the contribution of the spherical part of the Laplacian, the potential, and the integral operator $\hat{h}^L_{L0,L0}$ on the arc $\rho = \rho_i$.

The system (87) can be written in the form

$$(H - E)\,U + R_{N_\rho} U^{(N_\rho+1)} = I, \tag{88}$$

where H is a matrix of rank N_m. It follows from (87) that it is sparse and has a strip structure with strip width $2N_\theta$.

Thus, we have obtained the system (88) of N_m equations for $N_m + N_\theta$ unknowns. The subsidiary relation that selects a unique solution of this system follows from the asymptotic behavior (77). It specifies the connection between the values of the Faddeev component on two successive arcs:

$$U^{(N_\rho+1)} = CU^{(N_\rho)} + {}^4a_L \hat{I}^{(N_\rho)} + O(\rho_{N_\rho}^{-3/2}), \tag{89}$$

where $C = \text{diag}(c_1, \ldots, c_{N_\theta})$ is a diagonal matrix with the elements

$$c_i = Q_0(\rho_{N_\rho+1}, \theta_i)\,Q_0^{-1}(\rho_{N_\rho}, \theta_i),$$

and $\hat{I}^{(N_\rho)} = \Phi^{(N_\rho+1)} - C\Phi^{(N_\rho)}$, where $\Phi^{(i)} \in R^{N_\theta}$ are vectors with components

$$\Phi^{(i)}_j = \varphi_d(\rho_i \cos\theta_j) Q^L(\rho_i \sin\theta_j, k).$$

The condition (89) reduces Eq. (88) to the form

$$\tilde{H}U = I + {}^4a_L \hat{I}, \tag{90}$$

where the matrix $\tilde{H}$ differs from $H - E$ by replacement of the block M_{N_ρ} by $M_{N_\rho} + CR_{N_\rho}$, while the vector $\hat{I} \in R^{N_m}$ has a structure analogous to (86) but only the last component $\hat{I}^{(N_\rho)}$ is nonvanishing.

By virtue of the linearity, the solution of (90) is

$$U = U_0 + {}^4a_L U_1, \tag{91}$$

where the vectors $U_{0,1}$ are determined by the equations

$$\tilde{H}U_0 = I; \quad \tilde{H}U_1 = \hat{I}, \tag{92}$$

in which the inhomogeneous terms are known. Since the matrix $\tilde{H}$ has a strip structure, Gauss's algorithm can be used effectively to solve Eqs. (92).

Having determined the vectors, $U_{0,1}$, we can then, by means of the asymptotic behavior (77), find the elastic-scattering partial-wave amplitude ${}^4a_L$. For this, we compare the representations (77) and (91) in the region $|x| \ll |y|$ $(\theta \to 0)$. In this region, the third term in (77) is much smaller than the second if ρ_{N_ρ} is sufficiently large. Therefore, ignoring the third term in (77) we obtain from (91) the following expression for ${}^4a_L$:

$${}^4a_L = [U_0^{(N_\rho)}]_i (\Phi_i^{(N_\rho)} - [U_1^{(N_\rho)}]_i)^{-1},$$

where the index i corresponds to small angles θ_i.

Having calculated ${}^4a_L$ in this way, we can then, using (91), find the vector $U^{(N_\rho)}$ corresponding to the values of the function Ψ_L on the final arc $\rho = \rho_{N_\rho}$ and then determine the disintegration amplitude:

$${}^4A_L(\theta_i) = (U_i^{(N_\rho)} - {}^4a_L \Phi_i^{(N_\rho)})\,Q_0^{-1}(\rho_{N_\rho}, \theta_i).$$

To increase the accuracy of the calculations, the grid points must be distributed nonuniformly. The grid must be dense in the region adjoining the axis $|x| = 0$ and more dispersed as $|x| \to \infty$. However, it is here important to observe the consistency condition—the correction term of the asymptotic behavior (77) must be much smaller than the discretization error in the neighborhood of the boundary $\rho = \rho_{N_\rho}$, i.e., $(\Delta h)^2 \gg \rho_{N_\rho}^{-3/2}$. Here, Δh is the minimal step in the neighborhood of the boundary.

Results of calculations

In this section, we describe some results of calculations of the characteristics of *Nd* scattering in models with different types of *NN* interaction:

a) The potential model MT-I-III,

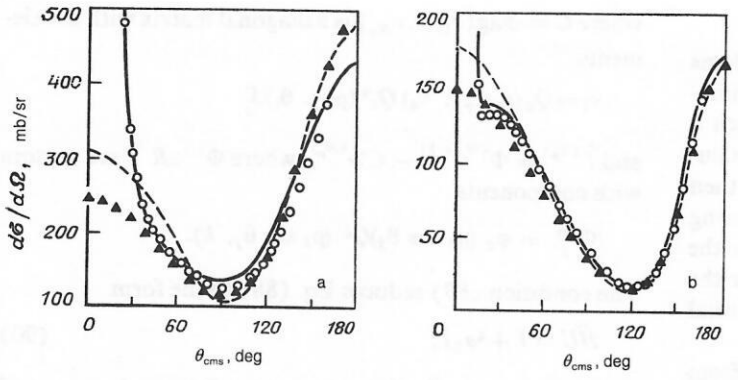

FIG. 3. Differential cross section of elastic *pd* and *nd* scattering at $E_{\text{lab}}^{N} = 2$ MeV (a) and $E_{\text{lab}}^{N} = 10$ MeV (b). The continuous and broken curves represent *pd* and *nd* scattering with the potential MT-I-III; the open circles and black triangles are, respectively, the experimental data for *pd* and *nd* scattering.[41]

$$V^{s,t}(x) = \sum_{k=1}^{2} \lambda_k^{s,t} \frac{\exp(-\mu_k |x|)}{|x|}, \qquad (93)$$

whose parameters are given in Ref. 38.

b) The boundary-condition model for the hard-core Yukawa potential (HCY)

$$V^{s,t}(x) = V_0^{s,t}\left(\frac{R^{s,t}}{|x|}\right)\exp\left(-\frac{|x|}{R^{s,t}}\right), \quad |x| > C$$

and for the hard-core exponential potential (HCE)

$$V^{s,t}(x) = V_0^{s,t} \exp[(C - |x|)/R^{s,t}], \quad |x| > C.$$

The parameters of the HCY and HCE models are given, for example, in Ref. 39.

c) The model with boundary conditions of the third kind without a potential (BCM).

