Hidden symmetries and their group structure for some two-dimensional models

R. P. Zaĭkov

Institute of Nuclear Research and Nuclear Energy, Sofia
Fiz. Elem. Chastits At. Yadra **16**, 1053–1090 (September–October 1985)

Two-dimensional classical relativistically invariant integrable (admitting infinitely many conservation laws) field-theory models are considered. It is shown that the local conserved currents for the Thirring model, the higher energy-momentum tensors for the sigma models, the sine-Gordon and Liouville systems, and some nonlocal conserved currents for the sigma models (including the supersymmetric case) have a Noether character. The corresponding symmetries are usually said to be hidden, because they are not imposed in the construction of the model. The explicit form of the hidden symmetry transformations is found for the listed models. For the sigma models, the Thirring model, and all conformally invariant two-dimensional models transformations whose generators form an infinite-dimensional closed algebra are also identified.

INTRODUCTION

There exists a large class of exactly solvable two-dimensional (with one time and one spatial coordinate) nonlinear equations. These equations are applicable to hydrodynamic processes as well as to various phenomena in plasma theory and nonlinear optics. The class includes the well-known Korteweg–de Vries equations, the nonlinear Schrödinger equation, and the sine-Gordon equation. Of particular interest are the relativistically invariant nonlinear equations that retain their exact integrability at the quantum level. For these, one can therefore construct at least two-dimensional exactly solvable quantum-field models. As a rule, there exist infinite series of conservation laws for the exactly solvable models. For some models, for example, the sigma models, there are not only local conserved quantities but also an infinite number of nonlocal conservation laws.[1,2] Then to find an exact solution for the corresponding quantum problem as well one can use either the quantum inverse scattering method or, in the case when higher conservation laws also exist at the quantum level,[3,7] use Zamolodchikov's method[8] to find the exact S matrix.

We here restrict the treatment to classical relativistic two-dimensional nonlinear exactly solvable equations that can be obtained from an invariant action and for which there exists an infinite number of conserved quantities. The first conserved quantity for these models is always obtained from Noether's theorem as a consequence of isotopic invariance of the action (for the current) or translational invariance (for the momentum). Then it is natural to assume that the remaining higher conserved quantities also have a Noether character. This was shown for the first time for nonlocal currents in the case of the nonlinear sigma models in Refs. 9–11, where the corresponding transformations were found. In Ref. 12, an infinite closed algebra of the generators of these transformations was constructed. Because symmetry with respect to such transformations was not postulated in the construction of the action, such a symmetry is usually said to be hidden.

In Ref. 13, the present author discussed the existence of hidden symmetries and found the explicit form of the transformations that generate the local conserved quantities for the Thirring model,[14,15] the sigma models,[15] and also for the sine-Gordon and Liouville systems. An analytic derivation of the commutation relations of the hidden symmetry transformations for chiral models was proposed in Refs. 16 and 17. The group structure of the transformations of the hidden symmetry that generates the additional series of nonlocal conserved currents for the chiral models found in Ref. 11 (see also Ref. 9 for the supersymmetric case) was investigated in Ref. 18.

Nonlocal conserved currents for supersymmetric sigma models were obtained in Refs. 19–22, and the corresponding hidden symmetry transformations were found in Ref. 9. The group structure of these transformations was investigated in Refs. 23 and 24.

It should be noted that the problem of finding a hidden symmetry of the considered type is very similar to the problem of linearizing the nonlinear equations considered in Refs. 25 and 26.

All our treatment will be given in Minkowski space with the metric tensor $g_{00} = -g_{11} = 1$. The transition to Euclidean space–time is made without difficulty.

1. LOCAL CONSERVED QUANTITIES

To demonstrate the existence of an infinite set of local conserved quantities, we consider first the massless Thirring model, whose Lagrangian is

$$\mathscr{L}(x) = i(\bar{\psi}\hat{\partial}\psi) - g(\bar{\psi}\gamma^{\mu}\psi)(\bar{\psi}\gamma_{\mu}\psi), \qquad (1)$$

where $\hat{\partial} = \gamma^{\mu}\partial_{\mu}$, and γ_{μ} are the Dirac matrices, for which $\{\gamma_{\mu}, \gamma_{\nu}\} = 2g_{\mu\nu}$. For convenience, we use here and in what follows the γ-matrix representation

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_1 = C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma_5 = \gamma_1\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (2)$$

where C is the matrix of charge conjugation. It is well known that the Lagrangian (1) is invariant with respect to ordinary and γ_5-global gauge transformations. This can be readily verified by going over to the light-cone variables

 0090-4759/85/050468-17$04.40

$$x_\pm = \frac{1}{2}(x_0 \mp x_1), \tag{3}$$

in which (1) takes the form

$$\mathcal{L}(x) = \frac{i}{2}(\psi_1^* \overleftrightarrow{\partial}_- \psi_1) + \frac{i}{2}(\psi_2^* \overleftrightarrow{\partial}_+ \psi_2) + g(\psi_1^*\psi_1)(\psi_2^*\psi_2). \tag{1a}$$

Here ψ_1, ψ_2 are the components of the two-component Dirac spinor $\psi = \binom{\psi_1}{\psi_2}$, and the asterisk denotes complex conjugation. It is then obvious that (1a) is invariant with respect to such global gauge transformations, which transform each of the components of ψ independently. The same applies when ψ transforms in accordance with a non-Abelian gauge group. Then by Noether's theorem we have one conserved vector current and one conserved axial-vector current:

$$j_\mu(x) = \bar{\psi}\gamma_\mu\psi(x), \quad j_\mu^5(x) = \bar{\psi}\gamma_5\gamma_\mu\psi = \varepsilon_{\mu\nu}j^\nu(x), \tag{4}$$

$$\partial^\mu j_\mu(x) = 0, \quad \partial^\mu j_\mu^5(x) = \varepsilon_{\mu\nu}\partial^\mu j^\nu(x) = 0. \tag{5}$$

In the light-cone variables (3),

$$j_+(x) = \psi_1^*\psi_1(x), \quad j_-(x) = \psi_2^*\psi_2(x), \tag{4a}$$

$$\partial_- j_+ x = 0, \quad \partial_+ j_-(x) = 0. \tag{5a}$$

Note that we do not discuss here the question of the existence of the corresponding conserved charges.

Then it follows from (5a) that there exists an infinite number of local conserved currents[3]:

$$\mathcal{Y}_+^{(k,\,m)}(x) = (\partial_+)^k (j_+)^m, \quad \mathcal{Y}_-^{(k,\,m)}(x) = (\partial_-)^k (j_-)^m, \tag{6}$$

$$\partial_+ \mathcal{Y}_-^{(k,\,m)} = 0, \quad \partial_- \mathcal{Y}_+^{(k,\,m)} = 0. \tag{7}$$

A second class of models that admit an infinite number of local conserved quantities is provided by the nonlinear sigma models. The action for these models has the form

$$S = \frac{1}{2}\int d^2x \,\mathrm{tr}\,\{\partial^\mu g^{-1}(x)\,\partial_\mu g(x)\}$$

$$= -\frac{1}{2}\int d^2x \,\mathrm{tr}\{A^\mu(x)\,A_\mu(x)\}, \tag{8}$$

where the field $g(x)$ takes its values in some compact group G or on the symmetric space $M = G/H$, where H is a subgroup of G.[27–29] In the first case, we are dealing with a principal chiral field, while the second case includes the $O(N)$, CP^{N-1}, and other chiral models, for which

$$g^2(x) = \mathrm{I}$$

and, therefore, $g^{-1}(x) = g(x)$, i.e., there exists a representation

$$g(x) = \mathrm{I} - 2\mathcal{P}(x),$$

where $\mathcal{P}^2(x) = \mathcal{P}(x)$ is a field with projective properties. In (8), we have used the notation

$$A_\mu(x) = g^{-1}(x)\,\partial_\mu g(x), \tag{9}$$

and this represents some conserved Noether current. Then the equations of motion can be written as the condition for conservation of the current (9):

$$\partial^\mu A_\mu(x) = 0. \tag{10}$$

The fact that (10) is indeed equivalent to the equation of motion is verified by substituting (9) in (10) with subsequent multiplication of both sides from the left by $g(x)$:

$$\Box g(x) + g\partial^\mu g^{-1}\partial_\mu g(x) = \Box g(x) - \partial^\mu g g^{-1}\partial_\mu g = 0. \tag{10a}$$

Thus, we have indeed obtained the equation of motion for the generalized sigma models.

We note that the action (8) is conformally invariant, and therefore the conserved energy–momentum tensor

$$T_{\mu\nu} = \mathrm{tr}\,\{\partial_\mu g^{-1}\partial_\nu g + \partial_\nu g^{-1}\partial_\mu g - g_{\mu\nu}\partial g^{-}\partial_\lambda g\}$$

is a symmetric traceless tensor, i.e.,

$$T_{\mu\nu} = T_{\nu\mu}, \quad T_\mu^\mu = 0, \tag{11}$$

$$\partial^\mu T_{\mu\nu} = 0. \tag{12}$$

In the light-cone variables, these conditions take the very simple form

$$T_{+-} = T_{-+} = 0, \tag{11a}$$

$$\partial_- T_{++} = 0, \quad \partial_+ T_{--} = 0. \tag{12a}$$

It follows from (12a) that[4]

$$\mathcal{T}_+^{(k,\,m)} = (\partial_+)^k (T_{++})^m, \quad \mathcal{T}_-^{(k,\,m)} = (\partial_-)^k (T_{--})^m \tag{13}$$

are also conserved, i.e.,

$$\partial_- \mathcal{T}_+^{(k,\,m)} = 0, \quad \partial_+ \mathcal{T}_-^{(k,\,m)} = 0.$$

Therefore, for all conformally invariant models in which the energy–momentum tensor is symmetric and has vanishing trace there exists an infinite number of local conserved quantities.

2. NONLOCAL CONSERVED QUANTITIES

It was shown in Ref. 1 that in the case of the $O(N)$ nonlinear sigma model an infinite number of conserved nonlocal currents exist. A simple constructive proof of the existence of an infinite number of nonlocal conserved currents for the classical nonlinear sigma models was given in Ref. 2. We give here this proof. We define the matrix covariant derivative

$$D_\mu = \partial_\mu + A_\mu(x), \tag{14}$$

where $A_\mu(x)$ is given by Eq. (9). Then it follows from (9) and (14) that

$$F_{\mu\nu}(x) = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0. \tag{15}$$

In addition, when the equations of motion are satisfied, the following operator identity holds:

$$\partial^\mu D_\mu X(x) = D^\mu \partial_\mu X(x), \tag{16}$$

where $X(x)$ is an arbitrary smooth matrix function.

Suppose that we are given the kth conserved current

$\mathscr{J}^k_\mu(x), \partial\mu\mathscr{J}^{(k)}_\mu = 0$. Then $\mathscr{J}^{(k)}_\mu$ can always be represented in the form

$$\mathscr{Y}^{(k)}_\mu(x) = \varepsilon_{\mu\nu}\partial^\nu\chi^{(k)}(x), \tag{17}$$

where $\chi^{(k)}(x)$ is a smooth matrix function; $\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}$ and $\varepsilon_{01} = 1$. We make the ansatz

$$\mathscr{Y}^{(k+1)}_\mu(x) = D_\mu\chi^{(k)}(x) = (\partial_\mu + A_\mu)\chi^{(k)}(x) \quad (k = 0, 1 \ldots). \tag{18}$$

We shall show that this current is also conserved if the equations of motion (10) are satisfied, i.e.,

$$\partial^\mu\mathscr{Y}^{(k+1)}_\mu(x) = \partial^\mu(D_\mu\chi^{(k)}) = D^\mu\partial_\mu\chi^{(k)}(x)$$
$$= -D_\mu\varepsilon^{\mu\nu}\mathscr{Y}^{(k)}_\nu(x) = \varepsilon^{\mu\nu}D_\mu D_\nu\chi^{(k-1)}(x) = 0.$$

Here, we have used the vanishing of the curvature (15), and also the identity (16), which is satisfied only when the equation of motion is. Therefore, beginning with $\chi^{(0)} = \mathbf{I}$, for which by virtue of (18)

$$\mathscr{Y}^{(1)}_\mu(x) = A_\mu(x),$$

we obtain an infinite series of conserved currents. For example, the k th nonlocal current has the form

$$\mathscr{Y}^{(k)}_\mu(x) = \varepsilon_{\mu\nu}A^\nu\chi^{(k-2)} + A_\mu\chi^{(k-1)}, \tag{19}$$

where

$$\chi^{(k)}(x) = \int_{-\infty}^{x_1} dy_1\mathscr{Y}^{(k)}_0(x_0, y_1). \tag{20}$$

It follows from (20) that the k th conserved charge is the value of $\chi^{(k)}(x)$ at the point $x_1 = \infty$, i.e., $Q^{(k)} = \chi^{(k)}(x_0, \infty)$. Finally, from (17) and (18) we obtain the following recursive system of differential equations:

$$\partial_\mu\chi^{(k+1)}(x) = \varepsilon_{\mu\nu}(\partial^\nu + A^\nu(x))\chi^{(k)}(x). \tag{18a}$$

The condition of integrability of this system is

$$\partial^\mu(\partial_\mu + A_\mu(x))\chi^{(k)}(x) = 0, \tag{20a}$$

where we have taken into account the vanishing curvature (15) and the identity (16).

