# Asymptotic LSZ condition and dynamical equations in quantum field theory

A. A. Arkhipov and V. I. Savrin
*Institute of High Energy Physics, Serpukhov*

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Methods that can be used to derive dynamical equations in quantum field theory are reviewed. A new method based on using the asymptotic LSZ condition is described. It is shown that by means of this method equations can be obtained for the wave functions of both scattering and bound states.

## INTRODUCTION

The present paper is devoted to the description of methods that can be used to derive dynamical equations in quantum field theory. Dynamical equations in quantum field theory occupy a central position. At the beginning of the fifties, Bethe and Salpeter, using Feynman's diagram technique, derived a four-dimensional completely relativistic equation for the wave function of a bound state of two Dirac particles with an arbitrary interaction.[1] Gell-Mann and Low soon after gave a formal derivation of the Bethe–Salpeter equation in quantum field theory.[2] Other methods of deriving the Bethe–Salpeter equation were proposed in a number of subsequent studies.[1)] The present paper considers some of these methods.

The methods of deriving the Bethe–Salpeter equation hitherto developed can all be divided into two groups. In the first, there are the methods by means of which the Bethe–Salpeter equation for the wave functions of scattering states can be derived. As a rule, the point of departure is the wave function expressed in terms of the matrix element of the field operators taken in the interaction representation. In the second group are the methods used to derive the Bethe–Salpeter equation for the wave functions of bound states. In these methods, the point of departure is the expression bound-state wave function in terms of the Heisenberg field operators, and these methods are based on investigation of the singularities of the four-point Green's functions with respect to the invariant mass of the two-particle system.

In the present paper, we describe a new universal method suitable for deriving the Bethe–Salpeter equation for both scattering and bound-state wave functions. In addition, the method is very convenient for deriving expressions for the amplitudes of scattering of elementary particles by composite systems.

In deriving the dynamical equations, we shall not use any particular field-theory model but only the facts that provide the basis of the axiomatic formulations of quantum field theory. This means that from the very beginning we avoid discussions associated with divergences of the theory and methods—renormalization procedures—of eliminating them. In addition, we shall assume that there exist vacuum expectation values of the Heisenberg field operators, or Green's functions and matrix elements determining the Bethe–Salpeter wave functions, leaving, of course, on one side the question of the rigorous justification of such an assumption. We shall see that the universality of the method derives from the use of the LSZ asymptotic condition,[4] the validity of which we also assume.

We give a brief summary of the content of the present paper. In Sec. 1, we consider the derivation of the dynamical Bethe–Salpeter equation for the wave function of a system of two interacting particles in the framework of the axiomatic formulation of quantum field theory in Bogolyubov's form. In Sec. 2, this method is generalized to a system of three interacting particles. We also discuss in detail the boundary conditions corresponding to different physical processes in the three-particle system. In Secs. 3 and 4, we formulate asymptotic conditions for elementary and composite particles and discuss the connection between the asymptotic conditions and the singularities of the many-particle Green's functions. Using the asymptotic conditions, we derive expressions for the amplitudes of physical processes in the three-particle system and develop a universal construction for deriving dynamical equations in quantum field theory, to the description of which Sec. 5 is devoted. In Sec. 6, we consider various iterative schemes for calculating the main physical quantities. Section 7 gives the derivation of a three-dimensional dynamical equation for the wave function of elastic scattering of a particle by a bound state of two other particles.

## 1. BETHE–SALPETER EQUATION FOR THE WAVE FUNCTION OF A TWO-PARTICLE SYSTEM IN QUANTUM FIELD THEORY

We define the two-particle Bethe–Salpeter wave function for scattering states by means of the matrix element

$$\Phi_{ab}(x_1x_2) = \langle 0 \mid T(\Phi_a(x_1)\,\Phi_b(x_2)) \mid \Phi_{ab};\ \text{in}\rangle, \quad (1)$$

where $\Phi_a(x_1)$ and $\Phi_b(x_2)$ are the Heisenberg field operators of particles $a$ and $b$; $|\Phi_{ab};\text{in}\rangle$ is the state vector corresponding to the asymptotic configuration as $t \to -\infty$ of the two free particles in the "in" basis. The vector $|\Phi_{ab};\text{in}\rangle$ can be represented as the result of applying creation operators to the vacuum state vector:

$$|\Phi_{ab};\ \text{in}\rangle = a^{+}_{\text{in}} b^{+}_{\text{in}} |0\rangle.$$

In the axiomatic formulation of quantum field theory,[4] the asymptotic "in" and "out" fields are expressed in terms of the Heisenberg fields by means of the Yang–Feldman equations,

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$$\Phi_a(x) = \varphi_a^{\rm in}(x) + \int dy D_a^{\rm ret}(x-y)\, j_a(y)$$
$$= \varphi_a^{\rm out}(x) + \int dy D_a^{\rm adv}(x-y)\, j_a(y),$$

where the current operator $j_a(x)$ is expressed in terms of the Heisenberg field by means of the equation $j_a(x) = \hat{K}_x^a \Phi_a(x)$; $\hat{K}_x^a$ is a differential operator (Klein-Gordon operator in the case of scalar particles, Dirac operator in the case of spinor particles, etc.) satisfying $\hat{K}_x^a D_a^{\rm ret}(x) = \hat{K}_x^a D_a^{\rm adv}(x) = \delta^4(x)$, where $D_a^{\rm ret}(x) = 0$ for $x^0 < 0$ and $D_a^{\rm adv}(x) = 0$ for $x^0 > 0$. From the Yang-Feldman equations, it is readily seen that the asymptotic "in" and "out" fields satisfy the equation

$$\hat{K}_x^a \varphi_a^{\rm in}(x) = 0, \quad \hat{K}_x^a \varphi_a^{\rm out}(x) = 0.$$

The creation and annihilation operators are defined in terms of the asymptotic "in" and "out" fields and smooth normalized solutions of the equation $\hat{K}_x^a f_a(x) = 0$.[4] Below, we describe explicit expressions for these operators.

A dynamical equation for the wave function (1) can be elegantly derived by using the formulation of axiomatic quantum field theory in Bogolyubov's form.[5,6] In this case, the Heisenberg field can be expressed in terms of the $S$ operator and asymptotic fields by[6]

$$\Phi_a(x) = T(\varphi_a^{\rm ex}(x) S) S^+, \tag{2}$$

where the symbol $T$ is understood as follows[2)]: It is assumed that the $S$ operator is a functional of the asymptotic fields that can be represented as an expansion with respect to normal products of the asymptotic fields, after which the $T$ product is defined in accordance with Wick's theorem on the expansion of a time-ordered product with respect to normal products; "ex" denotes "in" or "out," depending on the representation in which the $S$ operator is taken. In what follows, we shall use the "out" representation, and in Eq. (2) we therefore omit the index "out" of the asymptotic field. It can be readily shown that the Yang–Feldman equations follow from the relation (2) if the current operator $j_a(x)$ is identified with the first-order radiative operator[6]

$$j_a(x) = {\rm i}\,\frac{\delta S}{\delta \varphi_a(x)} S^+.$$

Using the expression (2), we obtain[3)]

$$T(\Phi_a(x_1)\Phi_b(x_2)\ldots) = T(\varphi_a(x_1)\varphi_b(x_2)\ldots S) S^+. \tag{3}$$

For the $T$ product of the asymptotic fields, we have the standard representation

$$T(\varphi_a(x)\varphi_a(y)) = :\varphi_a(x)\varphi_a(y): + \frac{1}{\rm i} D_a(x-y), \tag{4}$$

where $D_a(x)$, the causal Green's function, satisfies the equation

$$\hat{K}_x^a D_a(x) = \delta^4(x).$$

Using Eq. (3), we can rewrite the expression (1) for the Bethe–Salpeter wave function in the form

$$\Phi_{ab}(x_1x_2) = \langle 0 \mid T(\varphi_a(x_1)\varphi_b(x_2)S) \mid \Phi_{ab};\ {\rm out}\rangle, \tag{5}$$

where we have used the fact that $|\Phi_{ab};{\rm in}\rangle = S|\Phi_{ab};{\rm out}\rangle$. If we now use the Bogolyubov reduction formula

$$[F(\varphi),\ a^+] = \int dx \frac{\delta F(\varphi)}{\delta\varphi(x)} f_a(x),$$

where $F(\varphi)$ is some functional of the asymptotic fields, and $f_a(x)$ is a smooth normalized solution (of wave-packet type) of the equation $\hat{K}_x^a f_a(x) = 0$, then for the wave function (5) we can obtain the expression

$$\Phi_{ab}(x_1x_2) = \int dy_1\, dy_2 \left\langle 0 \left| \frac{\delta T\varphi_a(x_1)\varphi_b(x_2)S}{\delta\varphi_a(y_1)\delta\varphi_b(y_2)} \right| 0 \right\rangle$$
$$\times \Phi_{ab}^{(0)}(y_1y_2), \tag{6}$$

where $\Phi_{ab}^{(0)}(x_1x_2) = f_a(x_1) f_b(x_2)$ is the wave function of the initial state of the two noninteracting particles.

The two-particle (four-point) Green's function, which is defined by the equation[4)]

$$G_{ab}(x_1x_2;\ y_1y_2) = {\rm i}^2 \langle 0| T(\Phi_a(x_1)\Phi_b(x_2)\overline{\Phi}_a(y_1)\overline{\Phi}_b(y_2))|0\rangle$$
$$= {\rm i}^2 \langle 0| T(\varphi_a(x_1)\varphi_b(x_2)\overline{\varphi}_a(y_1)\overline{\varphi}_b(y_2)S) S^+|0\rangle,$$

can, after partial expansion of the time-ordered product, be represented in the form

$$G_{ab}(x_1x_2;\ y_1y_2)$$
$$= \int dz_1\, dz_2 \left\langle 0 \left| \frac{\delta T(\varphi_a(x_1)\varphi_b(x_2)S)}{\delta\varphi_a(z_1)\delta\varphi_b(z_2)} S^+ \right| 0 \right\rangle$$
$$\times D_a(z_1 - y_1) D_b(z_2 - y_2).$$

By means of this equation, it is now easy to see that the linear relation, (6) which connects the Bethe–Salpeter wave function to the initial wave function of the two free particles, is equivalent to the relation

$$\Phi_{ab}(x_1x_2) = [(G_{ab} * D_a^{-1} D_b^{-1}) * \Phi_{ab}^{(0)}](x_1x_2). \tag{7}$$

The operation * denotes convolution of the functions in the configuration space.

Expanding completely the time-ordered product in the four-point Green's function by means of the generalized Wick theorem,[5] we obtain

$$G_{ab} = G_{ab}^{(0)} + G_{ab}^{(0)} * R_{ab} * G_{ab}^{(0)}, \tag{8}$$

where $G_{ab}^{(0)} = D_a D_b$ is the free two-particle Green's function, and the function $R_{ab}$ has the structure

$$R_{ab} = R_a^{(2)} D_b^{-1} + R_b^{(2)} D_a^{-1} + R_{ab}^{(4)}, \tag{9}$$

where the functions

$$R_i^{(2)}(x;\ y) = \frac{1}{\rm i} \left\langle 0 \left| \frac{\delta^2 S}{\delta\overline{\varphi}_i(x)\delta\varphi_i(y)} S^+ \right| 0 \right\rangle, \quad i = a,\ b \tag{10}$$

are the vacuum expectation values of the second-order radiation operators, and the function

$$R_{ab}^{(4)}(x_1x_2;\ y_1y_2)$$
$$= \frac{1}{i^2} \left\langle 0 \left| \frac{\delta^4 S}{\delta\overline{\varphi}_a(x_1)\delta\overline{\varphi}_b(x_2)\delta\varphi_a(y_1)\delta\varphi_b(y_2)} S^+ \right| 0 \right\rangle \tag{11}$$

is the vacuum expectation value of the fourth-order radiation operator.

After substitution of the expression (8) for the two-particle Green's function in the linear relation (7), we find

$$\Phi_{ab}(x_1x_2)=\Phi^{(0)}_{ab}(x_1x_2)+(G^{(0)}_{ab}*R_{ab}*\Phi^{(0)}_{ab})(x_1x_2)$$
$$=\Phi^{(0)}_{ab}(x_1x_2)+(G^{(0)}_{ab}*R^{(4)}_{ab}*\Phi^{(0)}_{ab})(x_1x_2). \quad (12)$$

The second equation in the relation (12) is a consequence of the stability of the single-particle states. We note that from this stability there follow the relations[5)]

$$D_i*R^{(2)}_i*f_i=0,\quad i=a,\ b. \quad (13)$$

Therefore, the linear relation (7) can be rewritten in the form

$$\Phi_{ab}=(\overline{G}_{ab}*D_a^{-1}D_b^{-1})*\Phi^{(0)}_{ab}, \quad (14)$$

where

$$\overline{G}_{ab}=G^{(0)}_{ab}+G^{(0)}_{ab}*R^{(4)}_{ab}*G^{(0)}_{ab}$$
$$=G_{ab}-G^{(0)}_{ab}*(R^{(2)}_aD_b^{-1}+R^{(2)}_bD_a^{-1})*G^{(0)}_{ab}. \quad (15)$$

We define the function $V_{ab}$ by means of the relation

$$R^{(4)}_{ab}=V_{ab}+V_{ab}*G^{(0)}_{ab}*R^{(4)}_{ab}. \quad (16)$$

Substituting the relation (16) in (12), we arrive at the dynamical equation for the Bethe–Salpeter wave function:

$$\Phi_{ab}(x_1x_2)=\Phi^{(0)}_{ab}(x_1x_2)+(G^{(0)}_{ab}*V_{ab}*\Phi_{ab})(x_1x_2). \quad (17)$$

The inhomogeneous term in Eq. (17) is the wave function of the system of two free noninteracting particles and corresponds to the boundary condition of the scattering problem as $x_1^0\to\infty, x_2^0\to-\infty$. One can readily show that the function $R^{(4)}_{ab}$ is directly related to the amplitude of two-particle elastic scattering. Indeed, using Bogolyubov's reduction formulas, we obtain for the S-operator matrix element corresponding to the two-particle elastic scattering process the expression

$$\langle\Phi_{ab};\ \text{out}\,|S-1|\,\Phi_{ab};\ \text{out}\rangle$$
$$=\mathrm{i}^2\int dx_1\,dx_2\,dy_1\,dy_2\overset{*}{f}_a(x_1)\overset{*}{f}_b(x_2)R^{(4)}_{ab}(x_1x_2;\ y_1y_2)f_a(y_1)f_b(y_2).$$

The Bethe–Salpeter wave function for the bound state of two particles is determined by means of the matrix element

$$\Phi^A_{ab}(x_1x_2)=\langle 0\,|T(\Phi_a(x_1)\Phi_b(x_2))|\,\Phi^A_{ab}\rangle, \quad (18)$$

where $\Phi_a(x_1)$ and $\Phi_b(x_2)$ are, as before, the Heisenberg field operators of particles $a$ and $b$; $|\Phi^A_{ab}\rangle$ is the vector of the bound state of the same particles. For the bound-state vector, we shall also use the notation $|M_A;\mathbf{P},\sigma\rangle$, where $M_A$ is the mass of the bound state, $\mathbf{P}$ is the total momentum of the bound system, and $\sigma$ is the set of all the remaining quantum numbers, continuous and discrete, which together with the mass and the momentum completely characterize the bound state.