In this model, the parameters $\tau^{s,t}$ of the boundary condition in (85) are uniquely determined by the deuteron binding energy and the energy of the virtual state of the *np* system:

$$\tau^s = -0.04 \text{ F}^{-1}; \quad \tau^t = 0.231 \text{ F}^{-1}.$$

Figure 3 shows the differential cross sections (81) of elastic *Nd* scattering in the MT-I-III model. The discrepancy between the theory and experiment for *nd* scattering at small angles was already noted in the early studies of this subject (for example, Ref. 40) and was attributed to the unrealistic nature of potentials of the type (93).

Figure 4 gives the phase shifts ${}^{2S+1}\delta_0$ of *Nd* scattering for the models MT-I-III and HCY. Figure 5 gives the inelasticity coefficients ${}^{2S+1}\eta_0$ for the potential MT-I-III. This figure clearly illustrates the effect of the Pauli principle in *Nd* scattering. As is well known, the Pauli principle prevents the incident nucleon and the deuteron from approaching too close to each other. Therefore, the disintegration channel $N + d \to N + p + n$ is strongly suppressed for $S = 3/2$. As a result, $1 - {}^{4}\eta_0$, which characterizes the contribution of the disintegration amplitude to the total S matrix, is almost equal to 0 in a fairly wide range of energies above the disintegration threshold. For a similar reason, ${}^{2S+1}\eta_0|_{nd} < {}^{2S+1}\eta_0|_{pd}$, at all energies, since the Coulomb repulsion in the *pd* system also prevents close approach of the colliding particles.

Figure 6 gives the doublet disintegration amplitudes (82) for the potential MT-I-III. These same amplitudes for *nd* scattering in the HCY and BCM models are shown in Fig. 7. The core radius C in the BCM was taken from Ref. 44: $C = 1.095$ F.

To compare the models, Table II gives the phase shifts and inelasticity coefficients of *nd* scattering at a given energy. It can be seen that the MT-I-III, HCY, and HCE models give very similar results. Within reasonable limits, the results in the BCM with core $C = 0.4$ F differ from them. But for core radius $C = 1.4$ F, as proposed in Ref. 45, the BCM results differ strongly from the other calculations and the

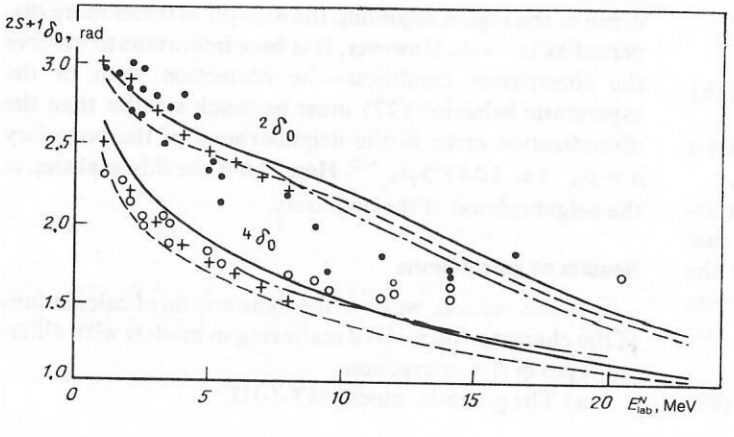

FIG. 4. Doublet and quartet phase shifts ${}^{2S+1}\delta_0$. The continuous and broken curves represent *pd* and *nd* scattering for the potential MT-I-III; the chain curves are for *nd* scattering for the hard-core Yukawa potential; the plus signs give the data of a *pd* experiment,[42] and the open circles and black circles give the data of an *nd* experiment.[43]

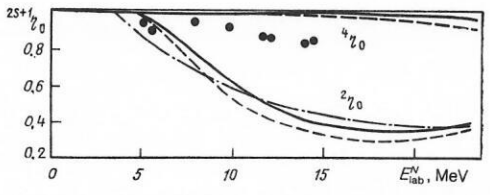

FIG. 5. Doublet and quartet inelasticity coefficients $^{2S+1}\eta_0$. The continuous and broken curves represent *pd* and *nd* scattering for the potential MT-I-III; the chain curve is the theoretical result of Ref. 40, and the points are from the experiment of Ref. 43.

experimental data. Evidently, such a large core radius does not correspond to the physics of the *NN* interaction at low energies.

The *Nd* scattering lengths

The *Nd* scattering lengths are included among the fundamental characteristics of three-nucleon systems. Their calculation can be naturally included in the framework of the differential formalism. We describe briefly the corresponding scheme.

The *Nd* scattering lengths are determined in accordance with the effective-range expansion for the phase shifts $^{2S+1}\delta_0$ of scattering with zero orbital angular momentum, i.e., as the limit at zero energy of the function

$$a_s(k) = -C_0^{-2}(\eta)k^{-1}\tan^{2S+1}\delta_0(k), \qquad (94)$$

where $C_0^2(\eta) = 2\pi\eta/(\exp(2\pi\eta) - 1)$ (in the case of *nd* scattering $\eta = 0$, i.e., $C_0 = 1$).