3. INFINITE NUMBER OF CONSERVED CURRENTS FOR TWO-DIMENSIONAL SUPERSYMMETRIC SIGMA MODELS

The constructive proof of the existence of nonlocal conserved currents given in Sec. 2 of Ref. 2 was generalized for supersymmetric nonlinear sigma models in Refs. 19–22. We give here the proof of Ref. 20. For this, we consider the action of the two-dimensional generalized supersymmetric sigma model, which has the form[31,32]

$$S = \frac{1}{2}\int d^2x\, d^2\theta\, \mathrm{tr}\,\{\mathscr{D}^2\mathscr{G}^{-1}(x;\theta)\,\mathscr{D}_\alpha\mathscr{G}(x;\theta)$$
$$= -\frac{1}{2}\int d^2x\, d^2\theta\, \mathrm{tr}\,\{(\mathscr{G}^{-1}\mathscr{D}^2\mathscr{G})(\mathscr{G}^{-1}\mathscr{D}_\alpha\mathscr{G})\}, \tag{21}$$

where

$$\mathscr{D}_\alpha = \mathrm{i}\frac{\partial}{\partial\theta^\alpha} + \bar{\theta}\hat{\partial}j_\alpha, \quad \hat{\partial} = \gamma^\mu\partial_\mu \quad (\alpha = 1, 2)$$

is the supercovariant derivative; for the "Dirac" matrices, we have used the representation (2). The superfield $\mathscr{G}(x;\theta)$ takes its values in a certain compact group G or symmetric space $M = G/H$, where H is a subgroup of G.[28–30] In the first case, we are dealing with a principal chiral superfield, while the second case includes the $O(N)$, CP^{N-1}, and other supersymmetric generalizations of the chiral models, for which

$$\mathscr{G}^2(x;\theta) = \mathbf{I}, \text{ i.e., } \mathscr{G}^{-1}(x;\theta) = \mathscr{G}(x;\theta).$$

From the action (21), we obtain the equation of motion

$$\mathscr{D}^\alpha\mathscr{A}_\alpha(x;\theta) = 0, \tag{22}$$

where

$$\mathscr{A}_\alpha(x;\theta) = \mathscr{G}^{-1}(x;\theta)\,\mathscr{D}_\alpha\mathscr{G}(x;\theta). \tag{23}$$

As in the ordinary case, it is readily verified that Eq. (22) is equivalent to the equation of motion for the supersymmetric chiral models:

$$\mathscr{D}^\alpha\mathscr{D}_\alpha\mathscr{G}(x;\theta) + \mathscr{G}(x;\theta)\,\mathscr{D}^\alpha\mathscr{G}^{-1}(x;\theta)\,\mathscr{D}_\alpha\mathscr{G}(x;\theta) = 0.$$

In terms of its components, the field $\mathscr{G}(x;\theta)$ can be expressed as

$$\mathscr{G}(x;\theta) = g(x) + \bar{\theta}\varphi(x) + \frac{1}{2}\bar{\theta}\theta\varkappa(x), \tag{24}$$

where $g(x)$ and $\varkappa(x)$ are scalar fields, and $\varphi(x)$ is a spinor field. Substituting (24) in (23), we express the components of the spinor supercurrent in terms of the field components,

$$\left.\begin{aligned}
&a_\alpha(x) = \mathrm{i}g^{-1}(x)\,\varphi_\alpha(x);\\
&r(x) = \frac{\mathrm{i}}{2}\bar{\varphi}^{-1}(x)\,\gamma_5\varphi(x);\\
&v_\mu(x) = g^{-1}(x)\,\partial_\mu g(x) - \frac{\mathrm{i}}{2}\varphi^{-1}\gamma_\mu\varphi(x);\\
&b_\alpha(x) = -g^{-1}(x)\,(\hat{\partial}\varphi(x))_\alpha - (\gamma^\mu\varphi^{-1})_\alpha\,\partial_\mu g(x)\\
&+\frac{\mathrm{i}}{2}[(\bar{\varphi}\varphi^{-1})\,\varphi_\alpha + \varphi_\alpha(\varphi^{-1}\varphi)],
\end{aligned}\right\} \tag{25}$$

where $a(x)$ and $b(x)$ are the spinor, $r(x)$ the pseudoscalar, and $v_\mu(x)$ the vector components of $\mathscr{A}_\alpha(x;\theta)$. By virtue of Eq. (22), the scalar component of $\mathscr{A}_\alpha(x;\theta)$ is annihilated, the pseudoscalar component remains completely arbitrary, and the remaining components satisfy the conditions

$$\mathrm{i}\hat{\partial}a(x) = b(x), \quad \partial^\mu v_\mu(x) = 0. \tag{26}$$

Equations (26) in the case when a, b, v_μ, and r are determined by the expression (25) are equivalent to the equations of motion (22). However, they are also satisfied for any conserved supercurrent.

As we saw in Sec. 2, a necessary condition for the existence of infinitely many conserved nonlocal currents was a vanishing curvature. In the supersymmetric case, the corresponding curvature tensor

$$\mathscr{F}_{\alpha\beta} = \{\nabla_\alpha, \nabla_\beta\} = \mathscr{D}_\alpha\mathscr{A}_\beta + \mathscr{D}_\beta\mathscr{A}_\alpha + \mathscr{A}_\alpha\mathscr{A}_\beta + \mathscr{A}_\beta\mathscr{A}_\alpha$$
$$= 2\mathrm{i}\,(C\gamma^\mu)_{\alpha\beta}\,\mathscr{G}^{-1}(x;\theta)\,\partial_\mu\mathscr{G}(x;\theta) \quad (\alpha, \beta = 1, 2) \tag{27}$$

does not vanish at all for any α and β. The nonvanishing of some of the components of the curvature tensor $\mathscr{F}_{\alpha\beta}$ is a manifestation of the torsion in superspace:

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = 2i\,(C\hat{\partial})_{\alpha\beta}.$$

In (27),

$$\nabla_\alpha = \mathcal{D}_\alpha + \mathcal{A}_\alpha\,(x;\,\theta)$$

denotes the matrix covariant derivative. It follows from (22) and (27) that

$$\mathcal{F}_{12} = \mathcal{F}_{21} = \mathcal{D}_1\mathcal{A}_2 + \mathcal{D}_2\mathcal{A}_1 + \mathcal{A}_1\mathcal{A}_2 + \mathcal{A}_2\mathcal{A}_1 = 0. \qquad (28)$$

It can be readily shown that (28) is the sufficient condition for the existence of conserved spinor supercurrents:

$$J^{(k)}_\alpha\,(x;\,\theta) = (\gamma_5\mathcal{D})_\alpha\,X^{(k)}\,(x;\,\theta) \quad (k=1,\,2,\,\ldots), \qquad (29)$$

where $X^{(k)}\,(x, \theta)$ can be obtained from the recursion relation

$$\mathcal{D}_\alpha X^{(k)}\,(x;\,\theta) = (\gamma_5\nabla)_\alpha\,X^{(k-1)}\,(x;\,\theta), \quad X^{(0)} = \mathrm{I} \quad (k=1,\,2\,\ldots). \qquad (30)$$

We begin here with $X^{(0)} = \mathrm{I}$ and then from (29) we also have

$$J^{(1)}_\alpha\,(x;\,\theta) = A_\alpha\,(x;\,\theta).$$

From (27), (28), and (30) we find that

$$\mathcal{D}^\alpha J^{(k)}_\alpha\,(x;\,\theta) = 0 \qquad (k=1,\,2\,\ldots),$$

if the equations of motion (29) are satisfied, i.e., $\mathcal{D}^\alpha\mathcal{A}_\alpha = 0$.

It should be noted that for smooth functions $X^{(k)}$ the following identity holds by virtue of the equation of motion (22) and the condition (28):

$$\nabla^\alpha\mathcal{D}_\alpha X^{(k)}\,(x;\,\theta) = \mathcal{D}^\alpha\nabla_\alpha X^{(k)}\,(x;\,\theta) = 0, \qquad (31)$$

which is the integrability condition of the system (30). If $X^{(k)}\,(x;\,\theta)$ is written in the superfield form

$$X^{(k)}\,(x;\,\theta) = \chi\,(x) + \theta^\alpha\varkappa_\alpha\,(x) + \theta^1\theta^2\xi\,(x),$$

the recursion relation (30) has in components the form

$$\left.\begin{aligned}
&\partial_\mu\chi^{(k+1)} = \varepsilon_{\mu\nu}\left\{(\partial^\nu + v^\nu(x))\,\chi^{(k)}\,(x) + \tfrac{1}{2}\overline{a}\,(x)\,\gamma^\nu\varkappa^{(k)}(x)\right\};\\
&\varkappa^{(k+1)}\,(x) = \gamma_5\varkappa^{(k)}\,(x) - i\,(\gamma_5 a)\,\chi^{(k)}\,(x);\\
&\xi^{(k+1)}\,(x) = ir\,(x)\,\chi^{(k)}\,(x) -\\
&-\tfrac{i}{2}\,\overline{a}\,(x)\,\gamma_5\varkappa^{(k)}\,(x) + \xi^{(k)}\,(x) \quad (k=0,\,1\,\ldots).
\end{aligned}\right\} \qquad (32)$$

We have here used the integrability condition (31), and $a(x)$, $v_\mu\,(x)$, and $r(x)$ are the components of $\mathcal{A}_\alpha\,(x;\theta)$ given by (25).

The solutions of Eq. (32) for $k=0$, $\chi^{(0)} = \mathrm{I}$, $\varkappa^{(0)} = \xi^{(0)} = 0$ have the form

$$\chi^{(1)}\,(x) = -\int\limits_{-\infty}^{x_1} dy_1\left(g^{-1}\partial_0 g - \frac{i}{2}\,\overline{\varphi}^{-1}\gamma_0\varphi\right)(x_0,\,y_1);$$

$$\varkappa^{(1)}\,(x) = g^{-1}\,(\gamma_5\varphi);\; \xi^{(1)}\,(x) = -\frac{1}{2}\,\varphi^{-1}\gamma_5\varphi\,(x).$$

Therefore, for the general term of the sequence we find

$$\left.\begin{aligned}
&\chi^{(k+1)}\,(x) = -\int\limits_{-\infty}^{x_1} dy_1\left\{\partial_0\chi^{(k)} + \left(g^{-1}\partial_0 g - \frac{i}{2}\,\overline{\varphi}^{-1}\gamma_0\varphi\right)\chi^{(k)}\right.\\
&\left.+\frac{i}{2}\,g^{-1}\overline{\varphi}\gamma_5\gamma_0\varkappa^{(k)}\right\}(x_0,\,y_1);\\
&\varkappa^{(k+1)}\,(x) = \gamma_5\varkappa^{(k)}\,(x) + \gamma_5\varphi^{-1}g\chi^{(k)}\,(x);\\
&\xi^{(k+1)}\,(x) = \xi^{(k)}\,(x) - \frac{1}{2}\,\overline{\varphi}^{-1}\gamma_5\varphi\chi^{(k)}\,(x) + \frac{1}{2}\,g^{-1}\overline{\varphi}\gamma_5\varkappa^{(k)}.
\end{aligned}\right\} \qquad (33)$$

Substituting (33) in (29), we obtain the corresponding conserved currents.

4. HIDDEN SYMMETRIES IN SOME TWO-DIMENSIONAL MODELS

As was shown in Secs. 1 and 3, the first conserved quantity in the obtained infinite series has a Noether character. It is a consequence of the translational or gauge invariance of the action that we postulated in constructing the model. A natural question then arises: Does there exist a symmetry of the action which generates the higher conserved quantities too? This symmetry is not postulated in constructing the model, and we shall therefore call it a hidden symmetry. We first consider the general conditions for the existence of such a symmetry.

Condition for the existence of hidden symmetries

Suppose that we are given a set of fields $\psi_\varkappa\,(x)$ $(\varkappa = 1,2,\ldots,M)$ that each transforms under space–time transformations

$$x'_\mu = x_\mu + \delta x_\mu \;(\mu = 0,\,1,\,\ldots,\,D-1), \quad k\in K \qquad (34)$$

in D-dimensional space–time and global gauge transformations $G\;[G = U(N),\; N = 1,2,\ldots]$ in accordance with the laws

$$\psi'_\varkappa\,(x) = \psi_\varkappa\,(x) + \delta_k\psi_\varkappa\,(x), \quad k\in K$$

and

$$\psi'_\varkappa\,(x) = U\,(g)\,\psi_\varkappa\,(x), \quad g\in G \qquad (35)$$

respectively. Here, the variations $\delta_k x_\mu$ and $\delta_k\psi_\varkappa$ are expressed in terms of the infinitesimal transformation parameters by

$$\delta_k x_\mu = X^{(l)}_\mu\delta\omega_l, \quad \delta_k\psi_\varkappa = \Omega^{(l)}_\varkappa\psi_\varkappa\delta\omega_l. \qquad (36)$$

We also assume that for these fields there exists an action

$$S = \int d^Dx L\,(\psi_\varkappa,\;\partial_\mu\psi_\varkappa) \qquad (37)$$

invariant with respect to transformations of the group $G\otimes K$. Here, L is an invariant Lagrange function which depends only on the fields $\psi_\varkappa$ and their first derivatives $\partial_\mu\psi_\varkappa$.

Then by the first part of Noether's theorem[33]

$$\Theta^{(l)}_\mu = -\frac{\partial L}{\partial\partial^\mu\psi}\,(\Omega^{(l)}\psi - \partial^\nu\psi X^{(l)}_\nu) - X^{(l)}_\mu L\,(x) \qquad (38)$$

is conserved if the equations of motion hold:

$$\frac{\partial L}{\partial \psi} - \partial^\mu \frac{\partial L}{\partial \partial^\mu \psi} = 0. \quad (39)$$

The number of these quantities is equal to the number of independent parameters $\delta\omega_1$.

With regard to the hidden symmetry transformations that extend the group $G \otimes K$, we make the following assumption.

Assumption. The generators of the extended group $G \otimes K$ are functions of the coordinates x determined by the condition of invariance

$$\delta S = 0 \quad (40)$$

of the action without the introduction of new fields.

Here, as usual, the condition (40) is ensured by the invariance of the Lagrange function with respect to the considered transformations. Thus, proceeding from the given finite-parameter transformation group, we arrive in general at certain infinite-parameter transformations. The explicit form of these transformations depends on the model and on the method of extending the original group. We here consider separately the extensions of the internal and space–time transformations.