Our derivation of the dynamical Bethe–Salpeter equation (17) is not suitable for deriving a dynamical equation for the wave function (18), though it is intuitively clear that the bound-state wave function must satisfy the homogeneous equation

$$\Phi^A_{ab}(x_1x_2)=(G^{(0)}_{ab}*V_{ab}*\Phi^A_{ab})(x_1x_2). \quad (19)$$

This result can be understood as follows. In Eq. (17), we introduce the variables $X=(x_1+x_2)/2, x=x_1-x_2$ and go over to the Fourier transform of the wave function with respect to the variable $X$:

$$\Phi_{ab}(x\mid P)=\int dX\exp(\mathrm{i}PX)\,\Phi_{ab}(X,\ x).$$

Then, with allowance for the translational invariance of the theory, Eq. (17) can be rewritten in the form

$$\Phi_{ab}(x|P)=\Phi^{(0)}_{ab}(x|P)$$
$$+\int dy\,dzG^{(0)}_{ab}(P|x;\ y)V_{ab}(P|y;\ z)\Phi_{ab}(z|P), \quad (20)$$

where

$$G^{(0)}_{ab}(Xx;\ Yy)=G^{(0)}_{ab}(X-Y|x;\ y)$$
$$=(2\pi)^{-4}\int dP\exp[-\mathrm{i}P(X-Y)]\,G^{(0)}_{ab}(P|x;\ y); \quad (21)$$
$$V_{ab}(Xx;\ Yy)=V_{ab}(X-Y\mid x;\ y)$$
$$=(2\pi)^{-4}\int dP\exp[-\mathrm{i}P(X-Y)]V_{ab}(P\mid x;\ y). \quad (21)$$

If $\sqrt{P^2}=M_A<m_a+m_b$, the inhomogeneous term in Eq. (20) obviously vanishes, and we thus arrive at a homogeneous equation for the bound-state wave function:

$$\Phi^A_{ab}(x|\mathbf{P})=\int dy\,dzG^{(0)}_{ab}(P_A|x;\ y)V_{ab}(P_A|y;\ z)\Phi^A_{ab}(z|\mathbf{P}),$$
$$P_A^2=M_A^2,\quad P_A^0\equiv E_A=E(\mathbf{P},\ M_A)=\sqrt{\mathbf{P}^2+M_A^2}. \quad (23)$$

It is, however, clear that the transition to the homogeneous equation also requires specification of the correct boundary conditions, these being obtained from the physical formulation of the problem of describing the bound system.

The rigorous derivation of Eq. (23) is based on investigating the singularities of the two-particle Green's function with respect to the invariant mass of the two-particle system, and we consider it briefly here.

We require the Fourier transform of the two-particle Green's function:

$$(2\pi)^4\delta^4(P-Q)\,\overline{G}_{ab}(P|x;\ y)$$
$$=\int dX\,dY\exp(\mathrm{i}PX-\mathrm{i}QY)\,\overline{G}_{ab}(Xx;\ Yy). \quad (24)$$

The expression for the inverse transformation is

$$\overline{G}_{ab}(Xx;\ Yy)=\overline{G}_{ab}(X-Y|x;\ y)$$
$$=(2\pi)^{-4}\int dP\exp[-\mathrm{i}P(X-Y)]\,\overline{G}_{ab}(P|x;\ y). \quad (25)$$

From the expression for the two-particle Green's function in terms of the vacuum expectation value of the Heisenberg field operators, it can be readily seen that $\overline{G}_{ab}(P|x;y)$ contains a pole singularity at the energy of the bound state. More precisely, for this quantity we can write down the representation

$$\overline{G}_{ab}(P|x;\ y)=\frac{1}{\mathrm{i}}\frac{\sum\limits_{\sigma}\Phi^{A}_{ab}(x|\mathbf{P},\ \sigma)\,\overline{\Phi}^{A}_{ab}(y|\mathbf{P},\ \sigma)}{2E(\mathbf{P},\ M_A)\,[P^0-E(\mathbf{P},\ M_A)+\mathrm{i}0]}+\overline{G}^{\mathrm{Reg}}_{ab}(P|x;\ y), \qquad (26)$$

where $\overline{G}^{\mathrm{Reg}}_{ab}(P|x;\ y)$ does not have a pole singularity at $P^0=E(\mathbf{P},M_A)=\sqrt{\mathbf{P}^2+M_A^2}$. From the representation (26) near the pole corresponding to the bound state, we obtain

$$\overline{G}_{ab}(P|x;\ y)=[\mathrm{i}\,(P^2-M_A^2+\mathrm{i}0)]^{-1}\sum_{\sigma}\Phi^{A}_{ab}(x|\mathbf{P},\ \sigma)\,\overline{\Phi}^{A}_{ab}(y|\mathbf{P},\ \sigma). \qquad (27)$$

The wave functions $\Phi^{A}_{ab}(x|\mathbf{P},\ \sigma)$ are related to the matrix element (18) as follows:

$$\langle 0|T(\Phi_a(x_1)\,\Phi_b(x_2))|M_A;\ \mathbf{P},\sigma\rangle=(2\pi)^{-3/2}\exp(-\mathrm{i}PX)\,\Phi^{A}_{ab}(x|\mathbf{P},\ \sigma);$$

$$\langle M_A;\ \mathbf{P},\ \sigma|T(\overline{\Phi}_a(x_1)\,\overline{\Phi}_b(x_2))|0\rangle=(2\pi)^{-3/2}\exp(\mathrm{i}PX)\,\overline{\Phi}^{A}_{ab}(x|\mathbf{P},\ \sigma).$$

We now note that substitution of (16) in the expression (15) leads to an equation for the two-particle Green's function:

$$\overline{G}_{ab}=G^{(0)}_{ab}+G^{(0)}_{ab}*V_{ab}*\overline{G}_{ab}. \qquad (28)$$

Making a Fourier transformation in this equation by means of (21), (22), and (25), we obtain

$$\overline{G}_{ab}(P|x;\ y)=G^{(0)}_{ab}(P|x;\ y)+\int dx'\,dy'\,G^{(0)}_{ab}(P|x;\ x')\,V_{ab}(P|x';\ y')\,\overline{G}_{ab}(P|y';\ y). \qquad (29)$$

We substitute in Eq. (29) the representation (26) for the two-particle Green's function, after which we multiply both sides of the equation by $(P^2-M_A^2)$ and go to the limit $P^2\to M_A^2$. As a result, we find

$$\Phi^{A}_{ab}(x|\mathbf{P},\ \sigma)=\int dx'\,dy'\,G^{(0)}_{ab}(P_A|x;\ x')\,V_{ab}(P_A|x';\ y')\,\Phi^{A}_{ab}(y'|\mathbf{P},\ \sigma), \qquad (30)$$

$$P_A^2=M_A^2.$$

In the derivation of Eq. (30), it is necessary to take into account the linear independence of the wave functions $\Phi^{A}_{ab}(x|\mathbf{P},\ \sigma)$ corresponding to the different quantum numbers $\sigma$.

Equation (30) is the required Bethe–Salpeter equation for the bound-state wave function. Our rigorous derivation of Eq. (30) can serve as a justification of the heuristic arguments for dropping the inhomogeneous term in Eq. (17) on the transition to the description of the bound states of the two-particle system. The order of the arguments can be reversed. For example, once Eq. (30) has been derived, one can say that to describe the scattering states it is necessary to go over from Eq. (30) to the inhomogeneous equation with a term outside the integral correctly describing the boundary conditions as $x_1^0,\ x_1^0\to-\infty$. In this case, the first method, by means of which Eq. (17) was derived, could serve as a justification of the correctness of such a transition.

In the following section, we shall show how both these methods can be used to investigate three-particle systems in quantum field theory.

## 2. DYNAMICAL EQUATIONS FOR A SYSTEM OF THREE PARTICLES IN QUANTUM FIELD THEORY

We define the wave function of a system of three interacting particles by means of the matrix element

$$\Phi_{abc}(x_1x_2x_3)=\langle 0\ |\ T\,(\Phi_a(x_1)\,\Phi_b(x_2)\,\Phi_c(x_3))\ |\ \Phi_{abc};\ \mathrm{in}\rangle=\langle 0\ |\ T\,(\varphi_a(x_1)\,\varphi_b(x_2)\,\varphi_c(x_3)\,S)\ |\ \Phi_{abc};\ \mathrm{out}\rangle, \qquad (31)$$

where $\Phi_i(x)$, $i=a,\ b,\ c$, are the Heisenberg operators of particles $a$, $b$, and $c$; $\varphi_i(x)$ are the asymptotic "out" fields of these particles, and $\Phi_{abc};\ \mathrm{in}\rangle$ is the state vector corresponding to the asymptotic configuration as $t\to-\infty$ of the three free particles. The vector $|\Phi_{abc};\ \mathrm{out}\rangle$ can be represented as the application of creation operators to the vacuum state vector:

$$|\Phi_{abc};\ \mathrm{out}\rangle=a^{+}_{\mathrm{out}}b^{+}_{\mathrm{out}}c^{+}_{\mathrm{out}}|0\rangle.$$

By means of the Bogolyubov reduction formulas, as in the previous case, we can readily find a linear relationship between the wave function (31) and the initial wave function of the three free particles:

$$\Phi_{abc}(x_1x_2x_3)=[(G_{abc}*D_a^{-1}D_b^{-1}D_c^{-1})*\Phi^{(0)}_{abc}]\,(x_1x_2x_3), \qquad (32)$$

where $\Phi^{(0)}_{abc}(x_1x_2x_3)=f_a(x_1)f_b(x_2)f_c(x_3)$ is the wave function of the initial state of the three noninteracting particles, $D_i(x)$, $i=a,\ b,\ c$, are the causal Green's functions defined above by the relation (4), and $G_{abc}$ is the three-particle (six-point) Green's function determined by means of the matrix element:

$$G_{abc}(x_1x_2x_3;\ y_1y_2y_3)=\mathrm{i}^3\,\langle 0|T\,\Phi_a(x_1)\,\Phi_b(x_2)\,\Phi_c(x_3)\,\overline{\Phi}_a(y_1)\,\overline{\Phi}_b(y_2)\,\overline{\Phi}_c(y_3))|0\rangle=\mathrm{i}^3\,\langle 0|T\,(\varphi_a(x_1)\,\varphi_b(x_2)\,\varphi_c(x_3)\,\overline{\varphi}_a(y_1)\,\overline{\varphi}_b(y_2)\,\overline{\varphi}_c(y_3)\,S)\,S^{+}|0\rangle.$$

Expanding the time-ordered product of the field operators by means of the generalized Wick theorem, we obtain the following expression for the three-particle Green's function:

$$G_{abc}=G^{(0)}_{abc}+G^{(0)}_{abc}*R_{abc}*G^{(0)}_{abc}, \qquad (33)$$

where $G^{(0)}_{abc}=D_aD_bD_c$ is the free three-particle Green's function, and $R_{abc}$ has the structure

$$R_{abc}=R^{(2)}_aD_b^{-1}D_c^{-1}+D_a^{-1}R^{(2)}_bD_c^{-1}+D_a^{-1}D_b^{-1}R^{(2)}_c+R^{(4)}_{ab}D_c^{-1}+R^{(1)}_{bc}D_a^{-1}+R^{(4)}_{ac}D_b^{-1}+R^{(6)}_{abc}.$$

The functions $R^{(2)}_i$, $i=a,\ b,\ c$, and $R^{(4)}_{ij}$, $(ij)=ab,\ bc,\ ac$, are determined by the expressions (10) and (11), and the function $R^{6}_{abc}$ is the vacuum expectation value of the sixth-order radiation operator:

$$R^{(6)}_{abc}(x_1x_2x_3;\ y_1y_2y_3) = \frac{1}{i^3}\left\langle 0 \left| \frac{\delta^6 S}{\delta\overline{\varphi}_a(x_1)\,\delta\overline{\varphi}_b(x_2)\,\delta\overline{\varphi}_c(x_3)\,\delta\varphi_a(y_1)\,\delta\varphi_b(y_2)\,\delta\varphi_c(y_3)} S^+ \right| 0 \right\rangle.$$

We substitute the expression (33) in the relation (32). We obtain

$$\Phi_{abc}(x_1x_2x_3) = \Phi^{(0)}_{abc}(x_1x_2x_3) + (G^{(0)}_{abc} * R_{abc} * \Phi^{(0)}_{abc})(x_1x_2x_3)$$
$$= \Phi^{(0)}_{abc}(x_1x_2x_3) + (G^{(0)}_{abc} * \overline{R}_{abc} * \Phi^{(0)}_{abc})(x_1x_2x_3), \qquad (34)$$

where $\overline{R}_{abc} = R^{(4)}_{ab} D_c^{-1} + R^{(4)}_{bc} D_a^{-1} + R^{(4)}_{ac} D_b^{-1} + R^{(6)}_{abc}$. In the second equation of the relation (34), we have again, as in the derivation of (12), used the stability of the single-particle states. With allowance for the stability of the single-particle states, the relation (32) can also be rewritten in the form

$$\Phi_{abc} = (\overline{G}_{abc} * D_a^{-1} D_b^{-1} D_c^{-1}) * \Phi^{(0)}_{abc}, \qquad (35)$$

where

$$\overline{G}_{abc} = G^{(0)}_{abc} + G^{(0)}_{abc} * \overline{R}_{abc} * G^{(0)}_{abc}$$
$$= G_{abc} - G^{(0)}_{abc} * (R^{(2)}_a D_b^{-1} D_c^{-1} + D_a^{-1} R^{(2)}_b D_c^{-1} + D_a^{-1} D_b^{-1} R^{(2)}_c) * G^{(0)}_{abc}. \qquad (33a)$$

We introduce the function $V_{abc}$ by means of the equation

$$\overline{R}_{abc} = V_{abc} + V_{abc} * G^{(0)}_{abc} * \overline{R}_{abc}. \qquad (36)$$

Then the relation (34) can be expressed in the form of a dynamical equation for the wave function of the three-particle system:

$$\Phi_{abc}(x_1x_2x_3) = \Phi^{(0)}_{abc}(x_1x_2x_3) + (G^{(0)}_{abc} * V_{abc} * \Phi_{abc})(x_1x_2x_3). \qquad (37)$$

Equation (37) is the basic integral equation in the theory of three-particle scattering. By means of Bogolyubov's reduction formulas, one can readily show that the function $\overline{R}_{abc}$ is directly related to the three-particle elastic scattering amplitude. More precisely, the matrix element of the $S$ operator corresponding to the three-particle elastic scattering process has the representation

$$\langle \Phi'_{abc};\ \text{out} | S-1 | \Phi_{abc};\ \text{out} \rangle = \int (dx)(dy)\, \overset{*}{f'_a}(x_1)\, \overset{*}{f'_b}(x_2)$$
$$\times \overset{*}{f'_c}(x_3)\, R^{(6)}_{abc}(x_1x_2x_3;\ y_1y_2y_3)\, f_a(y_1)\, f_b(y_2)\, f_c(y_3), \qquad (38)$$

where we have introduced the notation

$$(dx) = dx_1\, dx_2\, dx_3, \quad (dy) = dy_1\, dy_2\, dy_3.$$

The three-particle bound-state wave function, defined by means of the matrix element

$$\Phi^A_{abc}(x_1, x_2, x_3) = \langle 0 \mid T(\Phi_a(x_1)\, \Phi_b(x_2)\, \Phi_c(x_3)) \mid \Phi^A_{abc} \rangle,$$

where $|\Phi^A_{abc}\rangle$ is the vector of the bound state of particles $a$, $b$, and $c$, will satisfy the homogeneous equation

$$\Phi^A_{abc}(x_1x_2x_3) = (G^{(0)}_{abc} * V_{abc} * \Phi^A_{abc})(x_1x_2x_3). \qquad (39)$$

In confirmation of this result, we can bring forward intuitive arguments like those mentioned in the two-particle case. It is possible to give a rigorous derivation of Eq. (39) for the three-particle bound-state wave function, but we shall not dwell on it, since it is based on essentially the same arguments as those used in the previous section to derive a homogeneous equation for the two-particle bound-state wave function.