Thus, solving Eqs. (71) and (74) at several energies near the threshold, we can find the *Nd* scattering lengths $^{2S+1}A_{Nd}$ by extrapolating the function (94) to the point $k = 0$.

The *nd* scattering lengths were calculated on the basis of the differential Faddeev equations in Ref. 46. For the potential MT-I-III, for example, they were found to be $^2A_{nd} = 0.62$ F and $^4A_{nd} = 6.4$ F,[46] in very reasonable agreement with the experimental data. For the same nuclear potential, the *pd* scattering lengths were calculated in Ref. 47:

$$^2A_{pd} = 1.03 \text{ F}, \quad ^4A_{pd} = 11.96 \text{ F}. \qquad (95)$$

For comparison, we give the experimental values of these

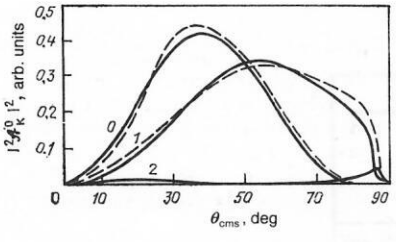

FIG. 6. Squares of the moduli of the disintegration amplitude $^2\mathscr{A}^0_k(\theta)$ for $S = 1/2$, $L = 0$ for the potential MT-I-III. The continuous and broken curves represent the *pd* and *nd* reactions at $E^N_{lab} = 14.1$ MeV. The numbers next to the curves are the values of k.

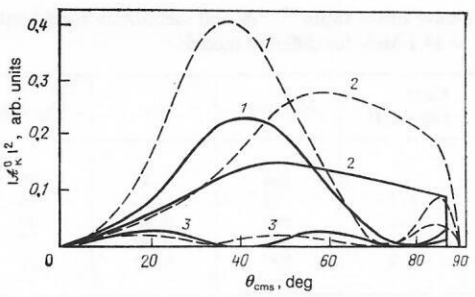

FIG. 7. Squares of the moduli of the doublet ($\mathscr{A}^0_k \equiv {}^2\mathscr{A}^0_{k-1}(\theta)$, $k = 1$, 2) and quartet ($\mathscr{A}^0_3 \equiv {}^4\mathscr{A}^0(\theta)$) disintegration amplitudes for *nd* scattering with $L = 0$ and $E^N_{lab} = 14.1$ MeV. The continuous curves are for the boundary-condition model without a potential, and the broken curves are for the hard-core Yukawa potential. The numbers next to the curves are the values of k.

scattering lengths:

(43): $^2A_{pd} = 1.3 - 0.2$ F, $\quad ^4A_{pd} = 11.4^{+1.8}_{-1.2}$ F

(42): $^2A_{pd} = 2.73 - 0.1$ F, $\quad ^4A_{pd} = 11.86^{+0.4}_{-0.1}$ F.

In Table III, we give the results of calculation of the *pd* scattering lengths by other authors. A two-particle approximation of the original problem was employed in Ref. 48, and therefore its results have the nature of a qualitative estimate. The studies of Refs. 49, 50, and 52 were based on the integral formulation of the Faddeev equations in the momentum space. In them, the nuclear interaction was specified by separable potentials of rank 1.

In Ref. 51, the differential Faddeev equations for the (*npp*) system were also used. However, in contrast to Ref. 47, which was based on rigorous results on the asymptotic behavior of the phase shifts in the limit $k \to 0$, in Ref. 51 the Faddeev equations were solved directly at zero energy. To do this, it was assumed that the components of the wave function with zero energy have the same coordinate asymptotic behavior as in the two-body problem. It should be noted that the doublet scattering length for the potential MT-I-III obtained in Ref. 51 differs appreciably from (95).

Polarization effects in *pd* scattering

As we noted above in the derivation of the *s*-wave Faddeev equations, these equations are approximate for the *pd* system. The approximation actually reduces to the replacement of the Coulomb interaction of the incident proton with the target proton by an interaction of the proton with the deuteron center of mass. The most important consequence of this approximation is that Eqs. (71) and (74) do not take into account the effect of the polarization of the deuteron in the *pd* scattering process.[36] In other words, the Hamiltonian of the *s*-wave equations corresponds to an effective potential V_{eff} of the proton–deuteron interaction, this potential containing asymptotically only exponentially decreasing corrections to the Coulomb interaction.[36] But the Hamiltonian of the original Faddeev equations (60) corresponds to a potential V_{eff} containing the well-known slowly decreasing polarization term:

TABLE II. The s-wave phase shifts ${}^{2S+1}\delta_0$ and inelasticity coefficients ${}^{2S+1}\eta_0$ for nd scattering at $E_{\text{lab}}^n = 14.1$ MeV for different models.

Model	Core radius, F	$2\delta_0$, deg	${}^2\eta_0$	$4\delta_0$, deg	${}^4\eta_0$
MT-I-III	—	104	0.38	69	0.99
BCM	1.4	52	0.88	59	1.01
BCM	0.4	149	0.48	90	0.95
HCY	0.4	103	0.40	77	1.01
HCE	0.4	100	0.40	79	1.01

$$V_{\text{eff}}(y) \underset{y\to\infty}{\sim} \frac{n}{|y|} - \frac{\gamma}{|y|^4} + \ldots, \quad \gamma = \frac{8\alpha}{9}\frac{e^2 m_N}{\hbar^2}, \qquad (96)$$

where α is the deuteron polarizability, $\alpha \simeq 0.6$ F^3.[53]

It is clear that the polarization potential (96) depends essentially on the characteristics of pd scattering at very low energies, when the Coulomb barrier at large distances suppresses the contribution of the short-range nuclear interaction. Therefore, the region of such energies actually determines the limit of applicability of the s-wave Faddeev equations.

