# Fundamentals of the relativistic theory of gravitation

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An extended exposition of the relativistic theory of gravitation (RTG) proposed by Logunov, Vlasov, and Mestvirishvili is presented. The RTG was constructed uniquely on the basis of the relativity principle and the geometrization principle by regarding the gravitational field as a physical field in the spirit of Faraday and Maxwell possessing energy, momentum, and spins 2 and 0. In the theory, conservation laws for the energy, momentum, and angular momentum for the matter and gravitational field taken together are strictly satisfied. The theory explains all the existing gravitational experiments. When the evolution of the universe is analyzed, the theory leads to the conclusion that the universe is infinite and flat, and it is predicted to contain a large amount of hidden mass. This missing mass exceeds by almost 40 times the amount of matter currently observed in the universe. The RTG predicts that gravitational collapse, which for a comoving observer occurs after a finite proper time, does not lead to infinite compression of matter but is halted at a certain finite density of the collapsing body. Therefore, according to the RTG there cannot be any objects in nature in which the gravitational contraction of matter to infinite density occurs, i.e., there are no black holes.

## INTRODUCTION

The present paper is devoted to an exposition of the fundamentals of the relativistic theory of gravitation (RTG) constructed in the papers of Ref. 1. Before we present the fundamentals of the theory, we briefly discuss some fundamental propositions of the general theory of relativity (GR).

When he created the general theory of relativity, Einstein took as his point of departure the principle of the equivalence of inertial and gravitational forces. He formulated this principle as follows (Ref. 2, p. 423): "...for an infinitesimally small region, coordinates can always be chosen in such a way that there will be no gravitational field in the region." In formulating the equivalence principle, Einstein already departed from the concept of the gravitational field as a Faraday-Maxwell field. This was later reflected in his introduction of the pseudotensorial characteristic $\tau_p^l$ of the gravitational field. Later, Schrödinger[4] showed that for an appropriate choice of the coordinate system all the components of the energy-momentum pseudotensor $\tau_p^l$ of the gravitational field outside a sphere vanish. In this connection, Einstein wrote (Ref. 2, p. 627): "With regard to Schrödinger's arguments, their conviction derives from the analogy with electrodynamics, in which the strengths and energy density of any field are nonzero. However, I can find no reason why it should be the same for gravitational fields. Gravitational fields can be specified without introducing strengths and an energy density." It can be seen from this that Einstein deliberately gave up the concept of the gravitational field as a physical Faraday-Maxwell field, since this field, as a material substance, cannot be eliminated by any choice of the frame of reference.

Since the concept of an energy–momentum-tensor density for the gravitational field is absent in GR, it is not possible in that theory to introduce a conservation law for the energy and momentum of the matter and the gravitational field taken together. It was Hilbert who first emphasized this circumstance. He wrote[5]: "I assert...that for the general theory of relativity, i.e., in the case of general invariance of the Hamilton function, energy equations that...correspond to the energy equations in orthogonally invariant theories do not exist at all, and I could even take this circumstance as a characteristic feature of the general theory of relativity." Some authors still do not understand this; others do and regard it as the most important fundamental step taken by the general theory of relativity, overthrowing concepts such as energy. The abandonment of the concepts of energy and momentum density of the gravitational field leads in GR to the impossibility of localization of the gravitational field energy. But the absence of such localization and of conservation laws leads to the absence of the concept of gravitational waves and a flux of gravitational radiation. This means that the transport of gravitational energy in space from one body to another is impossible.

According to the philosophy of GR, the relativity principle is invalid for gravitational phenomena. It was in this central point, almost 70 years ago, that Einstein and Hilbert, constructing GR, took the fundamental step from the special theory of relativity, leading to the rejection of conservation laws for energy, momentum, and angular momentum, and also to the appearance of unphysical concepts such as the nonlocalizability of gravitational energy and much else having no bearing on gravitation. These two great scientists abandoned the remarkable simplicity of Minkowski space, which possesses the maximal (10-parameter) group of motion of space and entered the jungle of Riemannian geometry, into which subsequent generations of physicists interested in gravitation have been inveigled.

Thus, in accepting GR, we must give up a fundamental principle—the energy-momentum conservation law for the matter and the gravitational field—and also the concept of a classical field. But this is a very great loss, and we should be

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irresponsible if we were to agree to it without the necessary experimental foundations. There is but one way out—to give up GR.

In Refs. 6–11 it was shown that since GR does not and cannot have energy-momentum conservation laws for the matter and the gravitational field taken together, the inertial mass defined in Einstein's theory does not have physical meaning, the flux of gravitational radiation, as defined in GR, can always be annihilated by an appropriate choice of an admissible frame of reference, and therefore Einstein's quadrupole formula for radiation of the gravitational field is not a consequence of GR. In principle, it does not follow from GR that a binary system loses energy through gravitational radiation. General relativity does not have the classical Newtonian limit, and therefore it does not satisfy one of the most fundamental principles of physics—the correspondence principle. This then is the consequence of the absence in GR of energy–momentum conservation laws; this is what one finds by eschewing dogmatism and seriously pondering the essence of the problem and making a detailed analysis.

All this shows that GR cannot be a physical theory, since from the point of view of physics it is logically contradictory, and therefore it also leads to a number of conclusions contradicting experience. On the other hand, it does have something attractive about it—the concept of the Riemannian geometry of space–time. The problem of constructing a theory of gravitation on the basis of field notions in the spirit of Faraday and Maxwell using an effective Riemannian space–time and satisfying all the requirements imposed on a physical theory is an urgent problem.

Our theory, in contrast to GR, is based on the relativity principle, which was advanced by Henri Poincaré as a universal principle for all physical processes and formulated as follows[12]: "The laws of physical phenomena will be the same for both an observer at rest and an observer in a state of uniform translational motion, so that we do not have and we cannot have any means of establishing whether we are in such motion or not."

In such a formulation, it would appear that the relativity principle cannot be applied to accelerated frames of reference. Moreover, Einstein asserted that in this case it would be necessary to go over to general relativity. However, this is not correct. As was shown in Ref. 13 (p. 126), the discovery by Poincaré and Minkowski of the pseudo-Euclidean geometry of space–time makes it possible to formulate a generalized relativity principle: "...whatever physical frame of reference we choose (inertial or noninertial), it is always possible to find an infinite set of other frames of reference such that in them all physical phenomena take place in the same way as in the original frame of reference, so that we do not have and we cannot have any experimental possibilities of distinguishing in which one of this infinite set of frames of reference we are." This means that for the description of physical phenomena in Minkowski space we can, depending on the physical problem, choose any appropriate frame of reference adequate for the problem and, therefore, specify a corresponding metric tensor $\gamma_{lp}$ of Minkowski space. Why did Einstein not understand this? This is evidently explained by the fact that he interpreted the theory of relativity only through the postulate of the constancy of the velocity of light in Galilean coordinates and identified accelerated frames of reference with gravitation on the basis of the equivalence principle.

Our theory is based on the idea of the gravitational field as a physical field in the spirit of Faraday and Maxwell—possessing energy, momentum, and angular momentum. Thus, the gravitational field is analogous to all other physical fields and is characterized by its own energy–momentum tensor of the system. We regard the gravitational field as a physical field with spins 2 and 0, the asymptotically free gravitational field having helicities $\pm 2$. The space–time geometry for all physical fields is pseudo-Euclidean (Minkowski space).

Thus, the conservation laws for energy, momentum, and angular momentum hold strictly for a closed system. This is another fundamental difference between our theory and Einstein's.

Another important question which arises in the construction of a theory of gravitation is that of the interaction of the gravitational field with matter. The gravitational field, as we now believe, is universal: It acts in the same manner on all forms of matter. We base our theory on the geometrization principle,[13,14] according to which the equations of motion of matter under the influence of the tensor gravitational field $\Phi^{ik}$ in Minkowski space with metric tensor $\gamma^{ik}$ can be represented identically as the equations of motion of the matter in an effective Riemannian space–time with metric tensor $g^{ik}$ that depends on the gravitational field $\Phi^{ik}$ and the metric tensor $\gamma^{ik}$. In this manner, we introduce the notion of an effective Riemannian space of a field nature. This force space is created in the RTG with strict observation of the conservation laws and arises because of the presence of the gravitational field and a definite universal nature of its action on matter. The curvature of this dynamical Riemannian space, as a secondary space, arises by virtue of the geometrization principle and is a consequence of the action of the gravitational field. Of course, this force Riemannian space will not in the general case have any group of motions.

On the basis of Minkowski space and the geometrization principle the Lagrangian density can be represented in the form

$$L = L_g(\tilde{\gamma}^{ik},\ \tilde{\Phi}^{ik}) + L_M(\tilde{g}^{ik},\ \Phi_A),$$

where $\tilde{\Phi}^{ik} = \sqrt{-\gamma}\Phi^{ik}$ is the tensor density of the field variable $\Phi^{ik}$ of the gravitational field, $\tilde{g}^{ik} = \sqrt{-g}g^{ik}$ is the density of the metric tensor $g^{ik}$ of the Riemannian space, $\tilde{\gamma}^{ik} = \sqrt{-\gamma}\gamma^{ik}$ is the metric-tensor density of the Minkowski space, and $\Phi_A$ are the matter fields.

In the RTG, the Lagrangian density $L_g$ of the gravitational field depends on the metric tensor $\gamma^{ik}$ and the gravitational field $\Phi^{ik}$, and therefore it differs in principle from GR, in which the Lagrangian density depends only on the metric tensor $g^{ik}$ of the Riemannian space. Thus, the Lagrangian density of the gravitational field is not fully geometrized in our theory, whereas in GR it is.

As will be shown below, the concept of the gravitational field as a field possessing an energy-momentum density and spins 2 and 0 makes it possible, in conjunction with the geometrization principle, to construct the relativistic theory of gravitation (RTG) uniquely. Such a theory changes the ideas about space and time formed under the influence of GR, leads us out of the jungle of Riemannian geometry, and in spirit corresponds to modern theories in the physics of elementary particles. As a consequence of our theory, Einstein's general principle of relativity is devoid of physical meaning and does not have any content.[15] In the exposition of a number of problems below, we follow Ref. 11.

## 1. CRITICAL REMARKS ABOUT THE EQUIVALENCE PRINCIPLE

In the Introduction, we drew attention to the logical presuppositions that must necessarily lead (and do indeed lead) to a number of difficulties in general relativity. These questions were discussed earlier in detail in Refs. 4–11 and 15. In this section, we intend, following Refs. 6–11, to consider some of them and prove the inability of GR to overcome these difficulties.

We begin with a discussion of the equivalence principle. In the scientific literature, there is still no common opinion about the content of the equivalence principle and the part which it plays in GR. Some regard it as the basis of GR, while others note its restricted nature. Einstein himself in the first stage in the creation of his theory used as a heuristic argument the analogy between fields of inertial forces and a gravitational field. It is true that in their effect on the mechanical motion of bodies these fields have much in common: The motion of bodies under the influence of a gravitational field is indistinguishable from their motion in an appropriately chosen noninertial frame of reference; in both fields, the acceleration of bodies does not depend on their mass and composition. This last circumstance gave Einstein the ground for asserting the exact equality of the passive gravitational and inertial masses of bodies, and also stimulated him to the formulation of the equivalence principle.

He wrote (Ref. 2, p. 227): "The theory presented here arose on the basis of the conviction that the proportionality of the inertial and gravitational masses is an exact law of nature that must find its reflection already in the bases of theoretical physics. I attempted to express this conviction in a number of earlier papers, in which I attempted to reduce gravitational mass to inertial mass; this attempt led me to the conjecture that a gravitational field (homogeneous in an infinitesimally small volume) can physically always be completely replaced by an accelerated frame of reference. This hypothesis can be formulated in a perspicuous manner as follows: An observer in a closed box cannot in any way establish whether the box is at rest in a static gravitational field or is in a space free of gravitational fields but moves with an acceleration due to forces applied to the box (equivalence hypothesis)."

Thus, from Einstein's point of view the only difference between fields of inertial forces and the gravitational field is in the different external cause producing them: The former are a consequence of the noninertiality of the frame of reference used by the observer, whereas material bodies are the source of the latter. However, in Einstein's opinion, these fields have an equivalent influence on all physical processes, and therefore in other respects they are indistinguishable. This assertion, in its turn, created the illusion of the possibility of eliminating the influence of a gravitational field on all physical phenomena, by analogy with the annihilation of fields of inertial forces, by a transformation of the space–time coordinates.

Characteristic in this sense is Pauli's statement (Ref. 16, p. 204): "Originally, the equivalence principle was established only for homogeneous gravitational fields. In the general case, it can be formulated as follows: For an infinitesimally small region of the four-dimensional world (i.e., for a region so small that spatial and temporal variations of the force of gravity in it can be ignored) there always exists a coordinate system $K_0(X_1, X_2, X_3, X_4)$ such that in it the force of gravity affects neither the motion of a material point nor any other physical processes. Putting it briefly, in an infinitesimally small region of the world any gravitational field can be annihilated by means of a coordinate transformation." A similar assertion can be found in Einstein's writings (Ref. 2, p. 423): "...for an infinitesimally small region, the coordinates can always be chosen in such a way that there will be no gravitational field in it. One can then assume that in such an infinitesimally small region the special theory of relativity holds. In this manner the general theory of relativity is related to the special theory of relativity, and the results of the latter can be transferred to the former." Subsequently, these erroneous assertions migrated almost unchanged into a number of textbooks. However, inertial forces and gravitational forces are completely different in their nature, since the curvature tensor for the former is identically equal to zero and for the latter is nonzero. Therefore, the influence of the former on all physical processes can be completely eliminated in the whole of space (globally) by going over to an inertial frame of reference, whereas the influence of the latter can be eliminated only in local regions of space and not for all physical processes but only for the simplest—those for which the curvature of space–time does not occur in the equations.

Therefore, on the one hand, the equivalence principle for processes with the participation of particles with higher spins is incorrect, since the curvature tensor occurs explicitly in the equations for these fields. On the other hand, the equivalence principle is also not applicable to extended bodies having a size sufficiently great for the deviation of the geodesics corresponding to the extreme points of the body to be manifested. Since the deviation equation contains the curvature tensor, inertial forces and gravitational forces will also not be equivalent for the mechanical motion of an extended body.

The merit of having played the main part in clarifying these questions is due to Eddington (Ref. 17, p. 74), who pointed out that "the equivalence principle played a large part in the construction of the general theory of relativity, but now that we have developed a new view of the nature of the world it has become less necessary...it is essentially a hypothesis that must be tested experimentally every time that is possible. In addition, this principle must be regarded as a guess rather than a dogma that admits no exceptions. It is possible that some phenomena are determined by comparatively simple equations that do not contain the components of the curvature of the world; these equations have the same form for flat and curved regions of the world. It is to such equations that the equivalence principle applies." However, one cannot assert the complete equivalence in the description of physical phenomena in a gravitational field and in a noninertial frame of reference of pseudo-Euclidean space–time, since "...there exist more complicated phenomena satisfying equations containing the components of the world curvature. The terms containing these components will be absent in the equations describing experiments made in flat regions; but on the transition to the general case these terms must be recovered anew. Obviously, there must exist such phenomena that permit one to distinguish a flat world from a curved one; otherwise, we could know nothing about the curvature of the world. To these phenomena, the equivalence principle does not apply."

Thus, the equivalence principle, understood as the possibility of eliminating the gravitational field in an infinitesimally small region, is not correct, since space–time curvature, if it exists, cannot be eliminated by any choice of the coordinate system, even to a specified accuracy. In addition, a gravitational field and fields of inertial forces do not have the same influence on all physical processes.

It should be noted that Einstein subsequently reviewed his point of view about the equivalence principle and no longer asserted complete equivalence of fields of inertial forces and the gravitational field, pointing out that fields of inertial forces (noninertial systems) are only a special case of gravitational fields satisfying the Riemann conditions $R^{i}_{nml} = 0$. He wrote (Ref. 3, p. 661): "There exists a special case of space whose physical structure (the field) we can assume to be known exactly, on the basis of the special theory of relativity. This is the case of flat space, in which there are neither electromagnetic fields nor matter. It is completely determined by its 'metric' property: let $dx_0, dy_0, dz_0, dt_0$ be the coordinate differences of infinitesimally close points (events); then

$$ds^2 = dx_0^2 + dy_0^2 + dz_0^2 - dt_0^2 \tag{1.1}$$

can be measured and its value does not depend on the concrete choice of the inertial system. If in this space we introduce new coordinates $x_1, x_2, x_3, x_4$ by a transformation of general form, then $ds^2$ for the same pair of points will have the form

$$ds^2 = g_{ik} dx^i dx^k \tag{1.2}$$

(here, summation over $i$ and $k$ from 1 to 4 is understood), and $g_{ik} = g_{ki}$. Then the quantities $g_{ik}$, which form a 'symmetric tensor' and are continuous functions of $x_1, ..., x_4$, describe, in accordance with the 'equivalence principle,' a special case of the gravitational field [namely, a field that one can again transform to the form (1.1)]. If we use Riemann's studies on metric spaces, the properties of a field $g_{ik}$ of such a kind can be exactly characterized (by 'Riemann's condition').

"However, we seek conditions that gravitational fields of 'general' form satisfy. It is natural to assume that they can also be described as tensor fields of the type $g_{ik}$ which do not, in general, admit transformation of the line element to the form (1.1), i.e., satisfy not Riemann's condition but weaker conditions, also, like Riemann's condition, independent of the choice of the coordinates (i.e., invariant with respect to a transformation of general form). Simple formal considerations lead to weaker conditions, which are intimately related to Riemann's condition. These conditions are the required equations for a purely gravitational field (in the absence of matter and electromagnetic fields)."

Thus, *Einstein changed the physical meaning of the equivalence principle, although this circumstance appears to have remained unremarked for many.*

However, in the period of the creation of general relativity Einstein was entirely guided by the equivalence principle in its original formulation, which, therefore, played a heuristic role in the construction of the theory (Ref. 2, p. 400): "The entire theory arose on the basis of the conviction that in a gravitational field all physical processes take place in exactly the same way as without a gravitational field but in an appropriately accelerated (three-dimensional) coordinate system ('equivalence hypothesis')."

Since at that time it was known, by virtue of Minkowski's discovery, that different (and in the general case nondiagonal) space–time metrics correspond to different frames of reference, Einstein and Grossman (Ref. 2, p. 399) concluded that the field variable for the gravitational field must be assumed to be the metric tensor of a Riemannian space–time, and it must be determined by the distribution and motion of matter.

Thus, there arose the idea of a connection between the space–time geometry and matter.

On the basis of these considerations, Einstein and Grossman attempted, purely intuitively, to establish the form of the equations connecting the components of the metric tensor of the Riemannian space–time to the matter energy–momentum tensor. After long unsuccessful attempts, such equations were found by Einstein at the end of 1915.

Since these equations were also obtained somewhat earlier by the mathematician Hilbert, on the basis of variational principles, we shall call them the Hilbert-Einstein equations.

It must here be particularly emphasized that the metric tensor of the Riemannian space cannot characterize the gravitational field, since its asymptotic behavior depends on the arbitrariness in the choice of the three-dimensional (spatial) coordinate system. It is precisely here that we have the sources of the delusion that for many decades has been a brake on the construction of a theory of gravitation.

## 2. ENERGY-MOMENTUM PSEUDOTENSORS OF THE GRAVITATIONAL FIELD IN GENERAL RELATIVITY

Einstein assumed that in general relativity the gravitational field together with the matter must possess a conservation law (Ref. 2, p. 299): "...it must certainly be required that the matter and energy together satisfy the momentum and energy conservation laws." In Einstein's opinion, this problem was completely solved on the basis of the "conservation laws" that use an energy-momentum pseudotensor as the energy-momentum characteristic of the gravitational field.

To obtain such conservation laws, one usually[18] proceeds as follows. If the Hilbert-Einstein equations are written in the form

$$-\frac{c^4}{8\pi G}g\left[R^{ik}-\frac{1}{2}g^{ik}R\right]=-gT^{ik}, \qquad (1)$$

where $g=\det g_{ik}$, $R^{ik}$ is the Ricci tensor, and $T^{ik}$, is the matter energy-momentum tensor, the left-hand side can be represented identically as a sum of two noncovariant quantities:

$$-\frac{c^4}{8\pi G}g\left[R^{ik}-\frac{1}{2}g^{ik}R\right]=\frac{\partial}{\partial x^l}h^{ikl}+g\tau^{ik}, \qquad (2)$$

where $\tau^{ik}=\tau^{ki}$ is the energy-momentum pseudotensor of the gravitational field, and $h^{ikl}=-h^{ilk}$ is the spin pseudotensor.

Using the identity (2), we can rewrite the Hilbert-Einstein equations (1) in a different, equivalent form:

$$-g(T^{ik}+\tau^{ik})=\frac{\partial}{\partial x^l}h^{ikl}. \qquad (3)$$

By virtue of the obvious equation

$$\frac{\partial}{\partial x^k\,\partial x^l}h^{ikl}=0 \qquad (4)$$

a differential conservation law follows from the Hilbert-Einstein equations (3):

$$\frac{\partial}{\partial x^k}[-g(T^{ik}+\tau^{ik})]=0, \qquad (5)$$

and from the formal point of view this is analogous to the energy-momentum conservation law in electrodynamics.

In accordance with this analogy, the "energy flux" of gravitational radiation through an infinitesimal area $dS_\alpha$ in GR is given by the expression

$$dI=c(-g)\tau^{0\alpha}dS_\alpha.$$

Choosing as a surface of integration a sphere of radius $r(dS_\alpha=-r^2n_\alpha d\Omega)$, we obtain the "intensity of the gravitational radiation" in the element of solid angle $d\Omega$:

$$\frac{\partial I}{\partial\Omega}=-cr^2(-g)\tau^{0\alpha}n_\alpha. \qquad (6)$$

In GR, the relation (5) is also used to obtain integral "energy-momentum conservation laws" for a system consisting of matter and the gravitational field. For this, one usually[2,18] integrates the expression (5) over a certain volume and assumes that there are no fluxes of matter through the surface bounding the volume of integration:

$$\frac{d}{dt}\int(-g)[T^{0i}+\tau^{0i}]\,dV=-\oint(-g)\,\tau^{\alpha i}\,dS_\alpha. \qquad (7)$$

Einstein (Ref. 2, p. 645) assumed that the right-hand side of this relation for $i=0$ "certainly represents the energy loss of the material system":

$$-\frac{dE}{dt}=\oint(-g)\,\tau^{0\alpha}\,dS_\alpha. \qquad (8)$$

In the absence of "energy-momentum fluxes" of the gravitational field through the surface bounding the volume of integration, an energy-momentum conservation law for the system is obtained from the expression (7):

$$P^i=\frac{1}{c}\int(-g)[T^{0i}+\tau^{0i}]\,dV=\text{const}. \qquad (9)$$

By means of the Hilbert-Einstein equations (3), the relation (9) can be rewritten as

$$P^i=\frac{1}{c}\oint h^{0i\alpha}\,dS_\alpha=\text{const}. \qquad (10)$$

In Einstein's opinion (Ref. 2, p. 652), the four quantities $P^i$ represent the energy ($i=0$) and the momentum ($i=1,2,3$) of the physical system. Moreover, it is usually asserted (Ref. 18, p. 362) that "the quantities $P^i$—the 4-momentum of the field and matter—have a quite definite meaning, being independent of the choice of the frame of reference to precisely the degree that is needed on the basis of physical considerations." However, as we shall show below, this assertion is incorrect.

On the basis of such a definition of the energy and momentum of a system consisting of matter and the gravitational field, the concept of the inertial mass $m_i$ of a system is introduced in GR:

$$m_i=\frac{1}{c}P^0=\frac{1}{c^2}\int(-g)[T^{00}+\tau^{00}]\,dV=\frac{1}{c^2}\oint h^{00\alpha}\,dS_\alpha. \qquad (11)$$

Expressions analogous to (5)–(11) can also be obtained by writing the Hilbert-Einstein equations in mixed components:

$$\sqrt{-g}\,[T_i^n+\tau_i^n]=\partial_m\sigma_i^{mn}.$$

To a large degree, the choice of the energy-momentum pseudotensors of the gravitational field depended on the inclinations of the authors and, as a rule, was made on the basis of secondary properties. Thus, choosing $h^{ikl}$ in the form

$$h^{ikl}=\frac{c^4}{16\pi G}\frac{\partial}{\partial x^m}[-g(g^{ik}g^{ml}-g^{il}g^{mk})], \qquad (12)$$

we obtain the symmetric Landau-Lifshitz pseudotensor, which contains only first derivatives of the metric tensor:

$$\begin{aligned}\tau^{ik}=\frac{c^4}{16\pi G}\{&(2\Gamma^n_{ml}\Gamma^p_{np}-\Gamma^n_{lp}\Gamma^p_{mn}-\Gamma^n_{nl}\Gamma^p_{mp})(g^{il}g^{mk}-g^{ik}g^{ml})\\&+g^{il}g^{mn}(\Gamma^k_{lp}\Gamma^p_{mn}+\Gamma^k_{mn}\Gamma^p_{lp}-\Gamma^k_{np}\Gamma^p_{ml}-\Gamma^k_{ml}\Gamma^p_{np})\\&+g^{kl}g^{mn}(\Gamma^i_{pl}\Gamma^p_{mn}+\Gamma^i_{mn}\Gamma^p_{pl}-\Gamma^i_{np}\Gamma^p_{ml}-\Gamma^i_{ml}\Gamma^p_{np})\\&+g^{ml}g^{np}(\Gamma^i_{nl}\Gamma^k_{mp}-\Gamma^i_{ml}\Gamma^k_{np})\},\end{aligned} \qquad (13)$$

where

$$\Gamma^k_{mn}=\frac{1}{2}g^{kp}(\partial_m g_{pn}+\partial_n g_{pm}-\partial_p g_{mn}).$$

Choosing

$$\sigma_k^{ni}=\frac{c^4 g_{km}}{6\pi G\sqrt{-g}}\cdot\frac{\partial}{\partial x^l}[-g(g^{mi}g^{nl}-g^{mn}g^{il})], \qquad (14)$$

we arrive at Einstein's pseudotensor,

$$\begin{aligned}\tau_k^i=\frac{c^4\sqrt{-g}}{16\pi G}\{&-2\Gamma^i_{ml}\Gamma^l_{kp}g^{mp}+\Gamma^l_{ml}\Gamma^m_{kp}g^{ip}+\Gamma^l_{lm}\Gamma^i_{kp}g^{mp}\\&+\Gamma^l_{kl}\Gamma^i_{mp}g^{mp}-\Gamma^l_{kl}\Gamma^p_{mp}g^{mi}-\delta^i_k[g^{mp}\Gamma^l_{mp}\Gamma^n_{ln}-g^{nl}\Gamma^p_{ml}\Gamma^m_{pn}]\},\end{aligned} \qquad (15)$$

which is identical to the canonical energy-momentum (pseudo)tensor obtained from the noncovariant Lagrangian density of the gravitational field:

$$L_g = \sqrt{-g}\, g^{li} [\Gamma^n_{pi}\Gamma^p_{nl} - \Gamma^n_{li}\Gamma^p{}_p].$$

For

$$\sigma^{ni}_h = \frac{c^4\sqrt{-g}}{16\pi G} g^{mi} g^{nl} [\partial_l g_{hm} - \partial_m g_{hl}] \quad (16)$$

we obtain Lorentz's pseudotensor

$$\tau^i_h = \frac{c^4\sqrt{-g}}{16\pi G} [\partial_h \Gamma^l_{pl} g^{pi} - \partial_h \Gamma^i_{mp} g^{mp} - \delta^i_h R], \quad (17)$$

which is identical to the canonical energy-momentum (pseudo)tensor obtained on the basis of the noncovariant method of infinitesimal displacements from the covariant Lagrangian density $L_g = \sqrt{-g}R$ of the gravitational field.

We shall study the properties of the "energy-momentum" quantities introduced in Einstein's theory for examples of the determination of the "inertial mass" of a spherically symmetric source. For definiteness, all calculations will be made using the symmetric Landau-Lifshitz pseudotensor (13).

## 3. INERTIAL MASS IN GENERAL RELATIVITY

The equality of the inertial and gravitational mass of a given body was regarded by Einstein as an exact law of nature that must find reflection in his theory. At the present time, it is assumed to be proven that in GR the gravitational mass of a system consisting of matter and the gravitational field is equal to its inertial mass. Such an assertion can be found in the works of Einstein,[2] Tolman,[19] and Weyl.[20] Subsequently, a "proof" of this theorem with various modifications was given by a number of other authors.[18,21,22]

However, this conclusion is incorrect. Following Ref. 9, we shall show where it is erroneous.

The gravitational mass $M$ of an arbitrary physical system at rest as a whole with respect to a Schwarzschild coordinate system Galilean at infinity was defined by Einstein (Ref. 2, p. 660) as the coefficient of the term $-2G/(c^2r)$ in the asymptotic expression ($r\to\infty$) for the component $g_{00}$ of the metric tensor of the Riemannian space-time:

$$g_{00} = 1 - \frac{2G}{c^2 r} M.$$

A somewhat different definition of the gravitational mass was given by Tolman[19]:

$$M = \frac{c^2}{4\pi G} \int R^0_0 \sqrt{-g}\, dV. \quad (18)$$

It follows directly from these definitions that the gravitational mass is not affected by transformations of the three-dimensional coordinates, since both the component $R^0_0$ of the Ricci tensor and the component $g_{00}$ of the metric tensor transform as scalars.

In the case of a static spherically symmetric source, these definitions are equivalent. We now show that they are equivalent for all static systems. To see this, we write the component $R^0_0$ in the form

$$R^0_0 = g^{0i}\left[\frac{\partial}{\partial x^l}\Gamma^l_{0i} - \frac{\partial}{\partial x^0}\Gamma^p_{pi} + \Gamma^n_{0i}\Gamma^p_{np} - \Gamma^n_{pi}\Gamma^p_{n0}\right].$$

After identical transformations, we obtain from this expression

$$R^0_0 = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}[\sqrt{-g}\, g^{0n}\Gamma^\alpha_{0n}] - g^{0i}\frac{\partial}{\partial x^0}\Gamma^n_{ni} - \frac{1}{2}\Gamma^0_{ni}\frac{\partial g^{ni}}{\partial x^0} + \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^0}[\sqrt{-g}\, g^{0n}\Gamma^0_{0n}]. \quad (19)$$

Since the last three terms in (19) can be ignored for static systems, from the expression (18) we have

$$M = \frac{c^2}{4\pi G}\oint dS_\alpha \sqrt{-g}\, g^{0n}\Gamma^\alpha_{0n}. \quad (20)$$

Since the metric of a static system sufficiently far from it must be described to a given accuracy by the Schwarzschild metric, the expression (20) takes the form

$$M = -\frac{c^2}{8\pi G}\lim_{r\to\infty}\oint dS^\alpha g^{00}\sqrt{-g}\frac{\partial}{\partial x^\alpha} g_{00}. \quad (21)$$

Since the integrand in (18) is a scalar under all transformations of the three-dimensional coordinate system, the gravitational mass $M$ will also be independent of the choice of the coordinates. In the Schwarzschild coordinates, we obtain from the expression (21)

$$M = \frac{c^2}{2G}\lim_{r\to\infty}\left(r^2\frac{\partial}{\partial r}g_{00}\right) = \frac{c^2}{2G}\lim_{r\to\infty}\left[r^2\frac{\partial}{\partial r}\left(1 - \frac{2G}{c^2 r}M\right)\right].$$

Thus, according to Tolman's definition, the gravitational mass of any static system is the coefficient of the term $-2G/(c^2r)$ in the asymptotic expression for the component $g_{00}$ of the metric tensor of the Riemannian space–time. Therefore, the definitions of the gravitational mass given by Einstein and Tolman are identical for static systems.

Einstein related the concept of inertial mass of a physical system in general relativity intimately to the concept of the energy of this system (Ref. 2, p. 660): "...the quantity that we have interpreted as the energy also plays the part of the inertial mass, in accordance with the special theory of relativity." Since Einstein proposed that the energy of a system in GR be calculated using energy-momentum pseudotensors, the inertial mass is also calculated on the basis of the expression (11).

We shall determine in accordance with this relation the inertial mass of a spherically symmetric source of the gravitational field and study its transformation properties under coordinate transformations.

In isotropic Cartesian coordinates, the metric of the Riemannian space–time has the form

$$g_{00} = \frac{(1 - r_g/4r)^2}{(1 + r_g/4r)^2};\quad g_{\alpha\beta} = \gamma_{\alpha\beta}(1 + r_g/4r)^4. \quad (22)$$

Here, $r_g = 2GM/c^2$.

These coordinates are asymptotically Galilean, since in the limit $r\to\infty$

$$g_{00} = 1 + O\left(\frac{1}{r}\right);\ g_{\alpha\beta} = \gamma_{\alpha\beta}\left(1 + O\left(\frac{1}{r}\right)\right). \quad (23)$$

Using the covariant components of the metric (22), from the expression (12) we obtain

$$h^{00\alpha} = \frac{c^4}{16\pi G}\frac{\partial}{\partial x^\beta}[g_{11}g_{22}g_{33}g^{\alpha\beta}].$$

Substituting this expression in (10), bearing in mind that

$$dS_\alpha = -\frac{x_\alpha}{r} r^2 \sin\theta \, d\theta \, d\varphi, \tag{24}$$

and integrating over an infinitely distant surface, we obtain

$$P^0 = \frac{c^3}{16\pi G} \lim_{r\to\infty} r^2 \int \frac{x_\alpha}{r} \frac{\partial}{\partial x^\beta} [-g_{11}g_{22}g_{33}g^{\alpha\beta}] \sin\theta \, d\theta \, d\varphi. \tag{25}$$

Thus, the component $P^0$ does not depend on the component $g_{00}$ of the metric tensor of the Riemannian space–time. From the expression (22) and (25), taking into account the relations

$$\frac{\partial}{\partial x^\beta} f(r) = -\frac{x_\beta}{r} \frac{\partial}{\partial r} f(r), \tag{26}$$

where $x_\alpha x^\alpha = -r^2$, we obtain for the energy–momentum component $P^0$ of the system

$$P^0 = c^3 r_g/2G = Mc. \tag{27}$$

It was this agreement between the inertial mass and the gravitational mass which provided the basis for the assertions of their equality in the general theory of relativity (Ref. 18, p. 424): "... $P^\alpha = 0, P^0 = Mc$ is a result that was naturally to be expected. It is an expression of the fact of the equality of what are known as the 'gravitational' and 'inertial' masses (the mass which determines the gravitational field produced by a body is called the 'gravitational' mass—it is the same mass that appears in the metric tensor in the gravitational field or, in particular, in Newton's law; in contrast, the 'inertial' mass determines the relationship between the momentum and energy of a body and, in particular, the rest energy of a body is equal to this mass multiplied by $c^2$)."

However, the assertion of Einstein (Ref. 2, p. 660) and other authors (see Refs. 17–19, 21, and 22) is incorrect. As is easy to show, the energy of a system and, therefore, its inertial mass (11) do not have any physical meaning, since they depend even on the choice of the three-dimensional coordinate system.

Indeed, an elementary requirement that must be satisfied by the definition of the inertial mass is that it be independent of the choice of the three-dimensional coordinate system, a condition that holds in any physical theory. However, in GR the definition (11) of the inertial mass does not satisfy this requirement.

We shall show, for example, that in the case of the Schwarzschild solution the inertial mass (11) can take arbitrary values, depending on the choice of the system of spatial coordinates. For this, we go over from the three-dimensional Cartesian coordinates $x_C^\alpha$ to other coordinates $x_H^\alpha$ related to the old coordinates by

$$x_C^\alpha = x_H^\alpha (1 + f(r_H)), \tag{28}$$

where

$$r_H = \sqrt{x_H^2 + y_H^2 + z_H^2},$$

and $f(r_H)$ is an arbitrary nonsingular function satisfying the conditions

$$f(r_H) \geqslant 0; \quad \lim_{r_H\to\infty} f(r_H) = 0; \quad \lim_{r_H\to\infty} r_H \frac{\partial}{\partial r_H} f(r_H) = 0. \tag{29}$$

It is easy to show that the transformation (28) corresponds to a change in the arithmetization of the points of the three-dimensional space along the radius:

$$r_C = r_H [1 + f(r_H)].$$

A necessary and sufficient condition for the transformation (28) to have an inverse and to be one-to-one is

$$\frac{\partial r_C}{\partial r_H} = 1 + f + r_H f' > 0,$$

where

$$f' = \frac{\partial}{\partial r_H} f(r_H).$$

Then the Jacobian of the transformation will also be nonzero:

$$J = \det \left\| \frac{\partial x_C}{\partial x_H} \right\| = (1+f)^2 \frac{\partial r_C}{\partial r_H} \neq 0.$$

In particular, all these requirements are satisfied by the function

$$f(r_H) = \alpha^2 \sqrt{\frac{8GM}{c^2 r_H}} [1 - \exp(-\varepsilon^2 r_H)], \tag{30}$$

where $\alpha$ and $\varepsilon$ are arbitrary nonvanishing numbers.

Since in the given case

$$\frac{\partial r_C}{\partial r_H} = 1 + \alpha^2 \sqrt{\frac{8GM}{c^2 r_H}} \left[\frac{1}{2} + \left(\varepsilon^2 r_H - \frac{1}{2}\right) \times \exp(-\varepsilon^2 r_H)\right] > 0,$$

it follows that $r_C$ is a monotonic function of $r_H$. It is easy to show that $f(r_H)$ is a non-negative nonsingular function in the whole of space. The Jacobian of the transformation is in this case strictly greater than unity:

$$J = (1+f)^2 \frac{\partial r_C}{\partial r_H} > 1.$$

Therefore, the transformation (28) with the function $f(r_H)$ defined by the expression (30) has an inverse and is one-to-one.

It is obvious that under the transformation (28) the gravitational mass (18) does not change. We now calculate the inertial mass (11) in the new coordinates $x_H^\alpha$. Using the transformation law for the metric tensor,

$$g_{ni}^H = \frac{\partial x_C^l}{\partial x_H^n} \frac{\partial x_C^m}{\partial x_H^i} g_{ml}^C (x_C(x_H)), \tag{31}$$

we find the components of the Schwarzschild metric (22) in the new coordinates:

$$\left.\begin{aligned} g_{00} &= \left[1 - \frac{r_g}{4r_H(1+f)}\right]^2 \left[1 + \frac{r_g}{4r_H(1+f)}\right]^{-2}; \\ g_{\alpha\beta} &= \left[1 + \frac{r_g}{4r_H(1+f)}\right]^4 \left\{-\delta_{\alpha\beta}(1+f)^2 - x_\alpha^H x_\beta^H \right. \\ &\quad \left. \times \left[(f')^2 + \frac{2}{r_H} f'(1+f)\right]\right\}. \end{aligned}\right\} \tag{32}$$

The determinant of the metric tensor (32) is

$$g = -g_{00}\left[1 + \frac{r_g}{4r_H(1+f)}\right]^{12} (1+f)^4 \times [(1+f)^2 + r_H^2 (f')^2 + 2r_H f'(1+f)]. \tag{33}$$

It should be noted particularly that the metric (32) is asymptotically Galilean:

$$\lim_{r_H\to\infty} g_{00}=1;\quad \lim_{r_H\to\infty} g_{\alpha\beta}=\gamma_{\alpha\beta}.$$

In the special case when the function $f$ is given by (30) and $r_H\to\infty$, the metric of the Riemannian space–time will have the asymptotic behavior

$$g_{00}\simeq 1+O\left(\frac{1}{r_H}\right);\quad g_{\alpha\beta}=\gamma_{\alpha\beta}\left[1+O\left(\frac{1}{\sqrt{r_H}}\right)\right]. \tag{34}$$

For the contravariant components of the metric (32), we have

$$g^{00}=\frac{1}{g_{00}};\quad g^{\alpha\beta}=\gamma^{\alpha\beta}A+x_H^\alpha x_H^\beta B, \tag{35}$$

where we have introduced the notation

$$A=(1+f)^{-2}\left[1+\frac{r_g}{4r_H(1+f)}\right]^{-4};$$

$$B=\frac{r_H(f')^2+2f'(1+f)}{r_H\left[1+\frac{r_g}{4r_H(1+f)}\right]^4(1+f)^2\left[(1+f)^2+r_H^2(f')^2+2r_Hf'(1+f)\right]}.$$

Substituting the expressions (33) and (35) in (25), we obtain

$$P^0=\frac{c^3}{16\pi G}\lim_{r_H\to\infty} r_H^2\int\frac{x_\alpha^H}{r_H}\frac{\partial}{\partial x_H^\beta}\left\{\gamma^{\alpha\beta}(1+f)^2\right.$$

$$\times\left[1+\frac{r_g}{4r_H(1+f)}\right]^8$$

$$\times\left[(1+f)^2+r_H^2(f')^2+2r_Hf'(1+f)\right]$$

$$+\frac{x_H^\alpha x_H^\beta}{r_H^2}(1+f)^2\left[1+\frac{r_g}{4r_H(1+f)}\right]^8$$

$$\left.\times\left[r_H^2(f')^2+2r_Hf'(1+f)\right]\right\}dV.$$

By virtue of the relations (26), we obtain from this

$$P^0=\frac{c^3}{2G}\lim_{r_H\to\infty}\left\{r_H^3(f')^2(1+f)^2\left[1+\frac{r_g}{4r_H(1+f)}\right]^8\right.$$

$$\left.+r_g(1+f)^2(1+f+r_Hf')\left[1+\frac{r_g}{4r_H(1+f)}\right]^7\right\}. \tag{36}$$

Taking into account the asymptotic expression (29) for $f$, we finally obtain[1)]

$$P^0=\frac{c^3}{2G}\lim_{r_H\to\infty}\{r_g+r_H^3(f')^2\}. \tag{37}$$

Thus, the inertial mass depends essentially on the rate at which $f'$ tends to zero as $r_H\to\infty$. In particular, choosing the function $f(r_H)$ in the form (30), we obtain from the expression (37)

$$m_i=M(1+\alpha^4). \tag{38}$$

It follows from this that for the inertial mass (11) of the system consisting of the matter and the gravitational field it is possible in GR, $\alpha$ being arbitrary, to obtain any preassigned number $m_i\geqslant M$, depending on the choice of the spatial coordinates, although the gravitational mass $M$ (18) of this system—and hence all three GR effects—are unaffected. We note also that for more complicated transformations of the spatial coordinates that leave the metric asymptotically Galilean the inertial mass (11) may take all preassigned values, both positive and negative.

Thus, we see that in GR the inertial mass, introduced for the first time by Einstein and taken over subsequently by many authors (Refs. 17–19, 21, and 22), depends on the choice of the three-dimensional coordinate system and therefore has no physical meaning. Therefore, the assertion of equality of the "inertial" and "gravitational" masses in Einstein's theory is also devoid of all physical meaning. This equality holds in a narrow class of three-dimensional coordinate systems, and since the inertial (11) and gravitational (18) masses have different transformation laws, their equality no longer holds on the transition to other three-dimensional coordinate systems.

In addition, this definition of the inertial mass in general relativity does not satisfy the principle of correspondence with Newton's theory. Indeed, because the inertial mass in Einstein's theory depends on the choice of the three-dimensional coordinate system, its expression in the general case of an arbitrary three-dimensional coordinate system does not go over into the corresponding expression of Newton's theory, in which the inertial mass does not depend on the choice of the spatial coordinates. Thus, in GR there is no classical Newtonian limit, and, therefore, it also does not satisfy the correspondence principle. It follows from this that GR is not only logically contradictory from the point of view of physics but also directly contradicts the experimental data on the equality of the inertial and active gravitational masses.

But this poses a question: Why has the meaninglessness of the definition (10) of the energy and momentum of a system and its inertial mass in GR not yet been revealed?

This can only be explained by the fact that all calculations of the energy, momentum, and inertial mass are usually made in a certain narrow class of three-dimensional coordinate systems, in which there is equality of the inertial and gravitational masses.

In the same class of coordinate systems, the expression for the inertial mass (11) in the Newtonian approximation is equal to the corresponding expression of Newton's theory, and this created the illusion that GR has a classical limit. It was evidently felt to be superfluous to consider the physical meaning of the inertial mass introduced by (11) in GR.

## 4. ENERGY-MOMENTUM CONSERVATION LAWS IN GENERAL RELATIVITY

The example of the baselessness of the definition of inertial mass given in Sec. 3 does not exhaust all the shortcomings of GR associated with the use of an energy-momentum pseudotensor. These shortcomings, with all the consequences of them, are considered in detail in Ref. 11. Without going into details of a technical nature, we discuss some of them.

In all physical theories describing different forms of matter, one of the most important characteristics of a field is the energy–momentum-tensor density, which is usually obtained by varying the Lagrangian density $L$ of the field with respect to the components $g_{mn}$ of the space–time metric tensor.

This characteristic reflects the existence of a field: The nonvanishing of the energy-momentum-tensor density in a certain region of space–time is a necessary and sufficient condition for the presence of a physical field in this region. Moreover, the energy and momentum of any physical field make a contribution to the total energy-momentum tensor of the system and do not vanish identically outside the field source. This makes it possible to treat the transport of energy by waves in the spirit of Faraday and Maxwell, to study the distribution of the field intensity in space, to determine the energy-momentum fluxes in emission and absorption processes, and to make other energy calculations.

In GR, the gravitational field does not possess the properties inherent in other physical fields, being devoid of such a characteristic.

Indeed, in Einstein's theory the Lagrangian density consists of two parts: the Lagrangian density $L_g = L_g(g_{mn})$ of the gravitational field, which depends only on the metric tensor $g_{mn}$, and the matter Lagrangian density $L_M = L_M(g_{mn}, \Phi_A)$, which depends on the metric tensor $g_{mn}$ and the remaining matter fields $\Phi_A$. Thus, in Einstein's theory the quantities $g_{mn}$ have a double meaning—that of field variables and that of the space–time metric tensor.

As a result of this physicogeometrical dualism, the density of the total symmetric energy-momentum tensor (the variation of the Lagrangian density with respect to the components of the metric tensor) is identical to the field equations (the variation of the Lagrangian density with respect to the components of the gravitational field). This has the consequence that the density of the total symmetric energy-momentum tensor of the system is strictly zero,

$$T^{ni} + T^{ni}_{(g)} = 0, \tag{39}$$

where $T^{ni} = -2\delta L_M/\delta g_{ni}$ is the density of the symmetric energy-momentum tensor of the matter (by matter we regard all matter fields except the gravitational field);

$$T^{ni}_{(g)} = -2\frac{\delta L_g}{\delta g_{ni}} = -\frac{c^4\sqrt{-g}}{8\pi G}\left[R^{ni} - \frac{1}{2}g^{ni}R\right].$$

It also follows from the expression (39) that all components of the density of the symmetric energy-momentum tensor $T^{ni}_{(g)}$ of the gravitational field are zero everywhere outside the matter.

Thus, it already follows from these results that in Einstein's theory the gravitational field does not have properties inherent in other physical fields, since outside the source it is devoid of a fundamental physical characteristic—the energy-momentum tensor.

The physical characteristic of the gravitational field in Einstein's theory is the curvature tensor $R^i_{\ klm}$. The clear recognition of this we owe to Synge (Ref. 23, p. 8). "...If we accept the idea that space–time is a Riemannian four-space (and if we are relativists we must), then surely our first task is to get the feel of it just as early navigators had to get the feel of a spherical ocean. And the first thing we have to get the feel of is the Riemann tensor, for it *is* the gravitational field—if it vanishes, and only then, there is no field. Yet, strangely enough, this most important element has been pushed into the background." And later he notes: "In Einstein's theory, either there is a gravitational field or there is none, according as the Riemann tensor does not or does vanish. This is an absolute property; it has nothing to do with any observer's world-line." The absence of such understanding leads to miscomprehension of the very essence of Einstein's theory.

Thus, since the gravitational field is characterized by the curvature tensor, and by it alone, it is not possible in GR to introduce any other simpler physical characteristic of this field, for example, an energy-momentum pseudotensor. Therefore, in Einstein's theory energy-momentum pseudotensors cannot in principle have any bearing on the existence of a gravitational field. This assertion has the nature of a theorem, a consequence of which is the possibility of situations in GR in which the curvature tensor is nonzero, i.e., a field exists, while the energy-momentum pseudotensor is zero, and, conversely, the curvature tensor may be zero but the energy-momentum pseudotensor is nonzero. Therefore, all kinds of calculation using energy-momentum pseudotensors are devoid of any meaning.

Einstein's general theory of relativity links matter and the gravitational field, the former being characterized, as in all theories, by an energy-momentum tensor, i.e., by a tensor of second rank, whereas the characteristic of the latter is the curvature tensor—a tensor of fourth rank. Because of the different dimensionalities of the physical characteristics of the gravitational field and matter in Einstein's theory it follows directly that in GR conservation laws linking the matter and the gravitational field cannot exist in principle. This fundamental fact, established in Refs. 5 and 8, means that Einstein's theory was constructed at the price of the abandonment of conservation laws of the matter and the gravitational field taken together.

The physical characteristic of the gravitational field in GR—the Ricci tensor—reflects in the first place the capacity of a gravitational field to change the energy and momentum of matter, i.e., it reflects the forces exerted by the gravitational field on the matter, but does not give any information about the flux of energy transported by a wave, so that in Einstein's theory there is no possibility of studying the distribution of the gravitational field intensity in space, determining energy fluxes of gravitational waves through a surface, etc.

The use of pseudotensors to find conserved quantities for the matter and gravitational field taken together in the framework of GR is a profound delusion.

For in GR the point of departure for obtaining the conservation laws is the identity

$$\partial_n (T^n_i + \tau^n_i) = 0. \tag{40}$$

If matter is concentrated solely in the volume $V$, we find from this relation

$$\frac{d}{dx^0}\int dV\,(T^0_i + \tau^0_i) = -\oint \tau^\alpha_i\, dS_\alpha. \tag{41}$$

There have by now been found numerous[10,24–28] exact solutions of the vacuum Hilbert-Einstein equations for which the stresses $\tau^\alpha_0$ are everywhere zero. Therefore, for exact wave solutions of the Hilbert-Einstein equations that make the components of the energy-momentum tensor vanish it follows from the relation (41) that

$$\frac{d}{dx^0}\left(\int dV\,[T^0_0+\tau^0_0]\right)=0,$$

i.e., the energy of the matter and the gravitational field in the volume $V$ is conserved. This means that there is no energy flux out of the volume $V$, and therefore there cannot be any effect on test bodies placed outside the volume. This is the conclusion that follows from Einstein's theory.

However, the exact wave solutions of the Hilbert-Einstein equations that make the components of the energy-momentum tensor vanish do not lead to a vanishing curvature tensor $R^i_{\ klm}$, so that by virtue of the equation

$$\frac{\delta^2 n^i}{\delta s^2}+R^i_{klm}u^k u^l n^m=0, \tag{42}$$

where $n^i$ is the infinitesimally small vector of the geodesic deviation and $u^i = dx^i/ds$ is the velocity 4-vector, curvature waves do act on test bodies outside the volume $V$, changing their energy.

Thus, from two different but exact relations of GR in Einstein's philosophy we arrive at completely mutually exclusive physical conclusions.

To understand the origin of these contradictory conclusions, we analyze the formalism of energy-momentum pseudotensors in Einstein's theory in more detail.

Since $\tau^{ni}$ is a pseudotensor, all the components of $\tau^{ni}$ can be made to vanish at any point of space by the choice of the coordinate system. This fact alone casts doubt on the interpretation of $\tau^{ni}$ as the energy-momentum stresses and density of the gravitational field.

In this connection, it is usually asserted[21] that in GR the energy of the gravitational field is in principle nonlocalizable, i.e., that a local distribution of the gravitational field energy does not have physical meaning, since it depends on the choice of the coordinate system, and that only the total energy of closed systems can be defined. But this assertion too does not stand up to criticism.

Indeed, the local energy distribution of the gravitational field determined using any energy-momentum pseudotensor depends on the choice of the coordinates and can be made to vanish at any point of space, this being usually interpreted as the absence of an "energy density" of the gravitational field at this point. But the gravitational field, described by the curvature tensor, cannot be made to vanish by the transition to any admissible coordinate system, and therefore, because of the effect of curvature waves on physical processes, one cannot assert that the gravitational field is absent in any coordinate system.

This is most clearly seen in the example of two exact wave solutions for which the components of the energy-momentum tensor are everywhere zero but curvature waves are present. Conversely, in the case of flat space–time, when the metric tensor $g_{ni}$ of the Riemannian space–time is equal to the metric tensor $\gamma_{ni}$ of pseudo-Euclidean space–time, the components of energy-momentum tensors need not vanish, although there is no gravitational field and all components of the curvature tensor are equal to zero in any coordinate system.

For example,[29] in a spherical coordinate system in pseudo-Euclidean space–time,

$$R^i_{klm}=0,\ g_{00}=1;\ g_{rr}=-1;\ g_{\theta\theta}=-r^2;\ g_{\varphi\varphi}=-r^2\sin^2\theta,$$

we obtain for the component $\tau^0_0$ of Einstein's pseudotensor

$$\tau^0_0=-\frac{1}{8\pi}\sin\theta.$$

It is obvious that $\tau^0_0<0$ and the total energy of the gravitational field in this coordinate system is infinite.

But in the same case the Landau-Lifshitz pseudotensor demonstrates a different energy distribution in space:

$$(-g)\,\tau^{00}=-\frac{r^2}{8\pi}(1+4\sin^2\theta).$$

It is evident from the examples given here that energy-momentum pseudotensors in Einstein's theory are not physical characteristics of the gravitational field and, therefore, do not have any physical meaning.

Thus, there do not and cannot exist energy-momentum conservation laws of the gravitational field and matter taken together. However, in the theories of the remaining physical fields we have a unified energy-momentum conservation law, and at the present time there are no experimental data indicating its violation. Therefore, we have no grounds for abandoning it.

What is the way out of this situation? What must we retain from the great creation of Einstein and Hilbert, and what must we give up to ensure that the new theory of gravitation possesses the fundamental laws of physics—the energy-momentum conservation law of the matter and the gravitational field taken together?

In order to find the answer to these questions, we must carefully analyze the existing very deep connection between energy-momentum conservation laws and the geometry of space–time.

