Problems of pion-nucleus interactions at low energies

M. Kh. Khankhasaev

Joint Institute for Nuclear Research, Dubna

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The theory of pion scattering by nuclei at low energies (below 80 MeV) is reviewed. The experimental situation is discussed. The present status of the optical model of the pion-nucleus interaction is analyzed. The problem of taking into account the pion-absorption channel and its influence on elastic scattering are considered. Despite considerable progress in the development of formal aspects of the optical model, there are serious shortcomings in its practical applications. The importance of making the theory consistent with the general requirements that follow from unitarity is emphasized. A unitary approach to the description of low-energy pion-nucleus scattering based on evolution with respect to the coupling constant is presented.

INTRODUCTION

The interaction of low-energy pions[1)] with nuclei has been intensively studied in recent years at the existing meson factories (high-current proton accelerators). An advantage of low-energy pions compared with resonance pions is that their mean free path in nuclear matter is long compared with nuclear dimensions. They are therefore more sensitive to nuclear structure and can be used for its study. In particular, simultaneous study of the interaction of π^+ and π^- mesons with nuclei can give valuable information about the proton and neutron distributions.

Knowledge of the dynamics of the pion-nucleus interaction is needed for the analysis of numerous reactions accompanied by the production of pions. For example, an important part in the description of data on the photoproduction and electroproduction of pions on nuclei is played by allowance for their interaction in the final state.

If pions are to become a genuinely effective tool for obtaining new information about nuclei, a reliable theory of the pion-nucleus interaction is needed. Significant progress has been achieved in this direction, and ways of creating the quantitative theory have been outlined (see, for example, the reviews of Refs. 1–4 and the monographs of Refs. 5–7). However, the practical realization is still far from complete. The main difficulty is in the correct description of the channel of pion absorption in nuclear matter. It is in this channel that pions differ in principle from other particles such as electrons and nucleons generally employed to study nuclear structure.

As a rule, interactions between particles and nuclei are described by some formulation of the theory of multiple scattering, which is fundamentally a potential theory.[8,9] It is only in recent years that the manner in which the absorption channel must be taken into account in the framework of such a theory has become clear. A fairly full analysis has so far been made only for the case of πd scattering.[10–12] The exact integral equations of Faddeev for a three-particle system[13] make it possible to calculate correctly the potential part of the problem, and these equations, generalized to the case when the particle number is not conserved,[10–12] give a real picture of the influence of the absorption channel on the other reaction channels. For nuclei with $A \geqslant 3$, the integral equations become effectively useless from the point of view of numerical calculations. In this case, various forms of the optical model are used to sum the multiple-scattering series.[1–7] The optical potential is a complicated many-particle operator and is constructed approximately in terms of the two-particle pion-nucleon scattering amplitudes. Only in special cases (see Refs. 3 and 14) is it possible to establish the inelastic channels that correspond to the imaginary part of the potential obtained in this manner. Therefore, the phenomenological allowance for the absorption channel is not determined uniquely, and the parameters that determine the correction for the absorption depend on the adopted approximate scheme. It is therefore necessary to make the theory consistent with the general requirements that follow from unitarity. Only then can one obtain a correct picture of the part played by the absorption channel.

Much attention has recently been paid (see, for example, Refs. 3, 5, 15, and 16) to questions related to the consistency of the theory with the unitarity condition. The present review analyzes various theoretical schemes employed to describe pion-nucleus scattering at low energies from this point of view. In particular, there is a detailed exposition of the unitary variant of the theory of pion scattering by nuclei[16–19] based on the method of evolution of systems with respect to the coupling constant.[20]

In this review, we consider mainly studies that have concrete results on the description of elastic pion-nucleus scattering data at low energies. Therefore, there is no discussion here of the many studies (see Ref. 5) which have developed formal aspects of the theory and field-theory models. Section 1 analyzes the experimental situation in low-energy pion-nucleus scattering. Sections 2 and 3 discuss the problems of the optical model of pion-nucleus interaction and the predictive power of the theory. Sections 4 and 5 present the unitary approach to the description of pion elastic scattering by nuclei. In the conclusions (Sec. 6), the prospects for the theory of the low-energy pion-nucleus interaction are discussed.

1. SCATTERING OF LOW-ENERGY PIONS BY NUCLEI (EXPERIMENT)

We consider the experimental data of recent years on the measurement of the cross section of pion elastic scattering by nuclei at low energies. The systematic study has been

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made basically by three groups working at the meson factories TRIUMF (Canada) and LAMPF (USA). At TRIUMF, measurements have been made of the differential cross sections of π^+ scattering by the carbon ^{12}C nucleus at 30, 40, and 50 MeV,[21] π^- scattering by the $^{12,13}C$ and $^{16,18}O$ nuclei at 29 MeV, and also by the ^{208}Pb nucleus at 50 MeV.[22] Recently,[23] the cross section of π^+ scattering by ^{12}C was measured there at the hitherto record low energy 13.9 MeV.

At LAMPF, one of the groups[24] has measured the differential cross sections of π^+ scattering by a whole series of nuclei from lithium to iron at 50 MeV. The other group at LAMPF has measured the differential cross sections of π^+ scattering by the ^{12}C, ^{16}O, ^{40}Ca, ^{90}Zr, and ^{208}Pb nuclei at the pion energies 30, 40, and 50 MeV,[25] and has recently obtained data on the same series of nuclei at 20 MeV.[26] In Ref. 27, the same group presented data on π^+ and π^- scattering by ^{40}Ca at 64.8 MeV. A detailed discussion of the experimental data up to 1979 is given in the review of Ref. 28.

In the low-energy region in which we are interested, there are data on $\pi^\pm$ scattering by the $^{3,4}He$ isotopes at 24 (Ref. 29), 25 (Ref. 30), 51 (Refs. 30 and 31), and 68 and 75 MeV,[31,32] and also by the deuteron at 47.7 and 65 MeV.[33]

Thus, much material has now been accumulated on the elastic scattering of low-energy pions by a considerable number of nuclei from $A = 2$ to $A = 208$. The range of scattering angles of the measured cross sections is in the interval from 20 to 160°. The most complete set of data is that on π^+ scattering by ^{12}C, the cross section of which has been measured at energies from 14 to 50 MeV. There are more data on π^+ than π^- scattering, the reason being the larger cross section for π^- capture in nuclear matter. Nevertheless, there are now available simultaneous data for both π^+ and π^- at low energies for deuterium nuclei at 47.7 and 65 MeV,[33] the $^{3,4}He$ isotopes in the interval from 25 to 75 MeV,[31,32] the ^{12}C nucleus at 30 and 50 MeV,[21,22] the ^{40}Ca nucleus at 65 MeV,[27] and ^{208}Pb at 50 MeV.[21,22] Data of such type are of particular interest from the point of view of obtaining information about the proton and neutron distributions in nuclei, testing the charge symmetry of strong interactions, etc.

Figure 1 presents experimental data for π^+ scattering by ^{12}C as a function of the energy of the incident pion. It can be seen that with increasing energy the characteristic minimum at $\theta_{cms} \sim 60°$ due to interference between the pion-nucleon s and p waves becomes more pronounced. The cross section has a basically simple nature. As a function of A, the form of the cross section also changes little (Fig. 2). For nuclei with $A \gtrsim 40$ at pion energies around 50 MeV a second diffraction minimum appears, this shifting to smaller angles with increasing energy in accordance with the law $qR = \text{const}$, where q is the momentum transfer and R is the radius of the nucleus. The characteristic difference between the differential cross sections for π^+ and π^- is shown in Fig. 3 for $\pi^\pm$ scattering by ^{40}Ca at 65 MeV.[27]

Despite this simple behavior of the differential cross section of pion scattering by nuclei at low energies, its theoretical description has proved to be a difficult problem. For this reason, low-energy pion-nucleus scattering is regarded as a test of the various models of the pion-nucleus interaction. The first-order optical potential (Kisslinger, 1955)[34]

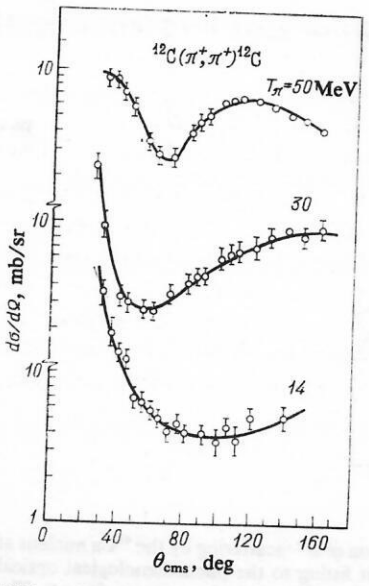

FIG. 1. Energy dependence of the differential cross section of π^+ elastic scattering by the ^{12}C nucleus. The data and curves (fitting to the phenomenological optical potential) are taken from Refs. 23 and 25.

obtained in the framework of multiple-scattering theory in terms of the free πN amplitudes proved to be inadequate for the description of low-energy pion-nucleus scattering (see, for example, Refs. 35 and 36). Thus, it was necessary to take into account the second-order effect in order to obtain at least a qualitative description. Very important in this energy range is the part played by the pion-absorption channel. Only when it is taken into account can one hope for a quantitative theory.

2. OPTICAL MODEL IN THE COORDINATE SPACE

The scattering problem and pionic atoms

When the importance of second-order effects was recognized, the idea arose of describing the scattering data at low energies phenomenologically by means of the optical

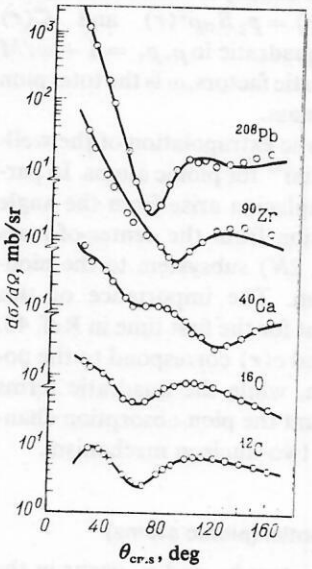

FIG. 2. The A dependence of the differential cross section of elastic pion-nucleus scattering at 50 MeV. The figure is taken from Ref. 25.

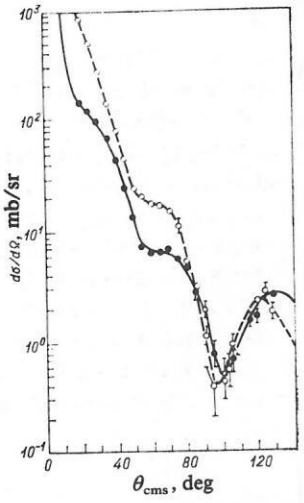

FIG. 3. Differential cross section of $\pi^{\pm}$ scattering by the ^{40}Ca nucleus at 65 MeV. The curves represent fitting to the phenomenological optical potential, the continuous one for π^+ and the broken one for π^-. The figure is taken from Ref. 27.

potentials used to describe the shifts and widths of pionic atoms.[36-41] The most complete investigation in this direction was made in Refs. 38 and 39. The potential proposed there for nuclei with zero spin and isospin has the form

$$2\omega U_{\text{opt}}(r) = -4\pi[b(r)+B(r)] + 4\pi\nabla\{L(r)[c(r)+C(r)]\}\nabla - 4\pi\left\{\frac{p_1-1}{2}\nabla^2 c(r) + \frac{p_2-1}{2}\nabla^2 C(r)\right\}, \quad (1)$$

where

$$L(r) = \left\{1 + \frac{4\pi}{3}\lambda[c(r)+C(r)]\right\}^{-1} \quad (2)$$

is the well-known[42] Lorentz-Lorenz correction (LL effect), $b(r) = p_1\bar{b}_0\rho(r)$ and $c(r) = c_0\rho(r)/p_1$ are the terms linear in the nuclear density $\rho(r)$ (the density is normalized by the number of nucleons), $B(r) = p_2 B_0\rho^2(r)$ and $C(r) = C_0\rho^2(r)/p_2$ are the terms quadratic in ρ, $p_1 = 1 + \omega/M$ and $p_2 = 1 + \omega/2M$ are kinematic factors, ω is the total pion energy, and M is the nucleon mass.

This potential is a kinematic extrapolation of the well-known Ericson-Ericson potential[42] for pionic atoms. In particular, the terms with the Laplacian arise from the angle transformation on the transition from the center-of-mass system of the (π, N) and $(\pi, 2N)$ subsystem to the pion-nucleus center-of-mass system. The importance of this transformation was pointed out for the first time in Ref. 43. The terms linear in ρ, $b(r)$, and $c(r)$ correspond to the potential part of the interaction, while the quadratic terms $B(r)$ and $C(r)$ take into account the pion-absorption channel under the assumption of a two-nucleon mechanism.

Parameters of the optical potential (pionic atoms)

The single-particle parameters b_0 and c_0 occur in the πN scattering amplitude

$$f_{\pi N} = b_0 + b_1(\mathbf{t}\cdot\boldsymbol{\tau}) + [c_0 + c_1(\mathbf{t}\cdot\boldsymbol{\tau})](\mathbf{k}\cdot\mathbf{k}') \quad (3)$$

and, therefore, can be obtained from the data of phase-shift analysis of πN scattering. Here, $\mathbf{t}$ and $\boldsymbol{\tau}/2$ are the isospin operators of the pion and nucleon, and $\mathbf{k}$ and $\mathbf{k}'$ are the pion momenta before and after scattering. It is known that at the threshold ($k\to 0$) the isoscalar scattering length is approximately zero, $b_0 \sim 0$. Therefore, the potential (1) contains the quantity[42]

$$\bar{b}_0 = b_0 + b^{(2)}, \quad b^{(2)} = -(b_0^2 + 2b_1^2)\langle 1/r\rangle, \quad (4)$$

which arises in the s-wave term when multiple scattering of the pion on a pair of nucleons is taken into account. The reciprocal correlation length of two nucleons is represented[38] by its expression in the Fermi-gas model: $\langle 1/r\rangle = 3k_F/2\pi$, where k_F is the Fermi momentum ($k_F = 1.4\ \text{F}^{-1}$).

The absorption-correction parameters B_0 and C_0, and also the parameter λ, which determines the strength of the LL effect, are usually regarded as free and are obtained by fitting to the data on pionic atoms (see the reviews of Refs. 1 and 44–46). Actually, the adjustable parameters also include $\bar{b}_0$, since a better description of the data is obtained at values of it much larger than suggested by (4). One of the parameter sets of the optical potential (1), obtained in Ref. 38 for the description of the shifts and widths of pionic atoms, is given in Table I (set 1). It differs only slightly from the standard set (see Ref. 2) on account of the terms proportional to $\nabla^2\rho$ contained in (1).

Unfortunately, it is not possible to determine the parameters of the optical potential uniquely from the data on pionic atoms. There is a correlation between the s-wave (Re $\bar{b}_0$, Re B_0) and p-wave (Re c_0, Re C_0) parameters.[38-40] Approximately constant are linear combinations of them[40]: Re b_{eff} = Re $\bar{b}_0 + \rho^s_{\text{av}}$ Re B_0 and Re c_{eff} = Re $c_0 + \rho^p_{\text{av}}$ Re C_0, where $\rho^s_{\text{av}} = \rho_0/2$ and $\rho^p_{\text{av}} = 3\rho_0/4$, and $\rho_0 = 0.15\mu^{-3}$ is the density of nuclear matter ($\mu^{-1} = 1.4$ F is the reciprocal pion mass). For the p-wave parameters, the situation is further complicated by the fact that they are also correlated with the parameter λ of the LL effect, and the value of this parameter is very uncertain.[1,3,4] In particular, it was noted in Refs. 1 and 40 that the data on pionic atoms do not give any indications of the existence of the LL effect, i.e., they are compatible with $\lambda = 0$. We recall that the classical value of this parameter[42] is 1. Thus, there is a whole set of phenomenological optical potentials that describe the pionic atoms equally well.

TABLE I. Parameters of the optical potential for pionic atoms.

Parameters	set 1 (Ref. 38)	set A (Ref. 39)
b_0, μ^{-1}	−0.028	−0.033
B_0, μ^{-4}	−0.043±i0.043	0.002±i0.048
c_0, μ^{-3}	0.266	0.234
C_0, μ^{-6}	−0.10+i0.10	0.036+i0.117
λ	1	1.4

Optical potential and theory (pionic atoms)

There have been several microscopic calculations[47–50] of the parameters B_0 and C_0 of the optical potential under the assumption of the two-nucleon absorption mechanism, for which these parameters can be expressed in terms of the diagrams that make the main contribution to the $\pi d \leftrightarrow NN$ process. The calculations are reviewed in Ref. 4 (see Sec. 4.2 there). The nucleon-nucleon correlations are calculated with allowance for the exchange of π and ρ mesons. The exchange of heavier mesons is simulated by the introduction of a repulsive core (core radius r_c). The corresponding matrix elements are calculated, as a rule, in the Fermi-gas model for infinite nuclear matter. The results of the calculations are very sensitive to the cutoff parameter r_c and to the parameters of the vertex functions. It should be noted that in such an approach an enhancement of the LL effect is predicted, this giving for the parameter λ a range of values from 1 to 2.[4,50] With regard to the numerical results for B_0 and C_0, a more or less stable value is obtained only for Im C_0: $(0.12–0.16)\mu^{-6}$, this being fairly close to the empirical value (see Table I). The value of Re C_0 depends strongly on the computational scheme, the value $0.036\mu^{-6}$ being obtained for it in Ref. 47 but $0.12\mu^{-6}$ in Ref. 48. It is interesting that Re C_0 and Im C_0 in the given model have the same sign. We recall that the standard choice gives Re $C_0 = -$ Im C_0.[1,44–46] This underlines the strong correlation between the parameters λ and Re C_0.

The s-wave parameters $\bar{b}_0$ and B_0 of the optical potential can be strongly influenced by the corrections associated with the influence of the medium on pion propagation.[49] Their values depend strongly on the parameter λ. Thus, the theoretical situation with regard to the parameters $\bar{b}_0$, B_0, and C_0 in the model of Ref. 4 is as yet uncertain.

In Ref. 39, the data on pionic atoms were fitted using the potential (1), in which some of the parameters were chosen in accordance with the theory of Refs. 47 and 48. Table I gives one of the sets (set A), in which the parameters λ, $\bar{b}_0$, Im B_0, and Im C_0 were regarded as free, while Re B_0 and Re C_0 were taken from Ref. 47. The two sets (set 1 and set A) describe the shifts and widths of the pionic atoms equally well.

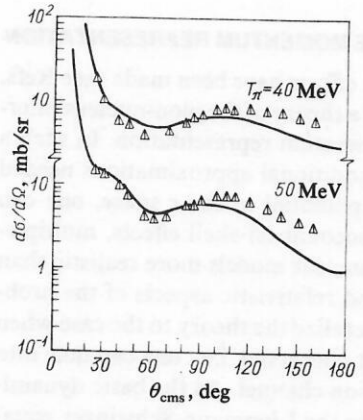

FIG. 4. Differential cross sections of π^+ scattering by the ^{16}O nucleus. The continuous curves represent the calculation with the optical potential (1) with the parameters of set 1 in Table II. The data are from Ref. 25, and the figure is taken from Ref. 38.

Low-energy pion-nucleus scattering

Can scattering data narrow the class of optical potentials equivalent from the point of view of pionic atoms? This question was investigated in Refs. 38–40 with different parameter sets (two of them given in Table I). No particular sensitivity was found—the differential cross sections hardly differ (see Fig. 3 in Ref. 39). Thus, in the scattering problem there is also a correlation between the parameters b_0 and B_0, and also c_0, C_0, and λ (see the detailed discussion in Ref. 40). However, the data at $T_\pi \sim 50$ MeV cannot be satisfactorily described with the optical-potential parameters determined from the data on the pionic atoms (Figs. 4 and 5); the dependence of these parameters on the energy must be taken into account.