In the three-particle case, there is a new feature, which is related to the fact that the homogeneous equation (39) is satisfied by not only the three-particle bound-state wave function. For example, the wave function corresponding to the $t \to -\infty$ asymptotic configuration in which two particles are in a bound state and the third is free also satisfies the homogeneous equation (39). Indeed, we define the wave function of the scattering state in which two particles in the initial state, say $a$ and $b$, form a composite system, and the third particle $c$ is free by means of the matrix element

$$\Phi^A_{(ab)c}(x_1x_2x_3)$$
$$= \langle 0 \mid T(\Phi_a(x_1)\, \Phi_b(x_2)\, \Phi_c(x_3)) \mid \Phi^A_{ab} \Phi_c;\ \text{in} \rangle, \qquad (40)$$

where $|\Phi^A_{ab}\Phi_c;\ \text{in}\rangle$ is the vector of the bound state of particles $a$ and $b$ and particle $c$, which does not interact with them.

In Eq. (37), we go over to the variables $X = (x_1 + x_2)/2$, $x = x_1 - x_2$, leave the third coordinate unchanged, and make a Fourier transformation with respect to the variable $X$. We obtain

$$\Phi_{abc}(xx_3 \mid P) = \Phi^{(0)}_{abc}(xx_3 \mid P)$$
$$+ (G^{(0)}_{abc} * V_{abc} * \Phi_{abc})(xx_3 \mid P). \qquad (41)$$

If $\sqrt{P^2} = M_A < m_a + m_b$, the inhomogeneous term in Eq. (41) vanishes and, therefore, the wave function (40) also satisfies the homogeneous equation (39). It is readily understood that the wave functions $\Phi^B_{a(bc)}$ and $\Phi^C_{b(ac)}$ will also satisfy the homogeneous equation (39). This circumstance indicates that in this case the equations of Bethe–Salpeter type do not have unique solutions, and subsidiary conditions must be imposed in the choice of them. In nonrelativistic scattering theory, this circumstance was noted for the first time by Faddeev.[7] He derived dynamical equations (Faddeev equations) free of this shortcoming.

Returning to the derivation of the dynamical equation (37), we can readily identify the origin of the nonuniqueness pointed out above. From the very beginning, by the manner of its derivation, the basic integral equation (37) is suited to the description of processes of elastic scattering of three particles and is ill suited to the investigation of processes in which bound systems participate. The inhomogeneous term in Eq. (37), the wave function of the system of three free particles, correctly describes the boundary condition in the problem of three-particle elastic scattering but is quite unsuitable for specifying the boundary condition corresponding to the asymptotic $t \to -\infty$ configuration in which there is a bound system of two particles and a third particle which does not interact with them. One of the ways to solve the problem of the correct specification of the boundary conditions is through rearrangement of the basic dynamical equation (37).

Bearing in mind this remark, we consider the derivation of a dynamical equation for the wave function (40). To this end, we introduce the function $V_{(ab)c}$ by means of the relation

$$\overline{R}_{abc}-R_{ab}^{(4)}D_c^{-1}=V_{(ab)c}+V_{(ab)c}*G_{abc}^{(0)}*\overline{R}_{abc}.$$

Substituting this in Eq. (34), we obtain

$$\Phi_{abc}(x_1x_2x_3)=\Phi_{ab}(x_1x_2)\,\Phi_c^{(0)}(x_3)+(G_{abc}^{(0)}*V_{(ab)c}*\Phi_{abc})(x_1x_2x_3), \quad (42)$$

where

$$\Phi_{ab}(x_1x_2)=\Phi_{ab}^{(0)}(x_1x_2)+[(D_aD_b)*R_{ab}^{(4)}*\Phi_{ab}^{(0)}](x_1x_2).$$

Equation (42) is equivalent to the original equation (37), but it is now easy to go over from Eq. (42) to a dynamical equation for the wave function (40) describing decay of the bound state. For this, it is sufficient to set the inhomogeneous term in Eq. (42) equal to $\Phi_{ab}^A(x_1x_2)\Phi_c^{(0)}(x_3)$, where $\Phi_{ab}^A$ is the wave function of the bound state of particles $a$ and $b$. As a result, we arrive at the equation

$$\Phi_{(ab)c}^A(x_1x_2x_3)=\Phi_{ab}^A(x_1x_2)\,\Phi_c^{(0)}(x_3)+(C_{abc}^{(0)}*V_{(ab)c}*\Phi_{(ab)c}^A)(x_1x_2x_3). \quad (43)$$

It is readily seen that the wave function (40), satisfying Eq. (43), will describe the breakup of the bound state of particles $a$ and $b$ as a result of its interaction with particle $c$. The inhomogeneous term in Eq. (43) specifies the correct boundary condition as $t\to-\infty$, and the presence of the free Green's function $G_{abc}^{(0)}$ in the kernel of this equation ensures the asymptotic configuration of three free particles in the limit $t\to+\infty$.

If we define the function $R_{(ab)c}$ by means of

$$R_{(ab)c}=V_{(ab)c}+V_{(ab)c}*G_{abc}^{(0)}*R_{(ab)c},$$

then Eq. (43) can be rewritten as

$$\Phi_{(ab)c}^A(x_1x_2x_3)=\Phi_{ab}^A(x_1x_2)\,\Phi_c^{(0)}(x_3)+(G_{abc}^{(0)}*R_{(ab)c}*\Phi_{ab}^A\Phi_c^{(0)})(x_1x_2x_3).$$

We show below that the function $R_{(ab)c}$ is directly related to the amplitude for the breakup of the bound state, the matrix element of the $S$ operator corresponding to this process being given by the expression

$$\langle\Phi_{abc};\ \text{out}|S-1|\Phi_{ab}^A\Phi_c;\ \text{out}\rangle=\frac{1}{i}(f_a^*f_b^*f_c^*)*R_{(ab)c}*(\Phi_{ab}^Af_c).$$

To describe the process of elastic scattering by the bound state, it is more convenient to make a different rearrangement of the basic integral equation for the wave function of the three-particle system. To be specific, we consider the elastic scattering of particle $c$ by the bound state of particles $a$ and $b$. If we substitute Eq. (36) in the expression for the three-particle Green's function $\overline{G}_{abc}$, we arrive at an equation for the Green's function,

$$\overline{G}_{abc}=G_{abc}^{(0)}+G_{abc}^{(0)}*V_{abc}*\overline{G}_{abc}, \quad (44)$$

which can be rewritten in the equivalent form

$$\overline{G}_{abc}=\overline{G}_c^{(0)}+\overline{G}_c^{(0)}*\overline{V}_c*\overline{G}_{abc}, \quad (45)$$

where $\overline{V}_c=V_{abc}-V_{ab}D_c^{-1}$, $\overline{G}_c^0=\overline{G}_{ab}D_c$, and $\overline{G}_{ab}$ satisfies Eq. (28). Substituting Eq. (45) in (35), we obtain an equation for the wave function of the three-particle system in the form

$$\Phi_{abc}(x_1x_2x_3)=\Phi_{(ab)c}^{(0)}(x_1x_2x_3)+(\overline{G}_c^{(0)}*\overline{V}_c*\Phi_{abc})(x_1x_2x_3), \quad (46)$$

where

$$\Phi_{(ab)c}^{(0)}(x_1x_2x_3)=[(\overline{G}_c^{(0)}*G_{abc}^{(0)-1})*\Phi_{abc}^{(0)}](x_1x_2x_3)=\Phi_{ab}(x_1x_2)\,\Phi_c^0(x_3).$$

To describe elastic scattering by the bound state, the inhomogeneous term in Eq. (46) must be replaced by $\Phi_{ab}^A\Phi_c^{(0)}$, where $\Phi_{ab}^A$ is the wave function of the bound state of particles $a$ and $b$; by doing this, we specify the correct boundary condition as $t\to-\infty$. The Green's function $\overline{G}_c^{(0)}$ in the kernel of Eq. (46) will ensure the boundary condition as $t\to+\infty$ required for the given problem.

We define the function $R^{c;c}$ by

$$R^{c;c}=\overline{V}_c+\overline{V}_c*\overline{G}_c^{(0)}*R^{c;c}.$$

Then Eq. (46) can be rewritten as

$$\Phi_{abc}=\Phi_{ab}^A\Phi_c^{(0)}+\overline{G}_c^{(0)}*R^{c;c}*(\Phi_{ab}^A\Phi_c^{(0)}).$$

The function $R^{c;c}$ is directly related to the amplitude of elastic scattering of particle $c$ by the bound state of particles $a$ and $b$. The corresponding matrix element of the $S$ operator can be expressed in terms of $R^{c;c}$ by means of

$$\langle\Phi_{(ab)c}^B;\ \text{out}|S-1|\Phi_{(ab)c}^A;\ \text{out}\rangle=-i\,(\overset{*}{\Phi_{ab}^B f_c})*R^{c;c}*(\Phi_{ab}^Af_c).$$

These examples are paradigms of the general scheme for constructing the dynamical equations. In the general case, the problem can be formulated as that of writing down a dynamical equation whose solution is directly related to the amplitude for the transition of the system from the state characterized by the wave function $\Phi_\beta^{(0)}$ to the state with the wave function $\Phi_\alpha^{(0)}$. The indices $\alpha$ and $\beta$ take values $a$, $b$, $c$, $abc$, so that, for example, $\alpha=c$ means that particle $c$ in the state $\Phi_c^{(0)}$ is free: $\Phi_c^{(0)}(x_1x_2x_3)=\Phi_{ab}^A(x_1x_2)f_c(x_3)$; similarly, $\Phi_a^{(0)}(x_1x_2x_3)=f_a(x_1)\Phi_{bc}^B(x_2x_3)$, $\Phi_b^{(0)}(x_1x_2x_3)=\Phi_{ac}^C(x_1x_3)f_b(x_2)$. The wave function $\Phi_{abc}^{(0)}(x_1x_2x_3)=f_a(x_1)f_b(x_2)f_c(x_3)$ describes the state of three free particles.

We proceed from the basic integral equation (37) for the wave function of the three-particle scattering state. We rearrange this equation by using the representation for the three-particle Green's function in the form

$$\overline{G}_{abc}=\overline{G}_\beta^{(0)}+\overline{G}_\alpha^{(0)}*R^{\alpha;\beta}*\overline{G}_\beta^{(0)}, \quad (47)$$

where $G_a^{(0)} = \overline{G}_{bc} D_a$, $\overline{G}_b^{(0)} = \overline{G}_{ac} D_b$, $\overline{G}_c^{(0)} = \overline{G}_{ab} D_c$, $\overline{G}_{abc}^{(0)} = D_a D_b D_c$.

We define the function $V^{\alpha;\beta}$ by

$$R^{\alpha;\,\beta} = V^{\alpha;\,\beta} + V^{\alpha;\,\beta} * \overline{G}_\alpha^{(0)} * R^{\alpha;\,\beta}.$$

Then (47) can be rewritten as an equation for the three-particle Green's function:

$$\overline{G}_{abc} = \overline{G}_\beta^{(0)} + \overline{G}_\alpha^{(0)} * V^{\alpha;\,\beta} * \overline{G}_{abc}.$$

Substituting this equation in Eq. (35), we obtain the equation for the wave function in the form

$$\Phi_{abc} = \Phi_\beta + \overline{G}_\alpha^{(0)} * V^{\alpha;\,\beta} * \Phi_{abc}, \tag{48}$$

where $\Phi_\beta = (\overline{G}_\beta^{(0)} * G_{abc}^{(0)\,-1}) * \Phi_{abc}^{(0)}$. The required equation is obtained from Eq. (48) if as $\Phi_\beta$ we take from among the wave functions described above the asymptotic-state wave function $\Phi_\beta^{(0)}$ specifying in the considered problem the boundary condition as $t \to -\infty$.

The function $R^{\alpha;\beta}$ is directly related to the amplitude for the transition of the system from the initial asymptotic state with wave function $\Phi_\beta^{(0)}$ to the final asymptotic state with wave function $\Phi_\alpha^{(0)}$. The matrix element of the $S$ operator corresponding to this process can be written as follows:

$$\langle \Phi_\alpha^{(0)};\ \text{out}\ | S - 1 | \Phi_\beta^{(0)};\ \text{out} \rangle = \frac{1}{i}\, \overline{\Phi}_\alpha^{(0)} * R^{\alpha;\,\beta} * \Phi_\beta^{(0)}. \tag{49}$$

We can readily extend to the relativistic case Faddeev's well-known scheme[7] in nonrelativistic scattering theory. For this, we express the function $V_{abc}$ defined by means of Eq. (36) in the form

$$V_{abc} = V_{ab} D_c^{-1} + V_{bc} D_a^{-1} + V_{ac} D_b^{-1} + V_0, \tag{50}$$

where $V_0$ is the part of the function $V_{abc}$ that does not reduce directly to two-particle interactions in the system. Given the functions $V_{abc}$, $V_{ab}$, $V_{bc}$, $V_{ac}$, the relation (50) essentially determines the function $V_0$. From each of these functions, we find a function $R_\alpha$, $\alpha = a, b, c, 0$, by means of the equations

$$R_\alpha = V_\alpha + V_\alpha * G_{abc}^{(0)} * R_\alpha = V_\alpha + R_\alpha * G_{abc}^{(0)} * V_\alpha, \tag{51}$$

so that $R_c = R_{ab}^{(4)} D_c^{-1}$, $R_b = R_{ac}^{(4)} D_b^{-1}$, $R_a = R_{bc}^{(4)} D_a^{(-1)}$, $R_0 = V_0 + V_0 * G_{abc}^{(0)} * R_0 = V_0 + R_0 * G_{abc}^{(0)} * V_0$.