Generalized gauge transformations. We extend the global gauge transformations (35) as follows:

$$\psi'_\varkappa(x) = \exp\{i\eta_a^{\varkappa, k}(x)\,\omega_a^{\varkappa, k}\}\,\psi_\varkappa(x), \quad (41)$$

where $\eta_a^{\varkappa,k}(x)$ are $N \times N$ matrix-valued functions ($k = 0,1,...;\ \varkappa = 1,...,M$). For simplicity, we consider the case of one (complex) field, i.e., $M = 2$ (here, the field and its complex conjugate are identified by different values of the index $\varkappa$). Then the variations of the Lagrangian under infinitesimal transformations (41) have the form

$$\delta L = \sum_{j=1}^{N} \sum_{\varkappa=1}^{2} \left(\frac{\partial L}{\partial \psi_{\varkappa, j}} \delta\psi_{\varkappa, j} + \frac{\partial L}{\partial \partial_\mu \psi_{\varkappa, j}} \partial_\mu (\delta\psi_{\varkappa, j}) \right)$$

$$= i \sum_{j=1}^{N} \sum_{\varkappa=1}^{2} \left\{ \left[\frac{\partial L}{\partial \psi_{\varkappa, j}} \psi_{\varkappa, l} + \frac{\partial L}{\partial \partial_\mu \psi_{\varkappa, j}} \partial_\mu \psi_{\varkappa, l} \right] (\eta_a^{\varkappa, k}(x))_{jl} \right.$$

$$\left. + \frac{\partial L}{\partial \partial_\mu \psi_{\varkappa, j}} \psi_{\varkappa, l} \partial_\mu (\eta_a^{\varkappa, k}(x))_{jl} \right\} \delta\omega_a^{k\varkappa} = \mathrm{tr}\,(j^\mu \partial_\mu \eta_a^k)\,\delta\omega_a^k. \quad (42)$$

Here $\delta\omega_a^{k,1} = -\delta\omega_a^{k,2} = \delta\omega_a^k$ and

$$j_\mu^{jl}(x) = i\left(\frac{\partial L}{\partial \partial^\mu \psi_j} \psi_l - \bar{\psi}_j \frac{\partial L}{\partial \bar{\psi}_l} \right) \quad (j,\ l = 1, \ldots, N)$$

is the Noether conserved current generated by the invariance of the Lagrangian with respect to the global gauge transformations of G. This amounts to fulfillment of the equation

$$\sum_{j,\ l=1}^{N} \sum_{\varkappa=1}^{2} \left(\frac{\partial L_j}{\partial \psi_{j,\varkappa}} \psi_{l,\varkappa} + \frac{\partial L}{\partial \partial_\mu \psi_{j,\varkappa}} \partial_\mu \psi_{l,\varkappa} \right) (\eta_a^{\varkappa, k}(x))_{jl}\,\delta\omega_a^{\varkappa, k} = 0,$$

which was taken into account in (42). Therefore, the condition of invariance of the Lagrangian with respect to the transformations (41) without recourse to compensating fields has the form

$$\mathrm{tr}\,\{j^\mu(x)\,\partial_\mu \eta_a^{(h)}(x)\} = 0. \quad (43)$$

We emphasize that the derivation of (43) did not use the equations of motion, i.e., (43) ensures invariance of the Lagrangian for all fields.

It is obvious that Eq. (43) always has the trivial solution $\eta_a^0 = T_a$ which leads to the original group G. The possible existence of nontrivial solutions is discussed in Sec. 2.

In this case, the following theorem holds.

Noether's theorem: For any generalized one-parameter transformation of the gauge type (41) for which the generator function satisfies (42) there exists the quantity

$$\mathscr{Y}_{\mu, a}^{(h)}(x) = \mathrm{tr}\,(j_\mu(x)\,\eta_a^{(h)}(x)), \quad (44)$$

which is conserved,

$$\partial^\mu \mathscr{Y}_{\mu, a}^{(h)}(x) = 0,$$

if the equations of motion (39) are satisfied.

Generalized Translations. We recall that the original Lagrange function $L(\psi, \partial_\mu\psi)$ does not depend explicitly on x. Then the condition of invariance of the action (34) with respect to generalized translations, for which δx_μ depends on x, and the total variation of the field (33) is zero, as for ordinary translations ($\Omega^{(k)} = 0$), has the form

$$\begin{aligned}
\delta S &= \int d^D x\,\{\partial^\mu(\delta x_\mu)\,L(x) + \partial^\mu L(x)\,\delta x_\mu + \bar{\delta}\,L(x)\} \\
&= \int d^D x \left\{ \partial^\mu(L\delta x_\mu) + \frac{\partial L}{\partial \partial_\mu \varphi} \partial_\mu \overline{\delta\varphi} + \frac{\partial L}{\partial \varphi} \bar{\delta}\varphi \right\} \\
&= \int d^D x \left\{ \partial^\mu(L\delta x_\mu) - \frac{\partial L}{\partial \partial_\mu \varphi} \partial^\mu \partial^\nu \varphi\,\delta x_\nu \right. \\
&\quad \left. - \frac{\partial L}{\partial \varphi} \partial^\nu \varphi\,\delta x_\nu - \frac{\partial L}{\partial \partial_\mu \varphi} \partial^\mu \varphi\,\partial_\mu \delta x_\nu \right\} \\
&= \int d^D x \left\{ L g^{\mu\nu} - \frac{\partial L}{\partial_\mu \varphi} \partial^\nu \varphi \right\} \partial_\mu \delta x_\nu = -\int d^D x\, T^{\mu\nu} \partial_\mu \delta x_\nu = 0,
\end{aligned} \quad (45)$$

where we consider generally a D-dimensional space–time and the translational invariance of the Lagrangian is taken into account, i.e.,

$$\partial_\mu L(x) - \frac{\partial L}{\partial \partial^\nu \varphi} \partial_\mu \partial^\nu \varphi - \frac{\partial L}{\partial \varphi} \partial_\mu \varphi = 0$$

and

$$\delta\varphi = \varphi'(x') - \varphi(x') = -\partial^\mu \varphi\,\delta x_\mu = -X_\mu^{(K)} j^\mu \varphi\,\delta\omega_k \quad (46)$$

is the form variation of the field. From (44) we obtain one sufficient condition on $X_\mu^{(\varkappa)}$, i.e.,

$$T^{\mu\nu}\,\partial_\mu X_\nu^{(\varkappa)}(x) = 0. \quad (45a)$$

Therefore, the following theorem holds.

Noether's theorem: For every one-parameter transformation (36) whose generators satisfy the condition (45), i.e., which conserves the action of the model, there exists a quantity

$$T^{\mu(\varkappa)}(x) = T^{\mu\nu} X_\nu^{(\varkappa)}(x), \quad (47)$$

that is conserved, i.e.,

$$\partial_\mu T^{\mu(\varkappa)}(x) = 0, \quad (48)$$

when the equations of motion are satisfied.

Equation (46) always has the trivial solution $X=\text{const}$, which gives the ordinary translations. As we shall see below, in some cases this equation also has nontrivial solutions.

Generalized Lorentz transformations. In this case, we make the additional assumption that

$$\left.\begin{aligned} X_\mu^{\nu,\lambda}(x) &= R^{(k)}(x)\,[x^\nu\delta_\mu^\lambda - x^\lambda\delta_\mu^\nu],\\ \Omega_{\mu\nu}^{(k)}(x) &= R^{(k)}(x)\,[x_\mu\partial_\mu - x_\nu\partial_\mu], \end{aligned}\right\} \tag{49}$$

where $R(x)$ is a function determined from the requirement of invariance of the action, which reduces in this case to

$$M_{\mu\nu\lambda}\partial^\mu R^{(k)}(x)=0 \quad (k=0,\ 1\ \ldots). \tag{50}$$

Here, $M_{\mu\nu\lambda}$ is the conserved tensor of the relativistic angular momentum and spin. Note that the derivation of the invariance condition (50) uses only the invariance of the action with respect to the ordinary Lorentz transformations. It is clear that $R^{(0)}=\text{const}$ is a trivial solution of (50). One can similarly extend the scale and special conformal transformations for conformally invariant models.

Therefore, here too we have the following theorem.

Noether's theorem: For any one-parameter generalized Lorentz transformation (36) whose generators (49) satisfy the action invariance conditions (50) there exists one quantity,

$$M_{\mu\nu\lambda}^{(k)} = M_{\mu\nu\lambda}R^{(k)}(x) \quad (k=0,\ 1\ \ldots), \tag{51}$$

which is conserved, i.e.,

$$\partial^\mu M_{\mu\nu\lambda}^{(k)}(x)=0,$$

when the corresponding equations of motion are satisfied.

On the generator functions $\eta_a^{(k)}(x)$, $X_\mu^{(k)}(x)$, and $R^{(k)}(x)$ we impose boundary conditions that ensure the existence of the corresponding integrals of the motion:

$$\begin{aligned} Q^{(k)} &= \int d^{D-1}x\mathcal{J}_0^{(k)}(x),\quad P_\lambda = \int d^{D-1}T_{0\lambda}^{(k)}(x),\\ M_{\mu\nu}^{(k)} &= \int d^{D-1}xM_{0\mu\nu}(x) \quad (k=0,\ 1\ \ldots), \end{aligned}$$

where $\mathcal{J}_\mu^{(k)}$, $T_{\mu\lambda}^{(k)}$, and $M_{\mu\nu\lambda}^{(k)}$ are given by Eqs. (44), (47), and (51), respectively. For this, it is sufficient to require $\eta^{(k)}(x)$, $X_\mu^{(k)}(x)$, and $R^{(k)}(x)$ to be bounded at spatial infinity, i.e.,

$$\begin{aligned} &\lim_{x_1\to\pm\infty}|\eta^{(k)}|\leqslant M_1<\infty,\quad \lim_{x_1\to\pm\infty}|X_\mu^{(k)}|\leqslant M_2<\infty,\\ &\lim_{x_1\to\pm\infty}|R^{(k)}|\leqslant M_3<\infty. \end{aligned} \tag{52}$$

We note finally that we have here required invariance of the action only in the strong sense, i.e., $\delta S=0$. However, as is well known, this restriction can be weakened by equating δS to the integral of a total divergence of some vector $W_\mu^{(k)}$, as, for example, in the nonlinear sigma model:

$$\delta S = \int d^2x\varepsilon_{\mu\nu}\,\mathrm{tr}\,\{\partial^\mu(A_\mu\chi_a^{(k)})\}\,\delta\omega_a^{(k)}$$

(see Sec. 5). The form of such a vector must be chosen in accordance with the particular model. In this case, the corresponding conserved quantities have the form

$$\tilde{\Theta}_\mu^{(k)} = \Theta_\mu^{(k)} - W_\mu^{(k)}, \tag{53}$$

where $\theta_\mu^{(k)}$ is given by the expression (38). In addition, we note that in some cases solutions of Eqs. (43), (45), and (50) can be found only on the subspace of the solutions of the equation of motion, i.e., on the extremals. As was shown in Ref. 33, such a restricted symmetry also ensures the existence of the conserved quantities (44), (47), and (51).

5. NOETHER CHARACTER OF SOME LOCAL CONSERVED QUANTITIES

Generalized "local" gauge transformations

We consider a two-dimensional model invariant with respect to global gauge and γ_5-gauge transformations, for which, therefore, the current j_μ and the axial current j_μ^5 are conserved, i.e., Eqs. (5) are satisfied. As examples here we can take the massless Thirring model, for which j_μ and j_μ^5 are given by the expressions (4), and also the free spinor field, the Schwinger model, etc. We note that for all models satisfying the continuity equations (5) there are infinitely many conservation laws, i.e., the quantities (6) also satisfy the continuity equations (7). To obtain the higher conserved currents (6) by Noether's theorem, we consider the gauge transformations

$$\psi_\alpha'(x) = U_\alpha(x)\,\psi_\alpha(x) = \exp\{i\zeta_\alpha(x)\}\,\psi_\alpha(x) \quad (\alpha=1,\ 2), \tag{54}$$

i.e., the spinor $\psi(x)$ transforms component by component. Note that the transformations (54) do not violate relativistic invariance, ψ transforming under Lorentz transformations in accordance with

$$\psi_1\to e^{\Lambda/2}\psi_1,\quad \psi_2\to e^{-\Lambda/2}\psi_2,$$

i.e., each component of the spinor ψ transforms separately. The condition (43) of invariance of the action with respect to the transformations (54) takes the form

$$\mathrm{tr}\,\{j_+(x)\,\partial_-\zeta_1(x)\} = 0,\quad \mathrm{tr}\,\{j_-(x)\,\partial_+\zeta_2(x)\} = 0. \tag{55}$$

The general solution of the condition (55) is given by

$$\zeta_1(x) = F_1(x_+,\ \omega_-^{(k)}),\quad \zeta_2(x) = F_2(x_-,\ \omega_+^{(k)}), \tag{56}$$

where $F_{1,2}$ are arbitrary scalar functions satisfying the boundary condition (52), and $\omega^{(k)}$ are infinitesimal tensor parameters of rank k (Lorentz dimension k). In particular, we choose $F_{1,2}$ in the form

$$\left.\begin{aligned} F_1 &= \sum_w h_1^{(-m)}(j_+,\ \ldots)\,\omega_+^{1(m)} + \sum_w h_1^{(m)}(j_+,\ \ldots)\,\omega_-^{1(m)},\\ F_2 &= \sum_w h_2^{(-m)}(j_-,\ \ldots)\,\omega_-^{2(m)} + \sum_w h_2^{(m)}(j_-,\ \ldots)\,\omega_-^{2(m)}, \end{aligned}\right\} \tag{57}$$

where, the scalar functions $F_{1,2}$ having Lorentz dimension zero, $h_1^{(-m)}$ and $h_2^{(m)}$ have Lorentz dimension $-m$, $h_2^{(-m)}$ and $h_1^{(m)}$ have dimension m, and $\omega_\pm^{(m)}$ are parameters with Lorentz dimension $\pm m$, respectively.

It follows from (57) that local polynomial conserved quantities are generated by the generators

$$h_1^{(m,n)}(x_+) = (\partial_+)^m (j_+)^n;\ h_2^{(m,n)}(x_-) = (\partial_-)^m (j_-)^n. \quad (58)$$

In the non-Abelian case, the currents j_μ are matrices. Then the generators of the "local" gauge transformations (54) have the form

$$\left.\begin{aligned} \zeta_1^{(m,n)}(x) &= h_{a,1}^{(m,n)}(x_+)\,\omega_-^{a(m+n)};\\ \zeta_2^{(m,n)}(x) &= h_{a,2}^{(m,n)}(x_-)\,\omega_+^{a(m+n)}, \end{aligned}\right\} \quad (59)$$

where

$$\left.\begin{aligned} h_{a,1}^{(m,n)} &= (\partial_+)^{k_1} j_+^{l_1} \ldots (\partial_+)^{k_q} j_+^{l_q} T_a (\partial_+)^{k_{q+1}} j_+^{l_{q+1}} \ldots (\partial_+)^{k_n} j_+^{l_n},\\ h_{a,2}^{(m,w)} &= (\partial_-)^{k_1} j_-^{l_1} \ldots (\partial_-)^{k_q} j_-^{l_q} T_a (\partial_-)^{k_{q+1}} j_-^{l_{q+1}} \ldots (\partial_-)^{k_n} j_-^{l_n}. \end{aligned}\right\} \quad (60)$$

Here, k_j and l_j are non-negative integers satisfying the equations

$$\sum_{j=1}^{n} k_j = m,\quad \sum_{j=1}^{n} l_j = n.$$

Then Noether's theorem indicates that the currents

$$\mathscr{Y}_{a,+}(x) = \mathrm{tr}\,(j_+ h_{a,1}),\ \mathscr{Y}_{a,-}(x) = \mathrm{tr}\,(j_- h_{a,2}), \quad (61)$$

where $h_{1,2}$ given by Eqs. (56)–(58) or (60) are conserved if the equations of motion (39) hold.