Energy dependence of the optical-potential parameters

The energy dependence of the potential parameters $b_{0,1}$ and $c_{0,1}$ was taken into account in Refs. 38 and 39 by the introduction of the pion-nucleon amplitudes (3) in place of the πN scattering lengths. It was shown in Ref. 38 that Re $b_{0,1}$ and Re $c_{0,1}$ hardly change in the range of energies $T_\pi \sim 50$ MeV. However, the parameters $b_{0,1}$ and $c_{0,1}$ acquire imaginary parts corresponding to quasielastic pion scattering. To take into account the influence of the medium on this process, the imaginary parts of the phase shifts were multiplied by a Pauli suppression factor $Q_p(k)$, which was calculated in Ref. 51 for infinite nuclear matter. The imaginary part of the parameter $\bar{b}_0$ (4) is formed not only by the πN phase shifts, but also by the correlation quantity $\langle 1/r \rangle$,

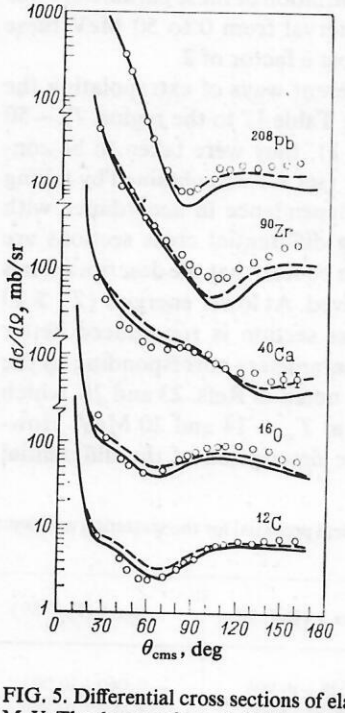

FIG. 5. Differential cross sections of elastic pion-nucleus scattering at 40 MeV. The data are from Ref. 25, and the curves represent the calculation with the optical potential (1), the continuous curves corresponding to the parameters of set A in Table I and the broken curves to set A in Table II. The figure is taken from Ref. 39.

which for low-energy pions in the Fermi-gas model has the form[36]

$$\langle 1/r\rangle = \frac{3k_F}{2\pi} + ik + O(k^2),$$

where k_F is the Fermi momentum, and k is the pion momentum in the pion-nucleus center-of-mass system. Note that when the imaginary parts of the parameters $\bar{b}_0$ and c_0 in (1) are determined in this manner the excitation spectrum of real finite nuclei is not taken into account. There is no correct quasi-two-particle limit of the theory, i.e., the imaginary parts do not vanish at pion energies below the threshold for excitation of the nuclear system.

The parameter λ arises in Ref. 42 through the short-range two-nucleon correlations. At $T_\pi \sim 50$ MeV, the pion momentum is $k \sim 0.6\ \mathrm{F}^{-1}$. Therefore, at core radius $r_c \sim 0.5$ F we have $kr_c \approx 0.3$, i.e., this parameter is fairly small at energies up to 50 MeV, and the dependence of λ on the energy can be ignored. For the same reason, the parameters B_0 and C_0 must depend weakly on the energy. Their energy dependence can be estimated more accurately (see Ref. 52) by considering the energy dependence of the cross section of the $\pi^+ d \leftrightarrow pp$ process:

$$\sigma(\pi^+ d \to pp) = \frac{2}{3}\left(\frac{p_p}{\mu}\right)^2 \left[\frac{\alpha_0}{\eta_\pi} + \alpha_1 + \alpha_2\eta_\pi + O(\eta_\pi^2)\right], \quad (5)$$

where $\eta_\pi = p_\pi/\mu$, p_π and p_p are the pion and proton momenta in the πd center-of-mass system, and $\alpha_{0,1,2}$ are energy-independent constants. It is found from this that B_0 and C_0 vary by only 33 and 21%, respectively, in the energy range $T_\pi = 0$–100 MeV. In Ref. 47, the energy dependence of Im B_0 and Im C_0 was calculated in the one-boson exchange model and a strong variation of these parameters was obtained, namely, in the interval from 0 to 50 MeV these parameters increase by almost a factor of 2.

Table II gives two different ways of extrapolating the parameters B_0 and C_0 (see Table I) to the region $T_\pi \sim 50$ MeV. In the first case (set 1), they were taken to be constant,[38] while the second set (set A) was obtained by taking into account their energy dependence in accordance with Ref. 47. The corresponding differential cross sections are shown in Figs. 4 and 5. It can be seen that the description has not been significantly improved. At lower energies ($T_\pi \lesssim 40$ MeV), the differential cross section is reproduced better with the optical-potential parameters corresponding to the pionic atoms. This was also noted in Refs. 23 and 26, which give data on πA scattering at $T_\pi = 14$ and 20 MeV. However, with increasing A the description of the differential cross sections at the given energy becomes less good (Fig. 5).

TABLE II. Parameters of the optical potential for the scattering problem: $T_\pi = 50$ MeV.

Parameters	set 1 (Ref. 38)	set A (Ref. 39)
b_0, μ^{-1}	−0.028+i0.004	−0.040+i0.004
B_0, μ^{-4}	−0.043+i0.043	−0.005+i0.063
c_0, μ^{-3}	0.266+i0.011	0.266+i0.011
C_0, μ^{-6}	−0.10+i0.10	0.045+i0.150
λ	1	1.4

Extremely sensitive to the imaginary part of the potential are the total scattering cross sections (σ_{tot}), the total reaction cross section (σ_r), the quasielastic scattering cross section (σ_{qe}), and the absorption cross section (σ_{abs}). These cross sections were calculated in Ref. 38 by the DWBA. In particular, for ($\pi^\pm$, ^{12}C) it was found that $\sigma_{abs}(\pi^+) = 126$ mb, while $\sigma_{abs}(\pi^-) = 159$ mb at 50 MeV. Recent measurements[53] of the total cross section of pion absorption by a number of nuclei at $T_\pi = 50$ MeV gave for ($\pi^\pm$, ^{12}C) the values $\sigma_{abs}(\pi^+) = 88 \pm 27$mb and $\sigma_{abs}(\pi^-) = 220 \pm 30$ mb.

The unsatisfactory situation at $T_\pi = 50$ MeV, and also the deteriorating description with increasing A at given energy, may arise not only from the procedure for extrapolating the parameters B_0 and C_0 responsible for the absorption to this region of energies. It is also necessary to take into account correctly the energy dependence of the potential parameters $\bar{b}_0$, c_0 and others, something that was done rather crudely in Refs. 38 and 39. Indeed, just as the real parts of the parameter pairs $\bar{b}_0$, B_0 and c_0, C_0 are correlated, so must their imaginary parts be,[40] since the differential cross sections are sensitive only to the total imaginary part of the optical potential.

Thus, at low energies it is necessary to take into account more correctly the potential part of the pion-nucleus interaction. In this connection, we must point out the important part played by off-shell effects in the construction of the optical potential (see Refs. 3 and 54). Potentials of the Kisslinger type ($\sim \nabla\rho\nabla$) correspond to the assumption of a zero range of the πN interaction, i.e., from the very beginning there is here an incorrect picture of the off-shell behavior of the πN scattering amplitude. Allowance for a finite range of the πN interaction ($r_{\pi N} \sim 0.3$–0.5 F) leads in particular to a significant weakening of the LL effect in nuclei.[55] It is also necessary to take into account the effects of the binding of the nucleons in nuclei in order to ensure the correct threshold behavior of the pion-nucleus scattering amplitude. None of this can be done at all easily if we remain in the coordinate space,[56] since the characteristic simplicity (locality) of the Kisslinger potentials is lost.

3. OPTICAL MODEL IN THE MOMENTUM REPRESENTATION

In recent years, great efforts have been made (see Refs. 1, 3, 5, and 6) to construct a theory of the pion-nucleus interaction directly in the momentum representation. In such a formalism, avoiding the additional approximations needed to obtain a local optical potential in the r space, one can systematically take into account off-shell effects, multiple-scattering effects (using nuclear models more realistic than the Fermi-gas model), and relativistic aspects of the problem, and one can also generalize the theory to the case when the particle number is not conserved, i.e., one can take into account the pion-absorption channel. As the basic dynamical equation, we have here the Lippmann-Schwinger equation for the many-particle T scattering matrix,[2)] on the basis of which the series for multiple scattering[8] of the pion on the nucleons of the nucleus is constructed. The optical model

arises as an effective method of summing this series when the elastic scattering amplitude is calculated. A detailed review of various versions of the optical model and its applications is given in Ref. 3. We here restrict ourselves to analyzing the problems that arise in the description of the scattering of low-energy pions by nuclei.

Potential pion-nucleus interaction

The theory of multiple scattering is a potential theory based on the Hamiltonian

$$H = K_\pi + H_A + V,\ V = \sum_{i=1}^{A} v^i, \tag{6}$$

where K_π is the operator of the pion kinetic energy, H_A is the Hamiltonian of the nucleus, and v^i is the potential of the interaction of the pion with nucleon i of the nucleus. The effects of the many-particle forces, which arise in any realistic theory of πA interaction (see, for example, Ref. 5) are here ignored.

The many-particle T matrix of pion-nucleus scattering is determined by means of the Lippmann-Schwinger equation

$$T = \Sigma v^i + \Sigma v^i G T, \tag{7}$$

where G is the Green's function

$$G(E) = (E + i\delta - K_\pi - H_A)^{-1}. \tag{8}$$

We introduce the auxiliary operator τ:

$$\tau_i = v_i + v_i G \hat{A} \tau_i, \tag{9}$$

which has the physical meaning of the t matrix of pion scattering by a bound nucleon of the nucleus. The operator $\hat{A}$ in (9) is responsible for antisymmetrization of the nuclear wave functions in the intermediate states. Eliminating the potentials v_i from Eqs. (7) and (9), we obtain the following expansion of the T matrix in powers (in collision multiplicities) of the τ matrices:

$$T(E) = \sum_i \tau_i(E) + \sum_{i \neq j} \tau_i(E) G \hat{A} \tau_j(E) + \ldots. \tag{10}$$

which is known as the Watson series.[8]

Optical model

For exact solution of the problem, there are two equivalent variants of the optical-potential method: the Watson formulation (W)[8] and the Kerman-McManus-Thaler formulation (KMT).[9] A detailed discussion of the difference between these approaches can be found in Refs. 1, 5, 14, and 57.

The equation for the T matrix of elastic pion-nucleus scattering in the W and KMT methods can be written[58] as follows:

$$T'(E) = U(E) + U(E) \hat{P} G(E) T'(E), \tag{11}$$

where $T' = (A - S)T/A$, $\hat{P} = |0\rangle\langle 0|$ is the operator of projection onto the ground state of the nucleus, $G(E)$ is the Green's function (8), and the parameter $S = 0$ corresponds to the W formulation, and $S = 1$ to the KMT formulation.

The optical potential $U(E)$ satisfies the equation

$$U(E) = (A - S)\tau(E) + (A - 1)\tau(E) G(E) \hat{Q} U(E). \tag{12}$$

Here, $\hat{Q} = \hat{A} - \hat{P}$ is the operator of projection onto the excited states of the nucleus, and the operator τ is determined by Eq. (9), where for $S = 0$ (W formulation) $\hat{A}$ must be replaced by the operator $\hat{Q}$. In writing down Eqs. (11) and (12), it is assumed that all the operators are defined on the space of antisymmetric nuclear states.

First-order optical potential

By the first-order optical potential one means the approximation linear in the τ matrix:

$$U^{(1)}(E) = (A - S)\tau(E). \tag{13}$$

In this approximation, the two approaches (W and KMT) are, as can be seen from (11) and (12), equivalent if the τ matrix is the exact solution of Eq. (9). In the usually employed impulse approximation,

$$\tau(E) \simeq t(\omega), \tag{14}$$

where $t(\omega)$ is the free πN scattering matrix, the equivalence is lost and the difference between W and KMT can be regarded as an indication of the importance of the second-order effects.

The τ matrix can be expressed more accurately in terms of the two-body t matrix as follows:

$$\tau(E) = t(\omega) + t(\omega)[\hat{Q}G(E) - g(\omega)]$$
$$\tau(E) + St(\omega)\hat{P}G(E)\tau(E). \tag{15}$$

Here $g(\omega)$ is the two-particle Green's function, S distinguishes the W and KMT approaches, and ω is the pion-nucleon collision energy, which is usually regarded as a parameter of the theory chosen on the basis of physical or mathematical considerations.

There exist different possibilities for choosing ω. In the spirit of the impulse approximation, the most natural is the so-called two-particle variant:

$$\omega = \omega_\pi(k) + E_N(p_i), \tag{16}$$

where k is the momentum of the incident pion, and p_i is the momentum of the nuclear nucleon. In this case, we have the half-off-shell matrix, and Eq. (15) takes the form ($S = 1$)[59]

$$\tau(E) = \sum_s \hat{p}_s [t(\omega_s) + t(\omega_s)(G(E) - g(\omega_s))\tau(E)], \tag{17}$$

where $\hat{p}_s = |\chi_s\rangle\langle\chi_s|$ is the operator of projection onto a definite state of the free Hamiltonian $H_0 = K_\pi + K_A$, and K_A is the kinetic-energy operator of the nucleons.

In the impulse approximation, the first-order potential has the form[52]

$$\langle \mathbf{k}' \mid U^{(1)}(E) \mid \mathbf{k} \rangle = (A-S)\,\rho_{00}(\mathbf{q}) \int dp F_{00}(\mathbf{p},\ \mathbf{p}) \times \langle \mathbf{k}',\ \mathbf{p}+\mathbf{p}_0-\mathbf{q} \mid t(\omega_0) \mid \mathbf{k},\ \mathbf{p}+\mathbf{p}_0 \rangle, \quad (18)$$

where $\mathbf{p}_0 = -\mathbf{k}/A + \mathbf{q}(A-1)/2A$, $\mathbf{q} = \mathbf{k}' - \mathbf{k}$, $\mathbf{k}$ and $\mathbf{k}'$ are the pion momenta before and after scattering, ρ_{00} is the nuclear form factor with parameters taken from electron scattering data, F_{00} is the single-particle density $\Sigma_\alpha \varphi_\alpha^2(p)$, φ_α is the single-particle wave function, and ω_0 is the collision energy of a pion with momentum $\mathbf{k}$ and a nucleon with momentum $\mathbf{p}+\mathbf{p}_0$. This expression is obtained in the factorization approximation and is well justified only for a wave function of the Gaussian form $\varphi_\alpha \sim \exp(-\alpha p^2)$. By averaging the two-particle t matrix of πN scattering in (18) by means of the single-particle density one can take into account the Fermi motion of the nuclear nucleons.

Angle transformation

The final aim is to express the optical potential in terms of the πN scattering phase shifts. For this, it is necessary to express the two-particle t matrix in (18), which is determined in the pion-nucleus center-of-mass system, in terms of the two-particle t matrix in the pion-nucleon center-of-mass system. In a fairly general form, this transformation can be expressed as

$$\langle \mathbf{k}',\ \mathbf{p}' \mid t(\omega) \mid \mathbf{k},\ \mathbf{p} \rangle = \gamma \langle \varkappa' \mid \tilde{t}(\tilde{\omega}) \mid \varkappa \rangle. \quad (19)$$

In the nonrelativistic case, the factor γ is unity, $\gamma = 1$, and the pion momenta $\varkappa$ and $\varkappa'$ in the pion-nucleon center-of-mass system can be expressed in terms of the corresponding quantities in the pion-nucleus center-of-mass system:

$$\varkappa = \bar{\mu}\left(\frac{\mathbf{k}}{\mu} - \frac{\mathbf{p}}{M}\right), \quad \varkappa' = \bar{\mu}\left(\frac{\mathbf{k}'}{\mu} - \frac{\mathbf{p}'}{M}\right),$$

where μ and M are the masses of the pion and the nucleon, and $\bar{\mu}$ is their reduced mass. The scattering angle in the pion-nucleon center-of-mass system can be expressed in terms of the scattering angle in the pion-nucleus system by means of the condition of invariance of the momentum transfer: $\mathbf{q}^2 = (\mathbf{k}'-\mathbf{k})^2 = (\varkappa - \varkappa')^2$.

In the case of pion-nucleus scattering, the pion cannot be regarded as nonrelativistic even in the low-energy region, since its kinetic energy is comparable with its mass when $T_\pi \sim 50$ MeV. There was therefore proposed[52] a simple generalization of the Lorentz transformation for the t matrix (see Ref. 8) to the off-shell case, the factor γ being chosen in the form

$$\gamma = [\omega_\pi(\varkappa)(\omega_\pi(\varkappa')E_N(\varkappa)E_N(\varkappa')/ (\omega_\pi(k)\,\omega_\pi(k')\,E_N(p)\,E_N(p'))], \quad (20)$$

where ω_π and E_N are the total energies of the pion and the nucleon, and the connection between the scattering angles was determined from the condition of invariance of the four-dimensional momentum transfer, $t = (k'-k)^2 = (\varkappa' - \varkappa)^2$, i.e.,

$$\varkappa\varkappa' = \omega_\pi(\varkappa)\,\omega_\pi(\varkappa') - \omega_\pi(k)\,\omega_\pi(k') + \mathbf{k}\cdot\mathbf{k}'. \quad (21)$$

The approximate nature of this transformation must be borne in mind. First, the transfer t is invariant only if the total 4-momentum of the πN subsystem is conserved in the scattering process. To be more correct, one must make the Lorentz transformation[61,63]

$$\left.\begin{aligned} &\varkappa = \mathbf{Q} - [(\mathbf{Q}\cdot\mathbf{K})/(\mathbf{K}_0(\mathbf{K}_0+\sqrt{s}))]\,\mathbf{K};\\ &2\mathbf{Q} = \mathbf{K} - \mathbf{P} - [(\mu^2 - M^2)/s]\,\mathbf{K};\\ &K = (K_0,\ \mathbf{K}) = (\omega_\pi(\mathbf{k}) + E_N(\mathbf{p}),\ \mathbf{k}+\mathbf{p}) \end{aligned}\right\} \quad (22)$$

separately for the initial and final states. Here, $s = (k+p)^2$; k and p are 4-momenta: $k = (\omega_\pi, \mathbf{k})$, $p = (E_N, \mathbf{p})$. The expression for the final momentum $\varkappa'$ is obtained by simple replacement: $k \to k'$, $p \to p'$, and s by $s' = (k'+p')^2$. However, as was shown in Ref. 3, in real calculations the transformations (21) and (22) lead to the same numerical results. Second, the transformation (19) for the off-shell t matrix does not reduce simply to the factor γ (see Ref. 63, and also Sec. 2.3 in Ref. 5). It has a more complicated form, differing by an additional kinematic factor as well as by an additional integral term that has a dynamical origin (for the off-shell continuation, it is necessary to use the equation for the t matrix). Thus, the expressions (19)–(21) must be regarded only as a certain approximation that is adequate in the nonrelativistic limit with respect to the nucleon. Allowance for the angle transformation is very important even at low energies. It significantly increases the s-wave repulsion due to the contribution from the P_{33} wave.

Summarizing, it can be said that the technique for calculating the first-order potential has been fairly well developed in the optical model in the p space.

At low energies, the first-order potential, like the Kisslinger potential (see Sec. 2), does not describe the scattering data satisfactorily. It is necessary to take into account second-order effects such as the corrections to the impulse approximation (correction for the binding energy of a nucleon in the nucleus), the corrections to coherent scattering [allowance for the second term in Eq. (12) for the optical potential], etc.

Three-particle model of the πA interaction

In order to take into account more accurately the nucleon binding energy in the nucleus, a three-particle model, in which the collision of the pion with the nuclear nucleon takes place in a certain average field formed by the remaining $A-1$ nucleons (core), was considered.[52] Such a generalization of multiple-scattering theory was proposed for the first time in Ref. 60. As in any three-body problem, the problem here arises of separating the disconnected diagrams corresponding to the free t_j^π matrix of πN scattering and the t_j^c matrix for scattering of nucleon j by the single-particle potential U_j^c.