Following Faddeev, we introduce the functions $M_{\alpha;\beta}$ by

$$M_{\alpha;\,\beta} = V_\alpha \delta_{\alpha\beta} + V_\alpha * \overline{G}_{abc} * V_\beta. \tag{52}$$

It is easy to see that

$$\sum_{\alpha;\,\beta} M_{\alpha;\,\beta} = \overline{R}_{abc}. \tag{53}$$

From the definition (52) of the functions $M_{\alpha;\beta}$ and Eq. (44) for the three-particle Green's function, we obtain

$$M_{\alpha;\,\beta} = V_\alpha \delta_{\alpha\beta} + V_\alpha * G_{abc}^{(0)} * \sum_\gamma M_{\gamma;\,\beta}.$$

We multiply the left- and right-hand sides of this equation by $(1 + R_\alpha * G_{abc}^{(0)})$. Using the relation (51), we obtain a system of equations for $M_{\alpha;\beta}$:

$$M_{\alpha;\,\beta} = R_\alpha \delta_{\alpha\beta} + R_\alpha * G_{abc}^{(0)} * \sum_{\gamma \neq \alpha} M_{\gamma;\,\beta}. \tag{54}$$

The functions $M_{\alpha;\beta}$, in contrast to the functions $R^{\alpha;\beta}$ introduced above, are not related by simple expressions to the amplitudes of the physical processes in the three-particle system. The one simple relation (53) reveals the direct connection of these quantities to the three-particle elastic scattering amplitude.

## 3. ASYMPTOTIC LSZ CONDITION AND AMPLITUDES OF PHYSICAL PROCESSES

We consider the theory of neutral scalar particles. Let $f_m(x)$ be a smooth negative-frequency solution of the Klein–Gordon equation

$$(\Box_x + m^2)\, f_m(x) = 0.$$

We denote by $f_{m;\alpha}(x)$ a complete orthonormal system of smooth negative-frequency solutions of the Klein–Gordon equation. The orthogonality condition can be written in the form

$$i \int d^3\mathbf{x}\, \overset{*}{f}_{m;\,\alpha}(x) \overleftrightarrow{\frac{\partial}{\partial x^0}} f_{m;\,\beta}(x) = \delta_{\alpha\beta}.$$

The completeness condition has the form

$$i \sum_\alpha f_{m;\,\alpha}(x)\, \overset{*}{f}_{m;\,\alpha}(y) = D^{(-)}(m;\ x - y),$$

where $D^{(-)}(m; x - y)$ is the negative-frequency part of the commutator function of the scalar field. For practical purposes, we shall use the limiting case of smooth solutions of the Klein–Gordon equation in the plane-wave form:

$$f_{m;\alpha}(x) \to f_{m;\mathbf{p}}(x) = (2\pi)^{-3/2} \exp(-i\, px),$$

$$p^0 = E(\mathbf{p},\ m) = \sqrt{\mathbf{p}^2 + m^2}. \tag{55}$$

The orthogonality condition is written in the form

$$i \int d^3\mathbf{x}\, \overset{*}{f}_{m;\,\mathbf{p}}(x)\, \overleftrightarrow{\partial_0} f_{m;\,\mathbf{q}}(x) = 2E(\mathbf{p},\ m)\, \delta^3(\mathbf{p} - \mathbf{q}),$$

and the completeness condition in the form

$$i \int \frac{d^3\mathbf{p}}{2E(\mathbf{p}, m)}\, f_{m;\,\mathbf{p}}(x)\, \overset{*}{f}_{m;\,\mathbf{p}}(y) = D^{(-)}(m;\ x - y).$$

We define smeared field operators by means of the expressions

$$a_t^{(-)}(f_{m;\,\alpha}) = i \int\limits_{x^0 = t} d^3\mathbf{x}\, \overset{*}{f}_{m;\,\alpha}(x)\, \overleftrightarrow{\partial_0} \Phi_a(x) \equiv a_t^{(-)\alpha}; \tag{56}$$

$$a_t^{(+)}(f_{m;\,\alpha}) = i \int\limits_{x^0 = t} d^3\mathbf{x}\, \Phi_a(x)\, \overleftrightarrow{\partial_0} f_{m;\,\alpha}(x) \equiv a_t^{(+)\alpha}, \tag{57}$$

where $\Phi_a(x)$ is the Heisenberg field operator of particle $a$, and $f_{m;\alpha}(x)$ is a smooth normalized negative-frequency solution of the Klein–Gordon equation. It is easy to show that the Heisenberg field operator can be expressed in terms of the smeared operators:

$$\Phi_a(x)=\sum_\alpha a_t^{(-)\alpha} f_{m;\alpha}(\mathbf{x},t)+a_t^{(+)\alpha} \overset{*}{f}_{m;\alpha}(\mathbf{x},t). \quad (x^0=t)$$

The asymptotic LSZ condition is formulated as the condition of weak convergence of the smeared field operators:

$$\left.\begin{aligned}\lim_{t\to+\infty}\langle\psi_1|a_t^{(\pm)}(f_{m;\alpha})|\psi_2\rangle&=\langle\psi_1|a_{\rm out}^{(\pm)}(f_{m;\alpha})|\psi_2\rangle;\\ \lim_{t\to-\infty}\langle\psi_1|a_t^{(\pm)}(f_{m;\alpha})|\psi_2\rangle&=\langle\psi_1|a_{\rm in}^{(\pm)}(f_{m;\alpha})|\psi_2\rangle\end{aligned}\right\} \quad (58)$$

for all normalized state vectors $|\psi_1\rangle$ and $|\psi_2\rangle$. In the formulation of the asymptotic condition, the operators $a^{(\pm)}_{\rm in,out}$ are defined in accordance with (56) and (57), in which the Heisenberg field operator $\Phi_a(x)$ must be replaced by the operators of the asymptotic "in" and "out" fields, $\varphi_a^{\rm ex}(x)$, which satisfy the free Klein–Gordon equation. From this, in particular, it follows that the operators $a^{(\pm)}_{\rm in,out}$ do not depend on the time.

By means of the asymptotic condition (58), Lehmann, Symanzik, and Zimmermann[4] obtained a reduction formula by means of which the matrix elements of the scattering operator $S$ can be expressed in terms of the vacuum expectation values of a time-ordered product of the Heisenberg field operators. For the special case when there are only two free particles in the initial and final states, the reduction formula has the form

$$\langle\Phi_{ab}^{\rm out};\ \mathbf{p}_1\mathbf{p}_2|S-1|\Phi_{ab}^{\rm out};\ \mathbf{q}_1\mathbf{q}_2\rangle={\rm i}^4\int dx_1\,dx_2\,dy_1\,dy_2 \overset{*}{f}_{m_a;\,\mathbf{p}_1}(x_1)$$
$$\times\overset{*}{f}_{m_b;\,\mathbf{p}_2}(x_2)\overrightarrow{K}^{m_a}_{x_1}\overrightarrow{K}^{m_b}_{x_2}\langle 0|T(\Phi_a(x_1)\Phi_b(x_2)\Phi_a(y_1)\Phi_b(y_2))|0\rangle$$
$$\times\overleftarrow{K}^{m_a}_{y_1}\overleftarrow{K}^{m_b}_{y_2}f_{m_a;\,\mathbf{q}_1}(y_1)f_{m_b;\,\mathbf{q}_2}(y_2), \quad (59)$$

where $K_x^m=(\Box_x+m^2)$ is the differential Klein–Gordon operator, and the direction of the arrows above it indicates the order of its application. If in (59) we substitute the relation (8) and use the property $\overrightarrow{K}^a_x D_a(x-y)=D_a(x-y)\overleftarrow{K}^a_y=\delta^4(x-y)$ of the causal Green's functions, for the matrix element of the $S$ operator corresponding to two-particle elastic scattering we obtain

$$\langle\Phi_{ab}^{\rm out};\mathbf{p}_1\mathbf{p}_2|S-1|\Phi_{ab}^{\rm out};\mathbf{q}_1\mathbf{q}_2\rangle$$
$$={\rm i}^2\int dx_1\,dx_2\,dy_1\,dy_2\overset{*}{f}_{m_a;\,\mathbf{p}_1}(x_1)\overset{*}{f}_{m_b;\,\mathbf{p}_2}(x_2)$$
$$\times R_{ab}^{(4)}(x_1x_2;\ y_1y_2)f_{m_a;\,\mathbf{q}_1}(y_1)f_{m_b;\,\mathbf{q}_2}(y_2). \quad (60)$$

It thus follows from the reduction formula (59) that the amplitudes of physical processes for which there are only free particles in the initial and final states are directly related to the single-particle singularities of the Green's functions. We emphasize once more that essential use is made of the asymptotic condition (58) in the derivation of the reduction formula (59), from which in turn the expression (60) is obtained.

The expression (60) for the element of the $S$ matrix can be obtained without recourse to the reduction formula (59). To this end, we write the relation (8) in the expanded form

$${\rm i}^2\langle 0\mid T(\Phi_a(x_1)\Phi_b(x_2)\Phi_a(y_1)\Phi_b(y_2))\mid 0\rangle$$
$$=D_a(x_1-y_1)D_b(x_2-y_2)+(2\pi)^{-16}$$
$$\times\int dp_1\,dp_2\,dk_1\,dk_2\exp(-{\rm i}\,p_1x_1-{\rm i}\,p_2x_2)$$
$$\times D_a(p_1)D_b(p_2)R_{ab}(p_1p_2;\ k_1k_2)$$
$$\times D_a(k_1)D_b(k_2)\exp({\rm i}\,k_1y_1+{\rm i}\,k_2y_2), \quad (61)$$

where $D_a(p)=m_a^2-p^2-i0)^{-1}$ is the Fourier transform of the causal Green's function. In this equation, we take a definite sequence of times $x_1^0>x_2^0>y_1^0>y_2^0$ and go over to the smeared field operators in accordance with (56) and (57), after which we go to the limit $x_1^0,x_2^0\to+\infty$ with $y_1^0$, $y_2^0\to-\infty$. Then, taking into account the asymptotic condition (58) on the left-hand side of the relation (61), we obtain

$$\lim_{\substack{x_1^0,\,x_2^0\to+\infty\\ y_1^0,\,y_2^0\to-\infty}}\langle 0|a_{x_1^0}^{(-)}b_{x_2^0}^{(-)}a_{y_1^0}^{(+)}b_{y_2^0}^{(+)}|0\rangle=\langle\Phi_{ab}^{\rm out};\ \mathbf{p}_1\mathbf{p}_2|\Phi_{ab}^{\rm in};\ \mathbf{q}_1\mathbf{q}_2\rangle.$$

On the right-hand side of (61), we use the limit relations

$$\lim_{t\to+\infty}{\rm i}\int_{x^0=t}d^3\mathbf{x}\int_{y^0=-t}d^3\mathbf{y}\overset{*}{f}_{m;\,\mathbf{p}}(x)\frac{\overleftrightarrow{\partial}}{\partial x^0}D(x-y)\frac{\overleftrightarrow{\partial}}{\partial y^0}f_{m;\,\mathbf{q}}(y)$$
$$=2E(\mathbf{p},\ m)\,\delta^3(\mathbf{p}-\mathbf{q}); \quad (62)$$

$$\lim_{t\to+\infty}{\rm i}\int_{x^0=t}d^3\mathbf{x}\overset{*}{f}_{m;\,\mathbf{p}}(x)\frac{\overleftrightarrow{\partial}}{\partial x^0}\int_{-\infty}^{\infty}dk^0D(k)\exp(-ikx)$$
$$=(2\pi)^{5/2}\,{\rm i}\delta^3(\mathbf{p}-\mathbf{k}). \quad (63)$$

$$\lim_{t\to-\infty}{\rm i}\int_{y^0=t}d^3\mathbf{y}\int_{-\infty}^{\infty}dk^0D(k)\exp(iky)\frac{\overleftrightarrow{\partial}}{\partial y^0}f_{m;\,\mathbf{q}}(y)$$
$$=(2\pi)^{5/2}\,{\rm i}\delta^3(\mathbf{k}-\mathbf{q}). \quad (64)$$

As a result, on the right-hand side of Eq. (61) we obtain

$${\rm i}^2 2E(\mathbf{p}_1,\ m_a)\,\delta^3(\mathbf{p}_1-\mathbf{q}_1)\,2E(\mathbf{p}_2,\ m_b)\,\delta^3(\mathbf{p}_2-\mathbf{q}_2)$$
$$+{\rm i}^4(2\pi)^{-6}R_{ab}^{(4)}(\tilde p_1\tilde p_2;\ \tilde q_1\tilde q_2),$$

where the tilde means that the corresponding momentum lies on the mass shell, $\tilde p_1^2=m_a^2=\tilde q_1^2,\tilde p_2^2=m_b^2=\tilde q_2^2$, and we have used the stability of the single-particle states. Thus, after the passage to the limit $x_1^0,x_2^0\to+\infty$, $y_1^0,y_2^0\to-\infty$ in Eq. (61) in accordance with the procedure described above, we find

$$\langle\Phi_{ab}^{\rm out};\ \mathbf{p}_1\mathbf{p}_2|\Phi_{ab}^{\rm in};\ \mathbf{q}_1\mathbf{q}_2\rangle$$
$$=2E(\mathbf{p}_1,\ m_a)\,\delta^3(\mathbf{p}_1-\mathbf{q}_1)\,2E(\mathbf{p}_2,\ m_b)\,\delta^3(\mathbf{p}_2-\mathbf{q}_2)$$
$$+{\rm i}^2(2\pi)^{-6}R_{ab}^{(4)}(\tilde p_1\tilde p_2;\ \tilde q_1\tilde q_2). \quad (65)$$

The last term in (65) is obviously equal to

$${\rm i}^2\int dx_1\,dx_2\,dy_1\,dy_2\overset{*}{f}_{m_a;\,\mathbf{p}_1}(x_1)\overset{*}{f}_{m_b;\,\mathbf{p}_2}(x_2)R_{ab}^{(4)}(x_1x_2;\ y_1y_2)$$
$$\times f_{m_a;\,\mathbf{q}_1}(y_1)f_{m_b;\,\mathbf{q}_2}(y_2),$$

in complete agreement with the expression (60).