## 5. CONNECTION BETWEEN THE ENERGY-MOMENTUM AND ANGULAR-MOMENTUM CONSERVATION LAWS AND THE GEOMETRY OF SPACE–TIME

The possibility of obtaining conservation laws for a closed system of interacting fields depends to a large degree on the nature of the space–time geometry.

As is well known,[30,31] the theory of any physical field can be constructed on the basis of the Lagrangian formalism. In it, the physical field is described by a certain function of the coordinates and the time, called the field function, the equations for which can be obtained from the variational principle of a stationary action. Besides the field equations, the Lagrangian method of constructing the classical theory of wave fields makes it possible to obtain a number of differential relations, called differential conservation laws. These relations are consequences of the invariance of the action functions under transformations of the space–time coordinates, and they connect the local dynamical characteristics of the fields and their covariant derivatives in the geometry which is natural for them.

It is currently usual in the literature to distinguish two types of differential conservation laws: strong and weak. By a strong conservation law one usually means a differential relation satisfied by virtue of the invariance of the action function under a coordinate transformation, and one does not require fulfillment of the equations of motion for the field. In contrast, weak conservation laws can be obtained from strong conservation laws if one takes into account the equations of motion for the system of interacting fields.

It must be particularly emphasized that despite the name the differential conservation laws do not in the general case assert that there is conservation, either locally or globally. They are simply differential identities connecting different characteristics of the field satisfied by virtue of the fact that the action function is not changed by an arbitrary coordinate transformation (i.e., is a scalar). These relations acquired their name by analogy with the corresponding differential conservation laws in pseudo-Euclidean space–time, in which corresponding integral laws can be obtained from the differential conservation laws. For example, writing the conservation law for the total energy-momentum tensor of a system of interacting fields in a Cartesian coordinate system of pseudo-Euclidean space–time, we obtain

$$\frac{\partial}{\partial x^0} t^{0i} + \frac{\partial}{\partial x^\alpha} t^{\alpha i} = 0.$$

Integrating this equation over a certain volume and using Gauss's theorem, we obtain

$$\frac{d}{dx^0} \int dV t^{0i} = -\oint t^{\alpha i}\, dS_\alpha.$$

This relation means that the change in the energy and momentum of the system of interacting fields in a certain volume is equal to their flux through the surface bounding the volume. If there is no energy-momentum flux through a certain surface,

$$\oint t^{\alpha i}\, dS_\alpha = 0,$$

then we arrive at a conservation law for the total 4-momentum of the isolated system:

$$\frac{d}{dx^0} P^i = 0, \text{ where } P^i = \frac{1}{c} \int t^{0i}\, dV.$$

Analogous integral relations in pseudo-Euclidean space–time can be obtained for the angular momentum.

But in an arbitrary Riemannian space–time the presence of a differential covariant conservation equation does not guarantee the possibility of obtaining a corresponding integral conservation law.

The possibility of obtaining integral laws in an arbitrary Riemannian space–time is entirely predetermined by its geometry and is intimately related to the existence of Killing vectors of the space–time or, as one sometimes says, the existence of a group of motions in the Riemannian space–time. We consider this question in somewhat more detail, since the formalism developed here can also be used to obtain integral conservation laws in arbitrary curvilinear coordinate systems in pseudo-Euclidean space–time.

In an arbitrary Riemannian space–time, it is possible to obtain a covariant conservation equation for the total energy-momentum tensor of the system:

$$\nabla_l T^{ml} = \partial_l T^{ml} + \Gamma^m_{nl} T^{nl} + \Gamma^l_{ln} T^{mn} = 0. \qquad (43)$$

We multiply this equation by a Killing vector, i.e., by a vector $\eta_m$ satisfying the Killing equations

$$\nabla_n \eta_m + \nabla_m \eta_n = 0. \qquad (44)$$

The tensor $T^{mn}$ being symmetric, the resulting expression can be written in the form

$$\eta_m \nabla_l T^{ml} = \nabla_l [\eta_m T^{ml}] = 0.$$

Using the properties of the covariant derivative, we obtain from this

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^l} [\sqrt{-g}\, \eta_m T^{ml}] = 0.$$

Since the left-hand side of this equation is a scalar, we can multiply it by $\sqrt{-g}dV$ and integrate over a certain volume. As a result, we obtain an integral conservation law in the Riemannian space–time:

$$\frac{d}{dx^0} \int \sqrt{-g}\, T^{0m} \eta_m\, dV = -\oint dS_\alpha \sqrt{-g}\, T^{\alpha m} \eta_m. \qquad (45)$$

If there is no flux of the three-dimensional vector through the surface bounding the volume, then

$$\int \sqrt{-g}\, T^{0m} \eta_m\, dV = \text{const}. \qquad (46)$$

Thus, in the presence of Killing vectors the integral conservation laws (45) and (46) can be obtained from the differential conservation law (43).

We now establish the restrictions on the metric of the Riemannian space–time which ensure that the Killing equations (44) have solutions, i.e., we find the conditions under which there exists a vector satisfying (44). We note first that the Killing equations (44) are a consequence of requiring the vanishing of the Lie variation of the metric tensor of the Riemannian space–time under infinitesimal coordinate transformations

$$x'^n = x^n + \eta^n(x), \qquad (47)$$

where $\eta^n(x)$ is an infinitesimally small 4-vector.

Indeed, under such a coordinate transformation the Lie variation of the metric tensor is

$$\delta_L g_{in} = -\nabla_i \eta_n - \nabla_n \eta_i.$$

Comparing this expression with (44), we see that the Killing equations require the vanishing of the Lie variation of the metric tensor $g_{ni}$:

$$\delta_L g_{in} = 0.$$

Thus, Killing vectors describe infinitesimally small coordinate transformations that leave the metric form-invariant.

The Killing equations (44) constitute a system of linear partial differential equations of first order. In accordance with the general theory,[32,33] to establish the conditions of integrability of a system of partial differential equations it is necessary to reduce it to the form

$$\frac{\partial \theta^a}{\partial x^i} = \Psi^a_i(\theta^b, x^n), \qquad (48)$$

where $\theta^a$ are unknown functions; $i, n = 1, 2,...,N$; $a, b = 1, 2,...,M$. Then the conditions of integrability of the system (48) can be obtained from the equation

$$\frac{\partial^2\theta^a}{\partial x^i\,\partial x^n}=\frac{\partial^2\theta^a}{\partial x^n\,\partial x^i},$$

the partial derivatives of first order being replaced by the right-hand side of (48):

$$\frac{\partial\Psi_i^a}{\partial x^n}+\frac{\partial\Psi_i^a}{\partial\theta^b}\Psi_n^b=\frac{\partial\Psi_n^a}{\partial x^i}+\frac{\partial\Psi_n^a}{\partial\theta^b}\Psi_i^b. \tag{49}$$

If the integrability conditions (49) are satisfied identically by virtue of (48), the system (48) is said to be completely integrable and its solution contains $M$ parameters—the maximal possible number of arbitrary constants for the given system.

But if the system (48) is not completely integrable, then its solution will contain fewer arbitrary constants. We shall determine the conditions under which the solution of the Killing equations (44) in the Riemannian space $V_N$ contains the maximal possible number of parameters and what the value of this number is.

We shall make all the calculations in a manifestly covariant form, this being the covariant generalization of the scheme given above for finding the conditions of integrability of the system of differential Killing equations (44) in the required form. We differentiate covariantly the Killing equations (44) with respect to the variable $x^n$, obtaining

$$\eta_{i;j;n}+\eta_{j;i;n}=0.$$

By virtue of this equation,

$$\eta_{i;jn}+\eta_{j;in}+\eta_{i;nj}+\eta_{n;ij}-\eta_{j;ni}-\eta_{n;ji}=0.$$

Regrouping the terms in this expression, we obtain

$$\eta_{i;jn}+\eta_{i;nj}+(\eta_{j;in}-\eta_{j;ni})+(\eta_{n;ij}-\eta_{n;ji})=0. \tag{50}$$

However, by virtue of the commutation rule for covariant derivatives,

$$\eta_{i;nj}-\eta_{i;jn}=\eta_k R^k_{inj}. \tag{51}$$

Substituting the expression (51) in (50), we obtain

$$2\eta_{i;jn}+\eta_k R^k_{inj}+\eta_k R^k_{jin}+\eta_k R^k_{nij}=0. \tag{52}$$

Using the Ricci identity

$$R^k_{inl}+R^k_{nli}+R^k_{lin}=0, \tag{53}$$

we obtain

$$\eta_k R^k_{inj}+\eta_k R^k_{jin}=\eta_k R^k_{nij}.$$

Therefore, the expression (52) can be written in the form

$$\eta_{i;jn}=-\eta_k R^k_{nij}.$$

Thus, we have the covariant equations

$$\eta_{i;n}+\eta_{n;i}=0;\quad \eta_{i;jn}=-\eta_k R^k_{nij}. \tag{54}$$

We transform this system of covariant differential equations into a system containing only first covariant derivatives. For this, we introduce in addition to the $N$ unknown components of the vector $\eta_m$ the unknown tensor $\lambda_{im}$ in accordance with the equations

$$\eta_{i;m}=\lambda_{im}. \tag{55}$$

This tensor contains $N^2$ unknown components, but among them only $N(N-1)/2$ components are independent, this tensor being antisymmetric by virtue of Eqs. (44) and (55):

$$\lambda_{mi}+\lambda_{im}=0. \tag{56}$$

With allowance for all this, the required system of covariant differential equations takes the form

$$\eta_{m;i}=\lambda_{mi};\quad \lambda_{mi;j}=\eta_k R^k_{jim}. \tag{57}$$

Thus, we have reduced the Killing equations (44) to a system of special form consisting of linear differential equations solved for the covariant derivatives of first order.

This system is the covariant generalization of the system (48), the part of the unknown functions $\theta^a$ being played by the $N(N+1)/2$ components of the tensors $\eta_m$ and $\lambda_{mi}$:

$$\theta^a=\{\eta_m,\ \lambda_{mi}\}.$$

The condition of integrability of the system (57) can be obtained from the commutation rule for the covariant derivatives, this being a consequence of the commutativity of the derivatives in partial differentiation. On the basis of this rule,

$$\left.\begin{aligned}&\eta_{i;mj}-\eta_{i;jm}=\eta_k R^k_{imj};\\&\lambda_{im,jl}-\lambda_{im,lj}=\lambda_{ik}R^k_{mjl}+\lambda_{km}R^k_{ijl}.\end{aligned}\right\} \tag{58}$$

Replacing on the left-hand sides of these equations the first covariant derivatives by their expressions (57) and using the antisymmetry property (56) of the tensor $\lambda_{im}$, we obtain the conditions of integration of the system (57) in the form

$$\lambda_{im;j}-\lambda_{ij;m}=\eta_k R^k_{imj}; \tag{59}$$

$$[\eta_k R^k_{jmi}]_{;l}-[\eta_k R^k_{lmi}]_{;j}=\lambda_{ik}R^k_{m;jl}+\lambda_{km}R^k_{ijl}. \tag{60}$$

It is easy to show that the first of these expressions is identically satisfied by virtue of (57) and the properties of the curvature tensor. Thus, if the condition (60) is identically satisfied by virtue of the symmetry properties of the Riemannian space–time alone, the system (57) will be completely integrable, and therefore the solution of the Killing equations (44) will contain the maximal possible number $M=N(N+1)/2$ of arbitrary constants. Since the unknown functions $\eta_m$, $\lambda_{mi}=-\lambda_{im}$ in the system (57) must at the same time be independent, the left-hand side of the expression (60) vanishes identically only when the following conditions are satisfied:

$$R^k_{mij;l}-R^k_{lij;m}=0; \tag{61}$$

$$\delta^n_j R^k_{iml}-\delta^k_j R^n_{iml}-\delta^n_i R^k_{jml}+\delta^k_i R^n_{jml}+\delta^n_l R^k_{mij}-\delta^k_l R^n_{mij}-\delta^n_m R^k_{lij}+\delta^k_m R^n_{lij}=0. \tag{62}$$

Contraction of the expression (62) with respect to the indices $l$ and $n$ with allowance for the relations

$$R^n_{lmn}=R_{lm};\quad R^n_{nml}=0$$

and the Ricci identity (53) gives

$$(N-1)\,R^k_{mij}=\delta^k_j R_{mi}-\delta^k_i R_{jm}.$$

It follows from this that

$$R_{lmij}=\frac{1}{N-1}\,(g_{jl}R_{mi}-g_{il}R_{jm}). \tag{63}$$

Multiplying this equation by $g^{mi}$, we obtain

$$NR_{jl}=g_{jl}R.$$

Substituting this relation in the expression (63), we obtain a condition by virtue of which Eq. (62) is satisfied identically:

$$R_{lmij}=\frac{R}{N(N-1)}\,[g_{jl}g_{mi}-g_{il}g_{jm}]. \tag{64}$$

From the expression (64) and Eq. (61) we obtain a requirement that the scalar curvature must satisfy:

$$[\delta^h_j g_{im}-\delta^h_i g_{jm}]\,\frac{\partial}{\partial x^l}\,R-[\delta^h_j g_{li}-\delta^h_i g_{lj}]\,\frac{\partial}{\partial x^m}\,R=0.$$

Multiplying this relation by $\delta^l_k g^{mi}$, we obtain

$$(N-1)\,\frac{\partial R}{\partial x^j}=0.$$

Since $N>1$ in the case considered, for this condition to be satisfied it is necessary and sufficient that $R=\text{const.}$ Therefore, the conditions of integrability (61) and (62) of the Killing equations (44) will be satisfied identically if and only if the curvature tensor of the Riemannian space–time has the form

$$R_{lmij}=\frac{R}{N(N-1)}\,[g_{jl}g_{mi}-g_{il}g_{jm}],$$

where $R=\text{const.}$

Thus, the Killing equations have solutions containing the maximal possible number $M=N(N+1)/2$ of arbitrary constants (parameters) if and only if the Riemannian space $V_N$ is a space of constant curvature. But if it is not, the number of parameters will be less.

Therefore, from the mathematical point of view the existence of integral conservation laws for energy, momentum, and angular momentum is a reflection of definite properties of space–time—its properties of homogeneity and isotropy. There exist three types of four-dimensional spaces possessing the properties of homogeneity and isotropy to such a degree that they admit the introduction of ten integrals of the motion for a closed system: a space of constant negative curvature (Lobachevskiĭ space), a space of zero curvature (pseudo-Euclidean space), and a space of constant positive curvature (Riemann space). The first two spaces are infinite, having infinite volume, while the third space is closed, having finite volume but no boundaries.

We now find a Killing vector in an arbitrary curvilinear coordinate system in pseudo-Euclidean space–time. For this, we first write the Killing equations in a Cartesian coordinate system:

$$\partial_i\eta_n+\partial_n\eta_i=0.$$

Therefore, to determine the Killing vectors we have a system of ten linear partial differential equations of first order.

Solving this system in accordance with the general rules, we obtain

$$\eta_i=a_i+\omega_{im}x^m, \tag{65}$$

where $a_i$ is an arbitrary constant infinitesimally small vector, and $\omega_{im}$ is an arbitrary constant infinitesimally small tensor satisfying

$$\omega_{im}=-\,\omega_{mi}.$$

Thus, the solution (65) contains all ten arbitrary parameters, as was to be expected.

Since the expression (65) contains ten independent parameters, we actually have ten independent Killing vectors, and the relation (65) is a linear combination of these ten independent vectors.

We shall find the meaning of these parameters. Substituting the expression (65) in the relation (47), we obtain

$$x'^n=x^n+a^n+\omega^n_m x^m. \tag{66}$$

It can be seen from this expression that the four parameters $a^n$ are the components of a 4-vector of infinitesimally small translations of the frame of reference. The three parameters $\omega_{\alpha\beta}$ are the components of a tensor of rotation through an infinitesimally small angle about a certain axis (so-called pure rotations). The three parameters $\omega_{0\beta}$ describe infinitesimally small rotations in the $x^0x^\beta$ plane; they are called Lorentz rotations. Since the metric tensor $\gamma_{mn}$ is form-invariant with respect to translations, pseudo-Euclidean space–time is homogeneous—its properties do not depend on the point of space–time at which the origin of the coordinates is placed. Similarly, from the form invariance of the metric tensor $\gamma_{mn}$ under transformations of four-dimensional rotations its isotropy follows. This means that in pseudo-Euclidean space–time all directions are on an equal footing.

Thus, pseudo-Euclidean space–time admits a ten-parameter group of motions consisting of the four-parameter subgroup of translations and the six-parameter subgroup of rotations. The existence of this group of motions and the existence of the corresponding Killing vectors guarantee the presence of ten integral energy-momentum and angular-momentum conservation laws for a system of interacting fields.

To see this, we note that since $\sqrt{-\gamma}=1$ in a Cartesian coordinate system, we obtain from the general relation (45) in the case of the subgroup of translations $\eta_i=a_i$

$$\frac{d}{dx^0}\int T^{0m}a_m\,dV=-\oint dS_\alpha T^{\alpha m}a_m.$$

Since $a_m$ is an arbitrary constant vector, from this equation we obtain

$$\frac{d}{dx^0}\int T^{0m}\,dV=-\oint dS_\alpha T^{\alpha m}.$$

For an isolated system of interacting fields, the expression on the right-hand side of this relation is zero, as a result of which the total 4-momentum of the system is conserved:

$$P^m = \int T^{0m} dV = \text{const.} \quad (67)$$

Similarly, for $\eta_n = \omega_{nm} x^m$ we obtain

$$\frac{d}{dx^0} \int dV T^{0m} x^n \omega_{mn} = - \int dS_\alpha T^{\alpha m} x^n \omega_{mn}.$$

Since the constant tensor $\omega_{mn}$ is antisymmetric, we obtain from this an integral conservation law for the angular momentum:

$$\frac{d}{dx^0} \int dV \, [T^{0m} x^n - T^{0n} x^m] = - \int dS_\alpha \, [T^{\alpha m} x^n - T^{\alpha n} x^m]. \quad (68)$$

For an isolated system, its total angular momentum is conserved through the vanishing of the right-hand side of Eq. (68),

$$M^{mn} = \int dV \, [T^{0m} x^n - T^{0n} x^m] = \text{const.} \quad (69)$$

It is only for pseudo-Euclidean space–time that separate conservation laws for energy, momentum, and angular momentum for a closed system hold. It should be noted that the solution of the Killing equations (44) in arbitrary curvilinear coordinates of pseudo-Euclidean space–time can, $x^i$ and $\eta^i$ being tensors, be obtained from the solution (66) of these equations in the Cartesian coordinate system. For this, we go over in the expression (66) from the Cartesian coordinates $x^i$ to arbitrary curvilinear coordinates $x^i_H$:

$$x^i = f^i (x_H).$$

We then obtain

$$\eta_m^H = \frac{\partial f^i}{\partial x_H^m} \eta_i \, [x (x_H)].$$

Thus, in an arbitrary curvilinear coordinate system of pseudo-Euclidean space–time the Killing vectors have the form

$$\eta_m^H = \frac{\partial f^i (x_H)}{\partial x_H^m} a_i + \frac{\partial f^i (x_H)}{\partial x_H^m} \omega_{in} f^n (x_H). \quad (70)$$

The generalization of the expressions (67)–(69) to the case of arbitrary curvilinear coordinates does not present any particular difficulty. Proceeding in the same way as above, we obtain for the 4-momentum of an isolated system

$$P^i = \int \sqrt{-\gamma (x_H)} \, dx_H^1 \, dx_H^2 \, dx_H^3 \, \frac{\partial f^i (x_H)}{\partial x_H^m} T^{0m} (x_H).$$

In this case, the antisymmetric angular-momentum tensor has the form

$$M^{im} = \int \sqrt{-\gamma (x_H)} \, dx_H^1 \, dx_H^2 \, dx_H^3 T^{0n} (x_H) \times \left[ f^m (x_H) \frac{\partial f^i (x_H)}{\partial x_H^n} - f^i (x_H) \frac{\partial f^m (x_H)}{\partial x_H^n} \right].$$

Thus, the possibility of obtaining integral conservation laws depends on the geometry of space–time. In the case of four dimensions (physical space–time) only spaces of constant curvature possess all ten integral conservation laws; in other spaces, their number is less than ten.

Our analysis shows that if we wish to have the maximal possible number of conserved quantities, we must give up Riemannian geometry of general form and for all fields, including the gravitational field, choose one of the three geometries of constant curvature as the natural geometry.

Since the experimental data obtained from the study of the strong, weak, and electromagnetic interactions indicate that the natural space–time geometry for the fields associated with these interactions is pseudo-Euclidean, it can be assumed, at least at the present level of our knowledge, that this geometry is the single natural geometry for all physical processes, including gravitational ones.

This assertion is one of the basic propositions of the approach which we have developed to the theory of the gravitational interaction. It is obvious that it leads to fulfillment of all the energy-momentum and angular-momentum conservation laws, ensuring the existence of all ten integrals of the motion for the system consisting of the gravitational field and the remaining matter fields.

In our approach, the gravitational field, like all other physical fields, is characterized by an energy-momentum tensor, which contributes to the total energy-momentum tensor of the system. This is the main fundamental difference between our approach and Einstein's theory. It should also be noted that besides the general simplicity in pseudo-Euclidean space–time integration of tensorial quantities always has a quite definite meaning.

Another key question that arises in the construction of the theory of the gravitational field is that of the interaction between it and matter. Influencing matter, the gravitational field can effectively change its geometry if the field occurs in terms with higher derivatives in the equations of motion of the matter. Then the motion of material bodies and other physical fields in pseudo-Euclidean space–time under the influence of the gravitational field will be indistinguishable from their motion in a certain effective Riemannian space–time. It follows from experimental data that the effect of the gravitational field on matter is universal, and therefore the effective Riemannian space–time is common to all matter.

This leads us to a proposition that we call the identity principle (geometrization principle), defining it as follows: The equations of motion of matter under the influence of the gravitational field in the pseudo-Euclidean space–time with metric tensor $\gamma_{ni}$ can be represented identically as the equations of motion of matter in a certain effective Riemannian space–time with metric tensor $g_{ni}$ that depends on the gravitational field and the metric tensor $\gamma_{ni}$.

This principle was introduced and formulated in Ref. 14, though essentially it was already stated in Ref. 35. It means that the description of the motion of matter under the influence of the gravitational field in pseudo-Euclidean space–time is physically identical to the description of the motion of matter in the corresponding effective Riemannian space–time. In such an approach, the gravitational field (as a physical field) is apparently eliminated in the description of the motion of the matter and, putting it figuratively, its energy is used to form the effective Riemannian space–time.

Thus, the effective Riemannian space–time becomes a "carrier" of energy and momentum. The creation of the Riemannian space–time requires as much energy as is contained in the gravitational field, and therefore the propagation of curvature waves in the Riemannian space–time reflects the ordinary transport of energy by gravitational waves in the pseudo-Euclidean space–time. This means that in our approach curvature waves in the Riemannian space–time are a direct consequence of the existence of gravitational waves in the spirit of Faraday and Maxwell, possessing an energy-momentum density.

The idea of the gravitational field as a physical field in the spirit of Faraday and Maxwell, possessing energy and momentum density and spins 2 and 0, changes our ideas about space–time and gravitation formed under the influence of GR. The RTG makes it possible to describe all the existing gravitational experiments, satisfies the correspondence principle, and leads to a number of fundamental consequences.

## 6. GEOMETRIZATION PRINCIPLE AND GENERAL RELATIONS IN THE RELATIVISTIC THEORY OF GRAVITATION

Without loss of generality, we shall assume that the tensor density of the metric tensor $\tilde{g}^{ik}$ of the Riemannian space–time is a local function depending on the density of the metric tensor $\tilde{\gamma}^{ik}$ of Minkowski space and the density of the gravitational field tensor $\tilde{\Phi}^{ik}$.

We shall assume that the matter Lagrangian density $L_M$ depends only on the fields $\Phi_A$, their covariant derivatives of first order, and also, by virtue of the geometrization principle, the density of the metric tensor $\tilde{g}^{ik}$. We shall assume that the Lagrangian density of the gravitational field depends on the density of the metric tensor $\tilde{\gamma}^{ik}$, its partial derivatives of first order, and the density of the gravitational field $\tilde{\Phi}^{ik}$ and its covariant derivatives of first order with respect to the Minkowski metric. To obtain the conservation laws, we use the invariance of the action with respect to infinitesimally small displacements of the coordinates. Since for any given Lagrangian density $L$ the action

$$J = \int L d^4x$$

is a scalar, the variation $\delta J$ corresponding to an arbitrary infinitesimally small coordinate transformation will be zero.

We calculate first the variation of the matter action

$$J_M = \int L_M d^4x$$

under the transformation

$$x'^i = x^i + \xi^i(x), \tag{71}$$

where $\xi^i(x)$ is an infinitesimally small displacement 4-vector:

$$\delta J_M = \int d^4x \left[ \frac{\delta L_M}{\delta \tilde{g}^{mn}} \delta_L \tilde{g}^{mn} + \frac{\delta L_M}{\delta \Phi_A} \delta_L \Phi_A + \mathrm{div} \right] = 0. \tag{72}$$

In (72), div denotes divergence terms, which in the present section are unimportant for our treatment.

As usual, the Eulerian variation is defined by

$$\frac{\delta L}{\delta \varphi} \equiv \frac{\partial L}{\partial \varphi} - \partial_n \frac{\partial L}{\partial(\partial_n \varphi)} + \partial_n \partial_k \frac{\partial L}{\partial(\partial_n \partial_k \varphi)} - \ldots$$

The variations $\delta_L \tilde{g}^{mn}$ and $\delta_L \Phi_A$ under the coordinate transformation (71) can be readily calculated by using their transformation law:

$$\delta_L \tilde{g}^{mn} = \tilde{g}^{kn} D_k \xi^m + \tilde{g}^{km} D_k \xi^n - D_k(\xi^k \tilde{g}^{mn}); \tag{73}$$

$$\delta_L \Phi_A = -\xi^k D_k \Phi_A + F_{A;\,k}^{B;\,n} \Phi_B D_n \xi^k. \tag{74}$$

Here and in what follows, $D_k$ is the covariant derivative with respect to the Minkowski metric. Substituting these expressions in (72) and integrating by parts, we obtain

$$\delta J_M = \int d^4x \Big\{ -\xi^m \Big[ D_k \Big( 2 \frac{\delta L_M}{\delta \tilde{g}^{mn}} \tilde{g}^{kn} \Big) - D_m \Big( \frac{\delta L_M}{\delta \tilde{g}^{lp}} \Big) \tilde{g}^{lp} + D_k \Big( \frac{\delta L_M}{\delta \Phi_A} F_{A;\,m}^{B;\,k} \Phi_m \Big) + \frac{\delta L_M}{\delta \Phi_A} D_m \Phi_A \Big] + \mathrm{div} \Big\} = 0.$$

The vector $\xi^m$ being arbitrary, the condition $\delta J_M = 0$ yields the strong identity

$$D_k \Big( 2 \frac{\delta L_M}{\delta \tilde{g}^{mn}} \tilde{g}^{kn} \Big) - D_m \Big( \frac{\delta L_M}{\delta \tilde{g}^{lp}} \Big) \tilde{g}^{lp} \equiv - D_k \Big( \frac{\delta L_M}{\delta \Phi_A} F_{A;\,m}^{B;\,k} \Phi_B \Big) - \frac{\delta L_M}{\delta \Phi_A} D_m \Phi_A, \tag{75}$$

which holds independently of the fulfillment of the equations of motion for the fields.

We introduce the notation

$$T_{mn} = 2 \frac{\delta L_M}{\delta g^{mn}}; \quad T^{mn} = -2 \frac{\delta L_M}{\delta g_{mn}} = g^{mk} g^{np} T_{kp}; \tag{76}$$

$$\tilde{T}_{mn} = 2 \frac{\delta L_M}{\delta \tilde{g}^{mn}}; \quad \tilde{T}^{mn} = -2 \frac{\delta L_M}{\delta \tilde{g}_{mn}} = \tilde{g}^{mk} \tilde{g}^{np} \tilde{T}_{kp}. \tag{77}$$

Here, $T_{mn}$ is the density of the matter energy-momentum tensor in the Riemannian space and is called the Hilbert tensor density.

Taking into account (77), we can represent the left-hand side of (75) in the form

$$D_k(\tilde{T}_{mn} \tilde{g}^{kn}) - \frac{1}{2} \tilde{g}^{kp} D_m \tilde{T}_{kp} = \partial_k(\tilde{T}_{mn} \tilde{g}^{kn}) - \frac{1}{2} \tilde{g}^{kp} \partial_m \tilde{T}_{kp}.$$

The right-hand side of this equation can be readily reduced to the form

$$\partial_k(\tilde{T}_{mn} \tilde{g}^{kn}) - \frac{1}{2} \tilde{g}^{kp} \partial_m \tilde{T}_{kp} = \tilde{g}_{mn} \nabla_k \Big( \tilde{T}^{kn} - \frac{1}{2} \tilde{g}^{kn} \tilde{T} \Big), \tag{78}$$

where $\tilde{T} = \tilde{g}_{kp} \tilde{T}^{kp}$, and $\nabla_k$ is the covariant derivative with respect to the metric of the Riemannian space.

On the basis of (78), the strong identity (75) can be written in the form

$$\tilde{g}_{mn} \nabla_k \Big( \tilde{T}^{kn} - \frac{1}{2} \tilde{g}^{kn} \tilde{T} \Big) = - D_k \Big( \frac{\delta L_M}{\delta \Phi_A} F_{A;\,m}^{B;\,k} \Phi_B \Big) - \frac{\delta L_M}{\delta \Phi_A} D_m \Phi_A. \tag{79}$$

By the principle of least action, the equations of motion for the matter fields have the form

$$\frac{\delta L_M}{\delta \Phi_A} = 0. \tag{80}$$

Taking into account these equations, we find from (79) the weak identity

$$\nabla_m \Big( \tilde{T}^{mn} - \frac{1}{2} \tilde{g}^{mn} \tilde{T} \Big) \equiv 0. \tag{81}$$

We note that the density of the matter energy-momentum tensor in the Riemannian space, $T^{mn}$, is related to $\tilde{T}^{mn}$ by

$$\sqrt{-g}\,T^{mn}=\tilde{T}^{mn}-\frac{1}{2}\tilde{g}^{mn}\tilde{T}. \tag{82}$$

Therefore, from the expression (81) we obtain a covariant conservation equation in the Riemannian space:

$$\nabla_m T^{mn}=0. \tag{83}$$

If the number of equations for the matter field is four, then in this and only in this case one can always use instead of the equations (80) for this field the equivalent equations (83).

The variation of the action integral (72) can be expressed in the equivalent form

$$\delta J_M=\int d^4x\left\{\frac{\delta L_M}{\delta\tilde{\Phi}^{mn}}\delta_L\tilde{\Phi}^{mn}+\frac{\delta L_M}{\delta\tilde{\gamma}^{mn}}\delta_L\tilde{\gamma}^{mn}+\frac{\delta L_M}{\delta\Phi_A}\delta_L\Phi_A+\mathrm{div}\right\}=0. \tag{84}$$

The variations $\delta_L\tilde{\Phi}^{mn}$ and $\delta_L\tilde{\gamma}^{mn}$ under the coordinate transformation (71) are

$$\delta_L\tilde{\Phi}^{mn}=\tilde{\Phi}^{kn}D_k\xi^m+\tilde{\Phi}^{km}D_k\xi^n-D_k(\xi^k\tilde{\Phi}^{mn}); \tag{85}$$

$$\delta_L\tilde{\gamma}^{mn}=\tilde{\gamma}^{kn}D_k\xi^m+\tilde{\gamma}^{km}D_k\xi^n-\tilde{\gamma}^{mn}D_k\xi^k. \tag{86}$$

Substituting the expressions for the variations $\delta_L\tilde{\Phi}^{mn}$, $\delta_L\tilde{\gamma}^{mn}$, $\delta_L\Phi_A$ in (84) and integrating, by parts, we obtain $\xi^m$ being arbitrary, the strong identity

$$D_k\left(2\frac{\delta L_M}{\delta\tilde{\Phi}^{mn}}\tilde{\Phi}^{kn}\right)-D_m\left(\frac{\delta L_M}{\delta\tilde{\Phi}^{kp}}\right)\tilde{\Phi}^{kp}+D_k\left(2\frac{\delta L_M}{\delta\tilde{\gamma}^{mn}}\tilde{\gamma}^{kn}\right)-D_m\left(\frac{\delta L_M}{\delta\tilde{\gamma}^{kp}}\tilde{\gamma}^{kp}\right)\equiv-D_k\left(\frac{\delta L_M}{\delta\Phi_A}F^{B\;k}_{A\;m}\Phi_B\right)-\frac{\delta L_M}{\delta\Phi_A}D_m\Phi_A, \tag{87}$$

which, like (75), holds independently of the fulfillment of the equations of motion of the matter and the gravitational field.

For any Lagrangian, we introduce certain notation and relations that will be used in what follows:

$$\tilde{t}^{mn}\equiv-2\frac{\delta L}{\delta\tilde{\gamma}_{mn}};\quad t^{mn}\equiv-2\frac{\delta L}{\delta\gamma_{mn}}; \tag{88}$$

$$t^{mn}\equiv\frac{1}{\sqrt{-\gamma}}\left(\tilde{t}^{mn}-\frac{1}{2}\tilde{\gamma}^{mn}\tilde{t}\right) \tag{89}$$

Since $L_M$ by virtue of the geometrization principle depends only on $\tilde{\gamma}^{mn}$ through $\tilde{g}^{mn}$, we can readily find the connection between $\tilde{t}_{(M)\,mn}$ and $\tilde{T}_{mn}$:

$$\tilde{t}_{(M)\,mn}=2\frac{\delta L_M}{\delta\tilde{\gamma}^{mn}}=\tilde{T}_{kp}\frac{\partial\tilde{g}^{kp}}{\partial\tilde{\gamma}^{mn}}. \tag{90}$$

Here, we have taken into account the definition (77).

Taking into account the identity

$$\frac{\partial\tilde{g}^{kp}}{\partial\tilde{\gamma}_{mn}}\equiv-\tilde{\gamma}^{ml}\tilde{\gamma}^{nq}\frac{\partial\tilde{g}^{pk}}{\partial\tilde{\gamma}^{lq}},$$

we obtain on the basis of (88)

$$\tilde{t}^{mn}_{(M)}=-\tilde{T}_{pk}\frac{\partial\tilde{g}^{pk}}{\partial\tilde{\gamma}_{mn}}. \tag{91}$$

Taking into account in (91) the identity (77), and also the relation

$$-\tilde{g}_{lp}\tilde{g}_{qk}\frac{\partial\tilde{g}^{lq}}{\partial\tilde{\gamma}_{mn}}=\frac{\partial\tilde{g}_{pk}}{\partial\tilde{\gamma}_{mn}},$$

we obtain from (91)

$$\tilde{t}^{mn}_{(M)}=\tilde{T}^{pk}\frac{\partial\tilde{g}_{pk}}{\partial\tilde{\gamma}_{mn}}. \tag{92}$$

Comparing now the identities (79) and (87) with allowance for (88), we find

$$\tilde{g}_{mn}\nabla_k\left(\tilde{T}^{kn}-\frac{1}{2}\tilde{g}^{kn}\tilde{T}\right)=\tilde{\gamma}_{mn}D_k\left(\tilde{t}^{kn}_{(M)}-\frac{1}{2}\tilde{\gamma}^{kn}\tilde{t}_{(M)}\right)+D_k\left(2\frac{\delta L_M}{\delta\tilde{\Phi}^{mn}}\tilde{\Phi}^{kn}\right)-D_m\left(\frac{\delta L_M}{\delta\tilde{\Phi}^{kp}}\right)\tilde{\Phi}^{kp}. \tag{93}$$

Similarly, from the invariance of the gravitational-field action $J_g=\int L_g\,d^4x$ with respect to the coordinate transformations (71) we have

$$\tilde{\gamma}_{mn}D_k\left(\tilde{t}^{kn}_{(g)}-\frac{1}{2}\tilde{\gamma}^{kn}\tilde{t}_{(g)}\right)+D_k\left(2\frac{\delta L_g}{\delta\tilde{\Phi}^{mn}}\tilde{\Phi}^{kn}\right)-D_m\left(\frac{\delta L_g}{\delta\tilde{\Phi}^{kp}}\right)\tilde{\Phi}^{kp}=0 \tag{94}$$

Adding the expressions (93) and (94), we find

$$\tilde{g}_{mn}\nabla_k\left(\tilde{T}^{kn}-\frac{1}{2}\tilde{g}^{kn}\tilde{T}\right)=\tilde{\gamma}_{mn}D_k\left(\tilde{t}^{kn}-\frac{1}{2}\tilde{\gamma}^{kn}\tilde{t}\right)+D_k\left(2\frac{\delta L}{\delta\tilde{\Phi}^{mn}}\tilde{\Phi}^{kn}\right)-D_m\left(\frac{\delta L}{\delta\tilde{\Phi}^{kp}}\right)\tilde{\Phi}^{kp}. \tag{95}$$

Here and in what follows,

$$\tilde{t}^{kn}=\tilde{t}^{kn}_{(g)}+\tilde{t}^{kn}_{(M)}. \tag{96}$$

From the principle of least action the equations for the gravitational field have the form

$$\frac{\delta L}{\delta\tilde{\Phi}^{mn}}=\frac{\delta L_g}{\delta\tilde{\Phi}^{mn}}+\frac{\delta L_M}{\delta\tilde{\Phi}^{mn}}=0. \tag{97}$$

Taking into account these equations, we obtain from (95) the very important equation

$$\tilde{g}_{mn}\nabla_k\left(\tilde{T}^{kn}-\frac{1}{2}\tilde{g}^{kn}\tilde{T}\right)=\tilde{\gamma}_{mn}D_k\left(\tilde{t}^{kn}-\frac{1}{2}\tilde{\gamma}^{kn}\tilde{t}\right). \tag{98}$$

Since the density of the total energy-momentum tensor in the Minkowski space is given by

$$\sqrt{-\gamma}\,t^{kn}=\tilde{t}^{kn}-\frac{1}{2}\tilde{\gamma}^{kn}\tilde{t}, \tag{99}$$

we can write the relation (98), using the expression (99) and also (82), in the form

$$D_m t^m_n\equiv\nabla_m T^m_n. \tag{100}$$

This last relation reflects the geometrization principle: The covariant divergence in the pseudo-Euclidean space of the sum of the densities of the energy-momentum tensors of the matter and the gravitational field is exactly equal to the covariant divergence in the effective Riemannian space of the density of the matter energy-momentum tensor alone. If the equations of motion for the matter hold,

$$D_m t^m_n=\nabla_m T^m_n=0. \tag{101}$$

From the covariant conservation equation for the matter in the Riemannian space it is not clear what is conserved, whereas from the conservation law of the total energy-momentum tensor $t^m_n$ in the Minkowski space it is clear that we have conservation of the energy and momentum of the matter and the gravitational field taken together. Thus, in this theory the Riemannian space arises as a result of the influence of the gravitational field on all forms of matter; it is therefore an effective Riemannian space of field origin. The Minkowski space finds its precise physical reflection in the conservation laws for the energy-momentum and angular-momentum tensors of the matter and the gravitational field taken together.

Since there exist ten Killing vectors in flat space, there are also ten conserved integral quantities for a closed system of fields.

Since the conservation equation for the total energy-momentum tensor in the Minkowski space

$$D_m t_n^m = D_m (t_{(g)n}^m + t_{(M)n}^m) = 0 \tag{102}$$

is equivalent to the covariant conservation equation for the matter in the Riemannian space, and the latter is equivalent to the equations of motion for the matter, we can use (102) instead of the equation of motion for the matter.

It must be particularly emphasized that both the matter and the gravitational field in this theory are characterized by energy-momentum tensors, so that in our case, in contrast to GR, it is in principle impossible for pseudotensors to arise, and therefore there are no unphysical concepts such as nonlocalizability of the gravitational energy.

If, following Hilbert and Einstein, we had taken the Lagrangian density of the gravitational field in a fully geometrized form, i.e., dependent only on the metric tensor of the Riemannian space $g^{ik}$ and its derivatives, for example,

$$L_g = \sqrt{-g}\, R$$

(where $R$ is the scalar curvature of the Riemannian space), the density of the energy-momentum tensor of the free gravitational field in the Minkowski space would always be equal to zero by virtue of the field equations:

$$\frac{\delta L_g}{\delta \gamma^{mn}} = \frac{\delta L_g}{\delta g^{pk}} \frac{\partial g^{pk}}{\partial \gamma^{mn}} = 0. \tag{103}$$

Thus, on the basis of the Minkowski space and using a tensorial physical field possessing energy and momentum it is in principle impossible to construct a fully geometrized Lagrangian of the gravitational field. Therefore, a theory constructed on the basis of a fully geometrized Lagrangian is in principle incapable of describing a physical gravitational field in the spirit of Faraday and Maxwell in Minkowski space. It has hitherto been asserted in the literature (see, for example, Ref. 36) that in Minkowski space one can, using a tensor field of spin 2, uniquely find the Lagrangian of the gravitational field in GR, which is equal to the scalar curvature $R$. However, these studies do not have any physical content, since the energy-momentum tensor for the gravitational field introduced in them is equal to zero, as can be seen from (103). Therefore, these studies are physically meaningless and their results are incorrect.

## 7. FUNDAMENTAL IDENTITY

As was shown in Ref. 37, a symmetric second-rank tensor $f^{ik}$ in Minkowski space can be decomposed into a direct sum of irreducible representations: one representation with spin 2, one with spin 1, and two with spin 0,

$$f^{lm} = [P_2 + P_1 + P_0 + P_{0'}]_{ik}^{lm} f^{ik}, \tag{104}$$

where $P_s$ $(s = 2, 1, 0, 0')$ are projection operators satisfying the standard relations

$$\left.\begin{aligned} & P_s P^t = \delta_s^t P_t \quad \text{(here no summation over } t\,!); \\ & P_{s;\,in}^{in} = (2s+1); \\ & \sum_s P_{s;\,ik}^{lm} = \frac{1}{2} (\delta_i^l \delta_k^m + \delta_i^m \delta_k^l) \equiv \delta_{ik}^{lm}. \end{aligned}\right\} \tag{105}$$

It is convenient to express the operators $P_s$ first in the momentum representation. To this end, we introduce the auxiliary (projection) quantities

$$X_{ik} = \frac{1}{\sqrt{3}} \left(\gamma_{ik} - \frac{q_i q_k}{q^2}\right); \; Y_{ik} = \frac{q_i q_k}{q^2}. \tag{106}$$

It is easy to show that the operators $P_s$ satisfying (105) can be expressed in terms of (106) as follows:

$$P_{0;\,ni}^{ml} = X_{ni} X^{lm}; \; P_{0';\,ni}^{ml} = Y_{ni} Y^{ml}; \tag{107}$$

$$P_{1;\,ni}^{ml} = \frac{\sqrt{3}}{2} [X_i^l Y_n^m + X_n^m Y_i^l + X_i^m Y_n^l + X_n^l Y_i^m]; \tag{108}$$

$$P_{2;\,ni}^{ml} = \frac{3}{2} [X_i^l X_n^m + X_i^m X_n^l] - X_{ni} X^{ml}. \tag{109}$$

It can be seen from (107)–(109) that the operators $P_{s;ni}^{ml}$ are symmetric with respect to the indices $ml$ and $ni$.

In the $x$ representation, the projection operators $P_s$ are nonlocal integro-differential operators:

$$(P_{s;\,ni}^{ml} f^{ni}) = \int d^4y P_{s;\,ni}^{lm} (x-y) f^{ni} (y).$$

The explicit expressions for $P_{0;ni}^{lm}(x)$ and $P_{2;ni}^{ml}(x)$ are

$$P_{0;\,ni}^{lm}(x) = \frac{1}{3} [\gamma^{lm} \gamma_{in} \delta(x) + (\gamma^{lm} \partial_i \partial_n + \gamma_{in} \partial^l \partial^m) D(x) + \partial_i \partial_n \partial^l \partial^m \Delta(x)]; \tag{110}$$

$$P_{2;\,ni}^{lm}(x) = \left(\delta_{in}^{lm} - \frac{1}{3} \gamma^{lm} \gamma_{in}\right) \delta(x) + \Big[\frac{1}{2} (\delta_i^l \partial^m \partial_n + \delta_n^m \partial^l \partial_i + \delta_n^l \partial^m \partial_i + \delta_i^m \partial^l \partial_n) - \frac{1}{3} (\gamma^{lm} \partial_i \partial_n + \gamma_{in} \partial^l \partial^m)\Big] D(x) + \frac{2}{3} \partial^l \partial^m \partial_i \partial_n \Delta(x). \tag{111}$$

In (110) and (111), $D(x)$ is the Green's function of the wave equation,

$$\Box D(x) = -\delta(x), \tag{112}$$

and $\Delta(x) = \int d^4y D(x-y) D(y)$, and we therefore have the equation

$$\Box \Delta(x) = -D(x). \tag{113}$$

On the basis of Eqs. (110)–(113), it is readily verified that the operators $P_0$ and $P_2$ are conserved, i.e., for these operators the following identities hold:

$$\left.\begin{aligned} & \partial_l P_{0;\,ni}^{lm}(x) = \partial^n P_{0;\,ni}^{lm}(x) \equiv 0; \\ & \partial_l P_{2;\,ni}^{lm}(x) = \partial^n P_{2;\,ni}^{lm}(x) = 0. \end{aligned}\right\} \tag{114}$$

But the operators $P_1$ and $P'_0$ do not possess this property.

It is clear from the decomposition (104) that if the tensor field satisfies the equation

$$\partial_l f^{lm} = 0, \tag{115}$$

the representations with spins 1 and 0′ do not occur in it. This means that such a tensor field describes only spins 2 and 0.

By virtue of (110)–(112), it is easy to show that the operator

$$\begin{aligned}\Box(2P_0 - P_2)^{mn}_{il} &= -(\delta^{mn}_{il} - \gamma^{mn}\gamma_{il})\,\Box\,\delta(x) \\ &- (\gamma^{mn}\partial_i\partial_l + \gamma_{il}\partial^m\partial^n)\,\delta(x) + \frac{1}{2}(\delta^n_i\partial^m\partial_l + \delta^m_l\partial^n\partial_i \\ &+ \delta^n_l\partial^m\partial_i + \delta^m_i\partial^n\partial_l)\,\delta(x)\end{aligned} \tag{116}$$

is the unique local and conserved operator of second order.

Applying this operator to the function

$$\varphi^{il} - \frac{1}{2}\gamma^{il}\varphi,$$

where $\varphi = \gamma_{pq}\varphi^{pq}$, we find

$$\begin{aligned}\Psi^{mn} &= \int \Box_y\,[2P_0(x-y) \\ &- P_2(x-y)]^{mn}_{il}\,(\varphi^{il}(y) - \frac{1}{2}\gamma^{il}\varphi(y))\,d^4y \\ &= \partial_k\partial_p\,[\gamma^{nk}\varphi^{pm} + \gamma^{mk}\varphi^{np} - \gamma^{pk}\varphi^{nm} - \gamma^{mn}\varphi^{pk}].\end{aligned} \tag{117}$$

The structure of (117) for any symmetric tensor field is remarkable in that it is local, linear, contains derivatives of only second order, and satisfies a conservation law, i.e., the divergence of $\Psi^{mn}$ is identically equal to zero:

$$\partial_m \Psi^{mn} = 0. \tag{118}$$

In what follows, we shall need the structure (117), expressed in terms of covariant derivatives with respect to the Minkowski metric for the density of the metric tensor $\tilde{g}^{lm}$:

$$J^{mn} = D_k D_p\,[\gamma^{np}\tilde{g}^{km} + \gamma^{pm}\tilde{g}^{kn} - \gamma^{kp}\tilde{g}^{mn} - \gamma^{mn}\tilde{g}^{kp}]. \tag{119}$$

It is obvious from (119) that the following identity holds:

$$D_m J^{mn} \equiv 0, \tag{120}$$

and we call it the fundamental identity, since it has fundamental importance for the construction of the RTG.

## 8. EQUATIONS OF THE RELATIVISTIC THEORY OF GRAVITATION

In general relativity, Einstein made the metric tensor $g^{ik}$ of Riemannian space the characteristic of the gravitational field. But this was a profound delusion and it must be abandoned, since it is impossible to impose physical boundary conditions on the behavior of $g^{ik}$, their asymptotic behavior depending on the arbitrariness in the choice of the spatial coordinate systems.

In the present section we construct in the framework of relativity theory and the geometrization principle relativistic equations for matter and the gravitational field.

The connection between the effective metric of the field Riemannian space and the gravitational field can always be chosen in the very simple form

$$\tilde{g}^{ik} = \sqrt{-g}\,g^{ik} = \sqrt{-\gamma}\,\gamma^{ik} + \sqrt{-\gamma}\,\Phi^{ik}. \tag{121}$$

In our theory, the field variable of the gravitational field is the tensor $\Phi^{ik}$. We shall assume that the gravitational field has in the general case only spins 2 and 0. As we saw in Sec. 7, such physical requirements lead in Galilean coordinates to the following four equations of the gravitational field:

$$\partial_i \Phi^{ik} = \partial_i \tilde{g}^{ik} = 0. \tag{122}$$

Similar conditions have sometimes been used already[15,38] in GR as a special class of harmonic coordinate conditions for solving problems of island type. The importance of harmonic coordinate conditions for the solution of island problems was especially emphasized by Fock (Ref. 15, p. 476). For example, he wrote: "The remarks made above about the privileged nature of the harmonic coordinate system must not in any way be understood in the sense of prohibition on the use of other coordinate systems. Nothing could be more alien to our point of view than such an interpretation of it..."; and further: "...although the existence of harmonic coordinates is a fact of the first importance, both theoretically and practically, it in no way precludes the possibility of using other nonharmonic coordinate systems." From the point of view of our theory, Fock, in solving island problems, was simply, without being aware of it, dealing with ordinary Galilean coordinates in an inertial frame of reference, and these, as is known from relativity theory, are, of course, distinguished. In Fock's calculations of island systems, the harmonic conditions were therefore not coordinate conditions, as he thought, but, as we see from our theory, field equations in the Galilean coordinates of an inertial frame of reference. It was for this reason that they played an important part in his actual calculations, something that Fock, of course, as incidentally, others, did not suspect.

Thus, Fock regarded harmonic conditions only as privileged coordinate conditions and nothing more, making a restriction, moreover, to problems of island type. This is understandable, for, like all his great predecessors, he was a prisoner of Riemannian geometry, and it in principle made impossible a deeper penetration to the essence of the problem. To make the fundamental step and advance these conditions it was necessary to abandon the ideology of general relativity, to extricate oneself from the jungle of Riemannian geometry, to extend, general relativity notwithstanding, the relativity principle to gravitational phenomena, and to introduce the concept of the gravitational field as a physical field in the spirit of Faraday and Maxwell possessing energy and momentum. And all this has been done in our theory, the choice, moreover, of the coordinate system in our theory being arbitrary and specified only by the metric tensor $\gamma^{ik}$ of Minkowski space, as is usually assumed in the theory of elementary particles. Moreover, in our theory Eqs. (122) are universal, being equations of the gravitational field. They do not have any bearing on the choice of the coordinate system. In Minkowski space, these equations are written in the covariant form

$$\sqrt{-\gamma}\, D_i \Phi^{ik} \equiv D_i \tilde{g}^{ik} = 0. \tag{123}$$

On the basis of the results of Sec. 7, we see that these field equations automatically eliminate from the gravitational tensor field the spins 1 and 0′. Thus, for the required 14 variables of the gravitational field and the matter we have already constructed the four covariant equations (123). To construct the following ten equations, we use a simple but far-reaching analogy with the electromagnetic field. Since any vector field $A^n$ contains spins 1 and 0, it can be decomposed into the direct sum of the corresponding irreducible representations. This decomposition can be implemented by means of the projection operators (106) introduced in Sec. 7:

$$A^n = X^n_m A^m + Y^n_m A^m, \tag{124}$$

in which the operator $X^n_m$ is conserved, i.e., satisfies the identities

$$\partial_n X^n_m = \partial^m X^n_m = 0, \tag{125}$$

while the operator $Y^n_m$ does not have this property.

It is well known from electrodynamics that the source of the electromagnetic field $A^n$ is the conserved electromagnetic current $j^n$. Therefore, to construct the equation of motion for the field it is also natural to use the conserved operator $X^n_m$. But this operator is nonlocal. On its basis, one can, however, construct a unique (containing second derivatives) local, linear, and conserved operator $\square X^n_m$. Applying this operator to $A^m$, we obtain an expression that in terms of covariant derivatives has the form

$$\gamma^{mk} D_m D_k A^n - D^n D_m A^m. \tag{126}$$

Postulating the equation

$$\gamma^{mk} D_m D_k A^n - D^n D_m A^m = \frac{4\pi}{c} j^n, \tag{127}$$

we obtain Maxwell's well-known equations.

One of the most important properties of the electrodynamic equations (127) is their invariance with respect to the gauge transformation

$$A^n \to A^n + D^n \varphi, \tag{128}$$

where $\varphi$ is an arbitrary scalar function.

None of the physical quantities are changed by the gauge transformation (128). This means that they do not depend on the presence of the spin 0 in the vector field $A^n$. Therefore, a gauge transformation can be chosen in such a way that the spin 0 is always eliminated from the vector field. This means that one can introduce the condition

$$D_m A^m = 0. \tag{129}$$

Thus, in electrodynamics the condition (129) can be introduced, though it need not be, since the spin 0 of the vector field does not, by virtue of the gauge invariance, affect the physical quantities.

Taking into account Eqs. (129) in (127), we find the system of equations

$$\gamma^{mk} D_m D_k A^n = \frac{4\pi}{c} j^n; \quad D_m A^m = 0,$$

which determine the vector potential $A^n$, which possesses only spin 1.

The Lagrangian formalism leading to these results is well known. We note that the idea of constructing the theory of interaction for vector fields (both Abelian and non-Abelian) on the basis of gauge invariance has proved to be extremely fruitful and is currently being successfully developed.

