Ground-state structure and properties of the Green's functions of colorless operators in the Schwinger model

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Explicit solution of the Schwinger model in the transverse gauge makes it possible to study the ground-state structure and the properties at both short and large distances of gauge-invariant operators and their products. The possibility of using in the model sum rules and the Wilson operator expansion is analyzed.

INTRODUCTION

Knowledge of the ground state of the system and the Green's functions of colorless operators has acquired great importance in connection with the description of hadron physics in the framework of QCD.

Weighty arguments have suggested in recent years that the requirement of gauge invariance leads to a complicated structure of the ground state in gauge field theories (see Refs. 1–4).[1)] Some hopes of solving the $U(1)$ problem,[5,6] describing quark confinement, and explaining the spontaneous breaking of chiral invariance[7,8] are associated with a complicated structure of the vacuum.

In unified models of elementary particles, the structure of the vacuum leads to the possibility of nonconservation of the baryon number,[9–11] which, as was shown by Rubakov,[12] will be strong in the presence of a magnetic monopole and leads to a significant enhancement of the proton decay probability (Rubakov effect).

The effects associated with the structure of the vacuum are basically studied in two directions. The first of them is based on the use of the semiclassical approximation in field theory, this being most readily realized in the framework of functional integration. The second direction is associated with the use of the Wilson expansion,[13] which makes it possible to take into account semiphenomenologically important properties of the vacuum such as the existence in it of quark, gluon, and scalar condensates.[14,15]

However, the use of these methods in QCD is not sufficiently well founded, and further advance in understanding the essence of the problems listed above requires a more detailed study of the ground-state structure.

Of great interest in this connection are exactly solvable models of quantum field theory, by means of which one can test the approximate methods used in realistic quantum field-theory models. Such is the Schwinger model[16]—two-dimensional massless quantum electrodynamics. This model has been studied from various points of view in Refs. 2–4 and 17–20.

In the present review, the explicit solution of the Schwinger model in the transverse gauge is used to consider questions related to the ground-state structure and the short- and long-distance properties of gauge-invariant operators and their products. The applicability of sum rules in the model is analyzed.

The paper is arranged as follows. In Sec. 1, we find an operator solution that expresses the initial fields $\psi(x)$ and $A_\mu(X)$ in terms of free pseudoscalar fields $\eta(x)$ and $\Sigma(x)$ and a fermion field $\psi_0(x)$.

When this solution is taken into account, Maxwell's equations reduce to a constraint in the form $L_\mu = 0$, where $L_\mu = \bar{\psi}_0\gamma_\mu\psi_0 - (1/\sqrt{\pi})\varepsilon_{\mu\nu}\partial^\nu\eta$.

The solution of this equation distinguishes in the extended space $\mathscr{H} = \mathscr{H}(\Sigma)\otimes\mathscr{H}(\eta)\otimes\mathscr{H}(\psi_0)$ the physical space $\mathscr{H}_{ph}$.

As a result, we construct a ground state of the system that does not have definite fermion and chiral numbers and is characterized by two arbitrary parameters $\theta_\pm$ $(0\leqslant\theta_\pm<2\pi)$.

The excitations of the system are described by a single neutral field $\Sigma(x)$, this signifying that there are no charged states in the spectrum.

In the model, the condition of confinement is formulated as the vanishing in physical space of the electric-charge operator. This condition is a direct consequence of Gauss's theorem and the fact that the transverse degree of freedom of the gauge field A_μ acquires a mass dynamically.

In Sec. 2, we establish a connection between the constraint equation and the invariance of the physical state space with respect to the local gauge transformations that remain after the Lorentz gauge has been imposed.

The construction of the ground state forces us to extend the class of gauge transformations. We must, in fact, include gauge functions which do not decrease at infinity.

In Sec. 3, we describe in detail the physical picture.

The operator solution of the model makes it possible to establish an exact connection between the physical field $\Sigma(x)$ and the gauge-invariant currents constructed from the fermion and gauge fields.

As a result, the mechanism of nonconservation of the axial current is revealed. The breaking of the axial symmetry is manifested in the appearance of a quark condensate, $\langle\bar{\psi}\psi\rangle\neq 0$, and in the acquisition by the pseudoscalar Σ particle of a nonzero mass [solution of the $U(1)$ problem].

A characteristic feature of the model is the possibility of regarding the Σ particle either as a quark–antiquark state or as pseudoscalar "gluonium."

Such treatment is based on operator identities that in the low-energy limit establish a direct proportionality between the pseudoscalar currents and the field of the physical particle Σ.

The significance of these approximate operator identities is demonstrated by the calculation of the two-point

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Green's function of the pseudoscalar current formed from the quark fields.