This derivation of Eq. (65) makes it possible to trace the connection between the asymptotic LSZ condition and the single-particle singularities of the Green's functions.

Proceeding from the representation (33) for the three-particle Green's function, we can in the same manner readily obtain an expression for the amplitude of three-particle elastic scattering. For this, we must carry out the smearing procedure by means of Eqs. (56) and (57) on the left- and right-hand sides of the relation (33) and go to the limit

$$x_1^0,\ x_2^0,\ x_3^0 \to +\infty,\ y_1^0,\ y_2^0,\ y_3^0 \to -\infty.$$

Using the asymptotic condition (58) and the limit relations (62)–(64), we obtain

$$\langle\Phi_{abc}^{\text{out}};\ \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|\Phi_{abc}^{\text{in}};\ \mathbf{q}_1\mathbf{q}_2\mathbf{q}_3\rangle$$

$$= 2E(\mathbf{p}_1,\ m_a)\,\delta^3(\mathbf{p}_1-\mathbf{q}_1)\,2E(\mathbf{p}_2,\ m_b)$$

$$\times\,\delta^3(\mathbf{p}_2-\mathbf{q}_2)\,2E(\mathbf{p}_3,\ m_c)\,\delta^3(\mathbf{p}_3-\mathbf{q}_3)$$

$$+\,\mathrm{i}^2(2\pi)^{-6}R_{ab}^{(4)}(\widetilde{p}_1\widetilde{p}_2;\ \widetilde{q}_1\widetilde{q}_2)\,2E(\mathbf{p}_3,\ m_c)\,\delta^3(\mathbf{p}_3-\mathbf{q}_3)$$

$$+\,\mathrm{i}^2(2\pi)^{-6}R_{bc}^{(4)}(\widetilde{p}_2\widetilde{p}_3;\ \widetilde{q}_2\widetilde{q}_3)\,2E(\mathbf{p}_1,\ m_a)\,\delta^3(\mathbf{p}_1-\mathbf{q}_1)$$

$$+\,\mathrm{i}^2(2\pi)^{-6}R_{ac}^{(4)}(\widetilde{p}_1\widetilde{p}_3;\ \widetilde{q}_1\widetilde{q}_3)\,2E(\mathbf{p}_2,\ m_b)\,\delta^3(\mathbf{p}_2-\mathbf{q}_2)$$

$$+\,\mathrm{i}^3(2\pi)^{-9}R_{abc}^{(6)}(\widetilde{p}_1\widetilde{p}_2\widetilde{p}_3;\ \widetilde{q}_1\widetilde{q}_2\widetilde{q}_3). \quad (66)$$

The last term in (66) is equal to the expression (38), in which the plane waves (55) are to be taken as the wave functions $f_a(x)$.

Using this method, we now show how expressions can be obtained for the amplitudes of other possible processes in the three-particle system. We consider the scattering of particle $c$ by a bound state of particles $a$ and $b$. In this case, it is convenient to proceed from the representation (47) for the three-particle Green's function with $\alpha=\beta=c$:

$$\overline{G}_{abc}=\overline{G}_c^{(0)}+\overline{G}_c^{(0)}*R^{c;\,c}*\overline{G}_c^{(0)}. \quad (67)$$

We recall that $\overline{G}_c^{(0)}=\overline{G}_{ab}D_c$, where $\overline{G}_{ab}$ is two-particle Green's function.

Following Ref. 8, we write down the expression for the three-particle Green's function, separating explicitly the poles corresponding to the bound states of particles $a$ and $b$:

$$\overline{G}_{abc}(x_1x_2x_3;\ y_1y_2y_3)$$

$$=\mathrm{i}\,(2\pi)^{-5}\int dP\,dQ\exp(-\mathrm{i}PX+\mathrm{i}QY)$$

$$\times\,[2E_{ab}^B(E_{ab}^B-P^0-\mathrm{i}0)$$

$$\times\,2E_{ab}^A(E_{ab}^A-Q^0-\mathrm{i}0)]^{-1}\sum_{\sigma'\sigma}\Phi_{ab}^B(x|\mathbf{P},\ \sigma')$$

$$\times\,\langle\Phi_{ab}^B;\ \mathbf{P},\ \sigma'|T(\Phi_c(x_3)\,\Phi_c(y_3))$$

$$|\Phi_{ab}^A;\ \mathbf{Q},\ \sigma\rangle\overline{\Phi}_{ab}^A(y|\mathbf{Q},\ \sigma)+\ldots, \quad (68)$$

where the wave functions of the bound states are determined by means of the matrix elements

$$\langle 0|T(\Phi_a(x_1)\,\Phi_b(x_2))|\Phi_{ab}^B;\ \mathbf{P},\ \sigma\rangle$$

$$=(2\pi)^{-3/2}\exp(-\mathrm{i}PX)\,\Phi_{ab}^B(x|\mathbf{P},\ \sigma);$$

$$\langle\Phi_{ab}^A;\ \mathbf{Q},\ \sigma|T(\Phi_a(y_1)\,\Phi_b(y_2))|0\rangle$$

$$=(2\pi)^{-3/2}\exp(\mathrm{i}QY)\,\overline{\Phi}_{ab}^A(y|\mathbf{Q},\ \sigma);$$

$$X=(x_1+x_2)/2,\ x=x_1-x_2;$$

$$Y=(y_1+y_2)/2,\ y=y_1-y_2;$$

$$E_{ab}^A=E(\mathbf{Q},\ M_A)=\sqrt{\mathbf{Q}^2+M_A^2};$$

$$E_{ab}^B=E(\mathbf{P},\ M_B)=\sqrt{\mathbf{P}^2+M_B^2}.$$

For the two-particle Green's function $\overline{G}_{ab}$, we shall use the representation in the form

$$\overline{G}_{ab}(x_1x_2;\ y_1y_2)=\mathrm{i}\,(2\pi)^{-4}\int dP$$

$$\times\exp(-\mathrm{i}P(X-Y))\,[2E_{ab}^B(E_{ab}^B-P^0-\mathrm{i}0)]^{-1}$$

$$\times\sum_\sigma\Phi_{ab}^B(x|\mathbf{P},\sigma)\,\overline{\Phi}_{ab}^B(y|\mathbf{P},\sigma)+\ldots$$

$$=\mathrm{i}\,(2\pi)^{-4}\int dQ\exp[-\mathrm{i}Q(X-Y)]$$

$$\times\,[2E_{ab}^A(E_{ab}^A-Q^0-\mathrm{i}0)]^{-1}\sum_\sigma\Phi_{ab}^A(x|\mathbf{Q},\ \sigma)\,\overline{\Phi}_{ab}^A(y|\mathbf{Q},\sigma)+\ldots$$

We introduce the Fourier transform of the three-particle Green's function with respect to the variables $X$ and $Y$:

$$\overline{G}_{abc}(P|xx_3;\ yy_3|Q)$$

$$=\int dX\,dY\exp(\mathrm{i}PX-\mathrm{i}QY)\,\overline{G}_{abc}(Xxx_3;\ Yyy_3). \quad (69)$$

Using the expression (68), we can readily see that the Fourier transform (69) of the three-particle Green's function has poles corresponding to bound states of the particles $a$ and $b$, and near these poles has the representation

$$\overline{G}_{abc}(P|xx_3;\quad yy_3|Q)\big|_{\substack{P^0\to E_{ab}^B\\ Q^0\to E_{ab}^A}}$$

$$=\mathrm{i}\,(2\pi)^3\,[(M_B^2-P^2-\mathrm{i}0)\,(M_A^2-Q^2-i0)]^{-1}$$

$$\times\sum_{\sigma'\sigma}\Phi_{ab}^B(x|\mathbf{P},\ \sigma')\,\langle\Phi_{ab}^B;\ \mathbf{P},\ \sigma'|T(\Phi_c(x_3)\,\Phi_c(y_3))|$$

$$\Phi_{ab}^A;\ \mathbf{Q},\ \sigma\rangle\,\overline{\Phi}_a^A(y|\mathbf{Q},\ \sigma).$$

We make a Fourier transformation with respect to the variables $X$ and $Y$ in Eq. (67). As a result, we arrive at a relation whose left- and right-hand sides have poles corresponding to the bound states of particles $a$ and $b$. Multiplying both sides of this relation by $(M_B^2-P^2)(M_A^2-Q^2)$ and going to the limit $P^2\to M_B^2$, $Q^2\to M_A^0$, we obtain

$$\langle\Phi_{ab}^B;\ \mathbf{P},\ \sigma'|T(\Phi_c(x_3)\,\Phi_c(y_3))|\Phi_{ab}^A;\ \mathbf{Q},\ \sigma\rangle$$

$$=\frac{1}{\mathrm{i}}D_c(x_3-y_3)\,\delta_{AB}\delta_{\sigma\sigma'}2E_{ab}^A\delta^3(\mathbf{P}-\mathbf{Q})$$

$$+\,\mathrm{i}\int(dx')\,(dy')\,\overline{\Phi}_{ab}^B(x_1'x_2'|\mathbf{P},\ \sigma')$$

$$\times\ D_c(x_3-x_3')\,R^{c;\,c}(x_1'x_2'x_3';\ y_1'y_2'y_3')$$

$$\times\,\Phi_{ab}^A(y_1'y_2'|\mathbf{Q},\ \sigma)\,D_c(y_3'-y_3). \quad (70)$$

It is now easy to find an expression for the amplitude for scattering of particle $c$ by a bound state of particles $a$ and $b$. In (70), we take the sequence of times $x_3^0 > y_3^0$ and carry out the smearing procedure by means of (56) and (57), after which we go to the limit $x_3^0 \to +\infty$, $y_3^0 \to -\infty$. Using the asymptotic condition (58) and the limit relation (62), we find

$$\langle \Phi^B_{ab(c)};\ \mathbf{P},\ \sigma',\ \mathbf{p};\ \text{out} | \Phi^A_{(ab)c};\ \mathbf{Q},\ \sigma,\ \mathbf{q};\ \text{in}\rangle = \delta_{AB}\delta_{\sigma\sigma'} 2E^A_{ab}\delta^3(\mathbf{P}-\mathbf{Q})\, 2E(\mathbf{p};\ m_c)\,\delta^3(\mathbf{p}-\mathbf{q}) - i\int (dx)(dy)\, \overline{\Phi}^B_{ab}(x_1x_2|\mathbf{P},\ \sigma') \overset{*}{f}_{m_c;\,\mathbf{p}}(x_3) \times R^{c;\,c}(x_1x_2x_3;\ y_1y_2y_3)\,\Phi^A_{ab}(y_1y_2|\mathbf{Q},\ \sigma)\, f_{m_c;\,\mathbf{q}}(y_3).$$

This expression for the amplitude of scattering by a bound state is obviously an expanded form of expression of (49) for $\alpha = \beta = c$. It is also readily seen that by means of the method described here it is possible to obtain expressions for the amplitudes of arbitrary processes in the three-particle system and, thus, to completely justify the expression (49).

Our derivation of expressions for the amplitudes of processes involving two-particle bound states was based on investigation of the singularities of the three-particle Green's function with respect to the variable that is the invariant mass of the two-particle subsystems. Investigating these singularities, we obtained explicit expressions for the amplitudes of these processes. In addition, we found that the amplitudes of physical processes with asymptotic states involving only free particles are related to the single-particle singularities of the Green's functions, and we showed how the single-particle singularities are related to the asymptotic LSZ condition. These results suggest the possibility of a generalization of the asymptotic LSZ condition that makes it possible to take into account the presence of bound states. The following exposition is devoted to formulation of such an asymptotic condition.

## 4. ASYMPTOTIC CONDITION FOR COMPOSITE PARTICLES

We consider first the normalization of the Bethe–Salpeter wave functions for bound states. The representation (27) for the two-particle Green's function near the pole corresponding to a two-particle bound state is bilinear in the Bethe–Salpeter wave functions. Therefore, the representation (27) already essentially fixes the normalization of the Bethe–Salpeter wave functions for the bound states.

We write down the formal identity

$$\overline{G}_{ab}(P|x;\ y) = \int dx'\, dy' \overline{G}_{ab}(P|x;\ x')\, \overline{G}^{-1}_{ab}(P|x';\ y')\, \overline{G}_{ab}(P|y';\ y), \quad (71)$$

where $\overline{G}_{ab}(P|x;y)$ is the translationally invariant part of the two-particle Green's function and can be expressed in terms of the total two-particle Green's function in accordance with (24). Substituting in (71) the representation (26) and equating the residues at the poles on the left- and right-hand sides of the resulting equation, we arrive at the normalization condition for the bound-state wave function:

$$\int dx\, dy \overline{\Phi}^A_{ab}(x|\mathbf{P},\ \sigma')\, N_A(\mathbf{P}|x;\ y)\, \Phi^A_{ab}(y|\mathbf{P},\ \sigma) = i\delta_{\sigma\sigma'}, \quad (72)$$

where $\Phi^A_{ab}(x|\mathbf{P},\sigma)$ is, obviously, the translationally invariant part of the Bethe–Salpeter wave function, and we have introduced the notation

$$N_A(\mathbf{P}|x;\ y) = \frac{\partial}{\partial P^2}\, \overline{G}^{-1}_{ab}(P|x;\ y)|_{P^2=M_A^2}.$$

In deriving the normalization condition (72), we used the fact that the wave functions $\Phi^A_{ab}(x|\mathbf{P},\sigma)$ corresponding to different quantum numbers $\sigma$ are linearly independent.