The problems which we encounter in constructing the remaining equations for the tensor gravitational field are of a quite different nature, since its source—the energy-momentum tensor—is noninvariant with respect to a gauge transformation of the field $\tilde{\Phi}^{ik}$. There will be a more detailed discussion of this below, while here, by analogy with Maxwellian electrodynamics, we construct the remaining equations for the tensor gravitational field. The unique conserved second-rank tensor is the energy-momentum tensor $t^{mn}$ of the matter and the gravitational field in the Minkowski space, and it is therefore natural to take it as the total source of the gravitational field. Since, as we established in Sec. 7, the simplest identically conserved tensor linear in $\tilde{g}^{mn}$ is $J^{mn}$, we postulate by analogy with electrodynamics the equations

$$J^{mn} \equiv D_k D_p [\gamma^{kn}\tilde{g}^{pm} + \gamma^{km}\tilde{g}^{pn} - \gamma^{kp}\tilde{g}^{mn} - \gamma^{mn}\tilde{g}^{kp}] = \lambda (t^{mn}_{(g)} + t^{mn}_{(M)}). \tag{130}$$

Such a form of the equations presupposes in general automatic fulfillment of the conservation law for the energy-momentum tensor of the matter and the gravitational field in the Minkowski space,

$$D_m (t^{mn}_{(g)} + t^{mn}_{(M)}) \equiv D_m t^{mn} = 0, \tag{131}$$

and also, as a consequence [see (101)], the fulfillment of the covariant conservation law for the matter in the Riemannian space:

$$\nabla_m T^{mn} = 0. \tag{83}$$

The Hilbert energy-momentum tensor $T^{mn}$ can be specified phenomenologically. In this case, (83) are the equations of motion for the matter.

Using Eqs. (123) and (130), we obtain

$$\left.\begin{aligned} &\gamma^{kp} D_k D_p \tilde{g}^{mn} = -\lambda (t^{mn}_{(g)} + t^{mn}_{(M)}); \\ &D_m \tilde{g}^{mn} = 0. \end{aligned}\right\} \tag{132}$$

The system of equations (132) is the required system for the RTG.

The part played by the second equation of the system (132) in the RTG is very different from the part played by (129) in electrodynamics. Indeed, although the left-hand side of (130) is invariant with respect to the gauge transformation

$$\tilde{g}^{mn} \to \tilde{g}^{mn} + D^m \xi^n + D^n \xi^m - \gamma^{mn} D_k \xi^k, \tag{133}$$

where $\tilde{\xi}^n = \sqrt{-\gamma}\xi^n$ is the density of the arbitrary 4-vector $\xi^n(x)$, the right-hand side of (130) being noninvariant with respect to the substitution (133), we do not have in the theory an arbitrariness of the type (133), and therefore (123) cannot be a consequence of (130).

Thus, in the RTG Eqs. (123) are additional independent dynamical equations of the gravitational field and not coordinate or gauge conditions.

The main question which must be answered in the construction of the theory is the following: Does there exist a Lagrangian density for a gravitational field with spins 2 and 0 that leads automatically on the basis of the principle of least action to the first equation of the system (132)?

The general Lagrangian density of a gravitational field $\tilde{\Phi}^{ik}$ describing spins 2 and 0 and quadratic in the first derivatives of the field has the form

$$L_g = a\tilde{g}_{km}\tilde{g}_{nq}\tilde{g}^{lp}D_l\tilde{g}^{kq}D_p\tilde{g}^{mn} + b\tilde{g}_{kq}D_m\tilde{g}^{pq}D_p\tilde{g}^{km} + c\tilde{g}_{km}\tilde{g}_{nq}\tilde{g}^{lp}D_l\tilde{g}^{km}D_p\tilde{g}^{nq}. \quad (134)$$

A characteristic feature of this Lagrangian is that the contraction of the covariant derivatives with respect to the Minkowski metric can be made by means of the effective metric tensor $\tilde{g}^{ik}$ of the Riemannian space.

We shall see below (see Sec. 10) that this requirement for the gravitational field is a consequence of the geometrization principle and the structure of a gravitational field possessing only the spins 2 and 0.

By the principle of least action, the system of equations for the gravitational field is

$$\frac{\delta L_g}{\delta\tilde{\Phi}^{ik}} + \frac{\delta L_M}{\delta\tilde{\Phi}^{ik}} \equiv \frac{\delta L_g}{\delta\tilde{g}^{ik}} + \frac{\delta L_M}{\delta\tilde{g}^{ik}} = 0. \quad (135)$$

We have here taken into account the constraint (121). In (135), $L_M$ is the Lagrangian density of the matter, and the Lagrangian density $L_g$ is given by (134).

If the system of equations (135) is to take the form of the first equation of the system (132), the constants $a$, $b$, and $c$ in the Lagrangian density (134) must be chosen in a definite and unique manner. To this end, on the basis of Eqs. (88), (89), (96), and (99) we find for the Lagrangian $L = L_g + L_M$ the density of the energy-momentum tensor of the matter and the gravitational field $t^{mn}$ in the Minkowski space. Calculating the variation of the total Lagrangian with respect to $\gamma_{mn}$, we obtain

$$t^{mn} = 2\sqrt{-\gamma}\left(\gamma^{nk}\gamma^{mp} - \frac{1}{2}\gamma^{mn}\gamma^{pk}\right)\frac{\delta L}{\delta\tilde{g}^{kp}} + 2bJ^{mn} + D_p\{(2a+b)[H_k^{pn}\gamma^{km} + H_k^{pm}\gamma^{kn} - H_k^{mn}\gamma^{kp}] - 2(a+2c)\gamma^{mn}\tilde{g}^{kp}\tilde{g}_{lq}D_k\tilde{g}^{lq}\}, \quad (136)$$

where $H_k^{pn} = (\tilde{g}^{pl}D_l\tilde{g}^{qn} + \tilde{g}^{nl}D_l\tilde{g}^{pq})\tilde{g}_{qk}$.

It can be seen from (136) that the equations

$$t^{mn} = 2bJ^{mn} + D_p\{(2a+b)[H_k^{pn}\gamma^{km} + H_k^{pm}\gamma^{kn} - H_k^{mn}\gamma^{kq}] - 2(a+2c)\gamma^{mn}\tilde{g}^{kp}\tilde{g}_{lq}D_k\tilde{g}^{lq}\} \quad (137)$$

are equivalent to the field equations (135).

To ensure that Eq. (131) does not yield any new equation for the field $\Phi^{ik}$, in which case we should have an overdetermined system of equations, it is necessary and sufficient that the coefficients $a$, $b$, and $c$ satisfy the conditions

$$a = -b/2; \quad c = b/4. \quad (138)$$

For this choice of the constants, we have the identity

$$D_m t^{mn} \equiv 0.$$

Thus, the equations of motion of the matter follow directly from the equations for the gravitational field. With allowance for the relations (138) between the coefficients, the expression (137) takes the form

$$D_pD_k(\gamma^{km}\tilde{g}^{pn} + \gamma^{kn}\tilde{g}^{pm} - \tilde{g}^{mn}\gamma^{kp} - \gamma^{mn}\tilde{g}^{kp}) = \frac{1}{2b}(t^{mn}_{(g)} + t^{mn}_{(M)}) \equiv \frac{1}{2b}t^{mn}, \quad (139)$$

which is identical to (130), which we wrote down earlier by analogy with electrodynamics, if we set

$$2b = 1/\lambda.$$

Thus, the Lagrangian density $L_g$ which leads us to field equations in the form (139) has the form

$$L_g = \frac{1}{2\lambda}\left[\tilde{g}_{kq}D_m\tilde{g}^{pq}D_p\tilde{g}^{km} - \frac{1}{2}\tilde{g}_{km}\tilde{g}_{nq}\tilde{g}^{lp}D_l\tilde{g}^{kq}D_p\tilde{g}^{mn} + \frac{1}{4}\tilde{g}_{km}\tilde{g}_{nq}\tilde{g}^{lp}D_l\tilde{g}^{km}D_p\tilde{g}^{nq}\right]. \quad (140)$$

It follows from the correspondence principle that the constant $\lambda$ has the value

$$\lambda = -16\pi. \quad (141)$$

With allowance for (141), the Lagrangian density (140) can be represented in the form

$$L_g = \frac{1}{32\pi}[\tilde{G}^l_{mn}D_l\tilde{g}^{mn} - \tilde{g}^{mn}\tilde{G}^k_{mk}\tilde{G}^l_{nl}], \quad (142)$$

where the third-rank tensor $\tilde{G}^k_{lm}$ is determined in accordance with the formula

$$\tilde{G}^k_{lm} = \frac{1}{2}\tilde{g}^{pk}(D_m\tilde{g}_{lp} + D_l\tilde{g}_{mp} - D_p\tilde{g}_{lm}). \quad (143)$$

The Lagrangian density can also be written in the form

$$L_g = -\frac{1}{16\pi}\sqrt{-g}\,g^{mn}[G^k_{lm}G^l_{nk} - G^l_{mn}G^k_{lk}]. \quad (144)$$

Such a Lagrangian was first considered by Rosen.[39] In (144), we have the third-rank tensor

$$G^k_{ml} = \frac{1}{2}g^{kp}(D_mg_{pl} + D_lg_{mp} - D_pg_{lm}). \quad (145)$$

With allowance for Eq. (123), the complete system of equations of the RTG for the matter and the gravitational field is

$$\left.\begin{aligned} &\gamma^{pk}D_pD_k\tilde{g}^{mn} = 16\pi t^{mn};\\ &D_m\tilde{g}^{mn} = 0.\end{aligned}\right\} \quad (146)$$

It is obvious that in a Galilean coordinate system Eqs. (146) take the form

$$\left.\begin{aligned} &\Box\tilde{g}^{mn} = 16\pi t^{mn};\\ &\partial_m\tilde{g}^{mn} = 0.\end{aligned}\right\} \quad (147)$$

If we were to restrict ourselves to Eq. (130), the division of the metric of the Riemannian space into the metric in the Minkowski space and the tensor gravitational field would be nominal only and devoid of any physical meaning. The system (123) of four field equations separates essentially everything that has to do with inertial forces from everything that has to do with the gravitational field. Both of the equations (146) are generally covariant. On the behavior of the gravitational field there are imposed, as usual, appropriate physical conditions in a given, for example, Galilean coordinate system. In GR, it is impossible to formulate physical conditions on the metric $g^{mn}$ if one remains in Riemannian space, since the asymptotic behavior of the metric always depends on the choice of the three-dimensional coordinate system.

We now find the explicit form of the system of equations (135). For the Lagrangian (140), it is possible to show that

$$\frac{\partial L_g}{\partial \tilde{g}^{mn}} = \frac{1}{16\pi}\,[G^k_{ml}G^l_{kn} - G^k_{mn}G^l_{kl}];$$

and

$$\frac{\partial L_g}{\partial(\partial_k \tilde{g}^{mn})} = \frac{1}{16\pi}\left[G^k_{mn} - \frac{1}{2}\,\delta^k_m G^l_{nl} - \frac{1}{2}\,\delta^k_n G^l_{ml}\right].$$

Therefore

$$\delta L_g/\delta\tilde{g}^{mn} = -\frac{1}{16\pi}\,R_{mn}, \qquad (148)$$

where $R_{mn}$ is the second-rank curvature tensor of the Riemannian space, equal to

$$R_{mn} = \partial_k G^k_{mn} - \partial_m G^l_{nl} + G^k_{mn}G^l_{kl} - G^k_{ml}G^l_{nk}. \qquad (149)$$

Since by virtue of (77) and (82)

$$\frac{\delta L_M}{\delta \tilde{g}^{mn}} = \frac{1}{\sqrt{-g}}\left(T_{mn} - \frac{1}{2}\,g_{mn}T\right), \qquad (150)$$

we find from (135)

$$\sqrt{-g}\,R_{mn} = 8\pi\left(T_{mn} - \frac{1}{2}\,g_{mn}T\right), \qquad (151)$$

i.e., we have arrived at the system of Hilbert-Einstein equations. It has long been known in the literature[19,39] that the Lagrangian (144) leads to the system (151). What we have established is that for a gravitational field with spins 2 and 0 the gravitational-field Lagrangian density (140) is unique, leading to the self-consistent system (146) of equations for the field and the matter. This means that the equations of the RTG are the unique simplest second-order equations.

In view of the importance of this fact, we give in a different form the proof of the equivalence of Eqs. (130) and (151), based in this case on direct calculation of the tensor densities $t^{nm}_{(g)}$ and $t^{mn}_{(M)}$ in the Minkowski space.

On the basis of the expressions (88) and (89) and taking into account the constraint (121), we find that the density of the gravitational-field energy-momentum tensor in the Minkowski space for the Lagrangian density (140) is

$$t^{mn}_{(g)} = -\frac{1}{16\pi}\,J^{mn} - \frac{\sqrt{-\gamma}}{8\pi}\left(\gamma^{mp}\gamma^{nk} - \frac{1}{2}\,\gamma^{mn}\gamma^{pk}\right)R_{pk}. \qquad (152)$$

As we see, the second-rank curvature tensor $R_{pk}$ of the Riemannian space has here automatically appeared. Similarly, using Eqs. (88), (89), and (121), and also the definition (76) of the Hilbert tensor density for the density of the matter energy-momentum tensor in the Minkowski space, we find

$$t^{mn}_{(M)} = \sqrt{\frac{\gamma}{g}}\left(\gamma^{mp}\gamma^{nk} - \frac{1}{2}\,\gamma^{mn}\gamma^{pk}\right)\left(T_{pk} - \frac{1}{2}\,g_{pk}T\right). \qquad (153)$$

Substituting (152) and (153) in the field equations (130), we obtain

$$\left(\gamma^{mp}\gamma^{nk} - \frac{1}{2}\,\gamma^{nm}\gamma^{pk}\right)\left[R_{pk} - \frac{8\pi}{\sqrt{-g}}\left(T_{pk} - \frac{1}{2}\,g_{pk}T\right)\right] = 0,$$

from which we can readily arrive at the system of equations for the gravitational field in the form (151).

Thus, the system of equations (130) is equivalent to the system of Hilbert-Einstein equations (151). And the complete system of equations (146) of the RTG for the matter and the gravitational field is equivalent to the system of equations

$$\left.\begin{aligned} &\sqrt{-g}\,R_{mn} = 8\pi\left(T_{mn} - \frac{1}{2}\,g_{mn}T\right);\\ &D_m\tilde{g}^{mn} = 0. \end{aligned}\right\} \qquad (154)$$

It must be emphasized particularly once more that the second equation of the system (154) is universal, since we have here field equations describing a gravitational field with spins 2 and 0. The choice of the frame of reference (or the coordinate system) is specified by the metric tensor of the Minkowski space. Therefore, these equations do not impose any restrictions on the choice of the coordinate system. Thus, Eqs. (123) eliminate from the density of the tensor field $\tilde{\Phi}^{ik}$ the spins 1 and 0′, leaving only the spins 2 and 0. The required six components of the gravitational field, corresponding to the spins 2 and 0, and the four matter components can be determined from the first equation of the field system (146) or the equivalent Hilbert-Einstein equations (151).

We note that some aspects of the theory of gravitation in Minkowski space were considered in Refs. 39 and 40. Even the authors who chose the correct direction did not at that time understand this and followed a different path in constructing the theory of gravitation. However, this path did not lead them to anything definitive.

It follows from the RTG that solutions satisfying the first equation of (154) but not the second equation of (154) do not have any physical meaning. Such is the fate of all the existing solutions of the Hilbert-Einstein equations. Therefore, it is necessary to seek solutions that satisfy the complete system of equations (154). It is only such solutions that have physical meaning. This will essentially lead to new physical predictions about the development of the universe, gravitational collapse, gravitational waves, etc.

To end this section, we make the following remark. The system of equations (123) is not a consequence of the principle of least action. Therefore, to apply this principle for the Lagrangian (134), we must take into account (123), introducing for this purpose in the action integral a term of the form $\eta_m D_n \tilde{g}^{mn}$, where $\eta_m$ are Lagrangian multipliers. The analysis made in the Appendix shows that the Lagrangian multipliers $\eta_m$ can be taken to be equal to zero by virtue of the conservation of the energy-momentum tensor. Therefore, both in this section and in what follows the principle of least action can be applied without the use of Lagrangian multipliers.

## 9. RELATIONS BETWEEN THE CANONICAL ENERGY-MOMENTUM TENSOR AND HILBERT'S TENSOR

The gravitational-field Lagrangian density depends on the density of the metric tensor $\tilde{\gamma}^{mn}$, the density of the tensorial gravitational field $\tilde{\Phi}^{mn}$, and their first derivatives. For the coordinate transformation (71), the variation $\delta J_g$ of the action is equal to zero, and, therefore,

$$\delta J_g = \int_\Omega d^4x \left[ D_k J^k + \frac{\delta L_g}{\delta \tilde{\Phi}^{mn}} \delta_L \tilde{\Phi}^{mn} + \frac{\delta L_g}{\delta \tilde{\gamma}^{mn}} \delta_L \tilde{\gamma}^{mn} \right] = 0. \quad (155)$$

Here

$$J^k = -\xi^p \tau_p^k + K_m^{pk} D_p \xi^m, \quad (156)$$

where the density of the canonical tensor is

$$\tau_p^k = -\delta_p^k L_g + \frac{\partial L_g}{\partial (\partial_k \tilde{\Phi}^{mn})} D_p \tilde{g}^{mn}$$

$$= -\delta_p^k L_g + \frac{\partial L_g}{\partial (\partial_k \tilde{g}^{mn})} D_p \tilde{g}^{mn}, \quad (157)$$

and the density of the third-rank tensor $K_m^{pk}$ is

$$K_m^{pk} = 2\frac{\partial L_g}{\partial (\partial_k \tilde{\Phi}^{mn})} \tilde{\Phi}^{pn} - \delta_m^p \frac{\partial L_g}{\partial (\partial_k \tilde{\Phi}^{lq})} \tilde{\Phi}^{lq} + 2\frac{\partial L_g}{\partial (\partial_k \tilde{\gamma}^{mn})} \tilde{\gamma}^{pn}$$

$$- \delta_m^p \frac{\partial L_g}{\partial (\partial_k \tilde{\gamma}^{lq})} \tilde{\gamma}^{lq}. \quad (158)$$

Substituting in (155) the expressions (85) and (86) for $\delta_L \tilde{\Phi}^{mn}$ and $\delta_L \tilde{\gamma}^{mn}$, we obtain, the volume of integration $\Omega$ being arbitrary, the identity

$$\xi^p \left[ D_k \tau_p^k + \frac{\delta L_g}{\delta \tilde{\Phi}^{mn}} D_p \tilde{\Phi}^{mn} \right] - K_m^{pk} D_p D_k \xi^m$$

$$+ D_p \xi^m \left[ \tau_m^p - D_k K_m^{pk} - 2\frac{\delta L_g}{\delta \tilde{\Phi}^{mn}} \tilde{\Phi}^{pn} + \delta_m^p \frac{\delta L_g}{\delta \tilde{\Phi}^{ql}} \tilde{\Phi}^{ql} \right.$$

$$\left. - 2\frac{\delta L_g}{\delta \tilde{\gamma}^{mn}} \tilde{\gamma}^{pn} + \delta_m^p \frac{\delta L_g}{\delta \tilde{\gamma}^{ql}} \tilde{\gamma}^{ql} \right] \equiv 0. \quad (159)$$

Since the displacement vector $\xi^p$ is arbitrary, from the last expression we obtain the strong identities

$$D_k \tau_p^k \equiv -\frac{\delta L_g}{\delta \tilde{\Phi}^{mn}} D_p \tilde{\Phi}^{mn}; \quad (160)$$

$$\tau_m^k - D_p K_m^{kp} \equiv 2\frac{\delta L_g}{\delta \tilde{\Phi}^{mn}} \tilde{\Phi}^{kn} - \delta_m^k \frac{\delta L_g}{\delta \tilde{\Phi}^{ql}} \tilde{\Phi}^{ql} + 2\frac{\delta L_g}{\delta \tilde{\gamma}^{mn}} \tilde{\gamma}^{kn}$$

$$- \delta_m^k \frac{\delta L_g}{\delta \tilde{\gamma}^{ql}} \tilde{\gamma}^{ql}; \quad (161)$$

$$K_m^{kp} \equiv -K_m^{pk}. \quad (162)$$

Since we have a connection between the density of the metric tensor of the effective Riemannian space, $\tilde{g}^{mn}$, and the density of the tensorial gravitational field $\tilde{\Phi}^{mn}$,

$$\tilde{g}^{mn} = \tilde{\gamma}^{mn} + \tilde{\Phi}^{mn}, \quad (163)$$

we find

$$\frac{\delta L_g}{\delta \tilde{\Phi}^{mn}} = \frac{\delta L_g}{\delta \tilde{g}^{mn}}; \quad \frac{\partial L_g}{\partial (\partial_p \tilde{\Phi}^{mn})} = \frac{\partial L_g}{\partial (D_p \tilde{g}^{mn})}.$$

Using these equations, we obtain

$$\frac{\partial L_g}{\partial (\partial_p \tilde{\gamma}^{mn})} = \frac{\partial L_g}{\partial (D_l \tilde{g}^{mn})} - \tilde{g}^{qj} \frac{\partial L_g}{\partial (D_k \tilde{g}^{qj})} \frac{\partial \gamma_{kl}^l}{\partial (\partial_p \tilde{\gamma}^{mn})}$$

$$+ \tilde{g}^{qj} \frac{\partial L_g}{\partial (D_k \tilde{g}^{lj})} \frac{\partial \gamma_{qk}^l}{\partial (\partial_p \tilde{\gamma}^{mn})}.$$

Here, $\gamma_{qk}^l$ are the Christoffel symbols of the Minkowski space:

$$\gamma_{qk}^l = \frac{1}{2} \gamma^{lj} (\partial_q \gamma_{kj} + \partial_k \gamma_{jq} - \partial_j \gamma_{qk}).$$

After elementary calculations, we obtain for $K_m^{kp}$ the expression

$$K_m^{kp} = \frac{\partial L_g}{\partial (D_p \tilde{g}^{mn})} \tilde{g}^{kn} - \frac{\partial L_g}{\partial (D_k \tilde{g}^{mn})} \tilde{g}^{pn}$$

$$+ \tilde{g}^{qn} \tilde{\gamma}_{qm} \left[ \frac{\partial L_g}{\partial (D_k \tilde{g}^{ln})} \tilde{\gamma}^{pl} - \frac{\partial L_g}{\partial (D_p \tilde{g}^{ln})} \tilde{\gamma}^{kl} \right]$$

$$+ \frac{\partial L_g}{\partial (D_l \tilde{g}^{ln})} \tilde{\gamma}_{lm} [\tilde{g}^{kn} \tilde{\gamma}^{pq} - \tilde{g}^{pn} \tilde{\gamma}^{kq}]. \quad (164)$$

Since the energy-momentum tensor density of the gravitational field is equal to

$$t_{(g)m}^k = 2\frac{\delta L_g}{\delta \tilde{\gamma}^{mn}} \tilde{\gamma}^{kn} - \delta_m^k \frac{\delta L_g}{\delta \tilde{\gamma}^{pl}} \tilde{\gamma}^{pq}, \quad (165)$$

the identity (161) can be written in the form

$$t_{(g)m}^k \equiv \tau_m^k - D_p K_m^{kp} - 2\frac{\delta L_g}{\delta \tilde{\Phi}^{mn}} \tilde{\Phi}^{kn} + \delta_m^k \frac{\delta L_g}{\delta \tilde{\Phi}^{pq}} \tilde{\Phi}^{pq}. \quad (166)$$

It establishes a connection between the density of Hilbert's tensor in the Minkowski space and the density of the canonical energy-momentum tensor.

For what follows, it is convenient to introduce as a characteristic of the gravitational field

$$\overset{(0)}{t}{}_{(g)m}^k = \tau_m^k - D_p K_m^{kp}, \quad (167)$$

which in the case of a free gravitational field is identical, by virtue of the identity (166), to the density of Hilbert's energy-momentum tensor.

The system of gravitational field equations (146) can be written in a somewhat different form in terms of the density of Hilbert's energy-momentum tensor in the Riemannian space. To this end, on the basis of the Lagrangian (142) and (143), we calculate the third-rank tensor density $K_m^{pk}$ in accordance with (164).

Taking into account the equation

$$\frac{\partial L_g}{\partial (D_k \tilde{g}^{mn})} = \frac{1}{16\pi} \left[ \tilde{G}_{mn}^k + \frac{1}{2} \tilde{g}^{kp} \tilde{g}_{mn} \tilde{G}_{lp}^l \right],$$

we obtain

$$16\pi K_m^{pk} = [\tilde{g}^{pn} \tilde{G}_{mn}^k - \tilde{g}^{kn} \tilde{G}_{mn}^p] + \tilde{g}^{nq} \tilde{\gamma}_{qm} [\tilde{\gamma}^{kl} \tilde{G}_{ln}^p - \tilde{\gamma}^{pl} \tilde{G}_{ln}^k]$$

$$+ \tilde{\gamma}_{ml} \tilde{G}_{qn}^l [\tilde{g}^{pn} \tilde{\gamma}^{kq} - \tilde{g}^{kn} \tilde{\gamma}^{pq}].$$

Substituting in this equation the expression for $\tilde{G}_{mn}^k$ [see (143)], we find

$$16\pi K_m^{pk} = \tilde{g}_{mn} D_q (\tilde{g}^{kq}\tilde{g}^{pn} - \tilde{g}^{pq}\tilde{g}^{kn})$$

$$-\gamma_{mn} D_q (\tilde{g}^{kq}\gamma^{pn} + \tilde{g}^{pn}\gamma^{kq} - \tilde{g}^{pq}\gamma^{kn} - \tilde{g}^{kn}\gamma^{pq}). \quad (168)$$

Using (168), on the basis of the definition (167) for $\overset{(0)}{t}{}^k_{(g)m}$ we obtain the formula

$$\overset{(0)}{t}{}^k_{(g)m} = \tau^k_m - \frac{1}{16\pi} D_p \sigma^{kp}_m - \frac{1}{16\pi} \gamma_{nm} J^{kn}, \quad (169)$$

with the antisymmetric tensor density

$$\sigma^{kp}_m = \tilde{g}_{mn} D_q (\tilde{g}^{pq}\tilde{g}^{kn} - \tilde{g}^{kq}\tilde{g}^{pn}), \quad (170)$$

and $J^{kn}$ is the known structure (119).

We shall need the expression (169) in what follows. As a preliminary, we now derive an identity frequently employed in the literature. In Galilean coordinates, the gravitational-field Lagrangian density (142) takes the form

$$L_g = \frac{1}{32\pi} [\tilde{G}^l_{mn} \partial_l \tilde{g}^{mn} - \tilde{g}^{mn} \tilde{G}^l_{mk} \tilde{G}^k_{nl}],$$

where in the given case

$$\tilde{G}^k_{mn} = \frac{1}{2} \tilde{g}^{kq} (\partial_m \tilde{g}_{qn} + \partial_n \tilde{g}_{qm} - \partial_q \tilde{g}_{mn}).$$

Quantities of the type $\tilde{G}^k_{mn}$ are third-rank tensors with respect to linear coordinate transformations, and therefore $L_g$ will be a scalar density under the same transformations. From the invariance of the action with respect to linear transformations, we obtain

$$\delta J_g = \int_\Omega d^4x \left[ \partial_h J^h + \frac{\delta L_g}{\delta (\tilde{g}^{mn})} \delta_L \tilde{g}^{mn} \right] = 0. \quad (171)$$

Here,

$$J^k = -\xi^p \tau^k_p + \tilde{K}^{pk}_m \partial_p \xi^m, \quad (172)$$

where the canonical tensor density is

$$\tau^k_p = -\delta^k_p L_g + \partial_p \tilde{g}^{mn} \frac{\partial L_g}{\partial (\partial_h \tilde{g}^{mn})}, \quad (173)$$

and the density of the third-rank tensor $\tilde{K}^{pk}_m$ in this case is equal to

$$\tilde{K}^{ph}_m = 2 \frac{\partial L_g}{\partial (\partial_h \tilde{g}^{mn})} \tilde{g}^{pn} - \delta^p_m \frac{\partial L_g}{\partial (\partial_h \tilde{g}^{ql})} \tilde{g}^{ql}. \quad (174)$$

Substituting in (171) the formula for the variation $\delta_L \tilde{g}^{mn}$ with respect to linear transformations, we obtain, the volume of integration $\Omega$ being arbitrary,

$$\xi^p \left[ \partial_h \tau^k_p + \frac{\delta L_g}{\delta (\tilde{g}^{mn})} \partial_p \tilde{g}^{mn} \right]$$

$$+ \partial_p \xi^m \left[ \tau^p_m - \partial_h \tilde{K}^{ph}_m - 2 \frac{\delta L_g}{\delta (\tilde{g}^{mn})} \tilde{g}^{pn} + \delta^p_m \frac{\delta L_g}{\delta (\tilde{g}^{kq})} \tilde{g}^{kq} \right] \equiv 0. \quad (175)$$

From this there follow directly the identities

$$\partial_h \tau^k_p \equiv - \frac{\delta L_g}{\delta (\tilde{g}^{mn})} \partial_p \tilde{g}^{mn}; \quad (176)$$

$$\tau^k_m - \partial_p \tilde{K}^{kp}_m \equiv 2 \frac{\delta L_g}{\delta (\tilde{g}^{mn})} \tilde{g}^{kn} - \delta^k_m \frac{\delta L_g}{\delta (\tilde{g}^{ql})} \tilde{g}^{ql}. \quad (177)$$

Since (148) holds, from the identity (177) we obtain

$$\tau^k_m - \partial_p \tilde{K}^{kp}_m = - \frac{\sqrt{-g}}{8\pi} \left[ R^k_m - \frac{1}{2} \delta^k_m R \right]. \quad (178)$$

Taking into account the equation

$$\frac{\partial L_g}{\partial (\partial_p \tilde{g}^{mn})} = \frac{1}{16\pi} \left[ \tilde{G}^p_{mn} + \frac{1}{2} \tilde{g}^{pq} \tilde{g}_{mn} \tilde{G}^l_{ql} \right]$$

and the expression for $\tilde{G}^e_{mn}$ in Galilean coordinates, we obtain after some manipulations

$$16\pi \tilde{K}^{kp}_m = \partial_n (\delta^p_m \tilde{g}^{kn} - \delta^n_m \tilde{g}^{kp}) + \sigma^{kp}_m, \quad (179)$$

where $\sigma^{kp}_m$ is the density of an antisymmetric tensor;

$$\sigma^{kp}_m = -\sigma^{pk}_m = \tilde{g}_{mn} \partial_q (\tilde{g}^{pq}\tilde{g}^{kn} - \tilde{g}^{kq}\tilde{g}^{pn}). \quad (180)$$

Substituting the expression for $\tilde{K}^{kp}_m$ in (178), we obtain the identity

$$\tau^k_m - \frac{1}{16\pi} \partial_p \sigma^{kp}_m \equiv - \frac{\sqrt{-g}}{8\pi} \left[ R^k_m - \frac{1}{2} \delta^k_m R \right]. \quad (181)$$

It is always possible to replace in the curvature tensor, without changing it, the ordinary derivatives by the covariant derivatives with respect to the Minkowski metric, and therefore the expression (181) can be written in the covariant form

$$\tau^k_m - \frac{1}{16\pi} D_p \sigma^{kp}_m = - \frac{\sqrt{-g}}{8\pi} \left[ R^k_m - \frac{1}{2} \delta^k_m R \right], \quad (182)$$

the density of the canonical tensor $\tau^k_m$ in (182) being equal to the expression (157), i.e.,

$$\tau^k_m = -\delta^k_m L_g + \frac{\partial L_g}{\partial (\partial_k \tilde{g}^{pq})} D_m \tilde{g}^{pq},$$

where the Lagrangian density $L_g$ will be expressed in terms of covariant derivatives with respect to the Minkowski metric and have the form (142).

Using the identity (182), we can write the expression for (169) in the form

$$\overset{(0)}{t}{}^k_{(g)m} = - \frac{\sqrt{-g}}{8\pi} \left[ R^k_m - \frac{1}{2} \delta^k_m R \right] - \frac{1}{16\pi} \gamma_{mn} J^{nk}. \quad (183)$$

We have established earlier that the system of equations (146) for the matter and the gravitational field is equivalent to the system of equations (154). By means of the expression (183), the system of equations of the matter and the gravitational field can also be written in the different equivalent form

$$\left. \begin{aligned} &\gamma_{ml} \gamma^{pq} D_p D_q \tilde{g}^{nl} = 16\pi (T^n_m + \overset{(0)}{t}{}^n_{(g)m}); \\ &D_m \tilde{g}^{mn} = 0. \end{aligned} \right\} \quad (184)$$

Here, $T^n_m$ is the density of the Hilbert energy-momentum tensor (76) for the matter in the Riemannian space. It is obvious that by virtue of Eqs. (184) the conservation law for the energy-momentum tensor of the matter and the gravitational field has the form

$$D_n (T^n_m + \overset{(0)}{t}{}^n_{(g)m}) = 0. \quad (185)$$

The covariant conservation law for the matter in the Riemannian space can be represented identically in the form

$$\nabla_n T^n_m = \partial_n T^n_m - \frac{1}{2} T^{nq} \partial_m g_{nq} \equiv D_n T^n_m - G^q_{mn} T^n_q = 0. \quad (186)$$

Comparing (185) and (186), we obtain

$$G^q_{mn}T^n_q = -D_n \overset{(0)}{t^n_{(g)m}}. \tag{187}$$

We see from this expression that the matter acquires energy and momentum directly from the gravitational field and that the total energy-momentum tensor of the matter and the gravitational field is always strictly conserved.

The construction of the RTG on the basis of Minkowski space and the geometrization principle has made it possible for us to consider in each stage of the arguments only covariant quantities.

## 10. SELF-INTERACTION OF THE GRAVITATIONAL FIELD AND UNIQUENESS OF THE RTG LAGRANGIAN

In Sec. 8, we considered the Lagrangian density of the gravitational field $\tilde{\Phi}^{ik}$ (134), which describes spins 2 and 0 and is quadratic in the first derivatives $D_m\tilde{\Phi}^{lk} \equiv D_m\tilde{g}^{lk}$.

A characteristic feature of this Lagrangian density is that the contraction of the covariant derivatives of $\tilde{g}^{lk}$ taken with respect to the Minkowski metric is realized in (134) solely by means of the effective metric tensor $\tilde{g}^{ij}$ of the Riemannian space. However, the contraction of the covariant derivatives of the field $\tilde{g}^{lk}$ taken with respect to the Minkowski metric can also be made by means of the density of the metric tensor $\tilde{\gamma}^{ij}$. Using for this the smallest[2)] of the possible numbers of tensor densities $\tilde{\gamma}^{ij}$ and $\tilde{g}^{ij}$, we arrive at the following Lagrangian densities different from (134) for the gravitational field $\tilde{\Phi}^{ik}$:

$$L_{g1} = \tilde{\gamma}_{mk}\tilde{g}_{nq}\tilde{g}^{lp}(a_1 D_l\tilde{g}^{kq}D_p\tilde{g}^{mn} + c_1 D_l\tilde{g}^{km}D_p\tilde{g}^{nq}); \tag{188}$$

$$L_{g2} = \tilde{g}_{km}\tilde{g}_{nq}\tilde{\gamma}^{lp}(a_2 D_l\tilde{g}^{kq}D_p\tilde{g}^{mn} + c_2 D_l\tilde{g}^{km}D_p\tilde{g}^{nq}); \tag{189}$$

$$L_{g3} = \tilde{\gamma}_{km}\tilde{\gamma}_{nq}\tilde{g}^{lp}(a_3 D_l\tilde{g}^{kq}D_p\tilde{g}^{mn} + c_3 D_l\tilde{g}^{km}D_p\tilde{g}^{nq}); \tag{190}$$

$$L_{g4} = \tilde{\gamma}_{km}\tilde{g}_{nq}\tilde{\gamma}^{lp}(a_4 D_l\tilde{g}^{kq}D_p\tilde{g}^{mn} + c_4 D_l\tilde{g}^{km}D_p\tilde{g}^{nq}); \tag{191}$$

$$L_{g5} = \tilde{\gamma}_{km}\tilde{\gamma}_{nq}\tilde{\gamma}^{lp}(a_5 D_l\tilde{g}^{kq}D_p\tilde{g}^{mn} + c_5 D_l\tilde{g}^{km}D_p\tilde{g}^{nq}). \tag{192}$$

The Lagrangian densities (188)–(192) also contain a gravitational field carrying only the spins 2 and 0, but the self-interactions of the gravitational field described by them differ from the self-interaction contained in the Lagrangian density (134).

Using the Lagrangian densities (188)–(192) to calculate on the basis of Eqs. (88), (89), (96), and (99) the contributions to $t^{mn}$ and requiring that the equation $D_m t^{mn} = 0$ should not give rise to a new equation for the field $\Phi^{ik}$, we find for the coefficients $a_i$ and $c_i$ the values

$$a_i = c_i = 0 \quad (i = 1, 2, 3, 4, 5). \tag{193}$$

It follows from this that among the simplest Lagrangian densities (134) and (188)–(192) considered above only the Lagrangian of the form (140) leads us to a compatible system of equations (146) describing a gravitational field possessing only spins 2 and 0.

Thus, the self-interaction of the gravitational field cannot be arbitrary. It is determined by the fact that the gravitational field is a Faraday-Maxwell field possessing energy and momentum and spins 2 and 0 and also by the fact that in accordance with (121) the Riemannian space arises as an effective space of field origin.

## 11. GENERALIZATION OF THE SYSTEM OF EQUATIONS OF THE RELATIVISTIC THEORY OF GRAVITATION

In Secs. 8 and 10, proceeding from the idea of the gravitational field as a physical field in the spirit of Faraday and Maxwell, possessing energy, momentum, and spins 2 and 0, we found uniquely in the framework of the relativity principle the Lagrangian density $L_g$ (140) of the gravitational field under the assumption that it must be a quadratic form in the first derivatives of the fields, and we constructed for the gravitational field the system of equations of motion (146).

In this section, we show that our theory of gravitation admits a rather interesting generalization, namely, that in the framework of the RTG it is possible to introduce a rest mass $m$ for the graviton. As we shall see later (see Sec. 16), an upper bound on the graviton mass can be obtained by assuming that the entire "hidden mass" of matter in the universe is associated with the existence of a gravitational field with nonvanishing rest mass. The bound is $m \leqslant 10^{-65}$ g.

Without violating the general requirements of the RTG, we can add to the Lagrangian (140) the simplest terms of the form

$$\Lambda\sqrt{-g} \tag{194}$$

and

$$\varkappa_0\sqrt{-\gamma},\ m^2\gamma_{mn}\tilde{g}^{mn} \text{ and } M^2 g_{mn}\tilde{\gamma}^{mn}. \tag{195}$$

Then the Lagrangian density of the gravitational field can be represented as the sum of Lagrangians

$$\tilde{L}_g = L_g + \lambda_g, \tag{196}$$

where $L_g$ is given by (140), and

$$\lambda_g = \frac{1}{16\pi}\Big(\varkappa_0\sqrt{-\gamma} + \Lambda\sqrt{-g} + \frac{1}{2}m^2\gamma_{mn}\tilde{g}^{mn} + \frac{1}{2}M^2 g_{mn}\tilde{\gamma}^{mn}\Big). \tag{197}$$

Note that, in contrast to GR, the inclusion in the Lagrangian of the gravitational field of terms of the form (195) is possible by virtue of the principal postulate of our theory, which asserts that the special relativity principle is the basis of the description of all physical phenomena, including gravitational ones. It might at first seem strange to include in the Lagrangian $\lambda_g$ the term $\varkappa_0\sqrt{-\gamma}$, which is essentially a constant. However, as we shall see below, the constant term is needed to ensure identical fulfillment of the field equation in the case when there is no matter or gravitational field.

In accordance with the geometrization principle, the matter Lagrangian $L_M$ will depend on the gravitational field $\Phi^{mn}$ only through $\tilde{g}^{mn}$. Thus, the total Lagrangian of the RTG is

$$L = \tilde{L}_g + L_M. \tag{198}$$

On the basis of the Lagrangian (198), where $\tilde{L}_g$ is given by

(196), we calculate, using the connection (121), the energy-momentum tensor density $t^{mn}$ of the matter and the gravitational field in Minkowski space:

$$t^{mn}=2\sqrt{-\gamma}\left(\gamma^{nk}\gamma^{mp}-\frac{1}{2}\gamma^{mn}\gamma^{pk}\right)\frac{\delta L}{\delta\tilde{g}^{pk}}-\frac{1}{16\pi}J^{mn}-\frac{1}{16\pi}\Big[m^2\tilde{g}^{mn}+\varkappa_0\tilde{\gamma}^{mn}-M^2\sqrt{-\gamma}\left(\gamma^{mp}\gamma^{nk}-\frac{1}{2}\gamma^{mn}\gamma^{pk}\right)g_{pk}\Big]. \quad (199)$$

In (199), $J^{mn}$ is the structure (119). By the principle of least action,

$$\delta L/\delta\tilde{g}^{pk}=0. \quad (200)$$

It is therefore clear from (199) that the equations

$$J^{mn}+m^2\tilde{g}^{mn}+\varkappa_0\tilde{\gamma}^{mn}-M^2\sqrt{-\gamma}\left(\gamma^{mp}\gamma^{nk}-\frac{1}{2}\gamma^{mn}\gamma^{pk}\right)g_{pk}=-16\pi t^{mn} \quad (201)$$

are completely equivalent to the system of equations (200).

To ensure that the equation

$$D_m t^{mn}=0, \quad (202)$$

where $t^{mn}$ is given by the expression (201), does not give rise to any new equation for the field $\Phi^{mn}$, it is necessary and sufficient to require that

$$M^2D_m\Big[\left(\gamma^{mp}\gamma^{nk}-\frac{1}{2}\gamma^{mn}\gamma^{pk}\right)g_{pk}\Big]=0$$

hold identically.

From this we find that

$$M^2=0. \quad (203)$$

The remaining constants in the Lagrangian (197), $\Lambda$, $\varkappa_0$, $m^2$, are not determined by the condition (202), and therefore we have

$$\tilde{L}_g=L_g+\frac{1}{16\pi}\left(\varkappa_0\sqrt{-\gamma}+\Lambda\sqrt{-g}+\frac{1}{2}m^2\gamma_{pk}\tilde{g}^{pk}\right). \quad (204)$$

Thus, if the relativistic theory of gravitation is based on a Lagrangian $\tilde{L}_g$ of the form (204), and the matter Lagrangian is chosen in accordance with the geometrization principle, we obtain the system of equations

$$\left.\begin{aligned}&\gamma^{pk}D_pD_k\tilde{g}^{mn}-m^2\tilde{g}^{mn}-\varkappa_0\tilde{\gamma}^{mn}=16\pi t^{mn};\\&D_m\tilde{g}^{mn}=0.\end{aligned}\right\} \quad (205)$$

A system of equations equivalent to (205), obtained from (200) by substitution of the Lagrangian (198), can be represented in the form

$$R^{mn}-\frac{1}{2}g^{mn}R+\frac{1}{2}\Lambda g^{mn}-\frac{m^2}{2}\left(g^{mp}g^{nk}-\frac{1}{2}g^{mn}g^{pk}\right)\gamma_{pk}=\frac{8\pi}{\sqrt{-g}}T^{mn}; \quad (206a)$$

$$D_m\tilde{g}^{mn}=0, \quad (206b)$$

Since the identities

$$\nabla_m\left(R^{mn}-\frac{1}{2}g^{mn}R\right)\equiv 0;\quad \nabla_m\tilde{g}^{mn}\equiv 0,$$

where $\nabla_m$ is the covariant derivative with respect to the metric $g^{mn}$, hold, on the basis of (206a) we have

$$\frac{m^2}{2}\sqrt{-g}\left(g^{mp}g^{nk}-\frac{1}{2}g^{mn}g^{pk}\right)\nabla_m\gamma_{pk}+8\pi\nabla_mT^{mn}=0. \quad (207)$$

We consider in (207) the term containing $\nabla_m\gamma_{pk}$.

Taking into account the equation

$$\nabla_m\gamma_{pk}=-G^q_{mp}\gamma_{qk}-G^q_{mk}\gamma_{pq},$$

where the tensor $G^q_{mn}$ is given by (145), we find

$$\left(g^{mp}g^{nk}-\frac{1}{2}g^{mn}g^{pk}\right)\nabla_m\gamma_{pk}\equiv\gamma_{mq}g^{mn}\left(D_pg^{pq}+G^l_{pl}g^{pq}\right). \quad (208)$$

In deriving this identity, we used the explicit expression (145) for $G^q_{mn}$. Using (208) in (207), we obtain

$$\frac{1}{2}m^2\sqrt{-g}\gamma_{qm}g^{mn}(D_pg^{pq}+G^l_{pl}g^{pq})+8\pi\nabla_mT^{mn}=0. \quad (209)$$

By virtue of the gravitational field equations (206b)

$$D_p\tilde{g}^{pq}\equiv\sqrt{-g}\,(D_pg^{pq}+G^l_{pl}g^{pq})=0$$

and (209), which are a consequence of (206a), we obtain necessarily the equations of motion for the matter:

$$\nabla_mT^{mn}=0. \quad (83)$$

Thus, the fulfillment of the matter equations (83) follows directly from the gravitational field equations (206). Equations (206b) ensure compatibility of (206a) with the matter equations. The converse is also true. If (206a) and the matter equations (83) hold, Eqs. (206b) are a necessary consequence of them, and, therefore, such a theory describes a massive gravitational field possessing only spins 2 and 0, since the system of equations (206b) eliminates from the tensor field $\Phi^{mn}$ the spins 1 and 0′.

In the absence of matter and the gravitational field, $\Phi^{mn}=0$, Eqs. (206) must vanish identically. This is possible only if $m^2$ and $\Lambda$ are related by

$$\Lambda=-m^2. \quad (210)$$

A further relation between the constants $\varkappa_0$ and $m^2$ can be obtained by analyzing the Lagrangian $\tilde{L}_g$. Taking in (204) the gravitational field equal to zero and using Eq. (210), we find

$$\tilde{L}_g=\lambda_g=\frac{\sqrt{-\gamma}}{16\pi}(\varkappa_0+m^2).$$

In the absence of matter and the gravitational field, it is natural to require identical vanishing of the Lagrangian (198). This leads us to

$$\varkappa_0=-m^2. \quad (211)$$

Thus, as the analysis made here shows, the Lagrangian (197) contains a single unique constant, which we denote by $m^2$.

We now clarify the physical meaning of the parameter $m^2$. To this end, we consider Eq. (206a) in the approximation of a weak field $\Phi^{mn}$. In Galilean coordinates, we can obtain on the basis of the connection (121) the following expansion for $g^{mn}$ and $g$:

$$g^{mn}=\gamma^{mn}+\Phi^{mn}-\frac{1}{2}\gamma^{mn}\Phi_k^k-\frac{1}{2}\Phi^{mn}\Phi_k^k$$
$$+\frac{1}{4}\gamma^{mn}\left(\Phi_{pq}\Phi^{pq}+\frac{1}{2}\Phi_p^p\Phi_q^q\right)+\ldots; \quad (212)$$
$$g=-1-\Phi_k^k+\frac{1}{2}\Phi_{pq}\Phi^{pq}-\frac{1}{2}\Phi_p^p\Phi_q^q+\ldots \quad (213)$$

Using these expansions, we obtain

$$R^{mn}\simeq\frac{1}{2}\left(\Box\Phi^{mn}-\frac{1}{2}\gamma^{mn}\Box\Phi_q^q\right),$$

and therefore

$$\sqrt{-g}\left(R^{mn}-\frac{1}{2}g^{mn}R\right)\simeq\frac{1}{2}\Box\Phi^{mn}. \quad (214)$$

In the first order in the field $\Phi^{mn}$, taking into account (210) and (211), we have

$$-\frac{1}{2}m^2\tilde{g}^{mn}-\frac{1}{2}m^2\sqrt{-g}\left(g^{mp}g^{nk}-\frac{1}{2}g^{mn}g^{pk}\right)\gamma_{pk}$$
$$=-\frac{1}{2}m^2\Phi^{mn}.$$

Thus, (206a) in the weak-field approximation takes the form

$$(\Box-m^2)\Phi^{mn}=0. \quad (215)$$

As we see, for a weak field the constant $m$ is the graviton rest mass. In passing, we also note that by virtue of (212) and (213) we obtain for $\lambda_g$ in the lowest order the expression

$$\lambda_g=\frac{m^2}{64\pi}\left(\Phi_{pq}\Phi^{pq}-\frac{1}{2}\Phi_p^p\Phi_q^q\right); \quad (216)$$

thus, in the framework of the RTG the system of equations describing the gravitational field $\Phi^{mn}$ with nonvanishing rest mass will have the form

$$\left.\begin{array}{l}\gamma^{pk}D_pD_k\tilde{\Phi}^{mn}-m^2\tilde{\Phi}^{mn}=16\pi t^{mn};\\ D_m\tilde{\Phi}^{mn}=0\end{array}\right\} \quad (217)$$

or, in a different form of expression,

$$\left.\begin{array}{l}\left(R^{mn}-\frac{1}{2}g^{mn}R\right)-\\ -\frac{m^2}{2}\left[g^{mn}+\left(g^{mp}g^{nk}-\frac{1}{2}g^{mn}g^{pk}\right)\gamma_{pk}\right]=\frac{8\pi}{\sqrt{-g}}T^{mn};\\ D_m\tilde{g}^{mn}=0.\end{array}\right\} \quad (218)$$

Whereas the flat metric $\gamma^{mn}$ occurred in the RTG hitherto only in (154), we now see that it occurs in both of the equations (218).

To conclude this section, we draw the following important conclusion. If the graviton rest mass is zero, $m=0$, i.e., if the gravitational interaction is a long-range interaction, then the gravitational field equations (218) go over into the RTG equations (154). Thus, for a long-range gravitational field the cosmological term is always equal to zero.

## 12. SOLUTION OF THE RTG EQUATIONS FOR A SPHERICALLY SYMMETRIC ISOLATED BODY

In this section, following Refs. 15, 19, and 41–43, we consider the solution of the RTG equations for a body possessing spherical symmetry in the case of a massless gravitational field.

In the Minkowski space, we choose spherical coordinates:

$$t;\ x=r\sin\theta\cos\varphi;\ y=r\sin\theta\sin\varphi;\ z=r\cos\theta. \quad (219)$$

In what follows, it is also convenient to use for the coordinates $t, r, \theta, \varphi$ the notation

$$t=x^0,\ r=x^1,\ \theta=x^2,\ \varphi=x^3. \quad (220)$$

In the coordinates (219), the metric tensor of the Minkowski space has the form

$$\left.\begin{array}{l}\gamma_{00}=1;\ \gamma_{11}=-1;\ \gamma_{22}=-r^2;\ \gamma_{33}=-r^2\sin^2\theta;\\ \gamma^{00}=1;\ \gamma^{11}=-1;\ \gamma^{22}=-\frac{1}{r^2};\ \gamma^{33}=-\frac{1}{r^2\sin^2\theta};\end{array}\right\} \quad (221)$$

$\gamma^{mn}=\gamma_{mn}=0$ if $m\neq n$; $\sqrt{-\gamma}=r^2\sin\theta$, and the nonvanishing Christoffel symbols are

$$\gamma_{22}^1=-r;\ \gamma_{33}^1=-r\sin^2\theta;\ \gamma_{12}^2=\gamma_{13}^3=\frac{1}{r};$$
$$\gamma_{33}^2=-\sin\theta\cos\theta;\quad \gamma_{23}^3=\cot\theta. \quad (222)$$

In Eqs. (221) and (222) and in what follows in this section, the numerical indices denote the spherical coordinates in accordance with (220).

The interval for a spherically symmetric and static mass in the Riemannian space can be written in the standard form

$$ds^2=U(r)\,dt^2-V(r)\,dr^2-W(r)\,(d\theta^2+\sin^2\theta d\varphi^2), \quad (223)$$

where $U(r)$, $V(r)$, and $W(r)$ are positive functions and depend only on the distance $r$ in the Minkowski space.