For the optical potential of first order the expression obtained in this scheme has the form

$$U^{(1)} = (A-1)\frac{1}{A}\sum_j \tau_c^j. \quad (23)$$

Here, τ_c is the solution of the Faddeev equations

$$\left.\begin{aligned}\tau_c^j &= t_j^\pi + t_j^\pi G_0^j \hat{\tau}_c;\\ \hat{\tau}_c &= t_j^c G_0^j \tau_c^j,\end{aligned}\right\} \qquad (24)$$

where $G_0^j = (E + i\delta - K_\pi - K_j - K_c - H_c)^{-1}$ is the Green's function; K_π, K_j, and K_c are the kinetic-energy operators of, respectively, the pion, nucleon, and core, described by the Hamiltonian H_c, and $\hat{\tau}_c$ is an auxiliary operator.

Of course, the solving of the Faddeev equations and the calculation with them of the potential with which one must then solve the Lippmann-Schwinger equation in order to obtain the required scattering amplitude is a technically most complicated problem.

Therefore, in Ref. 52 only the first term in (24) was taken but the collision energy of the free πN amplitude was chosen in accordance with the three-particle nature of the model, i.e.,

$$\omega_3 = \omega_\pi(k) + k^2/2M_A - |B| - (p + p_0 + k)^2/2M + M, \qquad (25)$$

where the momentum p_0 is defined in (18). Here, the nucleons are regarded as nonrelativistic: M_A is the mass of the nucleus and $\bar{M}$ is the reduced mass of the πN system and the core. The term $-|B|$ corresponds to the nucleon binding energy. The difference between ω_3 and ω_2 (16) is most clearly seen by comparing them in the limit $A \gg 1$ and $k \sim 0$:

$$\omega_3 = \omega_2 - |B| - p^2/2M,$$

i.e., the collision energy of the two-particle πN system is shifted compared with ω_2 by the binding energy and the nucleon kinetic energy, i.e., by almost 25 MeV for light nuclei, where $\langle p^2|2M\rangle \sim 20$ MeV and $|B| \sim 5$ MeV. This leads to a very strong decrease in the cross section at energies $T_\pi \leqslant 70$ MeV. Therefore, invoking the important part played by pion-rescattering effects, which are not taken into account in the quoted study, the authors regard $|B|$ as a free parameter of the theory. This is unsatisfactory from the point of view of the unitarity condition, since $|B|$ determines the correct threshold behavior of the optical potential, determining the upper limit of the range of variation of the energy $\omega_3 = (-\infty, T_\pi - |B|)$, i.e., for $T_\pi < |B|$ the potential is Hermitian. Even at $|B| \sim 0$ the energy shift is still large, about 20 MeV. It is compensated by introducing corrections for Pauli blocking and absorption (see below). Doing this, the authors obtain a very good description of the scattering data at low energies (see, for example, Fig. 6). It should, however, be noted that the data of the calculations are extremely sensitive to the balance of various corrections, and also that there is a free parameter $|B|$ in the theory.

A series of studies[64] developed a covariant optical model for describing elastic pion-nucleus scattering based on the relativistic variant of multiple-scattering theory. In this approach, only the first order of the theory (triangle diagram) for the optical potential has been studied in any detail. The second-order effects are taken into account phenomenologically by the introduction of a correction to the potential, the parameters of the correction being determined by fitting the data. In its practical realization, this approach is very close to the three-particle model presented above.

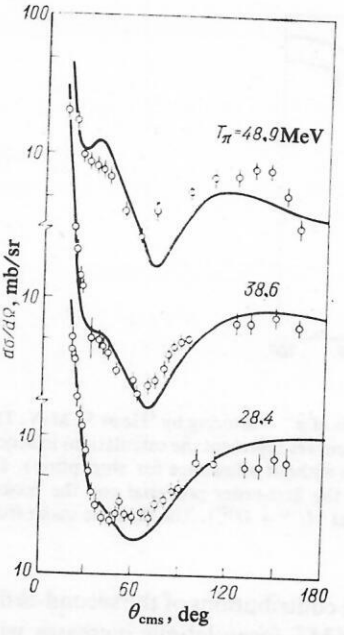

FIG. 6. Differential cross sections of elastic π^+ scattering by the ^{12}C nucleus. The data are taken from Ref. 21; the continuous curves represent the calculation in accordance with the optical model with the three-particle choice of the energy of the πN subsystem (25) and with allowance for the absorption channel. The figure is taken from Ref. 21.

Second-order potential

There have been very few second-order microscopic calculations[58,65]—such calculations are very laborious. Mainly, the second-order effects have been illustrated by the example of $\pi^{3,4}$He scattering. The general structure of the second-order potential in the impulse approximation and the completeness approximation has the form

$$\begin{aligned}&(2\pi)^3 \langle \mathbf{k}' | U^{(11)}(E) | \mathbf{k}\rangle\\ &= \int \frac{d\mathbf{k}''}{E - E(k'') + i\delta} \langle \varkappa' | t^1 | \varkappa''\rangle \langle \varkappa'' | t^2 | \varkappa\rangle\\ &\times [(A-1)(A-S) C_{00}(\mathbf{k}' - \mathbf{k}'', \mathbf{k}'' - \mathbf{k})\\ &- (A-S)^2 \rho_{00}(\mathbf{k}' - \mathbf{k}'') \rho_{00}(\mathbf{k}'' - \mathbf{k})],\end{aligned} \qquad (26)$$

where the indices 1 and 2 of the t matrices relate to the spin-isospin parts of these operators, averaging of which over the nuclear wave functions is understood in the given approximation; $\mathbf{k}$, $\mathbf{k}'$, and $\mathbf{k}''$ are the pion momenta in the pion-nucleus center-of-mass system, and $\varkappa$, $\varkappa'$, and $\varkappa''$ are the momenta in the pion-nucleon system. Further, $C_0(\mathbf{q}, \mathbf{q}')$ is the two-particle correlation function, and $\rho_{00}(\mathbf{q})$ is the form factor of the nucleus.

The expression (26) for the second-order potential is obtained by taking into account the correction to the impulse approximation [the second term in Eq. (15)] and to the coherent approximation [second term in Eq. (12)]. It was shown in Ref. 58 that these two corrections to a large degree cancel each other. The effect of the second-order potential on the differential cross section of π^4He scattering at 51 MeV is shown in Fig. 7, i.e., it is large. It was found that

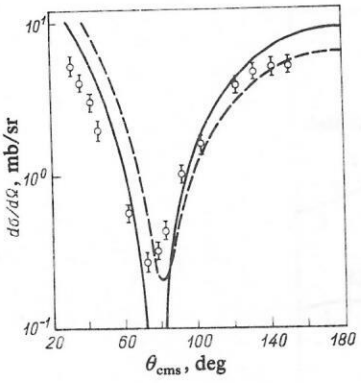

FIG. 7. Differential cross section of π^- scattering by ^{4}He at 51 MeV. The data are taken from Ref. 31; the curves represent the calculations in accordance with the optical model (without allowance for absorption), the continuous curve representing the first-order potential and the broken curve the second-order potential ($U^{(1)} + U^{(2)}$). The figure is taken from Ref. 58.

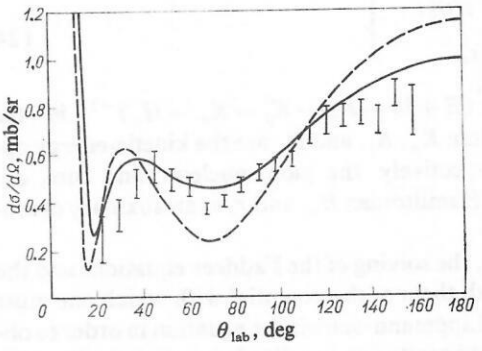

FIG. 8. Differential cross section of $\pi^+ d$ scattering at 47.7 MeV. The data are from Ref. 33, and the calculation is in accordance with the Faddeev equations, the broken curve corresponding to no allowance for the absorption channel and the continuous curve to the complete calculation. The figure is taken from Ref. 11.

the difference between the contributions of the second-order potential in the W and KMT formulations increases with decreasing energy, reducing at the same time the rather strong difference between these formulations in the first order of the theory. The results obtained in Refs. 58 and 65 agree qualitatively with one another. Quantitatively, their discrepancy in the contribution of the second-order potential is due mainly to the choice of the energy of the pion-nucleon t matrix. The two-particle variant (16) was used in Ref. 65, while in Ref. 58 use was made of an expression of the form

$$\omega = E - \frac{1}{8M}\frac{\bar{\mu}}{M}\frac{A-1}{A}(\mathbf{k}+\mathbf{k}')^2, \qquad (27)$$

where E is the total collision energy, and $\bar{\mu}$ and M are the reduced masses of the πN and πA systems. This expression is Galileo-invariant and effectively lowers the two-particle energy by 3–10 MeV compared with the two-particle choice. We note that the expression (27) does not ensure the correct threshold behavior of the optical potential.

Allowance for the pion-absorption channel

The main difficulty in the theory of the pion-nucleus interaction is the description of the channel of pion absorption in the nuclear medium. It is only recently that definite results have been achieved in this direction. A fairly full analysis has hitherto been made only for πd scattering on the basis of the Faddeev equations generalized to the case in which the particle number is not conserved.[10–12] The influence of the absorption channel on elastic πd scattering at $T = 47.7$ MeV is illustrated in Fig. 8. The formalism of Ref. 10 was generalized to the general case of pion-nucleus scattering in Ref. 66. A systematic exposition of the optical-potential method with allowance for the absorption channel is given in Ref. 5.

The expression for the T matrix of pion-nucleus scattering in this case has the form

$$T_{\pi A} = T_{\pi A}^{\text{pot}} + T_{\pi A}^{\text{abs}}, \quad T_{\pi A}^{\text{abs}} = \Omega_{\pi A}^{(-)}\Gamma_{10} g_A \Gamma_{01}\Omega_{\pi A}^{(+)}, \qquad (28)$$

where Γ_{10} and Γ_{01} are the many-particle vertex operators that determine the absorption and creation of a pion in the nuclear system, and g_A is the total Green's function of the system of interacting nucleons. The diagrammatic representation of these operators can be found in Refs. 5 and 66; $T_{\pi A}^{\text{pot}}$ is the scattering matrix that describes the purely potential interaction, i.e., all processes that conserve the pion in the intermediate states, and $\Omega_{\pi A}^{(-)}$ and $\Omega_{\pi A}^{(+)}$ are Møller operators (see Ref. 8) describing the distortion of the pion wave by the potential interaction. In the calculation of the potential quantities $T_{\pi A}^{\text{pot}}$ and $\Omega_{\pi A}^{(+)}$, it is necessary, in order to avoid double counting, to subtract the pole part in the p_{11} wave of the πN interaction, since it is precisely this term that is formed by the elementary (πN, N) vertex function, which is already included in the definition of the vertex operators Γ_{10} and Γ_{01}.

In the presence of the absorption channel, the optical potential has the form[5,66]

$$\left.\begin{aligned} U_0(E) &= U_0^{\text{pot}}(E) + U_0^{\text{abs}}(E); \\ U_0^{\text{abs}}(E) &= \hat{P}K(E)(1+g(E)K(E))^{-1}\hat{P}, \end{aligned}\right\} \qquad (29)$$

where

$$K(E) = \Omega_{\pi A}^{(-)}\hat{Q}\Gamma_{10} g_A(E)\Delta_{01}\hat{Q}\Omega_{\pi A}^{(+)};$$

the operators $\hat{P}$ and $\hat{Q}$ project onto the ground and excited states, respectively, of the nucleus, and $g(E)$ and $g_A(E)$ are the total Green's functions for the pion-nucleus and nucleon systems.

So far, no microscopic calculations of the optical potential have realized the rigorous formulation given above. The absorption channel is usually taken into account phenomenologically by means of an additional term in the optical potential, calculated in the framework of multiple-scattering theory as described above. The parameters of the correction for the absorption to describe the low-energy pion-nucleus scattering are taken from a fit to the data on pionic atoms. The expression for the correction for the absorption under the assumption of the two-particle absorption mechanism has the form

$$(2\pi)^3 U_0^{\text{abs}}(\mathbf{k}, \mathbf{k}'; E) = -\frac{2\pi}{M}A(A-1)\left\{B_0(E)\frac{g_0(k)\,g_0(k')}{g_0^3(k_0)}\right.$$

$$+C_0(E)\frac{g_1(k)\,g_1(k')}{g_1^2(k)}(\mathbf{k}\cdot\mathbf{k}')\Big\}\hat{\rho}^2(\mathbf{k}'-\mathbf{k}), \qquad (30)$$

where B_0 and C_0 are adjustable parameters having the same meaning as in the optical potential (1), $\hat{\rho}^2(q)$ is the Fourier transform of the square of the nuclear density $\rho^2(r)$, and $g_{0,1}$ are the form factors corresponding to the departure from the mass shell; the momentum k_0 corresponds to on-shell scattering, and the parameters of these form factors are taken to be the same as for the case of the off-shell t matrix of pion-nucleon scattering in the model with a separable interaction [see (31)].

The absorption channel strongly influences the elastic pion-nucleus scattering, and allowance for it by means of (30) significantly improves the description of the experimental data (see Refs. 52, 58, and 83, and also Figs. 6 and 11). One should, however, bear in mind a certain inconsistency in the existing calculations. For example, in Refs. 52 and 58 the potential part of the pion-nucleus interaction was calculated with different forms of the optical potential. Therefore, to determine the parameters of the correction for the absorption, one should first calculate the lengths and volumes of the pion-nucleus scattering and then, using the values obtained, fit to the data on pionic atoms. The values of the parameters B_0 and C_0 used in Refs. 52 and 58 were not made consistent in this manner.

Off-shell effects

As already noted (Sec. 2), the optical potentials of the Kisslinger type (1) correspond to the assumption of a zero range of the πN interaction. In Ref. 54, in which a separable model of the πN interaction was used, it was shown that such an assumption is very crude and that allowance for the finite range of the πN interaction significantly affects the results of the calculations.

The range of the πN interaction ($r_{\pi N}$) determines the off-shell behavior of the scattering matrix. This is most clearly seen in the model with a separable interaction, in which the off-shell t matrix is expressed in terms of the scattering t matrix by the simple relation

$$t_\alpha(k',\ k;\ E(k_0)) = t_\alpha(k_0,\ k_0;\ E(k_0))\,g_\alpha(k)\,g_\alpha(k')/g_\alpha^2(k_0). \qquad (31)$$

The index $\alpha\equiv(l,j,I)$ determines the partial wave of the πN system; l and j are the orbital and total angular momentum, and I is the isospin. The form factors g_α determine the πN potential

$$v_\alpha(k',\ k) = \sigma_\alpha g_\alpha(k')\,g_\alpha(k)$$

and are found by fitting to the data of πN phase-shift analysis. Their decrease in the r or p spaces gives us an estimate of the range of the πN interaction. A review of the specific expressions for the form factors $g_\alpha(k)$ and references to individual studies can be found in Ref. 3. In such potential models, the range is found to have a value $r_{\pi N}\approx 0.6$–0.8 F, i.e., to be comparable with the mean internucleon separation and the length of the short-range two-nucleon correlations ($r_c\sim 0.5$ F). A more detailed analysis of the πN interaction range gives $0.25\lesssim r_{\pi N}\lesssim 0.5$ F.[55]

Allowance for the finite range of the πN interaction leads (for a detailed discussion, see Ref. 1) to a significant suppression of the LL effect, making it practically unobservable in the analysis of data on pionic atoms. Therefore, in calculations of pion-nucleus observables using realistic models for the off-shell πN amplitude the short-range correlations leading to effects of the LL type are ignored. Since the range of the πN interaction is comparable with the length r_c of the short-range correlations, one must expect that at low energies ($T\lesssim 80$ MeV) there is a weak sensitivity to the definite value of $r_{\pi N}$, as is confirmed by numerical calculations.[54]

In this section, we have considered papers that gave definite results on the description of elastic pion-nucleus scattering data. It is evident from our exposition that, despite the significant progress in understanding the dynamics of the pion-nucleus interaction, the theory has difficulties in concrete applications. The description of the interaction of low-energy pions with nuclei has revealed the importance of the effects of second order relative to the lowest, fairly well-studied first-order optical potential. Many of the effects compensate each other to a considerable degree, and this makes the calculations highly sensitive to the various approximations employed in the calculations. Also unsolved is the problem of reconciling the optical potential with the unitarity condition, i.e., the requirement that the optical potential have the correct threshold behavior in the low-energy limit.

Most of the calculations in the optical model were made in the KMT formalism.[9] Analysis of the general unitarity condition shows[5,14,57] that in this formulation, in contrast to the Watson formulation,[8] the contribution of the elastic channel in the unitarity relation is not separated from the contribution of the inelastic processes. Therefore, in the KMT approach it is difficult, in an approximate solution to the problem, to determine the channels that are responsible for forming the imaginary part of the optical potential. Thus, for the description of the pion-nucleus interaction the Watson formulation is preferable. This is important for the consistent allowance for the channel with true pion absorption. The problem of microscopic calculation of the correction for the absorption in accordance with the program determined by Eqs. (28) and (29) still awaits its solution.

4. UNITARY THEORY OF PION-NUCLEUS SCATTERING

In the cases when the lowest approximation of the theory is insufficient and the problem of taking into account the higher corrections becomes acute, methods that from the very beginning are consistent with the unitarity condition are very helpful from the practical point of view. The unitarization method is successfully used to describe nucleon-nucleus scattering at low energies (see the review of Ref. 67). Figure 9, for the example of πd scattering at 14 MeV, shows the result[68] of unitarizing the impulse approximation in the framework of the R-matrix approach. Thus, a quantitative description can be obtained in the unitary scheme already in the lowest order. The unitarization mechanism consists of taking into account effectively the higher corrections—the

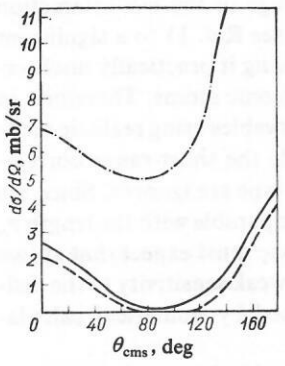

FIG. 9. Differential cross section of *nd* scattering at 14 MeV. The continuous curve represents the exact calculation in accordance with the Faddeev equations; the chain curve, the impulse approximation; and the broken curve, the unitarized impulse approximation. The figure is taken from Ref. 68.

so-called background diagrams, whose singularities are situated much further away than the nearest singularities (with respect to the energy and the scattering angle) of the lowest basic diagrams.

There is an analogous situation in the problem of pion-nucleus scattering at low energies, admittedly in a weaker form because of the relative weakness of the πN interaction. Indeed, the fact that, despite a strong quantitative discrepancy, the first-order optical potential reproduces qualitatively the differential cross section (see, for example, Fig. 7) as well as the energy dependence of the total cross section means that the analytic properties of the impulse graph (single scattering of a pion by a nucleon of the nucleus) correctly reproduce the analytic properties of the required amplitude with respect to the angle and energy variables. Pion-rescattering effects correspond to more complicated diagrams, whose singularities lie much further off. It can therefore be expected that the use of the unitarization method will significantly improve the convergence of the theory.

Agreement of the theory with the unitarity condition acquires particular importance in the case of the pion-nucleus interaction on account of the presence of the pion-absorption channel. It is only in the framework of such a scheme that one can obtain a correct picture of its influence on the other reaction channels.

Method of evolution with respect to the coupling constant (evolution method)

A simple and elegant unitary theory of multiple scattering can be constructed by the evolution method, which is based on the law of evolution of the system as the coupling constant is varied. The characteristic feature of this method is that the iterative scheme is constructed here directly for the calculation of the partial-wave phase shifts of particle scattering by a compound system. The evolution method was formulated as an independent approach by Kirzhnits in 1965.[20] A clear exposition of the essence of this method can be found in Refs. 69 and 70, where, in particular, some of its applications to problems of quantum field theory are considered. Below, we shall concentrate on the nonrelativistic variant of the method.