At the same time, the polarization effects at such energies lead to invalidity of the effective-range expansion for the phase shifts.[36] As a result, the function (94) in this case does not have a finite limit as $k\to 0$. By the same token, the pd scattering lengths do not, strictly speaking, exist. How then must we understand the scattering lengths (95) corresponding to the s-wave approximation?

To answer this question, we must establish the energies at which the polarization effects begin to play an important part and how they then affect the low-energy characteristics of pd scattering. It is clear that this influence will generate a significant deviation of the phase shift ${}^{2S+1}\delta_0$ calculated by solving the s-wave Faddeev equations from its exact value ${}^{2S+1}\tilde{\delta}_0$. The difference $\delta_p = {}^{2S+1}\tilde{\delta}_0 - {}^{2S+1}\delta_0$ will determine the contribution of the polarization potential to the phase shift.

At low energies, the function $\delta_p(k)$ can be approximated by an explicit expression[54] (see also Ref. 36):

$$\delta_p(k) \underset{k\to 0}{\sim} \gamma k^2 b(\eta), \qquad (97)$$

$$b(t) = 3\pi t^2(1 - \cot \pi t) + 2t + 2t^2 \operatorname{Im}\psi'(1 + it) - 1/3t,$$

where $\psi(z)$ is the logarithmic derivative of the gamma function.

We now consider the function a_S determined by Eq. (94). Without allowance for polarization, it has a finite limit at zero energy, equal to the scattering length ${}^{2S+1}A_{pd}$. But allowance for the polarization effects (${}^{2S+1}\delta_0 \to {}^{2S+1}\delta_0 + \delta_p$) leads, through the asymptotic behavior (97), to exponential growth of the function a_S:

$$a_S(k) \underset{k\to 0}{\sim} -\frac{\gamma}{15\pi\eta} k^2 \eta^{-3} e^{2\pi\eta}.$$

The dependence of the function $a_{1/2}$ on the proton laboratory energy is shown in Fig. 8. The polarization effects radically change the behavior of the function $a_{1/2}$ at superlow energies $E_{\text{lab}}^p \lesssim 10$ keV. At higher energies, the contribution of the polarization potential is negligibly small. As a result, the value of the function $a_{1/2}$ is stabilized in a wide range of energies. The quartet function $a_{3/2}$ behaves in exactly the same way.

From this it is clear that the pd scattering lengths are to be understood as the value of the function a_S at not too low energies $E_{\text{lab}}^p \gtrsim 15$ keV. It is this region of energies that determines the limit of applicability of the s-wave Faddeev equations.

3. BOUND STATES OF THREE-PARTICLE SYSTEMS

One of the important properties of the differential Faddeev equations is that they provide a unified approach for solving both scattering problems and problems of the bound states of few-particle systems. In this section, we use the differential formalism to calculate the characteristics of some three-particle systems differing very greatly in the nature of the interaction and the scale of energies. We consider the ^{3}H and ^{3}He nuclei, an atomic system with a purely Coulomb interaction—the positronium ion ($e^-e^+e^-$), and also three-quark systems—baryons in the spin-parity multiplets $J^P = 1/2^+, 3/2^+$.

TABLE III. Scattering lengths for pd scattering.

Reference	$2A_{pd}$, F	$4A_{pd}$, F
[48]	1.8±0.4	10.9
[49]	—	13.3
[50]	2,0	13.0
[51]	0,15	13.8
[52]	−0.68±0.17	14.0±0.2

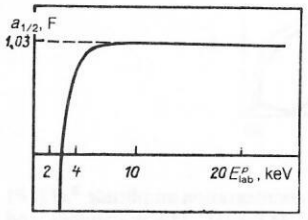

FIG. 8. Dependence of the function $a_{1/2}$ on the proton laboratory energy near the threshold.

TABLE IV. Binding energy of the ^{3}H nucleus and Coulomb energy ΔE_c of the ^{3}He nucleus for different models. For the experiment: $E(^3\text{H}) = 8.48$ MeV, $\Delta E_c = 0.764$ MeV.

Model	Reference	Core radius, F	$E(^3\text{H})$, MeV	ΔE_c, MeV
MT-I-III	*	—	8.58	0.670
	[56]	—	8.55	0.667
	[57]	—	8.5	—
	[38]	—	8.3	—
BCM	*	1.095	3.21	—
	[58]	1.095	4.05	—
HCY	*	0.4	9.46	—

The ^{3}H and ^{3}He nuclei

The ground state of the tritium and helium nuclei is characterized by zero total orbital angular momentum ($L = 0$) and total spin $S = 1/2$. The wave functions of these nuclei, as in the problem of *Nd* scattering, can be decomposed into three components Ψ_α, which satisfy the modified Faddeev equations (60). By means of partial-wave analysis, these equations can be reduced to the doublet *s*-wave equations (74) with $L = 0$. The only difference between the three-nucleon bound-state problem and the *Nd* scattering problem is that Eq. (74) is augmented by a different asymptotic condition for the partial-wave components Ψ_i^0 at large distances. Instead of (90), we must impose on them the condition of decrease

$$\Psi_i^0(\rho, \theta) \underset{\rho\to\infty}{\to} 0, \qquad i = 0, 1, 2, \tag{98}$$

which in conjunction with the regularity requirement (76) determines a unique solution of (74).