In accordance with (223), the components of the metric tensor of the Riemannian space are

$$\left.\begin{array}{l}g_{00}=U(r);\ g_{11}=-V(r);\ g_{22}=-W(r);\ g_{33}=-W(r)\sin^2\theta;\\ g^{00}=\frac{1}{U(r)};\ g^{11}=-\frac{1}{V(r)};\ g^{22}=-\frac{1}{W(r)};\ g^{33}=-\frac{1}{W(r)\sin^2\theta};\\ g^{mn}=g_{mn}=0,\quad \text{if}\quad m\neq n\quad \sqrt{-g}=\sqrt{UV}W\sin\theta.\end{array}\right\} \quad (224)$$

The functions $U(r)$, $V(r)$, and $W(r)$ must be determined from the RTG equations. In order to write these equations in expanded form, we find first the density of the metric tensor $\tilde{g}^{mn}$. It is easy to show that

$$\left.\begin{array}{l}\tilde{g}^{00}=\sqrt{U^{-1}V}W\sin\theta;\ \tilde{g}^{11}=-\sqrt{UV^{-1}}W\sin\theta;\\ \tilde{g}^{22}=-\sqrt{UV}\sin\theta;\ \tilde{g}^{33}=-\sqrt{UV}\frac{1}{\sin\theta}.\end{array}\right\} \quad (225)$$

Since the second equation of (154) can be represented in the form

$$D_l\tilde{g}^{lk}=\partial_l\tilde{g}^{lk}+\gamma_{mn}^k\tilde{g}^{mn}=0.$$

we find on the basis of (222) and (225)

$$\frac{d}{dr}\left(W\sqrt{UV^{-1}}\right)=2r\sqrt{UV}. \quad (226)$$

We now write the first equation of (154) in terms of the functions $U(r)$, $V(r)$, and $W(r)$. To this end, we find the nonvanishing components of the tensor

$$\Gamma^l_{mn}=\frac{1}{2}g^{lk}(\partial_m g_{kn}+\partial_n g_{km}-\partial_k g_{mn});$$

$$\left.\begin{aligned}
&\Gamma^0_{01}=\frac{1}{2U}\frac{dU}{dr};\ \Gamma^2_{33}=-\sin\theta\cos\theta;\\
&\Gamma^1_{00}=\frac{1}{2V}\frac{dU}{dr};\ \Gamma^2_{12}=\frac{1}{2W}\frac{dW}{dr};\\
&\Gamma^1_{11}=\frac{1}{2V}\frac{dV}{dr};\ \Gamma^3_{13}=\frac{1}{2W}\frac{dW}{dr};\\
&\Gamma^1_{22}=-\frac{1}{2V}\frac{dW}{dr};\ \Gamma^3_{23}=\cot\theta;\\
&\Gamma^1_{33}=-\frac{\sin^2\theta}{2V}\frac{dW}{dr}.
\end{aligned}\right\} \quad (227)$$

From the definition $R^l_k=p^{lp}R_{pk}$, where

$$R_{pk}=\partial_n\Gamma^n_{pk}-\partial_k\Gamma^n_{pn}+\Gamma^n_{pk}\Gamma^m_{nm}-\Gamma^n_{pm}\Gamma^m_{kn}, \quad (228)$$

by virtue of (227) we find

$$R^0_0=\frac{1}{2UV}\frac{d^2U}{dr^2}-\frac{1}{4UV^2}\frac{dU}{dr}\frac{dV}{dr}-\frac{1}{4U^2V}\left(\frac{dU}{dr}\right)^2+\frac{1}{2UVW}\frac{dU}{dr}\frac{dW}{dr}; \quad (229)$$

$$R^1_1=\frac{1}{2UV}\frac{d^2U}{dr^2}+\frac{1}{WV}\frac{d^2W}{dr^2}-\frac{1}{4VU^2}\left(\frac{dU}{dr}\right)^2-\frac{1}{2VW^2}\left(\frac{dW}{dr}\right)^2-\frac{1}{4UV^2}\frac{dV}{dr}\frac{dU}{dr}-\frac{1}{2V^2W}\frac{dV}{dr}\frac{dW}{dr}; \quad (230)$$

$$R^2_2=R^3_3=\frac{1}{2VW}\frac{d^2W}{dr^2}-\frac{1}{4V^2W}\frac{dV}{dr}\frac{dW}{dr}-\frac{1}{W}+\frac{1}{4UVW}\frac{dU}{dr}\frac{dW}{dr}; \quad (231)$$

$$R=R^l_l=\frac{1}{UV}\frac{d^2U}{dr^2}-\frac{1}{2UV^2}\frac{dV}{dr}\frac{dU}{dr}-\frac{1}{2U^2V}\left(\frac{dU}{dr}\right)^2+\frac{2}{VW}\frac{d^2W}{dr^2}+\frac{1}{UVW}\frac{dU}{dr}\frac{dW}{dr}-\frac{1}{2VW^2}\left(\frac{dW}{dr}\right)^2-\frac{1}{V^2W}\frac{dV}{dr}\frac{dW}{dr}-\frac{2}{W}; \quad (232)$$

$$R^l_k=0,\ k\neq l. \quad (233)$$

Then the first equation of the system (154) can be written in the form

$$\left.\begin{aligned}
&\sqrt{-g}\left(R^0_0-\frac{1}{2}R\right)=8\pi T^0_0;\\
&\sqrt{-g}\left(R^1_1-\frac{1}{2}R\right)=8\pi T^1_1;\\
&\sqrt{-g}\left(R^2_2-\frac{1}{2}R\right)=8\pi T^2_2.
\end{aligned}\right\} \quad (234)$$

In what follows, we restrict ourselves to finding only the exterior solution. It can be seen from (234) that outside the matter

$$R^0_0=R^1_1=R^2_2=0. \quad (235)$$

On the basis of (229)–(231), we can show that the system of equations (235) for the functions $U$, $V$, and $W$ is equivalent to the system of equations

$$\frac{dU}{dr}=2a\frac{1}{W}\sqrt{UV}; \quad (236)$$

$$\frac{dW}{dr}=2b\sqrt{UVW}; \quad (237)$$

$$\frac{2ab}{\sqrt{W}}+b^2U=1, \quad (238)$$

only two of which are independent. Therefore, we limit ourselves to studying Eqs. (237) and (238). From these equations, we find

$$U(r)=\frac{1}{b^2}\left(\frac{\sqrt{W}-2ab}{\sqrt{W}}\right); \quad (239)$$

$$V(r)=\left(\frac{d\sqrt{W}}{dr}\right)^2\frac{\sqrt{W}}{\sqrt{W}-2ab}. \quad (240)$$

Note that $a$ and $b$ in Eqs. (236)–(238) are arbitrary constants that must be determined subsequently. Taking into account (239) and (240), we obtain

$$\frac{d}{dr}\left[\frac{\sqrt{W}(\sqrt{W}-2ab)}{\frac{d\sqrt{W}}{dr}}\right]=2r\frac{d\sqrt{W}}{dr}. \quad (241)$$

It is convenient to solve this equation for $r$ as a function of $\sqrt{W}$.

Following Belinfante,[41] we introduce the new variables

$$\sqrt{W}=ab\,(x+1)\ \text{and}\ r=xy \quad (242)$$

and as the independent variable we choose $x$. Since

$$\frac{d\sqrt{W}}{dr}=\frac{d\sqrt{W}}{dx}\Big/\frac{dr}{dx}=\frac{ab}{[y+x\,dy/dx]},$$

we find from (241)

$$(4x^2-2)\frac{dy}{dx}+x(x^2-1)\frac{d^2y}{dx}=0. \quad (243)$$

The general solution of this equation is

$$y(x)=C_1\left(\frac{1}{x}+\frac{1}{2}\ln\frac{x-1}{x+1}\right)+C_2, \quad (244)$$

where $C_1$ and $C_2$ are constants of integration. Returning to the original variables $r$ and $\sqrt{W}$, we find

$$r=C_1\left(1+\frac{\sqrt{W}-ab}{2ab}\ln\frac{\sqrt{W}-2ab}{\sqrt{W}}\right)+\frac{C_2}{ab}(\sqrt{W}-ab). \quad (245)$$

Since in the limit $r\to\infty$ we must have $g_{22}\to\gamma_{22}$, from (221) and (224) we see that in the region of large $r$

$$W\simeq r^2.$$

Considering (245) for large $r$ and $W$ and taking into account the last relation, we find that

$$C_2=ab. \quad (246)$$

Now suppose $\sqrt{W}\to 2ab$. Then, as follows from (245), $r\to\infty$ if $C_1\neq 0$, a result that is not justified from the physical point of view. Therefore, it is necessary to set

$$C_1=0.$$

Thus, we finally find

$$W(r)=(r+ab)^2; \quad (247)$$

$$U(r)=\frac{1}{b^2}\frac{r-ab}{r+ab}; \quad (248)$$

$$V(r)=\frac{r+ab}{r-ab}. \qquad (249)$$

Since in the limit $r\to\infty$ we must have $U(r)\to 1$, from (248) we obtain

$$b=1. \qquad (250)$$

Taking into account the expressions (247)–(249) and Eq. (250) for the interval (223) outside the matter, we find

$$ds^2=\frac{r-a}{r+a}\,dt^2-\frac{r+a}{r-a}\,dr^2-(r+a)^2\,(d\theta^2+\sin^2\theta\,d\varphi^2). \qquad (251)$$

It follows from the correspondence principle that the constant $a$ is equal to the active gravitational mass of the body:

$$a=m. \qquad (252)$$

To demonstrate more clearly the role of the second equation of (154), we turn to the system (236)–(238), which is equivalent to the system of Hilbert-Einstein equations outside the matter. Of the three equations (236)–(238), only two are independent, and therefore one of the three functions $U(r)$, $V(r)$, and $W(r)$ is not determined by these equations. We identify $\sqrt{W}$ with $r$:

$$\sqrt{W}=r. \qquad (253)$$

Then it is easy to see that the system (236)–(238) with allowance for (250) and (252) admits the solutions

$$\left.\begin{aligned} &W(r)=r^2;\\ &U(r)=1-\frac{2m}{r};\\ &V(r)=\left(1-\frac{2m}{r}\right)^{-1} \end{aligned}\right\} \qquad (254)$$

and, therefore, the interval (223) can be written in the form

$$ds^2=\left(1-\frac{2m}{r}\right)dt^2-\left(1-\frac{2m}{r}\right)^{-1}dr^2-r^2\,(d\theta^2+\sin^2\theta\,d\varphi^2). \qquad (255)$$

This is the well-known Schwarzschild solution.

It can be seen from (255) that the parameter $r$ has acquired the meaning of distance in the Riemannian space. Thus, Eq. (253) means that we have identified the distance in the Minkowski space with the distance in the Riemannian space.

In the framework of the RTG, such a global identification of the distances is inadmissible. For in the effective Riemannian space, the distance is given by the metric tensor $g_{mn}$. By virtue of the geometrization principle, the effective Riemannian space has a field origin, and therefore its metric $g_{mn}$ must be determined from the system of equations of the RTG. But in the Minkowski space the distance is fixed by the metric tensor $\gamma_{mn}$. As soon as $r$ has been chosen in the Minkowski space, the connection between $r$ and the distance $\sqrt{W}$ of the Riemannian space must be uniquely determined by the RTG equations. Since the solutions (254) obtained by virtue of the connection (253) do not satisfy the field equation (226), the solution (254) found by Schwarzschild is not a solution of the system of RTG equations. For the interval (223), the RTG equations give an unambiguous result and lead to Eq. (251).

In GR, the concept of flat Minkowski space does not exist. Therefore, the quantity $r$ in the Hilbert-Einstein equations (236)–(238) is a parameter with no physical meaning. A connection between $W(r)$ and $r$ in GR can be established by means of so-called coordinate conditions. Fock was the first who ascribed a particular significance to the harmonic conditions as coordinate conditions for problems of island type in GR.

Following Ref. 43, we give here an analysis of the investigation of the system of Hilbert-Einstein equations for a spherically symmetric body in a sequence that most clearly demonstrates the restricted nature of Fock's ideas and how this restriction is eliminated in the framework of the RTG.

Bearing in mind the exigences of the following Secs. 13 and 14 of the present paper, we consider an interval of more general form than (223):

$$ds^2=g_{00}(t,r)\,dt^2+2g_{01}(t,r)\,dt\,dr+g_{11}(t,r)\,dr^2-B^2(t,r)\,(d\theta^2+\sin^2\theta\,d\varphi^2) \qquad (256)$$

and we introduce the notation

$$x^i=(t,\ r,\ \theta,\ \varphi). \qquad (257)$$

We now go over from $x^i$ to the variables $\xi^i$:

$$\xi^i=(\tau,\ R,\ \theta,\ \varphi), \qquad (258)$$

setting

$$t=t(R,\tau),\quad r=r(R,\tau). \qquad (259)$$

We require of the function (259) that in terms of $\xi^i$ the interval (256) take the form

$$ds^2=d\tau^2-e^{\omega(\tau,R)}\,dR^2-B^2(\tau,R)\,[d\theta^2+\sin^2\theta\,d\varphi^2]. \qquad (260)$$

It can be seen from (260) that $\tau$ is the proper time and $R$ is the radial variable in the comoving coordinate system, i.e., $\xi^i$ are the "proper" coordinates of the body.

We write the system of equations (123) in a somewhat different form. For this, we use the well-known equation

$$\Gamma^q_{kl}(x)\,g^{kl}(x)=-\frac{1}{\sqrt{-g}}\,\frac{\partial}{\partial x^p}\,[\sqrt{-g(x)}\,g^{pq}(x)], \qquad (261)$$

where $\Gamma^q_{kl}$ is determined by the formula

$$\Gamma^q_{kl}(x)=\frac{1}{2}\,g^{qp}\,(\partial_k g_{pl}+\partial_l g_{pk}-\partial_p g_{kl}),$$

and $x^i$ denote the variables (257). Using the transformation law of (259), we obtain for $\Gamma^q_{kl}(x)g^{kl}(x)$

$$\Gamma^q_{kl}(x)\,g^{kl}(x)=-\frac{1}{\sqrt{-g(\xi)}}\,\frac{\partial}{\partial \xi^p}\left[\sqrt{-g(\xi)}\,g^{pk}(\xi)\,\frac{\partial x^q}{\partial \xi^k}\right]. \qquad (262)$$

Comparing (261) and (262), we find

$$\Box x^q=\frac{1}{\sqrt{-g(x)}}\,\frac{\partial}{\partial x^p}\,[\sqrt{-g(x)}\,g^{pq}(x)];\quad q=0,\ 1,\ 2,\ 3. \qquad (263)$$

Here, $\square$ is the generalized d'Alembert operator:

$$\square\varphi=\frac{1}{\sqrt{-g(\xi)}}\frac{\partial}{\partial\xi^p}\left[\sqrt{-g(\xi)}\,g^{pq}(\xi)\frac{\partial\varphi}{\partial\xi^q}\right]. \quad (264)$$

Fock, and before him De Donder[38] used a noncovariant harmonic condition of the form

$$\frac{\partial}{\partial x^p}\left[\sqrt{-g(x)}\,g^{pq}(x)\right]=0 \quad (265)$$

as a privileged coordinate condition.

On the basis of Eq. (263), the condition (265) can be represented in the form

$$\square x^q=0 \quad q=0,\ 1,\ 2,\ 3. \quad (266)$$

The condition (266) cannot be made covariant in the framework of GR. In harmonic coordinates, the system of Hilbert-Einstein equations simplifies appreciably, and it was evidently this that prompted Fock to call this a privileged system. If we wish to keep the condition (265) and make it universal, we should have to give it a covariant nature. But this operation is not unique and cannot be implemented in the framework of GR. We solved the problem of finding new field equations on the basis of the physical structure of the gravitational field. Namely, it was on this basis that in Sec. 8 we obtained Eq. (123):

$$\frac{1}{\sqrt{-g(x)}}D_p\left[\sqrt{-g(x)}\,g^{pq}(x)\right]$$

$$=\frac{1}{\sqrt{-g(x)}}\partial_p\left[\sqrt{-g(x)}g^{pq}(x)\right]+\gamma^q_{mn}(x)\,g^{mn}(x)=0, \quad (267)$$

which can be cast by means of (263) in the form

$$\square x^q=-\gamma^q{}_n(x)\frac{\partial x^m}{\partial\xi^p}\frac{\partial x^n}{\partial\xi^l}g^{pl}(\xi). \quad (268)$$

Here, $\gamma^q_{mn}(x)$ is the Christoffel symbol of Minkowski space:

$$\gamma^q_{mn}(x)=\frac{1}{2}\gamma^{qp}(\partial_m\gamma_{pn}+\partial_n\gamma_{pm}-\partial_p\gamma_{mn}). \quad (269)$$

In the case of Galilean coordinates, $\gamma^q_{mn}=0$ and Eq. (268) becomes identical to Eq. (266). Equation (268) is covariant, reflects the structure of the gravitational field, and emphasizes the fundamental nature of Minkowski space. The part played by these equations is extremely important, since they change the nature of the predicted phenomena. A new physics arises, especially in the domain of strong fields. We shall see this in the following section, when we study the gravitational collapse of massive bodies, and also in Sec. 16 when we study the evolution of a homogeneous and isotropic universe in time.

We now turn to Eq. (268), which is a different form of expression of the generally covariant field equation (123) of the RTG.

Using for the metric tensor $\gamma^{mn}$ the value (221), for the connection $\gamma^q_{mn}$ the expression (222), and for $g^{pl}(\xi)$ the values in accordance with the expression of the interval (260),

$$\left.\begin{array}{l} g^{00}(\xi)=1;\quad g^{11}(\xi)=-e^{-\omega};\quad g^{22}(\xi)=-B^{-2};\\ g^{33}(\xi)=-B^{-2}\sin^{-2}\theta, \end{array}\right\} \quad (270)$$

we find from (268)

$$\frac{\partial}{\partial\tau}\left(e^{\frac{\omega}{2}}B^2\frac{\partial t}{\partial\tau}\right)=\frac{\partial}{\partial R}\left(e^{-\frac{\omega}{2}}B^2\frac{\partial t}{\partial R}\right); \quad (271)$$

$$\frac{\partial}{\partial\tau}\left(e^{\frac{\omega}{2}}B^2\frac{\partial r}{\partial\tau}\right)=\frac{\partial}{\partial R}\left(e^{-\frac{\omega}{2}}B^2\frac{\partial r}{\partial R}\right)-2re^{\frac{\omega}{2}}. \quad (272)$$

We now use the Hilbert-Einstein equations outside the matter,

$$R_{mn}=0.$$

In terms of $\omega(R,\tau)$ and $B(R,\tau)$ they have the form

$$\left(\frac{\partial B}{\partial R}\right)^2-e^{\omega}-\left(\left(\frac{\partial B}{\partial\tau}\right)^2+2B\frac{\partial^2B}{\partial\tau^2}\right)e^{\omega}=0; \quad (273)$$

$$\frac{\partial^2B}{\partial R^2}-\frac{1}{2}Be^{\omega}\left(\frac{\partial^2\omega}{\partial\tau^2}+\frac{1}{2}\left(\frac{\partial\omega}{\partial\tau}\right)^2\right)-e^{\omega}\frac{\partial^2B}{\partial\tau^2}$$
$$-\frac{1}{2}\left(\frac{\partial B}{\partial R}\frac{\partial\omega}{\partial R}+e^{\omega}\frac{\partial B}{\partial\tau}\frac{\partial\omega}{\partial\tau}\right)=0; \quad (274)$$

$$2\left(\left(\frac{\partial B}{\partial R}\right)^2+B\frac{\partial^2B}{\partial R^2}\right)-e^{\omega}-\left(\frac{\partial B}{\partial R}\right)^2-B\frac{\partial B}{\partial R}\frac{\partial\omega}{\partial R}$$
$$-\left(B\frac{\partial B}{\partial\tau}\frac{\partial\omega}{\partial\tau}+\left(\frac{\partial B}{\partial\tau}\right)^2\right)e^{\omega}=0. \quad (275)$$

Solutions of this system were found in Refs. 41 and 44:

$$B(R,\ \tau)=[R^{3/2}-b\tau]^{2/3}; \quad (276)$$

$$\omega(R,\ \tau)=\ln R-\frac{2}{3}\ln(R^{3/2}-b\tau). \quad (277)$$

The constant $b$ is determined from the solution of the system of Hilbert-Einstein equations inside the matter.

From (276) and (277) it is easy to show that

$$(\partial B/\partial R)^2=e^{\omega}. \quad (278)$$

Therefore, on the functions (276) and (277) the system of equations (271) and (272) has the form

$$\frac{\partial}{\partial\tau}\left(\frac{\partial B}{\partial R}B^2\frac{\partial t}{\partial\tau}\right)=\frac{\partial}{\partial R}\left(\left(\frac{\partial B}{\partial R}\right)^{-1}B^2\frac{\partial t}{\partial R}\right); \quad (279)$$

$$\frac{\partial}{\partial\tau}\left(\frac{\partial B}{\partial R}B^2\frac{\partial r}{\partial\tau}\right)=\frac{\partial}{\partial R}\left(\left(\frac{\partial B}{\partial R}\right)^{-1}B^2\frac{\partial r}{\partial R}\right)-2r\frac{\partial B}{\partial R}. \quad (280)$$

The solutions of Eqs. (279) and (280) establish an explicit connection between the coordinates $R$ and $\tau$ of the comoving coordinate system and the Minkowski-space coordinates $r$ and $t$.

In Eqs. (279) and (280), we go over from the variables $R$ and $\tau$ to the variables $R$ and $B$. Then, as was shown in Ref. 43, the solution of the system (279) and (280) in the variables $R$ and $B$ can be represented in the form

$$t=\frac{1}{b}R^{3/2}+\frac{3}{2b}\int\frac{B^{3/2}\,dB}{\left[\left(\frac{2}{3}b\right)^2-B\right]}; \quad (281)$$

$$r=B-\frac{1}{2}\left(\frac{2}{3}b\right)^2. \quad (282)$$

On the basis of Eqs. (276), (278), (281), and (282) and using the tensor transformation law for the metric coefficients of the interval (256), we find

$$\left.\begin{array}{l} g_{00}(r,\ t)=\dfrac{r-\frac{1}{2}\left(\frac{2}{3}b\right)^2}{r+\frac{1}{2}\left(\frac{2}{3}b\right)^2};\quad g_{01}=0;\\[2ex] g_{11}=-\dfrac{r+\frac{1}{2}\left(\frac{2}{3}b\right)^2}{r-\frac{1}{2}\left(\frac{2}{3}b\right)^2};\quad g_{22}=-\left[r+\frac{1}{2}\left(\frac{2}{3}b\right)^2\right]^2;\\[2ex] g_{33}=-\left[r+\frac{1}{2}\left(\frac{2}{3}b\right)^2\right]^2\sin^2\theta. \end{array}\right\} \quad (283)$$

Therefore, in the variables $r$ and $t$ the interval (256) takes the form (251) if we set

$$\frac{1}{2}\left(\frac{2}{3}b\right)^2=a, \qquad (284)$$

where by virtue of the correspondence principle $a$ is equal to the active gravitational mass $m$ of the body.

## 13. GRAVITATIONAL COLLAPSE

In the framework of GR, it is concluded (see, for example, Refs. 18, 42, and 45) that if a massive star, having exhausted its nuclear fuel, has not shed sufficient mass, then no forces can halt its further contraction under the influence of gravitation, and the density of the star will tend to infinity in a finite proper time. This process of evolution of a star is called gravitational collapse. Wheeler regarded gravitational collapse and the resulting singularity as "one of the greatest crises of all times" in fundamental physics.

In this section, following Ref. 43, we show that the RTG fundamentally changes the entire character of gravitational collapse and leads to the phenomenon of gravitational restraint, by virtue of which the contraction of a massive body in the comoving frame is halted after a finite proper time and, most importantly, the matter density remains finite.

Thus, the prediction of the RTG is radically different from that of GR.

We give briefly the collapse results that follow from GR. In the comoving frame for a spherically symmetric body, the interval can be written in the form (260).

Under the assumption that the pressure is zero, an exact solution of the Hilbert-Einstein equations was found in Ref. 44:

$$B=R\left(1-\frac{\tau}{\tau_0}\right)^{2/3}, \quad R\leqslant R_0; \qquad (285)$$

$$B=\left(R^{\frac{3}{2}}-R_0^{\frac{3}{2}}\frac{\tau}{\tau_0}\right)^{2/3}, \quad R\geqslant R_0; \qquad (286)$$

$$e^{\omega}=\left(\frac{\partial B}{\partial R}\right)^2,$$

where

$$R_0^3=\frac{9}{2}m\tau_0^2. \qquad (287)$$

Here, $m$ is the active gravitational mass of the body. It follows from (285) and (286) that the range of variation of $\tau$ is bounded above by $\tau=\tau_0$, while all values from 0 to $\infty$ are admissible for $B(R,\tau)$.

For the matter density in the comoving frame, we obtain

$$\varepsilon=\frac{1}{6\pi(\tau-\tau_0)^2}, \qquad (288)$$

from which it can be seen that the collapse of the matter takes place in a finite proper time $\tau_0$, i.e., the matter density $\varepsilon$ reaches an infinite value during a finite interval of proper time.

In accordance with the conception of the RTG, the natural geometry for the gravitational field is the geometry of Minkowski space. This means that the components of the gravitational field or, by virtue of the connection (121), the components of the metric tensor $g^{mn}$ of the Riemannian space satisfy not only the Hilbert-Einstein equations but also the universal field equations [see (123)]:

$$D_m\tilde{g}^{mn}=0.$$

Therefore, to study any problem in the RTG it is necessary to find a simultaneous solution satisfying not only the first system in (154) but also the second system of equations in (154). This has the consequence that the "proper" variables $R$ and $\tau$ will be functions of the Minkowski-space variables.

In Sec. 12, we have already found this connection for the region $R\geqslant R_0$, and it is expressed in terms of (281) and (282). We write them explicitly with allowance for (252) and (284):

$$t=\frac{2}{3\sqrt{2m}}[R^{3/2}-B^{3/2}]-2\sqrt{2mB}+2m\ln\frac{\sqrt{B}+\sqrt{2m}}{\sqrt{B}-\sqrt{2m}}; \qquad (289)$$

$$r=B-m. \qquad (290)$$

Note that (289) is identical to the formula obtained in Ref. 44 on the basis of different arguments.

It can be seen directly from (289) that for $R\geqslant R_0$ the RTG restricts the range of variation of $B(R,\tau)$ from below:

$$B(R,\tau)\geqslant 2m, \qquad (291)$$

where the point $B=2m$ corresponds to an infinite value of the variable $t$.

On the basis of (286) and (291) we conclude that $\tau$ never reaches the value $\tau_0$.

From the point of view of an external observer, for example, the surface of a spherical star of "radius" $R=R_0$ will approach the Schwarzschild sphere with radius $B(R_0,\tau)=2m$ during infinite time $t$, while from the point of view of the comoving frame this process will take place in a finite proper time $\tau_p$ equal to

$$\tau_p=\left[1-\left(\frac{2m}{R_0}\right)^{3/2}\right]\tau_0. \qquad (292)$$

This last formula can be readily obtained from (286) by taking into account (291).

Thus, the RTG equations (123) restrict the range of variation of the proper time $\tau$:

$$\tau\leqslant\tau_p<\tau_0.$$

We now calculate the limiting value of the density $\varepsilon$. In (288), we can use the expression (292) for $\tau_p$, since it is valid for $R=R_0$. Then

$$\varepsilon_{max}=3/32\pi m^2. \qquad (293)$$

It can be seen from this that the density $\varepsilon$ does not become infinite because the new field equations (123) necessarily prevent the proper time $\tau$ from reaching the value $\tau_0$. Therefore, in the comoving frame the process is halted at the finite time

$$\tau=\tau_p.$$

From the point of view of the external frame of reference, the brightness of the object decreases exponentially (it becomes black), but at the same time nothing untoward happens to it, since the matter density always remains finite. For example, for a mass of the body of the order of $10^8 M_\odot$ it is 2 g/cm$^3$.

Despite the fact that the gravitational contraction of the massive body to the Schwarzschild radius occurs in a finite proper time $\tau_p < \tau_0$, this does not by any means imply that objects at the present time can reach such a state. In the RTG, this is in principle impossible, since such a state is a limiting state and is reached subject to the condition that the time $t$ in the Minkowski space is equal to infinity. The possible existence of fossil objects of such type is not ruled out.

We now analyze the system of equations (279) and (280) within the body. We recall that this system is obtained from (123).

Within the body, one should take into account the dependence in $\varepsilon$ not only on $\tau$ but also on $R$. But this would greatly complicate the determination of an exact solution, and therefore we consider for methodological purposes the simple case when $\varepsilon$ depends only on $\tau$ and, hence, the solutions of the system of Hilbert-Einstein equations within the body have the form (285) and (278).

We require that (285) and (278) as functions of the Minkowski-space coordinates satisfy the system of equations (279) and (280). Solving this system, we find in this manner the explicit connection between the coordinates $R$ and $B(R, \tau)$ of the comoving frame and the Minkowski-space coordinates $r$ and $t$ within the body.

As was shown in Ref. 43, these solutions have the form

$$t = \frac{\xi}{2R\eta}\left[\frac{v_0(\xi+R_0-R)}{\xi+R_0-R} + \frac{v_0(\xi-R_0+R)}{\xi-R_0+R}\right] - \frac{(R_0-R)}{2R\eta}\int_{\xi-R_0+R}^{\xi+R_0-R} dz\, \frac{v_0(z)}{z^2} - \frac{1}{R\eta}\int_{\xi-R_0+R}^{\xi+R_0-R} dz\,\omega_0(z)\,\frac{\xi^2-(R_0-R)^2+z^2}{4z^2}; \quad (294)$$

$$r = B + \frac{m}{2R_0^3}R^3 - \frac{3m}{2R_0}R. \quad (295)$$

To shorten the expressions, we have introduced in (294) the notation

$$\eta = (1-\tau/\tau_0); \quad \xi = 3\tau_0\eta^{1/3}; \quad (296)$$

$$v_0(z) = R_0^2\left(\frac{z}{3\tau_0}\right)^3\left\{\frac{2m}{R_0}\ln\frac{z+2R_0}{z-2R_0} + \frac{2}{3}\left(\frac{R_0}{2m}\right)^{1/2}\left[1-\left(\frac{z}{3\tau_0}\right)^3\right] - 2\left(\frac{2m}{R_0}\right)^{1/2}\frac{z}{3\tau_0}\right\}; \quad (297)$$

$$\omega_0(z) = \frac{1}{R_0}v_0(z) - 2\left(\frac{R_0}{3\tau_0}\right)^2\frac{z^3}{z^2-(2R_0)^2}. \quad (298)$$

For $R = R_0$, the solutions (294) and (295) go over into the solutions (289) and (290), respectively, and therefore it is sufficient to analyze the expressions (294) and (295) in the region $R < R_0$. Since $t$ must be real, from (294) and (296)–(298) we find

$$\xi \geqslant \xi_{\min} = 3R_0 - R \geqslant 2R_0 \left(R_0 > \frac{9}{2}m\right). \quad (299)$$

The maximal value permitted for $\xi$ can be determined from (296) and is equal to $3\tau_0$. From (294) it can be shown that in the limit $\xi \to \xi_{\min}$ the value of $t$ tends to infinity in accordance with the law

$$t_{\xi\to\xi_{\min}} = \frac{\tau_0}{R(\tau_0-\tau)}\left(\frac{R_0}{3\tau_0}\right)^3 (3R_0-R)(4m+3\tau_0)\ln\frac{1}{\xi-\xi_{\min}(R)}.$$

For different values of $R$, we shall have different values of the limiting proper time $\tau_p(R)$ and, hence, different limiting densities $\varepsilon$ corresponding to an infinite value of the variable $t$. This means that the limiting density is a function of the radial variable $R$. Of course, our investigation of the solution of the problem within the body has only methodological interest. It merely demonstrates that the restriction for $\tau$ which arises in the solution of the RTG equations for the exterior problem also holds for the interior problem. Thus, if we retain the name "gravitational collapse," we must change its physical meaning, since in the RTG, in contrast to GR, it does not lead to an infinite matter density. Since the matter density in a collapsing star is not very great, and in a number of cases is even small, its interior region can in principle be observed from an external frame of reference.

We have considered a model in which the pressure is equal to zero, and even in this case there is no catastrophically strong contraction of the matter. But in real objects it must be assumed that the process of gravitational contraction is even weaker. Therefore, according to the RTG there cannot be in nature any objects (black holes) in which gravitational contraction of matter to infinite density occurs.

In the RTG, we encounter a new phenomenon—gravitational "restraint." It is by virtue of this phenomenon, with allowance for the mechanism of formation of neutron stars, that objects with density greater than $10^{16}$ g/cm$^3$ cannot arise in nature. Objects with greater density, if they exist at all in nature, can only be of a fossil origin. To study the evolution of a collapsing object, it is necessary to make a detailed study of the processes that take place within it, with allowance for the equation of state.

Thus, in the RTG it is in principle impossible for a singularity to arise as a result of gravitational collapse, and therefore there is no "greatest crisis of all times" in fundamental physics.

## 14. GRAVITATIONAL FIELD OF A NONSTATIC SPHERICALLY SYMMETRIC BODY IN THE RTG. BIRKHOFF'S THEOREM

In the general theory of relativity, it is proved that the gravitational field exterior to a nonstatic spherically symmetric body reduces to the static gravitational field determined by the Schwarzschild metric (255). This was proved by Birkhoff.[46] However, as was noted in Sec. 12, the Schwarzschild metric does not satisfy the RTG equations, and it is therefore necessary to prove an analogous theorem in the framework of the RTG. Following Ref. 47, we shall show that the exterior gravitational field of a nonstatic spherically symmetric body in the RTG is static.

Let the interval have the form (260). Then the functions $\omega(\tau, R)$ and $B(\tau, R)$ outside the considered body satisfy the following Hilbert-Einstein equations:

$$e^{\omega(\tau,R)}=\frac{1}{1+f(R)}\left(\frac{\partial B}{\partial R}\right)^2;\quad \left(\frac{\partial B}{\partial \tau}\right)^2=f(R)+\frac{2m}{B}, \quad (300)$$

where $f(R) > -1$ is an arbitrary function of the variable $R$, and $m$ is a positive constant number.

We note that the combination of variables $(\tau, R, \theta, \varphi)$ used in the representation (260) for the interval $ds^2$ forms a set of comoving coordinates $\xi^i = (\tau, R, \theta, \varphi)$ (see Sec. 12).

To find solutions satisfying not only (300) but also the RTG equations of the form (123), we must go over from the comoving coordinates to the Minkowski-space coordinates

$$x^i = (t, r, \theta, \varphi)$$

by means of the substitution

$$t = t(\tau, R) \text{ and } r = r(\tau, R), \quad (301)$$

and write the interval $ds^2$ in the form

$$ds^2 = g_{tt}\,dt^2 + 2g_{tr}dtdr + g_{rr}dr^2 - B^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (302)$$

By virtue of the tensor transformation law, on the basis of (301) we can establish the following connection between the metric coefficients of the representations (260) and (302):

$$\left.\begin{aligned} g^{tt} &= \left(\frac{\partial t}{\partial\tau}\right)^2-\left(\frac{\partial t}{\partial R}\right)^2 e^{-\omega};\quad g^{tr}=\frac{\partial t}{\partial\tau}\frac{\partial r}{\partial\tau}-\frac{\partial t}{\partial R}\frac{\partial r}{\partial R}e^{-\omega};\\ g^{rr} &= \left(\frac{\partial r}{\partial\tau}\right)^2-\left(\frac{\partial r}{\partial R}\right)^2 e^{-\omega};\quad g^{\theta\theta}=-\frac{1}{B^2};\quad g^{\varphi\varphi}=-\frac{1}{B^2\sin^2\theta}.\end{aligned}\right\} \quad (303)$$

Then from (123) we obtain equations of the form

$$\frac{\partial}{\partial\tau}\left[B^2 e^{\frac{\omega}{2}}\frac{\partial t}{\partial\tau}\right]=\frac{\partial}{\partial R}\left[B^2 e^{-\frac{\omega}{2}}\frac{\partial t}{\partial R}\right]; \quad (304)$$

$$\frac{\partial}{\partial\tau}\left[B^2 e^{\frac{\omega}{2}}\frac{\partial r}{\partial\tau}\right]=\frac{\partial}{\partial R}\left[B^2 e^{-\frac{\omega}{2}}\frac{\partial r}{\partial R}\right]-2re^{\frac{\omega}{2}}. \quad (305)$$

Note that the system of equations (300), (304), and (305) for $f(R) = 0$ is identical to the system (278)–(280).

We now find solutions of Eqs. (304) and (305) for all $f > -1$. We shall seek the metric $g^{ik}$ (303) in a form stationary with respect to the variable $t$:

$$\frac{\partial}{\partial t} g^{ik}(x) = 0. \quad (306)$$

Going over in (306) to differentiation with respect to $\tau$ and $R$, we obtain

$$\frac{\partial g^{ik}}{\partial\tau}\frac{\partial r}{\partial R}=\frac{\partial g^{ik}}{\partial R}\frac{\partial r}{\partial\tau}. \quad (307)$$

Hence, for $(i, k) = (\theta, \theta)$ we find

$$\frac{\partial B}{\partial\tau}\frac{\partial r}{\partial R}=\frac{\partial B}{\partial R}\frac{\partial r}{\partial\tau},$$

and this gives $r = r(B)$. Therefore, we seek a solution of (305) in the form $r = r(B)$. Then we obtain

$$\frac{\partial^2 r}{\partial B^2}(B^2-2mB)+\frac{\partial r}{\partial B}(2B-2m)-2r=0;$$

$$B \geqslant 2m.$$

This equation has a unique regular solution of the form[15,43]

$$r = B - m;\quad r \geqslant m. \quad (308)$$

Since by virtue of (306) $g^{tt}$ must be a function of $r$, and also setting $g^{tr} = 0$ with allowance for (303) and (308), we obtain for $\partial t/\partial\tau$ and $\partial t/\partial R$ the relations

$$\left.\begin{aligned}\left(\frac{\partial t}{\partial\tau}\right)^2-\left(\frac{\partial t}{\partial R}\right)^2\left(\frac{\partial B}{\partial R}\right)^{-2}(1+f) &= g^{tt}(r) \equiv H(B);\\ \frac{\partial t}{\partial\tau}\frac{\partial B}{\partial\tau}-\frac{\partial t}{\partial R}\left(\frac{\partial B}{\partial R}\right)^{-1}(1+f) &= 0.\end{aligned}\right\} \quad (309)$$

From these we find

$$\frac{\partial t}{\partial\tau}=\sqrt{1+f}\,\Psi(B);\quad \frac{\partial t}{\partial R}=\frac{\partial B}{\partial\tau}\frac{\partial B}{\partial R}\frac{\Psi(B)}{\sqrt{1+f}}, \quad (310)$$

where

$$\Psi(B)=H^{1/2}(B)\Big/\left(1-\frac{2m}{B}\right)^{1/2}.$$

The condition of compatibility

$$\frac{\partial}{\partial R}\left(\frac{\partial t}{\partial\tau}\right)=\frac{\partial}{\partial\tau}\left(\frac{\partial t}{\partial R}\right)$$

of the system (310) makes it possible to determine the function $\Psi(B)$, for which we obtain the equation

$$\frac{\partial\Psi}{\partial B}\left(1-\frac{2m}{B}\right)+\Psi\frac{2m}{B^2}=0.$$

Choosing as a solution of this equation the function

$$\Psi(B) = (1 - 2m/B)^{-1}, \quad (311)$$

we find for $H(B)$

$$H(B) = (1 - 2m/B)^{-1}.$$

Then from (310) we obtain for the function $t$ the system

$$\left.\begin{aligned}\frac{\partial t}{\partial\tau} &= \sqrt{1+f}\left(1-\frac{2m}{B}\right)^{-1};\\ \frac{\partial t}{\partial R} &= \frac{\partial B}{\partial\tau}\frac{\partial B}{\partial R}\left(1-\frac{2m}{B}\right)^{-1}\frac{1}{\sqrt{1+f}}.\end{aligned}\right\} \quad (312)$$

Quadrature gives

$$t(B, R) = -\sqrt{1+f}\int^{B} dB'\left[\left(f+\frac{2m}{B'}\right)^{-1/2}\left(1-\frac{2m}{B'}\right)^{-1}\right]. \quad (313)$$

Hence, for $f > 0$ we find[3)]

$$t(B, R) = 2m\ln\frac{\operatorname{th}\frac{\eta}{2}+\operatorname{th}y}{\operatorname{th}\frac{\eta}{2}-\operatorname{th}y} + 2m\operatorname{cth}y\left[(\eta-\operatorname{sh}\eta)\frac{1}{2\operatorname{sh}^2 y}-\eta\right],$$

where

$$f(R) \equiv \operatorname{sh}^2 y(R);\quad \operatorname{ch}\eta \equiv f\frac{B}{m}+1;$$

$$\operatorname{th}^2\frac{\eta}{2}-\operatorname{th}^2 y=\left(\frac{B}{2m}-1\right)\left(1+\frac{fB}{2m}\right)^{-1}\frac{f}{1+f}.$$

For $f = 0$, the expression for $t$ simplifies and takes the form (289), which we have already encountered in Sec. 13.

We do not give the explicit form of $t(B, R)$ when $f < 0$, since the expressions are cumbersome.

Substituting (308) and (312) in (303), we obtain the required exterior solution to the system of RTG equations

(283). Then for the interval $ds^2$, taking into account (284), we find

$$ds^2 = \frac{r-m}{r+m}\,dt^2 - \frac{r+m}{r-m}\,dr^2 - (r+m)^2\,(d\theta^2 + \sin^2\theta\,d\varphi^2). \tag{251}$$

But a static spherically symmetric body has such a solution (see Sec. 12). Therefore, in the RTG Birkhoff's theorem holds: A nonstatic spherically symmetric source produces a static gravitational field outside the body.

## 15. GRAVITATIONAL RADIATION

In the theory of gravitation, one of the most important problems is the emission and detection of gravitational waves.

Comprehensive theoretical study of this problem encounters a number of difficulties, which are mainly associated with the strongly nonlinear nature of the field equations. It is only in the weak-field approximation that the problem of gravitational radiation can be systematically studied to the end. As yet, nobody has succeeded in detecting gravitational radiation. It is natural to expect, gravitational radiation having an exceptionally low intensity, that the linearized field equations completely satisfy the requirements of investigators who study gravitational radiation reaching them from observable sources in the universe.

We assume that in the whole of space–time, including the region occupied by the source, the gravitational field $\Phi^{mn}(x)$ is weak:

$$|\Phi^{mn}(x)| \ll 1. \tag{314}$$

As was shown in Sec. 11, the generalized system (218) of RTG equations can be represented in a Cartesian coordinate system in the weak-field approximation in the form

$$\left.\begin{aligned} &\Box\Phi^{mn} - m^2\Phi^{mn} = -16\pi\overset{(0)}{T}{}^{mn};\\ &\partial_m\Phi^{mn} = 0. \end{aligned}\right\} \tag{315}$$

If in reality the graviton mass is nonzero, it must be very small and its influence should be manifested only on cosmological scales.

In this section, we intend to compare the results of GR and the RTG. It is therefore expedient to consider the system of equations

$$\left.\begin{aligned} &\Box\Phi^{mn} = -16\pi\overset{(0)}{T}{}^{mn};\\ &\partial_m\Phi^{mn} = 0, \end{aligned}\right\} \tag{316}$$

which are obtained from (315) for zero mass of the graviton. The quantity $\overset{(0)}{T}{}^{mn}$ on the right-hand side of (316) can be obtained from the Hilbert energy-momentum tensor for the matter by replacing in it $g^{mn}$ by $\gamma^{mn}$ and $\nabla_n$ by $\partial_n$. In the weak-field approximation, the covariant conservation law (83) takes the form

$$\partial_m\overset{(0)}{T}{}^{mn} = 0. \tag{317}$$

We use the standard scheme of solutions of (316). We represent the tensors $\Phi^{mn}(\mathbf{r}, t)$ and $\overset{(0)}{T}{}^{mn}(\mathbf{r}, t)$ in terms of Fourier integrals with respect to the time:

$$\Phi^{mn}(\mathbf{r},\ t) = \int_{-\infty}^{\infty} e^{-i\omega t}\Phi^{mn}(\mathbf{r},\ \omega)\,d\omega; \tag{318}$$

$$\overset{(0)}{T}{}^{mn}(\mathbf{r},\ t) = \int_{-\infty}^{\infty} e^{-i\omega t}\overset{(0)}{T}{}^{mn}(\mathbf{r},\ \omega)\,d\omega. \tag{319}$$

Because $\Phi^{mn}(\mathbf{r}, t)$ and $\overset{(0)}{T}{}^{mn}(\mathbf{r}, t)$ are real, we obtain from (318) and (319)

$$\overset{*}{\Phi}{}^{mn}(\mathbf{r},\ \omega) = \Phi^{mn}(\mathbf{r},\ -\omega);\quad \overset{(0)}{T}{}^{*mn}(\mathbf{r},\ \omega) = \overset{(0)}{T}{}^{mn}(\mathbf{r},\ -\omega).$$

Substituting in the first equation of the system (316) the integral representations (318) and (319), we obtain the equation

$$(\Delta + \omega^2)\,\Phi^{mn}(\mathbf{r},\ \omega) = 16\pi\overset{(0)}{T}{}^{mn}(\mathbf{r},\ \omega), \tag{320}$$

whose solution is well known:

$$\Phi^{mn}(\mathbf{r},\ \omega) = -4\int \frac{e^{i\omega R}}{R}\overset{(0)}{T}{}^{mn}(\mathbf{r}',\ \omega)\,dV. \tag{321}$$

Here

$$R = |\,\mathbf{r}' - \mathbf{r}\,|\,.$$

For the Fourier transforms $\Phi^{mn}(\mathbf{r}, \omega)$ and $\overset{(0)}{T}{}^{mn}(\mathbf{r}, \omega)$, we find from the second equation of the system (316) and (317)

$$i\omega\Phi^{0n}(\mathbf{r},\ \omega) = \partial_\alpha\Phi^{\alpha n}(\mathbf{r},\ \omega); \tag{322}$$

$$i\omega\overset{(0)}{T}{}^{0n}(\mathbf{r},\ \omega) = \partial_\alpha\overset{(0)}{T}{}^{\alpha n}(\mathbf{r},\ \omega). \tag{323}$$

Here and in what follows, the Greek indices take values from 1 to 3 inclusive. On the basis of (322), we can readily express $\Phi^{0n}(\mathbf{r}, \omega)$ in terms of the spatial components $\Phi^{\alpha\beta}(\mathbf{r}, \omega)$:

$$\Phi^{00}(\mathbf{r},\ \omega) = -\frac{1}{\omega^2}\,\partial_\alpha\partial_\beta\Phi^{\alpha\beta}(\mathbf{r},\ \omega); \tag{324}$$

$$\Phi^{0\alpha}(\mathbf{r},\ \omega) = -\frac{i}{\omega}\,\partial_\beta\Phi^{\alpha\beta}(\mathbf{r},\ \omega). \tag{325}$$

Thus, the solution of the system (321) of RTG equations has only six independent components.

The spatial components of the Fourier transforms $\Phi^{\alpha\beta}(\mathbf{r}, \omega)$ can be conveniently expressed in a form that will later enable us to demonstrate the quadrupole nature of the field $\Phi^{\alpha\beta}(\mathbf{r}, t)$ more clearly. With allowance for (323), we represent the expression (321) for the spatial components in the form

$$\begin{aligned}\Phi^{\alpha\beta}(\mathbf{r},\ \omega) = 2\omega^2\Big\{&\int \frac{e^{i\omega R}}{R}\overset{(0)}{T}{}^{00}(\mathbf{r}',\ \omega)\,x'^\alpha x'^\beta\,dV\\ &+\frac{2i}{\omega}\,\partial_\sigma\int \frac{e^{i\omega R}}{R}\overset{(0)}{T}{}^{0\sigma}(\mathbf{r}',\ \omega)\,x'^\alpha x'^\beta\,dV\\ &-\frac{1}{\omega^2}\,\partial_\sigma\partial_\tau\int \frac{e^{i\omega R}}{R}\overset{(0)}{T}{}^{\sigma\tau}(\mathbf{r}',\ \omega)\,x'^\alpha x'^\beta\,dV\Big\}.\end{aligned} \tag{326}$$

We now use the arbitrariness in the solution of the system of equations (316). In the weak-field approximation, $\overset{(0)}{T}{}^{mn}(\mathbf{r}, t)$ does not depend on $\Phi^{mn}(\mathbf{r}, t)$. Therefore, if $\Phi^{mn}(\mathbf{r}, t)$ is a solution of the system of equations (316), then so is

$$\begin{aligned}\Phi'^{mn}(\mathbf{r},\ t) = \Phi^{mn}(\mathbf{r},\ t) + \partial^m a^n(\mathbf{r},\ t) + \partial^n a^m(\mathbf{r},\ t)\\ - \gamma^{mn}\partial_k a^k(\mathbf{r},\ t),\end{aligned} \tag{327}$$

where the 4-vector $a^n(\mathbf{r}, t)$ satisfies the equation

$$\Box a^n(\mathbf{r}, t) = 0. \quad (328)$$

It is here appropriate to emphasize that (327) is a gauge transformation and has no relation to coordinate transformations.

Besides (328), we must impose on $a^n(\mathbf{r}, t)$ a condition which ensures weakness of the field $\Phi'^{mn}(\mathbf{r}, t)$. This means that for $\partial^m a^n(\mathbf{r}, t)$ we must require fulfillment of the inequality

$$|\partial^m a^n(\mathbf{r}, t)| \ll 1. \quad (329)$$

Then the observable physical quantities in the weak-field approximation can be calculated equally well on the basis of $\Phi^{mn}(\mathbf{r}, t)$ and $\Phi'^{mn}(\mathbf{r}, t)$.

From (327) and (328) we find for the Fourier transforms

$$\Phi'^{00}(\mathbf{r}, \omega) = \Phi^{00}(\mathbf{r}, \omega) - i\omega a^0(\mathbf{r}, \omega) - \partial_\alpha a^\alpha(\mathbf{r}, \omega); \quad (330)$$

$$\Phi'^{0\alpha}(\mathbf{r}, \omega) = \Phi^{0\alpha}(\mathbf{r}, \omega) - i\omega a^\alpha(\mathbf{r}, \omega) + \partial^\alpha a^0(\mathbf{r}, \omega); \quad (331)$$

$$\Phi'^{\alpha\beta}(\mathbf{r}, \omega) = \Phi^{\alpha\beta}(\mathbf{r}, \omega) + \partial^\alpha a^\beta(\mathbf{r}, \omega) + \partial^\beta a^\alpha(\mathbf{r}, \omega) - \gamma^{\alpha\beta}(\partial_\sigma a^\sigma(\mathbf{r}, \omega) - i\omega a^0(\mathbf{r}, \omega)); \quad (332)$$

$$(\omega^2 - \partial_\alpha \partial^\alpha) a^n(\mathbf{r}, \omega) = 0. \quad (333)$$

We choose the 4-vector $a^n(\mathbf{r}, \omega)$ in such a way as to make the transforms $\Phi'^{0\alpha}(\mathbf{r}, \omega)$ $(\alpha = 1, 2, 3)$ and the trace of the field $\Phi'^{mn}(\mathbf{r}, \omega)$, which is equal to $\Phi_0'^0(\mathbf{r}, \omega) + \Phi_\alpha'^\alpha(\mathbf{r}, \omega)$, vanish. Conditions of such form for the field $\Phi'^{mn}(\mathbf{r}, \omega)$ impose the so-called TT gauge.

On the basis of Eqs. (324), (325), and (330)–(333) we can show that the field $\Phi'^{mn}(\mathbf{r}, \omega)$ satisfies the conditions of the TT gauge if

$$a^0(\mathbf{r}, \omega) = -\frac{i}{2\omega}\left[\Phi^{00}(\mathbf{r}, \omega) - \frac{1}{2}\Phi_n^n(\mathbf{r}, \omega)\right]; \quad (334)$$

$$a^\alpha(\mathbf{r}, \omega) = -\frac{i}{\omega}\Phi^{0\alpha}(\mathbf{r}, \omega) - \frac{1}{2\omega^2}\partial^\alpha\left[\Phi^{00}(\mathbf{r}, \omega) - \frac{1}{2}\Phi_n^n(\mathbf{r}, \omega)\right]. \quad (335)$$

Taking into account the expressions (334) and (335), we obtain from (332) for $\Phi'^{\alpha\beta}(\mathbf{r}, \omega)$ outside the matter

$$\Phi'^{\alpha\beta}(\mathbf{r}, \omega) = S^{\alpha\beta}(\mathbf{r}, \omega) - \frac{1}{\omega^2}(\partial^\alpha\partial_\sigma S^{\sigma\beta}(\mathbf{r}, \omega) + \partial^\beta\partial_\sigma S^{\alpha\sigma}(\mathbf{r}, \omega)) + \frac{1}{2\omega^2}\gamma^{\alpha\beta}\partial_\sigma\partial_\tau S^{\sigma\tau}(\mathbf{r}, \omega) + \frac{1}{2\omega^4}\partial^\alpha\partial^\beta\partial_\sigma\partial_\tau S^{\sigma\tau}(\mathbf{r}, \omega), \quad (336)$$

where we have introduced the notation

$$S^{\alpha\beta}(\mathbf{r}, \omega) = \Phi^{\alpha\beta}(\mathbf{r}, \omega) - \frac{1}{3}\gamma^{\alpha\beta}\Phi_\sigma^\sigma(\mathbf{r}, \omega). \quad (337)$$

In (337), $\gamma^{\alpha\beta}$ is the spatial part of the Minkowski metric with elements $-1$ on the diagonal. It is readily noted that $S^{\alpha\beta}(\mathbf{r}, \omega)$ is a traceless tensor, i.e.,

$$\gamma_{\alpha\beta} S^{\alpha\beta}(\mathbf{r}, \omega) = 0. \quad (338)$$

We now turn to the solution of (326). Expanding $R^{-1}$ in powers of $r^{-1}$, where $r$ is the distance from the center of the source to the point of observation of the field, and assuming that the linear dimensions of the source are appreciably less than $r$, we find from (326)

$$\Phi^{\alpha\beta}(\mathbf{r}, \omega) = \frac{2\omega^2}{r}\int dV e^{i\omega R} x'^\alpha x'^\beta (\overset{(0)}{T}{}^{00}(\mathbf{r}', \omega) + 2e_\sigma \overset{(0)}{T}{}^{0\sigma}(\mathbf{r}', \omega) + e_\sigma e_\tau \overset{(0)}{T}{}^{\sigma\tau}(\mathbf{r}', \omega)). \quad (339)$$

Here $e^\sigma = x^\sigma/r$, $e_\sigma e^\sigma = -1$. On the transition from (326) to (339), we have omitted the nonwave terms, which decrease faster than $r^{-1}$. Substituting (339) in (337), we obtain

$$S^{\alpha\beta}(\mathbf{r}, \omega) = \frac{2\omega^2}{r}\int dV e^{i\omega R}\left(x'^\alpha x'^\beta - \frac{1}{3}\gamma^{\alpha\beta}x'_\sigma x'^\sigma\right) \times [\overset{(0)}{T}{}^{00}(\mathbf{r}', \omega) + 2e_\sigma \overset{(0)}{T}{}^{0\sigma}(\mathbf{r}', \omega) + e_\sigma e_\tau \overset{(0)}{T}{}^{\sigma\tau}(\mathbf{r}', \omega)]. \quad (340)$$

It is easy to show that apart from terms of order $r^{-2}$

$$\partial_\sigma S^{\alpha\beta}(\mathbf{r}, \omega) = -i\omega e_\sigma S^{\alpha\beta}(\mathbf{r}, \omega);$$
$$\partial^\sigma S^{\alpha\beta}(\mathbf{r}, \omega) = -i\omega e^\sigma S^{\alpha\beta}(\mathbf{r}, \omega).$$

This enables us to rewrite (336) in the form

$$\Phi'^{\alpha\beta}(\mathbf{r}, \omega) = S^{\alpha\beta}(\mathbf{r}, \omega) + e^\alpha e_\sigma S^{\sigma\beta}(\mathbf{r}, \omega) + e^\beta e_\sigma S^{\sigma\alpha}(\mathbf{r}, \omega) - \frac{1}{2}\gamma^{\alpha\beta}e_\sigma e_\tau S^{\sigma\tau}(\mathbf{r}, \omega) + \frac{1}{2}e^\alpha e^\beta e_\sigma e_\tau S^{\sigma\tau}(\mathbf{r}, \omega). \quad (341)$$

Introducing the projection operators

$$P_\beta^\alpha = \delta_\beta^\alpha + e^\alpha e_\beta, \quad (342)$$

which satisfy the conditions

$$P_\alpha^\alpha = 2; \quad P_\sigma^\alpha P_\beta^\sigma = P_\beta^\alpha, \quad (343)$$

we represent (341) in the compact form

$$\Phi'^{\alpha\beta}(\mathbf{r}, \omega) = \left[P_\tau^\alpha P_\sigma^\beta - \frac{1}{2}P^{\alpha\beta}P_{\tau\sigma}\right]S^{\sigma\tau}(\mathbf{r}, \omega). \quad (344)$$

Taking the Fourier integral of both sides of Eq. (344), we obtain

$$\Phi'^{\alpha\beta}(\mathbf{r}, t) = \left(P_\tau^\alpha P_\sigma^\beta - \frac{1}{2}P^{\alpha\beta}P_{\tau\sigma}\right)S^{\sigma\tau}(\mathbf{r}, t), \quad (345)$$

where $S^{\sigma\tau}(\mathbf{r}, t)$ has by virtue of (340) the form

$$S^{\sigma\tau}(\mathbf{r}, t) = -\frac{2}{r}\frac{d^2}{dt^2}\int dV\left(x'^\sigma x'^\tau - \frac{1}{3}\gamma^{\sigma\tau}x'_\alpha x'^\alpha\right) \times [\overset{(0)}{T}{}^{00}(\mathbf{r}', t') + 2e_\alpha \overset{(0)}{T}{}^{0\alpha}(\mathbf{r}', t') + e_\alpha e_\beta \overset{(0)}{T}{}^{\alpha\beta}(\mathbf{r}', t')]_{\text{ret}}. \quad (346)$$

Here and in what follows, the subscript ret of an integral means that the expression in the square brackets is to be taken at the retarded time $t' = t - R/c$.