It is shown that in the large-distance limit the exact Green's function is well approximated by the propagator of the Σ field, this particle appearing in the spectral density as an infinitely narrow resonance.

In Sec. 4, we study the behavior of the Green's functions at short distances. Having an exact operator solution of the model and, thus, an operator representation of all possible products in terms of free fields, we have analyzed the different forms of the operator expansion in the model. In general, the coefficient functions depend nonanalytically on e^2. By a redefinition of the operators, the nonanalyticity can be eliminated, and in one of the bases, in the leading logarithmic approximation, there is a correspondence between the exact operator expansion and Wilson's perturbation-theory expansion. Identity of these expansions is possible only if one knows the connection between the ultraviolet subtraction point for the operators and the infrared subtraction point for the coefficient functions.

In connection with the widespread use of finite-energy sum rules[14,21,22,29] in QCD, the accuracy with which they hold in the Schwinger model is analyzed in the present review.

Since the resonance in this model is formed in the strong-coupling region, where the effective coupling constant is near unity, $\alpha_s \sim 1$, the sum rules are naturally not well satisfied in the momentum space.

We have found that in the x space the resonance lies in the region of asymptotic freedom ($\alpha_s \sim 1/4$), and therefore the use of the sum rules in the x space has made it possible to calculate the parameters of the resonance with sufficient accuracy.

Another advantage of the sum rules in the x space is that they make it possible to take into account the contributions from the contact terms.

1. OPERATOR SOLUTION OF THE SCHWINGER MODEL

The Lagrangian of the Schwinger model is[16]

$$\mathcal{L} = -\frac{1}{4} F^2_{\mu\nu} + i\bar{\psi}\gamma_\mu (\partial_\mu - ieA_\mu)\psi, \qquad (1)$$

where A_μ and ψ are the electromagnetic and fermion fields, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\mu = 0, 1$, and the two-dimensional γ matrices are taken in the form

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The equations of motion that follow from the Lagrangian (1) are

$$\partial_\mu F_{\mu\nu} = -eJ_\nu; \qquad (2)$$

$$i\gamma_\mu (\partial_\mu - ieA_\mu)\psi = 0. \qquad (3)$$

The current J_μ is defined gauge-invariantly[16]:

$$\left. \begin{aligned} J_\mu(x) &= \frac{1}{2} \lim_{\varepsilon \to 0,\ \varepsilon^2 \neq 0} \{J_\mu(x|\varepsilon) + J_\mu(x|-\varepsilon)\}; \\ J_\mu(x|\varepsilon) &= \bar{\psi}(x)\gamma_\mu \exp\left\{-ie\int_x^{x+\varepsilon} A_\nu d\xi^\nu\right\}\psi(x+\varepsilon)\cdot \end{aligned} \right\} \qquad (4)$$

In this paper, we use the transverse gauge

$$\partial_\mu A_\mu = 0. \qquad (5)$$

In this gauge, we impose on the fields ψ and A the equal-time commutation relations

$$\{\psi_i(x^1, t), \psi_j^+(y^1, t)\} = \delta_{ij}\delta(x^1 - y^1),\ i, j = 1, 2;$$
$$[\partial_0 A_1(x^1, t), A_1(y^1, t)] = -i\delta(x^1 - y^1);$$
$$[\partial_0 A_0(x^1, t), \partial_0 A_1, (y^1, 1)] = i\frac{\partial}{\partial x^1}\delta(x^1 - y^1),$$

the remaining commutators vanishing.

The operator solution of the Schwinger model in the transverse gauge has the form[2]

$$\left. \begin{aligned} A_\mu(x) &= \frac{\sqrt{\pi}}{e}\varepsilon_{\mu\nu}\partial^\nu(\Sigma(x) + \eta(x)); \\ \psi(x) &= K{:}\ \exp\{-i\sqrt{\pi}\gamma_5(\Sigma(x) + \eta(x))\}{:}\ \psi_0(x), \end{aligned} \right\} \qquad (6)$$

where Σ is a free pseudoscalar field with mass $m = e/\sqrt{\pi}$, and η is a free massless field quantized with a negative metric, i.e.,

$$[\partial_0\eta(x^1, t), \eta(y^1, t)] = +i\delta(x^1 - y^1);$$

ψ_0 is a free massless fermion field (the free scalar and fermion fields in two-dimensional space–time are described in detail in Ref. 23). The negative-frequency part of the commutator function of the field η is

$$D^- = -\frac{i}{4\pi}\ln(-\mu^2 x^2 + i\varepsilon x_0),$$

where μ is an arbitrary parameter with the dimensions of mass.[23] The coefficient K in (6) can be expressed in terms of this parameter as follows:

$$K = (e\gamma/2\mu\sqrt{\pi})^{1/4},$$

where γ is Euler's constant ($\gamma \approx 1.781$).