If the interaction of the nucleons is specified by the boundary-condition model, (74) must also be augmented by boundary conditions of the type (84) or (85).

For the numerical calculation of the characteristics of the helium and tritium nuclei, a finite-difference approximation of (74) can be used. It is implemented in exactly the same way as in the *Nd* scattering problem (see Sec. 2). The condition of decrease (98) makes it possible to truncate the system of algebraic equations of the type (88) at a fairly large $\rho_{\max} = \rho_{N_\rho+1}$, i.e., to set $U^{(N_\rho+1)} = 0$ in (88). We then obtain a matrix eigenvalue problem:

$$(H - E)\, U = 0.$$

TABLE V. Three-boson binding energy in the BCM with parameters $C = 1.095$ F, $\tau^{s,t} = 0.231$ F^{-1}.

Reference	E, MeV
*	10.3
[44]	12.69
[29]	7.70
[58]	8.88

We recall that the matrix H is obtained by discretizing the Hamiltonian of (74), while the vector U specifies the values of the wave-function components at the points of the polar grid. The structure of the matrix H and the vector U is described in Sec. 2.

Thus, the bound-state problem for three nucleons reduces to determination of the eigenvalues and eigenvectors of the matrix H. It can be solved by any of the well-known methods of matrix algebra. In our calculations, we used the inverse iteration method.[55] The effectiveness of this method is due to the strip structure of the matrix H, which means that its inversion at each step of the method requires comparatively few computational resources.

Table IV gives the results of some calculations of the binding energy $-E(^3\text{H})$ of the tritium nucleus and the Coulomb energy $\Delta E_c = E(^3\text{H}) - E(^3\text{He})$ of the helium nucleus in the various models described in Sec. 2. Our result is identified by the asterisk. The differential Faddeev equations were also used in Refs. 56 and 57. The integral equations for the components was solved in Ref. 58 by means of an iterative method based on the construction of Padé approximants.

From the point of view of the analysis of a model of boundary conditions of the third kind, it is also of interest to calculate the binding energy of three bosons possessing nucleon masses in the BCM. For such a system, separation of the angular variables reduces the original differential Faddeev equations (58) to a single equation on a plane analogous to Eq. (71). This equation is augmented by the boundary conditions (85). Solving the resulting eigenvalue problem, we obtained the binding energy given in Table V. Also given there are the results obtained by other methods.

The positronium ion $e^-e^+e^-$

We consider the ground state of the $e^-e^+e^-$ system, which is characterized by zero total orbital angular momentum ($L = 0$) and total electron spin equal to 1. To be specific, we label the particles in such a way that pair $\alpha = 1$ contains the two electrons and the pairs $\alpha = 2$ and 3 contain an electron and a positron. In the standard manner [see (2)], we then decompose the wave function Ψ of the system into the three Faddeev components Ψ_α. On account of the Pauli principle, the coordinate part of the wave function must be

symmetric with respect to interchange of the electrons:

$$\Psi(r_1, r_2, r_3) = Q\Psi(r_1, r_2, r_3) = \Psi(r_1, r_3, r_2),$$

where Q is the operator that interchanges particles 2 and 3. Therefore, the components Ψ_α must satisfy the conditions

$$\Psi_1 = Q\Psi_1; \quad \Psi_3 = Q\Psi_2. \tag{99}$$

The relations (99) make it possible to reduce the Faddeev equations (3) to two equations for the components $\Psi_{1,2}$:

$$\left.\begin{aligned}(-\Delta + V_1 - E)\,\Psi_1 &= -V_1(1+Q)\,\Psi_2;\\ (-\Delta + V_2 - E)\,\Psi_2 &= -V_2(\Psi_1 + Q\Psi_2),\end{aligned}\right\} \tag{100}$$

where V_α is the Coulomb interaction in pair α.

To reduce Eq. (100) to a form suitable for numerical calculations, we must separate the angular variables. To this end, we expand the components $\Psi_{1,2}$ with respect to the hyperspherical basis (67). Since $L = 0$, only the term in these expansions with $\lambda = 1$ are nonzero. At the same time, by virtue of the symmetry condition (99), only terms with even l must occur in the partial-wave expansion (69) of the component Ψ_1.

When the angular variables have been separated in (100), we obtain an infinite system of integro-differential equations for the partial-wave components $\Phi^l_\alpha \equiv \Psi^{ll}_\alpha$:

$$\left.\begin{aligned}(H_{ll} + V_1 - E)\,\Phi^l_1 &= -V_1 \sum_k \hat{h}^0_{ll,\,kk}\Phi^k_2;\\ (H_{kk} + V_2 - E)\,\Phi^k_2 &= -\frac{1}{2}V_2\Big\{\sum_l (-1)^k\,\hat{h}^0_{kk,\,ll}\Phi^l_1\\ &+ \sum_m (-1)^m\,\hat{h}^0_{kk,\,mm}\Phi^m_2\Big\};\\ (l &= 0, 2, 4\ldots; \quad k, m = 0, 1, 2\ldots,\end{aligned}\right\} \tag{101}$$

where the operators $H_{\lambda l}$ are determined in (72), and $\hat{h}^L_{\lambda l,\lambda' l'}$ are the integral operators (68).

A unique solution of the system (101) is fixed by boundary conditions of the type (76) and (98).