We define the traceless tensor of the generalized quadrupole moment $\mathscr{D}^{\alpha\beta}$ by

$$\mathscr{D}^{\alpha\beta} = D^{\alpha\beta} + 2e_\sigma D^{\alpha\beta\sigma} + e_\sigma e_\tau D^{\alpha\beta\sigma\tau}, \quad (347)$$

where

$$D^{\alpha\beta} = \int dV(3x'^\alpha x'^\beta - \gamma^{\alpha\beta}x'_\sigma x'^\sigma)[\overset{(0)}{T}{}^{00}(\mathbf{r}', t')]_{\text{ret}}; \quad (348)$$

$$D^{\alpha\beta\sigma} = \int dV(3x'^\alpha x'^\beta - \gamma^{\alpha\beta}x'_\tau x'^\tau)[\overset{(0)}{T}{}^{0\sigma}(\mathbf{r}', t')]_{\text{ret}}; \quad (349)$$

$$D^{\alpha\beta\sigma\tau}=\int dV\,(3x'^{\alpha}x'^{\beta}-\gamma^{\alpha\beta}x'_{\nu}x'^{\nu})\,[\overset{(0)}{T}{}^{\sigma\tau}(\mathbf{r}',\,t')]_{\text{ret}}. \quad (350)$$

It then follows from (346) that

$$S^{\alpha\beta}(\mathbf{r},\,t)=-\frac{2}{3r}\,\frac{d^2}{dt^2}\,\mathscr{D}^{\alpha\beta}(\mathbf{r},\,t). \quad (351)$$

Substituting (351) in (345), for the field $\Phi'^{\alpha\beta}(\mathbf{r},\omega)$ satisfying the TT gauge we obtain outside the matter the basic formula

$$\Phi'^{\alpha\beta}(\mathbf{r},\,t)=-\frac{2}{3r}\left(P^{\alpha}_{\sigma}P^{\beta}_{\tau}-\frac{1}{2}\,P^{\alpha\beta}P_{\sigma\tau}\right)\frac{d^2}{dt^2}\,\mathscr{D}^{\sigma\tau}(\mathbf{r},\,t). \quad (352)$$

If for the source we have

$$\left|\frac{d^2}{dt^2}\overset{(0)}{T}{}^{00}\right|\gg\left|\frac{d^2}{dt^2}\overset{(0)}{T}{}^{0\tau}\right|\gg\left|\frac{d^2}{dt^2}\overset{(0)}{T}{}^{\tau\sigma}\right|,$$

then in (352) $\mathscr{D}^{\sigma\tau}(\mathbf{r},t)$ can be replaced by

$$D^{\sigma\tau}(\mathbf{r},\,t)=\int dV\,(3x'^{\sigma}x'^{\tau}-\gamma^{\sigma\tau}x'_{\nu}x'^{\nu})\left[\overset{(0)}{T}{}^{00}\left(\mathbf{r}',\,t-\frac{r}{c}\right)\right]. \quad (353)$$

In Sec. 11, on the basis of the connection (121), we obtained the expansions (212) and (213) in the weak-field approximation for the Riemannian metric $g^{mn}$ and the determinant $g$. It follows from these expansions in the first order in the field $\Phi^{mn}$ that

$$g=-1-\Phi^k_k \quad (354)$$

and

$$g^{mn}=\gamma^{mn}-h^{mn}, \quad (355)$$

where

$$h^{mn}=-\Phi^{mn}+\frac{1}{2}\,\gamma^{mn}\Phi^k_k. \quad (356)$$

If in (356) we take as the field $\Phi^{mn}(\mathbf{r},t)$ a field satisfying the TT gauge, then it follows from (356), (352), and (353) that

$$h^{\alpha\beta}(\mathbf{r},\,t)=\frac{2}{3r}\left(P^{\alpha}_{\sigma}P^{\beta}_{\tau}-\frac{1}{2}\,P^{\alpha\beta}P_{\sigma\tau}\right)\frac{d^2}{dt^2}\,D^{\sigma\tau}(\mathbf{r},\,t). \quad (357)$$

For completeness of the exposition, we study the polarization properties of the field $h^{\alpha\beta}(\mathbf{r},t)$. It is easy to show on the basis of (342) and (343) that

$$\gamma_{\alpha\beta}h^{\alpha\beta}(\mathbf{r},\,t)=0;$$
$$e_{\alpha}h^{\alpha\beta}(\mathbf{r},\,t)=0,\ \beta=1,\,2,\,3.$$

Therefore, because of these four relations only two of the six field components $h^{\alpha\beta}(\mathbf{r},t)$ are independent. We consider the field $h^{\alpha\beta}(\mathbf{r},t)$ at an appreciable distance from the source in the direction of the $z$ axis, this corresponding to the choice of the vector $e^{\sigma}$ in the form $e^1=e^2=0$, $e^3=1$. For the components $h^{\alpha\beta}$ we then obtain

$$h^{13}=h^{23}=h^{33}=0;\ h^{11}=-h^{22}. \quad (358)$$

Therefore, as independent components of the field $h^{\alpha\beta}$ we can choose $h^{11}$ and $h^{12}$.

We now establish the transformation properties of $h^{\alpha\beta}$ under rotation of the three-dimensional space around the $z$ axis through an angle $\theta$. Since the nonvanishing elements of the matrix of rotation about the $z$ axis have the form

$$\Omega_1^1=\cos\theta;\quad \Omega_1{}^2=\sin\theta;$$
$$\Omega_2{}^1=-\sin\theta;\quad \Omega_2^2=\cos\theta;\quad \Omega_3^3=1,$$

we find from (358) that

$$h'^{13}=h'^{23}=h'^{33}=0;$$
$$h'^{11}=-h'^{22}=\cos 2\theta h^{11}-\sin 2\theta h^{12}; \quad (359)$$
$$h'^{12}=\sin 2\theta h^{11}+\cos 2\theta h^{12}.$$

Since the field $h^{\alpha\beta}(\mathbf{r},t)$ outside the matter satisfies a linear homogeneous equation, any linear combination of the components $h^{\alpha\beta}$ will satisfy the same equation. We consider the combinations

$$h_{\pm}=h^{11}\pm ih^{12}.$$

For them, we find from (359) the transformation law

$$h'_{\pm}=e^{\pm 2i\theta}h_{\pm}. \quad (360)$$

It is well known that if the wave function $\Psi$ transforms under a rotation of space through an angle $\theta$ about the direction of propagation of the wave in accordance with the law

$$\Psi'=e^{i\lambda\theta}\Psi,$$

then $\Psi$ is an eigenfunction of the helicity operator $\hat{I}_{\lambda}$ with eigenvalue equal to $\lambda$.

Therefore, we conclude from (360) that the functions $h_{\pm}$ are eigenfunctions of the operator $\hat{I}_{\lambda}$ and describe a state of the gravitational field with helicity $\lambda=\pm 2$, respectively. The states of the gravitational field with helicities $\lambda=\pm 1$ and $\lambda=0$ do not have physical meaning, since they can always be made to vanish by an appropriate gauge transformation.

Thus, outside the matter the field $h^{\alpha\beta}(\mathbf{r},t)$ describes a physical gravitational field possessing only spin 2 and helicity $\pm 2$.

Following Einstein (Ref. 2, p. 631), we now calculate the intensity of gravitational radiation in the framework of GR. The method proposed by Einstein for calculating the intensity and various modifications of the method have been widely used and are given in many papers and monographs. In the present paper, we consider the variant presented by Landau and Lifshitz in their book.[18] It is well known that from the Hilbert-Einstein equations it is possible to obtain the differential conservation law

$$\partial_n\,[-g\,(T^{mn}+\tau^{mn})]=0, \quad (361)$$

where $\tau^{nm}=\tau^{mn}$ is the pseudotensor of the gravitational field. Integrating (361) over a sufficiently large volume and assuming that there is no flux of matter through the surface bounding the volume of integration, we obtain

$$\frac{d}{dt}\int(-g)\,[T^{0m}+\tau^{0m}]\,dV=-\oint(-g)\,\tau^{\alpha m}\,dS_{\alpha}. \quad (362)$$

According to Einstein (Ref. 2, p. 645), the right-hand side of (362) for $m=0$ "certainly represents the loss of energy of the material system," and, therefore,

$$\frac{dE}{dt}=-\oint(-g)\,\tau^{0\alpha}\,dS_{\alpha}. \quad (363)$$

Then the "energy flux" of the gravitational radiation through the infinitesimal area $dS_{\alpha}$ will be given by

$$dI = (-g)\,\tau^{0\alpha}\,dS_\alpha. \quad (364)$$

If as the surface of integration we take a sphere of radius $r(dS_\alpha = -r^2 e_\alpha d\Omega)$, then for the "intensity of the gravitational radiation" in the element $d\Omega$ of solid angle we obtain

$$\frac{dI}{d\Omega} = -r^2(-g)\,\tau^{0\alpha} e_\alpha. \quad (365)$$

Calculating $(-g)\tau^{0\alpha}$, for example, by means of the Landau-Lifshitz pseudotensor (Ref. 18, p. 360) in the weak-field approximation using (354) and (355) in the TT gauge, we find

$$(-g)\,\tau^{0\alpha} = \frac{e^\alpha}{32\pi}\left(\frac{dh^{\mu\nu}}{dt}\right)\left(\frac{dh_{\mu\nu}}{dt}\right). \quad (366)$$

Hence, the expression for the intensity of the gravitational radiation in the element $d\Omega$ of solid angle takes the form

$$\frac{dI}{d\Omega} = \frac{r^2}{32\pi}\left(\frac{dh^{\mu\nu}}{dt}\right)\left(\frac{dh_{\mu\nu}}{dt}\right). \quad (367)$$

After substitution of the expression (357) in (367), we obtain

$$\frac{dI}{d\Omega} = \frac{1}{36\pi}\left\{\frac{1}{4}(e_\alpha e_\beta \dddot{D}^{\alpha\beta})^2 + \frac{1}{2}\dddot{D}^{\alpha\beta}\dddot{D}_{\alpha\beta} + e_\alpha e^\beta \dddot{D}_{\beta\sigma}\dddot{D}^{\sigma\alpha}\right\}. \quad (368)$$

Here and in what follows, dots above $D$ denote derivatives with respect to the time $t$. Integrating (368) over the angular variables with allowance for the relations

$$\int d\Omega e_\alpha e_\beta = -\frac{4\pi}{3}\gamma_{\alpha\beta};$$

$$\int d\Omega e_\alpha e_\beta e_\sigma e_\tau = \frac{4\pi}{15}(\gamma_{\alpha\beta}\gamma_{\sigma\tau} + \gamma_{\alpha\sigma}\gamma_{\beta\tau} + \gamma_{\alpha\tau}\gamma_{\beta\sigma}),$$

we find the well-known quadrupole formula for the "total radiation" established for the first time by Einstein in Ref. 2 (p. 631):

$$I = \frac{1}{45}\dddot{D}_{\alpha\beta}\dddot{D}^{\alpha\beta}. \quad (369)$$

It is obvious from this that

$$I > 0. \quad (370)$$

It should, however, be emphasized particularly that although the expression (369) for the intensity of the gravitational radiation is correct, it does not follow from GR. For the derivation that we have reproduced here of the expression (369) in GR is based on the definition of the "energy flux" by means of the expression (365). This last expression contains $\tau^{0\alpha}$, which is not a tensor.

The analysis made in Refs. 6 and 11 showed that in GR, depending on the choice of the coordinate system, the intensity (368) of gravitational radiation through each element of a spherical surface of arbitrary radius $r$ and, therefore, the total intensity through the complete sphere during any finite preassigned interval of time can be not only zero but also negative, in contradiction to Einstein's assertions (Ref. 2, p. 631).

By the choice of an admissible frame of reference in GR, we found, on the basis of the expression (365) in the weak-field limit, the following expressions for the intensity of the gravitational radiation in the element $d\Omega$ of solid angle and for the total intensity:

$$\frac{dI}{d\Omega} = \frac{(1-a^2)}{36\pi}\left\{\frac{1}{4}(e_\alpha e_\beta \dddot{D}^{\alpha\beta})^2 + \frac{1}{2}\dddot{D}_{\alpha\beta}\dddot{D}^{\alpha\beta} + e_\alpha e^\beta \dddot{D}_{\beta\sigma}\dddot{D}^{\sigma\alpha}\right\} \quad (371)$$

and

$$I = \frac{(1-a^2)}{45}\dddot{D}_{\alpha\beta}\dddot{D}^{\alpha\beta}, \quad (372)$$

where $a$ is an arbitrary constant. It can be seen from (371) and (372) that it is only for $a = 0$ that we obtain the expressions (368) and (369). Taking, in particular, for $a$ the values $+1$ or $-1$, we find that the gravitational radiation intensity $dI/d\Omega$ and also the total intensity vanish.

Thus, we conclude that $dI/d\Omega$ and $I$, determined in GR following Einstein's method on the basis of (365), can be made to have arbitrary sign by an appropriate choice of an admissible coordinate system—they can be positive, negative, or zero. This fact by itself is physically meaningless, since radiation, as objective physical reality, cannot be destroyed by any admissible coordinate transformation.

In contrast to GR, we shall show below that in the framework of the RTG such difficulties do not arise, and the expressions (368) and (369) are rigorous consequences of our theory.

We base the calculation of the gravitational radiation intensity on the covariant conservation law of the RTG in the form (185):

$$D_m(T_n^m + \overset{(0)}{t}{}^m_{(g)n}) = 0.$$

In Sec. 9, it was shown that such a form of expression of the covariant conservation law for the energy-momentum tensor of the matter and the gravitational field taken together is completely equivalent to the covariant conservation law (102). We chose the form (185) for expressing the conservation law for purely technical reasons. For $\overset{(0)}{t}{}^m_{(g)n}$, we already have the representation (167), in which we have explicitly separated the term[4)] that is the covariant divergence of the tensor $K_m^{kp}$, which is antisymmetric with respect to the superscripts and therefore does not contribute to (185). With allowance for (167), the expression (185) can be written in the form

$$D_m(T_n^m + \tau_n^m) = 0, \quad (373)$$

where

$$\tau_n^m = -\delta_n^m L_g + \frac{1}{16\pi}\left[\tilde{G}_{pq}^m + \frac{1}{2}\tilde{g}^{mk}\tilde{g}_{pq}\tilde{G}_{kl}^l\right]D_n\tilde{g}^{pq}. \quad (374)$$

The left-hand side of the expression (373), in contrast to (361), is a true tensor, since it is a covariant divergence with respect to the Minkowski metric of the tensor quantities $T_n^m$ and $\tau_n^m$. Therefore, calculation of the intensity (or other characteristics of the gravitational field) based on the relation (373) will not depend on the choice of any particular coordinate system. Choosing a Cartesian system, we find from (373)

$$\partial_m(T_n^m + \tau_n^m) = 0. \quad (375)$$

Integrating (375) over a sufficiently large volume and assuming that there is no flux of matter through the surface bounding the volume of integration, we obtain

$$\partial_0 \int (T^0_n + \tau^0_n)\, dV = -\oint \tau^\alpha_n\, dS_\alpha. \qquad (376)$$

Since for $n = 0$ the left-hand side of Eq. (376) is the energy loss of the system, the energy flux of the gravitational radiation through the element of area $dS_\alpha$ is

$$dI = \tau^\alpha_0\, dS_\alpha. \qquad (377)$$

Choosing as the surface of integration a sphere of radius $r$, we arrive at the following expression for the intensity of the gravitational radiation in the element of solid angle $d\Omega$:

$$dI/d\Omega = -r^2 \tau^\alpha_0\, dS_\alpha. \qquad (378)$$

To find the explicit form of the intensity (378), it is necessary to calculate $\tau^\alpha_0$ in the weak-field approximation. Expressing first (374) in Cartesian coordinates and using the decompositions (354) and (355) in the weak-field approximation in the TT gauge, we find

$$\tau^\alpha_0 = -\frac{1}{16\pi}\, \partial_0 h^{\sigma\tau} \partial_\sigma h^\alpha_\tau + \frac{1}{32\pi}\, \partial_0 h^{\sigma\tau} \partial^\alpha h_{\sigma\tau}. \qquad (379)$$

To terms $O(1/r^2)$, we have the identity

$$\partial^\alpha h_{\sigma\tau} \equiv e^\alpha \partial_0 h_{\sigma\tau}, \qquad (380)$$

and in the TT gauge

$$\partial_\alpha h^{\alpha\beta} = 0,$$

so that from (379) we obtain

$$\tau^\alpha_0 e_\alpha = -\frac{1}{32\pi}\, \partial_0 h^{\sigma\tau} \partial_0 h_{\sigma\tau}. \qquad (381)$$

Therefore, the expression (378) for the gravitational radiation intensity takes the form

$$\frac{dI}{d\Omega} = \frac{r^2}{32\pi}\, \partial_0 h^{\alpha\beta} \partial_0 h_{\alpha\beta}.$$

With allowance for (357), this expression leads us to the expressions (368) and (369).

Einstein obtained a correct expression because of the fact that in Cartesian coordinates the expressions in the RTG for the energy-momentum tensor are identical to the expressions for Einstein's energy-momentum pseudotensor. On the transition to other general admissible coordinates, this equality no longer holds, and it therefore leads in GR to the possibility of annihilating the gravitational radiation by the choice of an admissible frame of reference. This indicates from the physical point of view the logical inconsistency of GR.

## 16. HOMOGENEOUS AND ISOTROPIC UNIVERSE. BOUND ON THE GRAVITON MASS

In this section, on the basis of (218), we consider a homogeneous and isotropic universe. As usual, we represent the interval for such a universe in the form

$$ds^2 = c^2 U(t)\, dt^2 - V(t, r)\, (dx^2 + dy^2 + dz^2), \qquad (382)$$

where

$$r = \sqrt{x^2 + y^2 + z^2}.$$

In the expression (382), $(ct, x, y, z)$ are coordinates of pseudo-Euclidean space and are chosen in accordance with the values $(1, -1, -1, -1)$ of the Minkowski metric.

For a given matter distribution, the functions $U(t)$ and $V(t, r)$ in (382) must be determined from the system of equations (218).

To simplify the calculations, we have hitherto used a system of units in which $c = \hbar = G = 1$. In this section, we return to the system of cgs units.

In what follows, it is convenient to base the treatment on the system of equations (218) written in mixed coordinates:

$$\left.\begin{aligned} &\sqrt{-g}\left(R^m_n - \frac{1}{2}\delta^m_n R\right) - \frac{\sqrt{-g}}{2}\left(\frac{mc}{\hbar}\right)^2 \Big[\delta^m_n + g^{mk}\gamma_{kn} - \\ &\qquad - \frac{1}{2}\delta^m_n g^{pk}\gamma_{pk}\Big] = \varkappa T^m_n; \\ &D_m \tilde{g}^{mn} = 0. \end{aligned}\right\} \qquad (383)$$

In (383), we have introduced the notation $\varkappa = 8\pi G/c^2$.

As $T^m_n$, we take the energy-momentum-tensor density of an ideal fluid[15]:

$$T^m_n = \sqrt{-g}\left[\left(\rho + \frac{1}{c^2}p\right) u^m u_n - \delta^m_n \frac{p}{c^2}\right], \qquad (384)$$

where $\rho(t)$ is the density, $p(t)$ is the isotropic pressure, and $u^n$ is the unit 4-vector of the velocity. In accordance with (382),

$$g_{00} = U(t);\ g_{11} = g_{22} = g_{33} = -V(t, r);$$
$$g_{mn} = 0;\ m \neq n; \qquad (385)$$

$$g^{00} = \frac{1}{U(t)};\ g^{11} = g^{22} = g^{33} = -\frac{1}{V(t, r)};\ g^{mn} = 0;\ m \neq n; \qquad (386)$$

$$\sqrt{-g} = \sqrt{U(t)\, V^3(t, r)}.$$

We first consider the conditions imposed by the second equation of (383) on $U(t)$ and $V(t, r)$. In a Cartesian coordinate system, we find

$$\partial_t \sqrt{U^{-1}(t)\, V^3(t, r)} = 0;$$

$$\partial_x V^{\frac{1}{2}}(t, r) = \partial_y V^{\frac{1}{2}}(t, r) = \partial_z V^{\frac{1}{2}}(t, r) = 0.$$

From this an important conclusion follows: The function $V$ does not depend on $r$ and

$$U(t) = V^3(t). \qquad (387)$$

Therefore, the expression for the interval (382) takes the form

$$ds^2 = c^2 V^3(t)\, dt^2 - V(t)\, (dx^2 + dy^2 + dz^2). \qquad (388)$$

Going over to the proper time $\tau$ in accordance with

$$V^{3/2}(t)\, dt = d\tau, \qquad (389)$$

we write the interval (388) in the form

$$\begin{aligned} ds^2 &= c^2 d\tau^2 - V(\tau)\, (dx^2 + dy^2 + dz^2) \\ &= c^2 d\tau^2 - V(\tau)\, (dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2). \qquad (390) \end{aligned}$$

Comparing (390) with the well-known general expression for the Robertson-Walker interval

$$ds^2 = c^2 d\tau^2 - V(\tau)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2\right], \quad (391)$$

where the constant $k$ takes the values 0, $\pm 1$, we arrive at the conclusion that in the RTG the constant $k$ is uniquely determined, its value being zero.

Thus, by virtue of (123) the RTG leads uniquely to this prediction: The universe is infinite and is "flat." Since this conclusion is a consequence of (123) alone, this general conclusion does not depend on the value of the graviton rest mass.

We now write the first equation of (383) in terms of $V(t)$, $\rho(t)$, and $p(t)$. By virtue of (384), (385), and (387),

$$\left(R_0^0 - \frac{1}{2}R\right) = \varkappa\rho(t) + \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\left[1 + \frac{1}{2}\left(\frac{1}{V^3} - \frac{3}{V}\right)\right]; \quad (392)$$

$$\left(R_1^1 - \frac{1}{2}R\right) = -\frac{\varkappa}{c^2}p(t) + \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\left[1 - \frac{1}{2}\left(\frac{1}{V^3} + \frac{1}{V}\right)\right]. \quad (393)$$

Parametrizing the function $V(t)$ in the form

$$V(t) = e^{\mu(t)}$$

for the left-hand sides of Eqs. (392) and (393) we can obtain the expressions

$$\left(R_0^0 - \frac{1}{2}R\right) = \frac{3}{4c^2}e^{-3\mu}\left(\frac{d\mu}{dt}\right)^2; \quad (394)$$

$$\left(R_1^1 - \frac{1}{2}R\right) = \frac{1}{c^2}e^{-3\mu}\left[\frac{d^2\mu}{dt^2} - \frac{3}{4}\left(\frac{d\mu}{dt}\right)^2\right]. \quad (395)$$

Going over in accordance with (389) to the proper time and setting

$$e^{\mu(\tau)} = R^2(\tau),$$

we obtain

$$\left(\frac{\dot R}{R}\right)^2 = \frac{1}{3}c^2\varkappa\rho(\tau) + \frac{1}{6}\left(\frac{mc^2}{\hbar}\right)^2\left[1 + \frac{1}{2}\left(\frac{1}{R^6} - \frac{3}{R^2}\right)\right]; \quad (396)$$

$$\left(\frac{\ddot R}{R}\right) = -\frac{1}{3}\varkappa c^2\rho(\tau) - \frac{1}{2}\varkappa p(\tau) + \frac{1}{6}\left(\frac{mc^2}{\hbar}\right)^2\left(1 - \frac{1}{R^6}\right). \quad (397)$$

Here and in what follows, the dots above $R$ denote the derivatives with respect to the proper time $\tau$.

Note that the expression in the square brackets in (396) is non-negative. Indeed, it is easy to show that

$$1 + \frac{1}{2}\left(\frac{1}{R^6} - \frac{3}{R^2}\right) = \frac{1}{R^6}(R^2-1)^2\left(R^2 + \frac{1}{2}\right) \geqslant 0.$$

By virtue of this inequality, Eq. (396) is defined for any $R \geqslant 0$.

We introduce the notation

$$H(\tau) = \left(\frac{\dot R}{R}\right). \quad (398)$$

For the contemporary epoch in the evolution of the universe $\tau = \tau_0$ the quantity $H(\tau_0)$ is known as the Hubble "constant" and is positive. Therefore, after the extraction of the root in (396) it is necessary to take the positive sign in front of the root:

$$\left(\frac{\dot R}{R}\right) = \left[\frac{1}{3}\varkappa c^2\rho(\tau) + \frac{1}{6R^6}\left(\frac{mc^2}{\hbar}\right)^2(R^2-1)^2\left(R^2 + \frac{1}{2}\right)\right]^{1/2}. \quad (399)$$

Differentiating (399) with respect to $\tau$ and taking into account (397), we obtain after some transformations

$$\frac{1}{R}\frac{dR}{d\tau} = -\frac{1}{3\left(\rho + \frac{1}{c^2}p\right)}\frac{d\rho}{d\tau}. \quad (400)$$

On the basis of Eqs. (399) and (400) we can draw some general conclusions about the development in time of a homogeneous and isotropic universe.

It is obvious from (399) that $\dot R > 0$. Therefore, $R(\tau)$ is a monotonically increasing function of the time $\tau$. Since $\rho + (1/c^2)p > 0$, from (400) we obtain $d\rho/d\tau < 0$ and, hence, $\rho(\tau)$ is a monotonically decreasing function of the time $\tau$.

If $m = 0$ and $\rho(\tau)$ for any finite $\tau$ is nonzero, then, as can be seen from (399) and (397), $\dot R > 0$, and the "acceleration" is negative, $\ddot R < 0$. Therefore, in this case the graph of the function $R = R(\tau)$ rises monotonically and is always convex upward. Therefore, after a finite time $\tau_{\min}$ in the past $R(\tau)$ reaches its minimal value $R_{\min}(\tau_{\min}) = 0$. In what follows, we take $\tau_{\min}$ as the origin of the proper time $\tau$, and we can therefore set $\tau_{\min} = 0$.

In the case when the graviton mass $m$ is nonzero, the function $R = R(\tau)$ also increases monotonically, and in the region $R(\tau) \leqslant 1$ it is convex upward. Therefore in this case too $R(\tau)$ vanishes at a certain value $\tau_{\min}$. But in the region $R(\tau) > 1$, an additional analysis is needed to determine the sign of $\ddot R(\tau)$, since the right-hand side of Eq. (397) contains not only a negative term for $\rho(\tau) + (3/2)(p(\tau)/c^2) > 0$ but also the positive term

$$\frac{1}{6}\left(\frac{mc^2}{\hbar}\right)^2\left(1 - \frac{1}{R^6}\right).$$

We determine for any $\tau$ the critical density by

$$\rho_c(\tau) = \frac{3}{\varkappa c^2}H^2(\tau). \quad (401)$$

Then by virtue of (398) and (399) we find

$$\rho_c(\tau) = \rho(\tau) + \frac{1}{2\varkappa R^6}\left(\frac{mc}{\hbar}\right)^2(R^2-1)^2\left(R^2 + \frac{1}{2}\right). \quad (402)$$

From this it can be seen that if $m \neq 0$, then

$$\rho_c(\tau) > \rho(\tau), \quad (403)$$

apart from the time value $\tau = \tau_1$, for which $R(\tau_1) = 1$. In this last case, i.e., when $R(\tau_1) = 1$,

$$\rho_c(\tau_1) = \rho(\tau_1). \quad (404)$$

But if $m = 0$, then for any $\tau$ the RTG leads to the equality

$$\rho_c(\tau) = \rho(\tau). \quad (405)$$

We consider the relation (402) for the present time $\tau = \tau_0$. It is natural to assume that $R(\tau_0) \gg 1$ at $\tau = \tau_0$, and, therefore,

$$\frac{1}{2}\left|\frac{1}{R^6}-\frac{3}{R^2}\right|\ll 1.$$

Then from (402) we find

$$\left(\frac{mc}{\hbar}\right)^2 \simeq 2\varkappa\,[\rho_c(\tau_0)-\rho_0]. \qquad (406)$$

Here, $\rho_0$ is the matter density at the present time $\tau=\tau_0$.

On the basis of the present observational data,

$$\rho_c(\tau_0)\approx 33\rho_0$$

and therefore from (406) we can obtain the estimate

$$2\varkappa\rho_0(\tau_0)\geqslant (mc/\hbar)^2. \qquad (407)$$

This inequality then determines an upper bound for the graviton mass. If we set

$$\rho_c(\tau_0)\simeq 10^{-29}\ \mathrm{g/cm^3},$$

then from (407) we find[5)]

$$m\leqslant 0{,}64\cdot 10^{-65}\ \mathrm{g}. \qquad (408)$$

We now turn to the investigation of the system of equations (399) and (400). This system is incomplete, since there are only two equations for the three unknown functions $R(\tau)$, $\rho(\tau)$, and $p(\tau)$. As a third equation, one usually takes the equation of state of the matter, which relates $p(\tau)$ to $\rho(\tau)$.

Assuming that at the initial stage of development at $\tau\sim 0$ the universe was in an ultraviolet state, we can use the relation

$$p(\tau)=\frac{c^2}{3}\rho(\tau). \qquad (409)$$

But at the present time $\tau\sim\tau_0$, the pressure can be ignored, and therefore in this stage of development of the universe we set

$$p(\tau)=0. \qquad (410)$$

We analyze Eqs. (399) and (400) separately for the cases when the graviton mass is zero, $m=0$, and when it is nonzero.

*A. Vanishing graviton mass, $m=0$.* For the initial stage in the development of the universe, we find from Eqs. (400), taking into account (409), the solution

$$\rho(\tau)=a/R^4(\tau), \qquad (411)$$

where $a$ is a constant of integration having the dimensions $\mathrm{g/cm^3}$. From Eq. (399) we obtain in this case, taking into account (411),

$$R(\tau)=\left(\frac{4}{3}\varkappa c^2 a\right)^{1/4}\tau^{1/2}. \qquad (412)$$

The expressions (411) and (412) are true for all times $\tau$ at which the universe was in an ultraviolet state.

In the region of times for which the pressure can be ignored, we find from Eqs. (400)

$$\rho(\tau)=b/R^3(\tau), \qquad (413)$$

and from Eqs. (399) we have in this case, taking into account (413),

$$R(\tau)=\left(\frac{3}{4}\varkappa c^2 b\right)^{1/3}\tau^{2/3}. \qquad (414)$$

In (413) and (414), $b$ is a constant of integration with the dimensions $\mathrm{g/cm^3}$.

For the purposes of cosmological measurements, we introduce the deceleration parameter

$$q(\tau)=-\left(\frac{\ddot R}{R}\right)\left(\frac{R}{\dot R}\right)^2. \qquad (415)$$

On the basis of (413) it is easy to show that

$$q(\tau)=1/2. \qquad (416)$$

In accordance with (405), at the present time the matter density must be equal to the critical density $\rho_c(\tau_0)\simeq 10^{-29}\ \mathrm{g/cm^3}$. Thus, the RTG predicts the existence of a large amount of "hidden mass" of the universe in some form of matter. This missing mass is almost 40 times greater than the mass of matter that we currently observe in the universe.

*B. Nonzero graviton mass, $m\neq 0$.* In an early stage in the evolution of the universe the solution of Eq. (399) can be represented in the form

$$\tau=\int_0^{R(\tau)} dx\,x^2\left[\frac{1}{3}\varkappa a c^2 x^2+\frac{1}{6}\left(\frac{mc^2}{\hbar}\right)^2\left(\frac{1}{2}+x^6-\frac{3}{2}x^4\right)\right]^{-1/2}. \qquad (417)$$

In deriving (417), we took into account the relation (411), which is a solution of Eq. (400).

For the range of values

$$\left|R^6(\tau)-\frac{3}{2}R^4(\tau)\right|\ll\frac{1}{2} \qquad (418)$$

we obtain from (417) after integration

$$\tau=\frac{1}{4\varkappa a}\left(\frac{mc}{\hbar}\right)^2\left(\frac{3}{4\varkappa a c^2}\right)^{1/2}\left(y\sqrt{1+y^2}-\operatorname{sh} y\right), \qquad (419)$$

where

$$y=2\sqrt{\varkappa a}\left(\frac{\hbar}{mc}\right)R(\tau). \qquad (420)$$

In the region of small $\tau$, we obtain from (419)

$$R(\tau)\simeq\left(\sqrt{3}\,\frac{mc^2}{\hbar}\right)^{1/3}\tau^{1/3}. \qquad (421)$$

Comparing this expression with (412), we see that the behavior of $R(\tau)$ as $\tau\to 0$ has been somewhat changed by the existence of the graviton mass.

Using (421), we obtain on the basis of (401) and (411)

$$\rho_c(\tau)\simeq\frac{1}{3\varkappa c^2}\frac{1}{\tau^2}; \qquad (422)$$

$$\rho(\tau)\simeq a\left(\frac{\hbar}{\sqrt{3}\,mc^2}\right)^{4/3}\frac{1}{\tau^{4/3}}. \qquad (423)$$

For times at which the pressure can be ignored, we obtain from Eq. (400) for $\rho(\tau)$ the formula (413). Then the solution of Eq. (399) for $R\gg 1$ has the form

$$\tau=\frac{1}{3\sqrt{\varepsilon}}\ln\Big[\Big(\frac{R(\tau)}{\sigma}\Big)^{3}$$

$$\times\Big(2\varepsilon+2\sqrt{\varepsilon^{2}+\frac{1}{3}\varepsilon\varkappa c^{2}\rho(\tau)+\frac{1}{3}\varkappa c^{2}\rho(\tau)}\Big)\Big], \quad (424)$$

where we have introduced the notation

$$\varepsilon=\frac{1}{6}\Big(\frac{mc^{2}}{\hbar}\Big)^{2}. \quad (425)$$

In (424), $\sigma$ is a constant of integration.

Solving (424) for $R(\tau)$, we obtain

$$R(\tau)=\sigma\Big[2\varepsilon+2\sqrt{\varepsilon^{2}+\frac{1}{3}\varepsilon\varkappa c^{2}\rho+\frac{1}{3}\varkappa c^{2}\rho}\Big]^{-1/3}e^{\sqrt{\varepsilon}\,\tau}. \quad (426)$$

On the basis of (426), it is easy to show that

$$H=\sqrt{\varepsilon+\frac{1}{3}\varkappa c^{2}\rho(\tau)}, \quad (427)$$

and by virtue of (415)

$$q(\tau)=-1+\frac{3}{2}\frac{\varkappa c^{2}\rho(\tau)}{\varkappa c^{2}\rho(\tau)+3\varepsilon}. \quad (428)$$

If for the graviton mass we take the value

$$m=0.64\cdot10^{-65}\ \text{g},$$

then from (425) we obtain

$$\varepsilon=4.3\cdot10^{-36}\ \sec^{-2}. \quad (429)$$

Then for the contemporary epoch $\tau=\tau_0$ in the evolution of the universe we obtain from (427) and (428), taking into account $\rho_0=3\times10^{-31}$ g/cm$^3$,

$$H_0=2\cdot10^{-18}\ \sec^{-1}\sim\sqrt{\varepsilon} \quad (430)$$

and

$$q(\tau_0)=-0.94. \quad (431)$$

As $\tau\to\infty$, bearing in mind that the matter density tends to zero, $\rho(\tau)\to0$, with increasing $\tau$, we find from (426)–(428)

$$R(\tau)\simeq\sigma\Big(\frac{1}{4\varepsilon}\Big)^{1/3}e^{\sqrt{\varepsilon}\,\tau}; \quad (432)$$

$$H\simeq\sqrt{\varepsilon}\simeq H_0; \quad (433)$$

$$q=-1. \quad (434)$$

Taking into account the relation (433), we obtain in the expression (432) for $R(\tau)$

$$R(\tau)\simeq\sigma\Big(\frac{1}{4H_0^{2}}\Big)^{1/3}e^{H_0\tau}.$$

This last expression differs appreciably from (414), which was obtained under the assumption that the graviton rest mass is zero.

## 17. POST-NEWTONIAN APPROXIMATION IN THE RTG

The post-Newtonian approximation is used to study systems of island type moving with small nonrelativistic velocities. In the post-Newtonian approximation, the velocity $v$, the gravitational potential $U$, the specific pressure $p/c^2\rho$, and the specific internal energy $\Pi$ have the orders of magnitude

$$v/c\simeq O(\varepsilon);\ U\simeq O(\varepsilon^{2});\ p/\rho c^{2}\simeq O(\varepsilon^{2});\ \Pi\simeq O(\varepsilon^{2}). \quad (435)$$

As experimental data show, the value of the dimensionless Newtonian interaction potential $GM/rc^2$ on, for example, the surface of the Sun (and, therefore, for other celestial bodies of the type of the Sun) does not exceed $2\times10^{-6}$, while on the surface of the Earth it is $6.95\times10^{-9}$. For the solar system, it is also known that the specific pressure $p/c^2\rho$ and the specific internal energy $\Pi$ have approximately the same order, this being $\varepsilon^2\sim10^{-8}$. This means that within the solar system $\varepsilon$ can be used as an expansion parameter for a perturbation-theory series of post-Newtonian form. One can expect that the first few terms of this series will describe the complete set of phenomena in the solar system with a sufficient degree of accuracy.

One of the characteristic features of the solar system is that in it the velocity of motion of the matter does not exceed $\varepsilon$ in units of $c=1$. Therefore, in order of magnitude we can use for the spatial and time derivatives the relation

$$\frac{\partial}{\partial t}\simeq\varepsilon\frac{\partial}{\partial x^{\alpha}}. \quad (436)$$

The connection (436) means that the variation of all quantities with the time is due in the first place to the motion of the matter.

In this section, we construct the post-Newtonian approximation for the RTG equations (154). Even if the graviton rest mass is nonzero, it cannot play any role within the solar system, being so exceptionally small, and therefore it will be sufficient to study only Eqs. (154).

In what follows, to simplify the calculations, it is convenient to work in a system of units in which $c=1$.

Our point of departure is the expansions

$$g_{00}+1+\overset{(2)}{g}_{00}+\overset{(4)}{g}_{00}+\dots; \quad (437)$$

$$g_{\alpha\beta}=\gamma_{\alpha\beta}+\overset{(2)}{g}_{\alpha\beta}+\overset{(4)}{g}_{\alpha\beta}+\dots; \quad (438)$$

$$g_{0\alpha}=\overset{(3)}{g}_{0\alpha}+\overset{(5)}{g}_{0\alpha}+\dots \quad (439)$$

Here, $\gamma_{\alpha\beta}$ is the spatial part of the Minkowski metric $\gamma_{mn}$. The symbols $\overset{(k)}{g}_{mn}$ $(k=2, 3, 4,...)$ on the right-hand sides of Eqs. (437)–(439) represent the terms of order $\varepsilon^k$ in the expansion of $g_{mn}$, respectively. It should be noted that if the sign of the time is reversed, $t\to-t$, it is necessary to require reversal of the sign of the parameter $\varepsilon$ as well. It is for this reason that the expansions (437) and (438) contain only even powers and (439) contains only odd powers of the parameter $\varepsilon$. The fact that $g_{0\alpha}$ does not contain the term $\overset{(1)}{g}_{0\alpha}$ is natural, since already the main (Newtonian) approximation for $g_{0\alpha}$ must be not lower than the second order in $\varepsilon$.

We now find the expansions for $g=\det g_{mn}$ and $g^{mn}$. On the basis of (437)–(439), it can be shown that

$$g=-1-\overset{(2)}{g}_{00}+\overset{(2)}{g}_{11}+\overset{(2)}{g}_{22}+\overset{(2)}{g}_{33}-\overset{(4)}{g}_{00}+\overset{(4)}{g}_{11}+\overset{(4)}{g}_{22}+\overset{(4)}{g}_{33}$$
$$+\overset{(2)}{g}_{00}(\overset{(2)}{g}_{11}+\overset{(2)}{g}_{22}+\overset{(2)}{g}_{33})-\overset{(2)}{g}_{11}\overset{(2)}{g}_{22}-\overset{(2)}{g}_{11}\overset{(2)}{g}_{33}$$
$$-\overset{(2)}{g}_{22}\overset{(2)}{g}_{33}+\overset{(2)}{g}{}^{2}_{12}+\overset{(2)}{g}{}^{2}_{13}+\overset{(2)}{g}{}^{2}_{23}+\dots \quad (440)$$

$$g^{00}=1+\overset{(2)}{g}{}^{00}+\overset{(4)}{g}{}^{00}+\dots; \tag{441}$$

$$g^{\alpha\beta}=\gamma^{\alpha\beta}+\overset{(2)}{g}{}^{\alpha\beta}+\overset{(4)}{g}{}^{\alpha\beta}+\dots; \tag{442}$$

$$g^{0\alpha}=\overset{(3)}{g}{}^{0\alpha}+\overset{(5)}{g}{}^{0\alpha}+\dots, \tag{443}$$

where $\overset{(k)}{g}{}^{mn}$ and $\overset{(k)}{g}_{mn}$ are given by

$$\left.\begin{aligned}
&\overset{(2)}{g}{}^{00}=-\overset{(2)}{g}_{00};\ \overset{(2)}{g}{}^{\alpha\beta}=-\overset{(2)}{g}_{\sigma\tau}\gamma^{\sigma\alpha}\gamma^{\tau\beta};\\
&\overset{(3)}{g}{}^{0\alpha}=-\overset{(3)}{g}_{0\beta}\gamma^{\alpha\beta};\ \overset{(4)}{g}{}^{00}=\overset{(2)}{g}{}^{2}_{00}-\overset{(4)}{g}_{00};\\
&\overset{(4)}{g}{}^{\alpha\beta}=-\gamma^{\alpha\sigma}\gamma^{\beta\tau}\overset{(4)}{g}_{\sigma\tau}+\gamma^{\alpha\omega}\gamma^{\beta\sigma}\gamma^{\lambda\tau}\overset{(2)}{g}_{\sigma\tau}\overset{(2)}{g}_{\omega\lambda};\\
&\overset{(5)}{g}{}^{0\alpha}=-\gamma^{\alpha\beta}\overset{(5)}{g}_{0\beta}+\gamma^{\alpha\beta}\overset{(2)}{g}_{00}\overset{(3)}{g}_{0\beta}+\gamma^{\alpha\tau}\gamma^{\sigma\beta}\overset{(3)}{g}_{0\sigma}\overset{(2)}{g}_{\tau\beta}.
\end{aligned}\right\} \tag{444}$$

To write down the second equation of (154) for the terms of the expansions (441)–(443), we first find $\tilde{g}^{mn}$. From (440)–(443),

$$\tilde{g}^{00}=1+\overset{(2)}{\tilde{g}}{}^{00}+\overset{(4)}{\tilde{g}}{}^{00}+\dots; \tag{445}$$

$$\tilde{g}^{0\alpha}=\overset{(3)}{\tilde{g}}{}^{0\alpha}+\overset{(5)}{\tilde{g}}{}^{0\alpha}+\dots; \tag{446}$$

$$\tilde{g}^{\alpha\beta}=\tilde{\gamma}^{\alpha\beta}+\overset{(2)}{\tilde{g}}{}^{\alpha\beta}+\overset{(4)}{\tilde{g}}{}^{\alpha\beta}+\dots, \tag{447}$$

where

$$\left.\begin{aligned}
&\overset{(2)}{\tilde{g}}{}^{00}=\overset{(2)}{g}{}^{00}+\frac{1}{2}\overset{(2)}{A};\ \overset{(4)}{\tilde{g}}{}^{00}=\overset{(4)}{g}{}^{00}+\frac{1}{2}\overset{(2)}{g}{}^{00}\overset{(2)}{A}+\frac{1}{2}\left(\overset{(4)}{A}-\frac{1}{4}\overset{(2)}{A}{}^{2}\right);\\
&\overset{(3)}{\tilde{g}}{}^{0\alpha}=\overset{(3)}{g}{}^{0\alpha};\ \overset{(5)}{\tilde{g}}{}^{0\alpha}=\overset{(5)}{g}{}^{0\alpha}+\frac{1}{2}\overset{(3)}{g}{}^{0\alpha}\overset{(2)}{A};\\
&\overset{(2)}{\tilde{g}}{}^{\alpha\beta}=\overset{(2)}{g}{}^{\alpha\beta}+\frac{1}{2}\gamma^{\alpha\beta}\overset{(2)}{A};\\
&\overset{(4)}{\tilde{g}}{}^{\alpha\beta}=\overset{(4)}{g}{}^{\alpha\beta}+\frac{1}{2}\overset{(2)}{g}{}^{\alpha\beta}\overset{(2)}{A}+\frac{1}{2}\gamma^{\alpha\beta}\left(\overset{(4)}{A}-\frac{1}{4}\overset{(2)}{A}{}^{2}\right).
\end{aligned}\right\} \tag{448}$$

In (448), we have introduced the notation

$$\overset{(2)}{A}=\overset{(2)}{g}_{00}-\overset{(2)}{g}_{11}-\overset{(2)}{g}_{22}-\overset{(2)}{g}_{33} \tag{449}$$

and

$$\begin{aligned}\overset{(4)}{A}=&\overset{(4)}{g}_{00}-\overset{(4)}{g}_{11}-\overset{(4)}{g}_{22}-\overset{(4)}{g}_{33}-\overset{(2)}{g}_{00}(\overset{(2)}{g}_{11}+\overset{(2)}{g}_{22}+\overset{(2)}{g}_{33})\\
&+\overset{(2)}{g}_{11}\overset{(2)}{g}_{22}+\overset{(2)}{g}_{11}\overset{(2)}{g}_{33}+\overset{(2)}{g}_{22}\overset{(2)}{g}_{33}-\overset{(2)}{g}{}^{2}_{12}-\overset{(2)}{g}{}^{2}_{13}-\overset{(2)}{g}{}^{2}_{23}.\end{aligned} \tag{450}$$

In a Galilean frame of reference, we find from the second equations of the system (154)

$$\frac{1}{2}\partial_0\overset{(2)}{g}_{00}-\frac{1}{2}\gamma^{\alpha\beta}\partial_0\overset{(2)}{g}_{\alpha\beta}=-\gamma^{\alpha\beta}\partial_\alpha\overset{(3)}{g}_{0\beta}; \tag{451}$$

$$\frac{1}{2}\partial_\alpha\overset{(2)}{g}_{00}+\frac{1}{2}\gamma^{\sigma\tau}\partial_\alpha\overset{(2)}{g}_{\sigma\tau}=\gamma^{\sigma\tau}\partial_\tau\overset{(2)}{g}_{\alpha\sigma}; \tag{452}$$

$$\begin{aligned}&\partial_0\left[\overset{(2)}{g}{}^{2}_{00}-\overset{(4)}{g}_{00}-\frac{1}{2}\overset{(2)}{g}_{00}\overset{(2)}{A}+\frac{1}{2}\left(\overset{(4)}{A}-\frac{1}{4}\overset{(2)}{A}{}^{2}\right)\right]\\
&=-\partial_\alpha\left(\overset{(5)}{g}{}^{0\alpha}+\frac{1}{2}\overset{(3)}{g}{}^{0\alpha}\overset{(2)}{A}\right);\end{aligned} \tag{453}$$

$$\partial_\beta\left[\overset{(4)}{g}{}^{\alpha\beta}+\frac{1}{2}\overset{(2)}{g}{}^{\alpha\beta}\overset{(2)}{A}+\frac{1}{2}\gamma^{\alpha\beta}\left(\overset{(4)}{A}-\frac{1}{4}\overset{(2)}{A}{}^{2}\right)\right]=\partial_0\overset{(3)}{g}_{0\beta}\gamma^{\beta\alpha}. \tag{454}$$

We now write down the first equation of the system (154) for $\overset{(k)}{g}_{mn}$. We first find the expansion of the tensor $\Gamma^{p}_{mn}$ in powers of $\varepsilon$. Since in the Galilean frame of reference

$$\Gamma^{p}_{mn}=\frac{1}{2}g^{pq}(\partial_m g_{qn}+\partial_n g_{qm}-\partial_q g_{mn}),$$

we find from this by virtue of (437)–(439) and (441)–(443)

$$\left.\begin{aligned}
&\Gamma^{0}_{00}=\frac{1}{2}\partial_0\overset{(2)}{g}_{00}+\frac{1}{2}\left(\overset{(2)}{g}{}^{00}\partial_0\overset{(2)}{g}_{00}-\overset{(3)}{g}{}^{0\alpha}\partial_\alpha\overset{(2)}{g}_{00}\right)+\dots;\\
&\Gamma^{0}_{0\alpha}=\frac{1}{2}\partial_\alpha\overset{(2)}{g}_{00}+\frac{1}{2}\left(\partial_\alpha\overset{(4)}{g}_{00}+\overset{(2)}{g}{}^{00}\partial_\alpha\overset{(2)}{g}_{00}\right)+\dots;\\
&\Gamma^{0}_{\alpha\beta}=\frac{1}{2}\left(\partial_\alpha\overset{(3)}{g}_{0\beta}+\partial_\beta\overset{(3)}{g}_{0\alpha}-\partial_0\overset{(2)}{g}_{\alpha\beta}\right)+\dots;\\
&\Gamma^{\alpha}_{00}=-\frac{1}{2}\gamma^{\alpha\beta}\partial_\beta\overset{(2)}{g}_{00}-\frac{1}{2}\gamma^{\alpha\beta}\partial_\beta\overset{(4)}{g}_{00}\\
&+\gamma^{\alpha\beta}\partial_0\overset{(3)}{g}_{0\beta}-\frac{1}{2}\overset{(2)}{g}{}^{\alpha\beta}\partial_\beta\overset{(2)}{g}_{00}+\dots;\\
&\Gamma^{\alpha}_{0\beta}=\frac{1}{2}\gamma^{\alpha\sigma}\partial_\beta\overset{(3)}{g}_{\sigma0}+\frac{1}{2}\gamma^{\alpha\sigma}\partial_0\overset{(2)}{g}_{\beta\sigma}-\frac{1}{2}\gamma^{\alpha\sigma}\partial_\sigma\overset{(3)}{g}_{0\beta}+\dots;\\
&\Gamma^{\alpha}_{\beta\omega}=\frac{1}{2}\gamma^{\alpha\sigma}\left(\partial_\beta\overset{(2)}{g}_{\sigma\omega}+\partial_\omega\overset{(2)}{g}_{\sigma\beta}-\partial_\sigma\overset{(2)}{g}_{\beta\omega}\right)+\dots
\end{aligned}\right\} \tag{455}$$

On the basis of (455), we can already find the expansion which we need for the curvature tensor $R_{mn}$ of second rank. Since $R_{mn}$ can be expressed in terms of $\Gamma^{p}_{kq}$ by means of (149), we find after simple calculations

$$\begin{aligned}R_{00}=&-\frac{1}{2}\gamma^{\alpha\beta}\partial_\alpha\partial_\beta\overset{(2)}{g}_{00}-\frac{1}{2}\gamma^{\alpha\beta}\partial_\alpha\partial_\beta\overset{(4)}{g}_{00}\\
&+\gamma^{\alpha\beta}\partial_0\partial_\alpha\overset{(3)}{g}_{0\beta}-\frac{1}{2}\partial_\alpha\left(\overset{(2)}{g}{}^{\alpha\beta}\partial_\beta\overset{(2)}{g}_{00}\right)-\frac{1}{2}\gamma^{\alpha\beta}\partial_0\partial_0\overset{(2)}{g}_{\alpha\beta}\\
&-\frac{1}{4}\gamma^{\alpha\beta}\partial_\beta\overset{(2)}{g}_{00}\gamma^{\sigma\tau}\partial_\alpha\overset{(2)}{g}_{\sigma\tau}+\frac{1}{4}\gamma^{\alpha\beta}\partial_\alpha\overset{(2)}{g}_{00}\partial_\beta\overset{(2)}{g}_{00}\quad\cdots\end{aligned} \tag{456}$$

$$\begin{aligned}R_{0\alpha}=&\frac{1}{2}\gamma^{\beta\sigma}\partial_0\partial_\beta\overset{(2)}{g}_{\alpha\sigma}-\frac{1}{2}\gamma^{\beta\sigma}\partial_0\partial_\alpha\overset{(2)}{g}_{\beta\sigma}\\
&+\frac{1}{2}\gamma^{\beta\sigma}\partial_\alpha\partial_\beta\overset{(3)}{g}_{\sigma0}-\frac{1}{2}\gamma^{\beta\sigma}\partial_\beta\partial_\sigma\overset{(3)}{g}_{0\alpha}+\dots;\end{aligned} \tag{457}$$

$$\begin{aligned}R_{\alpha\beta}=&-\frac{1}{2}\gamma^{\sigma\tau}\partial_\sigma\partial_\tau\overset{(2)}{g}_{\alpha\beta}+\frac{1}{2}\gamma^{\sigma\tau}\partial_\sigma\partial_\alpha\overset{(2)}{g}_{\tau\beta}\\
&+\frac{1}{2}\gamma^{\sigma\tau}\partial_\sigma\partial_\beta\overset{(2)}{g}_{\tau\alpha}-\frac{1}{2}\partial_\alpha\partial_\beta\overset{(2)}{g}_{00}-\frac{1}{2}\gamma^{\sigma\tau}\partial_\alpha\partial_\beta\overset{(2)}{g}_{\tau\sigma}+\dots\end{aligned} \tag{458}$$

Taking into account in these expressions Eqs. (451) and (452), we finally obtain

$$\begin{aligned}R_{00}=&-\frac{1}{2}\gamma^{\alpha\beta}\partial_\alpha\partial_\beta\overset{(2)}{g}_{00}-\frac{1}{2}\gamma^{\alpha\beta}\partial_\alpha\partial_\beta\overset{(4)}{g}_{00}-\frac{1}{2}\partial_0\partial_0\overset{(2)}{g}_{00}\\
&+\frac{1}{2}\gamma^{\sigma\alpha}\gamma^{\tau\beta}\overset{(2)}{g}_{\sigma\tau}\partial_\alpha\partial_\beta\overset{(2)}{g}_{00}+\frac{1}{2}\gamma^{\alpha\beta}\partial_\alpha\overset{(2)}{g}_{00}\partial_\beta\overset{(2)}{g}_{00}+\dots;\end{aligned} \tag{459}$$

$$R_{0\alpha}=-\frac{1}{2}\gamma^{\beta\sigma}\partial_\beta\partial_\sigma\overset{(3)}{g}_{0\alpha}+\dots; \tag{460}$$

$$R_{\alpha\beta}=-\frac{1}{2}\gamma^{\sigma\tau}\partial_\sigma\partial_\tau\overset{(2)}{g}_{\alpha\beta}+\dots \tag{461}$$

To complete the construction of the approximate equations of the RTG, we must expand the matter energy-momentum tensor in powers of $\varepsilon$. For what follows, it is convenient to use for $T^{mn}$ the expansion

$$\begin{aligned}&T^{00}=\overset{(0)}{T}{}^{00}+\overset{(2)}{T}{}^{00}+\dots;\ T^{0\alpha}=\overset{(1)}{T}{}^{0\alpha}+\overset{(3)}{T}{}^{0\alpha}+\dots;\\
&T^{\alpha\beta}=\overset{(2)}{T}{}^{\alpha\beta}+\overset{(4)}{T}{}^{\alpha\beta}+\dots\end{aligned} \tag{462}$$

Taking into account (437)–(439), from (462) we find for

$T_{mn}$

$$T_{00}=\overset{(0)}{T}_{00}+\overset{(2)}{T}_{00}+\cdots;\ T_{0\alpha}=\overset{(1)}{T}_{0\alpha}+\overset{(3)}{T}_{0\alpha}+\cdots;$$

$$T_{\alpha\beta}=\overset{(2)}{T}_{\alpha\beta}+\overset{(4)}{T}_{\alpha\beta}+\cdots, \tag{463}$$

where

$$\left.\begin{array}{l}\overset{(0)}{T}_{00}=\overset{(0)}{T}{}^{00};\ \overset{(2)}{T}_{00}=\overset{(2)}{T}{}^{00}+2\overset{(2)}{g}_{00}\overset{(0)}{T}{}^{00};\\ \overset{(1)}{T}_{0\alpha}=\gamma_{\alpha\beta}\overset{(1)}{T}{}^{0\beta};\ \overset{(3)}{T}_{0\alpha}=\overset{(3)}{g}_{0\alpha}\overset{(0)}{T}{}^{00}+(\overset{(2)}{g}_{\alpha\beta}+\gamma_{\alpha\beta}\overset{(2)}{g}_{00})\overset{(1)}{T}{}^{0\beta};\\ \overset{(2)}{T}_{\alpha\beta}=\gamma_{\alpha\sigma}\gamma_{\beta\tau}\overset{(2)}{T}{}^{\sigma\tau}.\end{array}\right\} \tag{464}$$

Since the right-hand side of the first equation of (154) contains the combination

$$S_{mn}=T_{mn}-\frac{1}{2}g_{mn}T, \tag{465}$$

on the basis of (463) and (464) we can readily find expansions for the components $S_{mn}$ in powers of $\varepsilon$:

$$S_{00}=\frac{1}{2}\overset{(0)}{T}{}^{00}+\frac{1}{2}(\overset{(2)}{T}{}^{00}+2\overset{(2)}{g}_{00}\overset{(0)}{T}{}^{00}-\gamma_{\alpha\beta}\overset{(2)}{T}{}^{\alpha\beta})+\cdots; \tag{466}$$

$$S_{0\alpha}=\gamma_{\alpha\beta}\overset{(1)}{T}{}^{0\beta}+\cdots; \tag{467}$$

$$S_{\alpha\beta}=-\frac{1}{2}\gamma_{\alpha\beta}\overset{(0)}{T}{}^{00}+\left(\gamma_{\alpha\sigma}\gamma_{\beta\tau}-\frac{1}{2}\gamma_{\alpha\beta}\gamma_{\sigma\tau}\right)\overset{(2)}{T}{}^{\sigma\tau}.$$
$$-\frac{1}{2}(\gamma_{\alpha\beta}\overset{(2)}{T}{}^{00}+\gamma_{\alpha\beta}\overset{(2)}{g}_{00}\overset{(0)}{T}{}^{00}+\overset{(2)}{g}_{\alpha\beta}\overset{(0)}{T}{}^{00})+\cdots \tag{468}$$

Substituting in the first equation of (154) the expansions (459)–(461) and (466)–(468) found above, we obtain

$$\gamma^{\alpha\beta}\partial_\alpha\partial_\beta\overset{(2)}{g}_{00}=-8\pi G\overset{(0)}{T}{}^{00}; \tag{469}$$

$$\gamma^{\alpha\beta}\partial_\alpha\partial_\beta\overset{(4)}{g}_{00}+\partial_0\partial_0\overset{(2)}{g}_{00}-\gamma^{\sigma\alpha}\gamma^{\tau\beta}\overset{(2)}{g}_{\sigma\tau}\partial_\alpha\partial_\beta\overset{(2)}{g}_{00}$$
$$-\gamma^{\alpha\beta}\partial_\alpha\overset{(2)}{g}_{00}\partial_\beta\overset{(2)}{g}_{00}=-8\pi G(\overset{(2)}{T}{}^{00}+2\overset{(2)}{g}_{00}\overset{(0)}{T}{}^{00}-\gamma_{\alpha\beta}\overset{(2)}{T}{}^{\alpha\beta}); \tag{470}$$

$$\gamma^{\beta\sigma}\partial_\beta\partial_\sigma\overset{(3)}{g}_{0\alpha}=-16\pi G\gamma_{\alpha\beta}\overset{(1)}{T}{}^{0\beta}; \tag{471}$$

$$\gamma^{\sigma\tau}\partial_\sigma\partial_\tau\overset{(2)}{g}_{\alpha\beta}=8\pi G\gamma_{\alpha\beta}\overset{(0)}{T}{}^{00}. \tag{472}$$

For given $\overset{(0)}{T}{}^{00}$, $\overset{(2)}{T}{}^{00}$, $\overset{(1)}{T}{}^{0\beta}$, and $\overset{(2)}{T}{}^{\alpha\beta}$, the system of equations (469)–(472) completely determines the effective Riemannian metric $g^{mn}$ in the Newtonian and post-Newtonian approximations.