We give the basic equations of the evolution method applicable to the case when there is only one type of interaction (see Ref. 70). In this case, the Hamiltonian of the system is

$$H = H_0 + gV, \tag{32}$$

where H_0 is the free Hamiltonian, gV is the interaction, and g is the coupling constant. We shall denote the eigenfunctions of H by $|\mu\rangle$, $|\nu\rangle$, etc., and the matrix elements of the interaction potential taken between them by $V_{\mu\nu}$, etc. The system of equations for the S matrix of scattering from the state $|\mu\rangle$ to $|\nu\rangle$, $S_{\mu\nu}$, has the form

$$\frac{d}{dg} S_{\mu\nu} = -2\pi i \sum_{\sigma} S_{\mu\sigma} V_{\sigma\nu} \delta(E_\mu - E_\sigma),\ E_\mu = E_\nu; \tag{33}$$

$$\frac{d}{dg} V_{\mu\nu} = \sum_{\sigma} V_{\mu\sigma} V_{\sigma\nu} \left(\frac{1}{E_\mu - E_\sigma - i\delta} + \frac{1}{E_\nu - E_\sigma + i\delta} \right) \tag{34}$$

with the obvious boundary conditions $S_{\mu\nu}(g=0) = 1$, $V_{\mu\nu}(g=0) = V_{\mu\nu}^{\text{Born}}$, which is the Born approximation for the matrix element.

We draw particular attention to the matrix elements $V_{\mu\nu}$. These quantities occupy a central position in the evolution method, since observables such as the phase shifts and the energies of the bound states are expressed in terms of them. In particular, the energies of the bound states are determined by means of the well-known Hellmann-Feynman relation[72]

$$dE_\mu/dg = V_{\mu\mu}.$$

It is important that $V_{\mu\nu}$ is a Hermitian matrix for a Hermitian interaction V. A consequence of this is unitarity of the S matrix. Therefore, all iterative schemes of solution of (34) that preserve Hermiticity of $V_{\mu\nu}$ will lead to a unitary S matrix in each successive approximation.

We consider a simple problem—the elastic scattering of two particles. In this case, one can make a partial-wave expansion in each channel $\alpha = (l, j, I)$ (l and j are the orbital and total angular momentum, and I is the isospin) of all quantities in Eqs. (33) and (34):

$$S(\mathbf{k}', \mathbf{k}; g) = 4\pi \sum_{\alpha} Y_\alpha^*(\hat{\mathbf{k}}')\, Y_\alpha(\hat{\mathbf{k}})\, S_\alpha(k, g),\ k = k';$$

$$V(\mathbf{k}', \mathbf{k}; g) = 4\pi \sum_{\alpha} Y_\alpha^*(\hat{\mathbf{k}}')\, Y_\alpha(\hat{\mathbf{k}})'\, V_\alpha(k', k; g).$$

Here, $\mathbf{k}$ and $\mathbf{k}'$ are the momenta of the particles before and after scattering in their center-of-mass system. Setting $S_\alpha = \exp(2i\delta_\alpha)$, where δ_α is the partial-wave phase shift, we obtain an important relation for determining the phase shifts,

$$\frac{d}{dg} \delta_\alpha(k, g) = -\pi \varepsilon(k)\, V_\alpha(k, k; g), \tag{35}$$

where $\varepsilon(k) = k^2/[2\pi^2 dE(k)/dk]$ is the density of the scattering states, and $E(k)$ is the total collision energy of the two particles in the center-of-mass system.

The two-body problem was considered in detail in the

evolution method in Ref. 70. It is important that Eq. (34) can be solved exactly for a separable interaction. Such a class of potentials is widely used in nuclear-physics problems, in particular to describe the πN interaction.

In Ref. 71, a study was made in the framework of this approach of the problem of the interactions of three nucleons with the Hamiltonian (32), the potential V corresponding to two-body forces. An important result of Ref. 71 was the expansion obtained there for the many-particle matrix element $V_{\mu\nu}$ in a series in powers of the two-particle matrix elements $v^i_{\mu\nu}$ (i is the spectator particle). We shall use this result below to construct a unitary theory of multiple pion scattering by nuclei. The rapid convergence of this expansion was demonstrated for the example of *nd* scattering in Ref. 71.

Pion-nucleus scattering: basic equations of the approach

We here consider the problem of the elastic scattering of a particle by a composite system when the scattered particle is different from those that form the bound complex. Pion scattering by nuclei is such a case.

We assume for the moment[3)] that the pion-nucleus interaction has a potential nature, and we write the Hamiltonian in the form [cf. (6)]

$$H = K_\pi + H_A + \lambda V,\ V = \sum_{i=1}^{A} v^i_{\pi N}, \tag{36}$$

where H_A is the Hamiltonian of the nucleus, K_π is the kinetic-energy operator of the pion, $v^i_{\pi N}$ is the potentiai of the interaction of the pion with nucleon i of the nucleus, and λ plays the part of the πN coupling constant. We assume known the solution to the purely nuclear problem with the Hamiltonian

$$h = K_\pi + H_A. \tag{37}$$

We consider the evolution of the system as λ is varied from 0 to the real value $\lambda = 1$. The parameter λ is here introduced formally and is allowed to tend to unity at the end of the calculations. A central part is here played by the matrix elements $V_{\mu\nu} = \langle\mu|V|\nu\rangle$ of the interaction potential $\Sigma_i v^i_{\pi N}$ taken between the eigenfunctions $|\mu\rangle$, $|\nu\rangle$, etc., of the Hamiltonian H (36). It is in terms of them that the pion-nucleus phase shifts are ultimately expressed.

It is convenient to introduce the eigenfunctions of the channel Hamiltonian h: $|\psi_\mu\rangle$, $|\psi_\nu\rangle$, etc. In the space of the given asymptotic, λ-independent, states, the equation for the S matrix has the form[18]

$$\frac{d}{d\lambda} S(E, \lambda) = -2\pi i S(E, \lambda)\, \delta(E-h)\, V(\lambda) \tag{38}$$

with the boundary condition $S(E, \lambda = 0) = 1$. Accordingly, for the T matrix, defined on the mass shell by the relation

$$S = I + 2\pi i \delta(E-h)\, T,$$

we obtain the equation

$$\frac{d}{d\lambda} T(E, \lambda) = -V(\lambda) - 2\pi i T(E, \lambda)\, \delta(E-h)\, V(\lambda) \tag{39}$$

with the boundary condition $T(E, \lambda = 0) = 0$. The Hermitian operator $V(\lambda)$ is defined so that

$$V_{\mu\nu} = \langle\mu|V|\nu\rangle \equiv \langle\Psi_\mu|V(\lambda)|\Psi_\nu\rangle. \tag{40}$$

It is obvious that for $\lambda = 0$ the operator $V(\lambda)$ is identical to the potential V.

Equations (38) and (39) admit an iterative solution in the form of a series in powers of $V(\lambda)$. These expressions can be represented compactly in the form

$$S(E, \lambda) = T_\lambda \exp\left[-2\pi i \int_0^\lambda d\lambda_1 \delta(E-h)\, V(\lambda_1)\right];$$

$$T(E, \lambda) = -T_\lambda \int_0^\lambda d\lambda_1 V(\lambda_1) \times \exp\left[-2\pi i \int_{\lambda_1}^\lambda d\lambda_2 \delta(E-h)\, V(\lambda_2)\right], \tag{41}$$

where T_λ is the operator of "antiordering"[69] with respect to the variable λ. Under this symbol, the operators are arranged in order of ascending arguments from the left to the right.

Optical-potential method

We concentrate on the study of elastic scattering. We introduce more detailed notation for the asymptotic pion-nucleus states: $|\psi\rangle \equiv |\mathbf{k}, n\rangle$, where $\mathbf{k}$ is the pion momentum in the pion-nucleus center-of-mass system, and n denotes the quantum numbers of the states of the nucleus ($n = 0$ corresponds to the ground state). To elastic scattering there corresponds the transition from the state $|\mathbf{k}, 0\rangle$ to $|\mathbf{k}', 0\rangle$, where $\mathbf{k}$ and $\mathbf{k}'$ are the pion momenta before and after the scattering.

By means of the operator of projection $\hat{P} = |0\rangle\langle 0|$ onto the ground state we define the submatrix T_0 of the total scattering matrix T: $T_0 = \hat{P}T\hat{P}$, describing the elastic reaction channel. Rearranging the right-hand side of Eq. (39) by means of the operators $\hat{P}$ and $\hat{Q} = 1 - \hat{P}$, we can obtain the following equation for the matrix T_0:

$$\frac{d}{d\lambda} T_0(E, \lambda) = -U_0(E, \lambda) - 2\pi i T_0(E, \lambda)\, \hat{P}\delta(E-h)\, U_0(E, \lambda) \tag{42}$$

with the boundary condition $T_0(E, \lambda = 0) = 0$. The effective energy-dependent operator $U_0(E, \lambda)$ is determined by the system of equations

$$U_0(E, \lambda) = V(\lambda) + 2\pi i \mathscr{K}(E, \lambda)\, \hat{Q}\delta(E-h)\, V(\lambda); \tag{43}$$

$$\frac{d}{d\lambda} \mathscr{K}(E, \lambda) = -U_0(E, \lambda) + 2\pi i U_0(E, \lambda)\, \hat{P}\delta(E-h)\, \mathscr{K}(E, \lambda) \tag{44}$$

with a boundary condition for the auxiliary operator $\mathscr{K}$: $\mathscr{K}(E, \lambda = 0) = 0$. A brief proof of the equivalence of Eqs. (42)–(44) to the original relation (39) is given in Appendix 1.

In matrix form, the equation for the elastic-scattering matrix T_0 is

$$\frac{d}{d\lambda}\langle \mathbf{k}',\ 0 \mid T_0(E,\lambda) \mid \mathbf{k},\ 0\rangle = -\langle \mathbf{k}',\ 0 \mid U_0(E,\lambda) \mid \mathbf{k},\ 0\rangle$$
$$-2\pi i \int \frac{d\mathbf{k}''}{(2\pi)^3}\langle \mathbf{k}',\ 0 \mid T_0(E,\lambda) \mid \mathbf{k}'',\ 0\rangle\, \delta(E-E_0(k''))$$
$$\times \langle \mathbf{k}'',\ 0 \mid U_0(E,\lambda) \mid \mathbf{k},\ 0\rangle. \qquad (45)$$

Here, $E = E_0(k) = \omega_\pi(k) + \omega_A(k)$ is the collision energy in the pion-nucleus center-of-mass system, measured from the ground state of the nucleus;

$$\omega_\pi(k) = (\mu^2 + k^2)^{1/2};\quad \omega_A(k) = (M_A^2 + k^2)^{1/2};$$

and μ and M_A are, respectively, the masses of the pion and the nucleus. Here and below, relativity is taken into account at the kinematic level.

It can be seen from Eqs. (42) and (45) that the complicated many-channel problem has been reduced effectively by means of the energy-dependent operator $U_0(E,\lambda)$ to a two-particle problem (presence of the projection operator $\hat{P}$ on the right-hand sides of these equations). Thus, the operator $U_0(E,\lambda)$ plays in this approach the part of an optical potential.

By a partial-wave expansion of the quantities that occur in Eq. (45) it is possible to obtain [in complete analogy with (35)] for the partial-wave phase shifts of pion-nucleus scattering the simple formula[4)]

$$\delta(k) = -\pi\varepsilon_A(k)\int_0^1 d\lambda\ \langle k',\ 0 \mid U_0(E,\lambda) \mid k,\ 0\rangle. \qquad (46)$$

Here, $\varepsilon_A(k) = k^2/[2\pi^2 dE_0(k)/dk]$ is the level density of the scattering states. In the nonrelativistic case, $\varepsilon_A(k) = k\mathcal{M}/2\pi^2$, where $\mathcal{M}$ is the reduced mass of the πA system.

The problem of finding the pion-nucleus phase shifts has been reduced to the calculation of the matrix element $\langle \mathbf{k}', 0|U_0|\mathbf{k}, 0\rangle$. It can be seen from Eq. (43) that the operator $U_0(E,\lambda)$ is non-Hermitian. Its non-Hermitian part reflects the contribution of the inelastic channels to the elastic channel [presence of the operator $\hat{Q}$ on the right-hand side of (43)]. In the low-energy limit, when the pion energy is less than the energy of the first excited state of the nucleus, $E_0(k) - E_1(k=0) < 0$, the second term in (43) disappears. In this limit, the operator $U_0 = V$ is Hermitian, and T_0, the scattering matrix, is unitary [the phase shifts (46) are real]. Therefore, in this approach the condition of two-particle unitarity is automatically satisfied.

Iterative solution of the equations for $U_0(E,\lambda)$

The system of equations (43) and (44) can be solved iteratively,

$$U_0^{(N)} = V + 2\pi i \mathscr{K}^{(N-1)}\hat{Q}\delta(E-h)V;$$
$$\frac{d}{d\lambda}\mathscr{K}^{(N)} = -U_0^{(N)} + 2\pi i U_0^{(N)}\hat{P}\delta(E-h)\mathscr{K}^{(N-1)}, \qquad (47)$$

where $N = 1,2,3,\ldots$ is the number of the iteration, and $\mathscr{K}^{(N)}$ and $U_0^{(N)}$ are set equal to zero for $N \leqslant 0$. Such a scheme gives an expansion of U_0 in powers of the operator V, the first two terms being

$$U_0(E,\lambda) = V(\lambda) - 2\pi i\int_0^\lambda d\lambda_1 V(\lambda_1)\hat{Q}\delta(E-h)V(\lambda). \qquad (48)$$

Accordingly, for the phase shifts (46) we obtain in this approximation

$$\delta(k) = -\pi\varepsilon_A(k)\int_0^1 d\lambda\,[\langle \mathbf{k}',\ 0 \mid V(\lambda) \mid \mathbf{k},\ 0\rangle - 2\pi i\int_0^\lambda d\lambda_1 \sum_{n>0}$$
$$\times \int \frac{d\mathbf{k}''}{(2\pi)^3}\langle \mathbf{k}',\ 0 \mid V(\lambda_1) \mid \mathbf{k}'',\ n\rangle\, \delta(E_0(k) - E_n(k''))$$
$$\times \langle \mathbf{k}'',\ n \mid V(\lambda) \mid \mathbf{k},\ 0\rangle, \qquad (49)$$

where $E_n(k) = \omega_\pi(k) + \omega_A(k) + \varepsilon_n$, in which ε_n is the energy of the excited state, measured from the ground state, i.e., $\varepsilon_n > 0$.

Multiple-scattering series

As a result of the iterative solution of the equation for $U_0(E,\lambda)$, we have obtained the expansion (49) for the pion-nucleus phase shifts in powers of the many-particle Hermitian operator $V(\lambda)$, which is defined by the relation (40). The problem is to express this quantity in terms of the two-particle scattering matrices $u^i(\lambda)$ that correspond to interaction of the pion with nucleon i of the nucleus. Thus, in contrast to the standard theory of multiple scattering, in which a series is constructed for the T matrix, the analogous expansion is here obtained for the scattering operator $V(\lambda)$.

The required expansion was obtained in Ref. 17. The first two terms of the series have the form

$$V(\lambda) = V^{(1)}(\lambda) + V^{(2)}(\lambda),\quad V^{(2)} = V_1^{(2)} + V_2^{(2)}, \qquad (50)$$

where

$$V^{(1)}(\lambda) = \sum_{i=1}^{A} u^i(\lambda); \qquad (51)$$

$$V_1^{(2)}(\lambda) = \sum_{i\neq j=1}^{A}\int_0^\lambda d\lambda_1\Big[\sum_s \hat{p}_s u^i(\lambda_1)\, G_0^{(+)}(E_s)\, u^j(\lambda_1) + \text{h.c.}\Big]; \qquad (52)$$

$$V_2^{(2)}(\lambda) = \sum_{i,\ j=1}^{A}\int_0^\lambda d\lambda_1\Big\{\Big[\sum_s \hat{p}_s u^i(\lambda_1)\, G_0^{(+)}(E_s)\, u^j(\lambda)$$
$$-\sum_\sigma \hat{p}_\sigma u^i(\lambda_1)\, G^{(+)}(E_\sigma)\, u^j(\lambda)\Big] + \text{h.c.}\Big\}. \qquad (53)$$

Here, $\hat{p}_s \equiv |\chi_s\rangle\langle\chi_s|$ and $\hat{p}_\sigma \equiv |\psi_\sigma\rangle\langle\psi_\sigma|$ are the operators of projection onto the eigenstates $|\chi_s\rangle$ and $|\psi_\sigma\rangle$, respectively, of the free, $H_0 = K_\pi + K_A$ (K_A is the kinetic-energy operator of the nucleons), and the channel, h (37), Hamiltonians, and G_0 and G are the Green's functions

$$G_0^{(\pm)}(E) = (E \pm i\delta - H_0)^{-1} \text{ and } G^{(\pm)}(E) = (E \pm i\delta - h)^{-1}. \qquad (54)$$

The diagrammatic representation of this expansion is

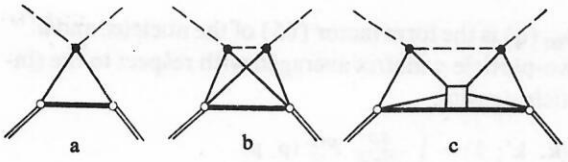

FIG. 10. Graphical representation of the multiple-scattering series (50).

shown in Fig. 10, in which the double line denotes the nucleus; the thin continuous line, the nucleon; the thick line, the residual nucleus; the broken line, the pion; the black dot, the interaction vertex; the open circle, the nuclear wave function; and the square, the multiple scattering of the nucleons. The diagram of Fig. 10a corresponds to the first-order approximation (51), and the diagrams of Figs. 10b and 10c to multiple scattering of the pion without the interaction (52) and with the interaction (53) of the nucleons in the intermediate states, respectively.

The two-particle scattering matrix $u^i(\lambda)$ in (51)–(53) is the matrix element of the given two-particle potential v^i [see (36)] between the exact wave functions $\psi^{(1)}_{\mathbf{k},\mathbf{p}}(\lambda)$ of the problem of pion scattering by a free nucleon with the Hamiltonian

$$H^i = K_\pi + K_N^i + \lambda v^i, \tag{55}$$

where K_N is the kinetic-energy operator of the nucleon, i.e.,

$$\langle \mathbf{k}',\ \mathbf{p}' \mid u(\lambda) \mid \mathbf{k},\ \mathbf{p}\rangle \equiv \langle \psi^{(+)}_{\mathbf{k}',\mathbf{p}'}(\lambda) \mid v \mid \psi^{(-)}_{\mathbf{k},\mathbf{p}}(\lambda)\rangle. \tag{56}$$

Here, $\mathbf{k}$, $\mathbf{p}$ and $\mathbf{k}'$, $\mathbf{p}'$ are the momenta of the pion and of the nucleon before and after the collision. In the pion-nucleon center-of-mass system, these matrix elements determine the πN phase shifts by means of the relation (35). In (51)–(53), the matrix u^i acts on the $(A+1)$-particle space and includes the δ functions for the conservation laws of the momentum, spin, etc., of all the spectator nucleons.

The structure of the resulting series is analogous to the Watson series (10). It is important that in each successive approximation the operator $V(\lambda)$ is Hermitian. Therefore, the condition of two-particle unitarity is not violated in this iterative scheme.