In numerical calculations, the system (101) must be truncated, i.e., partial-wave components Φ^l_α with only the first few values of l are taken into account in it. The results of our calculations showed that to achieve an error in the calculations of order 1% it is sufficient to retain in (101) only the three equations for the components Φ^0_1, Φ^0_2, and Φ^1_2. The kernels of the corresponding integral operators $\hat{h}^0_{\lambda l,\lambda' l'}$ in these equations have the form

$$h^0_{00,\,00} = \frac{4}{\sqrt{3}}, \quad h^0_{11,\,00}(\theta, \theta') = 4P_1(u);$$

$$h^0_{00,\,11}(\theta, \theta') = \frac{2}{\sin 2\theta'}\{\sqrt{3}\cos 2\theta - P_1(u)\sin 2\theta\};$$

$$h^0_{11,\,11}(\theta, \theta') = \frac{2}{\sqrt{3}\sin 2\theta'}\{\sin 2\theta + 3\sqrt{3}P_1(u)\cos 2\theta + 2P_2(u)\sin 2\theta\},$$

where P_k are Legendre polynomials, and the parameter u is defined in (73).

Solving numerically the system (101) by the method described briefly at the beginning of this section, we can calculate the binding energy $-E$ and the wave function of the $e^-e^+e^-$ system, and also all its physically interesting characteristics, for example, the width of two-photon annihilation

$$\Gamma_{2\gamma} = 2\pi\alpha^4\left(\frac{c}{a_0}\right)\left[1 - \alpha\left(\frac{17}{\pi} - \frac{19\pi}{12}\right)\right]\frac{\langle\Psi\,|\,\delta(r_1 - r_3)\,|\,\Psi\rangle}{\langle\Psi\,|\,\Psi\rangle},$$

where α is the fine-structure constant, and a_0 is the Bohr radius.

TABLE VI. Binding energy and width of two-photon annihilation of the positronium ion. The experimental value: $\Gamma_{2\gamma} = 2.09 \pm 0.09$ nsec^{-1} (Ref. 59).

Reference	E, a.e.	$\Gamma_{2\gamma}$, nsec^{-1}
*	0.267	2.17
[60]	0.26451	—
[61]	0.26208	—
[62]	0.2620045	2.0908
[63]	0.2620045	2.0861

Our values of E and $\Gamma_{2\gamma}$ are identified in Table VI by the asterisk. Also given there are the results of other authors. Integral Faddeev equations in which the two-particle t matrix was approximated by a series of Sturm functions were used in Ref. 60. The adiabatic technique of two-center expansions was used in Ref. 61. In the variational calculations of Refs. 62 and 63, a Hylleraas basis with $N = 125$ and $N = 250$ terms, respectively, was used.

Baryons in the nonrelativistic quark model

One of the important applications of the differential Faddeev equations is to calculations of the static characteristics of baryons in the nonrelativistic quark model (NRQM).[18,19] The use of equations based on the original dynamical formulation of the problem makes it possible, in particular, to separate the difficulties associated with the technique of three-particle calculations from the restrictions inherent in the NRQM itself. Omitting a discussion of these restrictions, which are mainly associated with the extrapolation of the NRQM to the region of quarks of light flavors, we analyze briefly the most popular methods of solution of the spectral problem for the three-quark energy operator.

The main shortcoming of the harmonic-oscillator model (HOM) is the systematic use of oscillator wave functions despite the fact that the quark–quark potential at large distances is evidently not quadratic.[64] Another difficulty of the HOM is the need to eliminate the defects of the wave functions and the extreme degeneracy of the spectrum by introducing into the model anharmonic corrections,[65] which lead to a complicated technique of decouplings and mixings. As a result, the connection between the properties of the baryons and the dynamics on which the model is based is lost.

The alternative variational method of Ref. 66 is fairly effective, especially for baryons constructed from quarks of different masses, because the asymmetry of the system can be parametrized in advance. However, it is difficult to estimate the convergence of the variational calculations, since the control is realized only for one fixed parametrization of the trial functions.

Finally, the method of hyperspherical expansions,[64,67] which can be used for calculations in the NRQM, has, as is well known, rapid convergence only for qq interactions that are not too singular at short interquark distances. The convergence of the method is already much less good for potentials $V_{qq}(x)$ possessing a Coulomb singularity as $|x|\to 0$. This circumstance greatly restricts the class of potentials for which the method is effective.

The Faddeev method in the configuration space is free of the shortcomings listed above; it is a convenient tool for investigating the bound-state problem in three-quark systems with two-body or three-body interactions of any type. A further advantage of the Faddeev method is the fact that the Faddeev components are smoother functions of the coordinates than the total wave function. This feature of the differential Faddeev equations is decisive for the attainment of a high accuracy of numerical solution of the three-quark spectral problem on a computer.

In the present section, we describe some results obtained by solving the Faddeev equations for baryons of the spin–parity multiplets $J^P = 1/2^+, 3/2^+$ under the assumption $V_{qq} = 1/2\, V_{\bar{q}q}$. Omitting here a discussion of the validity of this last relation,[67,68] we consider some of the most frequently employed potentials: the Bhaduri potentials,[66] the Richardson potential,[69] and the Martin potential.[70] Besides the mass spectrum, we also calculate structural parameters of the baryons which characterize the "quality" of the wave functions: the charge radii, the electromagnetic form factors, and the quark distribution functions.

For simplicity, we consider baryons consisting of quarks of equal masses, and introduce the operators $\mathscr{P}^{\pm}$ of cyclic permutation (63) of the particles. Defining the operator $\mathscr{P} = \mathscr{P}^+ + \mathscr{P}^-$, $\mathscr{P}^* = \mathscr{P}$, we write the Faddeev equations in the form

$$(-\Delta + V_\alpha - E)\,\Psi_\alpha = -V_\alpha \mathscr{P} \Psi_\alpha, \quad \alpha = 1, 2, 3, \quad (102)$$

where V_α and $\mathscr{P}$ are operators in the complementary spin–isospin–color space $\mathscr{H} = \mathscr{H}_s \otimes \mathscr{H}_\tau \otimes \mathscr{H}_c$. The components $\Psi_\alpha \in \mathscr{H}$ are related to each other by the operators $\mathscr{P}^{\pm}$ and generate the total wave function Ψ in accordance with (62).