Setting

$$\overset{(2)}{g}_{00}=-2U, \tag{473}$$

where $U$ is the Newtonian interaction potential, we obtain from (469)

$$\nabla^2 U=-4\pi G\overset{(0)}{T}{}^{00}. \tag{474}$$

The solution of this equation under the assumption that $U$ vanishes at infinity can be represented in the form

$$U=G\int\frac{d^3x'\overset{(0)}{T}{}^{00}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}. \tag{475}$$

Similarly, from (471) and (472) we find

$$\overset{(3)}{g}_{0\alpha}=-4G\gamma_{\alpha\beta}\int\frac{d^3x'\overset{(1)}{T}{}^{0\beta}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|} \tag{476}$$

and

$$\overset{(2)}{g}_{\alpha\beta}=2G\gamma_{\alpha\beta}\int\frac{d^3x'\overset{(0)}{T}{}^{00}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}=2\gamma_{\alpha\beta}U. \tag{477}$$

Equation (470) with allowance for (473), (474), and (477) can be written in the form

$$\nabla^2(\overset{(4)}{g}_{00}-2U^2)=-2\partial_0^2U+8\pi G(\overset{(2)}{T}{}^{00}-\gamma_{\alpha\beta}\overset{(2)}{T}{}^{\alpha\beta}). \tag{478}$$

Since $\overset{(4)}{g}_{00}$ must vanish at infinity, from (478) we find

$$\overset{(4)}{g}_{00}=2U^2+\frac{1}{2\pi}\partial_0^2\int\frac{d^3x'U(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}$$
$$-2G\int\frac{dx'(\overset{(2)}{T}{}^{00}-\gamma_{\alpha\beta}\overset{(2)}{T}{}^{\alpha\beta})}{|\mathbf{x}-\mathbf{x}'|}. \tag{479}$$

We note that by virtue of (451) and (477) the potential $U$ and $\overset{(3)}{g}_{0\beta}$ are related by

$$\partial_0 U=\frac{1}{4}\gamma^{\alpha\beta}\partial_\alpha\overset{(3)}{g}_{0\beta} \tag{480}$$

Thus, we have found solutions of the RTG equations for the components of the effective Riemannian metric $g_{mn}$ in the following orders:

$g_{00}$ up to $\varepsilon^4$,

$g_{\alpha\beta}$ up to $\varepsilon^2$, and

$g_{0\alpha}$ up to $\varepsilon^3$.

As we shall see below, such accuracy in the determination of $g_{mn}$ is practically sufficient for the description of all gravitational experiments made within the solar system. Therefore, from the system of equations (451)–(454) it is sufficient to use only the first two. We note that by virtue of (473) and (474) Eq. (452) is satisfied automatically.

Before we turn to the study of gravitational effects in the post-Newtonian approximation, we must choose a "model" for the matter. We assume a body in the state of an ideal fluid. Then as $T^{mn}$ we can take the expression for the energy-momentum tensor for an ideal fluid:

$$T^{mn}=(p+\rho(1+\Pi))u^n u^m-pg^{mn}. \tag{481}$$

In (481), as usual, $p$, $\rho$, and $\Pi$ denote the isotropic pressure, the density of the ideal fluid, and the specific self-energy, respectively, and $u^n$ is the 4-vector of the velocity.

For the energy-momentum tensor $T^{mn}$ and for the invariant density $\rho$ we have the following exact relations: the covariant conservation law

$$\nabla_m T^{mn}=\partial_m T^{mn}+\Gamma^m_{nk}T^{kn}+\Gamma^n_{nk}T^{mk}=0 \tag{482}$$

and the covariant continuity equation

$$\frac{1}{\sqrt{-g}}\partial_n(\sqrt{-g}\rho u^n)=0. \tag{483}$$

In the Newtonian approximation, i.e., when we ignore the gravitational forces,

$$u^0=1+O(\varepsilon^2);\ u^\alpha=v^\alpha(1+O(\varepsilon^2)) \tag{484}$$

and we therefore find from (481)

$$\overset{(0)}{T}{}^{00}=\rho(1+O(\varepsilon^2)); \tag{485}$$

$$\overset{(0)}{T}{}^{\alpha\beta} = O(\varepsilon^2); \qquad (486)$$

$$\overset{(1)}{T}{}^{0\alpha} = \rho v^\alpha (1 + O(\varepsilon^2)). \qquad (487)$$

In deriving (485)–(487), we used the fact that the specific isotropic pressure $p/\rho$ has the order $\varepsilon^2$.

Ignoring in Eqs. (482) and (483) the terms of order higher than $\varepsilon$, we obtain for $\rho$

$$\partial_0 \rho + \partial_\alpha (\rho v^\alpha) = 0. \qquad (488)$$

From this we see that in the Newtonian approximation the total mass of the body is equal to the integral

$$M = \int \rho d^3x$$

and is a conserved quantity.

Taking into account (485) and (487) in Eqs. (475), (476), and (477), we find

$$\overset{(2)}{g}_{00} = -2U; \ \overset{(2)}{g}_{\alpha\beta} = 2\gamma_{\alpha\beta} U; \ \overset{(3)}{g}_{0\alpha} = 4\gamma_{\alpha\beta} V^\beta, \qquad (489)$$

where

$$U = G \int \frac{\rho(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} d^3x'; \qquad (490)$$

$$V^\beta = -G \int \frac{\rho(\mathbf{x}', t) v^\beta}{|\mathbf{x} - \mathbf{x}'|} d^3x'. \qquad (491)$$

Therefore, in the approximation (489) the metric coefficients $g_{mn}$ of the effective Riemannian space can be represented in the form

$$g_{00} = (1 - 2U); \ g_{\alpha\beta} = \gamma_{\alpha\beta}(1 + 2U); \ g_{0\alpha} = 4\gamma_{\alpha\beta} V^\beta. \qquad (492)$$

Knowing the metric in this approximation on the basis of Eqs. (482) and (483), we can determine the components of the matter energy-momentum tensor in the following approximation. But for this it is first necessary to find $\sqrt{-g}u^0$ and the components of the tensor $\Gamma^p_{mn}$ in the Newtonian approximation. By virtue of (492) and (455),

$$\left.\begin{aligned}
&\sqrt{-g} = 1 + 2U;\\
&u^0 = 1 + U - \frac{1}{2} v_\alpha v^\alpha;\\
&\Gamma^0_{00} = -\partial_0 U; \ \Gamma^0_{0\alpha} = -\partial_\alpha U; \ \Gamma^\alpha_{00} = \gamma^{\alpha\beta}\partial_\beta U;\\
&\Gamma^0_{\alpha\beta} = -\gamma_{\alpha\beta}\partial_0 U + 2(\gamma_{\beta\sigma}\partial_\alpha + \gamma_{\alpha\sigma}\partial_\beta) V^\sigma;\\
&\Gamma^\alpha_{0\beta} = 2\partial_\beta V^\alpha + \delta^\alpha_\beta \partial_0 U - 2\gamma^{\alpha\sigma}\gamma_{\beta\tau}\partial_\sigma V^\tau;\\
&\Gamma^\alpha_{\beta\omega} = \delta^\alpha_\omega \partial_\beta U + \delta^\alpha_\beta \partial_\omega U - \gamma^{\alpha\sigma}\gamma_{\beta\omega}\partial_\sigma U.
\end{aligned}\right\} \qquad (493)$$

Then the covariant conservation equation (482) can be written in the form

$$\partial_0 \overset{(2)}{T}{}^{00} + \partial_\alpha \overset{(3)}{T}{}^{0\alpha} - \rho\partial_0 U - 2\rho v^\alpha \partial_\alpha U = O(\varepsilon^5); \qquad (494)$$

$$\partial_\beta \overset{(2)}{T}{}^{\alpha\beta} + \partial_0(\rho v^\alpha) + \gamma^{\alpha\beta}\rho\partial_\beta U = O(\varepsilon^4), \qquad (495)$$

and the continuity equation (483) takes the form

$$\frac{1}{\sqrt{-g}}\left[\partial_0\left(\rho + 3U\rho - \frac{1}{2}\rho v_\alpha v^\alpha\right) + \partial_\alpha\left(\rho v^\alpha + 3\rho v^\alpha U + \frac{1}{2}\rho v^2 v^\alpha\right)\right] = O(\varepsilon^3). \qquad (496)$$

To these equations, we must add the equations of motion of an ideal fluid[15]:

$$\hat{\rho}(\partial_0 v^\alpha + v^\beta \partial_\beta v^\alpha) = \gamma^{\alpha\beta}(-\rho\partial_\beta U + \partial_\beta p) + O(\varepsilon^4); \qquad (497)$$

$$\hat{\rho}(\partial_0 \Pi + v^\beta \partial_\beta \Pi) = -p\partial_\alpha v^\alpha + O(\varepsilon^5), \qquad (498)$$

where

$$\hat{\rho} = \sqrt{-g}\rho u^0. \qquad (499)$$

In accordance with (483), $\hat{\rho}$ is the conserved mass density.

In the approximation which we require, $\hat{\rho}$ admits the expansion

$$\hat{\rho} = \rho\left(1 + 3U - \frac{1}{2} v_\alpha v^\alpha\right), \qquad (500)$$

and therefore $\hat{\rho}$ in (497) and (498) can be replaced by the invariant density $\rho$. From the system of equations (494)–(498), we readily find solutions for $\overset{(2)}{T}{}^{00}$, $\overset{(3)}{T}{}^{0\alpha}$ and $\overset{(2)}{T}{}^{\alpha\beta}$:

$$\left.\begin{aligned}
&\overset{(2)}{T}{}^{00} = \rho(2U + \Pi - v_\alpha v^\alpha);\\
&\overset{(3)}{T}{}^{0\alpha} = \rho v^\alpha (2U + \Pi - v_\beta v^\beta) + p v^\alpha;\\
&\overset{(2)}{T}{}^{\alpha\beta} = \rho v^\alpha v^\beta - \gamma^{\alpha\beta} p.
\end{aligned}\right\} \qquad (501)$$

Therefore, from (479) we find for $\overset{(4)}{g}_{00}$

$$\overset{(4)}{g}_{00} = 2U^2 + \frac{1}{2\pi}\partial_0^2 \int \frac{d^3x' U(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} - 4\Phi_1 - 4\Phi_2 - 2\Phi_3 - 6\Phi_4. \qquad (502)$$

The quantities $\Phi_i$ $(i = 1, 2, 3, 4)$ have the form

$$\Phi_1 = -G \int \frac{\rho v_\alpha v^\alpha}{|\mathbf{x} - \mathbf{x}'|} d^3x'; \quad \Phi_2 = G \int \frac{\rho U}{|\mathbf{x} - \mathbf{x}'|} d^3x';$$

$$\Phi_3 = G \int \frac{\rho \Pi}{|\mathbf{x} - \mathbf{x}'|} d^3x'; \ \Phi_4 = G \int \frac{p\, d^3x'}{|\mathbf{x} - \mathbf{x}'|} \qquad (503)$$

and are generalized gravitational potentials.

Since we have the identity[48]

$$\frac{1}{2\pi}\int \frac{d^3x' U}{|\mathbf{x} - \mathbf{x}'|} \equiv \frac{G}{2\pi}\int \rho(\mathbf{x}'', t) d^3x'' \int \frac{d^3x'}{|\mathbf{x} - \mathbf{x}'||\mathbf{x}' - \mathbf{x}''|} \equiv -G \int \rho(\mathbf{x}'', t)|\mathbf{x} - \mathbf{x}''| d^3x'',$$

we finally obtain for $\overset{(4)}{g}_{00}$

$$\overset{(4)}{g}_{00} = 2U^2 - G\partial_0^2 \int \rho(\mathbf{x}', t)|\mathbf{x} - \mathbf{x}'| d^3x' - 4\Phi_1 - 4\Phi_2 - 2\Phi_3 - 6\Phi_4. \qquad (504)$$

Combining the expressions (492) and (504) for the metric coefficients of the tensor $g_{mn}$ of the effective Riemannian space–time up to the post-Newtonian approximation, we obtain the expressions

$$g_{00} = 1 - 2U + 2U^2 - 4\Phi_1 - 4\Phi_2 - 2\Phi_3 - 6\Phi_4 - G\partial_0^2 \int \rho(\mathbf{x}', t)|\mathbf{x} - \mathbf{x}'| d^3x' + O(\varepsilon^6); \qquad (505)$$

$$g_{0\alpha} = 4\gamma_{\alpha\beta} V^\beta + O(\varepsilon^5); \qquad (506)$$

$$g_{\alpha\beta} = \gamma_{\alpha\beta}(1 + 2U) + O(\varepsilon^4). \qquad (507)$$

Until recently, the requirements imposed on possible theories of gravitation reduced to the need to obtain Newton's law of gravity in the weak-field limit and also to describe the three effects accessible to observation: gravitational red shift in the field of the Sun, bending of a light ray passing near the Sun, and advance of Mercury's perihelion. The insufficient accuracy of measurement in these experiments, and also their paucity were the reason why we have at the present time a large number of different theories of gravitation that successfully explain all these effects.

For further selection of the theory of gravitation, it is necessary, on the one hand, to increase the accuracy in the measurement of the old experiments and to propose qualitatively new experiments, and, on the other, to develop an appropriate theoretical formalism, since the requirements on the possible theories of gravitation are clearly inadequate, a very large number of theories satisfying these requirements.

More recently, in connection with the development of experimental techniques, above all space physics, and the increased accuracy of measurements, new possibilities have appeared for more accurate measurement of the orbital parameters of the planets (and above all the Moon), for measurement of the delay of radio signals in the gravitational field of the Sun, and for new experiments within the solar system. These experiments make it possible to restrict the class of viable theories of gravitation. To facilitate the comparison of the results of experiments made within the solar system with the predictions of the various theories of gravitation for which Riemannian geometry is the natural geometry for the motion of the matter, Nordtvedt and Will[49] developed the so-called parametrized post-Newtonian (PPN) formalism.

In this formalism, the metric of the Riemannian space–time produced by a body that consists of an ideal fluid is expressed as the sum of all possible generalized gravitational potentials with arbitrary coefficients, which are called the post-Newtonian parameters. Using the revised Will-Nordtvedt parameters, we can write the metric of the Riemannian space–time in the form

$$g_{00}=1-2U+2\beta U^2-(2\gamma+2+\alpha_3+\xi_1)\,\Phi_1+\xi_1 A+2\xi_\omega\Phi_\omega$$
$$-2\,[(3\gamma+1-2\beta+\xi_2)\,\Phi_2+(1+\xi_3)\,\Phi_3+3\,(\gamma+\xi_4)\,\Phi_4]$$
$$-(\alpha_1-\alpha_2-\alpha_3)\,W^\alpha W_\alpha U+\alpha_2 W^\alpha W^\beta U_{\alpha\beta}-(2\alpha_3-\alpha_1)\,W^\alpha V_\alpha; \quad (508)$$

$$g_{0\alpha}=\frac{1}{2}\,(4\gamma+3+\alpha_1-\alpha_2+\xi_1)\,\gamma_{\alpha\beta}V^\beta+\frac{1}{2}\,(1+\alpha_2-\xi_1)\,N_\alpha$$
$$-\frac{1}{2}\,(\alpha_1-2\alpha_2)\,W_\alpha U+\alpha_2 W^\beta U_{\alpha\beta}; \quad (509)$$

$$g_{\alpha\beta}=\gamma_{\alpha\beta}\,(1+2\gamma U). \quad (510)$$

Here, $W^\alpha$ are the spatial components of the velocity of the frame of reference relative to some universal rest frame. For some theories of gravitation, this is the velocity of the center of mass of the solar system with respect to the rest frame of the universe. The expressions (508) and (509) contain in addition to the generalized gravitational potentials (490), (491), and (503) introduced above the potentials

$$\left.\begin{aligned} A&=G\int\frac{\rho v_\alpha v_\beta\,(x^\alpha-x'^\alpha)\,(x^\beta-x'^\beta)}{|\mathbf{x}-\mathbf{x}'|^3}\,d^3x';\\ N_\alpha&=\gamma_{\alpha\sigma}G\int\frac{\rho v_\beta\,(x^\beta-x'^\beta)\,(x^\sigma-x'^\sigma)}{|\mathbf{x}-\mathbf{x}'|^3}\,d^3x';\\ U_{\alpha\beta}&=G\int\frac{\rho\,(x_\alpha-x'_\alpha)\,(x_\beta-x'_\beta)}{|\mathbf{x}-\mathbf{x}'|^3}\,d^3x';\\ \Phi_\omega&=G\int\frac{\rho\,(\mathbf{x}',t)\,\rho\,(\mathbf{x}''t)}{|\mathbf{x}'-\mathbf{x}|^3}\left[\frac{\mathbf{x}'-\mathbf{x}''}{|\mathbf{x}-\mathbf{x}''|}-\frac{\mathbf{x}-\mathbf{x}''}{\mathbf{x}'-\mathbf{x}''}\right]\\ &\quad\times(\mathbf{x}-\mathbf{x}')\,d^3x'\,d^3x''.\end{aligned}\right\} \quad (511)$$

To each theory of gravitation for which the natural geometry for describing the motion of the matter is Riemannian geometry there will correspond a set of values of the ten post-Newtonian parameters $\beta, \gamma, \alpha_1, \alpha_2, \alpha_3, \xi_1, \xi_2, \xi_3, \xi_4, \xi_\omega$. From the point of view of experiments made in the solar system, one theory of gravitation will differ from another only by the values of these parameters. It should be noted that when different theories of gravitation are compared the metric tensor $g_{mn}$ in each theory must be specified in the same coordinate system as are the components (508)–(510), for otherwise the comparison of the post-Newtonian parameters becomes meaningless, different sets of parameters corresponding to different coordinate systems. Therefore, after the determination of the metric tensor $g_{mn}$ produced by the gravitational field of the solar system, it is necessary to go over to the "canonical" coordinate system, in which the metric tensor $g_{mn}$ takes the form (508)–(510).

A characteristic feature of the standard post-Newtonian expansion (508)–(510) is that in canonical coordinates the nondiagonal components of the spatial part of the metric tensor $g_{mn}$ are equal to zero and the nonvanishing components do not contain terms of the form

$$\partial_0^2\int\rho|\mathbf{x}-\mathbf{x}'|\,d^3x'. \quad (512)$$

Our solution (505)–(507) for $g_{mn}$, in contrast to Eqs. (508)–(510), contains the expression (512) in $g_{00}$. Therefore, it is necessary to go over to the coordinate system in which the solutions (505)–(507) take the form (508)–(510). Making the coordinate transformation

$$\left.\begin{aligned} x'^0&=x^0+\xi^0\,(x);\\ x'^\alpha&=x_\alpha,\end{aligned}\right\} \quad (513)$$

where $\xi^0\,(x)\simeq O(\varepsilon^3)$, we obtain

$$\left.\begin{aligned} g'_{00}&=g_{00}-2\partial_0\xi_0\,(x);\\ g'_{0\alpha}&=g_{0\alpha}-\partial_\alpha\xi_0\,(x);\\ g'_{\alpha\beta}&=g_{\alpha\beta}.\end{aligned}\right\} \quad (514)$$

Choosing $\xi_0\,(x)$ in the form

$$\xi_0\,(x)=-\frac{G}{2}\,\partial_0\int\rho\,(\mathbf{x}',\,t)|\mathbf{x}-\mathbf{x}'|\,d^3x'$$

we find from (514), taking into account (505)–(507), the following expressions for the metric coefficients $g'_{mn}$ in the canonical coordinates:

$$g'_{00}=1-2U+2U^2-4\Phi_1-4\Phi_2-2\Phi_3-6\Phi_4+O\,(\varepsilon^6); \quad (515)$$

$$g'_{0\alpha}=\frac{7}{2}\,\gamma_{\alpha\beta}V^\beta-\frac{1}{2}\,N_\alpha+O\,(\varepsilon^5); \quad (516)$$

$$g'_{\alpha\beta}=\gamma_{\alpha\beta}\,(1+2U)+O\,(\varepsilon^4). \quad (517)$$

In deriving (516), we used the identity

$$\partial_\alpha \xi_0 \equiv +\frac{1}{2}(\gamma_{\alpha\beta}V^\beta - N_\alpha). \quad (518)$$

Comparing (515)–(517) obtained in the post-Newtonian approximation for the metric coefficients in the framework of the RTG with Eqs. (508)–(510), we find for the post-Newtonian parameters the values

$$\gamma = 1;\ \beta = 1;$$

$$\alpha_1 = \alpha_2 = \alpha_3 = \xi_1 = \xi_2 = \xi_3 = \xi_4 = \xi_\omega = 0. \quad (519)$$

We note in passing that for the case when the source of the gravitational field is a static spherically symmetric body of radius $r_0$ the metric (515)–(517) takes the form

$$\left.\begin{aligned} g'_{00} &= 1 - \frac{2MG}{r} + \frac{2M^2G^2}{r^2} + O\left(\frac{G^3M^3}{r^3}\right); \\ g'_{0\alpha} &= 0; \\ g'_{\alpha\beta} &= \gamma_{\alpha\beta}\left(1 + \frac{2MG}{r}\right) + O\left(\frac{G^2M^2}{r^2}\right), \end{aligned}\right\} \quad (520)$$

with total mass of the source

$$M = 4\pi \int_0^{r_0} \rho\left[1 + \Pi + \frac{3p}{\rho} + 2U\right] r^2\, dr. \quad (521)$$

To find the theories of gravitation that in the post-Newtonian limit make it possible to describe all experiments performed in the solar system, it is sufficient to determine from all these experiments the values of the ten post-Newtonian parameters and to select only those theories of gravitation whose post-Newtonian approximation leads to parameter values agreeing with the values obtained from the experiments. Then all such theories of gravitation will be indistinguishable from the point of view of all experiments made with post-Newtonian accuracy. Further selection of the theory of gravitation requires either an increase in the accuracy of measurement to the post-post-Newtonian level or the search for possibilities of studying the properties of gravitational waves and also phenomena in strong gravitational fields.

As was shown in Ref. 50, the vanishing of the three parameters $\alpha_1, \alpha_2, \alpha_3$ has a definite physical meaning: Any theory of gravitation in which $\alpha_1 = \alpha_2 = \alpha_3 = 0$ does not possess a preferred universal rest frame in the post-Newtonian limit. In this case, on the transition from the universal rest frame to a moving system the metric of the effective Riemannian space–time in the post-Newtonian limit is form-invariant, and the velocity $W^\alpha$ of the new coordinate system relative to the universal rest frame will not occur explicitly in the metric. Since (519) holds, both GR and the RTG belong to the class of such theories.

A linear dependence of the parameters $\xi$ and $\alpha$ also has a certain physical meaning. As was shown in Ref. 51, if [6)]

$$\left.\begin{aligned} \alpha_1 = \alpha_2 = \alpha_3 = 0;\ \xi_3 = 0;\ \xi_2 = \xi_\omega; \\ 3\xi_4 + 2\xi_\omega = 0;\ \xi_1 + 2\xi_\omega = 0, \end{aligned}\right\} \quad (522)$$

then from the post-Newtonian equations of motion it is possible to determine quantities that in the post-Newtonian approximation do not depend on the time. Generally speaking, one can interpret these quantities as energy, momentum, and angular momentum of the system (i.e., as integrals of the motion) only in the theories of gravitation which possess conservation laws for the energy-momentum tensor of the matter and the gravitational field taken together.

For example, in GR the relations (522) are satisfied, but, as detailed analysis shows, the corresponding time-independent quantities in the post-Newtonian approximation are not integrals of the motion of the system consisting of the matter and the gravitational field.

In the RTG, an isolated system has in pseudo-Euclidean space–time all ten conservation laws in their usual sense, and these lead in the post-Newtonian approximation to ten integrals of the motion of the system. The fulfillment of the relations (522) in the RTG confirms this conclusion.

## 18. GRAVITATIONAL EXPERIMENTS IN THE SOLAR SYSTEM

The experiments of Bessel and Eötvös, which were already made in the last century, established that for bodies of laboratory sizes the ratio of the gravitational mass to the inertial mass can differ from unity by not more than $10^{-9}$, irrespective of the matter of which the body is made. This result made a deep impression on Einstein and prompted his formulation of the equivalence principle.

The more recent gravitational experiments have established that the deviation from unity of the ratio of the gravitational mass to the inertial mass for bodies of laboratory size does not exceed $10^{-12}$ (experiments of Braginskiĭ's group[52]).

Although these results are regarded as experimental confirmation of the hypothesis of the equality of the gravitational and inertial masses to a very high accuracy, this does not mean that for large bodies the gravitational and inertial masses are equal to the same accuracy.

According to Nordtvedt's estimates,[53] for bodies of laboratory size the ratio of the gravitational self-energy to the total energy of a body is a quantity in order of magnitude not greater than $10^{-25}$. Therefore, at an accuracy of measurement equal to $10^{-12}$, nothing can be said about the distribution of the gravitational self-energy between the inertial and gravitational masses of the body.

To solve the problem of the equality of the gravitational and inertial masses of an extended body experimentally it is necessary either to increase appreciably the accuracy of gravimetric experiments with bodies of laboratory size or to make measurements with bodies, for example, planets, for which the ratio of the gravitational self-energy to the total energy must be appreciably greater than $10^{-25}$. In the latter case, small perturbations in the planetary orbits may be revealed if the gravitational and inertial masses do indeed differ.

What is the answer given by GR and the RTG with regard to the question of the equality of the gravitational and inertial masses?

It does not follow from GR that the inertial mass is equal to the active gravitational mass. In GR, as was shown in Sec. 3, the value of the inertial mass depends on the choice of the coordinate axes in three-dimensional space, something that is meaningless from the physical point of view.

It follows from the RTG that the active gravitational mass and the inertial mass of a body are equal.

For since the RTG is based on the special relativity principle, the inertial mass of an island system is strictly defined and is equal to

$$m_i = \int d^3x\, (t^{00}_{(g)} + t^{00}_{(M)}). \tag{523}$$

By virtue of the conservation law for the total energy-momentum tensor in Minkowski space,

$$\partial_m\, (t^{mn}_{(g)} + t^{mn}_{(M)}) = 0\,,$$

it is obvious that $m_i$ does not depend on the time. Note also that (523) is a scalar with respect to transformations of the spatial coordinates.

We now write down the first equation of (146) for the field $\tilde{\Phi}^{00}$ in a Cartesian coordinate system. Since in this case $\tilde{\Phi}^{00} = \Phi^{00}$,

$$\Box\, \Phi^{00} = 16\pi\, (t^{00}_{(g)} + t^{00}_{(M)}).$$

When $t^{00}_{(g)} = t^{00}_{(M)}$ is constant in time or varies very weakly in time, so that the gravitational radiation is negligibly small, for $\Phi^{00}$ we obtain the equation

$$\Delta\Phi^{00} = -16\pi\, (t^{00}_{(g)} + t^{00}_{(M)}),$$

whose solution we represent in the form

$$\Phi^{00} = 4 \int \frac{d^3x'\,(t^{00}_{(g)} + t^{00}_{(M)})}{|\,\mathbf{x} - \mathbf{x}'\,|}.$$

Hence, in the limit $|\mathbf{x}| = r \to \infty$ we find

$$\Phi^{00} \simeq 4m_i/r. \tag{524}$$

By virtue of the connection (121), for $\tilde{g}^{00}$ we have the expression

$$\tilde{g}^{00} \simeq 1 + 4m_i/r.$$

At the same time, determining $\tilde{g}^{00}$ from (445), (448), and (489), we obtain

$$\tilde{g}^{00} \simeq 1 + 4U,$$

where $U$ is the Newtonian potential. Now far from the source $U$ (in the units $c = G = 1$) has the representation

$$U = M/r,$$

where $M$ is by definition the gravitational mass of the body, and therefore we arrive at the identity

$$m_i \equiv M, \tag{525}$$

which is what we wanted to prove.

In passing, we note that in the framework of the RTG the quantity

$$P^n = \int d^3x\, (t^{0n}_{(g)} + t^{0n}_{M}) \tag{526}$$

is the energy-momentum 4-vector of the system with respect to arbitrary transformations of the coordinates. Similarly, the angular momentum of the system in the RTG is a tensor with respect to all coordinate transformations in the four-dimensional Minkowski space.

In order to obtain bounds on the values of the post-Newtonian parameters imposed by the experiments, we consider them in the following order. We first consider the standard effects—the deflection of light and radio waves in the field of the Sun, the advance of Mercury's perihelion, and measurement of the time delay of radio signals in the gravitational field of the Sun. We then analyze the experiment planned to measure the precession of a gyroscope in a near-Earth orbit. After this, we consider the Nordtvedt effect, and also effects with nonvanishing parameters $\alpha_1$, $\alpha_2$, $\alpha_3$, and $\xi_\omega$.

*Deviation of light and radio waves in the gravitational field of the Sun.* To calculate the standard effects in the gravitational field of the Sun, one usually takes as an idealized model of the Sun a spherically symmetric sphere of radius $R_\odot$. Then in this case the metric coefficients of the effective Riemannian space are to post-Newtonian accuracy in the region $r > R_\odot$

$$\left.\begin{aligned} g_{00} &= 1 - \frac{2GM_\odot}{r} + 2\beta\frac{G^2M_\odot^2}{r^2}\,;\\ g_{\alpha\beta} &= \gamma_{\alpha\beta}\left(1 + 2\gamma\frac{GM_\odot}{r}\right);\\ g_{0\alpha} &= 0, \end{aligned}\right\} \tag{527}$$

where $M_\odot$ is the mass of the Sun. On the basis of (527) we readily obtain expressions for $g^{mn}$:

$$\left.\begin{aligned} g^{00} &= 1 + \frac{2GM_\odot}{r} + 2\,(2-\beta)\frac{G^2M_\odot^2}{r^2}\,;\\ g^{\alpha\beta} &= \gamma^{\alpha\beta}\left(1 - 2\gamma\frac{GM_\odot}{r}\right);\\ g^{0\alpha} &= 0. \end{aligned}\right\} \tag{528}$$

Note that we have chosen the spherical coordinates in the pseudo-Euclidean space in accordance with the Minkowski metric, which has the form (221), and therefore the Greek indices in (527) and (528) take the values $(r, \theta, \varphi)$.

We write the equations of motion for a material particle or a photon in the form of geodesic equations in the effective Riemannian space with the metric (527):

$$\frac{D}{d\sigma}\left(\frac{dx^m}{d\sigma}\right) + G^m_{pq}\,\frac{dx^p}{d\sigma}\,\frac{dx^q}{d\sigma} = 0, \tag{529}$$

where $\sigma$ is the parameter describing the trajectory.

For the metric (527), the components of the tensor $G^m_{pq}$ which are nonvanishing to the post-Newtonian accuracy are

$$\left.\begin{aligned} G^0_{0r} &= \frac{1}{2}\,g^{00}\partial_r g_{00};\\ G^r_{00} &= -\frac{1}{2}\,g^{rr}\partial_r g_{00};\\ G^r_{rr} &= \frac{1}{2}\,g^{rr}\partial_r g_{rr};\\ G^r_{\theta\theta} &= -\frac{1}{2}\,g^{rr}\partial_r g_{\theta\theta};\\ G^r_{\varphi\varphi} &= -\frac{1}{2}\,g^{rr}\partial_r g_{\varphi\varphi};\\ G^\theta_{r\theta} &= \frac{1}{2}\,g^{\theta\theta}\partial_r g_{\theta\theta};\\ G^\varphi_{r\varphi} &= \frac{1}{2}\,g^{\varphi\varphi}\partial_r g_{\varphi\varphi}. \end{aligned}\right\} \tag{530}$$

For what follows, it is convenient to express the metric coefficients (527) and (528) in a different form, equivalent to the post-Newtonian accuracy:

$$\left.\begin{aligned}g_{00}&=(g^{00})^{-1}=1-2U+2\beta U^2;\\ g_{rr}&=(g^{rr})^{-1}=-(1+2\gamma U);\\ g_{\theta\theta}&=(g^{\theta\theta})^{-1}=-r^2(1+2\gamma U)\simeq-(r+\gamma rU)^2;\\ g_{\varphi\varphi}&=(g^{\varphi\varphi})^{-1}=-r^2\sin^2\theta\,(1+2\gamma U)\simeq-\sin^2\theta\,(r+\gamma rU)^2,\end{aligned}\right\}\tag{531}$$

where

$$U=GM_\odot/r.\tag{532}$$

Let

$$R=r+\gamma rU=r+\gamma GM_\odot.\tag{533}$$

Then

$$g_{\theta\theta}=(g^{\theta\theta})^{-1}=-R^2;\quad g_{\varphi\varphi}=(g^{\varphi\varphi})^{-1}=-R^2\sin^2\theta.\tag{534}$$

Introducing the notation

$$\begin{aligned}g_{00}&=(g^{00})^{-1}=(1-2U+2\beta U^2)=B(R);\\ g_{rr}&=(g^{rr})^{-1}=-(1+2\gamma U)\simeq-(1+\gamma U)^2\equiv-A(R)\end{aligned}\tag{535}$$

and noting that $\partial/\partial r=\partial/\partial R$, we find from (530)

$$\left.\begin{aligned}&G^0_{0r}=\frac{1}{2B(R)}\frac{dB(R)}{dR};\\ &G^r_{00}=\frac{1}{2A(R)}\frac{dB(R)}{dR};\\ &G^r_{rr}=\frac{1}{2A(R)}\frac{dA(R)}{dR};\\ &G^r_{\theta\theta}=-\frac{R}{A(R)};\ G^r_{\varphi\varphi}=-\frac{R\sin^2\theta}{A(R)};\\ &G^\theta_{r\theta}=G^\varphi_{r\varphi}=\frac{1}{R}.\end{aligned}\right|\tag{536}$$

From this point on, we shall follow the scheme set forth in Ref. 42. We write the geodesic equations (529) in the expanded form

$$\frac{d^2t}{d\sigma^2}+\frac{1}{B(R)}\frac{dB(R)}{dR}\frac{dR}{d\sigma}\frac{dt}{d\sigma}=0;\tag{537}$$

$$\frac{d^2R}{d\sigma^2}+\frac{1}{2A(R)}\frac{dA(R)}{dR}\left(\frac{dR}{d\sigma}\right)^2+\frac{1}{2A(R)}\frac{dB(R)}{dR}\left(\frac{dt}{d\sigma}\right)^2-\frac{R}{A(R)}\left(\frac{d\theta}{d\sigma}\right)^2-\frac{R\sin^2\theta}{A(R)}\left(\frac{d\varphi}{d\sigma}\right)^2=0;\tag{538}$$

$$\frac{d^2\theta}{d\sigma^2}+\frac{2}{R}\frac{dR}{d\sigma}\frac{d\theta}{d\sigma}=0;\tag{539}$$

$$\frac{d^2\varphi}{d\sigma^2}+\frac{2}{R}\frac{dR}{d\sigma}\frac{d\varphi}{d\sigma}=0.\tag{540}$$

Since the field is isotropic, without loss of generality we can consider only those trajectories that lie in the equatorial plane. Therefore, in Eqs. (538) and (539) we can set $\theta=\pi/2$. Then in this case Eq. (539) is satisfied identically, and Eq. (538) takes the form

$$\frac{d^2R}{d\sigma^2}+\frac{1}{2A(R)}\frac{dA(R)}{dR}\left(\frac{dR}{d\sigma}\right)^2+\frac{1}{2A(R)}\frac{dB(R)}{dR}\left(\frac{dt}{d\sigma}\right)^2-\frac{R}{A(R)}\left(\frac{d\varphi}{d\sigma}\right)^2=0.\tag{541}$$

From (537) and (540), we can readily find that

$$dt/d\sigma=C/B(R)\tag{542}$$

and

$$R^2d\varphi/d\sigma=J,\tag{543}$$

where $C$ and $J$ are constants of integration and are integrals of the motion of the problem. However, by a redefinition of the parameter $\sigma$ it is always possible to make the constant $C$ in (542) equal to unity. Therefore, in what follows, without loss of generality, we shall use the equation

$$dt/d\sigma=1/B(R).\tag{544}$$

Substituting the relations (543) and (544) in (541) and multiplying the resulting expression by $2A(R)dR/d\sigma$, we obtain after simple manipulation

$$\frac{d}{d\sigma}\left[A(R)\left(\frac{dR}{d\sigma}\right)^2+\frac{J^2}{R^2}-\frac{1}{B(R)}\right]=0.$$

It follows from this that

$$A(R)\left(\frac{dR}{d\sigma}\right)^2+\frac{J^2}{R^2}-\frac{1}{B(R)}=-E\tag{545}$$

is also an integral of the motion of the problem. We now find the connection between the proper time $\tau$ and the parameter $\sigma$ of the trajectory. We determine the proper time $\tau$ from the interval

$$d\tau^2=g_{00}\,dt^2+g_{rr}\,dR^2+g_{\theta\theta}\,d\theta^2+g_{\varphi\varphi}\,d\varphi^2,\tag{546}$$

where the metric coefficients are specified in terms of Eqs. (534) and (535). Since $\theta=\pi/2$, taking into account the relations (543)–(545) we obtain from (546)

$$d\tau^2=Ed\sigma^2.\tag{547}$$

Since $d\tau^2=0$ for massless particles, we see from this that for a photon

$$E=0.\tag{548}$$

If the rest mass of the particles is nonzero, then from (547) there follows the inequality $E>0$. On the basis of (544) and (547), we can relate the proper time $\tau$ to the time coordinate $t$ of the Minkowski space:

$$d\tau^2=EB^2(R)\,dt^2.\tag{549}$$

Similarly, from (543) and (545), eliminating the parameter $\sigma$ by means of (544), we find

$$R^2\frac{d\varphi}{dt}=JB(R);\tag{550}$$

$$\frac{A(R)}{B^2(R)}\left(\frac{dR}{dt}\right)^2+\frac{J^2}{R^2}-\frac{1}{B(R)}+E=0.\tag{551}$$

Since $A(R)>0$, the following inequality must hold if Eqs. (551) are to be solvable:

$$J^2/R^2+E\leqslant 1/B(R).\tag{552}$$

The system of equations (550) and (551) can be solved by quadrature if the variable $t$ is eliminated. Determining $dt$ from (550) and substituting in (551), we obtain

$$\frac{A(R)}{R^4}\left(\frac{dR}{d\varphi}\right)^2+\frac{1}{R^2}-\frac{1}{J^2B(R)}+\frac{E}{J^2}=0. \quad (553)$$

Hence, the solution, which expresses $\varphi$ in terms of $R$, can be represented in the form

$$\varphi=\pm\int\frac{A^{1/2}(R)\,dR}{R^2}\left[\frac{2}{J^2B(R)}-\frac{E}{J^2}-\frac{1}{R^2}\right]^{-1/2}. \quad (554)$$

We must now express (553) and (554) in terms of $r$. By virtue of (533) and (535),

$$\frac{1}{r^4}\left(\frac{dr}{d\varphi}\right)^2+\frac{1}{r^2}-\frac{(1+\gamma U)^2}{J^2B(r)}+\frac{E(1+\gamma U)^2}{J^2}=0 \quad (555)$$

and

$$\varphi=\pm\int\frac{dr}{r^2}\left[\frac{(1+\gamma U)^2}{J^2B(r)}-\frac{E(1+\gamma U)^2}{J^2}-\frac{1}{r^2}\right]^{-1/2}. \quad (556)$$

The relation (556) determines the shape of the trajectory of the particles in the gravitational field of the Sun in the post-Newtonian approximation.

Suppose that we are interested in the problem of describing the trajectories of particles which arrive from distant regions and pass by the Sun. We place the origin of the coordinate system at the center of the Sun. We assume that the particle in which we are interested moves in the equatorial plane $(xy)$ parallel to the $x$ axis, from positive infinity in the direction of negative infinity. Since the angle $\varphi$ is measured from the positive direction of the $x$ axis, it is obvious that $\varphi(+\infty)=0$.

If there were no deflection of the particle trajectory in the gravitational field of the Sun, the change in the angle $\varphi$ during the complete motion would be $\pi$, since $\varphi(-\infty)-\varphi(\infty)=\pi$. However, the gravitational field deflects the particles from rectilinear motion. Therefore, we shall characterize the measure of this deflection by

$$\delta\varphi=\Delta\varphi-\pi. \quad (557)$$

At an appreciable distance from the Sun, i.e., in the limit $r\to\infty$, we find from (535) that $A(\infty)=B(\infty)=1$. Then from (551) we obtain

$$(dr/dt)^2=1-E. \quad (558)$$

Since the particle moves freely in the region $r\to\infty$, its velocity $dr/dt=v$ is constant and, therefore, $E=\text{const}$, in complete agreement with the significance of $E$ as an integral of the motion. Since for a photon (in units with $c=1$) $v=1$, we have $E=0$. For a material particle $v<1$ and therefore $E<1$.

Let $r_0$ be the distance of closest approach to the Sun along the trajectory of the particle. Then for the integral of the motion $J$ we find from (555) by virtue of $(dr/d\varphi)_{r=r_0}=0$

$$J=r_0\,[1+\gamma U(r_0)]\left(\frac{1}{B(r_0)}-1+v^2\right)^{1/2}. \quad (559)$$

Taking into account this expression in (556), we obtain

$$\varphi(r)=\int_r^\infty\frac{dr'}{r'}\left[\frac{B(r_0)}{[1-(1-v^2)B(r_0)]}\left(\frac{r'}{r_0}\right)^2\frac{A^2(r')}{A^2(r_0)}\right.$$
$$\left.\times\left(\frac{1}{B(r')}-E\right)-1\right]^{-1/2}. \quad (560)$$

For a photon, we find from this

$$\varphi(r)=\int_r^\infty\frac{dr'}{r'}\left[\left(\frac{r'}{r_0}\right)^2\frac{B(r_0)}{B(r')}\frac{A^2(r')}{A^2(r_0)}-1\right]^{-1/2}. \quad (561)$$

In the first order in $M_\odot G/r_0$ we obtain from (561)

$$\varphi(r_0)=(M_\odot G/r_0)(1+\gamma)+\frac{\pi}{2}.$$

Note that the complete change in the angle $\varphi(r)$ when the photon travels from infinity to the point $r=r_0$ and then out to infinity again is $2\varphi(r_0)$. Therefore, for $\Delta\varphi$ we have

$$\Delta\varphi=2[\varphi(r_0)-\varphi(+\infty)]=\frac{2M_\odot G}{r_0}(1+\gamma)+\pi.$$

Substituting this expression in (557), we obtain for the deflection $\delta\varphi$, apart from terms of order $GM_\odot/r_0$, the final expression

$$\delta\varphi=\frac{4M_\odot G}{r_0}\left(\frac{1+\gamma}{2}\right). \quad (562)$$

Taking $r_0$ equal to the radius of the Sun, we find from (562)

$$\delta\varphi=1{,}75''\left(\frac{1+\gamma}{2}\right). \quad (563)$$

Analysis of the experimental results obtained from observation in the gravitational field of the Sun of the bending of the light rays of distant stars and also radio waves emitted by quasars suggests[54] that the post-Newtonian parameter is $\gamma=1\pm0.2$.

*Advance of Mercury's perihelion*. Suppose that a particle with mass $m\neq0$ moves in a closed curve around the Sun. To describe this motion, we use the general expressions (555) and (556). As in the preceding subsection, we place the origin of the coordinate system at the center of the Sun and take the equatorial plane $(xy)$ coincident with the plane of the motion of the particles. Since the trajectory is closed, there exist two values of $r(\varphi)$ for which $dr/d\varphi=0$. We denote them by $r_\pm$. Then from (555) we have

$$\frac{1}{r_\pm^2}-\frac{A^2(r_\pm)}{J^2B(r_\pm)}+\frac{EA^2(r_\pm)}{J^2}=0.$$

From this we can determine the integrals of the motion $J$ and $E$:

$$J^2=\frac{r_+^2r_-^2A(r_+)A(r_-)}{B(r_+)B(r_-)}\,\frac{B(r_+)-B(r_-)}{r_+^2A(r_+)-r_-^2A(r_-)}; \quad (564)$$

$$E=\frac{1}{r_+^2A(r_+)-r_-^2A(r_-)}\left[\frac{r_+^2A(r_+)}{B(r_+)}-\frac{r_-^2A(r_-)}{B(r_-)}\right]. \quad (565)$$

In accordance with (556), the angle $\varphi(r)$ through which the particle-position radius vector $\mathbf{r}$ is turned (measured from the direction $\mathbf{r}=\mathbf{r}_-$) can be calculated in accordance with

$$\varphi(r)=\varphi(r_-)+\int_{r_-}^{r}\frac{dr'}{r'^2}\left[\frac{A(r')}{J^2B(r')}-\frac{EA(r')}{J^2}-\frac{1}{r'^2}\right]^{-1/2}. \quad (566)$$

Setting here $r=r_+$ and taking into account the relations (564) and (565), we find after simple but lengthy calculations in the post-Newtonian approximation

$$\varphi(r_+)-\varphi(r_-)\approx\pi+\frac{\pi M_\odot G}{(1-e^2)L}(2+2\gamma-\beta). \quad (567)$$

Here, $L$ is the semimajor axis and $e$ is the eccentricity of the orbit.

The changes in the angle $\varphi$ for motion of the particle from the position $r=r_-$ to the position $r=r_+$ and back from the position $r=r_+$ to the position $r=r_-$ must be equal. Therefore, the total change in the angle $\varphi$ in one revolution is

$$\Delta\varphi = 2\pi + \frac{2\pi M_\odot G}{(1-e^2)\,L}(2+2\gamma-\beta). \tag{568}$$

From this it can be seen that the orbit of the motion is not closed. It precesses in the direction of the motion of the particle, and the measure of the precession in the post-Newtonian approximation is

$$\delta\varphi = \Delta\varphi - 2\pi = \frac{2\pi G M_\odot}{(1-e^2)\,L}(2+2\gamma-\beta). \tag{569}$$

The advance of Mercury's perihelion is subject to the influence of various other factors besides the post-Newtonian corrections in the equation of motion. These factors include attraction by the planets in the solar system and the existence of a quadrupole moment of the Sun. The only undetermined factor among them is the quadrupole moment of the Sun; the influence of all the other factors can be calculated with sufficient accuracy.

The total displacement of Mercury's perihelion due to the presence of the quadrupole moment $J_2$ of the Sun and the post-Newtonian corrections to the equations of motion is

$$\delta\varphi = 42.98\,[(2+2\gamma-\beta)/3] + 1.3\cdot 10^5 J_2$$

(in units of seconds of arc per century). It follows from the results of the observations[54] that $\delta\varphi = 41.4 \pm 0.9$ seconds of arc per century.

Measurements of the apparent shape of the Sun made by Dicke and Goldenberg[55] gave for $J_2$ the value $J_2 = (2.5 \pm 0.2)\times 10^{-5}$, while later measurements of Hill *et al.*[56,57] showed that $J_2 < 0.5\times 10^{-5}$. Comparison of the observed displacements of the perihelia of Mercury and Mars[58] gave the estimate $J_2 < 3\times 10^{-5}$.

Thus, because of the absence of direct measurements of the quadrupole moment of the Sun there remains a large uncertainty in the value of $\beta$ as determined from the advance of Mercury's perihelion: $\beta = 1^{+0.4}_{-0.2}$ subject to the condition that $\gamma = 1$.