One can obtain similarly the orthogonality condition for wave functions corresponding to different values of the square of the invariant mass of the two-particle system. For this, we write the equation for the bound-state wave function in the form

$$\overline{G}^{-1}_{ab}(P_B) * \Phi^B_{ab} = 0, \quad P_B^2 = M_B^2 \quad (73)$$

and consider the formal identity

$$\overline{G}_{ab}(P) * \overline{G}^{-1}_{ab}(P) * \Phi^B_{ab} = \Phi^B_{ab}. \quad (74)$$

Multiplying (73) from the left by $\overline{G}_{ab}(P)$ and subtracting the equation then obtained from the identity (74), we obtain

$$\overline{G}_{ab}(P) * \frac{\overline{G}^{-1}_{ab}(P) - \overline{G}^{-1}_{ab}(P_B)}{P^2 - M_B^2} * \Phi^B_{ab} = \frac{\Phi^B_{ab}}{P^2 - M_B^2}.$$

Substituting here the representation (27) and going to the limit $P^2 \to M_A^2$, we obtain the orthogonality condition

$$\int dx\, dy \overline{\Phi}^A_{ab}(x|\mathbf{P},\ \sigma')\, W_{AB}(\mathbf{P}|x;\ y)\, \Phi^B_{ab}(y|\mathbf{P},\ \sigma) = i\delta_{AB}\delta_{\sigma\sigma'},$$

where

$$W_{AB}(\mathbf{P}|x;\ y) = \frac{\overline{G}^{-1}_{ab}(P_A|x;\ y) - \overline{G}^{-1}_{ab}(P_B|x;\ y)}{P_A^2 - P_B^2};\quad P_A^2 = M_A^2,\quad P_B^2 = M_B^2.$$

It is easy to see that

$$W_{AA}(\mathbf{P}|x;\ y) = W_{AB}(\mathbf{P}|x;\ y)|_{P_A^2 \to P_B^2} = N_A(\mathbf{P}|x;\ y).$$

For the total Bethe–Salpeter wave function, the normalization condition can thus be written in the form

$$\int dx\, dy N_A(\mathbf{P}|x;\ y) \int d^3\mathbf{X} \overline{\Phi}^A_{ab}(Xx|\mathbf{P},\ \sigma') \frac{\overleftrightarrow{\partial}}{\partial X^0} \Phi^A_{ab}(Xy|\mathbf{Q},\ \sigma) = 2E(\mathbf{P},\ M_A)\,\delta^3(\mathbf{P}-\mathbf{Q})\,\delta_{\sigma\sigma'}.$$

We define new functions $\overline{U}^A_{ab}$ and $V^A_{ab}$ by means of the relations

$$\overline{U}^A_{ab}(Xx|\mathbf{P},\ \sigma) = \int dy \overline{\Phi}^A_{ab}(Xy|\mathbf{P},\ \sigma)\, N_A(\mathbf{P}|y;\ x); \quad (75)$$

$$V^A_{ab}(Xx|\mathbf{P},\ \sigma) = \int dy N_A(\mathbf{P}|x,\ y)\, \Phi^A_{ab}(Xy|\mathbf{P},\ \sigma). \quad (75a)$$

Then the normalization condition can be rewritten in the two equivalent forms

$$\int dx \int d^3\mathbf{X}\, \overline{U}^A_{ab}(Xx|\mathbf{P},\ \sigma') \frac{\overleftrightarrow{\partial}}{\partial X^0} \Phi^A_{ab}(Xx|\mathbf{Q},\ \sigma) = 2E(\mathbf{P},\ M_A)\,\delta^3(\mathbf{P}-\mathbf{Q})\,\delta_{\sigma\sigma'} = \int dx \int d^3\mathbf{X}\, \overline{\Phi}^A_{ab}(Xx|\mathbf{P},\ \sigma') \frac{\overleftrightarrow{\partial}}{\partial X^0} V^A_{ab}(Xx|\mathbf{Q},\ \sigma).$$

We now turn directly to the formulation of the asymptotic condition for the composite particles. We express the Bethe–Salpeter wave function for bound states by means of the matrix element

$$\left.\begin{aligned}&\langle 0|T(\Phi_a(X+x/2)\Phi_b(X-x/2))|\Phi_{ab}^A\rangle=\Phi_{ab}^A(Xx);\\&\langle\Phi_{ab}^A|T(\Phi_a(X+x/2)\Phi_b(X-x/2))/0\rangle=\overline{\Phi}_{ab}^A(Xx),\end{aligned}\right\}\quad(76)$$

where $|\Phi_{ab}^A\rangle$ is the vector of the bound state of the two particles $a$ and $b$ with invariant mass $M_A$.

Let $\overline{U}_{ab}^A$ and $V_{ab}^A$ be functions smooth with respect to the variable $X$ and satisfying the normalization condition

$$\int dx\int d^3\mathbf{X}\overline{U}_{ab}^A(Xx)\frac{\overleftrightarrow{\partial}}{\partial X^0}\Phi_{ab}^A(Xx)$$

$$=\int dx\int d^3\mathbf{X}\overline{\Phi}_{ab}^A(Xx)\frac{\overleftrightarrow{\partial}}{\partial X^0}V_{ab}^A(Xx)=1.$$

We consider a bilocal construction of the form

$$A(Xx)=T(\Phi_a(X+x/2)\Phi_b(X-x/2)).$$

A construction of such type was used in Refs. 9–11 to construct local field operators of composite particles. The "in" and "out" fields of the composite particles are determined, as in the case of elementary particles, by means of the Yang–Feldman equation:

$$\begin{aligned}A(Xx)&=A_{\rm in}(Xx)+\int dY D_A^{\rm ret}(X-Y)J(Yx)\\&=A_{\rm out}(Xx)+\int dY D_A^{\rm adv}(X-Y)J(Yx),\end{aligned}$$

where the current operator is determined by

$$J(Xx)=(\Box_X+M_A^2)A(Xx).$$

We define the smeared field operators for the composite particle by

$$A_t^{(-)}(\Phi_{ab}^A)=\int dx\int_{X^0=t} d^3\mathbf{X}\overline{U}_{ab}^A(Xx)\frac{\overleftrightarrow{\partial}}{\partial X^0}A(Xx);\quad(77)$$

$$A_t^{(+)}(\Phi_{ab}^A)=\int dx\int_{X^0=t} d^3\mathbf{X}A(Xx)\frac{\overleftrightarrow{\partial}}{\partial X^0}V_{ab}^A(Xx)\quad(78)$$

and require fulfillment of the conditions of weak convergence[6]:

$$\left.\begin{aligned}&\lim_{t\to+\infty}\langle\psi_1|A_t^{(\pm)}(\Phi_{ab}^A)|\psi_2\rangle=\langle\psi_1|A_{\rm out}^{(\pm)}(\Phi_{ab}^A)|\psi_2\rangle;\\&\lim_{t\to-\infty}\langle\psi_1|A_t^{(\pm)}(\Phi_{ab}^A)|\psi_2\rangle=\langle\psi_1|A_{\rm in}^{(\pm)}(\Phi_{ab}^A)|\psi_2\rangle\end{aligned}\right\}\quad(79)$$

for all normalized vectors $|\psi_1\rangle$ and $|\psi_2\rangle$. The operators $A_{\rm in,out}^{(\pm)}(\Phi_{ab}^A)$ are determined by means of (77) and (78), in which the bilocal operator $A(Xx)$ must be replaced by the operator $A_{\rm in,out}(Xx)$. As is readily seen, the operators $A_{\rm in,out}(Xx)$ satisfy the free Klein–Gordon equation

$$(\Box_X+M_A^2)A_{\rm in,\,out}(Xx)=0.$$

The smeared operators $A_{\rm in,out}^{(\pm)}(\Phi_{ab}^A)$ can be interpreted as the operators of creation and annihilation of bound "in" and "out" states with wave function $\Phi_{ab}^A$.

For practical purposes, it will be sufficient to use the Bethe–Salpeter wave functions in the plane-wave representation. In other words, for the bound-state wave functions we shall use the expression (76), in which the bound-state vector has the form

$$|\Phi_{ab}^A\rangle=|M_A;\ \mathbf{P},\ \sigma\rangle.$$

Accordingly, for the functions $\overline{U}_{ab}^A$ and $V_{ab}^A$ we shall use the expressions (75) and (75a).

It is of interest to compare the formulation of the asymptotic condition in the form (79) with the asymptotic condition formulated in Ref. 9.

Let $f_{M_A}(X)$ be a smooth negative-frequency solution of the Klein–Gordon equation $(\Box_x+M_A^2)f_{M_A}(X)=0$ normalized by the condition

$$i\int d^3\mathbf{X}\overset{*}{f}_{M_A}(X)\frac{\overleftrightarrow{\partial}}{\partial X^0}f_{M_A}(X)=1.$$

We introduce smeared field operators in accordance with

$$\left.\begin{aligned}&A_t^{(-)}(x)=i\int_{X^0=t}d^3\mathbf{X}\overset{*}{f}_{M_A}(X)\frac{\overleftrightarrow{\partial}}{\partial X^0}A(Xx)\equiv A_t^{(-)}(f_{M_A};\ x);\\&A_t^{(+)}(x)=i\int_{X^0=t}d^3\mathbf{X}A(Xx)\frac{\overleftrightarrow{\partial}}{\partial X^0}f_{M_A}(X)\equiv A_t^{(+)}(f_{M_A};\ x).\end{aligned}\right\}\quad(80)$$

The asymptotic condition formulated in Ref. 9 presupposes the existence of the weak limits

$$\left.\begin{aligned}&\langle\psi_1|A_t^{(\pm)}(x)|\psi_2\rangle\xrightarrow[t\to+\infty]{}\langle\psi_1|A_{\rm out}^{(\pm)}(x)|\psi_2\rangle;\\&\langle\psi_1|A_t^{(\pm)}(x)|\psi_2\rangle\xrightarrow[t\to-\infty]{}\langle\psi_1|A_{\rm in}^{(\pm)}(x)|\psi_2\rangle\end{aligned}\right\}\quad(81)$$

for all normalized vectors $|\psi_1\rangle$ and $|\psi_2\rangle$. The operators $A_{\rm in,out}^{(\pm)}(x)$ are determined by means of the expressions (80), in which the operator $A(Xx)$ must be replaced by the asymptotic fields $A_{\rm in,out}(Xx)$, which satisfy the free Klein–Gordon equation with respect to the variable $X$.

It was shown in Ref. 9 that the operators $A_{\rm in,out}^{(\pm)}(f_{M_A};x)$ admit a factorized representation in the form

$$A_{\rm in,\,out}^{(-)}(f_{M_A};\ \mathbf{P};\ x)=A_{\rm in,\,out}^{(-)}(\mathbf{P})\Phi_{ab}^A(x|\mathbf{P});$$

$$A_{\rm in,\,out}^{(+)}(f_{M_A};\ \mathbf{P};\ x)=A_{\rm in,\,out}^{(+)}(\mathbf{P})\overline{\Phi}_{ab}^A(x|\mathbf{P}),$$

where $A_{\rm in,out}^{(\pm)}(\mathbf{P})$, the operators of creation and annihilation of the bound state with momentum $\mathbf{P}$, satisfy the canonical commutation relations

$$[A_{\rm in,out}^{(-)}(\mathbf{P}),\ A_{\rm in,out}^{(+)}(\mathbf{Q})]=2E(\mathbf{P},\ M_A)\delta^3(\mathbf{P}-\mathbf{Q}),$$

all the remaining commutators vanishing; $\Phi_{ab}^A(x|\mathbf{P})$ is the Bethe–Salpeter wave function. Thus, the asymptotic condition (81) in the plane-wave representation can be rewritten in the form

$$\left.\begin{aligned}&\langle\psi_1|A_t^{(-)}(f_{M_A};\ \mathbf{P};\ x)|\psi_2\rangle\xrightarrow[t\to\mp\infty]{}\langle\psi_1|A_{\substack{\rm in\\ \rm out}}^{(-)}(\mathbf{P})|\psi_2\rangle\Phi_{ab}^A(x|\mathbf{P});\\&\langle\psi_1|A_t^{(+)}(f_{M_A};\ \mathbf{P};\ x)|\psi_2\rangle\xrightarrow[t\to\mp\infty]{}\langle\psi_1|A_{\substack{\rm in\\ \rm out}}^{(+)}(\mathbf{P})|\psi_2\rangle\overline{\Phi}_{ab}^A(x|\mathbf{P}).\end{aligned}\right\}\quad(82)$$

If it is borne in mind that the asymptotic condition (81) is satisfied uniformly with respect to the variable $x$, and allowance is made for the normalization condition for the bound-state wave function, it is readily seen that from the asymptotic condition (82) it is possible to obtain the asymptotic condition in the form (79). In other words, we have the following important proposition: In the plane-wave representation, the validity of the asymptotic condition (79) follows from the asymptotic condition (81).

We rewrite the expressions (77) and (78) in the plane-wave representation:

$$A_t^{(-)}(\Phi_{ab}^A(|\mathbf{P},\sigma)) \equiv A_t^{(-)}(\mathbf{P},\sigma)$$
$$= \int dx\,dy N_A(\mathbf{P}|x;y) \int_{X^0=t} d^3\mathbf{X}\overline{\Phi}_{ab}^A(Xx|\mathbf{P},\sigma)\frac{\overleftrightarrow{\partial}}{\partial X^0} A(Xy); \tag{83}$$

$$A_t^{(+)}(\Phi_{ab}^A(|\mathbf{P},\sigma)) \equiv A_t^{(+)}(\mathbf{P},\sigma)$$
$$= \int dx\,dy N_A(\mathbf{P}|x;y) \int_{X^0=t} d^3\mathbf{X} A(Xx)\frac{\overleftrightarrow{\partial}}{\partial X^0}\Phi_{ab}^A(Xy|\mathbf{P},\sigma). \tag{83a}$$

The operators $A^{(\pm)}_{\rm in,out}(\mathbf{P},\alpha)$ can be constructed in accordance with the same formulas with the operator $A(Xx)$ replaced by the asymptotic field operators $A_{\rm in,out}(Xx)$. It is readily seen that they have the representation

$$A_{\substack{\rm in\\ \rm out}}(Xx) = \sum_\sigma \int \frac{d^3\mathbf{P}}{2E(\mathbf{P},M_A)}[A^{(-)}_{\substack{\rm in\\ \rm out}}(\mathbf{P},\sigma)\Phi_{ab}^A(Xx|\mathbf{P},\sigma)$$
$$+ A^{(+)}_{\substack{\rm in\\ \rm out}}(\mathbf{P},\sigma)\overline{\Phi}_{ab}^A(Xx|\mathbf{P},\sigma)]. \tag{84}$$