Note that the operator $H = -\Delta + V_\alpha + V_\alpha \mathscr{P}$ is not symmetric, since the operators $\mathscr{P}$ and V_α do not commute. However, the readily verified relation $\mathscr{P} H = H^* \mathscr{P}$ makes it possible to show that the eigenvalues of the operator H are real.

We fix the coordinate system in R^6, choosing the Jacobi coordinates x_α, y_α corresponding to pair $\alpha = 1$, and introduce the operator Q, $Q(123) = (132)$; $Q\mathscr{P}^{\pm} = \mathscr{P}^{\pm} Q$. In accordance with the Pauli principle, the total wave function Ψ is antisymmetric and, therefore, after separation of the color degrees of freedom the component $U \equiv \Psi_1$ must be symmetric with respect to interchange of the quarks of pair $\alpha = 1$:

$$QU = U. \quad (103)$$

We shall denote the Jacobi coordinates in pair $\alpha = 1$ by x, y, and the corresponding orbital angular momenta by l and λ: $l \oplus \lambda = L$. The total spin S and isospin T take the values 1/2 and 3/2. The total orbital angular momentum L is combined with the total spin S into the total angular momentum $J = L \oplus S$. The conservation of the spin S in the strong interactions makes it possible to consider the multiplets $J^P = 1/2^+, 3/2^+$ separately. On the other hand, the parity $P = (-1)^{\lambda + l}$ of the state imposes a restriction on the possible values of l and λ. For example, for the multiplet $J^P = 1/2^+$ we obtain from (103) the condition

$$\langle {}^2e_{ik} \mid Q \mid {}^2e_{ik} \rangle = (-1)^l, \quad (104)$$

where $\{{}^2e_{ik}\}$ is the basis of spin–isospin states of the system that we described in Sec. 2.

Using the decomposition of the component U with respect to the orthogonal basis $\{{}^2e_{ik} \otimes |\lambda l L\rangle\}^J$ [see (67)], we obtain for the partial-wave components $U^{\lambda l}_{ik}$ corresponding to the baryons of the $J^P = 1/2^+$ multiplet the system of integro-differential equations

$$(H_{\lambda l} + V^{qq}_{ik} - E)\, U^{\lambda l}_{ik} = -V^{qq}_{ik} \sum_{i'k',\, l'\lambda'} \hat{h}^L_{\lambda l,\, \lambda' l'} P^+_{ik,\, i'k'} U^{\lambda' l'}_{i'k'}, \quad (105)$$

where the operator $H_{\lambda l}$ is determined in (72),

$$V^{qq}_{ik} = \langle {}^2e_{ik} \mid V_{qq} \mid {}^2e_{ik} \rangle;\ P_{ik,\, i'k'} = \langle {}^2e_{ik} \mid \mathscr{P}^+ \mid {}^2e_{i'k'} \rangle.$$

and $\hat{h}^L_{\lambda l, \lambda' l'}$ is the integral operator (72).

In numerical calculations, the system (105) must be truncated with respect to the angular momenta at certain $l = l_0$, $\lambda = \lambda_0$. For s-wave baryons, $L = 0$ and, therefore, $\lambda = l$. The minimal truncation of the system (105) corre-

TABLE VII. Baryon mass spectrum (GeV) for different models of the quark–quark potential.

Baryon, J^P	Model					
	M		BI		BII	
	*	[73]	*	[66]	*	[66]
Δ, $3/2^+$	1.241	—	1.229	1.231	1.205	1.234
N, $1/2^+$	1.241	—	0.896	0.939	0.912	0.939
Ω, $3/2^+$	1.617	1.621	1.694	1.695	1.655	1.668
Ω^*, $3/2^+$	2.112	2.139				
ccc, $3/2^+$	4.770	4.776				
ccc^*, $3/2^+$	5.237	5.254				
bbb, $3/2^+$	14.323	14.326				

TABLE VIII. Structural characteristics of baryons for different models of the quark–quark potential.

Characteristics	RE	RM	BII
$m_{u,d}$, MeV	282	405	336
m_s, MeV	616	642	582
Λ, MeV	448	335	—
m_N, MeV	939	939	912
m_Δ, MeV	1232	1232	1205
m_Ω, MeV	1672	1672	1655
$\langle r^2_{ch}\rangle^{1/2}_p$, F	0.469	0.511	0.620
$\langle r^2_{ch}\rangle_n$, F^2	−0.021	−0.022	−0.029
v^2/c^2	1.93	0.59	0.50

sponds to retention of only the components U^{00}_{ik}. In the $J^P = 1/2^+$ multiplet, the condition (104) entails $U^{00}_{ik} = 0$ for $i \neq k$, $k < 2$; in addition, the component U^{00}_{12}, corresponding to $T = 3/2$, vanishes. Thus, the system (105) reduces to a system of two equations for the components U^{00}_{00}, U^{00}_{11}. Similar equations can be obtained for the baryons of spin and parity $J^P = 3/2$.

A unique solution of the system (105) is fixed by the boundary conditions of regularity (76) and decrease (98) of the partial-wave components.