Note that (57)–(59) satisfy the condition of invariance of the Lagrangian only on the extremals. However, one can identify a class of functions that satisfy Eq. (55) up to a total derivative for arbitrary field configurations. It is obvious that in the Abelian case all the $h^{(0,n)}(x)$ (58) are here suitable. In the non-Abelian case, the combinations

$$h_{\pm,a}^{(0,n)} = j_\pm^n T_a + T_a j_\pm^n + (n-1)\, j_\pm T_a j_\pm^{n-1}, \quad (60a)$$

constructed from (60), also change the Lagrangian by a total derivative for arbitrary fields $\partial^\mu j_\mu \neq 0$.

Generalized translations

In two-dimensional conformally invariant field-theory models the condition (11) is satisfied for the energy-momentum tensor, i.e., it is symmetric ("improved") and traceless. As can be seen from Sec. 1, in this case there is an infinite number of conserved quantities of the type (13).

To obtain the quantities (13) in accordance with Noether's theorem (47), we write down the action invariance condition (45) in the light-cone variables In the case considered here, Eqs. (45) take the very simple form

$$T_{++}\partial_- X_-^{(k)}(x) = 0,\ T_{--}\partial_+ X_+^{(k)}(x) = 0. \quad (61a)$$

Obviously,

$$X_\pm = \mathrm{const}$$

is a trivial solution of (61). The general solution of Eq. (61) can be expressed in the form

$$X_+^{(k)} = X_+^{(k)}(x_-),\ X_-^{(k)} = X_-(x_+), \quad (62)$$

where X_+ and X_- are arbitrary functions of x_+ and x_-, respectively, that satisfy the boundary conditions (52).

In the special case, (62) can be chosen in the form

$$X_-^{(k,m)}(x) = (\partial_+)^k (T_{++})^m,\ X_+^{(k,m)}(x) = (\partial_-)^k (T_{--})^m. \quad (63)$$

It follows from (36) and (63) that $\omega_{+(-)}^{(k,m)}$ transforms as the $-$ and $+$ component, respectively, of a Lorentz tensor of rank $k+2m+1$.

Noether's theorem gives an infinite number of conserved quantities,

$$\mathscr{T}_+^{(k,m)} = T_{++} X_-^{(k,m)},\ \mathscr{T}_-^{(k,m)} = T_{--} X_+^{(k,m)},$$

where $X_\pm^{(k,m)}$ are given by the expressions (62) or (63). The corresponding integrals of the motion have the form

$$P_{+(-)}^{(k,m)} = \int_{-\infty}^{\infty} dx_1\, T_{++(--)} X_{-(+)}^{(k,m)}.$$

Thus, we have shown that the higher conserved energy–momentum tensors can also be obtained from Noether's theorem. They are generated by the coordinate transformations (36), where the generators $X_\mu^{(k)}$ are given by (62) or (63), with respect to which the fields $\varphi_j(x)$ transform in accordance with the law (46).

It should be noted that for the integrals of the motion $P_\pm^{(k,m)}$ the canonical Poisson brackets vanish,

$$\{P_\pm^{(k,m)},\ P_\pm^{(k',m')}\} = 0,$$

i.e., the integrals are in involution. However the generators of the field transformations (46),

$$\mathscr{P}_\pm^{(k,m)} = X_\pm^{(k,m)}\partial_\pm,$$

do not commute with one another at all with respect to the ordinary commutator:

$$[\mathscr{P}_\pm^{(k,m)}(x),\ \mathscr{P}_\pm^{k',m'}(x)] \neq 0.$$

Therefore, here $\mathscr{P}_\pm^{(k,m)}$ are not a representation of the algebra of the integrals of the motion. In Sec. 8, we shall discuss this algebra. If the model is not conformally invariant, the invariance condition can be written in the weak form, i.e.,

$$T^{\mu\nu}\partial_\mu X_\nu^{(k)}(x) = \partial^\mu W_\mu^{(k)}(x), \quad (64)$$

where W_μ is an arbitrary vector function.

As examples, we consider scalar-field models for which the Lagrangian has the form

$$L(x) = \frac{1}{2}\partial_+\varphi\partial_-\varphi - V(\varphi),$$

where $V(\varphi)$ is an arbitrary function of φ. The corresponding equations of motion and the energy–momentum tensor can be expressed in the form

$$\varphi_{+-} = -\partial V/\partial\varphi, \quad (65)$$

$$T_{++} = \frac{1}{2}(\varphi_+)^2,\ T_{--} = \frac{1}{2}(\varphi_-)^2,\ T_{+-} = T_{-+} = V(\varphi). \quad (66)$$

We seek solutions of Eq. (64) in the form

$$X_+^{(k)} = g_{+-}\varkappa_-^{(2k)},\ X_-^{(k)} = g_{-+}\varkappa_+^{(2k)},$$

where $G_{+-} = g_{-+} = 1$ and $\varkappa_\pm^{(2k)}$ are the $\pm$ components of a tensor with Lorentz dimension $\pm 2k$. The explicit form of $\varkappa_\pm$ depends on the model.

We consider two special cases. The first is specified by the interaction $V=\alpha \exp \beta\varphi(x)$, where α is a coupling constant with the dimensions of the square of a mass, and β is a dimensionless constant. Then from (64), taking into account (65) and (66), we find

$$\varkappa_\pm^{(2)}(x) = -\frac{(\varphi_{\pm\pm})}{(\varphi_\pm)^2} + \frac{1}{4}\beta^2(\varphi_\pm)^2 - \beta\varphi_{\pm\pm}. \quad (67)$$

Substituting the latter in (47), we obtain $T_{\pm\pm\pm\pm} = (\varphi_{\pm\pm} - (1/2)\beta(\varphi_\pm)^2)^2, T_{\mp\pm\pm\pm} = 0$. The remaining generators $\varkappa_\pm^{(2k)}$ are obtained similarly.

The second case is the sine-Gordon model,[35] for which $V=\alpha(\cos\beta\varphi - 1)$. Then

$$\left.\begin{aligned} \varkappa_\pm^{(2)} &= \frac{(\varphi_{\pm\pm})^2}{(\varphi_\pm)^2} - \frac{\beta^2}{4}(\varphi_\pm)^2, \\ T_{\pm\pm\pm\pm} &= (\varphi_{\pm\pm})^2 - \frac{\beta^2}{4}(\varphi_\pm)^2, \quad T_{\mp\pm\pm\pm} = \alpha(\varphi_\pm)^2 \cos\beta\varphi. \end{aligned}\right\} \quad (68)$$

Note that the translations with the generators (67) and (68) are symmetries (in the weak sense) only on the extremals.

6. NOETHER CHARACTER OF NONLOCAL CONSERVED QUANTITIES

Generalized nonlocal Abelian gauge transformations

If G is an Abelian group, Eq. (43) in two-dimensional space-time can in principle always be integrated. Note that, as was shown above, to find the conserved currents it is sufficient to have the explicit form of these solutions only in the case $\partial^\mu j_\mu = 0$, i.e., on the extremals. In this case, the differential equation of the characteristics that corresponds to Eq. (43) (for $D=2$) reduces directly to the equation for a total differential. The first integral of this equation can be written in the form

$$\Phi(x) = \int_{-\infty}^{x_1} dy_1 j_0(y_1, x_0). \quad (69)$$

As is well known, the general solution of Eq. (43) has the form $\eta(x) = F[\Phi(x)]$, where F is an arbitrary function. It follows from (69) that $\Phi(x_0,\infty) = Q$ is the first conserved charge, and therefore $\eta^{(1)} = \Phi$ satisfies the boundary condition (52) if the first charge exists.

It can be verified that the Poisson brackets of the generator functions (69) vanish, i.e., they generate an infinite-parameter Abelian group. Therefore, the charges corresponding to them are in involution.

Explicit form of the generator functions in the non-Abelian case

In the two-dimensional case, if the gauge group G is non-Abelian, the invariance condition (43) can be written in the following equivalent forms:

$$\partial_\mu \eta_a^{(k)}(x) = \varepsilon_{\mu\nu}\{j^\nu(x), \chi_a^{(k)}(x)\}, \quad (70)$$

or

$$\partial_\mu \eta_a^{(k)}(x) = [j_\mu(x), \widetilde{\chi}_a^{(k)}(x)], \quad (71)$$

where $\eta_a^{(k)}(x)$ and $\chi_a^{(k)}$ are matrix functions, and { , } and [,] are the matrix anticommutator and commutator, respectively. We obtain a restriction on the functions χ and $\bar{\chi}$ from the conditions of integrability of (70) and (71), which have the form

$$\partial^\mu\{j_\mu(x), \chi_a^{(k)}(x)\} = 0, \quad (72)$$

$$\varepsilon^{\mu\nu}\partial_\mu\{j_\nu(x), \widetilde{\chi}_a^{(k)}(x)\} = 0. \quad (73)$$

On the subspace of the solutions of the equations of motion (when $\partial^\mu j_\mu = 0$ or $\partial^\mu j_\mu^5 = 0$) we denote the trivial solutions of Eq. (43) by $\mathbf{C}_a$ and $\widetilde{\mathbf{C}}_a$, respectively. Substituting these constants on the right-hand sides of (70) and (71) [$\mathbf{C}_a$ and $\widetilde{\mathbf{C}}_a$ satisfy the integrability conditions (72) and (73)], we obtain the following solutions of these equations:

$$\eta_a^{(1)}(x) = \int_{-\infty}^{x_1} dy_1\{j_0(x_0, y_1), T_a\}; \quad (74)$$

$$\widetilde{\eta}_a^{(1)}(x) = \int_{-\infty}^{x_1} dy_1 [j_0(x_0, y_1), T_a], \quad (75)$$

where we have substituted $\mathbf{C}_a = \widetilde{\mathbf{C}}_a = T_a$. Substituting (74) and (75) in the expression for the conserved current (44), we obtain

$$\mathcal{Y}_{\mu,a}^{(1)}(x) = \mathrm{tr}\,(j_\mu(x)\,\eta_a^{(1)}(x)), \quad \widetilde{\mathcal{Y}}_{\mu,a}^{(1)}(x) = \mathrm{tr}\,(j_\mu(x)\,\widetilde{\eta}^{(1)}(x)).$$

We also consider the Gross-Neveu model, for which the Lagrangian is

$$L(x) = \frac{i}{2}\bar{\psi}\hat{\partial}\psi_j + \frac{1}{8N}g_0(\bar{\psi}_j\psi_j)^2 \ (j=1, 2, \ldots, N)$$

and the conserved current is given by $[j_\mu(x)]_{jk} = \bar{\psi}_j\gamma_\mu\psi_k$. The corresponding axial current

$$(j_\mu^5)_{jk} = \bar{\psi}_j\gamma_5\gamma_\mu\psi_k = \varepsilon_{\mu\nu}(j^\nu)_{jk}$$

satisfies on the extremals $\partial^\mu j_\mu(x) = 0$ the relation

$$\partial^\mu j_\mu^5(x) = -\frac{g_0}{N}j^\mu(x)\,j_\mu^5(x) = -\frac{g_0}{N}\varepsilon_{\mu\nu}j^\mu(x)\,j^\nu(x). \quad (76)$$

Then by analogy with the chiral models,

$$\begin{aligned} \delta S &= \int d^2x\,\mathrm{tr}\,\{j^\mu(x)\,\partial_\mu\eta_{(k)}^a(x)\}\,\delta\omega_a^k \\ &= \varepsilon^{\mu\nu}\int d^2x\,\mathrm{tr}\,\{\partial_\mu(j_\nu(x)\,\chi_a^{(k)}(x))\}\,\delta\omega_a^k \\ &= \varepsilon^{\mu\nu}\int d^2x\,\mathrm{tr}\,\Big\{j_\mu\Big(\partial_\nu\chi_a^{(k)} + \frac{g_0}{N}j_\nu\chi_a^{(k)}\Big)\Big\}\,\delta\omega_a^{(k)}, \end{aligned} \quad (77)$$

where we have used the identity (76), i.e., δS reduces to an integral of a divergence only on the extremals, and $\chi_a^{(k)}$ is a new set of functions (see Ref. 35). At the same time, there exists a set of functions satisfying (77):

$$\chi_a^{(k+1)}(x) = \int_{-\infty}^{x_1} dy_1\Big\{\partial_0\chi_a^{(k)}(x_0, y_1) + \frac{g_0}{N}[j_0, \chi_a^{(k)}](x_0, y_1),\quad (78)$$

where $\chi_a^{(0)} = T_a$, $\chi_a^{(-1)} = 0$. It follows from (53) that the corresponding nonlocal conserved currents have the form

$$\mathcal{Y}_{\mu,a}^{(k)}(x) = \mathrm{tr}\,\{j_\mu(x)\,\chi_a^{(k)}(x) + \varepsilon_{\mu\nu}[j^\nu(x), \chi^{(k-1)}(x)]\}.$$

Note that the transformations (41) with the generators (78) are a symmetry in the weak sense, i.e., only on the extremals.

Explicit form of the generator functions for the generalized nonlinear sigma models

As already noted in Sec. 3, there is a simple proof of the existence of an infinite number of nonlocal conserved currents for the generalized nonlinear sigma models; it was proposed in Ref. 2. The main assumptions are the recursion relation (18a), the vanishing curvature (15), and the condition of integrability of Eq. (18a). Note that (20a) is satisfied only on the extremals $\partial^\mu A_\mu = 0$ for any smooth function $\chi^{(k)}(x)$.