*Time delay of radio signals in the field of the Sun.* Another independent method of determining the post-Newtonian parameter $\gamma$ is through measurement of the time delay of radio signals in the field of the Sun.[59] This effect takes the form that the propagation time of radio signals sent from the Earth to a reflector situated somewhere else in the solar system and back, measured by a clock on the Earth, differs from the time obtained when the process takes place in the absence of a gravitational field.

If one takes the reflector to be Mercury when in superior conjunction and the path of the radio signal touches the Sun, the delay time is maximal and can be calculated by means of (551).

As was shown in Ref. 57, it is

$$\Delta t_{\max} \simeq 4GM_\odot(1+\gamma)\left[1+\ln\frac{2\sqrt{r_3 r_M}}{r_\odot}\right] \simeq \frac{1+\gamma}{2}\cdot 260\ \mu\text{sec},$$

which exceeds the corresponding GR result by 20 $\mu$sec.

Thus, already in the solar system the RTG predicts a result amenable to experimental verification and different from the GR prediction.

In view of the fundamental importance of this conclusion, it is extremely desirable to make careful experimental measurements of the delay time.

*Precession of a gyroscope in orbit.* If the parameters $\alpha_1$, $\alpha_2$, $\alpha_3$ are zero, measurement of the precession of a gyroscope moving in orbit around the Earth provides a third independent way of measuring the parameter $\gamma$.

According to Ref. 60, the angular velocity of precession of a gyroscope in a circular orbit in the Earth's field is

$$\Omega = \frac{2\gamma+1}{2}\, m\,\frac{[\mathbf{r}\mathbf{v}]}{r^3} + \frac{\gamma+1}{2r^3}\left(-\mathbf{J} + \frac{3\mathbf{r}\,(\mathbf{J}\mathbf{r})}{r^2}\right),$$

where $m$ is the mass of the Earth, $\mathbf{v}$ is the linear velocity of the gyroscope relative to the center of the Earth, $\mathbf{J}$ is the angular momentum of the Earth, and $\mathbf{r}$ is the radius vector of the point at which the gyroscope is situated. The level of the present development of technology offers hope that this experiment will be realized in the near future.

*Nordtvedt effect and lunar laser ranging.* In recent years, several investigators[61–63] have concentrated their attention on establishing relationships between the inertial and gravitational masses of a given body in different theories of gravitation and the search for possibilities of testing these relationships in experiments.

In any theory of gravitation, we can, following Bondi,[64] distinguish three forms of mass according to the measurements by which they are determined: the inertial mass $m_i$, the passive gravitational mass $m_p$, and the active gravitational mass $m_a$.

The inertial mass is the mass that occurs (and is defined by it) in Newton's second law:

$$m_i a^\alpha = F^\alpha.$$

The passive gravitational mass is the mass on which the gravitational field acts, i.e., it is the mass determined by the expression

$$F^\alpha = -m_p \nabla^\alpha U.$$

The active gravitational mass is the mass that is the source of the gravitational field.

In Newtonian mechanics, Newton's third law requires equality of the active and passive masses, $m_a = m_p$, irrespective of the size and composition of the bodies; the equality of the inertial mass and the other two is taken as an empirical fact.

In Einstein's theory, the inertial and passive gravitational masses are equal for point bodies. However, equality of the active and passive gravitational masses is not postulated.

In some theories of gravitation, all three masses of a given body may be different. It is therefore necessary to establish the correspondence between these three masses by means of experiment.

As was already pointed out at the beginning of this section, the attempt to measure the ratio of the passive gravitational mass to the inertial mass for bodies of laboratory sizes[52] provided only a partial answer to the question posed, since the accuracy of the experiments was certainly insufficient to determine the ratio in which the gravitational self-energy of the body, the energy of elastic deformations, etc., occur in these masses.

Since the ratio of the gravitational self-energy of a body to its mass increases with increasing size of the body, it is more expedient to use extended bodies—planets—for these purposes.

However, since gravimetric measurement of the ratio of the passive gravitational mass of an extended body (a planet) to its inertial mass is impossible, it was necessary to study theoretically the motion of extended bodies in the gravitational field of other bodies with a view to establishing what features in the motion of an extended body can result from a possible inequality of its inertial and gravitational masses.

One such effect is the possible deviation at the post-Newtonian level of the motion of the center of mass of an extended body from motion along a geodesic of the Riemannian space–time. The possibility of such an effect was pointed out by Dicke,[61] who suggested that the ratio of the gravitational mass to the inertial mass for astronomical bodies could differ slightly from unity if the gravitational self-energy of all these bodies changes when their position in the gravitational field of other bodies changes. Subsequently, this effect was investigated by Nordtvedt,[63] Will,[65] and Dicke.[62]

Using a model of coherent particles, Nordtvedt[63] made a very detailed study of this effect (which subsequently became known as the Nordtvedt effect) and demonstrated its possibility in certain metric theories of gravitation.

Calculating the motion of an extended body in the gravitational field of a massive point source at rest, Will[65] concluded that the tensor of the passive gravitational mass of an extended body in the post-Newtonian approximation of an arbitrary metric theory of gravitation has the form

$$\frac{m^{\alpha\beta}}{M} = -\gamma^{\alpha\beta}\left[1-(4\beta-\alpha_1-\gamma-3-\xi_1+\alpha_2)\,\Omega_\nu^\nu\right] - (\alpha_2+\xi_2-\xi_1)\,\Omega^{\alpha\beta}, \qquad (570)$$

where $\Omega_\nu^\nu$ and $\Omega^{\alpha\beta}$ are the post-Newtonian corrections:

$$\left.\begin{aligned}\Omega^{\alpha\beta} &= -\frac{1}{2M}\int\frac{\rho\rho'\,(x^\alpha-x'^\alpha)\,(x^\beta-x'^\beta)}{|\mathbf{x}-\mathbf{x}'|^3}\,d^3x\,d^3x';\\ \Omega_\nu^\nu &= \frac{1}{2M}\int\frac{\rho\rho'\,d^3x\,d^3x'}{|\mathbf{x}-\mathbf{x}'|}.\end{aligned}\right\} \qquad (571)$$

In such an approach, the presence of post-Newtonian corrections in the expression (570) was interpreted as the result of violation in certain theories of gravitation of the equality of the gravitational and inertial masses of the extended body at the post-Newtonian level. In addition, it was asserted that equality of the inertial and passive gravitational masses in the post-Newtonian approximation will mean that the center of mass of an extended body moves along a geodesic of the Riemannian space–time.

However, it is also rather difficult under the conditions of a real experiment to determine whether or not the center of mass of an extended body moves along a geodesic of the Riemannian space–time. Therefore, it was proposed to determine from experiments the values of all the necessary post-Newtonian parameters and, using Will's formula (570), to answer the questions, already becoming academic, of the relationship between the tensor of the passive gravitational mass of an extended body and its inertial mass, and also of the nature of the motion of the center of mass of this body relative to a geodesic of the Riemannian space–time.

As a result of calculation of the motion of the Earth–Moon system in the gravitational field of the Sun, Nordtvedt[63] drew attention to a number of possible anomalies in the motion of the Moon whose observation could permit the measurement of different combinations of the post-Newtonian parameters. One of these anomalies is a polarization of the Moon's orbit in the direction of the Sun with amplitude $\delta r = \eta L$, where $L$ is a constant of order 10 m, and

$$\eta = -\frac{1}{3}(\xi_2+\alpha_2-\xi_1)+(4\beta-\xi_1-\gamma-\alpha_1+\alpha_2-3)-\frac{10}{3}\xi_\omega. \qquad (572)$$

To detect this effect, data obtained by lunar laser ranging were analyzed. As a result of this analysis, one of the groups[66] arrived at the conclusion that

$$\eta = 0 \pm 0.03. \qquad (573)$$

Another group[67] obtained the very similar result

$$\eta = -0.001 \pm 0.015. \qquad (574)$$

Using these estimates and Will's theoretical formula (570) for the tensor of the passive gravitational mass, the authors of the groups concluded that the ratio of the passive gravitational mass of an extended body to its inertial mass is near unity:

$$\left|\frac{m_p^{\alpha\beta}}{M}-\delta^{\alpha\beta}\right| < 1.5\cdot10^{-11}.$$

Thus, the data obtained by the lunar laser ranging appeared to permit the assertion (as was made in the literature[54,66,67]) that in the post-Newtonian approximation the passive gravitational mass of an extended body is equal to its inertial mass and that the center of mass of an extended body moves along a geodesic of the Riemannian space–time.

However, as was shown in Ref. 51, Eq. (570), obtained by Will, is incorrect. Therefore, the interpretation of the results of the measurement of $\eta$, (573) and (574), given in Refs. 66 and 67 and based on the use of this formula is incorrect.

*Effects associated with the presence of a preferred frame of reference.* Theories of gravitation in which at least one of the parameters $\alpha_1$, $\alpha_2$, $\alpha_3$ is nonzero possess a preferred frame of reference. The predictions of such theories of gravitation for the standard effects can agree with the results of the observations only if the solar system is a preferred frame of reference. However, it is more reasonable to assume that the solar system, moving relative to other star systems, is in no way distinguished compared with them, and therefore cannot be a preferred universal rest frame for such theories.

Since the preferred rest frame must be distinguished in some manner from other systems, it is more reasonable to associate this system with the center of mass of the Galaxy or even the universe. In this case, the solar system will be in motion relative to the preferred frame of reference with velocity $v \simeq 10^{-3}c$, of the same order of magnitude as the orbital velocity of the solar system relative to the center of the Galaxy. In this case, it is possible to observe a number of effects associated with the motion relative to the preferred frame of reference,[54] permitting estimation of the parameters $\alpha_1, \alpha_2, \alpha_3$.

In theories of gravitation with a preferred rest frame, the gravitational constant $G$ measured in gravimetric experiments will depend on the motion of the Earth relative to such a system.

For the relative quantity $\Delta G/G$ we have[54]

$$\frac{\Delta G}{G} = \left(\frac{\alpha_2}{2} + \alpha_3 - \alpha_1\right) \boldsymbol{\omega}\mathbf{v} + \frac{1}{4}\alpha_2 [(\mathbf{v}\mathbf{e}_r)^2 + 2(\boldsymbol{\omega}\mathbf{e}_r)(\mathbf{v}\mathbf{e}_r) + (\boldsymbol{\omega}\mathbf{e}_r)^2],$$

where $\mathbf{v}$ is the orbital velocity of the Earth around the Sun, $\boldsymbol{\omega}$ is the velocity of the Sun relative to the preferred rest frame, and $\mathbf{e}_r$ is the unit vector from the gravimeter to the center of the Earth.

Because of the rotation of the Earth about its axis, the vector $\mathbf{e}_r$ changes its orientation relative to the vectors $\mathbf{v}$ and $\boldsymbol{\omega}$, and this leads to a periodic change in the scalar products $\mathbf{v}\cdot\mathbf{e}_r$ and $\boldsymbol{\omega}\cdot\mathbf{e}_r$ with a period of about 12 h. This leads to corresponding periodic changes in the acceleration of free fall, and for a point of observation at latitude $\theta$

$$\frac{\Delta g}{g} \approx 3\alpha_2 \cdot 10^{-8} \cos^2\theta.$$

Will,[68,69] analyzing the results of gravimetric experiments, found that the relative changes in $g$ do not exceed $10^{-11}$:

$$\left|\frac{\Delta g}{g}\right| < 10^{-11}.$$

From this we obtain the estimate

$$|\alpha_2| < 3\cdot 10^{-4}.$$

Motion of the Earth around the Sun also leads to a periodic change in $\boldsymbol{\omega}\cdot\mathbf{v}$ with a period of the order of a year. This variation is due to contraction and expansion of the Earth, this leading, in its turn, to periodic changes in the angular velocity of the Earth's rotation produced by the change in its moment of inertia:

$$\frac{\Delta\omega}{\omega} = 3\cdot 10^{-9}\left(\alpha_3 + \frac{2}{3}\alpha_2 - \alpha_1\right).$$

It follows from the results of observations that

$$\left|\alpha_3 + \frac{2}{3}\alpha_2 - \alpha_1\right| < 0.02.$$

Motion of the solar system relative to the center of the universe can lead to an anomalous displacement of the perihelia of the planets. For Mercury,[69] the additional contribution to the advance of the perihelion (in seconds of arc per century) has the form

$$\delta\varphi_0 = -123\alpha_1 + 92\alpha_2 + 1.4\cdot 10^5\alpha_3 + 63\xi_\omega.$$

Comparison with the observations and combination of all the estimates of the parameters $\alpha$ gives

$$|\xi_\omega| < 2\cdot 10^{-3};\ |\alpha_1| < 10^{-3};$$
$$|\alpha_2| < 4\cdot 10^{-2};\ |\alpha_3| < 2\cdot 10^{-7}.$$

In the RTG, $\alpha_1 = \alpha_2 = \alpha_3 = 0$, and therefore all these effects are absent.

*Ratio of the active and passive gravitational masses.* For the active gravitational mass in the post-Newtonian approximation, Nordtvedt[70] obtained the expression

$$m_a = m_i + 2\left(\frac{\alpha_3}{2} + \frac{\xi_1}{2} - \xi_4\right)E_{\text{kin}} + \xi_3 E_{\text{int}} + (4\beta - 2\xi_2 - 3 - \gamma)\Omega - \xi_t E^{\alpha\beta} e_\alpha e_\beta, \quad (575)$$

where $\mathbf{e}_\alpha$ is a unit vector along the line connecting the massive body to the point at which its field is measured;

$$E_{\text{kin}} = -\frac{1}{2}\int \hat{\rho} v_\alpha v^\alpha d^3x;\quad E_{\text{int}} = \int \hat{\rho}\Pi\, d^3x;$$

$m_i$ is the inertial mass of the body,

$$m_i = \int \hat{\rho}\left(1 - \frac{1}{2}v_\alpha v^\alpha + \Pi - \frac{1}{2}U\right)d^3x;$$
$$E^{\alpha\beta} = -\frac{1}{2}\int \hat{\rho} v^\alpha v^\beta d^3x.$$

In the RTG, we obtain from (575)

$$m_a = m_i.$$

*Effects of anisotropy relative to the center of the Galaxy.* In theories of gravitation for which $\xi_\omega \neq 0$, anisotropy effects due to the influence of the gravitational field of the Galaxy are possible.[71]

If we assume that the mass $M$ of the Galaxy is concentrated at its center at distance $R$ from the solar system, then the gravitational field of the Galaxy leads to periodic changes in the readings of a gravimeter with period 12 h:

$$\frac{\Delta G}{G} = \xi_\omega\left(1 - \frac{3K}{mr^2}\right)\frac{M}{R}(\mathbf{e}_r\mathbf{e}_R)^2,$$

where $K$ is the moment of inertia, $m$ is the mass, and $r$ is the radius of the Earth; $\mathbf{e}_r$ is the unit vector directed from the gravimeter to the center of the Earth; and $\mathbf{e}_R = \mathbf{R}/R$.

Another effect is the anomalous displacement of the perihelia of the planets due to the anisotropy produced by the Galaxy:

$$\delta\varphi_0 = \frac{\pi\xi_\omega}{2}\frac{M}{R}\cos^2\beta\cos^2(\omega - \lambda),$$

where $\lambda$ and $\beta$ are the angular coordinates of the center of the Galaxy, and $\omega$ is the angle of the planet's perihelion in geocentric coordinates.

The comparison with observations gives as an upper limit for $\xi_\omega$ the estimate

$$|\xi_\omega| < 10^{-3}.$$

In the RTG, $\xi_\omega = 0$ and all anisotropy effects due to the gravitational field of the Galaxy are absent.

## 19. POST-NEWTONIAN INTEGRALS OF THE MOTION IN THE RTG

We write the covariant conservation law (102) for the density $t^{mn}$ of the total energy-momentum tensor in the pseudo-Euclidean space–time in Cartesian coordinates:

$$\partial_m t^{mn} = \partial_m [t^{mn}_{(g)} + t^{mn}_{(M)}] = 0. \quad (576)$$

On the basis of this relation, we readily obtain the corresponding integral conservation law

$$-\partial_0 \int dV t^{0n} = \oint dS_\alpha t^{\alpha n}. \quad (577)$$

If there is no energy flux of the matter and gravitational field through the surface bounding the volume of integration in (577), then

$$\oint dS_\alpha t^{n\alpha} = 0, \quad (578)$$

and we arrive at the conservation law for the total 4-momentum of the isolated system:

$$dP^n/dt = 0,$$

where

$$P^n = \int dV t^{0n}. \quad (579)$$

In this case, because the density $t^{mn}$ of the total energy-momentum tensor is symmetric, the angular-momentum tensor of the system is also conserved:

$$dM^{ni}\, dt = 0,$$

where

$$M^{ni} = \int dV\, [x^n t^{0i} - x^i t^{0n}]. \quad (580)$$

It follows from the conservation of the components $M^{0\alpha}$ that the center of mass of an isolated system, determined by the formula

$$X^\alpha = \frac{\int x^\alpha t^{00}\, dV}{\int t^{00}\, dV} = \frac{P^\alpha t - M^{0\alpha}}{P^0}, \quad (581)$$

executes uniform rectilinear motion with velocity

$$\frac{d}{dt} X^\alpha = \frac{P^\alpha}{P^0}. \quad (582)$$

Thus, to describe the motion of an isolated system consisting of matter and the gravitational field it is necessary to determine the 4-momentum $P^n$ (579). It should be noted that in any real system gravitational waves may be emitted because of the motion of its constituent parts; any real system exchanges matter with other systems both in the form of electromagnetic radiation and in the form of particles, atoms, etc. Therefore, in the general case the energy fluxes of the matter and the gravitational radiation cannot be ignored. There are astrophysical processes in which these energy fluxes play a leading role, and allowance for them makes it possible to understand and predict many astrophysical phenomena. At the same time, for the systems for which the energy fluxes of the matter and the gravitational field are small the condition (578) of being isolated is satisfied to a certain degree of accuracy. Then to the same degree of accuracy we can assert that the 4-momentum of this system is conserved. Systems for which the post-Newtonian formalism apply belong to the class of such systems.

To find the explicit form of the 4-momentum $P^n$ in the post-Newtonian approximation, we shall proceed from the identity (100). We write this identity in a Cartesian coordinate system:

$$\partial_m t^m_n \equiv \nabla_m T^m_n. \quad (583)$$

Multiplying both sides of (583) by $dV$ and integrating over some sufficiently large volume, we obtain

$$\partial_0 \int t^0_n dV + \oint t^\alpha_n\, dS_\alpha = \int \nabla_m T^m_n\, dV. \quad (584)$$

Setting

$$\oint t^\alpha_n dS_\alpha = 0, \quad (585)$$

we find

$$\partial_0 P_n = \int \nabla_m T^m_n\, dV, \quad (586)$$

where by virtue of the definition (579)

$$P_n = \gamma_{nk} \int dV t^{0k}. \quad (587)$$

To the accuracy that there is no energy flux of matter and gravitational radiation through the surface bounding the volume of integration in (584), i.e., (585) holds, the relation (586) is exact.

We now use (586) to determine the explicit form of the 4-momentum $P_n$. This will be possible if we represent the right-hand side of Eq. (586) in the form of the time derivative of some expression.

Since in this section our aim is to find integrals of the motion in the post-Newtonian approximation, we consider the right-hand side of (586) in this approximation.

We write down the expression $\nabla_m T^{mk}$ component by component:

$$\nabla_m T^{m0} = \partial_0 T^{00} + \partial_\alpha T^{\alpha 0} + \Gamma^0_{00} T^{00} + 2\Gamma^0_{0\alpha} T^{\alpha 0} + \Gamma^0_{\alpha\beta} T^{\alpha\beta}; \quad (588)$$

$$\nabla_m T^{m\alpha} = \partial_0 T^{0\alpha} + \partial_\nu T^{\nu\alpha} + \Gamma^\alpha_{00} T^{00} + 2\Gamma^\alpha_{0\nu} T^{\nu 0} + \Gamma^\alpha_{\beta\nu} T^{\beta\nu}. \quad (589)$$

Here

$$\Gamma^k_{pq} = \frac{1}{2} g^{kl} (\partial_p g_{lq} + \partial_q g_{lp} - \partial_l g_{pq}). \quad (590)$$

Using the post-Newtonian expansion (505)–(507) of the metric, and also the relation

$$\partial^\alpha \partial_0 \int dV \rho |\mathbf{x} - \mathbf{x}'| = N^\alpha - V^\alpha + O(\varepsilon^5),$$

we find from (590) to the required accuracy

$$\left.\begin{aligned}
&\Gamma^0_{00} = -\frac{\partial U}{\partial t} + O(\varepsilon^5);\ \Gamma^0_{0\alpha} = -\partial_\alpha U + O(\varepsilon^4);\\
&\Gamma^\alpha_{\beta\nu} = (\delta^\alpha_\beta \partial_\nu U + \delta^\alpha_\nu \partial_\beta U - \gamma^{\alpha\sigma}\gamma_{\beta\nu}\partial_\sigma U) + O(\varepsilon^4);\\
&\Gamma^\alpha_{0\beta} = \delta^\alpha_\beta \frac{\partial U}{\partial t} + 2(\partial_\beta V^\alpha - \gamma^{\alpha\sigma}\gamma_{\beta\tau}\partial_\sigma V^\tau) + O(\varepsilon^5);\\
&\Gamma^0_{\alpha\beta} = 0(\varepsilon^3);\quad \Gamma^\alpha_{00} = 4\frac{\partial V^\alpha}{\partial t} + (1-2U)\,\partial^\alpha U\\
&-\frac{1}{2}\partial^\alpha[2U^2 - 4\Phi_1 - 4\Phi_2 - 2\Phi_3 - \Phi_4]\\
&+\frac{1}{2}\frac{\partial}{\partial t}(N^\alpha - V^\alpha) + O(\varepsilon^6).
\end{aligned}\right\} \quad (591)$$

In addition, from (485)–(487), (500), and (501) we have in the post-Newtonian approximation

$$\left.\begin{aligned}
&T^{00} = \hat\rho\left[1 + U + \Pi - \frac{1}{2}v_\nu v^\nu\right] + O(\varepsilon^4);\\
&T^{0\alpha} = \hat\rho v^\alpha\left[1 + \Pi + U - \frac{1}{2}v_\nu v^\nu\right] + pv^\alpha + O(\varepsilon^5);\\
&T^{\alpha\beta} = \hat\rho v^\alpha v^\beta\left(1 - \frac{1}{2}v_\nu v^\nu + \Pi + U + \frac{p}{\hat\rho}\right) - \gamma^{\alpha\beta}p + O(\varepsilon^6).
\end{aligned}\right\} \quad (592)$$

Substituting the expressions (591) and (592) in (588), we obtain

$$\begin{aligned}
\nabla_m T^{m0} = {}& \frac{\partial}{\partial t}\left[\hat\rho\left(1 + \Pi + U - \frac{1}{2}v_\alpha v^\alpha\right)\right]\\
&+ \partial_\alpha\left[\hat\rho v^\alpha\left(1 + U + \Pi - \frac{1}{2}v_\nu v^\nu\right) + pv^\alpha\right] - \hat\rho\frac{\partial U}{\partial t}\\
&- 2\hat\rho v^\alpha\partial_\alpha U + O(\varepsilon^5).
\end{aligned} \quad (593)$$

Similarly, after substitution in (589) of the expressions (591) and (592), we find

$$\begin{aligned}
\nabla_m T^{m\alpha} = {}& \frac{\partial}{\partial t}\left[\hat\rho v^\alpha\left(1 + \Pi + U - \frac{1}{2}v_\nu v^\nu\right) + pv^\alpha\right]\\
&+ \partial_\beta\left[pv^\alpha v^\beta - p\gamma^{\alpha\beta} + \hat\rho v^\alpha v^\beta\left(1 + \Pi + U - \frac{1}{2}v_\nu v^\nu\right)\right]\\
&+ \frac{7}{2}\hat\rho\frac{\partial V^\alpha}{\partial t} + \hat\rho\left(1 + \Pi + U - \frac{1}{2}v_\nu v^\nu\right)\partial^\alpha U\\
&- 4\hat\rho U\partial^\alpha U + \hat\rho\partial^\alpha[2\Phi_1 + 2\Phi_2 + \Phi_3 + 3\Phi_4]\\
&+ 2\hat\rho v^\alpha\frac{\partial U}{\partial t} + \frac{1}{2}\hat\rho\frac{\partial N^\alpha}{\partial t} + 4\hat\rho v^\beta(\partial_\beta V^\alpha - \partial^\alpha V_\beta)\\
&+ p\partial^\alpha U + 2\hat\rho v^\alpha v^\beta\partial_\beta U + \hat\rho v^2\partial^\alpha U + O(\varepsilon^6).
\end{aligned} \quad (594)$$

The expressions (593) and (594) can be simplified if we take into account the continuity equation (483) and the Newtonian equations of motion for an elastic body,

$$\hat\rho\frac{dv^\alpha}{dt} = -\hat\rho\partial^\alpha U + \partial^\alpha p;\ \hat\rho\frac{d\Pi}{dt} = -p\partial_\nu v^\nu \quad (595)$$

where $d/dt = \partial/\partial t + v_\beta\partial^\beta$, and also the relations

$$\left.\begin{aligned}
&\partial_\beta V_\alpha - \partial_\alpha V_\beta = \partial_\beta N_\alpha - \partial_\alpha N_\beta;\\
&\hat\rho\frac{\partial U}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}[\hat\rho U]\\
&+ \frac{1}{8\pi}\partial^\alpha\left[\partial_\alpha U\frac{\partial U}{\partial t} - U\partial_\alpha\frac{\partial U}{\partial t}\right];
\end{aligned}\right\} \quad (596)$$

As a result, we obtain

$$\begin{aligned}
\nabla_m T^{m0} = {}& \frac{\partial}{\partial t}\left[\hat\rho\left(1 + \Pi - \frac{1}{2}U - \frac{1}{2}v_\nu v^\nu\right)\right]\\
&+ \partial_\alpha\left[\hat\rho v^\alpha\left(1 + \Pi + U - \frac{1}{2}v_\nu v^\nu\right) + pv^\alpha - \frac{1}{8\pi}\partial^\alpha U\frac{\partial U}{\partial t}\right.\\
&\left. + \frac{1}{8\pi}U\partial^\alpha\frac{\partial U}{\partial t}\right] + O(\varepsilon^4);
\end{aligned} \quad (597)$$

$$\begin{aligned}
\nabla_m T^{m\alpha} = {}& \frac{\partial}{\partial t}\left[\hat\rho v^\alpha\left(1 + \Pi + U - \frac{1}{2}v_\nu v^\nu\right)\right.\\
&\left. + pv^\alpha + \frac{1}{2}\hat\rho(N^\alpha - V^\alpha)\right]\\
&+ \partial_\beta\left[\hat\rho v^\alpha v^\beta\left(1 + \Pi + U - \frac{1}{2}v_\nu v^\nu\right) + pv^\alpha v^\beta - p(1 + 2U)\gamma^{\alpha\beta}\right.\\
&\left. + \hat\rho v^\beta(N^\alpha - V^\alpha)\right] + 4\hat\rho\frac{dV^\alpha}{dt} + 2\hat\rho\frac{d}{dt}(Uv^\alpha) + \hat\rho\partial^\alpha U + 2\hat\rho U\partial^\alpha U\\
&+ \hat\rho\left(\Pi + 2v^2 + \frac{3p}{\hat\rho}\right)\partial^\alpha U - 4\hat\rho v^\beta\partial^\alpha V_\beta - \frac{1}{2}\hat\rho v^\beta\partial_\beta(N^\alpha - V^\alpha)\\
&+ \hat\rho\partial^\alpha[2\Phi_1 + 2\Phi_2 + \Phi_3 + 3\Phi_4] + O(\varepsilon^6).
\end{aligned} \quad (598)$$

Substituting these expressions on the right-hand side of (586) and taking into account the identities

$$\left.\begin{aligned}
&\int dV\rho\,[\Pi\partial_\beta U + \partial_\beta\Phi_3] \equiv 0;\\
&\int dV\rho\,[U\partial_\beta U + \partial_\beta\Phi_2] \equiv 0;\\
&\int dV\,[p\partial_\beta U + \rho\partial_\beta\Phi_4] \equiv 0;\\
&\int dV\,[V^2\partial_\beta U + \partial_\beta\Phi_1] \equiv 0;\\
&\int dV\rho v^\beta\partial_\beta(N^\alpha - V^\alpha) \equiv 0;\\
&\int dV\frac{\partial U}{\partial t}\partial^\alpha U = 2\int dV\hat\rho\,[v^\alpha U + N^\alpha] \equiv 0;\\
&\int dV\hat\rho V^\alpha \equiv -\int dV\hat\rho v^\alpha U;\\
&\int dV\rho\partial_\beta U \equiv 0;\\
&\int dV\rho v_\nu\partial_\beta V^\nu = \int dV\rho v_\nu\partial_\beta N^\nu \equiv 0,
\end{aligned}\right\} \quad (599)$$

and also the fact that the volume integrals of a spatial divergence vanish after they have been transformed into surface integrals, we obtain

$$\frac{\partial}{\partial t}P^0 = \frac{\partial}{\partial t}\int dV\hat\rho\left(1 + \Pi - \frac{1}{2}U - \frac{1}{2}v_\nu v^\nu\right); \quad (600)$$

$$\begin{aligned}
&\frac{\partial}{\partial t}P_\alpha\\
&= \frac{\partial}{\partial t}\int dV\left[\hat\rho v_\alpha\left(1 + \Pi - \frac{1}{2}U - \frac{1}{2}v_\alpha v^\nu\right) + pv_\alpha + \frac{1}{2}\hat\rho N_\alpha\right].
\end{aligned} \quad (601)$$

In deriving this last formula, we have used the fact that in the expression (598) the indices can be raised and lowered by the metric tensor of the Minkowski space.

From (600) and (601), we finally find

$$P^0 = \int dV\hat\rho\left[1 + \Pi - \frac{1}{2}U - \frac{1}{2}v_\nu v^\nu\right] \quad (602)$$

and

$$P^\alpha = \int dV\left[\hat\rho v^\alpha\left(1 + \Pi - \frac{1}{2}U - \frac{1}{2}v_\nu v^\nu\right) + pv^\alpha + \frac{1}{2}\hat\rho N^\alpha\right]. \quad (603)$$

## 20. POST-NEWTONIAN MOTION OF BINARY SYSTEMS

We now study the post-Newtonian motion of two extended bodies in the gravitational field that they produce. We shall assume that both bodies consist of an ideal fluid, occupy volumes $V_1$ and $V_2$, and are separated by a distance $R$ appreciably exceeding their linear dimensions $L_1$ and $L_2$: $L_1/R \sim L_2/R \sim \varepsilon$. For simplicity, we assume that the matter in each of the bodies is distributed spherically symmetrically and that their internal motions are negligibly small.

The conserved mass density of the ideal fluid in the case which we consider can be written in the form

$$\hat{\rho}(x, t) = \hat{\rho}_1(x, t) + \hat{\rho}_2(x, t), \quad (604)$$

where the density $\hat{\rho}_1(x, t)$ is nonzero in the volume $V_1$ and the density $\hat{\rho}_2(x, t)$ is nonzero in the volume $V_2$.

Taking into account the relation

$$\frac{d}{dt}\int \hat{\rho} f(v^\alpha, U, V^\alpha)\, dV = \int \hat{\rho}\left[\frac{df}{dt} + fO(\varepsilon^2)\right] dV, \quad (605)$$

and also Eq. (604), we integrate the post-Newtonian equations of motion (598) of the ideal fluid over the volume occupied by the first body. Since $\nabla_m T^{m\alpha} = 0$, we obtain

$$\frac{d}{dt} P^\alpha_{(1)} = F^\alpha_{(1)}, \quad (606)$$

where the post-Newtonian expressions for the momentum $P^\alpha_{(1)}$ of the first body and the force $F^\alpha_{(1)}$ acting on it are

$$\left.\begin{aligned} P^\alpha_{(1)} &= \int \hat{\rho}_1 \Big\{ v^\alpha \Big[1 + \Pi + 3U + \frac{1}{2} v^2 + \frac{p_1}{\hat{\rho}_1} + O(\varepsilon^4)\Big] \\ &+ \frac{1}{2} N^\alpha + \frac{7}{2} V^\alpha \Big\} d^3x; \\ F^\alpha_{(1)} &= -\int \hat{\rho}_1 \Big\{ \Big[1 - U + \Pi + \frac{3}{2} v^2 + \frac{3p_1}{\hat{\rho}_1}\Big] \partial^\alpha U \\ &+ 2\partial^\alpha \Phi_1 + 2\partial^\alpha \Phi_2 + \partial^\alpha \Phi_3 + 3\partial^\alpha \Phi_4 - \\ &- \frac{7}{2} v^\beta \partial^\alpha V_\beta - \frac{1}{2} v^\beta \partial^\alpha N_\beta \Big\} d^3x. \end{aligned}\right\} \quad (607)$$

Integrating (598) over the volume occupied by the second body, we obtain

$$\frac{d}{dt} P^\alpha_{(2)} = F^\alpha_{(2)}. \quad (608)$$

The expressions for the momentum of the second body and the force acting on it are analogous to the relations (607):

$$\left.\begin{aligned} P^\alpha_{(2)} &= \int \hat{\rho}_2 \Big\{ v^\alpha \Big[1 + \Pi + 3U + \frac{1}{2} v^2 + \frac{p_2}{\hat{\rho}_2} + O(\varepsilon^4)\Big] \\ &+ \frac{1}{2} N^\alpha + \frac{7}{2} V^\alpha \Big\} d^3x; \\ F^\alpha_{(2)} &= -\int \hat{\rho}_2 \Big\{ \Big[1 - U + \Pi + \frac{3}{2} v^2 + \frac{3p_2}{\hat{\rho}_2}\Big] \partial^\alpha U \\ &+ 2\partial^\alpha \Phi_1 + 2\partial^\alpha \Phi_2 + \partial^\alpha \Phi_3 + 3\partial^\alpha \Phi_4 - \frac{7}{2} v^\beta \partial^\alpha V_\beta \\ &- \frac{1}{2} v^\beta \partial^\alpha N_\beta \Big\} d^3x. \end{aligned}\right\} \quad (609)$$

Using Eqs. (599) and (605), and also (500), we can readily show that $F^\alpha_{(1)} + F^\alpha_{(2)} = 0$. Therefore, as was to be expected, the total momentum of the binary system is conserved.

We place the coordinate origin at the center of mass of the binary system. We denote the radius vector of the center of mass of the first body in this coordinate system by $Y^\alpha_{(1)}$, that of the second body by $Y^\alpha_{(2)}$, and their difference by $Y^\alpha_{(1)} - Y^\alpha_{(2)} = R^\alpha$. Then for the radius vector $X^\alpha_{(1)}$ of an arbitrary point of the first body we have $X^\alpha_{(1)} = x^\alpha_{(1)} + Y^\alpha_{(1)}$, where $x^\alpha_{(1)}$ is the radius vector of the same point but in the coordinate system with origin at the center of mass of the first body. A similar relation can be written down for the radius vector of any point of the second body.

In the coordinate system associated with the center of mass of the binary system,

$$\begin{aligned} &\int \hat{\rho}_1 X^\alpha_{(1)} \Big[1 + \Pi + \frac{v^2}{2} - \frac{1}{2} U + O(\varepsilon^4)\Big] d^3x \\ &+ \int \hat{\rho}_2 X^\alpha_{(2)} \Big[1 + \Pi + \frac{v^2}{2} - \frac{1}{2} U + O(\varepsilon^4)\Big] d^3x = 0. \end{aligned} \quad (610)$$

Since the radius vectors $Y^\alpha_{(1)}$ and $Y^\alpha_{(2)}$ satisfy by virtue of their definitions

$$\begin{aligned} Y^\alpha_{(1)} &\int \hat{\rho}_1 \Big[1 + \Pi - \frac{1}{2} U + \frac{1}{2} v^2 + O(\varepsilon^4)\Big] d^3x \\ &= \int \hat{\rho}^1 X^\alpha_{(1)} \Big[1 + \Pi - \frac{1}{2} U + \frac{1}{2} v^2 + O(\varepsilon^4)\Big] d^3x; \\ Y^\alpha_{(2)} &\int \hat{\rho}_2 \Big[1 + \Pi - \frac{1}{2} U + \frac{1}{2} v^2 + O(\varepsilon^4)\Big] d^3x \\ &= \int \hat{\rho}_2 X^\alpha_{(2)} \Big[1 + \Pi - \frac{1}{2} U + \frac{1}{2} v^2 + O(\varepsilon^4)\Big] d^3x, \end{aligned}$$

it follows that

$$\left.\begin{aligned} &\int \hat{\rho}_1 x^\alpha_1 \Big[1 + \Pi + \frac{v^2}{2} - \frac{1}{2} U + O(\varepsilon^4)\Big] d^3x = 0; \\ &\int \hat{\rho}_2 x^\alpha_2 \Big[1 + \Pi + \frac{v^2}{2} - \frac{1}{2} U + O(\varepsilon^4)\Big] d^3x = 0; \\ &Y^\alpha_{(1)} \int \hat{\rho}_1 \Big[1 + \Pi + \frac{v^2}{2} - \frac{1}{2} U + O(\varepsilon^4)\Big] d^3x \\ &+ Y^\alpha_{(2)} \int \hat{\rho}_2 \Big[1 + \Pi + \frac{v^2}{2} - \frac{1}{2} U + O(\varepsilon^4)\Big] d^3x = 0. \end{aligned}\right\} \quad (611)$$

We shall find the equations of motion of the centers of mass of each body. Taking into account the relations (604) and (500), we rewrite the generalized potentials in the expressions (607) in the form

$$\left.\begin{aligned} U &= \tilde{U} - 3\tilde{\Phi}_2 - \frac{1}{2}\tilde{\Phi}_1 + \int \frac{\hat{\rho}_2 d^3y}{|\mathbf{X}-\mathbf{y}|} + \frac{1}{2}\int \frac{\hat{\rho}_2 v_\alpha v^\alpha}{|\mathbf{X}-\mathbf{y}|} d^3y \\ &- 3\int \frac{\hat{\rho}_1'\hat{\rho}_2'' + \hat{\rho}_2'\hat{\rho}_1'' + \hat{\rho}_2'\hat{\rho}_2''}{|\mathbf{X}-\mathbf{X}'|\,|\mathbf{X}'-\mathbf{X}''|} d^3X' d^3X'' + O(\varepsilon^6); \\ \Phi_1 &= \tilde{\Phi}_1 - \int \frac{\hat{\rho}_2 v_\alpha v^\alpha}{|\mathbf{X}-\mathbf{y}|} d^3y + O(\varepsilon^6); \\ \Phi_2 &= \tilde{\Phi}_2 + \int \frac{[\hat{\rho}_1'\hat{\rho}_2'' + \hat{\rho}_2'\hat{\rho}_1'' + \hat{\rho}_2'\hat{\rho}_2'']\, d^3X' d^3X''}{|\mathbf{X}-\mathbf{X}'|\,|\mathbf{X}'-\mathbf{X}''|} + O(\varepsilon^6); \\ \Phi_3 &= \tilde{\Phi}_3 + \int \frac{\hat{\rho}_2 \Pi d^3y}{|\mathbf{X}-\mathbf{y}|} + O(\varepsilon^6); \\ \Phi_4 &= \tilde{\Phi}_4 + \int \frac{p_2 d^3y}{|\mathbf{X}-\mathbf{y}|} + O(\varepsilon^6); \\ V^\alpha &= \tilde{V}^\alpha - \int \frac{\hat{\rho}_2 v^\alpha d^3y}{|\mathbf{X}-\mathbf{y}|} + O(\varepsilon^5); \\ N^\alpha &= \tilde{N}^\alpha + \int \frac{\hat{\rho}_2 v_\beta (X^\beta - y^\beta)(X^\alpha - y^\alpha)}{|\mathbf{X}-\mathbf{y}|^3} d^3y + O(\varepsilon^5), \end{aligned}\right\} \quad (612)$$

where for the gravitational self-potentials of the first body we have introduced the notation

$$\left.\begin{aligned}
&\widetilde{U}=\int\frac{\hat{\rho}_1}{|\mathbf{X}-\mathbf{y}|}\,d^3y;\quad \widetilde{\Phi}_1=-\int\frac{\hat{\rho}_1 v_\alpha v^\alpha}{|\mathbf{X}-\mathbf{y}|}\,d^3y;\\
&\widetilde{\Phi}_2=\int\frac{\hat{\rho}_1\widetilde{U}}{|\mathbf{X}-\mathbf{y}|}\,d^3y;\quad \widetilde{\Phi}_3=\int\frac{\hat{\rho}_1\Pi d^3y}{|\mathbf{X}-\mathbf{y}|};\\
&\widetilde{\Phi}_4=\int\frac{p_1 d^3y}{|\mathbf{X}-\mathbf{y}|};\quad \widetilde{V}^\alpha=-\int\frac{\hat{\rho}_1 v^\alpha}{|\mathbf{X}-\mathbf{y}|}\,d^3y;\\
&\widetilde{N}^\alpha=\int\frac{\hat{\rho}_1 v_\beta (X^\beta-y^\beta)(X^\alpha-y^\alpha)}{|\mathbf{X}-\mathbf{y}|^3}\,d^3y.
\end{aligned}\right\}\qquad(613)$$

We introduce the notation

$$m_1=\int\hat{\rho}_1 d^3x;\quad m_2=\int\hat{\rho}_2 d^3x \qquad (614)$$

for the rest masses of the first and second body respectively. By virtue of the continuity equation (483), these masses do not depend on the time.

We expand each term in the expressions (607) in powers of $1/R$ to order $O(1/R^4)$. For this, we use the expansions

$$\left.\begin{aligned}
&\frac{1}{|\mathbf{X}_1-\mathbf{X}_2|}=\frac{1}{R}+\frac{R_\nu(x_1^\nu-x_2^\nu)}{R^3}+\frac{1}{2}\frac{(x_{1\nu}-x_{2\nu})(x_1^\nu-x_2^\nu)}{R^3}\\
&+\frac{3}{2}\frac{[R_\nu(x_1^\nu-x_2^\nu)]^2}{R^5}+O\left(\frac{1}{R^4}\right);\\
&\frac{(X_1^\alpha-X_2^\alpha)(X_1^\beta-X_2^\beta)}{|\mathbf{X}_1-\mathbf{X}_2|^3}=\frac{R^\alpha R^\beta}{R^3}+\frac{3R^\alpha R^\beta R_\nu(x_1^\nu-x_2^\nu)}{R^5}\\
&+\frac{3}{2}\frac{R^\alpha R^\beta(x_{1\nu}-x_{2\nu})(x_1^\nu-x_2^\nu)}{R^5}+\frac{15}{2}\frac{R^\alpha R^\beta[R_\nu(x_1^\nu-x_2^\nu)]^2}{R^7}\\
&+\frac{R^\alpha(x_1^\beta-x_2^\beta)+R^\beta(x_1^\alpha-x_2^\alpha)}{R^3}+\frac{3R^\alpha R_\nu(x_1^\nu-x_2^\nu)(x_1^\beta-x_2^\beta)}{R^5}\\
&+\frac{3R^\beta R_\nu(x_1^\nu-x_2^\nu)(x_1^\alpha-x_2^\alpha)}{R^5}+\frac{(x_1^\alpha-x_2^\alpha)(x_1^\beta-x_2^\beta)}{R^3}+O\left(\frac{1}{R^4}\right);\\
&\frac{X_1^\alpha-X_2^\alpha}{|\mathbf{X}_1-\mathbf{X}_2|^3}=\frac{R^\alpha}{R^3}+\frac{3R^\alpha R_\nu(x_1^\nu-x_2^\nu)}{R^5}+\frac{x_1^\alpha-x_2^\alpha}{R^3}+O\left(\frac{1}{R^4}\right);\\
&\frac{(X_1^\alpha-X_2^\alpha)(X_1^\beta-X_2^\beta)(X_1^\nu-X_2^\nu)}{|\mathbf{X}_1-\mathbf{X}_2|^5}=\frac{R^\alpha R^\beta R^\nu}{R^5}+\frac{R^\alpha R^\beta(x_1^\nu-x_2^\nu)}{R^5}\\
&+\frac{R^\alpha R^\nu(x_1^\beta-x_2^\beta)}{R^5}+\frac{R^\beta R^\nu(x_1^\alpha-x_2^\alpha)}{R^5}+\\
&+\frac{5R^\alpha R^\beta R^\nu R_\lambda(x_1^\lambda-x_2^\lambda)}{R^7}+O\left(\frac{1}{R^4}\right).
\end{aligned}\right\}\qquad(615)$$

Introducing the notation

$$\left.\begin{aligned}
&\Omega_{(1)}^{\alpha\beta}=-\frac{1}{2m_1}\int\frac{\hat{\rho}_1\hat{\rho}_1'(x^\alpha-x'^\alpha)(x^\beta-x'^\beta)}{|\mathbf{x}-\mathbf{x}'|^3}\,d^3x d^3x';\\
&\Omega_{(2)}^{\alpha\beta}=-\frac{1}{2m_2}\int\frac{\hat{\rho}_2\hat{\rho}_2'(x^\alpha-x'^\alpha)(x^\beta-x'^\beta)}{|\mathbf{x}-\mathbf{x}'|^3}\,d^3x d^3x';\\
&P_{(1)}=\frac{1}{m_1}\int p_1 d^3x;\quad P_{(2)}=\frac{1}{m_2}\int p_2 d^3x;\\
&\Pi_{(1)}=\frac{1}{m_1}\int\hat{\rho}_1\Pi d^3x;\quad \Pi_{(2)}=\frac{1}{m_2}\int\hat{\rho}_2\Pi d^3x;\\
&n^\alpha=R^\alpha/R;\quad \Omega=\Omega_\alpha^\alpha,
\end{aligned}\right\}\qquad(616)$$

using the expressions (612) and (615), and taking into account the trivial relations

$$\left.\begin{aligned}
&\frac{1}{m}\int\frac{\hat{\rho}\hat{\rho}'x^\beta(x^\alpha-x'^\alpha)}{|\mathbf{x}-\mathbf{x}'|^3}\,d^3x d^3x'=-\Omega^{\alpha\beta};\\
&\int\hat{\rho}_1 x_1^\alpha d^3x=M_1L_1O(\varepsilon^2),
\end{aligned}\right\}\qquad(617)$$

we obtain a number of relations needed in what follows:

$$\left.\begin{aligned}
&\int\hat{\rho}_1 v^\alpha\left[1+\Pi+\frac{1}{2}v^2+\frac{p_1}{\hat{\rho}_1}\right]d^3x=\\
&=m_1V_{(1)}^\alpha\left[1+\Pi_{(1)}+P_{(1)}+\frac{V_{(1)}^2}{2}+O(\varepsilon^4)\right];\\
&\int\hat{\rho}_1 v^\alpha U d^3x=m_1V_{(1)}^\alpha\left[2\Omega_{(1)}+\frac{m_2}{R}+O(\varepsilon^4)\right];\\
&\int\hat{\rho}_1 V^\alpha d^3x=-m_1\left[2\Omega_{(1)}V_{(1)}^\alpha+\frac{m_2}{R}V_{(2)}^\alpha+O(\varepsilon^4)\right];\\
&\int\hat{\rho}_1 N^\alpha d^3x=m_1\left[-2\Omega_{(1)}^{\alpha\beta}V_{(1)\beta}+\frac{m_2}{R}\,n^\alpha n_\beta V_{(2)}^\beta+O(\varepsilon^5)\right];\\
&\int\hat{\rho}_1\partial^\alpha U d^3x=\frac{m_1m_2}{R^2}\left\{n^\alpha\left[1-6\Omega_{(2)}-\frac{1}{2}V_{(2)}^2-\frac{3m_2}{R}+O(\varepsilon^3)\right]\right.\\
&\left.-3n_\beta\Omega_{(1)}^{\alpha\beta}\right\}-3\int\hat{\rho}_1\partial^\alpha\widetilde{\Phi}_2 d^3x-\frac{1}{2}\int\hat{\rho}_1\partial^\alpha\widetilde{\Phi}_1 d^3x;\\
&\int\hat{\rho}_1 v^\beta\partial^\alpha N_\beta d^3x=\frac{m_1m_2}{R^2}\{[V_{(1)}^\alpha V_{(2)}^\nu+V_{(2)}^\alpha V_{(1)}^\nu]\,n_\nu\\
&+3n^\alpha n_\nu V_{(1)}^\nu n_\beta V_{(2)}^\beta+O(\varepsilon^4)\};\\
&\int\hat{\rho}_1 v^\beta\partial^\alpha V_\beta d^3x=-\frac{m_1m_2}{R^2}[n^\alpha V_{(1)}^\beta V_{(2)\beta}+O(\varepsilon^4)];\\
&\int\hat{\rho}_1 U\partial^\alpha U d^3x=\frac{m_1m_2}{R^2}\left[2n^\alpha\Omega_{(1)}-n_\nu\Omega_{(1)}^{\nu\alpha}+n^\alpha\frac{m_2}{R}+O(\varepsilon^4)\right]\\
&+\int\hat{\rho}_1\partial^\alpha\widetilde{U}\widetilde{U}d^3x;\\
&\int\hat{\rho}_1\Pi\partial^\alpha U d^3x=\frac{m_1m_2}{R^2}[n^\alpha\Pi_{(1)}+O(\varepsilon^4)]+\int\hat{\rho}_1\Pi\partial^\alpha\widetilde{U}d^3x;\\
&\int\hat{\rho}_1 v^2\partial^\alpha U d^3x=\frac{m_1m_2}{R^2}[n^\alpha V_{(1)}^2+O(\varepsilon^4)]+\int\hat{\rho}_1 v^2\partial^\alpha\widetilde{U}d^3x;\\
&\int p_1\partial^\alpha U d^3x=\frac{m_1m_2}{R^2}[n^\alpha P_{(1)}+O(\varepsilon^4)]+\int p_1\partial^\alpha\widetilde{U}d^3x;\\
&\int\hat{\rho}_1\partial^\alpha\Phi_1 d^3x=\frac{m_1m_2}{R^2}[n^\alpha V_{(1)}^2+O(\varepsilon^4)]+\int\hat{\rho}_1\partial^\alpha\widetilde{\Phi}_1 d^3x;\\
&\int\hat{\rho}_1\partial^\alpha\Phi_2 d^3x=\frac{m_1m_2}{R^2}\left[n^\alpha\left(2\Omega_{(2)}+\frac{m_1}{R}\right)+n_\nu\Omega_{(1)}^{\nu\alpha}+O(\varepsilon^4)\right]\\
&+\int\hat{\rho}_1\partial^\alpha\widetilde{\Phi}_2 d^3x;\\
&\int\hat{\rho}_1\partial^\alpha\Phi_3 d^3x=\frac{m_1m_2}{R^2}[n^\alpha\Pi_{(2)}+O(\varepsilon^4)]+\int\hat{\rho}_1\partial^\alpha\widetilde{\Phi}_3 d^3x;\\
&\int\hat{\rho}_1\partial^\alpha\Phi_4 d^3x=\frac{m_1m_2}{R^2}[n^\alpha P_{(2)}+O(\varepsilon^4)]+\int\hat{\rho}_1\partial^\alpha\widetilde{\Phi}_4 d^3x,
\end{aligned}\right\}\qquad(618)$$

where $V_{(1)}^\alpha$ and $V_{(2)}^\alpha$ are the velocities of the centers of mass of the first and second body, respectively:

$$V_{(1)}^\alpha=\frac{d}{dt}Y_{(1)}^\alpha;\quad V_{(2)}^\alpha=\frac{d}{dt}Y_{(2)}^\alpha.$$

Substituting the expansions (618) in the expressions (607) and taking into account the relations (599), which are satisfied by the self-potentials (613) of the first body, we obtain

$$\left.\begin{aligned}
&P_{(1)}^\alpha=m_1\left\{\left[1+\Pi_{(1)}-\Omega_{(1)}+P_{(1)}+\frac{1}{2}V_{(1)}^2+\frac{3m_2}{R}\right]V_{(1)}^\alpha\right.\\
&\left.-\Omega_{(1)}^{\alpha\beta}V_{(1)\beta}+\frac{1}{2}\frac{m_2}{R}[n^\alpha n_\nu V_{(2)}^\nu-7V_{(2)}^\alpha]+O(\varepsilon^5)\right\};\\
&F_{(1)}^\alpha=-\frac{m_1m_2}{R^2}\left\{n^\alpha\left[1-2\Omega_{(1)}-2\Omega_{(2)}+\Pi_{(1)}+\Pi_{(2)}+3P_{(1)}\right.\right.\\
&+3P_{(2)}+\frac{3}{2}V_{(1)}^2+\frac{3}{2}V_{(2)}^2-\frac{3}{2}n_\nu V_{(1)}^\nu n_\beta V_{(2)}^\beta+\frac{7}{2}V_{(1)}^\beta V_{(2)\beta}\\
&\left.\left.-\frac{m_1+m_2}{R}\right]-\frac{1}{2}[V_{(1)}^\alpha V_{(2)}^\beta+V_{(2)}^\alpha V_{(1)}^\beta]\,n_\beta+O(\varepsilon^4)\right\}.
\end{aligned}\right\}\qquad(619)$$

We have similar expressions for the second body:

$$P^{\alpha}_{(2)}=m_2\left\{\left[1+\Pi_{(2)}-\Omega_{(2)}+P_{(2)}+\frac{1}{2}V^2_{(2)}+\frac{3m_1}{R}\right]V^{\alpha}_{(2)}\right.$$
$$\left.-\Omega^{\alpha\beta}_{(2)}V_{(2)\beta}+\frac{1}{2}\frac{m_1}{R}\left[n^{\alpha}n_{\beta}V^{\beta}_{(1)}-7V^{\alpha}_{(1)}\right]+Q(\varepsilon^5)\right\};$$
$$F^{\alpha}_{(2)}=-F^{\alpha}_{(1)}. \quad (620)$$