Pion-nucleus phase shifts

Substituting (50) in (49), for the first two terms in the expansion of the phase shifts in powers of the two-particle u matrix we obtain the expressions

$$\delta(k) = \delta^{(1)}(k) + \delta^{(2)}(k), \quad \delta^{(2)} = \delta^{(2)}_{\mathrm{Re}} + i\delta^{(2)}_{\mathrm{Im}}, \tag{57}$$

where $\delta^{(1)}$ is the approximation of the first order,

$$\delta^{(1)}(k) = -\pi\varepsilon_A(k) \int_0^1 d\lambda \langle \mathbf{k}',\ 0 \mid V^{(1)}(\lambda) \mid \mathbf{k},\ 0\rangle; \tag{58}$$

$\delta^{(2)}_{\mathrm{Re}}$ is the real part of the second-order correction,

$$\delta^{(2)}_{\mathrm{Re}}(k) = -\pi\varepsilon_A(k) \int_0^1 d\lambda \langle \mathbf{k}',\ 0 \mid V^{(2)}(\lambda) \mid \mathbf{k},\ 0\rangle, \tag{59}$$

and $\delta^{(2)}_{\mathrm{Im}}$ is its imaginary part:

$$\delta^{(2)}_{\mathrm{Im}}(k) = 2\pi^2\varepsilon_A(k) \int_0^1 d\lambda \int_0^\lambda d\lambda_1 \langle \mathbf{k}',\ 0 \mid V^{(1)}(\lambda_1) \times \hat{Q}\delta(E-h)\, V^{(1)}(\lambda) \mid \mathbf{k},\ 0\rangle. \tag{60}$$

The operators $V^{(1),(2)}$ are defined in (51)–(53), and the operator $\hat{Q}$ projects onto the excited states of the nucleus. The structure of the matrix elements in (58)–(60) is analogous to that of the expressions for the first- and second-order optical potentials [see (13), (14), and (26)]. A difference is in the replacement of the two-particle t matrix by the u matrix. Therefore, in the actual calculations below of the quantities (58)–(60) we shall use a number of approximations that have already been tested in the optical model (Sec. 3).

First-order approximation

In this approximation, with allowance for (58) and (51), the pion-nucleus phase shifts are determined by the expression

$$\delta^{(1)}(k) = -\pi\varepsilon_A(k) \int_0^1 d\lambda \left\langle \mathbf{k}',\ 0 \left| \sum_{i=1}^A v^i(\lambda) \right| \mathbf{k},\ 0 \right\rangle. \tag{61}$$

By means of the single-particle overlap function

$$F^{(1)}_{00}(\mathbf{k}_1,\ \mathbf{k}_1') = \int \prod_{i=2}^A \frac{d\mathbf{k}_i}{(2\pi)^3} \delta\left(\mathbf{k} + \sum_{j=1}^A \mathbf{k}_j\right) \overline{\Psi}_0(\mathbf{k}_1,\ \mathbf{k}_2,\ \ldots,\ \mathbf{k}_A) \times \Psi_0(\mathbf{k}_1',\ \mathbf{k}_2,\ \ldots,\ k_A), \tag{62}$$

where ψ_0 is the wave function of the ground state of the nucleus, $\delta^{(1)}(k)$ can be represented in the form

$$\delta^{(1)}(k) = -A\pi\varepsilon_A(k) \int_0^1 d\lambda \int \frac{d\mathbf{p}}{(2\pi)^3} F^{(1)}_{00}(\mathbf{p},\ \mathbf{p}-\mathbf{q}) \langle \mathbf{k},\ \mathbf{p} \mid u^1(\lambda) \mid \mathbf{k}',\ \mathbf{p}-\mathbf{q}\rangle, \tag{63}$$

where $\mathbf{q} = \mathbf{k}' - \mathbf{k}$ is the momentum transfer. Besides integration over the momenta in (62) and (63), summation over the spin-isospin variables is also understood. The index 1 of the u matrix corresponds to the spin-isospin part of this operator.

In order to gain a better understanding of the expressions obtained in this approach, it is helpful to calculate (63) in the static limit of the theory, i.e., as $\mu/M \to 0$, where μ and M are the pion and nucleon masses. In this limit, as is shown in Ref. 73, the matrix element $\langle \mathbf{k},0 \mid \Sigma_i u^i \mid \mathbf{k}',0\rangle$ factorizes and has the form

$$\left\langle \mathbf{k},\ 0 \mid \sum_{i=1}^A u^i(\lambda) \mid \mathbf{k}',\ 0 \right\rangle = A\rho_{00}(\mathbf{q}) \langle \mathbf{k},\ -\mathbf{k} \mid u^1(\lambda) \mid \mathbf{k}',\ -\mathbf{k}'\rangle, \tag{64}$$

where

$$\rho_{00}(\mathbf{q}) = \int \frac{d\mathbf{p}}{(2\pi)^3} F_{00}^{(1)}(\mathbf{p},\ \mathbf{p}-\mathbf{q}) = \langle 0|e^{i\mathbf{q}\mathbf{r}_1}|0\rangle \qquad (65)$$

is the nuclear form factor. Thus, for the phase shifts we obtain

$$\delta^{(1)}(k) = -\pi A \varepsilon_A(k)\, \rho_{00}(\mathbf{q}) \int_0^1 d\lambda\, \langle \mathbf{k},\ -\mathbf{k} \mid u^1(\lambda) \mid \mathbf{k}',\ -\mathbf{k}' \rangle. \qquad (66)$$

The two-particle u matrix is here found on the mass shell ($|\mathbf{k}| = |\mathbf{k}'|$), and therefore the integration over λ can be performed exactly by means of the relation (35) after a preliminary expansion of the matrix element with respect to the partial waves. We give the result[73] for the case of pion scattering by a nucleus with total isospin zero:

$$\delta_L^{(1)}(k) = A \frac{\mathscr{M}}{\bar{\mu}} \sum_{\substack{l,\, l' \\ j=l\pm\frac{1}{2}}} \left(j+\frac{1}{2}\right) \times \begin{pmatrix} L & l' & l \\ 0 & 0 & 0 \end{pmatrix}^2 \rho_{l'}(k) \frac{1}{3}\left(2\delta_{lj}^{3/2} + \delta_{lj}^{1/2}\right), \qquad (67)$$

where $\mathscr{M}$ and $\bar{\mu}$ are the reduced masses of the pion-nucleus and pion-nucleon systems, $\rho_l(k)$ is the partial-wave harmonic of the nuclear form factor,

$$\rho_{00}(q) = \sum_{l=0}^{\infty} (2l+1)\, \rho_l(k)\, P_l(\hat{k}\hat{k}'),$$

and $\delta_{ij}^I(k)$ is the πN phase shift in the channel $\alpha = (l, j, I)$ with definite orbital (l) and total (j) angular momenta and isospin ($I = 1/2,\ 3/2$).

For the scattering length, we obtain in this approximation

$$a_{\pi A}^{(1)} = \lim_{k\to 0} \delta^{(1)}(k)/k = A \frac{\mathscr{M}}{\bar{\mu}} \frac{1}{3}(2a_3 + a_1), \qquad (68)$$

where a_τ are the s-wave πN scattering lengths in the isotopic states, $\tau = 2I = 1$ and 3. This expression is identical to the first order of multiple-scattering theory[1] in the impulse approximation.

It can be seen from (67) that the partial-wave pion-nucleus phase shifts have been expressed in terms of the πN phase shifts and the nuclear form factor. This makes possible a semiphenomenological analysis of pion-nucleus scattering in which the experimentally determined values and parameters are used as input data.

In the description of pion-nucleus scattering, the corrections to the static approximation are not small. The p-wave πN interaction plays an important part. It is therefore necessary to take into account the Fermi motion of the nucleons in the nucleus and the angle transformation (see Sec. 3) that arises when the two-particle u matrix is transformed from the pion-nucleon center-of-mass system to the pion-nucleus system. Therefore, in real calculations we use the factorization approximation (see Sec. 3, first-order potential), in which the following expression is obtained for $\delta^{(1)}(k)$:

$$\delta^{(1)}(k) = -A\pi\varepsilon_A(k)\, \rho_{00}(\mathbf{q}) \int_0^1 d\lambda \bar{u}^1(\mathbf{k},\ \mathbf{k}';\ \lambda), \qquad (69)$$

where $\rho_{00}(\mathbf{q})$ is the form factor (65) of the nucleus, and $\bar{u}^{(1)}$ is the two-particle u matrix averaged with respect to the single-particle density:

$$\bar{u}^1(\mathbf{k},\ \mathbf{k}';\ \lambda) = \int \frac{dp}{(2\pi)^3} F_{00}^{(1)}(\mathbf{p},\ \mathbf{p}) \times \langle \mathbf{k},\ \mathbf{p}+\mathbf{p}_0 | u^1(\lambda) | \mathbf{k}',\ \mathbf{p}+\mathbf{p}_0-\mathbf{q}\rangle. \qquad (70)$$

Here, $\mathbf{p}_0 = -\mathbf{k}/A + [(A-1)/2A]\mathbf{q}$, and $F_{00}^{(1)}$ is defined in (62). We give below the result of integrating $\bar{u}^1$ over λ and its expression in terms of the πN phase shifts (see "Two-particle u matrix").

The results of numerical calculations of the differential cross sections of pion scattering by the light nuclei ^{4}He and ^{12}C in the first order of the theory are shown in Figs. 11 and 12. It can be seen that in this approach, in contrast to the optical model, a satisfactory description of the data is obtained already in the first approximation, this being a consequence of its consistency with the unitarity condition.

Second-order effects

The restriction to the first-order approximation (69) is invalid for two reasons. First, in the first order of the theory the pion-nucleus phase shifts are real. The processes with excitation of the nucleus in the intermediate states corresponding to the occurrence of imaginary parts in the phase shifts (inelasticity parameters) appear only in the second order [see (60)]. Second, it is necessary to take into account the effects of multiple scattering of the pion, i.e., the term (59), particularly at low energies. This is due, not to the slow convergence of the method, but to the well-known isotopic effect of the almost vanishing of the isoscalar πN scattering lengths

$$b_0 = \frac{1}{3}(a_1 + 2a_3),$$

where a_{2I} are the s-wave πN scattering lengths. For example, in Ref. 74 the following values were obtained for $a_{1,3}$: $a_1 = 0.170\mu^{-1}$ and $a_3 = -0.092\mu^{-1}$. For this reason, the pion scattering length (68) for scattering on an isoscalar nucleus is also very small. As can be seen from Table III, allowance for the second order in the case of πd scattering practically reproduces the exact result[75] in accordance with the Faddeev equations.

A. The real part of the second-order correction (59) contains two terms, $\delta_{\mathrm{Re}}^{(2)} = \delta_{\mathrm{Re},1}^{(2)} + \delta_{\mathrm{Re},2}^{(1)}$, where

$$\delta_{\mathrm{Re},\,i}^{(2)}(k) = -\pi\varepsilon_A(k) \int_0^1 d\lambda\, \langle \mathbf{k},\ 0 | V_i^{(2)}(\lambda) | \mathbf{k}',\ 0\rangle,\quad i = 1,\ 2. \qquad (71)$$

The operators $V_{1,2}^2$ are defined in (52) and (53) and correspond to the diagrams of Figs. 10b and 10c.

As is shown in Ref. 19, the following expression is obtained in the factorization approximation for $\delta_{\mathrm{Re},1}^{(2)}$:

$$\delta_{\mathrm{Re},\,1}^{(2)}(k) = -A(A-1)\pi\varepsilon_A(k) \int \frac{d\mathbf{k}''}{(2\pi)^3} 2P\left(\frac{1}{E_0(k) - E_0(k'')}\right) \times C_{00}(\mathbf{q}',\ \mathbf{q}'') \int_0^1 d\lambda \int_0^\lambda d\lambda_1 \bar{u}^1(\mathbf{k},\ \mathbf{k}'';\ \lambda_1)\, \bar{u}^2(\mathbf{k}'',\ \mathbf{k}';\ \lambda_1), \qquad (72)$$

TABLE III. The πd scattering length calculated for different sets of πN scattering lengths.

πN scattering lengths, F			πd scattering lengths, F		
a_1	a_3	Reference	$a_{\pi d}^{(1)}$	$a_{\pi d}^{(1)}+a_{\pi d}^{(2)}$	$a_{\pi d}^{\text{Fad}}$
0.2573	−0.1541	[81]	−0.0365	−0.0725	−0.0746
0.2566	−0.1260	[84]	0.0032	−0.0284	−0.0309
0.2404	−0.1300	[74]	−0.0141	−0.0485	−0.0460

Note. The values of $a_{\pi d}^{\text{Fad}}$ are taken from Ref. 75. The experimental value of the πd scattering length is $-(0.073^{+0.031}_{-0.024})$ F.[80]

where $\mathbf{q}'=\mathbf{k}-\mathbf{k}''$, $\mathbf{q}''=\mathbf{k}''-\mathbf{k}'$; P denotes integration in the principal-value sense; $\bar{u}^{1,2}$ are the elements of the u matrix (70) averaged over the single-particle density, and

$$C_{00}(\mathbf{q},\ \mathbf{q}')=\langle 0\mid \exp(i\mathbf{q}\mathbf{r}_1+i\mathbf{q}'\mathbf{r}_2)\mid 0\rangle \tag{73}$$

is the two-particle correlation function.

We now consider the correction $\delta_{\text{Re},2}^{(2)}$. It can be expressed in terms of the matrix element of $V_2^{(2)}$ (53). In Ref. 17, it was shown that this correction vanishes strictly in the static limit of the theory, i.e., as $\mu/M\to 0$. This is due to the fact that in this limit the energy denominators of the Green's functions G and G_0 (54) do not depend on the nucleon variables, and $G=G_0+O(\mu/M)$. Therefore, taking into account the completeness condition for the wave functions of the Hamiltonians h and H_0, we obtain the desired result. From this it is to be expected that the contribution from $\delta_{\text{Re},2}^{(2)}$ will be not more than 20% $\delta_{\text{Re},1}^{(2)}$. The estimate can be made more precise by using the completeness approximation in the calculation of the term proportional to G in (53), i.e., by setting $E_n(k'')=E_0(k'')+\Delta$, where E_n is the eigenvalue of the Hamiltonian h, and Δ is a certain mean excitation energy of the nuclear system. In Ref. 65 (Wakamatsu, 1978) it was shown by numerical calculation for the example of π^4He scattering that variation of Δ from 0 to 20 MeV changes the value of the integral by only 10%. Therefore, we shall assume that $\delta_{\text{Re}}^{(2)}\simeq\delta_{\text{Re},1}^{(2)}$.

B. In accordance with (60), the imaginary part of the second-order correction $\delta_{\text{Im}}^{(2)}$ has the form

$$\delta_{\text{Im}}^{(2)}(k)=2\pi^2\varepsilon_A(k)\sum_{n>0}\int\frac{d\mathbf{k}''}{(2\pi)^3}\delta(E_0(k)-E_n(k'')) \tag{74}$$

$$\times\int_0^1 d\lambda\int_0^\lambda d\lambda_1\langle k,\ 0|V^{(1)}(\lambda_1)|k'',\ n\rangle\langle k'',\ n|V^{(1)}(\lambda)|k',\ 0\rangle,$$

where

$$V^{(1)}=\sum_i u^i;\ E_0(k)=\omega_\pi(k)+\omega_A(k);\ E_n(k'')=E_0(k'')+\varepsilon_n;$$

ε_n is the energy of the excited state of the nucleus, measured from the ground-state energy.

The matrix elements in (74) are analogous to the matrix element that arises in the first-order approximation (61). Therefore, by analogy with (69) they have in the factorization approximation the form

$$\langle \mathbf{k},\ 0\mid\sum_{i=1}^{A}u^i(\lambda)\mid\mathbf{k}',\ n\rangle=A\rho_{0n}(\mathbf{q}')\,\bar{u}^1(\mathbf{k},\ \mathbf{k}'';\ \lambda), \tag{75}$$

where $\rho_{0n}(\mathbf{q}')=\langle 0|\exp(i\mathbf{q}\mathbf{r}_1)|n\rangle$ is the transition form factor, $\mathbf{q}'=\mathbf{k}-\mathbf{k}''$, and $\bar{u}^1$ is the u matrix (70) averaged with respect to the single-particle density. Substituting (75) in (74) and integrating explicitly over $d\mathbf{k}''$ with allowance for the δ function, we obtain

$$\delta_{\text{Im}}^{(2)}(k)=2\pi^2A^2\varepsilon_A(k)\sum_{n>0}\varepsilon_A(k_n)\rho_{0n}(\mathbf{q}')\rho_{n0}(\mathbf{q}'')$$

$$\times\int\frac{d\mathbf{n}''}{4\pi}\int_0^1 d\lambda\int_0^\lambda d\lambda_1\bar{u}^1(\mathbf{k},\ \mathbf{k}'';\ \lambda_1)\bar{u}^2(\mathbf{k}'',\ \mathbf{k}';\ \lambda), \tag{76}$$

where $\mathbf{q}'=\mathbf{k}-\mathbf{k}''$, $\mathbf{q}''=\mathbf{k}''-\mathbf{k}'$, $\mathbf{k}''=k_n\mathbf{n}''$. The pion momentum k_n in the intermediate state is determined by the equation

$$E_0(k)-E_0(k_n)-\varepsilon_n=0. \tag{77}$$

In the nonrelativistic case, $\varepsilon_A(k_n)=\mathcal{M}k_n/2\pi^2$, $k_n=\sqrt{k^2-2\mathcal{M}\varepsilon_n}$, where $\mathcal{M}$ is the reduced mass of the pion-nucleus system. Thus, the expression (76) gives the correct threshold behavior of the scattering amplitude, making it possible to take into account consistently the channels that become open as the pion energy is raised.

To estimate the inelasticity parameters $\eta=\exp(-2\delta_{\text{Im}})$, we can use the completeness approximation, setting $E_n(k)=E_0(k)+\Delta$, where Δ is a certain mean excitation energy of the nuclear system. Then, integrating explicitly over $d\mathbf{k}''$ and summing over the intermediate states with allowance for the identity

$$\sum_{n>0}\rho_{0n}(\mathbf{q}')\rho_{n0}(\mathbf{q}'')=A^{-1}\rho_{00}(\mathbf{q}'+\mathbf{q}'')$$
$$+(1-A^{-1})C_{00}(\mathbf{q}',\ \mathbf{q}'')-\rho_{00}(\mathbf{q}')\rho_{00}(\mathbf{q}''),$$

we obtain

$$\delta_{\text{Im}}^{(2)}(k)=2A\pi^2\varepsilon_A(k)\varepsilon_A(k_\Delta)\int\frac{d\mathbf{n}''}{4\pi}[\rho_{00}(\mathbf{q})-A\rho_{00}(\mathbf{q}')\rho_{00}(\mathbf{q}'')$$
$$+(A-1)C_{00}(\mathbf{q}',\ \mathbf{q}'')]\int_0^1 d\lambda\int_0^\lambda d\lambda_1\bar{u}^1$$
$$(\mathbf{k},\ \mathbf{k}'';\ \lambda_1)\bar{u}^2(\mathbf{k}'',\ \mathbf{k}';\ \lambda), \tag{78}$$

where $\mathbf{q} = \mathbf{k}' - \mathbf{k}$ is the momentum transfer, $\mathbf{q}' = \mathbf{k} - \mathbf{k}''$, $\mathbf{q}'' = \mathbf{k}'' - \mathbf{k}'$, $\mathbf{k}'' = k_\Delta \mathbf{n}''$, $\mathbf{n}''$ is a unit vector, and the momentum k_Δ is determined by the equation $E_0(k) - E_0(k_\Delta) - \Delta = 0$.

In this approximation, the theory contains the parameter Δ, which depends, in general, on the energy of the incident pion. It follows from (76) and (77) that at a pion energy insufficient for excitation of the lowest (ε_1) state of the nucleus the imaginary part of the phase shift must vanish. Therefore, as a first approximation it is reasonable to choose $\Delta = \varepsilon_1$, i.e., to take Δ equal to the experimental value of the first excited state. As numerical calculations show, such a choice gives an upper bound for $\delta_{\text{Im}}^{(2)}$.