We shall show here that all these currents have a Noether character, i.e., they are determined by symmetry of the action in the weak sense (up to a total divergence). We shall solve this problem first only on the extremals and then show that there exist nonlocal transformations that are a symmetry of the action for all field configurations $\partial^\mu A_\mu(x) = 0$.

We consider the infinitesimal transformations

$$g'(x) = g(x) + \delta g(x) \tag{79}$$

and require that $A_\mu(x)$ transform as a gauge field, i.e.,

$$\begin{aligned}\delta A_\mu(x) &= \delta g^{-1}(x)\,\partial_\mu g(x) + g^{-1}(x)\,\partial_\mu \delta g(x)\\ &= \partial_\mu \alpha(x) + [A_\mu, \alpha(x)].\end{aligned} \tag{80}$$

We have here substituted

$$\alpha(x) = g^{-1}(x)\,\delta g(x) = \zeta^{(k)}(x)\,\delta\omega_k, \tag{81}$$

where $\zeta^{(k)}(x)$ are generators of the transformation (79), ω_k are the corresponding parameters, and [,] is the matrix commutator. The variation of the action (8) with respect to the infinitesimal transformations (79) has the form

$$\begin{aligned}\delta S &= -\int d^D x\ \mathrm{tr}\,\{A^\mu(x)\,\delta A_\mu(x)\}\\ &= -\int d^D x\ \mathrm{tr}\,\{A^\mu(x)\,(\partial_\mu \zeta^{(k)}(x)\\ &+ [A_\mu(x), \zeta^{(k)}(x)])\}\,\delta\omega_k = -\int d^D x\ \mathrm{tr}\,\{A^\mu(x)\,\partial_\mu \zeta^{(k)}(x)\}\,\delta\omega_k.\end{aligned} \tag{82}$$

Therefore, only transformations for which $\mathrm{tr}(A^\mu \partial_\mu \zeta^{(k)}) = 0$ preserve the action (8), i.e., $\delta S = 0$. In this section, we consider the more general class of transformations (79), for which δS can be expressed in terms of an integral of a total divergence. The explicit form of these transformations can be determined from the following assumptions.

Assumption I. The variation of the action (8) under the transformations (79) can be represented as the integral of the total divergence of some arbitrary function, i.e.,

$$\delta S = \int d^D x\ \mathrm{tr}\,\{\partial^\mu K^j_\mu(x, g, \partial_\nu g)\}\,\delta\omega_j, \tag{83}$$

where K^j_μ is a *D*-vector function with explicit form given by the second assumption.

Assumption II. We restrict ourselves to *D*-vector matrix functions K^j_μ that can be represented in the form

$$K^j_\mu(x) = A^\nu(x)\,\chi^j_{\mu\nu}(x) \quad (j = 1, 2\ldots), \tag{84}$$

where $\chi^j_{\mu\nu}(x) = -\chi^j_{\mu\nu}(x)$, i.e., $\chi^j_{\mu\nu}$ is a second-rank antisymmetric tensor with $N \times N$ matrix components. The functions $\chi^j_{\mu\nu}(x)$ are related to $\zeta_j(x)$ by the following theorem.

Theorem I: A necessary and sufficient condition for the equivalence of (82) and (83), where K^j_μ is given by (84), is that $\zeta^j(x)$ and $\chi^j_{\mu\nu}(x)$ be related by the system of differential equations

$$\partial_\mu \zeta^j(x) = D^\nu \chi^j_{\mu\nu}(x) = (\partial^\nu + A^\nu(x))\,\chi^j_{\mu\nu}(x). \tag{85}$$

To see this, we substitute (80) in (82) and take into account the vanishing curvature, i.e., $F_{\mu\nu}(x) = 0$; we then obtain (83), where K_μ is determined in (84). Conversely, substituting (84) in (83), again taking into account the vanishing curvature, $F_{\mu\nu} = 0$, and comparing with (82), we obtain (85).[1)]

For the transformation (79), whose generators (81) satisfy Eq. (85), i.e., which change the action by the integral of a total divergence, the generalized Noether theorem (53) holds.

Noether's theorem: To any one-parameter transformation whose generators satisfy (85), i.e., for which the variation of the action can be represented in the form of an integral of a total divergence, there corresponds

$$\mathcal{J}^{(k)}_\mu(x) = \mathrm{tr}\,\{A_\mu(x)\,\zeta^{(k)}(x) + A^\nu \chi^{(k)}_{\mu\nu}(x)\}, \tag{86}$$

which is conserved in the weak form, i.e., $\partial^\mu \mathcal{J}^{(k)}_\mu(x) = 0$, when the equations of motion $\partial^\mu A_\mu = 0$ are satisfied.

Therefore, for all functions $\zeta^{(k)}(x)$ and $\chi^{(k)}_{\mu\nu}(x)$ satisfying (85) there is one conserved current. Equation (85) is a first-order partial differential equation and, therefore, has infinitely many solutions. As will be seen in what follows, these solutions are nonlinear and nonlocal functionals of the field $g(x)$. It follows from this that we are dealing with nonlinear and nonlocal transformations (79) whose generating functions, specified by ζ, are obtained as solutions of Eq. (85). Therefore, the problem of finding the nonlocal currents (86) and the generators that determine these currents reduces to the problem of solving (85). These last equations have the trivial solution

$$\zeta^{(0)}(x) = \mathrm{const}, \quad \chi^{(0)}_{\mu\nu} = 0. \tag{87}$$

For this solution, the current (86) is identical to (9) apart from a constant factor. The transformations (79) corresponding to (87) are linear and local.

To find the remaining solutions of the system (85), it is necessary to take into account the subsidiary conditions

$$D^\mu \partial_\mu \zeta^{(k)}(x) = 0 \tag{88}$$

and

$$\partial_\mu D^\lambda \chi_{\nu\lambda} - \partial_\nu D^\lambda \chi_{\mu\lambda} = 0 \tag{89}$$

on the functions $\zeta^{(k)}(x)$ and $\chi^{(k)}_{\mu\nu}$, respectively. The first of them is a consequence of the vanishing curvature (15), while the second is the condition of integrability of the system (85).

In the following treatment, we restrict ourselves to the two-dimensional case, for which

$$\chi_{\mu\nu}(x)=\varepsilon_{\mu\nu}\chi(x),\quad \varepsilon_{\mu\nu}=-\varepsilon_{\nu\mu},\quad \varepsilon_{01}=\varepsilon^{10}=1 \tag{90}$$

and, therefore, (85) and (89) take the form

$$\partial_\mu\zeta(x)=\varepsilon_{\mu\nu}D^\nu\chi(x) \tag{91}$$

and

$$D^\mu\partial_\mu\chi(x)=0,\quad \partial^\mu D_\mu\zeta(x)=D^\mu\partial_\mu\zeta(x)=0. \tag{92}$$

Therefore, in the two-dimensional case the subsidiary conditions (88) and the integrability conditions (89) have the same form. The fact that in the two-dimensional case $\zeta(x)$ and $\chi(x)$ are Lorentz scalar matrices satisfying the same second-order equations (88) and (92) is very helpful for solving Eq. (91). Indeed, if we know one solution $\chi^{(0)}(x)$ of Eq. (92), then, substituting it in the right-hand side of Eq. (91), we determine the function $\zeta=\chi^{(1)}$ from the equation

$$\partial_\mu\chi^{(1)}(x)=\varepsilon_{\mu\nu}D^\nu\chi^{(0)}(x).$$

It follows from Eq. (91) that $\chi^{(1)}(x)$ also satisfies (92), and, therefore, we can substitute it in the right-hand side of Eq. (91). Thus, we can construct an infinite sequence of functions $\chi^{(k)}(x)$ $(k=0,1,\ldots)$ satisfying (92), and any two functions $\chi^{(k)}$ and $\chi^{(k-1)}$ satisfy (91), i.e.,

$$\partial_\mu\chi^{(k)}(x)=\varepsilon_{\mu\nu}D^\nu\chi^{(k-1)}(x)\ (k=0,\ 1\ldots). \tag{93}$$

If we find M linearly independent[2)] solutions $\chi_m^{(0)}(x)$ $(m=1,\ldots,M)$ that are not related by multiple application of (91), we can construct by the above method M linearly independent infinite sequences $\{\chi_m^{(k)}(x)\ \ (k=0,1,\ldots)\}$. Note that, given $\chi^{(k)}$ we can determine $\chi^{(k-1)}(x)$ from Eq. (91). Thus, the sequence $\{\chi^{(k)}\}$ can be continued to negative k, i.e.,

$$\ldots\chi_m^{(-k)},\ \ldots,\ \chi_m^{(-1)},\ \chi_m^{(0)},\ \chi_m^{(1)},\ \ldots,\ \chi_m^{(k)}\ldots \tag{94}$$

Substituting (94) in (86), we obtain M infinite sequences of conserved currents:

$$\mathscr{Y}_\mu^{(m,\,k)}(x)=A_\mu(x)\,\chi_m^{(k)}(x)+\varepsilon_{\mu\nu}A^\nu(x)\,\chi_m^{(k-1)}(x). \tag{95}$$

To find the solutions $\chi_m^{(0)}$ of Eq. (92) with which the construction of the sequence (94) commences, we consider the stronger condition

$$D_\mu\chi_m^{(0)}(x)=\partial_\mu\chi_m^{(0)}+A_\mu(x)\,\chi_m^{(0)}(x)=0. \tag{96}$$

Note that Eq. (92) is equivalent to the first-order equation

$$D_\mu\chi_m^{(0)}(x)=\mathbf{C}\mathscr{Y}_\mu^{(m)}(x), \tag{97}$$

where $\mathbf{C}$ is a constant $N\times N$ matrix and $\mathscr{J}_\mu^{(m)}(x)$ is some conserved current, $\partial^\mu\mathscr{J}_\mu^{(m)}=0$. We shall discuss this possibility later, restricting ourselves initially to the case $\mathbf{C}=0$, i.e., we consider Eq. (96). We note that the condition of integrability of Eq. (96) is a consequence of the vanishing curvature (15).

For the system (96), we have three linearly independent solutions unrelated by multiple application of (93):

a) the trivial solution

$$\chi_1^{(0)}(x)=0;$$

b) nontrivial solutions that can be written in the form

$$\chi_a^{(0)}(x)=U_\alpha\mathbf{C}_\alpha\quad (a=2,\ 3), \tag{98}$$

where U_a $(a=2,3)$ satisfy Eq. (96) and $\mathbf{C}_a$ is a constant $N\times N$ matrix. Equation (96) has the following two nontrivial solutions:

$$U_2(x)=g^{-1}(x) \tag{99}$$

and

$$U_3(x)=V(x)\,W(x), \tag{100}$$

where

$$\left.\begin{aligned} V(x)&=P\exp\Big\{-\int_{-\infty}^{x_0}dy_0A_0(y_0,\ x_1)\Big\},\\ W(x)&=P\exp\Big\{-\int_{-\infty}^{x_1}dy_1\,[V^{-1}\partial_1V+V^{-1}A_1V](x_0,\ y_1)\Big\}.\end{aligned}\right\} \tag{101}$$

Here, P is the Wilson ordering operator.

From (98)–(101), using the method developed above, we construct three linearly independent sequences of functions $\{\chi_m^{(k)}\}$. It follows from the fact that $\chi_m^{(0)}$ $(m=1,2,3)$ satisfy Eq. (96) that for $k\geqslant1$ all these sequences are identical. However, these sequences differ for $k\leqslant0$, and they are therefore linearly independent.

We begin by constructing the first series. It can be verified that the general form of $\{\chi_1^{(k)}\}$ is given by

$$\left.\begin{aligned} &\chi_1^{(0)}=0;\\ &\chi_1^{(1)}=\mathbf{C}_1;\\ &\vdots\\ &\chi_1^{(k)}(x)=\int_{-\infty}^{x_1}dy_1\{\partial_0\chi^{(k-1)}+A_0\chi^{(k-1)}\}(x_0,\ y_1)+\mathbf{C}_k,\end{aligned}\right\} \tag{102}$$

where $\mathbf{C}_1,\mathbf{C}_2,\ldots$ are constant $N\times N$ matrices. The constants $\mathbf{C}_k$ from (102) in the currents (95) lead to the terms $A_\mu\mathbf{C}_k$, which are conserved separately. Therefore, without loss of generality the constants $\mathbf{C}_k$ for $k\geqslant2$ can be omitted in (102).

The functions $X^{(k)}(x_0,\infty)$ are identical to the nonlocal conserved charges found in Refs. 1 and 2.

To find the remaining two series, we use the following symmetry of Eqs. (91) and (93). Suppose that there exists a nonsingular matrix U satisfying Eq. (96). Then by means of the ansatz

$$\zeta=U\tilde\zeta\ \text{ and }\ \chi=U\tilde\chi$$

Eq. (91) can be written in the form

$$\partial_\mu\tilde\chi=\varepsilon_{\mu\nu}(\partial^\nu+\tilde A^\nu)\,\tilde\zeta(x), \tag{103}$$

where

$$\tilde A_\mu(x)=-U^{-1}A_\mu(x)\,U. \tag{104}$$

From (91) and (104) we find that

$$\partial^\mu \tilde{A}_\mu (x) = -U^{-1}\partial^\mu A_\mu U = 0,$$

if the equation of motion (10) is satisfied and

$$\tilde{F}_{\mu\nu} = [\tilde{D}_\mu, \tilde{D}_\nu] = U^{-1}[D_\mu, D_\nu] U = U^{-1}F_{\mu\nu}U = 0.$$

Therefore, $\tilde{\zeta}$ and $\tilde{\chi}$, like ζ and χ, satisfy (92).

Then it follows from (103) that

$$\{\chi_2^{(-k)} = U_2\tilde{\chi}_2^{(k)}\}, \ \{\chi_3^{(-k)} = U_3\tilde{\chi}_3^{(k)}\}, \qquad (105)$$

where $\tilde{\chi}_a^{(k)}$ can be obtained from (102) by the substitution

$$A_\mu (x) \to -U_\alpha A_\mu U_a^{-1} \quad (a=2,\ 3),$$

and U_a has the form (99)–(101).

Substituting (102) and (105) in (95), we find three linearly independent infinite series of conserved currents.

It should be noted that the derivation of (102) makes explicit use of the equations of motion, and therefore the symmetry of the action so far considered exists only on the extremals, i.e., when $\partial^\mu A_\mu (x) = 0$.