Using the Newtonian virial theorem

$$\Omega^{\alpha\beta}_{(1)}=\gamma^{\alpha\beta}P_{(1)};\ \Omega_{(1)}=3P_{(1)};\ \Omega^{\alpha\beta}_{(2)}=\gamma^{\alpha\beta}P_{(2)};\ \ \Omega_{(2)}=3P_{(2)}$$

and introducing the notation

$$M_1=m_1\left[1+\Pi_{(1)}-\Omega_{(1)}\right];\ M_2=m_2\left[1+\Pi_{(2)}-\Omega_{(2)}\right] \quad (621)$$

for the total rest masses of each of the bodies, we write the expressions (619) and (620) in the form

$$P^{\alpha}_{(1)}=M_1\left\{V^{\alpha}_{(1)}\left[1+\frac{1}{2}V^2_{(1)}-\frac{1}{2}\frac{M_2}{R}\right]+\frac{7}{2}\frac{M_2}{R}(V^{\alpha}_{(1)}-V^{\alpha}_{(2)})\right.$$
$$\left.+\frac{1}{2}\frac{M_2}{R}n^{\alpha}n_{\beta}V^{\beta}_{(2)}+O(\varepsilon^5)\right\};$$
$$P^{\alpha}_{(2)}=M_2\left\{V^{\alpha}_{(2)}\left[1+\frac{1}{2}V^2_{(2)}-\frac{1}{2}\frac{M_1}{R}\right]+\frac{7}{2}\frac{M_1}{R}(V^{\alpha}_{(2)}-V^{\alpha}_{(1)})\right.$$
$$\left.+\frac{1}{2}\frac{M_1}{R}n^{\alpha}n_{\beta}V^{\beta}_{(1)}+O(\varepsilon^5)\right\};$$
$$F^{\alpha}_{(1)}=-\frac{M_1M_2}{R^2}\left\{n^{\alpha}\left[1+\frac{3}{2}V^2_{(1)}+\frac{3}{2}V^2_{(2)}+\frac{7}{2}V^{\beta}_{(1)}V_{(2)\beta}\right.\right.$$
$$\left.-\frac{M_1+M_2}{R}-\frac{3}{2}V^{\nu}_{(1)}n_{\nu}V^{\beta}_{(2)}n_{\beta}\right]$$
$$\left.-\frac{1}{2}\left[V^{\alpha}_{(1)}V^{\beta}_{(2)}+V^{\alpha}_{(2)}V^{\beta}_{(1)}\right]n_{\beta}+O(\varepsilon^3)\right\}. \quad (622)$$

The characteristics of the first and second body in these expressions are not independent, being related not only by the equations of motion (606) and (608) but also by the third of the relations (611). With allowance for the expressions (612), (618), and (621) this last relation takes the form

$$M_1Y^{\alpha}_{(1)}\left[1+\frac{1}{2}V^2_{(1)}-\frac{1}{2}\frac{M_2}{R}+O(\varepsilon^4)\right]$$
$$+M_2Y^{\alpha}_{(2)}\left[1+\frac{1}{2}V^2_{(2)}-\frac{1}{2}\frac{M_1}{R}+O(\varepsilon^4)\right]=0. \quad (623)$$

With allowance for the expressions (622) and (623), the equations of motion (606) and (608) make it possible to determine the motion of each of the bodies in the post-Newtonian approximation. As in Newtonian mechanics, it is convenient in the case that we consider to reduce the equations of motion of the two bodies to the equation of motion of a single body in the field of a fixed gravitating center. For this, we must express all the quantities (622) in terms of the relative coordinates $R^{\alpha}$ and $v^{\alpha}$:

$$Y^{\alpha}_{(1)}=R^{\alpha}+Y^{\alpha}_{(2)};\quad V^{\alpha}_{(1)}=v^{\alpha}+V^{\alpha}_{(2)}$$

Substituting these relations in the expression (623), we obtain

$$Y^{\alpha}_{(2)}=-\frac{M_1}{M_1+M_2}R^{\alpha}\left[1+\frac{M_2(M_2-M_1)}{2(M_1+M_2)^2}v^2\right.$$
$$\left.-\frac{M_2-M_1}{2(M_1+M_2)}\frac{M_2}{R}+O(\varepsilon^4)\right]. \quad (624)$$

Then for the radius vector of the center of mass of the first body we have

$$Y^{\alpha}_{(1)}=\frac{M_2}{M_1+M_2}R^{\alpha}\left[1+\frac{M_1(M_1-M_2)}{2(M_1+M_2)^2}v^2\right.$$
$$\left.-\frac{M_1-M_2}{2(M_1+M_2)}\frac{M_1}{R}+O(\varepsilon^4)\right]. \quad (625)$$

The center-of-mass velocities of the bodies can be expressed in terms of the relative coordinates in two ways: either by directly differentiating the relations (624) and (625) with respect to the time and taking into account the equations of motion of the centers of mass of the bodies in the Newtonian approximation, or from the condition of vanishing of the total momentum of the bodies. Since the equation $P^{\alpha}_{(1)}+P^{\alpha}_{(2)}=0$ is a consequence of the expression (623), in the two cases we naturally arrive at the same result:

$$V^{\alpha}_{(1)}=\frac{M_2}{M_1+M_2}v^{\alpha}\left[1+\frac{M_1(M_1-M_2)}{2(M_1+M_2)^2}\left(v^2-\frac{M_1+M_2}{R}\right)\right.$$
$$\left.+\frac{M_1M_2(M_1-M_2)}{2(M_1+M_2)^2R}n^{\alpha}n_{\beta}v^{\beta}+O(\varepsilon^5)\right];$$
$$V^{\alpha}_{(2)}=-\frac{M_1}{M_1+M_2}v^{\alpha}\left[1+\frac{M_2(M_2-M_1)}{2(M_1+M_2)^2}\left(v^2\right.\right.$$
$$\left.\left.-\frac{M_1+M_2}{R}\right)+\frac{M_1M_2(M_2-M_1)}{2(M_1+M_2)^2R}n^{\alpha}n_{\beta}v^{\beta}+O(\varepsilon^5)\right]. \quad (626)$$

Substituting the relations (626) in the expressions (622), we obtain

$$P^{\alpha}_{(1)}=-P^{\alpha}_{(2)}=\frac{M_1M_2}{M_1+M_2}v^{\alpha}\left[1+\frac{M_1^3+M_2^3}{2(M_1+M_2)^3}v^2+\right.$$
$$\left.+\frac{3M_1^2+7M_1M_2+3M_2^2}{(M_1+M_2)R}\right]+\frac{M_1^2M_2^2}{(M_1+M_2)^2R}n^{\alpha}n_{\beta}v^{\beta}+O(\varepsilon^5);$$
$$F^{\alpha}_{(1)}=-F^{\alpha}_{(2)}=-\frac{M_1M_2}{R^2}\left\{n^{\alpha}\left[1+\frac{3M_1^2+7M_1M_2+3M_2^2}{2(M_1+M_2)^2}v^2\right.\right.$$
$$\left.+\frac{3M_1M_2}{2(M_1+M_2)^2}(n_{\beta}v^{\beta})^2-\frac{M_1+M_2}{R}\right]$$
$$\left.+\frac{M_1M_2}{(M_1+M_2)^2}v^{\alpha}n_{\beta}v^{\beta}+O(\varepsilon^5)\right\}. \quad (627)$$

Thus, in the relative variables the equations of motion (606) and (608) take the same form. Introducing the notation

$$M=M_1+M_2;\ m=M_1M_2/(M_1+M_2),$$

we obtain

$$m\frac{d}{dt}\left\{v^{\alpha}\left[1+\frac{1}{2}\left(1-\frac{3m}{M}\right)v^2+\frac{3M+m}{R}\right]\right.$$
$$\left.+\frac{m}{R}n^{\alpha}n_{\nu}v^{\nu}+O(\varepsilon^5)\right\}$$
$$=-\frac{Mm}{R^2}\left\{n^{\alpha}\left[1-\frac{M}{R}+\frac{1}{2}\left(3+\frac{m}{M}\right)v^2\right.\right.$$
$$\left.\left.+\frac{3m}{2M}(n_{\nu}v^{\nu})^2\right]+\frac{m}{M}v^{\alpha}n_{\nu}v^{\nu}+O(\varepsilon^4)\right\}.$$

Using the Newtonian approximation

$$m\frac{dv^{\alpha}}{dt}=-\frac{Mm}{R^2}\left[n^{\alpha}+O(\varepsilon^2)\right], \quad (628)$$

we find from this

$$\frac{dv^{\alpha}}{dt}=-\frac{M}{R^2}\left\{n^{\alpha}\left[1-\left(4+\frac{2m}{M}\right)\frac{M}{R}+\left(1+\frac{3m}{M}\right)v^2\right.\right.$$
$$\left.\left.-\frac{3m}{2M}(n_{\nu}v^{\nu})^2\right]+\left(4-\frac{2m}{M}\right)v^{\alpha}n_{\nu}v^{\nu}+O(\varepsilon^4)\right\}. \quad (629)$$

In three-dimensional vector form, this equation gives

$$\frac{d\mathbf{v}}{dt}=-\frac{M\mathbf{R}}{R^3}\left[1-\left(4+\frac{2m}{M}\right)\frac{M}{R}+\left(1+\frac{3m}{M}\right)v^2-\frac{3m}{2M}\left(\frac{\mathbf{Rv}}{R}\right)^2\right]+\left(4-\frac{2m}{M}\right)\frac{M}{R^3}\mathbf{v}(\mathbf{Rv})+O(\varepsilon^6). \quad (630)$$

Multiplying Eq. (630) by the vector $\mathbf{R}$ vectorially, we obtain

$$\left[\mathbf{R}\frac{d\mathbf{v}}{dt}\right]=\left(4-\frac{2m}{M}\right)\frac{M}{R^3}(\mathbf{Rv})\,[\mathbf{Rv}]+O(\varepsilon^6).$$

With allowance for the Newtonian approximation of the equations of motion (628), this relation can be written in the form

$$\frac{d}{dt}\left\{[\mathbf{Rv}]\left[1+\frac{2M}{R}\left(2-\frac{m}{M}\right)+O(\varepsilon^4)\right]\right\}=0.$$

It follows from this that the expression in the curly brackets is a first integral of the equations of motion (629):

$$[\mathbf{Rv}]\left[1+\frac{2M}{R}\left(2-\frac{m}{M}\right)+O(\varepsilon^4)\right]=\mathbf{C}_5=\text{const}. \quad (631)$$

Multiplying this integral scalarly by the radius vector $\mathbf{R}$, we obtain

$$\mathbf{C}_5\mathbf{R}=0.$$

Therefore, in the RTG the trajectory of the post-Newtonian motion of the binary system lies in a plane whose normal is directed along the vector $\mathbf{C}_5$.

We shall find the equation of this trajectory. Introducing in the plane of the trajectory the polar coordinates $R$ and $\varphi$, we obtain from the relation (631)

$$R^2\dot{\varphi}\left[1+\frac{2M}{R}\left(2-\frac{m}{M}\right)+O(\varepsilon^4)\right]=C_5, \quad (632)$$

where $C_5=|\mathbf{C}_5|$. We also write down the equation for the radial component of the acceleration:

$$\ddot{R}-R\dot{\varphi}^2=-\frac{M}{R^2}\left[1-\left(4+\frac{2m}{M}\right)\frac{M}{R}+\left(\frac{7m}{2M}-3\right)\dot{R}^2+\left(1+\frac{3m}{M}\right)R^2\dot{\varphi}^2+O(\varepsilon^4)\right]. \quad (633)$$

Using the expression (632) and introducing the notation $u=M/R$, $u'=du/d\varphi$, we transform Eq. (633) to a form analogous to the Binet formula:

$$u''\left[1-4u\left(2-\frac{m}{M}\right)\right]-u'^2\left(1+\frac{3m}{M}\right)+u+\left(\frac{m}{M}-9\right)u^2-\frac{M^2}{C_5^2}\left[1-\left(4+\frac{2m}{M}\right)u\right]=O(\varepsilon^5). \quad (634)$$

Since the quantities that occur in this equation are small, it can be solved by successive approximation. We expand $u$ and $M^2/C_5^2$ in series in the small parameter $\varepsilon^2$:

$$\left.\begin{aligned} u&=\overset{(1)}{u}+\overset{(2)}{u}+\ldots,\\ \frac{M^2}{C_5^2}&=\frac{M^2}{C_6^2}[1+C_7+\ldots],\end{aligned}\right\} \quad (635)$$

where

$$\overset{(1)}{u}\sim O(\varepsilon^2);\quad \overset{(2)}{u}\sim O(\varepsilon^4);\quad \frac{M^2}{C_6^2}\sim O(\varepsilon^2);\quad C_7\sim O(\varepsilon^2).$$

Substituting the expansions (635) in Eq. (634), we obtain in the first order

$$\overset{(1)}{u}{}''+\overset{(1)}{u}=\frac{M^2}{C_6^2}.$$

It follows from this that

$$\overset{(1)}{u}=\frac{M^2}{C_6^2}[1+\mathrm{e}\cos\varphi].$$

Thus, in the first approximation and for $e<1$ the trajectory of the motion of the binary system is an ellipse. Expressing the constant of integration $C_6^2$ in terms of the parameter $p$ of the ellipse, we obtain

$$\overset{(1)}{u}=\frac{M}{p}[1+\mathrm{e}\cos\varphi].$$

In the second approximation, Eq. (634) gives

$$\overset{(2)}{u}{}''+\overset{(2)}{u}=\frac{M}{p}C_7+\frac{M^2}{p^2}\left(5-\frac{3m}{M}\right)+\frac{M^2\mathrm{e}^2}{p^2}\left(1+\frac{3m}{M}\right)+\frac{6M^2\mathrm{e}}{p^2}\cos\varphi+\frac{3Mm\mathrm{e}^2}{4p^2}\cos 2\varphi.$$

The solution of this equation has the form

$$\overset{(2)}{u}=\frac{M}{p}[C_7+C_8\cos\varphi]+\frac{M^2}{p^2}\left(5-\frac{3m}{M}\right)+\frac{M^2\mathrm{e}^2}{p^2}\left(1+\frac{3m}{4M}\right)+\frac{3M^2\mathrm{e}}{p^2}\varphi\sin\varphi-\frac{mM\mathrm{e}^2}{4p^2}\cos 2\varphi.$$

Redetermining the Newtonian expressions for $p$ and $e$, we finally obtain

$$u=\frac{M}{R}=\frac{M}{p}\left\{1+\mathrm{e}\cos\varphi+\frac{M}{p}\left[-3+\frac{m}{M}+\left(1+\frac{9m}{4M}\right)\mathrm{e}^2+\frac{1}{2}\left(7-\frac{2m}{M}\right)\mathrm{e}\cos\varphi+3\mathrm{e}\varphi\sin\varphi-\frac{m\mathrm{e}^2}{4M}\cos 2\varphi\right]\right\}. \quad (636)$$

Since this solution is valid only under the condition $|me\varphi/p|\ll 1$, it can be written in the form

$$u=\frac{M}{p}\left\{1+\mathrm{e}\cos\left(\varphi-\frac{3M}{p}\varphi\right)+\frac{M}{p}\left[-3+\frac{m}{M}+\left(1+\frac{9m}{4M}\right)\mathrm{e}^2+\frac{1}{2}\left(7-\frac{2m}{M}\right)\mathrm{e}\cos\varphi-\frac{m\mathrm{e}^2}{4M}\cos 2\varphi\right]\right\}. \quad (637)$$

It follows from this expression that the trajectory is a curve that is not closed, its pericenters (i.e., the points of the trajectory at the shortest distance from the origin of the polar coordinate system) being systematically displaced with increasing polar angle $\varphi$ into the region of larger values of the angle. Indeed, it is easy to show that the position of the pericenters of the trajectory is determined to post-Newtonian accuracy by the condition

$$\varphi_n-\frac{3M}{p}\varphi_n=2\pi n,$$

where $n$ is the number of the pericenter. From this it is readily found that the $n$th pericenter of the trajectory is at a polar angle equal to

$$\varphi_n\simeq 2\pi n\left(1+\frac{3M}{p}\right).$$

Thus, in accordance with the RTG the displacement of the pericenter of the binary system in one revolution is

$$\delta\varphi=\varphi_n-\varphi_{n-1}-2\pi=6\pi M/p,$$

in agreement with the results of measurements of the displacements of the perihelia of Mercury and Mars. Substituting the expression (637) in the relation (632), we obtain

$$R^2\dot{\varphi}=\sqrt{Mp}\left[1-\frac{M}{p}e\left(4-\frac{2m}{M}\right)\cos\varphi\right].$$

We can now find the relative velocity of the binary system as a function of the polar angle. If the Cartesian components of the radius vector $R^\alpha$ are written in the form

$$R^\alpha=R\,[\delta_1^\alpha\cos\varphi+\delta_2^\alpha\sin\varphi],$$

then after simple calculations the expression for the relative velocity can be reduced to the form

$$v^\alpha=\sqrt{\frac{P}{M}}\left[1-\frac{M}{p}e\left(4-\frac{2m}{M}\right)\cos\varphi\right]$$
$$\times\{\delta_1^\alpha[-u\sin\varphi-u'\cos\varphi]+\delta_2^\alpha[u\cos\varphi-u'\sin\varphi]\}.$$

Substituting in this expression the relation (636) and restricting ourselves to the post-Newtonian terms, we obtain

$$v^\alpha=\sqrt{\frac{M}{p}}\Big\{-\delta_1^\alpha\sin\varphi+\delta_2^\alpha(e+\cos\varphi)$$
$$+\delta_1^\alpha\frac{M}{p}\Big[-3e\varphi+\left(3-\frac{m}{M}\right)\sin\varphi-\left(1+\frac{21m}{8M}\right)e^2\sin\varphi$$
$$+\frac{1}{2}\left(1-\frac{2m}{M}\right)e\sin 2\varphi-\frac{me^2}{8M}\sin 3\varphi\Big]$$
$$+\delta_2^\alpha\frac{M}{p}\Big[-\left(3-\frac{m}{M}\right)\cos\varphi-\left(3-\frac{31m}{8M}\right)e^2\cos\varphi$$
$$+\frac{1}{2}\left(1-\frac{2m}{M}\right)e\cos 2\varphi+\frac{me^3}{8M}\cos 3\varphi\Big]\Big\}. \qquad (638)$$

The trajectories of each of the bodies and the velocities of their centers of mass can be obtained from the expressions (624)–(626), (636), and (638).

## 21. PETERS-MATHEWS COEFFICIENTS IN THE RTG

It is well known that in the scientific literature there is much discussion of not only GR but also other variants of the theory of gravitational interaction. In recent years, theories of gravitation have been analyzed mainly in two directions. The studies of the first direction have considered their agreement with various general theoretical requirements: completeness of the theory and its consistency, covariance of the basic equations and subsidiary conditions, analysis of the solution of the energy-momentum problem for the gravitational field, and similar questions. These studies have made it possible to reduce considerably the number of theories worthy of further study, and have also revealed logical contradictions (in the first place, the absence in principle of conservation laws for the matter and gravitational field taken together) in GR from the point of view of physics.

The studies of the other direction have analyzed the agreement between the predictions of the various theories of gravitation and the results of gravitational experiments and have also looked for the experimental situations in which the different theories must give different predictions. The interest of investigators in these questions increased considerably after the experimental technology had achieved the post-Newtonian level of accuracy and the parametrized post-Newtonian formalism—the main theoretical formalism used to analyze post-Newtonian effects—had been constructed.

The studies in this direction made possible a further restriction of the class of viable theories of gravitation claiming to describe physical reality.

Following the discovery of the binary pulsar system PSR 1913 + 16 and in view of the possible existence of other such systems, the prospects of using the results of observations of them as a new experimental test for the various theories of gravitation have been considered in the literature. These systems are of interest for two main reasons. First, in the case of compact binary systems containing a pulsar statistical analysis of the pulsar radiation makes it possible to determine with high accuracy the parameters of the orbit of each of the components of the system. Second, the characteristics of compact binary systems (the masses of the components, which are comparable with the mass of the Sun, the dimensions of the orbits, which are of the order of the radius of the Sun, the short revolution periods, and the fairly large values of the eccentricity) make these systems most favorable objects for observing a number of small gravitational effects; in particular they make possible an indirect measurement of the energy loss of such a system through gravitational radiation.

In addition, there is a prospect of direct detection of gravitational waves emitted by binary systems. Then determination of the directional diagram and the spectral characteristics of the gravitational radiation extends still further the use of the results of observation of these systems as a decisive test for the majority of theories of gravitation.

To analyze the emission power of compact binary systems in the various theories of gravitation and make a comparison with the results of observations, Will[72] proposed the use of the following general expression for the energy loss of a binary system through gravitational radiation:

$$-\frac{dE}{dt}=\frac{8m^2M^2}{15R^4}\left[k_1v^2-k_2\left(\frac{R_\alpha v^\alpha}{R}\right)^2+\frac{5}{8}k_d(\Omega_1-\Omega_2)^2\right], \qquad (639)$$

where $k_1$ and $k_2$ are the Peters-Mathews coefficients, $k_d$ is the coefficient of dipole radiation, $m$ and $M$ are, respectively, the reduced and total mass of the system, $R$ is the distance between the bodies, $v$ and $R_\alpha v^\alpha/R$ are the total and radial relative velocities of the bodies of the system, and $\Omega_1$ and $\Omega_2$ are, as usual, determined by the relation (571).

In such an approach, each theory of gravitation will correspond to a certain set of values of the coefficients $k_1$, $k_2$, $k_d$, which characterize it in the approximation of weak gravitational waves to the same degree that the set of post-Newtonian parameters characterizes its post-Newtonian limit. And comparison of these coefficients with the coefficients found experimentally will make it possible to establish the correspondence between the predictions of each theory of gravitation and the results of observations.

As follows from the expression (639), the energy losses of a binary system on gravitational radiation do not in the general case have a positive sign: For $k_1<k_2$ or for $k_d<0$, the right-hand side of the relation (639) can become negative in the case of certain binary systems. Therefore, in the theories of gravitation for which $k_1<k_2$ or $k_d<0$ it is possible to have emission of gravitational waves carrying negative energy, something that is physically meaningless, and therefore such theories must be immediately rejected.

Once the values of the coefficients $k_1$, $k_2$, and $k_d$ have been found from the results of observations of binary systems of the type of PSR 1913 + 16, the requirements on the possible theories of gravitation will be greatly strengthened.

We shall determine the values of these coefficients in the RTG. For this, we consider two neutron stars moving in an orbit in the gravitational field which they produce, and we calculate the energy losses of this system through gravitational radiation. In accordance with the model of the system adopted in such cases, we shall assume that both stars are spherically symmetric and are static. In addition, we assume that the gravitational fields produced by them have values that permit the use of the post-Newtonian formalism for the determination of the motion of the bodies of the system.

Since the gravitational self-potential $U_s$ on the surface of each of the bodies in compact binary systems is appreciably greater than the potential $U_{\rm int}$ of the gravitational interaction, we shall in what follows assume that $\Omega \sim U_s \sim \varepsilon$, $U_{\rm int} \sim \varepsilon^2$. This means that the ratio of the characteristic dimension $L$ of each of the bodies to the distance $R$ between them must be equal to $\varepsilon$ in order of magnitude: $L/R \sim \varepsilon$.

It is convenient to write the gravitational field components (345) in the form

$$\Phi^{\alpha\beta}(r,\ t) = -\frac{2}{r}\left[P^\alpha_\nu P^\beta_\mu - \frac{1}{2}P^{\alpha\beta}P_{\mu\nu}\right]\ddot{E}^{\mu\nu}(r,\ t), \qquad (640)$$

where $r$ is the distance from the center of mass of the binary system to the point of observation, and

$$E^{\mu\nu} = \int dV\left(X^\nu X^\mu - \frac{1}{3}\gamma^{\mu\nu}X_\alpha X^\alpha\right)[q]_{\rm ret}. \qquad (641)$$

Here

$$q = T^{00} + 2n_\alpha T^{0\alpha} + n_\alpha n_\beta T^{\alpha\beta}, \text{ and } n^\alpha = X^\alpha/|\mathbf{X}|. \qquad (642)$$

We shall find the post-Newtonian expansion of $q$. Using the expansion (592) for the energy-momentum tensor, we obtain

$$q = \hat{\rho}\left[1 + \frac{1}{2}v^2 + \Pi + U + \frac{p}{\hat{\rho}} + 2n_\nu v^\nu + (n_\nu v^\nu)^2 + O(\varepsilon^4)\right]. \qquad (643)$$

Since the expression (641) contains the retarded value of (643), we must, taking into account the estimate $v \sim \varepsilon$, expand $[q]_{\rm ret}$ in the neighborhood of the retarded time $t' = t - r$:

$$[q]_{\rm ret} = q(t') - \dot{q}(t')\, n_\nu X^\nu + \frac{1}{2}\ddot{q}(t')\,(n_\nu X^\nu)^2 + \rho O(\varepsilon^4),$$

where $X^\nu$ is the radius vector of the points of the bodies in the frame of reference associated with the center of mass of the binary system. It follows from the expressions (643) and (595) that

$$[q]_{\rm ret} = \hat{\rho}\left[1 + \frac{1}{2}v^2 + \Pi + U + \frac{p}{\hat{\rho}} + 2n_\nu v^\nu + (n_\nu v^\nu)^2\right.$$
$$\left. + 2n_\beta X^\beta n_\nu \partial^\nu U\right] + n_\nu X^\nu \partial_\beta(\hat{\rho}v^\beta) + \frac{1}{2}(n_\nu X^\nu)^2\partial_\beta\partial_\alpha(\hat{\rho}v^\alpha v^\beta)$$
$$- \frac{1}{2}(n_\nu X^\nu)^2\partial_\beta\left[-\hat{\rho}\partial^\beta U + \partial^\beta p\right]$$
$$+ 2n_\nu n_\beta X^\beta\partial_\alpha(\hat{\rho}v^\alpha v^\nu) - 2n_\beta X^\beta n_\nu\partial^\nu p + \hat{\rho}O(\varepsilon^3). \qquad (644)$$

Substituting the relation (644) in the expression (641), integrating it, and bearing in mind that for static spherically symmetric bodies

$$P_{(i)} = \frac{1}{3}\Omega_{(i)},\quad \Omega^{\alpha\beta}_{(i)} = \frac{1}{3}\gamma^{\alpha\beta}\Omega_{(i)},$$

we obtain in the relative variables (624)–(626)

$$E^{\alpha\beta} = m\left[1 + \frac{4}{3}\left(\frac{M_2\Omega_1 + M_1\Omega_2}{M}\right)\right]$$
$$\times R^\alpha R^\beta\left\{1 + \left(1 - \frac{3m}{M}\right)\frac{v^2}{2} + \frac{M_2 - M_1}{M}n_\nu v^\nu\right.$$
$$\left. + \left(1 - \frac{2m}{M}\right)\frac{M}{R}\right\}$$
$$+ m\left(1 - \frac{3m}{M}\right)v^\alpha v^\beta(n_\nu R^\nu)^2$$
$$+ m\frac{M_1 - M_2}{M}n_\nu R^\nu(v^\alpha R^\beta + v^\beta R^\alpha) + mR^2O(\varepsilon^3), \qquad (645)$$

where

$$M = M_1 + M_2;\ m = M_1M_2/M.$$

Using the Newtonian energy integral

$$\frac{v^2}{2} - \frac{M}{R} = E = \text{const} + O(\varepsilon^4) \qquad (646)$$

and introducing the notation

$$\tilde{m} = m\left[1 + \left(1 - \frac{3m}{M}\right)E + \frac{4}{3}\left(\frac{M_2\Omega_1 + M_1\Omega_2}{M}\right)\right], \qquad (647)$$

it will be convenient for what follows to write the expression (645) in the form

$$E^{\alpha\beta} = \tilde{m}R^\alpha R^\beta\left\{1 + \left[2 - \frac{5m}{M}\right]\frac{M}{R} + \frac{M_2 - M_1}{M}n_\nu v^\nu\right\}$$
$$+ m\frac{(M_1 - M_2)}{M}n_\nu R^\nu(v^\alpha R^\beta + v^\beta R^\alpha)$$
$$+ m\left(1 - \frac{3m}{M}\right)v^\alpha v^\beta(n_\nu R^\nu)^2 + mR^2O(\varepsilon^3). \qquad (648)$$

To determine the components of the gravitational field, we differentiate the expression (648) twice with respect to the time and, taking into account the post-Newtonian equations of motion (629), substitute it in the relation (640). We obtain

$$\Phi^{\alpha\beta} = -\frac{2\tilde{m}}{r}\left[P^\alpha_\mu P^\beta_\varepsilon - \frac{1}{2}P^{\alpha\beta}P_{\mu\varepsilon}\right]$$
$$\times\left\{2v^\mu v^\varepsilon\left[1 + \left(2 - \frac{5m}{M}\right)\frac{M}{R} - 2\left(1 - \frac{3m}{M}\right)\frac{M}{R^3}(n_\nu R^\nu)^2\right.\right.$$
$$\left. + \left(1 - \frac{3m}{M}\right)(n_\nu v^\nu)^2 + \frac{M_1 - M_2}{M}n_\nu v^\nu\right]$$
$$+ \frac{M}{R^3}(R^\mu v^\varepsilon + R^\varepsilon v^\mu)$$
$$\times\left[\left(3 - \frac{10m}{M}\right)R_\nu v^\nu - 4\left(1 - \frac{3m}{M}\right)R_\nu v^\nu n_\delta v^\delta\right.$$
$$\left. - \frac{3(M_1 - M_2)}{M}n_\nu R^\nu - 3\left(1 - \frac{3m}{M}\right)(n_\nu R^\nu)^2\frac{R_\delta v^\delta}{R^2}\right]$$
$$+ \frac{MR^\mu R^\varepsilon}{R^3}\left[-2 + 3\left(2 + \frac{3m}{M}\right)\frac{M}{R} - \left(4 + \frac{m}{M}\right)v^2\right.$$
$$+ 6\left(1 - \frac{3m}{M}\right)\frac{(R_\nu v^\nu)^2}{R^2} + 2\left(1 - \frac{3m}{M}\right)M\frac{(n_\nu R^\nu)^2}{R^3}$$
$$\left.\left. - \frac{3(M_1 - M_2)}{M}\frac{R_\nu v^\nu n_\delta R^\delta}{R^2}\right] + O(\varepsilon^5)\right\}. \qquad (649)$$

To determine the coefficients $k_1$, $k_2$, and $k_d$ in the RTG, we must retain in the expression (649) only the terms having the order of magnitude $m\varepsilon^2/r$:

$$\Phi^{\alpha\beta} = -\frac{4m}{r}\left[P^{\alpha}_{\tau}P^{\beta}_{\sigma} - \frac{1}{2}p^{\alpha\beta}P_{\tau\sigma}\right] \times \left\{v^{\tau}v^{\sigma} - \frac{MR^{\tau}R^{\sigma}}{R^3} + O(\varepsilon^3)\right\}.$$

Differentiating this equation with respect to the time and noting that in the TT gauge $\Phi^{mn} = -h^{mn}$ [see (356)], we obtain after substitution in the relation (367) the following expression for the radiation intensity of the gravitational waves of a compact binary system in the RTG:

$$\frac{dI}{d\Omega} = \frac{m^2M^2}{\pi R^6}\Big\{4v^2R^2 - 4R^2(n_\nu v^\nu)^2 - 4v^2(n_\nu R^\nu)^2 + 4(n_\nu v^\nu)^2(n_\beta R^\beta)^2 - \frac{15}{4}(R_\nu v^\nu)^2 - 6R_\nu v^\nu n_\beta v^\beta R_\alpha n^\alpha + \frac{6}{R^2}R_\nu v^\nu n_\beta v^\beta(R_\alpha n^\alpha)^3 + \frac{3}{2R^2}(R_\nu v^\nu)^2(R_\alpha n^\alpha)^2 + \frac{9}{4R^4}(R_\nu v^\nu)^2(R_\alpha n^\alpha)^4\Big\}.$$

Integrating this relation over the solid angle and bearing in mind that

$$\left.\begin{aligned} &\int d\Omega n^\alpha n^\beta = -\frac{4\pi}{3}\gamma^{\alpha\beta};\\ &\int d\Omega n^\alpha n^\beta n^\nu n^\delta = \frac{4\pi}{15}[\gamma^{\alpha\beta}\gamma^{\nu\delta} + \gamma^{\alpha\nu}\gamma^{\beta\delta} + \gamma^{\alpha\delta}\gamma^{\beta\nu}];\\ &\int d\Omega n^\alpha n^\beta n^\nu = 0, \end{aligned}\right\}$$

we obtain an expression for the energy loss of the compact binary system through gravitational radiation:

$$-\frac{dE}{dt} = \frac{8}{15}\frac{m^2M^2}{R^6}[12v^2R^2 - 11(R_\nu v^\nu)^2]. \qquad (650)$$

Comparing the expressions (639) and (650), we obtain $k_1 = 12$, $k_2 = 11$, $k_d = 0$.

Thus, in the RTG there is no dipole gravitational radiation, and the Peters-Mathews coefficients are in agreement with the results of observation of the binary pulsar system PSR 1913 + 16.

We note that for the energy loss of a compact binary system in Cartesian coordinates in GR we also arrive at Eq. (650), since in these coordinates the expression for Einstein's energy-momentum pseudotensor is identical to the energy-momentum tensor of the gravitational field in the RTG. However, on the transition to other admissible coordinates this equality no longer holds, and therefore the expression (650) for the energy loss does not follow from GR.

Indeed, of what formula for the radiation in GR can we speak if the gravitational field can be annihilated in GR by the choice of an admissible frame of reference? We are here dealing with one of the deepest and most fundamental delusions in theoretical physics. This is evidently due to the fact that dogmatism and belief have penetrated so deeply into GR and have become so strongly rooted there that Einstein's ideas have for long not received the critical analysis and necessary creative development. But there lacks but a little time and this dogmatism will become a thing of history.

## 22. GAUGE TRANSFORMATION AND UNIQUENESS OF THE RTG LAGRANGIAN

We shall formulate a gauge principle and on its basis construct the Lagrangian density of the gravitational field in the RTG. The general route to the construction of the theory proposed here leads us to the Lagrangian density (140) (see below). In addition, the following important fact will be established: The Riemannian geometry of space–time is uniquely determined only in the presence of matter; otherwise, because of the gauge freedom, it cannot in principle be fixed by gravitational fields.

We introduce a gauge transformation of the gravitational field:

$$\delta_\varepsilon\tilde{\Phi}^{mn} = \delta_\varepsilon\tilde{g}^{mn} = \tilde{g}^{ml}D_l\varepsilon^n + \tilde{g}^{nl}D_l\varepsilon^m - D_l(\varepsilon^l\tilde{g}^{mn}). \qquad (651)$$

We note that the gauge transformation for the field $\tilde{\Phi}^{mn}$ is quite different from its coordinate transformation (85). Therefore, the gauge transformations (651) are hypercoordinate transformations. One can show that the operators $\delta_\varepsilon$ determined by (651) form a Lie algebra. It is easy to show that under the gauge transformation (651) the Lagrangian density (140) changes in accordance with the law

$$L_g \to L_g + D_kQ^k(x), \qquad (652)$$

where

$$Q^k(x) = -\varepsilon^kL_g - \frac{1}{16\pi}[D_n\varepsilon^pD_p\tilde{g}^{nk} + \varepsilon^kD_nD_p\tilde{g}^{np} - D_p(\varepsilon^pD_n\tilde{g}^{nk})]. \qquad (653)$$

The requirement that under the gauge transformation (651) the Lagrangian density of the gravitational field change only by a divergence can be advanced as a *gauge principle*. The RTG Lagrangian (140) satisfies this principle. We shall show below that if the original requirements of the RTG are satisfied, then the gauge principle advanced here leads uniquely, when applied to Lagrangian densities of general form but quadratic in the first derivatives $D_p\tilde{g}^{mn}$, to the Lagrangian (140).

The general relation that must be satisfied by a Lagrangian density which transforms in accordance with the gauge principle can be obtained from the condition of vanishing of the action under a variation (651) of the field $\tilde{\Phi}^{mn}$. It has the form[73]

$$\nabla_l\left(2\frac{\delta L_g}{\delta g_{il}}\right) \equiv 0. \qquad (654)$$

The identity (654) reflects a requirement on the structure of the Lagrangian density $L_g$ of the gravitational field. It is easy to show that this identity is satisfied by any scalar density which depends only on $g_{ik}$ and its derivatives. The simplest densities of such form are

$$L_g = \sqrt{-g} \qquad (655)$$

and

$$L_g = \sqrt{-g}R, \qquad (656)$$

where $R$ is the scalar density of the Riemannian space–time. One can show that under the transformation (651) the expressions (655) and (656) change, respectively, in accordance with the law

$$\sqrt{-g} \to \sqrt{-g} - D_k(\varepsilon^k \sqrt{-g}); \quad (657)$$

$$\sqrt{-g}R \to \sqrt{-g}R - D_k(\varepsilon^k \sqrt{-g}R). \quad (658)$$

By virtue of Eq. (103), the choice of a Lagrangian density that depends only on $g_{ik}$ and its derivatives does not satisfy our original requirements, since in this case the gravitational field will not be a field of Faraday-Maxwell type. Therefore, in accordance with our ideas it is necessary to construct a Lagrangian density that depends on both $g_{ik}$ and $\gamma_{ik}$ and their first derivatives. It can be shown that such a solution exists and is unique.

From the requirement of relativistic invariance, the total Lagrangian density of the gravitational field, quadratic in the first derivatives $D_p\tilde{g}^{mn}$, can be represented in the form[7]

$$\mathscr{L}_g = \sum_{i=1}^{5} a_i L_i + \sum_{j=1}^{6} L_{gj}, \quad (659)$$

where

$$\left.\begin{aligned} &L_1 = \tilde{g}_{hq} D_m \tilde{g}^{pq} D_p \tilde{g}^{mh}; \quad L_2 = \tilde{g}^{lp} D_l \tilde{g}^{mn} D_p \tilde{g}_{mn}; \\ &L_3 = \tilde{g}_{km} \tilde{g}_{nq} \tilde{g}^{lp} D_l \tilde{g}^{hm} D_p \tilde{g}^{nq}; \quad L_4 = \tilde{g}_{hm} D_p \tilde{g}^{ph} D_n \tilde{g}^{mn}; \\ &L_5 = \tilde{g}_{mn} D_p \tilde{g}^{pk} D_k \tilde{g}^{mn}; \end{aligned}\right\} \quad (660)$$

$$L_{g6} = b_6 \gamma_{mk} D_q \tilde{g}^{kq} D_n \tilde{g}^{mn} + c_6 \gamma_{mn} D_p \tilde{g}^{pq} D_q \tilde{g}^{mn}, \quad (661)$$

and the Lagrangian densities $L_{gj}$ $(j=1,...,5)$ have the form (188)–(192); $a_i$, $b_j$, and $c_j$ are arbitrary numbers.

Under the gauge transformation (651), the change in the Lagrangian densities (660) can be represented in the form

$$\delta_\varepsilon L_i = D_k Q^k_{(i)} + \varepsilon^k(x)(\alpha_k^{(i)} + \beta_k^{(i)} + \sigma_k^{(i)}); \quad i=1, \ldots, 5, \quad (662)$$

and the change in the Lagrangian densities (188)–(192) and (661) in the form

$$\delta_\varepsilon L_{gj} = D_k \theta^k_{(j)} + \varepsilon^k(x)\,[b_j(U_k^{(j)} + V_k^{(j)} + W_k^{(j)}) + c_j(X_k^{(j)} + Y_k^{(j)} + Z_k^{(j)})]; \quad j=1, 2, \ldots, 6. \quad (663)$$

In (662), the structures $\alpha_k^{(i)}$ are equal to[73]

$$\left.\begin{aligned} &\alpha_k^{(1)} = 2\tilde{g}^{pl} g_{hm} D_l D_n D_p \tilde{g}^{mn}; \\ &\alpha_k^{(2)} = -2\alpha_k^{(1)} - \frac{1}{2}\alpha_k^{(3)}; \\ &\alpha_k^{(3)} = -4\tilde{g}^{pl} \tilde{g}_{mn} D_k D_p D_l \tilde{g}^{mn}; \\ &\alpha_k^{(4)} = \alpha_k^{(1)}; \\ &\alpha_k^{(5)} = -\frac{1}{4}\alpha_k^{(3)} - 2 D_k D_l D_p \tilde{g}^{pl}. \end{aligned}\right\} \quad (664)$$

We do not give the remaining structures here because the expressions for them are cumbersome. We note only that $U_k^{(j)}$ and $X_k^{(j)}$ contain covariant third-order derivatives of $\tilde{g}^{mn}$; $\beta_k^{(i)}$, $V_k^{(j)}$, and $Y_k^{(j)}$ contain second and first derivatives, and, finally, the structures $\sigma_k^{(i)}$, $W_k^{(j)}$, and $Z_k^{(j)}$ contain derivatives of only first order.

On the basis of (662) and (663), we obtain from (659)

$$\delta_\varepsilon \mathscr{L}_g = D_k\left[\sum_{i=1}^{5} a_i Q^k_{(i)} + \sum_{j=1}^{6} \theta^k_{(j)}\right] + \varepsilon^k(x)\left\{\sum_{i=1}^{5} a_i(\alpha_k^{(i)} + \beta_k^{(i)} + \sigma_k^{(i)}) + \sum_{j=1}^{6}[b_j(U_k^{(j)} + V_k^{(j)} + W_k^{(j)}) + c_j(X_k^{(j)} + Y_k^{(j)} + Z_k^{(j)})]\right\}. \quad (665)$$

From this it can be seen that a necessary and sufficient condition for the Lagrangian density $\mathscr{L}_g$ to change under the gauge transformation (651) by only a divergence is the identical vanishing of the expression

$$\sum_{i=1}^{5} a_i(\alpha_k^{(i)} + \beta_k^{(i)} + \sigma_k^{(i)}) + \sum_{j=1}^{6} b_j(U_k^{(j)} + V_k^{(j)} + W_k^{(j)}) + c_j(X_k^{(j)} + Y_k^{(j)} + Z_k^{(j)}) \equiv 0. \quad (666)$$

Since the groups of structures $(\alpha_k^{(i)}, U_k^{(j)}, X_k^{(j)})$, $(\beta_k^{(i)}, V_k^{(j)}, Y_k^{(j)})$, and $(\sigma_k^{(i)}, W_k^{(j)}, Z_k^{(j)})$ differ from one another in containing different orders of derivatives of $\tilde{g}^{mn}$, separate identities follow from (666):

$$\sum_{i=1}^{5} a_i \alpha_k^{(i)} + \sum_{j=1}^{6} (b_j U_k^{(j)} + c_j X_k^{(j)}) \equiv 0; \quad (667)$$

$$\sum_{i=1}^{5} a_i \beta_k^{(i)} + \sum_{j=1}^{6} (b_j V_k^{(j)} + c_j Y_k^{(j)}) \equiv 0; \quad (668)$$

$$\sum_{i=1}^{5} a_i \sigma_k^{(i)} + \sum_{j=1}^{6} (b_j W_k^{(j)} + c_j Z_k^{(j)}) \equiv 0. \quad (669)$$

The structures $U_k^{(j)}$ and $X_k^{(j)}$ contain the metric tensor of the Minkowski space,[73] but $\alpha_k^{(i)}$ do not contain it, and therefore, by virtue of the absence of possible compensation of the terms of these structures, the identity (667) decomposes into two identities:

$$\sum_{i=1}^{5} a_i \alpha_k^{(i)} \equiv 0; \quad (670)$$

$$\sum_{j=1}^{6} (b_j U_k^{(j)} + c_j X_k^{(j)}) \equiv 0. \quad (671)$$

Taking into account (664), we obtain from (670)

$$\alpha_k^{(1)}(a_1 - 2a_2 + a_4) + \alpha_k^{(3)}\left(a_3 - \frac{1}{2}a_2\right) + a_5\alpha_k^{(5)} \equiv 0.$$

Since the structures $\alpha_k^{(1)}$, $\alpha_k^{(3)}$, and $\alpha_k^{(5)}$ are independent, it follows from this that

$$a_1 - 2a_2 + a_4 = 0, \quad a_3 - \frac{1}{2}a_2 = 0, \quad a_5 = 0. \quad (672)$$

In Ref. 73, it was shown that the structures $U_k^{(j)}$ and $X_k^{(j)}$ $(j=1,...,6)$ are also independent, and from (671) we therefore obtain

$$b_j = c_j = 0 \quad (j=1, \ldots, 6). \quad (673)$$

On the basis of (672) and (673), we obtain from the identities (668) and (669)

$$a_1\left(\beta_k^{(1)} + \frac{1}{2}\beta_k^{(2)} + \frac{1}{4}\beta_k^{(3)}\right) + a_4\left(\frac{1}{2}\beta_k^{(2)} + \frac{1}{4}\beta_k^{(3)} + \beta_k^{(4)}\right) \equiv 0; \quad (674)$$

$$a_1\left(\sigma_k^{(1)} + \frac{1}{2}\sigma_k^{(2)} + \frac{1}{4}\sigma_k^{(3)}\right) + a_4\left(\frac{1}{2}\sigma_k^{(2)} + \frac{1}{4}\sigma_k^{(3)} + \sigma_k^{(4)}\right) \equiv 0. \quad (675)$$

Since we have the identities[73]

$$\beta_k^{(1)} + \frac{1}{2}\beta_k^{(2)} + \frac{1}{4}\beta_k^{(3)} \equiv 0; \quad (676)$$

$$\sigma_k^{(1)}+\frac{1}{2}\sigma_k^{(2)}+\frac{1}{4}\sigma_k^{(3)}\equiv 0, \qquad (677)$$

the coefficient $a_1$ in (674) and (675) is not determined and is arbitrary. Its value must be found from the correspondence principle.

Using (677) in (675), we obtain

$$a_4(\sigma_k^{(4)}-\sigma_k^{(1)})\equiv 0. \qquad (678)$$

The expression $(\sigma_k^{(4)}-\sigma_k^{(1)})$ does not vanish identically,[73] and it therefore necessarily follows from (678) that

$$a_4\equiv 0.$$

Then from (672) we find

$$a_2=\frac{1}{2}a_1,\quad a_3=\frac{1}{4}a_1,\quad a_4=0,\quad a_5=0. \qquad (679)$$

Obviously, for the values of the coefficients (673) and (679) we obtain from (659)

$$\mathscr{L}_g=a_1\Big[\tilde{g}_{kq}D_m\tilde{g}^{pq}D_p\tilde{g}^{mk}+\frac{1}{2}\tilde{g}^{lp}D_l\tilde{g}^{mn}D_p\tilde{g}_{mn}+\frac{1}{4}\tilde{g}_{km}\tilde{g}_{nq}\tilde{g}^{lp}D_l\tilde{g}^{km}D_p\tilde{g}^{nq}\Big], \qquad (680)$$

and from (665) we have

$$\delta_\varepsilon\mathscr{L}_g=a_1D_k\Big(Q^k_{(1)}+\frac{1}{2}Q^k_{(2)}+\frac{1}{4}Q^k_{(3)}\Big). \qquad (681)$$

Choosing the coefficient $a_1$ on the basis of the correspondence principle,

$$a_1=-\frac{1}{32\pi},$$

we arrive at the RTG Lagrangian (140). One can show that

$$-\frac{1}{32\pi}\Big(Q^k_{(1)}+\frac{1}{2}Q^k_{(2)}+\frac{1}{4}Q^k_{(3)}\Big)=Q^k(x),$$

where $Q^k(x)$ is given by (653).

Thus, the gauge transformation (651) of the gravitational field has made it possible to establish uniquely the nature of the self-interaction of the field $\Phi^{ik}$ and the structure of the Lagrangian—results that were obtained in Secs. 8 and 10 on the basis of different arguments.

In the presence of matter, the RTG equations do not admit the gauge transformations (651). In this, they differ from the gauge transformations in electrodynamics, which also hold for interacting fields. In the absence of matter, the gauge transformations (651) do not change the gravitational field equations but lead to a change in the interval of the Riemannian space–time and, therefore, in its geometrical characteristics. One can show that

$$\left.\begin{aligned}&\delta_\varepsilon ds^2=dx^i\,dx^k\delta_\varepsilon g_{ik};\\&\delta_\varepsilon R_{ik}=-R_{il}D_k\varepsilon^l-R_{kl}D_i\varepsilon^l-\varepsilon^lD_lR_{ik};\\&\delta_\varepsilon R_{iklm}=-R_{qklm}D_i\varepsilon^q-R_{iqlm}D_k\varepsilon^q-R_{ikqm}D_l\varepsilon^q\\&\quad-R_{iklq}D_m\varepsilon^q-\varepsilon^qD_qD_{iklm}.\end{aligned}\right\} \qquad (682)$$

It is in this respect that we see the main difference between the transformations (651) and gauge invariance in electrodynamics, in which gauge transformations do not change the physically observable quantities. *The geometry of space–time is uniquely determined only in the presence of matter, since it is in this case that there is no gauge freedom.* In our theory, the equations of motion of the matter are consequences of the ten equations (130) for the gravitational field. If we were to restrict ourselves solely to the equations (130), then for the ten gravitational field variables $\Phi^{ik}$ and the four variables that characterize the matter we should have only ten equations. For completeness of the system, we require four further covariant field equations. Such equations were introduced in Sec. 8. They have the form (123) and determine the structure of the gravitational field as a field of Faraday-Maxwell type, possessing spins 2 and 0. Equations (123) are universal and bear no relation to the coordinate conditions of Fock and De Donder, since the choice of the coordinate system in the RTG is completely specified by the metric tensor $\gamma^{ik}$ of Minkowski space. For the gravitational field outside matter, Eqs. (123) restrict the class of possible gauge transformations, imposing on $\varepsilon^l(x)$ the condition

$$g^{mn}D_mD_n\varepsilon^l(x)=0. \qquad (683)$$

Thus, the gauge principle in conjunction with the concept of the gravitational field as a physical Faraday-Maxwell field possessing energy, momentum, and spins 2 and 0 leads uniquely to the system of RTG equations.

In conclusion, we note that the gauge freedom (682) considered in this section is also present in GR. Therefore, our conclusions about the extent to which the geometry is determined also apply to GR.

## APPENDIX

The system of equations (123) that we have postulated for the gravitational field is not a consequence of the principle of least action. Therefore, these equations, as universal equations, must be taken into consideration as "subsidiary conditions" when the principle of least action is applied. In other words, we must vary the action

$$J=\int L\,d^4x \qquad (A1)$$

with respect to the fields $\tilde{\Phi}^{mn}$ or, in view of the connection (121), with respect to $\tilde{g}^{mn}$ on the manifold determined by the system of equations (123).

It is well known that such a variational problem can be solved by the method of Lagrangian multipliers. The standard device of this method is to add to the Lagrangian $L$ in the integrand of (A1) a term of the form $\eta_mD_n\tilde{g}^{mn}$, where $\eta_m$ is a Lagrangian multiplier, and apply the principle of least action to the expression

$$J=\int(L+\eta_mD_n\tilde{g}^{mn})\,d^4x. \qquad (A2)$$

Since the variation of (A2) is made independently with respect to $\eta_m$ as well as all the components of the tensor density $\tilde{g}^{mn}$, we find

$$\frac{\delta L}{\delta\tilde{g}^{mn}}-\frac{1}{2}(D_n\eta_m+D_m\eta_n)=0, \qquad (A3)$$

$$D_n\tilde{g}^{mn}=0. \qquad (A4)$$

We make the further analysis for the example of the Lagrangian $L=L_g+L_M$, where $L_g$ is given by (140), and $L_M$ is the matter Lagrangian.

Calculating for $L+\eta_m D_n \tilde{g}^{mn}$ the total energy-momentum tensor $t^{mn}$ in accordance with (88) and (89), we obtain

$$t^{mn}=(-1/16\pi)J^{mn}+D_k\{[\tilde{g}^{kn}\gamma^{ml}+\tilde{g}^{km}\gamma^{nl}-\tilde{g}^{mn}\gamma^{kl}]\,\eta_l\}$$
$$+2\sqrt{-\gamma}\left(\gamma^{ml}\gamma^{nk}-\frac{1}{2}\gamma^{mn}\gamma^{lk}\right)\left[\frac{\delta L}{\delta\tilde{g}^{kl}}-\frac{1}{2}D_l\eta_k-\frac{1}{2}D_k\eta_l\right].$$

From this we see that if (A3) holds, then

$$t^{mn}=(-1/16\pi)\,J^{mn}+D_k\,[\eta_l\,(\tilde{g}^{kn}\gamma^{ml}+\tilde{g}^{km}\,\gamma^{nl}-\tilde{g}^{mn}\,\gamma^{k}{}_{l}]. \quad (A5)$$

Since

$$D_m t^{mn}=0,$$

we find from (A5), taking into account (A4),

$$\tilde{g}^{km}D_m D_k\eta^n=0.$$

Therefore, the Lagrangian multipliers $\eta^n$ can be taken to be equal to zero.

[1)] See also C. Møller, Atti del Convegno sulla Relativitá Generale: Problemi dell'Energia e Ondre Gravitazionali, Vol. II, Rome (1964), t. 1, p. 21.

[2)] The most general Lagrangian density of the gravitational field quadratic in the derivatives $D_m\tilde{g}^{lk}$ can be constructed by taking also contractions by means of the expressions $\gamma_{kp}\tilde{g}^{pl}\gamma_{ln}$, $\tilde{g}_{kp}\gamma^{pl}\tilde{g}_{ln}$, $\gamma^{kp}\tilde{g}_{pl}\gamma^{ln}$, etc. However, this does not introduce anything fundamentally new, and therefore we restrict ourselves to studying the Lagrangians (134) and (188)–(192).

[3)] *Translator's Note.* The Russian notation for the trigonometric, inverse trigonometric, hyperbolic functions, etc., is retained here and throughout the article in the displayed equations.

[4)] Terms whose divergences vanish identically by virtue of their structure do not lead to conservation laws for any quantities.

[5)] A similar estimate was found earlier on the basis of other arguments by F. M. Gomide and D. F. Kurdgelaidze (see F. M. Gomide, Nuovo Cimento **30**, 672 (1963); D. F. Kurdgelaidze, Zh. Eksp. Teor. Fiz. **47**, 2313 (1964) [Sov. Phys. JETP **20**, 1546 (1965)]).

[6)] See also D. L. Lee, A. P. Lightman, and W.-T. Ni, Phys. Rev. D **10**, 1685 (1974).

[7)] As in Sec. 10 (see footnote 2), for the contraction of indices we choose the minimal possible number of tensors: $\gamma_{mn}$ and $\tilde{g}_{mn}$.

*Note.* All paginations given in the body of the text refer to the Russian editions and translations.


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