Two-body matrix elements

The final aim of the calculations is to express the pion-nucleus phase shifts in terms of the πN phase shifts. For this, the two-body u matrices in (69), (72), and (78), determined in the pion-nucleus center-of-mass system, must be expressed in terms of the two-body $\tilde{u}$ matrices in the pion-nucleon center-of-mass system. Bearing in mind that the u matrix has the same transformation properties as the t matrix, we obtain

$$\langle \mathbf{k}_s\ \mathbf{p} \mid u(\lambda) \mid \mathbf{k}',\ \mathbf{p}' \rangle = \gamma\ \langle \varkappa \mid \tilde{u}(\lambda) \mid \varkappa' \rangle, \tag{79}$$

where the factor γ and the pion momenta in the pion-nucleon center-of-mass system are given in (19)–(22).

The two-body u matrix, as an operator on the spin-isospin space, has the form

$$\langle \varkappa | \tilde{u}(\lambda) | \varkappa' \rangle = \sum_{\beta=0}^{3} \langle \varkappa | u_\beta(\lambda) \varkappa' \rangle O_\beta. \tag{80}$$

Here, $O_\beta = 1,\ \mathbf{t}\cdot\boldsymbol{\tau},\ i(\boldsymbol{\sigma}\cdot\mathbf{n}),\ i(\boldsymbol{\sigma}\cdot\mathbf{n})(\mathbf{t}\cdot\boldsymbol{\tau})$; $\mathbf{t}$ is the pion isospin operator; $\boldsymbol{\tau}/2$ and $\boldsymbol{\sigma}/2$ are the nucleon isospin and spin operators, and $\mathbf{n} = [\varkappa \times \varkappa']/|[\varkappa \times \varkappa']|$. If we ignore the spin-dependent terms ($\beta = 2, 3$), the partial-wave expansion of the matrix elements of $u_{0,1}$ can be written in the form

$$\langle \varkappa | u_\beta(\lambda) | \varkappa' \rangle = \sum_I c_I^\beta \sum_{l,j} \left(j + \frac{1}{2}\right) u_{lj}^I(\varkappa,\ \varkappa';\ \lambda)\, P_l(\hat{\varkappa}\hat{\varkappa}'), \tag{81}$$

where $I = 1/2,\ 3/2$ are the isospin values of the πN system, and $c_{1/2}^0 = 1/3$, $c_{3/2}^0 = 2/3$, $c_{3/2}^1 = -c_{1/2}^1 = 1/3$, $j = l \pm 1/2$.

On the mass shell $\varkappa = \varkappa'$ the partial-wave matrix element $u_\nu(\varkappa, \varkappa; \lambda)$ ($\nu \equiv l, j, I$) determines the corresponding πN phase shift.

$$\delta_\nu(\varkappa) = -\pi \varepsilon_2(\varkappa) \int_0^1 d\lambda u_\nu(\varkappa,\ \varkappa;\ \lambda), \tag{82}$$

where $\varepsilon_2(\varkappa) = \varkappa^2/[2\pi^2 dE_2(\varkappa)/d\varkappa]$, $E_2(\varkappa) = \omega_\pi(\varkappa) + E_N(\varkappa)$.

The off-shell behavior of the two-body u matrices can be taken into account in the model of a separable πN interaction [see (31)]:

$$u_\nu(\varkappa,\ \varkappa';\ \lambda) = u_\nu(\varkappa,\ \varkappa;\ \lambda)\, g_\nu(\varkappa')/g_\nu(\varkappa), \tag{83}$$

where $g_\nu(\varkappa)$ is the form factor of the separable interaction, which can be determined by fitting to πN scattering data.

We consider, finally, the integrals over λ in (72) and (78) of the products of the two-particle matrix elements. With allowance for (83), they reduce to

$$A_{\nu\nu'} \equiv \int_0^1 d\lambda \int_0^\lambda d\lambda_1 [u_\nu(\varkappa,\ \varkappa;\ \lambda_1)\, u_{\nu'}(\varkappa',\ \varkappa';\ \lambda) + (\nu \rightleftarrows \nu')];$$

$$B_{\nu\nu'} \equiv \int_0^1 d\lambda \int_0^\lambda d\lambda_1 [u_\nu(\varkappa,\ \varkappa;\ \lambda_1)\, u_{\nu'}(\varkappa',\ \varkappa';\ \lambda) + (\nu \rightleftarrows \nu')]. \tag{84}$$

Integrating by parts using (82), we can obtain an expansion whose leading term is

$$A_{\nu\nu'} = B_{\nu\nu'} = \delta_\nu(\varkappa)\, \delta_{\nu'}(\varkappa')/[\pi^2 \varepsilon_2(\varkappa)\, \varepsilon_2(\varkappa')]. \tag{85}$$

One can show that the corrections to (85) are about 20% (see, for example, Eq. (31) in Ref. 17). Therefore, they can be ignored to the accuracy of our calculations of $\delta_{\text{Re}}^{(2)}$ and $\delta_{\text{Im}}^{(2)}$ in (72) and (78).

The relations (79)–(85) complete the determination of the first two iterations of (69), (72), and (78) in terms of the πN phase shifts, the correlation functions, and the nuclear form factors.

Convergence of the iterative scheme

The calculation and analysis of the second-order corrections (72) and (78) make it possible to establish the domain of convergence of the iterative scheme. This was investigated in Refs. 17 and 19. In Ref. 17, it was shown (see Table III) that the πd scattering length calculated in the second order of the theory is practically equal to the exact value of Ref. 75 obtained by means of the Faddeev equations.[13] In Ref. 17, an analysis was made of the contribution of the second-order corrections, and also partly of the higher iterations for $\delta_{\text{Im}}^{(2)}$ in the case of π^4He scattering. It was shown

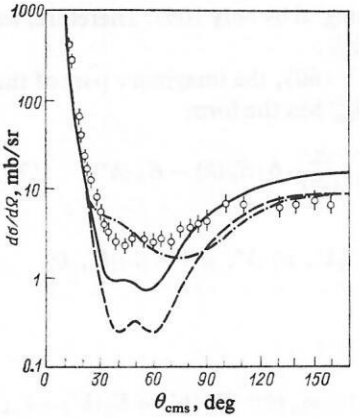

FIG. 11. Differential cross section of π^+ scattering by ^{12}C at 28.4 MeV. The data are taken from Ref. 21; the continuous curve represents the first-order approximation ($\delta^{(1)}$) of the unitary approach, and the broken and chain curves are the calculations of Ref. 5 with the optical potential in the three-particle model (23) without allowance and with allowance for the absorption channel, respectively.

TABLE IV. Phase shifts of π^4He scattering, deg, and the total and total elastic cross sections at 51 MeV.

Phase shifts	Re δ_s	Im δ_s	Re δ_p	Im δ_p	σ_{el}, mb	σ_{tot}, mb	Re $f(0)$, F
$\delta_{\pi A}=\delta_{\pi A}^{pot}$	−7.56	0.98	12.82	0.51	53.1	68.4	1.04
$\delta_{\pi A}=\delta_{\pi A}^{pot}+\delta_{\pi A}^{abs}$	−9.60	1.89	8.17	2.22	29.0	77.44	0.59
Data of phase-shift analysis[78]	−8.40	1.75	9.05	2.30	30.6	79.6	0.61

that for $T_\pi \lesssim 70$–80 MeV the contribution from the second-order correction $\delta^{(2)}$ (72) is about 10% of the first correction $\delta^{(1)}$ (69) at $T_\pi \lesssim 70$ MeV. However, with decreasing energy, when $T_\pi \lesssim 50$ MeV, the part played by multiple-scattering effects becomes important, particularly for the s-wave phase shift. The scattering length $a_{\pi^4\mathrm{He}}$ in the first order with the πN phase shifts from Ref. 76 is $-0.023\mu^{-1}$, but with allowance for the second order $a_{\pi^4\mathrm{He}} = -0.071\mu^{-1}$, this satisfactorily reproducing the experimental value[77]: Re $a_{\pi^4\mathrm{He}} = -(0.098 \pm 0.028)\mu^{-1}$. The increased importance of the second-order effects is due to the isotopic structure of the problem, large quantities canceling in the lowest approximation [see (68)].

The results of numerical calculations of the π^4He phase shifts and the differential scattering cross section are given, respectively, in Table IV and Fig. 12. The details of the calculation—the actual choice of the form factor, correlation function, etc.—are described in Ref. 19. It can be seen from Table IV that the real parts of the π^4He phase shifts agree to within 20% with the data of the phase-shift analysis of Ref. 78. In the potential theory, it is not possible to describe the imaginary parts. This is the main reason for the discrepancy between the theory and experiment (Fig. 12). The characteristic qualitative feature of the potential behavior of the inelasticity parameters at low energies is[19] their increase (at a given energy) with increasing orbital angular momentum, i.e., $\eta_S < \eta_P < \eta_P$, etc. [$\eta_L = \exp(-2\,\mathrm{Im}\,\delta_L)$]. This is naturally explained by the increasing importance of the centrifugal barrier. Therefore, the deviation observed at $T_\pi \sim 50$ MeV (see Table IV, and also Ref. 79) in the data of the phase-shift analysis can be regarded as a manifestation of the pion-absorption channel.

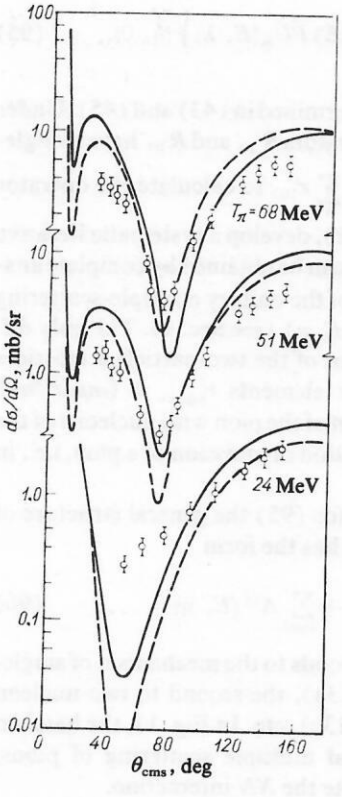

FIG. 12. Differential cross sections of π^+ scattering by ^{4}He. The data are taken from Refs. 29 and 31; the curves are the results of the potential calculation in the unitary approach of Ref. 19, the broken curves representing the first-order approximation ($\delta^{(1)}$), and the continuous curves $\delta^{(1)}+\delta^{(2)}$.

5. ALLOWANCE FOR THE ABSORPTION CHANNEL IN THE UNITARY APPROACH

We now generalize the formalism of Sec. 4 to the case in which the pion number is not conserved. We shall assume that the solution to the potential problem determined by the Hamiltonian (36) is known and restrict ourselves to collision energies insufficient for the production of a second pion, i.e., $T_\pi < 250$ MeV. In this case, the problem of taking into account the absorption channel can be reduced[16] to the two-channel problem with the Hermitian Hamiltonian

$$\mathcal{H} = \hat{H} + \eta\hat{R}, \tag{86}$$

where

$$\hat{H} = \begin{pmatrix} H_1 & 0 \\ 0 & \tilde{H}_0 \end{pmatrix}, \quad \hat{R} = \begin{pmatrix} 0 & R_{10} \\ R_{01} & 0 \end{pmatrix}. \tag{87}$$

Here, the subscript 1 denotes the channel with one pion in the complete configuration space, while 0 denotes the purely nucleon states. The quantities R_{10} and R_{01} are the many-particle pion absorption and creation operators, and η is a dynamical variable, which can vary in the range from 0 to 1. The potential pion-nucleus interaction is described by the Hamiltonian

$$H_1 = K_\pi + H_A + V, \quad V = \sum_{i=1}^{A} v^i, \tag{88}$$

where K_π is the pion kinetic-energy operator, H_A is the nuclear Hamiltonian, and v^i is the potential of the pion interaction with nucleon i of the nucleus. To avoid double counting in the potential scheme, it is necessary to subtract the pole

part of the p_{11}-wave πN interaction. The channel Hamiltonian $\tilde{H}_0$ does not, in contrast to the physical H_A, contain renormalizations of the masses and interactions generated by the operators R_{10} and R_{01}.

The Hamiltonian (86) is obtained by projecting the complete state space of the pion-nucleus system, which contains an arbitrary number of mesons (π, ρ, ω, etc.) in the intermediate states, onto the subspace of states with not more than one pion. A microscopic derivation of this Hamiltonian and a diagrammatic representation of the quantities contained in it are given in Refs. 10 and 66 (see also Chap. 6 in Ref. 5).

Basic equations

We denote by $|m, \mu\rangle$, $E_{m\mu}$, etc., the eigenfunctions and energies of the Hamiltonian $\mathscr{H}$, where the Latin letters distinguish asymptotic states with respect to the number of pions ($m = 0, 1$). We shall denote the corresponding matrix elements of the operator $\hat{R}$ by $R_{m\mu,n\nu}$. The equation for the T scattering matrix has the form[70,16]

$$\frac{d}{d\eta} T_{m\mu,\, n\nu} = -R_{m\mu,\, n\nu} - 2\pi i \sum_{s,\,\sigma} T_{m\mu,\, s\sigma}\delta(E - E_{s\sigma}) R_{s\sigma,\, n\nu} \tag{89}$$

with the boundary condition for the elastic channel

$$T_{1\mu,\, 1\nu}(E,\ \eta = 0) = T^{\text{pot}}_{\mu\nu}(E), \tag{90}$$

where $T^{\text{pot}}_{\mu\nu}$ is the potential scattering T matrix corresponding to the Hamiltonian H_1. The matrix elements $R_{m\mu,n\nu}$ are determined by the system of equations

$$\frac{d}{d\eta} R_{m\mu,\, n\nu} = \sum_{s,\,\sigma} R_{m\mu,\, s\sigma} R_{s\sigma,\, n\nu} \left(\frac{1}{E_{m\mu} - E_{s\sigma} - i\delta} + \frac{1}{E_{n\nu} - E_{s\sigma} + i\delta} \right) \tag{91}$$

with the boundary condition $R_{m\mu,n\nu}(\eta = 0) = \langle \psi_{m\mu} | R | \psi_{n\nu} \rangle$, where $|\psi_{m\mu}\rangle$ are the eigenfunctions of the channel Hamiltonian $\hat{H}$ (87).

In the problem of pion elastic scattering, our aim is to reduce the multichannel problem effectively to a two-particle problem. This will make it possible to obtain an explicit expression for the correction to the pion-nucleus phase shifts corresponding to allowance for the pion absorption. In the presence of the absorption channel, this procedure, in contrast to the optical-potential method presented in Sec. 4, consists of two stages. In the first, the purely nucleon channel (index 0) is separated from the channel with one pion (index 1), and in the second the elastic channel is separated from the inelastic channels in the subspace of wave functions of the pion-nucleus system. We note that in the second stage the problem is reduced to the one already solved in Sec. 4. If this program is implemented (see Appendix 2), the following expression can be obtained for the pion-nucleus phase shifts[16]:

$$\delta_{\pi A}(k) = \delta^{\text{pot}}_{\pi A}(k) + \delta^{\text{abs}}_{\pi A}(k) \tag{92}$$

with allowance for the boundary condition (90). Here, $\delta^{\text{pot}}_{\pi A}$ is the phase shift that arises from taking into account the potential scattering (46). The method of calculating it was presented in Sec. 4. The correction for the absorption has the form

$$\delta^{\text{abs}}_{\pi A}(k) = -\pi\varepsilon_A(k) \int_0^1 d\eta \langle \mathbf{k},\ 0 | \Lambda(E,\ \eta) | \mathbf{k}',\ 0 \rangle, \tag{93}$$

where $\mathbf{k}$ and $\mathbf{k}'$ are the momenta of the pion before and after scattering, and

$$\Lambda(E,\ \eta) = \Omega^{(+)+} \mathscr{R}_0(E,\ \eta)\, \Omega^{(+)}, \tag{94}$$

where $\Omega^{(+)}$ is the Møller operator that describes the distortion of the pion wave:

$$|\psi^{(+)}_{\mathbf{k},\alpha}\rangle = \Omega^{+} |\mathbf{k},\ \alpha\rangle,$$

$|\psi^{(+)}_{\mathbf{k},\alpha}\rangle$ are the scattering states of the Hamiltonian H_1, $|\mathbf{k},\ \alpha\rangle$ are the eigenfunctions of the Hamiltonian $h = K_\pi + H_A$, and $\alpha = 0,1,2,...$ are the states of the nucleus. The operator $\mathscr{R}_0(E, \eta)$ includes all diagrams with intermediate nucleon states. It can be expressed (see Appendix 2) in terms of the original operator $\hat{R}$ (87) by means of a system of integral equations.

In the framework of this formalism, the distortion of the pion wave is taken into account by an expansion with respect to the multiplicity of the πN collisions, i.e., in powers of the two-particle u matrices of the πN scattering (56):

$$|\psi^{(+)}_{k,\,0}\rangle = \left(1 + \int_0^1 d\lambda G^{+}(E)\, \hat{P} U_0(E,\ \lambda)\right) |k,\ 0\rangle, \tag{95}$$

where $G(E)$ and U_0 are determined in (43) and (45). Under the assumption that the operators R_{10} and R_{01} have a single-nucleon nature, i.e., $R_{10} = \sum_i r^i_{10}$, to calculate the operator $\mathscr{R}_0(E, \eta)$ we can, using (91), develop a systematic iterative scheme. Such an expansion can be obtained by complete analogy with the construction of the unitary multiple-scattering series for the operator $U_0(E, \eta)$ (see Sec. 4). The only difference is in the replacement of the two-particle u matrices by the two-particle matrix elements $r^i_{m\mu,n\nu} = \langle m\mu | r^i | n\nu \rangle$ corresponding to interaction of the pion with nucleon i of the nucleus through the absorption or emission of a pion, i.e., in terms of vertex functions.

Thus, with allowance for (95) the general structure of the operator $\Lambda(E, \eta)$ (94) has the form

$$\Lambda(E,\ \eta) = \sum_i \Lambda^i(E,\ \eta) + \sum_{i \neq j} \Lambda^{ij}(E,\ \eta) + \ldots, \tag{96}$$

where the first term corresponds to the mechanism of single-nucleon absorption (Fig. 13a), the second to two-nucleon absorption (Figs. 13b and 13c), etc. In Fig. 13, the hatched circles denote the potential multiple scattering of pions, while the open circles denote the NN interaction.

Two-nucleon absorption mechanism

We obtain an expression for $\delta^{\text{abs}}_{\pi A}$ by assuming that the main contribution is made by the two-nucleon absorption

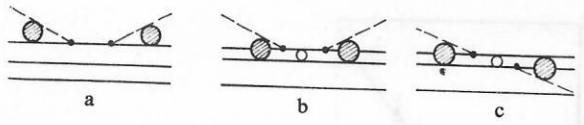

FIG. 13. Graphical representation of the series (96) for the operator $\Lambda(E, \eta)$.

mechanism (Figs. 13b and 13c). For this, it is necessary to calculate the matrix element of the two-particle operator Λ^{12} between the ground-state wave functions of the nucleus. We have already encountered such a problem in calculating the second-order correction $\delta_{\mathrm{Re}}^{(2)}$ (see Sec. 4). A simplifying circumstance in this case is the fact that the range of the absorption operators is appreciably less than the mean internucleon separation. This makes it possible to use the approximation of a local density. As a result, we obtain[85]

$$\delta_{\pi A}^{\mathrm{abs}}(k) = A(A-1)\,\gamma\,[\varepsilon_A(k)/\varepsilon_{2N}(\varkappa)]\,M(\varkappa,\varkappa')\hat{\rho}^2(\mathbf{q});$$

$$M(\varkappa,\varkappa') = -\pi\varepsilon_{2N}(\varkappa)\int_0^1 d\eta \int d\mathbf{p}\,d\mathbf{p}'\,\mathcal{F}(\mathbf{p},\mathbf{p}') \times \Lambda^{12}(\mathbf{p},\mathbf{k};\mathbf{p}',\mathbf{k}';\eta), \qquad (97)$$

where ε_A and ε_{2N} are the level densities for the pion-nucleus and pion–two-nucleon systems, respectively; $\varkappa = |\varkappa| = |\varkappa'|$ is the pion momentum in the $(\pi, 2N)$ center-of-mass system; the factor γ arises as a result of the transition from the $(\pi, 2N)$ center-of-mass system to the pion-nucleus system; $\hat{\rho}^2(q)$ is the Fourier transform of the square of the nuclear density $\rho(r)$, normalized to unity; $\mathbf{q} = \mathbf{k}' - \mathbf{k}$ is the momentum transfer; and $\mathcal{F}(\mathbf{p}, \mathbf{p}')$ is a correlation function of Jastrow type. In the region of low energies, nonrelativistic kinematics for the nucleon can be employed. Then

$$\varkappa = (\mathbf{k} - \epsilon\mathbf{P}_0)/(1+\epsilon),\ \varkappa' = (\mathbf{k}' - \epsilon\mathbf{P}_0')/(1+\epsilon),$$
$$\epsilon = \omega_\pi(k)/2M,\ \omega_\pi(k) = (k^2+\mu^2)^{1/2},$$

where μ and M are the pion and nucleon masses, $\mathbf{P}_0$ and $\mathbf{P}_0'$ are the total momenta of the pair of nucleons (in the approximation in which they are "frozen": $\mathbf{P}_0 = -\mathbf{k}/2A$ and $\mathbf{P}_0' = \mathbf{P}_0 - \mathbf{q}$). Finally, $\gamma\varepsilon_A/\varepsilon_{2N} = (1+\epsilon)/(1+2\epsilon/A)$.