To generalize this symmetry for an arbitrary case, we use the method proposed in Refs. 11 and 12. This will be done in Sec. 9.

To end this section, we consider in more detail Eq. (97), from which the initial functions $\chi_m^{(0)}(x)$ can be determined. From the condition of integrability of the system of equations (97) we obtain the following restrictions on the conserved currents $\mathscr{J}_\mu^{(m)}$ on the right-hand sides of these equations:

$$\varepsilon^{\mu\nu} (\partial_\mu + A_\mu) \mathscr{Y}_\nu^{(m)} (x) = 0. \qquad (106)$$

As follows from (17), any of the conserved currents obtained above can be represented in the form $\mathscr{J}_\mu^{(m)} = \varepsilon_{\mu\nu}\partial^\nu \chi^{(m)}(x)$, where the function $\chi^{(m)}(x)$ is determined from Eq. (18a).

Substituting (17) in (106), we obtain

$$(\partial^\mu + A^\mu (x))\, \partial_\mu \chi^{(m)} (x) = 0.$$

This is the condition of integrability of Eq. (18a), i.e., for all conserved currents constructed from $\chi^{(m)}$ the condition of integrability of the system (97) is satisfied. Then the solution of Eq. (97) has the form

$$\chi_m^{(m)} (x) = \chi^{(0)} \int_{-\infty}^{x_1} dy_1\, (\chi^{(0)})^{-1} \mathscr{Y}_1^{(m)} = -\chi^{(0)} \int_{-\infty}^{x_1} dy_1\, (\chi^{(0)})^{-1}\partial_0\chi^{(m)},$$

where $\chi^{(0)}$ is a nontrivial solution of the corresponding homogeneous equation (96) having the form (99) or (100). Using Eqs. (18) and (96), we obtain

$$\chi_m^{(m)} = \chi^{(m-1)},$$

i.e., if in the right-hand side of (97) we substitute any of the conserved currents (17), we obtain the original sequence of functions (102). To find the new sequence, we must find conserved quantities of some other origin satisfying the integrability condition (106).

Solution of Eq. (43) for a subspace with an arbitrary number of dimensions in the Abelian case

For $D>2$, Eq. (43) does not always have a nontrivial solution in the complete space. As an example, we consider the three-dimensional case $D=3$. Then the corresponding characteristic system of equations (43) can be written in the form

$$j_0\, dx_1 + j_1\, dx_0 = 0,\ j_0\, dx_2 + j_2\, dx_0 = 0. \qquad (107)$$

One first integral of the system (107) can be found if

$$j_\mu (x) = j_\mu (x_0,\ x_1 + \alpha x_2)\ (\mu = 0,\ 1,\ 2) \qquad (108)$$

or

$$j_1 (x) + j_2 (x) = \frac{j_0}{g} \int_{-\infty}^{x_1} dy_1 \partial_0 g\, (x_0,\ y_1 + \alpha x_2), \qquad (109)$$

where α is an arbitrary parameter and g is an arbitrary function. Then the first integral of (107) has the form

$$\Phi_1 (x) = \int_{-\infty}^{x_1} dy_1 j_0\, (x_0,\ y_1 + \alpha x_2),$$

$$\Phi_2 (x) = \int_{-\infty}^{x_1} dy_1 g\, (x_0,\ y_1 + \alpha x_2)$$

respectively. As in the two-dimensional case, any function of these first integrals satisfies the invariance equation (43).

For the four-dimensional case, the conditions (108) and (109) have the form

$$j_\mu (x) = j_\mu (x_0,\ x_1 + \alpha x_2 + \beta x_3) \quad (\mu = 0,\ 1,\ 2,\ 3);$$

$$j_1 + \alpha j_2 + \beta j_3 = \frac{j_0}{g} \int_{-\infty}^{x_1} dy_1 \partial_0 g\, (x_0,\ y_1 + \alpha x_2 + \beta x_3)$$

respectively. The first integrals are given by

$$\Phi_1 = \int_{-\infty}^{x_1} dy_1 j_0\, (x_0,\ y_1 + \alpha x_2 + \beta x_3),$$

$$\Phi_2 = \int_{-\infty}^{x_1} dy_1 g\, (x_0,\ y_1 + \alpha x_2 + \beta x_3),$$

where α and β are arbitrary parameters and g is an arbitrary function.

We can similarly obtain solutions of Eqs. (45) and (50) in a space with an arbitrary number of dimensions.

7. NOETHER CHARACTER OF THE NONLOCAL CONSERVED CURRENTS IN THE SUPERSYMMETRIC NONLINEAR SIGMA MODELS

As we have seen in Sec. 3, for the supersymmetric nonlinear sigma models there is also an infinite number of conserved nonlocal currents. In this section, we generalize the results of the previous section to the supersymmetric case.

We note that, as in the ordinary case, the first, $k=1$, conserved spinor supercurrent (29),

$$\mathscr{Y}_\alpha^{(1)} (x;\ \theta) = \mathscr{A}_\alpha (x;\ \theta) = \mathscr{G}^{-1} (x;\ \theta)\, \mathscr{D}_\alpha \mathscr{G} (x;\ \theta) \quad (\alpha = 1,\ 2), \qquad (110)$$

is identical to (23) and, therefore, can be obtained by Noether's theorem as a consequence of the invariance of the action (21) with respect to global gauge transformations. To investigate the origin of the higher conserved nonlocal currents, we make the following assumption.

Assumption I. There exist two functions $X(x;\theta)$ and $Y(x;\theta)$, which are $N\times N$ matrices whose elements transform as scalar superfields, related by the equation

$$\mathscr{D}_\alpha X(x;\ \theta)=(\gamma_5\nabla)_\alpha\, Y(x;\ \theta)\quad(\alpha=1,\ 2). \tag{111}$$

Using these functions, we define the spinor quantity

$$\mathscr{Y}_\alpha^{(X,\,Y)}(x;\ \theta)=\mathscr{A}_\alpha\,(x;\ \theta)\,X\,(x;\ \theta)+(\gamma_5\mathscr{A})_\alpha\,(x;\ \theta)\,Y\,(x;\ \theta). \tag{112}$$

Then the following theorem is readily verified.

Theorem I. Assumption I is the necessary and sufficient condition for conservation of the spinor supercurrent (112) in the weak sense, i.e.,

$$\mathscr{D}^\alpha\mathscr{Y}_\alpha^{X,\,Y}\,(x;\ \theta)=0,$$

if the equations of motion (22) are satisfied.

It is readily verified that after the substitution

$$X=X^{(k)}\ (x;\ \theta),\ Y=X^{(k-1)}\,(x;\ \theta)\ (k=1,\ 2\ldots) \tag{113}$$

Assumption I becomes equivalent to (30), and the supercurrent (112) can be expressed in terms of a linear combination of the currents (29). However, Assumption I is more general than (30) because it holds not only for the countable set of functions (113). In addition, as we shall see, Assumption I makes it possible to obtain the supercurrents (112) from the generalized Noether theorem.[36] To this end, we consider one "global" transformation of the field $\mathscr{G}(x;\theta)$.

$$\mathscr{G}'\,(x;\ \theta)=\mathscr{G}\,(x;\ \theta)\,U\,(x;\ \theta). \tag{114}$$

For this transformation, we assume the following property.

Assumption II. The variation of the action (21) under the infinitesimal transformations (114) does not in general vanish but reduces to the integral of the total superdivergence of some spinor quantity.

It can be readily verified that Assumption I is a sufficient condition for fulfillment of Assumption II. Indeed, the variation of the action for the infinitesimal transformations (114) has the form

$$\delta S=-\int d^2xd^2\theta\ \mathrm{tr}\,\{\mathscr{A}^\alpha\,(x;\ \theta)\,\delta\mathscr{A}_\alpha\,(x;\ \theta)\}$$

$$=-\int d^2xd^2\theta\ \mathrm{tr}\,\{\mathscr{A}^\alpha\,(x;\ \theta)\,\mathscr{D}_\alpha X^{(a)}\,(x;\ \theta)\}\,\delta\omega_a$$

$$=-\int d^2xd^2\theta\ \mathrm{tr}\,\{\mathscr{D}^\alpha\,[(\gamma_5\mathscr{A})_\alpha\,Y^{(a)}\,(x;\ \theta)]\}\,\delta\omega_a,$$

where we have used Assumption I and made the ansatz

$$\delta\mathscr{A}_\alpha=\delta\mathscr{G}^{-1}\mathscr{D}_\alpha\mathscr{G}+\mathscr{G}^{-1}\mathscr{D}_\alpha\mathscr{G}=\{\mathscr{D}_\alpha X^{(a)}+[\mathscr{A}_\alpha,\ X^{(a)}]\}\,\delta\omega_a.$$

Here, $\delta\omega_a$ are infinitesimal parameters that do not depend on x and θ, and $X^{(a)}$ are the generator functions of the transformations (114), i.e.,

$$X^{(a)}\,(x;\ \theta)=\left.\frac{\partial U}{\partial\omega_a}\right|_{\omega=0}\quad(a=1,\ 2\ \ldots). \tag{115}$$

As we see, these generators are in general nonlinear and nonlocal functionals of the field $\mathscr{G}$.

It follows from Assumption II that the generalized Noether theorem[36] holds.

Noether's theorem: To every one-parameter nonlinear and nonlocal transformation satisfying Assumption II there corresponds one conserved quantity (112).

The explicit form of the generator functions (115) of the transformations (114) is obtained from (111). Note that the transformations with the generators $X=\text{const}$ are an exact symmetry of the action, i.e., $\delta S=0$, and the currents corresponding to them have the form (110). The functions $X=\text{const}$, $Y=0$ are a trivial solution of (111). To find nontrivial solutions of Eq. (111), we use the following properties of these equations:

$$\nabla^\alpha\mathscr{D}_\alpha X\,(x;\ \theta)=\mathscr{D}^\alpha\nabla_\alpha X\,(x;\ \theta)=0 \tag{116}$$

and

$$\mathscr{D}^\alpha\nabla_\alpha Y\,(x;\ \theta)=0, \tag{116a}$$

where X and Y are smooth functions. The first of them, (116), is a consequence of (22) and (28), and the second, (116a), is the condition of integrability of Eq. (111).

Equations (116) are partial differential equations. As is known from the general theory of differential equations, they have a nondenumerable set of solutions. Therefore, from these solutions it is possible to construct an infinite number of conserved currents (112). To each of these solutions there corresponds a field transformation (114). Thus, from Assumptions I and II we find the conserved quantities (112) and the transformations (114) generating these quantities. As we have seen, the determination of an infinite number of nonlocal conserved supercurrents was reduced to solutions of Eq. (111) under the restriction (116) on the functions X and Y. We shall here make essential use of the fact that Eqs. (116) for X and Y are identical. Then the problem of finding the explicit form of X and Y can be solved in two ways:

a) find all solutions of (116) and seek among them pairs satisfying (111);

b) using the ansatz (113), write the assumption (111) in the form (30), i.e., restrict our treatment to the countable set of functions $X^{(k)}$. Then, beginning with certain independent solutions[3)] $X_m^{(0)}$ $(m=1,\ldots,M)$ of Eqs. (116), we substitute them in the right-hand side of (30). Solving these last equations, we obtain the functions $X_m^{(1)}$, which also satisfy (116). Therefore, $X_m^{(1)}$ can again be substituted in the right-hand side of (30), and so forth.

Thus, we obtain the infinite sequences

$$X_m^{(0)},\ X_m^{(1)},\ X_m^{(2)},\ \ldots,\ X_m^{(k)}\ \ldots\quad(m=1,\ 2,\ \ldots,\ M),$$

where each element satisfies (116) and each pair of neighboring elements $X_m^{(k)}$ and $X_m^{(k+1)}$ is, by construction, related by (30). The constant correction $\mathbf{C}_k$ to $X^{(k)}$, being the general solution of the homogeneous equation

$$\mathscr{D}_\alpha X^{(k)}\,(x;\ \theta)=0$$

need, the term $\mathscr{A}_\alpha\mathbf{C}_k$ being conserved separately, be taken into account only once, for example, for $k=1$.

It must be noted that Assumption I (30) is a two-sided connection, i.e., from given X we can also determine Y. Then the general solution (30) contains solutions of the homogeneous equation

$$(\mathscr{D}_\alpha+\mathscr{A}_\alpha(x;\theta))\,X_m^{(k)}(x;\theta)=0. \tag{117}$$

The solutions of this equation are no longer constants, and therefore they must be taken into account.

Thus, from given $X_m^{(0)}$ we determine $X_m^{(-1)}, X_m^{(-2)},\ldots$ and, therefore, we have in general the sequences

$$\ldots,\ X_m^{(-k)},\ \ldots,\ X_m^{(-1)},\ X_m^{(0)},\ X_m^{(1)},\ \ldots,\ X_m^{(k)}\ \ldots \tag{118}$$

If for some element $X_m^{(l)}$ $(-\infty<l<\infty)$ the sequences (118) satisfy Eq. (117), then the sequence (117) can be terminated at this element $X_m^{(l)}$.

For our purposes, only the linearly independent sequences (117) are of interest. We shall assume that two sequences $\{X_l^{(k)}\}$ and $\{X_m^{(k)}\}$ are linearly independent if each of them contains not more than one element $X_l^{(p)}$ (respectively, $X_m^{(p)}$) linearly independent of an element of the sequence $\{X_m^{(k)}\}$ (respectively, $\{X_l^{(k)}\}$).

Using (112), for each sequence (118) we associate one infinite series of conserved supercurrents. In the case of linearly independent sequences, the currents corresponding to them are also linearly independent.

Using the procedure described above, we can find the explicit form of each term of the three linearly independent infinite series of conserved supercurrents. For this purpose, we consider (116). Suppose that we have found a certain number of conserved currents $\mathscr{J}_\alpha^{(m)}$ $(m=1,\ldots,M)$. Then Eq. (116) is equivalent to the system of first-order equations

$$\{\mathscr{D}_\alpha+\mathscr{A}_\alpha(x;\theta)\}\,X_w^{(0)}(x;\theta)=\mathscr{Y}_\alpha^{(m)}(x;\theta). \tag{119}$$

The condition of integrability of the system (119) has the form

$$(\mathscr{D}^\alpha+\mathscr{A}^\alpha(x;\theta))\,(\gamma_5)_\alpha^\beta\mathscr{Y}_\beta^{(m)}(x;\theta)=0,$$

where we have used the fact that $\mathscr{F}_{\alpha\beta}=0$. Note that if, as in the ordinary case, $\mathscr{J}_\alpha^{(m)}$ is one of the conserved currents (112), then we obtain from the solution of (119) the same sequence $\{X_m^{(k)}\}$ from which the currents $\mathscr{J}_\alpha^{(m)}$ are constructed. Therefore, we restrict our treatment to the homogeneous equation (117). This last equation has three linearly independent solutions.