The representation (84) agrees with the result of Ref. 9 mentioned above if it is remembered that the Bethe–Salpeter wave function admits the factorized representation

$$\Phi_{ab}^A(Xx\mid\mathbf{P},\sigma) = f_{M_A;\mathbf{P}}(X)\,\Phi_{ab}^A(x\mid\mathbf{P},\sigma),$$

where

$$f_{M_A;\mathbf{P}}(X) = (2\pi)^{-3/2}\exp(-\mathrm{i}\,PX);$$

$$P^0 = E(\mathbf{P},M_A) = \sqrt{\mathbf{P}^2+M_A^2}.$$

By means of the asymptotic condition (79), using the method of Ref. 4, we can obtain different reduction formulas. For example, we have

$$\langle\psi_1|A^{(-)}_{\rm out}(\mathbf{P},\sigma)\,T(\Phi_1(y_1)\dots\Phi_n(y_n))|\psi_2\rangle$$
$$-\langle\psi_1|T(\Phi_1(y_1)\dots\Phi_n(y_n))\,A^{(-)}_{\rm in}(\mathbf{P},\sigma)|\psi_2\rangle$$
$$= \int dx\,dy N_A(\mathbf{P}|x;y)$$
$$\times\int dX\overline{\Phi}_{ab}^A(Xx|\mathbf{P},\sigma)\overrightarrow{K}_X^{M_A}\langle\psi_1|T(\Phi_a(X+y/2)$$
$$\times\langle\Phi_b(X-y/2)\,\Phi_1(y_1)\dots\Phi_n(y_n))|\psi_2\rangle$$

or

$$\langle\psi_1|A^{(+)}_{\rm out}(\mathbf{P},\sigma)\,T(\Phi_1(y_1)\dots\Phi_n(y_n))|\psi_2\rangle$$
$$-\langle\psi_1|T(\Phi_1(y_1)\dots\Phi_n(y_n))\,A^{(+)}_{\rm in}(\mathbf{P},\sigma)|\psi_2\rangle$$
$$= \int dx\,dy N_A(\mathbf{P}|y;x)$$
$$\times\int dx\langle\psi_1|T(\Phi_a(X+y/2)\,\Phi_b(X-y/2)\,\Phi_1(y_1)\dots$$
$$\dots\Phi_n(y_n))|\psi_2\rangle\overleftarrow{K}_X^{M_A}\Phi_{ab}^A(Xx|\mathbf{P},\sigma).$$

As another example of the use of the asymptotic condition (79), we consider the derivation of the amplitude for scattering by a bound state. To this end, we write the relation (67) in the expanded form

$$\overline{G}_{abc}(x_1x_2x_3;\,y_1y_2y_3) = \overline{G}_{ab}(x_1x_2;\,y_1y_2)\,D_c(x_3-y_3)$$
$$+\int(dx')(dy')\,\overline{G}_{ab}(x_1x_2;\,x_1'x_2')$$
$$\times D_c(x_3-x_3')\,R^{c;c}(x_1'x_2'x_3';\,y_1'y_2'y_3')$$
$$\times G_{ab}(y_1'y_2';\,y_1y_2)\,D_c(y_3'-y_3). \tag{85}$$

Using the representation for Green's functions in terms of the vacuum expectation value of the time-ordered product of the Heisenberg field operators of the particles, we can readily show by means of the asymptotic condition (79) that

$$\lim_{t\to-\infty}\int dx\,dy N_A(\mathbf{Q}|y;x)\int_{Y^0=t} d^3\mathbf{Y}\overline{G}_{ab}(y_1'y_2';\,Yy)$$
$$\times\frac{\overleftrightarrow{\partial}}{\partial Y^0}\Phi_{ab}^A(Yx\,|\mathbf{Q},\sigma) = \mathrm{i}^2\Phi_{ab}^A(y_1'y_2'|\mathbf{Q},\sigma); \tag{86}$$

$$\lim_{t\to+\infty}\int dx\,dy N_B(\mathbf{P}|y;x)$$
$$\times\int_{X^0=t} d^3\mathbf{X}\overline{\Phi}_{ab}^B(Xy|\mathbf{P},\sigma')\frac{\overleftrightarrow{\partial}}{\partial X^0}\overline{G}_{ab}(Xx;\,x_1'x_2')$$
$$= \mathrm{i}^2\overline{\Phi}_{ab}^B(x_1'x_2'/\mathbf{P},\sigma'). \tag{87}$$

In (85) we consider the sequence of times $X^0 > x_3^0 > y_3^0 > Y^0$ and carry out the procedure of smearing with respect to the coordinates $\mathbf{X}$ and $\mathbf{Y}$ by means of (83) and (83a) and with respect to the coordinates $\mathbf{x}_3$ and $\mathbf{y}_3$ by means of (56) and (57), after which we go to the limits $X^0, x_3^0\to+\infty$ and $Y^0, y_3^0\to-\infty$. Using the limit relations (62), (86), and (87), we obtain

$$\langle\Phi_{(ab)c}^B;\,\mathbf{P},\sigma',\mathbf{p};\,\mathrm{out}|\Phi_{(ab)c}^A;\,\mathbf{Q},\sigma,\mathbf{q};\,\mathrm{in}\rangle$$
$$=\delta_{AB}\delta_{\sigma\sigma'}2E(\mathbf{P},M_A)\,\delta^3(\mathbf{P}-\mathbf{Q})\,2E(\mathbf{p},m_c)\,\delta^3(\mathbf{p}-\mathbf{q})$$
$$-\mathrm{i}\int(dx)(dy)\,\overline{\Phi}_{ab}^B(x_1x_2|\mathbf{P},\sigma')\,\overset{*}{f}_{m_c;\mathbf{p}}(x_3)$$
$$\times R^{c;c}(x_1x_2x_3;\,y_1y_2y_3)\,\Phi_{ab}^A(y_1y_2|\mathbf{Q},\sigma)\,f_{m_c;\mathbf{q}}(y_3),$$

in agreement with the expression obtained earlier for the amplitude of scattering by a bound state. It is obvious that in the same manner one can obtain expressions for the amplitudes of arbitrary processes in the three-particle system. Thus, the asymptotic condition in the form (79) completely solves the problem posed at the end of the previous section.

## 5. ASYMPTOTIC LSZ CONDITION AND DERIVATION OF DYNAMICAL EQUATIONS IN QUANTUM FIELD THEORY

In the previous sections, we have established how the asymptotic conditions formulated in the form of (58) and (79) are related to definite singularities of the Green's functions. In this section, we wish to show that by using the asymptotic conditions it is, in addition, possible to arrive at a universal method for deriving dynamical equations in quantum field theory. The universality of the method is expressed in the fact that it can be used to obtain equations for the wave functions of both scattering states and bound states. Since the Green's functions contain complete information about the system, it is natural to construct the derivation of the equations for the wave functions on the basis of the equations for the Green's functions.

The essence of the method that we wish to describe here is, proceeding from the equations for the Green's functions and applying appropriately asymptotic conditions, to obtain dynamical equations for the wave functions. It should b noted that the dynamical equations that we considered i Secs. 1 and 2 were also essentially obtained as consequence of the corresponding equations for the Green's functions.

We consider the derivation of the Bethe–Salpeter equation. We proceed from Eq. (28) for the two-particle Green's function, which we rewrite in the expanded form

$$\overline{G}_{ab}(x_1x_2;\ y_1y_2)=D_a(x_1-y_1)\,D_b(x_2-y_2)$$
$$+\int(dx')\,(dy')\,D_a(x_1-x_1')\,D_b(x_2-x_2')$$
$$\times V_{ab}(x_1'x_2';\ \ y_1'y_2')\,\overline{G}_{ab}(y_1'y_2';\ y_1y_2).\qquad(88)$$

In this equation, we carry out the procedure of smearing with respect to the variables $\mathbf{y}_1$ and $\mathbf{y}_2$ by means of (57), after which we go to the limit $y_1^0\to-\infty$, $y_2^0\to-\infty$. Using the limit relations

$$\lim_{y^0\to-\infty}\int d^3\mathbf{y}D(x-y)\,\overleftrightarrow{\frac{\partial}{\partial y^0}}\,f_{m;\,\mathbf{q}}(y)=f_{m;\,\mathbf{q}}(x);\qquad(89)$$

$$\lim_{\substack{y_1^0\to-\infty\\ y_2^0\to-\infty}}\int d^3\mathbf{y}_1\int d^3\mathbf{y}_2\overline{G}_{ab}(x_1x_2;\ \ y_1y_2)$$
$$\times\overleftrightarrow{\frac{\partial}{\partial y_1^0}}\,\overleftrightarrow{\frac{\partial}{\partial y_2^0}}\,f_{m_a;\,\mathbf{q}_1}(y_1)\,f_{m_b;\,\mathbf{q}_2}(y_2)$$
$$=\Phi_{ab}(x_1x_2|\mathbf{q}_1\mathbf{q}_2),\qquad(90)$$

we obtain the Bethe–Salpeter equation for the scattering-state wave function:

$$\Phi_{ab}(x_1x_2\mid\mathbf{q}_1\mathbf{q}_2)=\Phi_{ab}^{(0)}(x_1x_2\ \ \mathbf{q}_1\mathbf{q}_2)$$
$$+\int(dx')\,(dy')\,D_a(x_1-x_1')\,D_b(x_2-x_2')$$
$$\times V_{ab}(x_1'x_2';\ y_1'y_2')\,\Phi_{ab}(y_2'y_2'\mid\mathbf{q}_1\mathbf{q}_2),$$

where

$$\Phi_{ab}^{(0)}(x_1x_2|\mathbf{q}_1\mathbf{q}_2)=f_{m_a;\ \mathbf{q}_1}(x_1)\,f_{m_b;\,\mathbf{q}_2}(x_2);$$

Proceeding from the same equation, Eq. (88), we can obtain a dynamical equation for the bound-state Bethe–Salpeter wave function. For this, we carry out in Eq. (88) the procedure of smoothing with respect to the variable $\mathbf{Y}=(\mathbf{y}_1+\mathbf{y}_2)/2$ by means of (83a) and go to the limit $Y^0\to-\infty$. Using the limit relation (86) and bearing in mind that the inhomogeneous term in Eq. (88) makes a vanishing contribution in the limit $Y^0\to-\infty$, we obtain the homogeneous Bethe–Salpeter equation

$$\Phi_{ab}^A(x_1x_2|\mathbf{P},\ \sigma)=\int(dx')\,(dy')\,D_a(x_1-x_1')$$
$$\times D_b(x_2-x_1')\,V_{ab}(x_1'x_2';\ y_1'y_2')$$
$$\times\Phi_{ab}^A(y_1'y_2'|\mathbf{P},\ \sigma).$$

As another example, we consider the derivation of Eq. (43) for the wave function describing the breakup of a bound state. Substituting the definition of the function $V_{(ab)c}$ in the expression (33a), we obtain an equation for the three-particle Green's function in the form

$$\overline{G}_{abc}(x_1x_2x_3;\ y_1y_2y_3)=\overline{G}_{ab}(x_1x_2;\ y_1y_2)\,D_c(x_3-y_3)$$
$$+\int(dx')\,(dy')\,D_a(x_1-x_1')\,D_b(x_2-x_2')$$
$$\times D_c(x_3-x_3')\,V_{(ab)c}(x_1'x_2'x_3';\ y_1'y_2'y_3')\overline{G}_{abc}(y_1'y_2'y_3';\ y_1y_2y_3).$$

In this equation, we carry out the procedure of smearing with respect to the variable $\mathbf{Y}=(\mathbf{y}_1+\mathbf{y}_2)/2$ by means of (83a) and with respect to the variable $\mathbf{y}_3$ by means of (57) and go to the limit $Y^0\to-\infty$, $y_3^0\to-\infty$. Taking into account the limit relations (89) and (86), we obtain

$$\Phi_{(ab)c}^A(x_1x_2x_3|\mathbf{Q},\ \sigma,\ \mathbf{q})=\Phi_{ab}^A(x_1x_2|\mathbf{Q},\sigma)\,f_{m_c;\,\mathbf{q}}(x_3)$$
$$+\int(dx')\,(dy')\,D_a(x_1-x_1')\,D_b(x_2-x_2')$$
$$\times D_c(x_3-x_3')\,V_{(ab)c}(x_1'x_2'x_3';\ y_1'y_2'y_3')\,\Phi_{(ab)c}^A(y_1'y_2'y_3'|\mathbf{Q},\ \sigma,\ \mathbf{q}),\qquad(91)$$

where

$$\Phi_{(ab)c}^A(x_1x_2x_3|\ \mathbf{Q},\ \sigma,\ \mathbf{q})$$
$$=\langle 0|T(\Phi_a(x_1)\,\Phi_b(x_2)\,\Phi_c(x_3))|\Phi_{(ab)c}^A;\mathbf{Q},\ \sigma,\ \mathbf{q};\ \text{in}\rangle.$$

In Sec. 2, we obtained Eq. (91) by heuristic arguments. The method based on the use of the asymptotic conditions provides us with a rigorous derivation of Eq. (91). It is obvious that by this method one can also obtain a rigorous derivation of the other dynamical equations found in Sec. 2.

## 6. ITERATIVE SCHEMES FOR CALCULATING THE FUNCTIONS $R^{\alpha;\beta}$

It was shown above that the functions $R^{\alpha,\beta}$ introduced in Sec. 2 by means of the relation (47) are directly related to the amplitudes of physical processes in which the system goes over from the initial state characterized by the wave function $\Phi_\beta^{(0)}$ to the final state with $\Phi_\alpha^{(0)}$. Through the relation (47), the functions $R^{\alpha,\beta}$ specify a certain representation for the Green's function of the three-particle system. We consider some equivalent representations for the three-particle Green's function:

$$\overline{G}_{abc}=\overline{G}_{\alpha}^{(0)}+\overline{G}_{\alpha}^{(0)}*R^{(+)\alpha;\beta}*\overline{G}_{\beta}^{(0)}; \quad (92)$$

$$\overline{G}_{abc}=\overline{G}_{\beta}^{(0)}+\overline{G}_{\alpha}^{(0)}*R^{(-)\alpha;\beta}*\overline{G}_{\beta}^{(0)}; \quad (93)$$

$$G_{abc}=\overline{G}_{\alpha}^{(0)}\delta_{\alpha\beta}+\overline{G}_{\alpha}^{(0)}*R^{\alpha;\beta}*\overline{G}_{\beta}^{(0)}. \quad (94)$$

Each of these representations determines a corresponding function $R^{\alpha,\beta}$ and, conversely, each of the functions $R^{\alpha,\beta}$ determines through Eqs. (92), (93), and (94) a certain representation for the three-particle Green's function. Comparing the different representations for the three-particle Green's function, we find the connection between the different functions $R^{\alpha;\beta}$:

$$R^{\alpha;\beta}=(1-\delta_{\alpha\beta})\,\overline{G}_{\beta}^{(0)-1}+R^{(+)\alpha;\beta}=(1-\delta_{\alpha\beta})\,\overline{G}_{\alpha}^{(0)-1}+R^{(-)\alpha;\beta}. \quad (95)$$

It follows from this that

$$\overline{\Phi}_{\alpha}^{(0)}*R^{\alpha;\beta}*\Phi_{\beta}^{(0)}=\overline{\Phi}_{\alpha}^{(0)}*R^{(+)\alpha;\beta}*\Phi_{\beta}^{(0)}=\overline{\Phi}_{\alpha}^{(0)}*R^{(-)\alpha;\beta}*\Phi_{\beta}^{(0)},$$

i.e., to calculate the amplitudes of the physical processes we can use any of the functions $R^{\alpha,\beta}$. This means that the functions $R^{\alpha;\beta}$, $R^{(+)\alpha;\beta}$, $R^{(-)\alpha;\beta}$ are physically equivalent. Comparing, for example, the representation (94) with (33a), we find

$$R^{\alpha;\beta}=(1-\delta_{\alpha\beta})\,G_{abc}^{(0)-1}+\overline{R}_{abc}-R_{\alpha}\delta_{\alpha\beta}-R_{\alpha}*G_{abc}^{(0)}*R^{\alpha;\beta}-R^{\alpha;\beta}*G_{abc}^{(0)}*R_{\beta}-R_{\alpha}*G_{abc}^{(0)}*R^{\alpha;\beta}*G_{abc}^{(0)}*R_{\beta}.$$

This relation can be used to construct an iterative scheme making it possible to calculate the functions $R^{\alpha,\beta}$ in terms of the fundamental functions of the theory—the vacuum expectation values of the radiation operators.