Besides the integration over the momenta in (97), appropriate averaging of the matrix element of Λ^{12} over the spin-isospin variables is assumed. The spin-isospin structure of Λ^{12} is completely analogous to the structure of the $(\pi, 2N)$ amplitude (see Ref. 42). Restricting ourselves to the s and p waves in the $\pi 2N$ scattering and taking the corresponding partial wave from the right-hand side of (97), for nuclei with zero spin and isospin we obtain[16]

$$\delta_l^{\mathrm{abs}}(k) = A(A-1)\,k\,\frac{1+\epsilon}{1+2\epsilon/A}\,[\hat{\rho}_l^2(k)\,(\tilde{B}_0(k) + \alpha k^2\tilde{C}_0(k)) + \beta k^2\tilde{C}_0(k)\,((l+1)\,\hat{\rho}_{l+1}^2(k) + l\hat{\rho}_{l-1}^2(k))/ \times(2l+1)],\quad l = 0, 1, 2\ldots \qquad (98)$$

Here, $\hat{\rho}_l^2$ is the partial-wave harmonic of $\hat{\rho}^2(q)$, and the parameters α and β correspond to the angle transformation from the pion-nucleus center-of-mass system to the $(\pi, 2N)$ system; $\alpha = (1 - 1/2A)(1+\epsilon/2A)/(1+\epsilon)^2$, $\beta = (1+\epsilon/2A)/(1+\epsilon)$. The complex quantities $\tilde{B}_0$ and $\tilde{C}_0$ can be expressed in terms of the πN phase shifts and the $(\pi N, N)$ vertex functions. They have not yet been calculated microscopically in the framework of this formalism.

Energy dependence of the parameters $\tilde{B}_0$ and $\tilde{C}_0$

From the discussion in Sec. 2 it can be seen that as yet little is known about the energy dependence of the parameters B_0 and C_0, which determine the correction for absorption in the optical potential. One can only expect that at low energies ($T_\pi \lesssim 50$ MeV) they will vary weakly, since the relative momentum corresponding to the short-range two-nucleon correlations ($p_c \sim 400$ MeV/c) is appreciably greater than the momentum of a low-energy pion ($p_\pi \sim 120$ MeV/c at $T_\pi \sim 50$ MeV). This assumption was used, in particular, to obtain the expressions (97) and (98) in the approximation of a local density. Therefore, as a first step one can assume that the parameters $\tilde{B}_0$ and $\tilde{C}_0$ are constant in the region of low energies and choose them in such a way as to describe the experimental data for the scattering lengths ($a_0 = \lim \delta_0(k)/k$, $k \to 0$) and volumes ($a_1 = \lim \delta_1(k)/k^3$, $k \to 0$), i.e.,

$$a_0^{\mathrm{exp}} - a_0^{\mathrm{pot}} = \gamma\hat{\rho}_0^2(0)\,\tilde{B}_0;\ \gamma = A(A-1)(1+\epsilon)/(1+2\epsilon/A),$$
$$a_1^{\mathrm{exp}} - a_1^{\mathrm{pot}} = \gamma\,(\delta\tilde{B}_0 + \beta\tilde{C}_0\hat{\rho}_0^2(0)/3), \qquad (99)$$

where β is determined in (98); $\delta = \lim \hat{\rho}_1^2(k)/k^2$, $k \to 0$.

Note that the parameters $\tilde{B}_0$ and $\tilde{C}_0$ differ from the corresponding parameters of the optical potential by inclusion (see Fig. 13) of the effects associated with the distortion of the pion wave in the elastic channel.

Low-energy π^4He scattering

In Ref. 16, the differential and total cross sections of π^4He scattering were calculated with parameters $\tilde{B}_0$ and $\tilde{C}_0$ determined in accordance with (99) (Figs. 14 and 15). It can be seen from Fig. 14 that allowance for the absorption channel makes it possible to describe quantitatively the differential cross sections in the region up to 50 MeV. In Fig. 15, the calculation (continuous curve) satisfactorily reproduces the data of the energy-independent phase-shift analysis of Refs. 31 and 86 for σ_{el} and σ_{tot} at energies below 80 MeV, except for the point for σ_{tot} at 24 MeV.[29] At this energy, the phase-shift analysis given in Ref. 29 is significantly underdetermined, since data are available only for scattering angles larger than 50°. New data for π^4He scattering at 25 MeV were recently given in Ref. 30. The differential cross sections obtained there for π^+ mesons agree well with the old data of Ref. 29, but for π^- mesons they differ in the region of small angles (they are significantly smaller). As was shown in Ref. 87, our calculations describe the new data better. The broken curve in Fig. 15 shows the $1/v$ law (v is the pion velocity) for the total cross section. A deviation from the $1/v$ law is observed at energies higher than 5 MeV.

Thus, the results obtained confirm the assumption made above of the approximate constancy of the parameters $\tilde{B}_0$ and $\tilde{C}_0$ in the region of energies from 0 to 50 MeV. These parameters are directly related to the scattering lengths and volumes (99). One can therefore successfully solve the in-

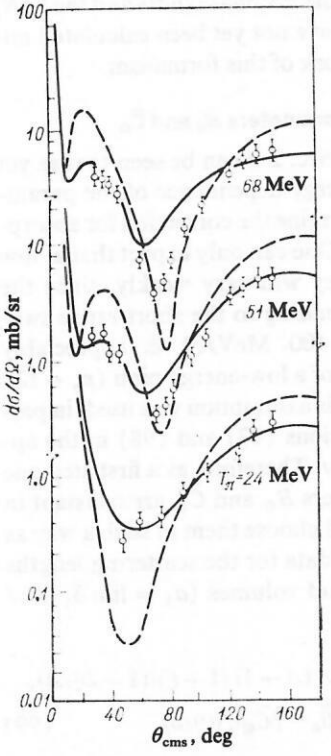

FIG. 14. Differential cross sections of π^+ elastic scattering by ^{4}He. The data are taken from Refs. 29 and 31; the broken curves represent the potential calculation ($\delta^{(1)} + \delta^{(2)}$); the continuous curves take into account the correction for absorption. The figure is taken from Ref. 16.

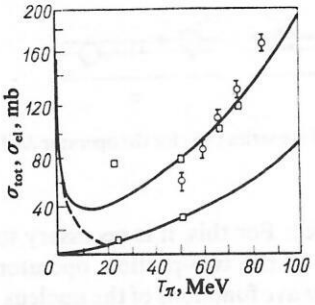

FIG. 15. Total (upper curve) and total elastic (lower curve) cross section of π^4He scattering calculated in the unitary approach with allowance for the absorption channel. The open squares are the results of the phase-shift analysis of Refs. 29 and 31, the open circles are the data of the phase-shift analysis for π^- scattering by ^{4}He from Ref. 86, and the broken curve is the $1/v$, dependence for σ_{tot}. The figure is taken from Ref. 16.

verse problem of determining the shifts and widths of the p orbits of the pionic atoms of light nuclei from data on low-energy scattering.

Effect of subtracting the pole term in the p_{11} wave

We have considered three variants of potential calculation of the quantities $a_{0,1}$ (Table V). In the first row, we give the results using the data of the phase-shift analysis of Ref. 76 without subtraction of the pole term of the p_{11}-wave πN interaction. The value obtained for a_0 agrees well with the result of the calculation of the π^4He scattering length made in Ref. 90 in the framework of multiple-scattering theory. The parameters $\widetilde{B}_0$ and $\widetilde{C}_0$ agree qualitatively with the standard values for B_0 and C_0 in the optical potential (see Table I), and $\mathrm{Re}\,\widetilde{B}_0/\mathrm{Im}\,\widetilde{B}_0 \simeq -1$. The second row corresponds to exclusion of the p_{11} wave. It can be seen that the results are changed little, this reflecting the small value of the p_{11} phase shifts ($\lesssim 2^0$) up to energies of 200 MeV. Finally, the third row corresponds to exclusion of the pole term in the p_{11} wave. In this case, the values of $a_{0,1}$ have changed appreciably, and, as a result, the parameters $\widetilde{B}_0$ and $\widetilde{C}_0$ have changed too. This is a consequence of the fact that in the p_{11} wave there is a strong mutual cancellation of the pole and the non-pole term. For example, the scattering volume in the p_{11} wave is $-0.082\mu^{-3}$,[91] whereas in the first Born approximation, corresponding to the pole approximation, the value $-0.243\mu^{-3}$ is obtained for it, i.e., a value three times greater than the experimental value. The calculations shown in Fig. 14 correspond to the last variant.

6. CONCLUSIONS

We have considered problems in the theory of low-energy pion-nucleus scattering. We have paid much attention to the analysis of the optical model (Secs. 2 and 3) from the point of view of its applications to the description of experimental data. Despite great progress in the development of the formal aspects of the optical model, the practically employed schemes are not yet free of adjustable parameters. This is due to the important part played by the second-order effects, allowance for which greatly complicates the computational schemes. Since a number of the second-order effects have a tendency to mutual cancellation, the results of the calculations are sensitive to the approximations employed. Consistent allowance for the pion-absorption channel remains a problem. Of great importance here is the consistency between the theory and the unitarity condition. Formally, consistency can be achieved in, for example, the framework

TABLE V. Length a_0, volume a_1, and parameters $\widetilde{B}_0$ and $\widetilde{C}_0$ of the absorption correction for π^4He scattering.

Phase shifts of πN scattering	a_0^{pot}, μ^{-1}	a_1^{pot}, μ^{-3}	$\widetilde{B}_0$, μ^{-4}	$\widetilde{C}_0$, μ^{-6}
RSL (Ref. 76)	−0.071	0.367	−0.024+i0.026	−0.30+i0.146
RSL ($-p_{11}$)	−0.067	0.394	−0.027+i0.026	−0.38+i0.146
RSL ($-p_{11}^{pole}$)	−0.055	0.508	−0.038+i0.026	−0.694+i0.146

Note. The experimental value for $a_0^{exp} = (-0.098 - i0.030)\mu^{-1}$,[77] and the value $a_1^{exp} = (0.258 + i0.054)\mu^{-3}$ is taken from Ref. 38.

of a three-particle model (see Sec. 3) for the optical potential. However, in the practical implementation, in which the parameter characterizing the nucleon binding energy becomes a free parameter, the optical potential ceases to reproduce correctly the threshold properties of the scattering amplitude.

In this connection, the unitary method of describing the pion-nucleus interaction expounded in Secs. 4 and 5 is promising. Besides the correct separation of the potential interaction from the nonpotential interaction, the unitarization method effectively takes into account the higher corrections, this leading to rapidly converging iterative series. In the method of evolution with respect to the coupling constant,[20,69,70] such a series can be constructed directly to calculate the pion-nucleus phase shifts. This, in particular, makes it possible to avoid the additional approximations associated with the need for off-shell continuation of the optical potential in the solution of the Lippmann-Schwinger equation for the scattering amplitude.

The rigorous formulation of the optical model,[5,66] (see also Sec. 5) prescribes, in particular, subtraction of the pole term of the p_{11} wave of the πN interaction in the construction of the potential block of the optical potential in order to avoid double counting when allowance is made for the pion-absorption channel. Because of the strong mutual cancellation of the pole and nonpole parts of the πN scattering amplitude in the p_{11} wave, this procedure can strongly influence the relationship between the potential term of the optical potential and the term in it responsible for the absorption (see Ref. 11, and also Table V). In this connection, one must exercise caution with respect to the comparison of the microscopically calculated parameters B_0 and C_0, which determine the absorption correction, with their phenomenological values determined from the shifts and widths of the levels of the pionic atoms. In the variants of the optical potentials considered above, the pole term of the p_{11} wave was not separated.

Further investigation is required to study the influence of the short-range NN correlations on the pion propagation in the nucleus that arise as a result of ρ-meson exchange. In Refs. 47–50, this mechanism was ascribed a decisive role in the dynamics of the pion-nucleus interaction. In the optical models developed in the p space (Sec. 3), a satisfactory description of the data at low energies can be achieved without allowance for the short-range correlations. The energy dependence of the parameters that determine the correction for pion absorption is still little studied.

In this review, we have not touched on the problem of taking into account the Coulomb interaction, which strongly influences the scattering of low-energy pions. Rigorous allowance for this interaction is a complicated problem, and significant progress has been achieved here only recently for the three-body case (see Ref. 92). An analysis of the problems that arise when the Coulomb potential is introduced in the Watson multiple scattering expansion can be found in Ref. 93. In the majority of calculations in the optical model (see, for example, Refs. 40 and 52), the Coulomb potential is simply added to the strong potential, this corresponding to neglect of the Coulomb excitation and Coulomb interaction in the intermediate states in the construction of the optical potential.[93] These effects can play an important part in the description of the interaction of slow pions with heavy nuclei, when the characteristic scales of the strong and Coulomb (Bohr radius) interactions become comparable.[93,94] These remarks also apply to the unitary method of describing pion-nucleus scattering (Secs. 4 and 5). Here, the Coulomb interaction is taken into account by means of the well-known formula for the amplitude of scattering by two potentials.[8,18] The Coulomb corrections to the phase shifts are calculated in accordance with the procedure developed recently in Ref. 95. Correct allowance for the Coulomb interaction is particularly important in the analysis of simultaneous data for π^+ and π^- mesons, such data being a source of information on the proton and neutron distributions in nuclei,[27,88] on the possible violation of charge symmetry,[89] and also in the description of the shifts and widths in heavy pionic atoms.[93,96]

The final aim of the theoretical schemes considered here was to express the pion-nucleus characteristics in terms of the two-body πN scattering amplitudes (or the two-body u matrices in the unitary approach). To describe the πN interaction, potential models with a separable interaction having parameters determined by fitting to the data of phase-shift analysis in each partial wave have been used. A defect of such a description is the lack of consistency with the crossing-symmetry condition,[97] which relates the partial-wave amplitudes in different channels. When allowance is made for the absorption channel, the theory also contains the form factor of the πNN vertex, which cannot be determined uniquely from data on on-shell πN scattering. At the same time, the results of calculations of the characteristics of the πNN systems[11,98] are sensitive to the actual choice of the πNN form factors. To eliminate these uncertainties, it is necessary to go beyond the framework of the potential description of the πN interaction. Great hopes in this direction are now placed in the chiral theory of πN interactions based on the fundamental principles of quantum chromodynamics. In its semiphenomenological realization—the chiral bag model (see Ref. 99)—a natural cutoff parameter arises; this is the radius of the quark bag, which determines the required vertex functions.

Recently, a start has been made on the experimental study of various inelastic processes at low energies such as pion absorption,[53,100] quasielastic knockout,[101] inelastic scattering with the excitation of nuclear states,[88] etc. Therefore, in what follows great attention will be devoted to simultaneous analysis of both the elastic and the inelastic interactions of pions with nuclei. The inelastic processes are more sensitive to the details of the nuclear structure and to the dynamics of the pion-nucleus interaction.

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APPENDIX 1

We shall prove that the system of equations (42)–(44) is equivalent to the original equation (39) for the T matrix.

Substituting (43) in (42), we obtain by a simple regrouping of the terms

$$\frac{d}{d\lambda}T_0(E,\ \lambda)=-V(\lambda)-2\pi i T_0(E,\ \lambda)\hat{P}\delta(E-h)V(\lambda)$$
$$-2\pi i M(E,\ \lambda)\hat{Q}\delta(E-h)V(\lambda), \qquad (A1.1)$$

where

$$M(E,\ \lambda)=(1+2\pi i T(E,\ \lambda)\delta(E-h))\hat{P}K(E,\ \lambda). \qquad (A1.2)$$

Equation (A1.1) will be equivalent to (39) if $M(E, \lambda) = T(E, \lambda)$. Using (42) and (44), for the derivative of $M(E, \lambda)$ with respect to $d\lambda$ we obtain the expression

$$\frac{d}{d\lambda}M(E,\ \lambda)=[1+2\pi i T(E,\ \lambda)\delta(E-h)]\hat{P}U_0(E,\ \lambda). \qquad (A1.3)$$

Using (42), we find from this equation that on the subspace of functions formed by the operator $\hat{P}$ (ground state of the nucleus)

$$\hat{P}M(E,\ \lambda)\hat{P}=T_0(E,\ \lambda). \qquad (A1.4)$$

In the general case, substitution of (43) in (A1.3) and the use of (A1.2) and (A1.4) lead to an equation of the form

$$\frac{d}{d\lambda}M(E,\ \lambda)=-V(\lambda)-2\pi i M(E,\ \lambda)\delta(E-h)V(\lambda) \qquad (A1.5)$$

with the boundary condition $M(E, \lambda = 0) = 0$, which follows from (A1.2) and (44). It follows from (A1.5) [cf. (39)] that $M(E, \lambda) = T(E, \lambda)$, and therefore Eq. (A1.1) is indeed equivalent to the original equation (39).

APPENDIX 2

We obtain a system of equations for the operator $\mathscr{R}_0(E, \eta)$, which determines the absorption correction (93). It is convenient to introduce the wave functions $|\psi_\mu^m\rangle$, $|\psi_\nu^n\rangle$, etc., of the channel Hamiltonian H (87) and regard all the required quantities as operators that act on this space of functions. In particular, instead of the original operator $\hat{R}$ (87) we obtain the new η-dependent operator $R(\eta)$ determined by

$$R_{m\mu,\ n\nu}\equiv\langle m\mu\mid\hat{R}\mid n\nu\rangle=\langle\psi_\mu^m\mid R(\eta)\mid\psi_\nu^n\rangle. \qquad (A2.1)$$

It is obvious that $R(\eta = 0) = \hat{R}$. The equation for the T matrix (89) can be written in the form

$$\frac{d}{d\eta}T(E,\ \eta)=-R(\eta)-2\pi i T(E,\ \eta)\hat{\delta}(E-H)R(\eta), \qquad (A2.2)$$

where $\hat{\delta}(E-H)$ is the 2×2 matrix

$$\hat{\delta}(E-H)=\begin{pmatrix}\delta(E-H_1) & 0\\ 0 & \delta(E-\tilde{H}_0)\end{pmatrix}. \qquad (A2.3)$$

1. Separation of the nucleon channel

We decompose the space of the state vectors of the Hamiltonian $\mathscr{H}$ (86) into two subspaces by means of the projection operators $\hat{q}_1$ and $\hat{q}_0$, where $\hat{q}_1$ projects onto the states with one pion at each point of the space, and $\hat{q}_0$ onto the pionless states. In complete analogy with the derivation of the system of equations (42)–(44), we can obtain for the submatrix $T_{11}\equiv\hat{q}_1\hat{T}\hat{q}_1$ in which we are interested the equation

$$\frac{d}{d\eta}T_{11}(E,\ \eta)=-R_1(E,\ \eta)-2\pi i T_{11}(E,\ \eta)\hat{q}_1\delta(E-H_1)R_1(E,\ \eta), \qquad (A2.4)$$

where the effective energy-dependent operator $R_1(E, \eta)$ is determined by the system of equations

$$R_1(E,\ \eta)=R(\eta)+2\pi i B(E,\eta)\hat{q}_0\delta(E-\tilde{H}_0)R(\eta); \qquad (A2.5)$$

$$\frac{d}{d\eta}B(E,\ \eta)=-R_1(E,\ \eta)+2\pi i R_1(E,\ \eta)\hat{q}_1\delta(E-H_1)B(E,\ \eta). \qquad (A2.6)$$

The boundary conditions for Eqs. (A2.4) and (A2.6) have the form

$$T_{11}(E,\ \eta=0)=T^{\text{pot}}(E),\ B(E,n=0)=0, \qquad (A2.7)$$

where T^{pot} is the matrix of potential scattering corresponding to the Hamiltonian H_1. The non-Hermitian part of the operator $R_1(E, \eta)$ arises from the presence of the absorption channel.