The first solution is the trivial one

$$X_1^{(0)}=0; \tag{120}$$

the second has the form

$$X_2^{(0)}(x;\theta)=\mathscr{G}^{-1}(x;\theta)\,\mathbf{C}. \tag{121}$$

To find the third solution, we use the expression of Assumption I (111) for the components [see the expression (32)].

Then, with allowance for (22) and (25), Eqs. (117) take the form (for $\mathbf{C}=0$)

$$(\partial_\mu+g^{-1}\partial_\mu g)\,\chi^{(0)}(x)=0; \tag{122a}$$

$$\varkappa^{(0)}(x)=-\varphi^{-1}(x)\,g(x)\,\varphi(x)\,\chi^{(0)}(x); \tag{122b}$$

$$\xi^{(0)}(x)=-\frac{1}{2}g^{-1}(x)\,\overline{\varphi}(x)\,\varphi^{-1}(x)\,g(x)\,\chi^0(x). \tag{122c}$$

It is readily verified that (120) and (121) satisfy (122). Moreover, integrating Eq. (122a), we find the further solution

$$\chi_3^{(0)}=VW\mathbf{C}_3, \tag{123}$$

where $\mathbf{C}_3$ is a constant matrix, and

$$\left.\begin{aligned} V&=P\exp\left\{-\int_{-\infty}^{x_0}dy_0v_0(y_0,x_1)\right\},\\ W&=P\exp\left\{-\int_{-\infty}^{x_1}dy_1\,(V^{-1}\partial_1V+V^{-1}v_1V)\right\}.\end{aligned}\right| \tag{124}$$

Here, P is the Wilson ordering operator. Substituting (124) in (122a), we obtain $\varkappa_3^{(0)}$ and $\xi_3^{(0)}$, respectively.

The solutions of Eq. (30) are given by Eq. (33). The conserved currents (112) corresponding to this sequence in the case when $\mathbf{C}_1=\mathbf{I}$ are, apart from linear combinations, identical to the expressions found in Refs. 19–22.

To construct the sequences $\{X_2^{(k)}\}$ and $\{X_3^{(k)}\}$, we use the following symmetry of Assumption I.

Assumption (30) has the following symmetry. Suppose that there exists a nonsingular matrix $Y_0(x;\theta)$ satisfying the equation

$$\nabla_\alpha Y_0(x;\theta)=\{\mathscr{D}_\alpha+\mathscr{A}_\alpha(x;\theta)\}\,Y_0(x;\theta)=0. \tag{125}$$

Then we represent $X(x;\theta)$ and $Y(x;\theta)$ in the form

$$X=Y_0\widetilde{X},\quad Y=Y_0\widetilde{Y}. \tag{126}$$

Substituting (126) in Assumption I, we obtain

$$\mathscr{D}_\alpha\widetilde{Y}(x,\theta)=(\gamma_5)_\alpha^\beta\,[\mathscr{D}_\alpha+\mathscr{A}_\beta(x;\theta)]\,\widetilde{X}(x;\theta), \tag{127}$$

where

$$\widetilde{\mathscr{A}}_\alpha(x;\theta)=-Y_0^{-1}\mathscr{A}_\alpha(x;\theta)\,\widetilde{Y}. \tag{128}$$

It is readily verified that

$$\mathscr{D}^\alpha\widetilde{\mathscr{A}}_\alpha=-Y_0^{-1}\mathscr{D}^\alpha\mathscr{A}_\alpha(x;\theta)\,Y_0, \tag{129a}$$

if the equations of motion (22) are satisfied and

$$\widetilde{\mathscr{F}}_{\alpha\beta}=Y_0^{-1}\mathscr{F}_{\alpha\beta}Y_0=\{\widetilde{\nabla}_\alpha,\ \widetilde{\nabla}_\beta\}. \tag{129b}$$

Therefore, $\widetilde{X}$ and $\widetilde{Y}$ must also satisfy the second-order equation (116). Note that (126) is not a similarity transformation.

It is then sufficient to require invertibility of the matrices $\mathbf{C}_2$ and $\mathbf{C}_3$ for the sibstitution (126), i.e.,

$$X^{(-k)}(x;\theta)=X_a^{(0)}\widetilde{X}^{(k)}(x;\theta)\quad(a=2,3). \tag{130}$$

Assumption I takes the form [see (127)]

$$\mathscr{D}_\alpha\widetilde{X}_a^{(k)}(x;\theta)=(\gamma_5)_\alpha^\beta\,(\mathscr{D}_\beta+\widetilde{\mathscr{A}}_\beta)\,\widetilde{X}^{(k-1)}(x;\theta)$$

$$(a=2,3;\ k=1,2\ldots).$$

Therefore, each term of the sequence $\{\widetilde{X}_\alpha^{(k)}\}$ can be obtained from the corresponding term $\{X_1^{(k)}\}$ by the substitution (128): $A_\alpha \to -\hat{A}_\alpha$.

Thus, we can construct the three sequences

$$\{X_1^{(k)} \mid k \geqslant 0\}, \quad \{X_2^{(-k)} = X_2^{(0)} \widetilde{X}_2^{(k)} \mid k \geqslant 0\}$$

and $\{X_3^{(-k)} = X_3^{(0)} \widetilde{X}_3^{(k)} \mid$

which are linearly independent by virtue of (121), (124), and (125) and are not related by a similarity transformation. Indeed, from (121) and (126) we find the sequence

$$X_2^{(1)}(x; \theta) = \mathcal{G}^{-1}(x; \theta)\, C_2,$$

which is linearly independent of the sequence $\{X_1^{(k)}\}$, whose members are given by (125). To these sequences there correspond three infinite series of conserved nonlocal supercurrents.

The conserved charges obtained from the supercurrents $\mathcal{J}_\alpha^{(k)}$ have the form

$$Q^{(k)} = \int_{-\infty}^{\infty} dx_1 V_0^{(k)}(x), \tag{131}$$

i.e., are determined by the vector component of the spinor supercurrent. It follows from (131) that $Q^{(k)}$ are the values of the functions $\chi^{(k)}$ (33) at the point $x_1 = \infty$, i.e., $Q^{(k)} = \chi^{(k)}(x_0, \infty)$.

8. GROUP STRUCTURE OF LOCAL HIDDEN SYMMETRY TRANSFORMATIONS

In this section, we study the group structure of hidden symmetry transformations that generate higher local conserved quantities, as considered in Sec. 5. We discuss the question of gauge transformations.[54] For this purpose, we consider the commutator of two infinitesimal transformations:

$$\begin{aligned}
&(U_1U_2 - U_2U_1)_{jh}\psi_h(x)\\
&= \{\delta_{jl} + i(\eta_a^{(p)})_{jl}[\psi_m + i(\eta_b^{(q)})_{mn}\psi_n\delta\omega_{2,\,b}^{(q)}]\,\delta\omega_{1,\,a}^{(p)}\}\\
&\quad \times \{\delta_{lh} + i(\eta_a^{(q)})_{lh}\delta\omega_{2,\,b}^{(q)}\}\psi_h(x)\\
&- \{\delta_{jl} + i(\eta_b^{(q)})_{jl}[\psi_m + i(\eta_a^{(p)})_{mn}\psi_n\delta\omega_{1,\,a}^{(p)}]\,\delta\omega_{2,\,b}^{(q)}\}\\
&\quad \times \{\delta_{lh} + i(\eta_a^{(p)})_{lh}\delta\omega_{1,\,a}^{(p)}\}\psi_h(x)\\
&= \Big\{[\eta_a^{(p)}(x),\ \eta_b^{(q)}(x)]_{jh} + \frac{\partial(\eta_a^{(p)})_{jh}}{\partial\psi_m}(\eta_b^{(q)}\psi)_m\\
&\quad - \frac{\partial(\eta_b^{(q)})_{jh}}{\partial\psi_m}(\eta_a^{(p)}\psi)_m - (\bar{\psi}(\eta_b^{(q)})^+)_n\frac{\partial(\eta_a^{p})_{jh}}{\partial\bar{\psi}_n}\\
&\quad + (\bar{\psi}(\eta_a^{(p)})^+)_n\frac{\partial(\eta_b^{q})_{jh}}{\partial\bar{\psi}_n}\Big\}\psi_h(x)\,\delta\omega_{a,\,1}^{(p)}\delta\omega_{b,\,2}^{(q)}\\
&= [\![\eta_a^{(p)}(x),\ \eta_b^{(q)}(x)]\!]_{jh}\psi_h(x)\delta\omega_{1,\,a}^{(p)}\,\delta\omega_{2,\,b}^{(p)},
\end{aligned} \tag{132}$$

where we have introduced the notation $[\![\,,\,]\!]$ for the commutator of two generator functions, and $[\,,\,]$ denotes the ordinary matrix commutator. It is obvious that if the generators do not depend on the fields, then (132) can be expressed in terms of the ordinary matrix commutator. From (132) we find that each original symmetry group of the considered model is Abelian, as is the corresponding group of hidden symmetries.

We consider the following generator functions of the type (56):

$$\eta_{\pm,\,a}^{(\nu)}(x) = (x_\pm)^\nu T_a, \tag{133}$$

where ν is any, in general complex, number. Substituting (133) in the commutator (132), we obtain

$$\begin{aligned}
&[\![\eta_{\pm,\,a}^{(\nu)}(x),\ \eta_{\pm,\,b}^{(\nu')}(x)]\!] = [\eta_{\pm,\,a}^{(\nu)}(x),\ \eta_{\pm,b}^{(\nu')}(x)]\\
&= (x_\pm)^{\nu+\nu'}[T_a, T_b] = C_{abc}(x_\pm)^{\nu+\nu'}T_c = C_{abc}\eta^{\pm,c+\nu')}(x),
\end{aligned}$$

where C_{abc} are the structure constants of the group G. Therefore, (133) realizes the infinite Lie algebra considered in Refs. 37 and 38. Note that the transformations (133) with $\mathrm{Re}\,\nu > 0$ do not satisfy the boundary condition (52).

We consider also the transformations with the generators

$$(\eta_a^{(0,\,m)})_{jh} = (j^m)_{jl}(T_a)_{lh} = \psi_j(\psi^* T_a)_h(\psi^*\psi)^{m-1}, \tag{134}$$

$$(\widetilde{\eta}_a^{(0,\,m)})_{jh} = \mathrm{tr}\,(j^m)(T_a)_{jh}. \tag{135}$$

It is readily verified that

$$[\![\eta_a^{(0,\,m)}(x),\ \eta_b^{(0,\,n)}(x)]\!] = C_{abc}\eta_c^{(0,\,m+n)}(x),$$

$$[\![\widetilde{\eta}_a^{(0,\,m)}(x),\ \widetilde{\eta}_b^{(0,\,n)}(x)]\!] = C_{abc}\widetilde{\eta}_c^{(0,\,m+n)}(x),$$

i.e., the generators $\eta_a^{(0,k)}$ and $\widetilde{\eta}_a^{(0,k)}$ form the same infinite Lie algebra as the generators (133).

In conclusion, we note that we have not succeeded in obtaining a closed Lie algebra for generators (60) realizing a symmetry for arbitrary fields. We note also that our treatment is valid for both cases of spinors with commuting or anticommuting components. In the latter case, the number of generators $\eta_a^{(0,k)}$ (134) and (135) is finite $(k = 0,1,\dots,N)$.

We also consider the group structure of some of the transformations of the generalized translations (46). To obtain the commutator of the generators of the translations (62), we take the commutator

$$\begin{aligned}
(T_1T_2 - T_2T_1)\,\varphi(x) &= \varphi[x + \delta_2(x + \delta_1 x)] - \varphi[x + \delta_1(x + \delta_2 x)]\\
&= \Big(\frac{\partial X_\mu^{(m)}}{\partial x_\nu}X_\nu^{(n)} - \frac{\partial X_\mu^{(n)}}{\partial x_\nu}X_\nu^{(m)}\Big)\partial^\mu\varphi(x)\,\delta\omega_1^{(m)}\delta\omega_2^{(n)},
\end{aligned} \tag{136}$$

where T is the translation operator, which acts on $\varphi(x)$ in accordance with (62). We choose the functions $X_\pm^{(k)}$ in the form

$$X_\pm^{(\nu)} = (x_\pm)^\nu, \tag{137}$$

where ν is an arbitrary number. Substituting (137) in (136), we obtain the following infinite Lie algebra:

$$[\mathcal{P}_\pm^{(\nu)},\ \mathcal{P}_\pm^{(\nu')}] = (\nu - \nu')\,\mathcal{P}_\pm^{(\nu+\nu'-1)}, \tag{138}$$

where

$$\mathcal{P}_\pm^{(\nu)} = (x_\pm)^\nu\partial_\pm.$$

Note that the corresponding Lie algebra that can be obtained from (138) by substituting there the polynomial generators (63) has a more complicated form.

9. GROUP STRUCTURE OF NONLOCAL TRANSFORMATIONS FOR CHIRAL MODELS

In this section, we shall investigate the group structure of the nonlocal transformations (79) in the case of ordinary chiral models. We shall show that from the generator functions (102) and (105) we can construct several series of generators, each of them forming an infinite-dimensional closed algebra. Further, on each of these series there is imposed the natural boundary condition

$$S_a(x;\lambda)\,|_{x_+=-\infty}=T_a. \qquad (139)$$

Here T_a are the generators of the group G:

$$S_a(x;\lambda)=\chi(x;\lambda)\,T_a\chi^{-1}(x;\lambda), \qquad (140)$$

$$\chi(x;\lambda)=\sum_{k=-\infty}^{\infty}\lambda^k\chi^{(k)}(x), \qquad (141)$$

where $\chi^{(k)}(x)$ are given by (102) if $k>0$ and by (105) if $k<0$. For $\chi^{(0)}$, we assume that

$$\chi^{(0)}(x)=\alpha I+\beta U_3(x), \qquad (142)$$

where $U_3(x)$ is given by (100) and α and β are arbitrary parameters satisfying the restriction $\alpha+\beta=1$, which is a consequence of the boundary condition (139). In the case when $\chi^{(k)}=0$ for $k<0$, $\alpha=1$, and $\beta=0$, we obtain the series considered in Ref. 12 (see also Refs. 16 and 17). The case when $\chi^{(k)}=0$ for $k>0$, $\alpha=0$, and $\beta=1$ was discussed in Ref. 18.