To calculate the functions $R^{\alpha,\beta}$, it is also possible to use dynamical schemes taken from nonrelativistic three-particle scattering theory. For example, proceeding from the definitions of the functions $R^{(\pm)\alpha,\beta}$ and the equation for the three-particle Green's function in the form

$$\overline{G}_{abc}=\overline{G}_{\alpha}^{(0)}+\overline{G}_{\alpha}^{(0)}*\overline{V}_{\alpha}*\overline{G}_{abc}=\overline{G}_{\alpha}^{(0)}+\overline{G}_{abc}*\overline{V}_{\alpha}*\overline{G}_{\alpha}^{(0)},$$

where $\overline{V}_{\alpha}=V_{abc}-V_{\alpha}$ (we recall that the index $\alpha$ takes the values $a, b, c, 0$), we can readily establish that the functions $R^{(\pm)\alpha,\beta}$ have the representation

$$R^{(+)\alpha;\beta}=\overline{V}_{\alpha}+\overline{V}_{\alpha}*\overline{G}_{abc}*\overline{V}_{\beta};$$

$$R^{(-)\alpha;\beta}=\overline{V}_{\beta}+\overline{V}_{\alpha}*\overline{G}_{abc}*\overline{V}_{\beta}.$$

Substituting here the equations

$$\overline{V}_{\alpha}*\overline{G}_{abc}*V_{\gamma}=R^{(+)\alpha;\gamma}*G_{abc}^{(0)}*R_{\gamma};$$

$$V_{\gamma}*\overline{G}_{abc}*\overline{V}_{\beta}=R_{\gamma}*G_{abc}^{(0)}*R^{(-)\gamma;\beta};$$

we arrive at equations[7)] for the functions $R^{(\pm)\alpha,\beta}$:

$$R^{(+)\alpha;\beta}=\overline{V}_{\alpha}+\sum_{\gamma\neq\beta}R^{(+)\alpha;\gamma}*G_{abc}^{(0)}*R_{\gamma}; \quad (96)$$

$$R^{(-)\alpha;\beta}=\overline{V}_{\beta}+\sum_{\gamma\neq\alpha}R_{\gamma}*G_{abc}^{(0)}*R^{(-)\gamma;\beta}. \quad (97)$$

Hence, using the relation (95), we obtain equations[8)] for the functions $R^{\alpha;\beta}$:

$$R^{\alpha;\beta}=(1-\delta_{\alpha\beta})\,G_{abc}^{(0)-1}+\sum_{\gamma\neq\alpha}R_{\gamma}*G_{abc}^{(0)}*R^{\gamma;\beta}=(1-\delta_{\alpha\beta})\,G_{abc}^{(0)-1}+\sum_{\gamma\neq\beta}R^{\alpha;\gamma}*G_{abc}^{(0)}*R_{\gamma}. \quad (98)$$

Equations (96)–(98) can serve as the basis for the construction of iterations for the functions $R^{\alpha;\beta}$. The system of equations (98) is particularly convenient for constructing an iterative scheme when the function $R_0$ can be ignored. In this case, iterating the system of equations (98), we obtain expressions for $R^{\alpha,\beta}$ solely in terms of the functions $R_{\alpha}\,\alpha=(a,b,c)$, which are the vacuum expectation values of the radiation operators of fourth order.

## 7. DERIVATION OF THREE-DIMENSIONAL DYNAMICAL EQUATION FOR THE WAVE FUNCTION OF ELASTIC SCATTERING BY A BOUND STATE

The dynamical equations considered in the present paper are four-dimensional and manifestly relativistically invariant. Therefore, to investigate the structure of the kernels of these equations it is possible to use the methods of relativistic quantum field theory. For example, in the case of a small coupling constant it is possible to use the method, well known from quantum electrodynamics, of invariant perturbation theory. Many of the relations which we have considered can serve as the basis for the derivation of three-dimensional equations if one uses for this purpose the method of one-time reduction developed in Refs. 12, 16, and 17, the sources of which go back to Logunov and Tavkhelidze's quasipotential approach in quantum field theory.[18]

In Ref. 12, the present authors used the method of one-time reduction to obtain a one-time equation for the wave function of elastic scattering of a particle by a bound state of two other particles. We shall now rederive this equation, using for this purpose the method described above for deriving dynamical equations in quantum field theory.

We proceed from the relation (85), in which we smear with respect to the variables $\mathbf{X}=(\mathbf{x}_1+\mathbf{x}_2)/2$ and $\mathbf{Y}=(\mathbf{y}_1+\mathbf{y}_2)/2$ by means of (83) and (83a) and with respect to the coordinate $\mathbf{y}_3$ by means of (57), going then to the limit $X^0\to+\infty$, $Y^0, y_3^0\to-\infty$. Using the limit relations (86), (87), and (89), we obtain

$$\langle\Phi_{ab}^{A};\ \mathbf{P};\ \mathrm{out}\,|\Phi_c(x_3)|\,\Phi_{(ab)c}^{A};\ \mathbf{Q},\ \mathbf{q};\ \mathrm{in}\rangle=2E(\mathbf{P},\ M_A)\,\delta^3(\mathbf{P}-\mathbf{Q})\,f_{m_c;\,\mathbf{q}}(x_3)-\int dx_3'\,dy_3'D_c(x_3-x_3')\,R_A^{c;\,c}(\mathbf{P}|x_3';\ y_3'|\mathbf{Q})\,f_{m_c;\,\mathbf{q}}(y_3'), \quad (99)$$

where we have introduced the notation

$$R_A^{c;\,c}(\mathbf{P}|x_3';\ y_3'|\mathbf{Q})=\int dx_1'\,dx_2'\,dy_1'\,dy_2'\overline{\Phi}_{ab}^{A}(x_1'x_2'|\mathbf{P})\times R^{c;\,c}(x_1'x_2'x_3';\ y_1'y_2'y_3')\,\Phi_{ab}^{A}(y_1'y_2'|\mathbf{Q})$$

and considered for simplicity the case of a bound state with zero quantum numbers $\sigma$.

We introduce the Fourier transforms

$$D_c(x_3-y_3)=(2\pi)^{-4}\int dpD_c(p)\exp[-i\,p\,(x_3-y_3)];$$

$$R_A^{c;\,c}(\mathbf{P}\mid p;\ q\mid\mathbf{Q})=\int dx_3\,dy_3\exp(ipx_3-iqy_3)\,R_A^{c;\,c}(\mathbf{P}\mid x_3;\ y_3\mid\mathbf{Q})$$

and make a three-dimensional Fourier transformation in the relation (99) with respect to the variable $\mathbf{x}_3$, obtaining

$$\langle \Phi^A_{ab};\ \mathbf{P};\ \text{out} \mid \Phi_c(\mathbf{p},\ t) \mid \Phi^A_{(ab)\,c};\ \mathbf{Q},\ \mathbf{q};\ \text{in}\rangle = 2E(\mathbf{P},\ M_A)\,\delta^3(\mathbf{P}-\mathbf{Q}) \times (2\pi)^{3/2}\delta^3(\mathbf{p}-\mathbf{q})\exp[-iE(\mathbf{q},\ m_c)\,t] - (2\pi)^{-5/2}\int_{-\infty}^{\infty} dp^0 D_c(p)\exp(-ip^0t)\,R_A^{c;\,c}(\mathbf{P}\mid p;\ \tilde{q}\mid \mathbf{Q}). \quad (100)$$

We introduce the new functions

$$\left.\begin{aligned}&\langle \Phi^A_{ab};\mathbf{P};\text{out} \mid \Phi_c(\mathbf{p},t) \mid \Phi^A_{(ab)\,c};\ \mathbf{Q},\ \mathbf{q},\ \text{in}\rangle \exp[-iE(\mathbf{P},\ M_A)\,t]\\ &\quad = (2\pi)^3\,\delta^3(\mathbf{P}+\mathbf{p}-\mathbf{Q}-\mathbf{q})\,2E(\mathbf{P},\ M_A)\,\chi_{\mathbf{Q};\,\mathbf{q}}(\mathbf{p},\ t),\\ &\quad -\frac{1}{2\pi}\int_{-\infty}^{\infty} dp^0 D_c(p)\,R_A^{c;\,c}(\mathbf{P}\mid p;\ \tilde{q}\mid \mathbf{Q})\\ &\quad = \frac{(2\pi)^3\,\delta^3(\mathbf{P}+\mathbf{p}-\mathbf{Q}-\mathbf{q})\,T(\mathbf{p};\ \mathbf{q}\mid \mathbf{Q}+\mathbf{q})}{E^2(\mathbf{p},\ m_c)-[E-E(\mathbf{P},\ M_A)]^2-i0},\end{aligned}\right\} \quad (101)$$

where $E = E(\mathbf{q}, m_c) + E(\mathbf{Q}, M_A)$ is the energy of the initial state, and we have used the fact that, translational invariance holding, the function $R_A^{c;c}$ contains the four-dimensional $\delta$ function $\delta^4(P + p - Q - q)$. The relation (100) can be rewritten in terms of the functions that we have introduced in the form

$$\chi_{\mathbf{Q},\,\mathbf{q}}(\mathbf{p},\ t) = \chi^{(0)}_{\mathbf{Q},\,\mathbf{q}}(\mathbf{p},\ t) + \{2E(P,\ M_A)\,[E^2(\mathbf{p},\ m_c) - (E-E(\mathbf{P},\ M_A))^2 - i0]\}^{-1}\int d\mathbf{k}\,T(\mathbf{p};\ \mathbf{k}\mid \mathbf{Q}+\mathbf{q})\,\chi^{(0)}_{\mathbf{Q},\,\mathbf{q}}(\mathbf{k},\ t), \quad (102)$$

where $X^{(0)}_{\mathbf{Q}},\ \mathbf{q}(\mathbf{p}, t) = (2\pi)^{-3/2}\delta^3(\mathbf{p}-\mathbf{q})\exp(-iEt)$, and $\mathbf{P} = \mathbf{Q} + \mathbf{q} - \mathbf{p}$. The relation (102) can be conveniently written in the symbolic form

$$\chi_{\mathbf{Q},\,\mathbf{q}} = \chi^{(0)}_{\mathbf{Q},\,\mathbf{q}} + \tilde{G}^{(0)}_c T \chi^{(0)}_{\mathbf{Q},\,\mathbf{q}},$$

where to the operator $\tilde{G}^{(0)}_c$ there corresponds the kernel

$$\tilde{G}^{(0)}_c(\mathbf{p};\ \mathbf{k}\mid \mathbf{Q}+\mathbf{q}) = \frac{\delta^3(\mathbf{p}-\mathbf{k})}{2E_A\,[E^2(\mathbf{p},\ m_c)-(E-E_A)^2-i0]}.$$

We introduce the function $V$ by means of the relation

$$T = V + V\tilde{G}^{(0)}_c T.$$

Then the wave function $\chi_{\mathbf{Q},\mathbf{q}}$ will satisfy the equation

$$\chi_{\mathbf{Q},\,\mathbf{q}} = \chi^{(0)}_{\mathbf{Q},\,\mathbf{q}} + \tilde{G}^{(0)}_c V \chi_{\mathbf{Q},\,\mathbf{q}},$$

which, as already noted, was obtained in Ref. 12 by a different method. It is readily seen that on the energy shell

$$E(\mathbf{P},\ M_A) + E(\mathbf{p},\ m_c) = E(\mathbf{K},\ M_A) + E(\mathbf{k},\ m_c) = E,$$

where $\mathbf{K} = \mathbf{Q} + \mathbf{q} - \mathbf{k}$, $\mathbf{P} = \mathbf{Q} + \mathbf{q} - \mathbf{p}$, the function $T(\mathbf{p}; \mathbf{k}|\mathbf{Q} + \mathbf{q})$ introduced by the relation (101) is equal to the amplitude of elastic scattering of particle $c$ by the bound state of particles $a$ and $b$ expressed in the invariant normalization. For this reason, we call the function $\chi_{\mathbf{Q},\mathbf{q}}$ the wave function of elastic scattering by the bound state.

## CONCLUSIONS

In this paper, we have attempted to show how, proceeding from the most general structure of local quantum field theory, it is possible to obtain simple and perspicuous relations that can be interpreted as dynamical equations. To this end, we have proposed an original method for deriving dynamical equations of Bethe–Salpeter type and have demonstrated the universality of this method, universal in that it is suitable for deriving the dynamical equations for the wave functions of either scattering states or bound states. In addition, by means of the method which we have developed it is possible to obtain correct expressions for the amplitudes for scattering of elementary particles by composite systems and composite systems by one another, so that this method can be used not only in relativistic nuclear physics but also in the quark physics of elementary particles.

In formulating the method, we did not adopt any particular quantum field-theory model, nor did we have recourse to concepts such as an interaction Lagrangian; instead, we used only those facts that lie at the basis of the axiomatic formulations of quantum field theory. In such an approach, the kernels of the dynamical equations are determined outside the framework of perturbation theory, something that ultimately may present an attractive possibility for the construction of a dynamical formalism of the theory of strong interactions.

1) For a review of work on the Bethe–Salpeter equation, see Ref. 3, in which there is an extensive bibliography on the subject.

2) We shall not here go into the subtleties related to the difference between the time-ordered product defined in this way and the standard Dyson definition. Such a discussion can be found in Ref. 6. Our terminology is also taken from Ref. 6.

3) The expression (3) can be justified in any renormalizable perturbation theory.

4) The tilde above a field operator denotes the Hermitian conjugate for the operators of charged scalar fields and the Dirac adjoint for the operators of spinor fields. It must, of course, be borne in mind that when spinor fields are present the Green's functions and related quantities are operators in the spin space.

5) The condition (13) is automatically satisfied if the function $R^{(2)}$ has in the momentum space a zero of second order at the point $p^2 = m^2$.

6) Such an asymptotic condition was used implicitly in Ref. 12. See also Ref. 13.

7) Equations (96) and (97) are obviously a relativistic generalization of the equations of Ref. 14.

8) Equations (98) were obtained in nonrelativistic three-particle scattering theory in Ref. 15.


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