2. Separation of the inelastic channels

Equation (A2.4) is completely analogous to Eq. (42) for the T matrix of potential scattering. Therefore, the separation of the elastic channel from the inelastic channels is made by a system of equations of the form (42)–(44). We denote by $\mathscr{T}_0$ the elastic-scattering submatrix of the T_{11} matrix, i.e., $\hat{P}T_{11}\tilde{P}$ ($\tilde{P}$ is the operator of projection onto the ground state of the nucleus). The equation for this quantity has the form

$$\frac{d}{d\eta}\mathscr{T}_0(E,\ \eta)=-\mathscr{R}_0(E,\ \eta)-2\pi i\mathscr{T}_0(E,\ \eta)\hat{P}\delta(E-H_1)\mathscr{R}_0(E,\ \eta), \qquad (A2.8)$$

where the operator $\mathscr{R}_0(E, \eta)$ is expressed in terms of the operator $R_1(E, \eta)$ by the system of equations

$$\mathscr{R}_0(E,\ \eta)=R_1(E,\ \eta)+2\pi i D(E,\ \eta)\hat{Q}\delta(E-H_1)R_1(E,\ \eta); \qquad (A2.9)$$

$$\frac{d}{d\eta}D(E,\ \eta)=-\mathscr{R}_0(E,\ \eta)+2\pi i\mathscr{R}_0(E,\ \eta)\hat{P}\delta(E-H_1)D(E,\ \eta) \qquad (A2.10)$$

with the boundary condition $D(E, \eta = 0) = 0$. The operator $\hat{Q}=\hat{q}_1-\hat{P}$ projects onto the excited states of the nucleus. In matrix form, Eq. (A2.8) can be written as

$$\frac{d}{d\eta}\mathscr{T}_0(\mathbf{k},\ \mathbf{k}';\ E,\ \eta)=-\langle\psi^{(+)}_{\mathbf{k},\ 0}\mid\mathscr{R}_0(E,\ \eta)\mid\psi^{(+)}_{\mathbf{k}',\ 0}\rangle$$
$$-2\pi i\varepsilon_A(k)\int\frac{d\mathbf{n}''}{4\pi}\mathscr{T}_0(\mathbf{k},\ \mathbf{k}'';\ E,\ \eta)\langle\psi^{(+)}_{\mathbf{k}'',\ 0}\mid\mathscr{R}_0(E,\ \eta)\mid\psi^{(+)}_{\mathbf{k}',\ 0}\rangle, \qquad (A2.11)$$

where $\mathbf{k}''=kn''$, $k=|\mathbf{k}|=|\mathbf{k}'|$, $\mathbf{k}$ and $\mathbf{k}'$ are the momenta of the pion before and after the collision, and $\varepsilon_A(k)$ is the density of the scattering states. The partial-wave expansion of the quantities that occur in this equation in conjunction with allowance for the boundary condition (A2.7) gives the expression (92) for the π-nucleus phase shifts.

The system of equations (A2.9)–(A2.10) in conjunction with (A2.1), (A2.5), and (A2.6) completes the determination of the operator $\mathscr{R}_0(E, \eta)$ in terms of the original operator $R(\eta)$. Using it, we can construct an iterative solution, expanding $\mathscr{R}_0(E, \eta)$ in powers of the operator $R(\eta)$. The first two terms of this expansion have the form

$$\mathscr{R}_0(E, \eta) = R(\eta) - 2\pi i \int_0^\eta d\eta_1 R(\eta_1)\, \hat{q}_0 \delta(E - \tilde{H}_0)\, R(\eta)$$
$$- 2\pi i \int_0^\eta d\eta_1 R(\eta_1)\, \hat{q}_1 \hat{Q} \delta(E - H_1)\, R(\eta). \qquad \text{(A2.12)}$$

[1)]By low-energy pions we mean particles whose kinetic energy in the laboratory system satisfies $T_\pi \leqslant 70$–80 MeV.

[2)]In the relativistic case, the theory can be constructed[64] on the basis of the Bethe-Salpeter equation.

[3)]Section 5 is devoted to the problem of taking into account the pion-absorption channel.

[4)]In relations like (46) that have a general nature, we do not indicate explicitly the angular momentum, spin, etc., indices.

[1]J. Hufner, Phys. Rep. **21C**, 1 (1975).
[2]T. I. Kopaleĭshvili, Fiz. Elem. Chastits At. Yadra **10**, 429 (1979) [Sov. J. Part. Nucl. **10**, 167 (1979)].
[3]A. W. Thomas and R. H. Landau, Phys. Rep. **58**, 121 (1980).
[4]E. Oset, H. Toki, and W. Weise, Phys. Rep. **83**, 281 (1982).
[5]T. I. Kopaleĭshvili, *Voprosy teorii pion-yadernogo vzaimodeĭstviya* (*Problems in the Theory of the Pion-Nucleus Interaction*), Énergoatomizdat, Moscow (1984).
[6]J. M. Eisenberg and D. S. Koltun, *Theory of Meson Interactions with Nuclei*, Wiley, New York (1980).
[7]V. G. Kirillov-Ugryumov, Yu. P. Nikitin, and F. M. Sergeev, *Atomy i mezony* (*Atoms and Mesons*), Atomizdat, Moscow (1980).
[8]M. L. Goldberger and K. M. Watson, *Collision Theory*, Wiley, New York (1964) [Russian translation published by Mir, Moscow (1967)].
[9]A. K. Kerman, H. McManus, and R. M. Thaler, Ann. Phys. (N.Y.) **8**, 551 (1959).
[10]T. Mizutani and D. S. Koltun, Ann. Phys. (N.Y.) **109**, 1 (1977).
[11]B. Blankleider and I. R. Afnan, Phys. Rev. C **24**, 1572 (1981); I. R. Afnan and B. Blankleider, Phys. Lett. **B93**, 367 (1980).
[12]Y. Avishai and T. Mizutani, Nucl. Phys. **A326**, 352 (1979); **A338**, 377 (1980); **A352**, 399 (1981).
[13]L. D. Faddeev, Zh. Eksp. Teor. Fiz. **39**, 1459 (1961) [Sov. Phys. JETP **12**, 1014 (1961)]; Tr. Mat. Inst. V. A. Steklov **69**, 1 (1963).
[14]P. C. Tandy, E. F. Redish, and D. Bolle, Phys. Rev. C **16**, 1924 (1977).
[15]F. Cannata, J. P. Dedonder, and F. Lenz, Ann. Phys. (N.Y.) **143**, 84 (1982).
[16]V. B. Belyaev and M. Kh. Khankhasayev, Phys. Lett. **B137**, 299 (1984).
[17]V. B. Belyaev, D. A. Kirzhnits, N. Zh. Takibaev, and M. Kh. Khankhasaev, Yad. Fiz. **32**, 1124 (1980) [Sov. J. Nucl. Phys. **32**, 578 (1980)].
[18]M. Kh. Khankhasaev, Yad. Fiz. **36**, 633 (1982) [Sov. J. Nucl. Phys. **36**, 369 (1982)].
[19]M. Kh. Khankhasaev, Yad. Fiz. **37**, 1196 (1983) [Sov. J. Nucl. Phys. **37**, 710 (1983)].
[20]D. A. Kirzhnits, Zh. Eksp. Teor. Fiz. **49**, 1544 (1965) [Sov. Phys. JETP **22**, 1059 (1966)].
[21]R. R. Johnson *et al.*, Nucl. Phys. **A296**, 444 (1978).
[22]R. R. Johnson *et al.*, Phys. Rev. Lett. **43**, 844 (1979); Can. J. Phys. **57**, 775 (1979).
[23]D. R. Gill *et al.*, Phys. Rev. C **26**, 1306 (1982).
[24]J. Amman *et al.*, Phys. Rev. Lett. **35**, 426 (1975); S. A. Dytman *et al.*, Phys. Rev. Lett. **38**, 1059 (1977); Phys. Rev. C **18**, 2316 (1978); **19**, 971 (1979).
[25]D. J. Malbrough *et al.*, Phys. Rev. C **17**, 1395 (1978); M. A. Moinester *et al.*, Phys. Rev. C **18**, 2678 (1978); F. E. Bertrand *et al.*, Phys. Rev. C **20**, 1884 (1979); B. M. Preedom *et al.*, Phys. Rev. C **23**, 1134 (1981).
[26]F. E. Obeshain *et al.*, Phys. Rev. C **27**, 2753 (1983).
[27]S. E. Dam *et al.*, Phys. Rev. C **25**, 2574 (1982).
[28]R. P. Redwine, "Low-energy pion-nucleus scattering," in: AIP Conf. Proc. No. 54: Meson-Nuclear Physics, 1979, Houston (ed. E. V. Hungerford, III), AIP, New York (1979), p. 501.
[29]M. E. Nordberg and K. F. Kinsey, Phys. Lett. **20**, 692 (1966).
[30]G. Fournier *et al.*, Nucl. Phys. **A426**, 542 (1984).
[31]K. M. Crowe *et al.*, Nucl. Phys. **180**, 1349 (1969).
[32]Yu. A. Shcherbakov *et al.*, Nuovo Cimento **A31**, 249, 262 (1976).
[33]D. Axen *et al.*, Nucl. Phys. **A256**, 387 (1976); B. Balestri *et al.*, Nucl. Phys. **A392**, 217 (1983).
[34]L. Kisslinger, Phys. Rev. **98**, 761 (1955).
[35]G. E. Brown, B. K. Jennings, and V. I. Rostokin, Phys. Rep. **50**, 229 (1979).
[36]M. Thies, Phys. Lett. **B 63**, 43 (1976).
[37]G. G. Bunatyan and Yu. S. Pol', Yad. Fiz. **25**, 535 (1977) [Sov. J. Nucl. Phys. **25**, 287 (1977)].
[38]K. Stricker, H. McManus, and J. A. Carr, Phys. Rev. C **19**, 929 (1979).
[39]K. Stricker, J. A. Carr, and H. McManus, Phys. Rev. C **22**, 2043 (1980).
[40]R. Seki and K. Masutani, Phys. Rev. C **27**, 2799 (1983); R. Seki, K. Masutani, and K. Yazaki, Phys. Rev. C **27**, 2817 (1983).
[41]N. J. Digiacomo *et al.*, Phys. Lett. **B66**, 421 (1977).
[42]M. Ericson and T. E. O. Ericson, Ann. Phys. (N.Y.) **36**, 32 (1966).
[43]R. Mach, Nucl. Phys. **A205**, 56 (1973).
[44]G. Backenstoss, Ann. Rev. Nucl. Sci. **20**, 467 (1970) [Russian translation published in Usp. Fiz. Nauk **107**, 405 (1972)].
[45]L. Tauscher, in: *Proc. of the First Course of the Intern. School of Physics of Exotic Atoms*, Erice (1977).
[46]C. J. Batty, Fiz. Elem. Chastits At. Yadra **13**, 164 (1982) [Sov. J. Part. Nucl. **13**, 71 (1982)].
[47]J. Chai and D. O. Riska, Nucl. Phys. **A329**, 429 (1979).
[48]E. Oset, W. Weise, and R. Brockman, Phys. Lett. **B82**, 344 (1979); K. Shimizu and A. Faessler, Nucl. Phys. **A333**, 495 (1980); R. Brockman, B. K. Jennings, and R. Rockmore, Phys. Lett. **B98**, 236 (1981).
[49]H. McManus and D. O. Riska, J. Phys. G **7**, L153 (1981); D. O. Riska and H. Sarafian, Phys. Lett. **B95**, 185 (1980).
[50]G. Baym and G. E. Brown, Nucl. Phys. **A247**, 395 (1975).
[51]R. H. Landau and M. McMillan, Phys. Rev. C **8**, 2094 (1973).
[52]R. H. Landau and A. W. Thomas, Nucl. Phys. **A302**, 461 (1978); R. H. Landau, S. C. Phatak, and F. Tabakin, Ann. Phys. (N.Y.) **78**, 299 (1973).
[53]I. Navon *et al.*, TRIUMF-Preprint TPI-PP-83-14, Vancouver (1983).
[54]R. H. Landau and F. Tabakin, Nucl. Phys. **A231**, 445 (1974).
[55]J. Hufner and F. Iachello, Nucl. Phys. **A247**, 441 (1975).
[56]L. S. Kisslinger and F. Tabakin, Phys. Rev. C **9**, 188 (1974).
[57]D. J. Ernst, C. M. Shakin, and R. M. Thaler, Phys. Rev. C **9**, 1374 (1974).
[58]R. Mach and M. G. Sapozhnikov, J. Phys. G **10**, 147 (1984).
[59]G. F. Chew and M. L. Goldberger, Phys. Rev. **87**, 778 (1952); G. F. Chew and G. C. Wick, Phys. Rev. **85**, 637 (1952).
[60]J. Revai, Nucl. Phys. **A208**, 20 (1973).
[61]R. Aaron, R. D. Amado, and J. E. Young, Phys. Rev. **174**, 2022 (1968).
[62]G. Miller, Phys. Rev. C **10**, 1242 (1974).
[63]L. Heller, G. E. Bohanon, and F. Tabakin, Phys. Rev. C **13**, 742 (1976).
[64]L. Celenza, L. C. Liu, and C. M. Shakin, Phys. Rev, C **11**, 1593 (1975); L. C. Liu and C. M. Shakin, Phys. Rev. C **20**, 2339 (1979); Nuovo Cimento **A53**, 142 (1979); J. De Kam and C. K. Wafelbakker, Phys. Rev. C **26**, 570 (1982).
[65]T. S. Lee and S. Charkavarti, Phys. Rev. C **16**, 273 (1977); M. Wakamatsu, Nucl. Phys. **A312**, 427 (1978); **A340**, 289 (1980).
[66]D. S. Koltun and T. Mizutani, Phys. Rev. C **22**, 1657 (1980).
[67]A. M. Badalyan and Yu. A. Simonov, Fiz. Elem. Chastits At. Yadra **6**, 299 (1975) [Sov. J. Part. Nucl. **6**, 119 (1975)].
[68]J. H. Sloan, Phys. Rev. **165**, 1578 (1968).
[69]D. A. Kirzhnits, in: *Problemy teoreticheskoĭ fiziki, pamyati I. E. Tamma* (*Problems of Theoretical Physics, Dedicated to the Memory of I. E. Tamm*), Nauka, Moscow (1972), p. 74.
[70]D. A. Kirzhnits, G. Yu. Kryuchkov, and N. Zh. Takibaev, Fiz. Elem. Chastits At. Yadra **10**, 741 (1979) [Sov. J. Part. Nucl. **10**, 289 (1979)].
[71]D. A. Kirzhnits and N. Zh. Takibaev, Yad. Fiz. **35**, 700 (1982) [Sov. J. Nucl. Phys. **35**, 405 (1982)].
[72]H. Hellman, *Einführung in die Quantenchemie*, Leipzig (1937); R. P. Feynman, Phys. Rev. **56**, 340 (1939).
[73]M. Kh. Khankhasayev, Preprint E4-80-691 [in English], JINR, Dubna (1980).

[74]D. V. Bugg, A. A. Carter, and J. R. Carter, Nucl. Phys. **B58**, 378 (1973).
[75]N. M. Petrov and V. V. Peresypkin, Ukr. Fiz. Zh. **23**, 117 (1978).
[76]G. Rowe, M. Solomon, and R. H. Landau, Phys. Rev. C **18**, 584 (1978).
[77]G. Backenstoss *et al.*, Nucl. Phys. **A232**, 519 (1974).
[78]K. M. Crowe *et al.*, Nucl. Phys. **180**, 1349 (1969).
[79]F. Nichitiu *et al.*, Nuovo Cimento **A67**, 1 (1982).
[80]J. Bailey *et al.*, Phys. Lett. **B58**, 403 (1974).
[81]T. E. O. Ericson, Preprint TH-1098, CERN (1969).
[82]M. Gmitro, S. S. Kamalov, R. Mach, and M. G. Sapozhnikov, Yad. Fiz. **40**, 107 (1984) [Sov. J. Nucl. Phys. **40**, 68 (1984)].
[83]R. A. Eramzhyan, M. Gmitro, S. S. Kamalov, T. D. Kaipov, and R. Mach, Nucl. Phys. **A429**, 403 (1984).
[84]V. K. Samaranayake and W. S. Woolcock, Nucl. Phys. **B48**, 205 (1972).
[85]M. Kh. Khankhasaev, Preprint R4-83-593 [in Russian], JINR, Dubna (1983).
[86]F. Binon *et al.*, Nucl. Phys. **A298**, 499 (1978).
[87]M. Kh. Khankhasaev, in: *Sb. dokl. IX Evropeĭskoĭ konf. po probl. nesk. tel v fizike* (*Proc. of the Ninth European Conf. on Few-Body Problems in Physics*), Tbilisi, 25-31 August (1984), p. 67.
[88]R. Tacik *et al.*, Phys. Rev. Lett. **52**, 1276 (1984); R. J. Sobie *et al.*, TRIUMF-Preprint TH-PP-84-8, Vancouver (1984).
[89]E. Pedroni *et al.*, Nucl. Phys. **A300**, 321 (1978).
[90]K. P. Lohs, Nucl. Phys. **A312**, 297 (1978).
[91]H. Pilkuhn *et al.*, Nucl. Phys. **B65**, 460 (1973).
[92]S. P. Merkur'ev, Yad. Fiz. **24**, 289 (1976) [Sov. J. Nucl. Phys. **24**, 150 (1976)]; Teor. Mat. Fiz. **38**, 201 (1979).
[93]F. Cannata and J. D. Dedonder, Nuovo Cimento **A76**, 468 (1983).
[94]D. A. Kirzhnits and F. M. Pen kov, Zh. Eksp. Teor. Fiz. **82**, 657 (1982) [Sov. Phys. JETP **55**, 393 (1982)].
[95]J. Frolich, H. Pilkuhn, and H. G. Schaile, Phys. Lett. **B121**, 235 (1983); J. Frolich *et al.*, J. Phys. G **6**, 841 (1980).
[96]d'Achard van Ensehut *et al.*, Phys. Lett. **B136**, 24 (1984).
[97]S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory*, Row, Peterson and Co., Evanston, Ill. (1961) [Russian translation published by Izd. Inostr. Lit., Moscow (1963), p. 380].
[98]A. S. Rinat and Y. Starkand, Nucl. Phys. **A393**, 381 (1983).
[99]M. M. Musakhanov, ITEP-179, Moscow (1980); A. W. Thomas, CERN/TH-3368 (1982).
[100]G. Backenstoss *et al.*, in: *Proc. of the Tenth Intern. Conf. on Few Body Problems in Physics*, Vol. II, Karlsruhe (1983), p. 239; K. Aniol *et al.*, in: *Proc. of the Tenth Intern. Conf. on Few Body Problems in Physics*, Vol. II, Karlsruhe (1983), p. 243.
[101]G. W. Butler *et al.*, Phys. Rev. C **26**, 1737 (1982).

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