We recall that the explicit form of $\chi^{(k)}(x)$ was obtained as the solution of Eq. (18a). Substituting (141) in (18a), we obtain (in the light-cone variables)

$$(1\mp\lambda)\,\partial_\pm\chi(x;\lambda)=\pm\lambda A_\pm\chi(x;\lambda).$$

From the condition $\chi(x;\lambda)\chi^{-1}(x;\lambda)=\chi^{-1}(x;\lambda)\chi(x;\lambda)=I$ we now have the following equation for $\chi^{-1}(x;\lambda)$:

$$(1\mp\lambda)\,\partial_\pm\chi^{-1}(x;\lambda)=\mp\lambda\chi^{-1}(x;\lambda)\,A_\pm(x).$$

The last two equations enable us to obtain the equations for S_a:

$$(1-\lambda)\,\partial_+S_a(x;\lambda)=\lambda\,[A_+(x),\ S_a(x;\lambda)], \qquad (143)$$

$$(1+\lambda)\,\partial_-S_a(x;\lambda)=-\lambda\,[A_-(x),\ S_a(x;\lambda)]. \qquad (143a)$$

We find the explicit form of $\chi^{-1}(x;\lambda)$ as the solution of the corresponding equation for the coefficient functions $\eta^{(k)}(x)$ of the Laurent expansion of χ^{-1}:

$$\partial_\pm\eta^{(k+1)}(x)=\pm\partial_\pm\eta^{(k)}(x)\mp\eta^{(k)}(x)\,A_\pm.$$

Note that the transformations with generators $S_a(x;\lambda)$ satisfying both equations of (143) are a symmetry only on the extremals. As was shown in Refs. 16 and 17, if it is required that $S_a(x;\lambda)$ satisfy only one of the equations of (143), then we are dealing with a symmetry for arbitrary field configurations.

Indeed, suppose that S_a satisfies only one equation of (143). Then

$$\delta_aS=\int d^2x\,\mathrm{tr}\,\{A^+(x)\,\partial_+S_a(x;\lambda)+A^-(x)\,\partial_-S_a(x;\lambda)\}$$
$$=\varepsilon^{\mu\nu}\int d^2x\partial_\mu\,\mathrm{tr}\left\{[A_\nu,\ S_a]+\left(\frac{1}{\lambda}-\lambda\right)\chi^{-1}\partial_\nu\chi T_a\right\}.$$

Therefore, in this case too the transformations (79) are a symmetry of the action (in the weak sense) for arbitrary field configurations.

To investigate the group structure of the transformation (79) with generators $S_a(x;\lambda)$, we consider the commutator of two infinitesimal transformations:

$$g(x)\,(U_1^aU_2^b-U_2^bU_1^a)=g(x)\,\{S_a(x;\lambda)\,S_b(x;\tau)$$
$$-S_b(x;\tau)\,S_a(x;\lambda)+\delta_aS_b(x;\tau)-\delta_bS_a(x;\lambda)\}. \qquad (144)$$

Here, $\delta_aS_b(x;\tau)$ are the transformations of the generator $S_b(x;\tau)$ under the transformation $U_1=I+S_a(x;\lambda)\delta\omega_a^1$, i.e.,

$$\delta_aS_b(x;\tau)\,\delta\omega_a^1=S_b(g+gS_a\delta\omega_a^1)-S_b(g).$$

To calculate the commutator (144), it is necessary to find $\delta_aS_b(x;\tau)$. For this purpose, we obtain from (143) the following equation for δ_aS_b:

$$(1-\tau)\,\partial_+\delta_aS_b(x;\tau)=\tau\delta_a([A_+,\ S_b(x;\tau)])$$
$$\times\frac{\tau}{1-\lambda}[[A_+,\ S_a(x;\lambda)]\,S_b(x;\tau)]+\tau\,[A_+,\ \delta_aS_b(x;\tau)], \qquad (145)$$

where we have used the gauge nature of the transformation (82) of A_μ and the Jacobi identity. The solution of Eq. (145) satisfying the boundary condition

$$\delta_aS_b(x;\tau)\,|_{x_+=-\infty}=\delta_aT_b=0$$

has the form

$$\delta_aS_b(x;\tau)=\frac{\tau}{\lambda-\tau}[S_a(x;\lambda)-S_a(x;\tau),\ S_b(x;\tau)]$$
$$=\frac{\tau}{\lambda-\tau}\{[S_a(x;\lambda),\ S_b(x;\tau)]-C_{abc}S_c(x;\tau)\}, \qquad (146)$$

We have here used the fact that

$$[S_a(x;\tau),\ S_b(x;\tau)]=\chi(x;\tau)\,[T_a,\ T_b]\,\chi^{-1}(x;\tau)$$
$$=C_{abc}S_c(x;\tau),$$

where C_{abc} are the structure constants of the original group G. Introducing now the quantity[12]

$$\mathcal{T}_a(\lambda)=\int d^2xg(x)\,\dot{S}_a(x;\lambda)\,\frac{\delta}{\delta g(x)},$$

we obtain

$$[\mathcal{T}_a(\lambda),\ \mathcal{T}_b(\tau)]=C_{abc}\int d^2xg\,\frac{\tau S_c(\tau)-\lambda S_c(\lambda)}{\lambda-\tau}\,\frac{\delta}{\delta g}. \qquad (147)$$

Note that the commutator (147) has the same form as in the cases considered in Ref. 12 when the summation in (145) is over $k\geqslant0$ and in Ref. 18 when over $k\leqslant0$. Expanding both sides of (147) in powers of λ and τ, we find

$$[\mathcal{T}_a^{(k)},\ \mathcal{T}_b^{(m)}]=C_{abc}\mathcal{T}_c^{(k+m)}\quad(k,\ m\geqslant0), \qquad (148)$$

and this is the algebra obtained in Ref. 12. In addition, we obtain the further nonvanishing commutators

$$[\mathscr{T}_a^{(k)},\ \mathscr{T}_b^{(-k)}]=[\mathscr{T}_a^{(-k)},\ \mathscr{T}_b^{(k)}]=C_{abc}\mathscr{T}_c^{(0)}\quad (k=0,\ 1,\ 2\ \ldots); \tag{149a}$$

$$[\mathscr{T}_a^{(-k)},\ \mathscr{T}_b^{(-m)}]=-C_{abc}\mathscr{T}_c^{(-k-m)}\quad (k,\ m=0,\ 1\ \ldots); \tag{149b}$$

$$[\mathscr{T}_a^{(k)},\ \mathscr{T}_b^{(-m)}]=C_{abc}(\Theta(m-k)\mathscr{T}_c^{-(m-k)}-\Theta(k-m)\mathscr{T}_c^{(k-m)})\quad (k,\ m=1,\ 2\ \ldots);$$

$$[\mathscr{T}_a^{(k)},\ \mathscr{T}_b^{(-)}]=[\mathscr{T}_a^{(0)},\ \mathscr{T}_b^{(-k)}]=0\quad (k=0,\ 1\ \ldots). \tag{149c}$$

Here, $\Theta(k)$ is the Heaviside function: $\Theta(k)=1$ for $k\geqslant 0$ and $\Theta(k)=0$ for $k<0$. The commutation relations (149a) were obtained for the first time in Ref. 12, and also in Refs. 16 and 17, in which a parametric representation was used for the generators $\mathscr{T}_a$. The algebraic structure of the transformations (79) with generators $\mathscr{T}_a^{(-k)}$ was investigated in Ref. 18. Finally, we note that the second series of generator functions (130) gives the same algebra (149) if we require additionally $g(-\infty, x_-)=\mathbf{I}$. This is a consequence of the identity of the differential equations satisfied by the generator functions $\chi_j^{(k)}$ $(j=1,2,3)$. We note also that the transformation $U_2=g^{-1}(x)$ carries the current $A_\mu=g^{-1}\partial_\mu g$ into $-U_2^{-1}A_\mu U_2=g\partial_\mu g^{-1}$, i.e., carries the left-handed current into the right-handed current.

10. GROUP STRUCTURE OF NONLOCAL TRANSFORMATIONS FOR SUPERSYMMETRIC CHIRAL MODELS

Here, we shall merely outline the generalization of the results of the previous section to the supersymmetric case. We introduce the generators

$$\Omega_a(x;\ \theta,\ \lambda)=X(x;\ \theta,\ \lambda)\,T_aX^{-1}(x;\ \theta,\ \lambda), \tag{150}$$

where

$$X(x;\ \theta,\ \lambda)=\sum_{k=-\infty}^{\infty}\lambda^kX^{(k)}(x;\ \theta), \tag{151}$$

and require for $\Omega_a(x;\theta,\lambda)$ fulfillment of the boundary condition (139). Here, $X^{(k)}$ are given by (33) for $k>0$ and by (130) for $k<0$. For $k=0$, we use the combination

$$X^{(0)}=\alpha I+\beta X_3^{(0)}(x;\ \theta), \tag{152}$$

where $X_3^{(0)}$ has the form (125). From the equation

$$(1-\lambda)\,\mathscr{D}_1X(x;\ \theta,\ \lambda)=\lambda\mathscr{A}_1(x;\ \theta)\,X(x;\ \theta,\ \lambda)$$

we obtain the following equation for $\Omega_a(x;\theta,\lambda)$:

$$(1-\lambda)\,\mathscr{D}_1\Omega_a(x;\ \theta,\ \lambda)=\lambda\,[\mathscr{A}_1(x;\ \theta),\ \Omega_a(x;\ \theta,\ \lambda)]. \tag{153}$$

On the extremals, $\Omega_a(x;\theta,\lambda)$ also satisfies the equations

$$(1+\lambda)\,\mathscr{D}_2\Omega_a(x;\ \theta,\ \lambda)=-\lambda\,[\mathscr{A}_2(x;\ \theta),\ \Omega_a(x;\ \theta,\ \lambda)].$$

For arbitrary field configurations, we obtain the variation of the action

$$\delta_aS=-\int d^2xd^2\theta\,\mathrm{tr}\,\{\mathscr{A}^\alpha(x;\ \theta)\,\mathscr{D}_\alpha\Omega_a(x;\ \theta,\ \lambda)\}$$
$$=-\int d^2xd^2\theta\,\mathrm{tr}\left(\mathscr{D}^\alpha\left\{\lambda\left[(\gamma_5\mathscr{A})_\alpha\,\Omega_a+\left(\frac{1}{\lambda}-\lambda\right)X^{-1}\right]\times(\gamma_5\mathscr{D}_\alpha)\,XT_a\right\}\right),$$

and, therefore, the transformations (134) with generators Ω_a satisfying (153) are a symmetry (in the weak sense) of the action for arbitrary field configurations.

As in the ordinary case, we consider the commutator of two infinitesimal transformations (114):

$$\mathscr{G}(x;\ \theta)\,\{U_1(x;\ \theta)\,U_2(x;\ \theta)-U_2(x;\ \theta)\,U_1(x;\ \theta)\}$$
$$=\mathscr{G}(x;\ \theta)\,\{[\Omega_a(x;\ \theta,\ \lambda),\ \Omega_b(x;\ \theta,\ \tau)]+\delta_a\Omega_b(x;\ \theta,\ \tau)-\delta_b\Omega_a(x;\ \theta,\ \lambda)\}\,\delta\omega_a^1\,\delta\omega_b^2.$$

Here, $\delta_a\Omega_b(x;\theta,\tau)$ is obtained as the solution of the differential equation that we find from Eq. (153):

$$(1-\lambda)\,\mathscr{D}_1\,(\delta_a\Omega_b\,(x;\ \theta,\ \tau))$$
$$=\tau\delta_a\,([\mathscr{A}_1(x,\ \theta),\ \Omega_b\,(x;\ \theta,\ \tau)])$$
$$=\frac{\tau}{1-\lambda}\,[[\mathscr{A}_1(x;\theta),\Omega_a\,(x;\theta,\lambda)],\Omega_b\,(x;\theta,\tau)]$$
$$+\tau\,[\mathscr{A}_1\,(x;\theta),\ \Omega_b\,(x;\theta,\tau)]$$

for the boundary condition

$$\delta_a\Omega_b\,(x;\ \theta,\ \tau)\,|_{x_+=-\infty}=\delta_aT_b=0.$$

This solution has the form

$$\delta_aS_b\,(x;\ \theta,\ \tau)=\frac{\tau}{\lambda-\tau}\,\{[\Omega_a\,(x;\ \ \theta,\ \lambda),\ \Omega_b\,(x;\ \theta,\ \tau)]-C_{abc}\Omega_c\,(x;\ \theta,\ \tau)\},$$

and therefore has the same form as (146). Then denoting

$$\mathscr{T}_a(\lambda)=\int d^2xd^2\theta\mathscr{G}(x;\ \theta)\,\Omega_a(x;\ \theta,\ \lambda)\,\frac{\delta}{\delta\mathscr{G}(x;\ \theta)}$$

and expanding the commutator $[\mathscr{T}_a(\lambda),\mathscr{T}_b(\tau)]$ in powers of λ and τ, we obtain the algebra (148) and (149).

Note that if we choose $\alpha=1$, $\beta=0$ in (152) and $\chi^{(k)}=0$ for $k<0$ in (151), we obtain the algebra (148) found in Refs. 23 and 24.

[1)]Note that all the propositions proved above are also valid if (84) is replaced by $\widetilde{K}_\mu^j(x)=[A^\nu(x),\tilde{\chi}_{\mu\nu}]$.

[2)]We say that the sequences $\{\chi_m^{(k)}\}$ and $\{\chi_l^{(k)}\}$ are linearly independent if each of them contains not less than one element $\chi_{l(m)}^{(p)}$ linearly independent of all the elements of the other sequence.

[3)]I.e., not related by multiple application of (116).

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