The system of equations (105) was solved numerically by the method of finite-difference approximations for the Bhaduri potentials I, II, (BI, BII), the Martin potential (M), and the Richardson potential (R). The parameters of the potentials are described in Refs. 66, 69, and 70; we fitted the parameters of the Richardson potential by means of the N, Δ, and Ω^- masses. We shall discuss in more detail the Richardson model:

$$V(x) = \frac{4\pi}{(33-2n_f)}\Lambda\left(\Lambda\,|\,x\,| - \frac{f(\Lambda\,|\,x\,|)}{\Lambda\,|\,x\,|}\right), \qquad (106)$$

where

$$f(t) = \frac{4}{\pi}\int_0^\infty dq\,\frac{\sin qt}{q}\left\{\frac{1}{\ln(1+q^2)} - \frac{1}{q^2}\right\},$$

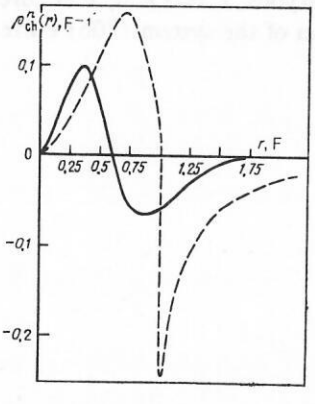

FIG. 9. Density of the charge distribution in the neutron. The continuous curve corresponds to the potential BII in the nonrelativistic quark model, and the broken curve to the result of the bag model of Ref. 74.

and n_f is the number of flavors. We choose $n_f = 3$ (see Ref. 71), and there then remains the single adjustable parameter Λ (in Richardson's original study of Ref. 69, $\Lambda = 398$ MeV). In accordance with (106), we can make the decomposition

$$V_s = a\,|\,x\,|,\quad V_v = V - V_s,$$

where V_v is the fourth component of the vector potential, and V_s is the scalar part of the potential.

In accordance with this decomposition, we determine two types of spin–spin interaction: "electric" V^{ss}_v and "magnetic" V^{ss}_s:

$$V^{ss}_{v,s} = \frac{2}{3m_q^2}(\vec{S}_i\vec{S}_j)\,\Delta\begin{cases} V_v; \\ V_v + V_s. \end{cases}$$

The two types of spin–spin interaction generate, respectively, two Richardson models; the first is the RE model

$$V^e_{qq} = V + V^{ss}_v + V_0$$

and the second is the RM model

$$V^m_{qq} = V + V^{ss}_s + V_0,$$

where $V_0 = -(2/3)\sqrt{2\pi}/3\,\Delta e^{1/2-\gamma}$; γ is Euler's constant.[72]

With the potentials listed above, we calculated the mass spectrum of some s-wave baryons (Table VII). Our results

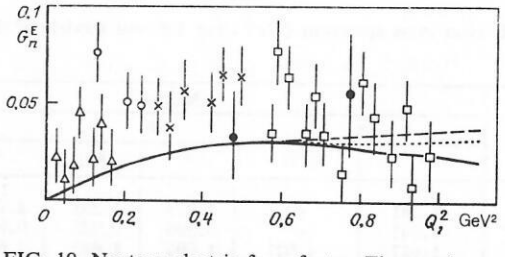

FIG. 10. Neutron electric form factor. The continuous, broken, and dotted curves represent the potentials BII, RE, and RM in the nonrelativistic quark model; the open triangles, open circles, crosses, open squares, and black circles are data from the experiments of Refs. 75 and 76.

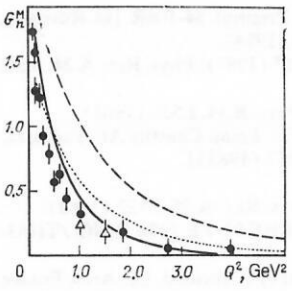

FIG. 11. Neutron magnetic form factor. The continuous, broken, and dotted curves correspond to the potentials BII, RE, and RM; the black circles and open triangles represent the data of the experiments of Refs. 76 and 77.

are in satisfactory agreement with the calculations of other authors. It should be emphasized that for the potential BI the three-particle problem for the baryons Δ and Ω^- can be solved exactly. Thus, these baryons served for us as "bench marks" in the model BI. The practically exact coincidence of our numerical results with the exact solution indicates the high efficiency of the Faddeev method and makes it possible to estimate the error of our calculations at not above 0.5%.

The wave functions obtained as a result make it possible to calculate characteristics of the baryons that are more differentiated than the mass spectrum: the charge radii and $\beta^2 = (v/c)^2$ for the neutron and proton (Table VIII), and also the charge distribution in the neutron (Fig. 9). For comparison, Fig. 9 also gives the charge-distribution density in the neutron calculated in the bag model.[74] With the known charge densities of the nucleon, we obtain the electromagnetic form factors of the neutron (Figs. 10 and 11) and proton (Fig. 12). For small momentum transfers $Q^2 \lesssim 1$–2 GeV^2, the results of the calculations are in satisfactory agreement with the experimental data.

We emphasize that in the present section we have investigated only the s-wave baryons and some of their characteristics. It would clearly also be interesting to investigate the effects of including the pion field in the original version of the NRQM, to calculate the amplitudes of the radiative transition $\Delta \to N\gamma$ and weak transition $\Lambda \to pe\nu$, to take into account the effects of the inclusion of three-particle and tensor forces, etc. All these effects can be calculated with guaranteed accuracy on the basis of the differential Faddeev equations.

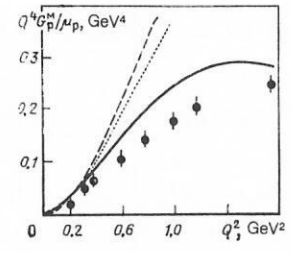

FIG. 12. Proton magnetic form factor. The continuous, broken, and dotted curves correspond to the potentials BII, RE, and RM. The experimental points are from Ref. 78.

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