The ansatz (6) satisfies the condition (5) and Eq. (3) identically, and Eq. (2) reduces to the operator equation

$$L_\mu = 0, \qquad (7)$$

where

$$L_\mu = j_\mu + \frac{1}{\sqrt{\pi}}\varepsilon_{\mu\nu}\partial_\eta^\nu, \qquad (8)$$

and $j_\mu = {:}\bar{\psi}_0(x)\gamma_\mu\psi_0(x){:}$, the free-fermion current, satisfies the equation $\Box j_\mu = 0$.

Since the original operators A_μ and $\psi(x)$ are expressed by the relation (6) in terms of the free operators Σ, η, ψ_0, we have at our disposal the Hilbert state space

$$\mathcal{H} = \mathcal{H}(\Sigma) \otimes \mathcal{H}(\eta) \otimes \mathcal{H}(\psi_0).$$

Since the fields η and ψ_0 are independent in the whole of this space, the relation (7) is not satisfied in $\mathcal{H}$, but there must exist a subspace $\mathcal{H}_{ph} \subset \mathcal{H}$, in which the condition (7) is satisfied identically. It is in this subspace that the ansatz (6) is a solution of the equations of motion (2) and (3). What can be said directly about the space $\mathcal{H}_{ph}$?

The original Lagrangian (1) is invariant with respect to global transformations of the form

$$\psi(x) \to e^{i\alpha}\psi(x); \qquad (9)$$

$$\psi(x) \to e^{i\alpha'\gamma_5}\psi(x). \qquad (10)$$

To the group of transformations (9) there corresponds the locally conserved current $J_\mu = \bar{\psi}\gamma_\mu\psi$, and to the group of chiral transformations (10) the current $J^5_\mu = \overline{\psi}\gamma_\mu\gamma_5\psi$. These currents are related by

$$J^5_\mu = \varepsilon_{\mu\nu}J^\nu. \tag{11}$$

To the current J_μ there corresponds the conserved electric charge $Q = -\int J_0 dx^1$, which in what follows (by analogy with QCD) we shall call color, and to the current J^5_μ there corresponds the chiral charge $Q^5 = \int J^5_0 dx^1$ which is conserved at the classical level.

The calculation of J_μ in accordance with (4) leads[2,24] to the expression

$$J_\mu = j_\mu + \frac{e}{\pi} A_\mu \tag{12}$$

or

$$J_\mu = L_\mu + \frac{1}{\sqrt{\pi}} \varepsilon_{\mu\nu}\partial^\nu \Sigma,$$

from which it can be seen that J_μ is conserved at the quantum level, $\partial_\mu J_\mu = 0$, and in the physical space $\mathscr{H}_{\text{ph}}$, determined by the condition $L_\mu = 0$, this operator has the form

$$J_\mu = \frac{1}{\sqrt{\pi}} \varepsilon_{\mu\nu}\partial^\nu \Sigma. \tag{13}$$

It follows from the upper equation that the color operator in $\mathscr{H}_{\text{ph}}$ is the zeroth operator

$$Q = -\int J_0\, dx^1 = \frac{1}{\sqrt{\pi}} \int \partial_1 \Sigma\, dx^1 = 0. \tag{14}$$

The integral is zero, since the field $\Sigma(x)$ is massive and therefore decreases at large distances.

The identity (14) can be obtained without using the explicit form (13) of the current operator. Integrating the Maxwell equation (2) over x^1, we obtain

$$eQ = \int \partial_1 F^{10} dx^1 = F^{10}(\infty) - F^{10}(-\infty).$$

This equation expresses Gauss's theorem in the two-dimensional space.

If it is assumed that the gauge fields have acquired mass in some manner, this theorem immediately yields[2]

$$eQ \equiv 0\,.$$

The essence of this result is that $\mathscr{H}_{\text{ph}}$ is a space of colorless states and in the physical region the model has the confinement property.

The physical aspect of the result is also clear—the electric field, decreasing exponentially at large distances, can only produce a system of charges that is overall neutral.

Using Eq. (11), we obtain the following expression for the axial current:

$$J^5_\mu = \varepsilon_{\mu\nu} L^\nu(x) + \frac{1}{\sqrt{\pi}} \partial_\mu \Sigma.$$

Thus, at the quantum level this current is not conserved:

$$\partial_\mu J^5_\mu = -\frac{m^2}{\sqrt{\pi}} \Sigma(x), \tag{15}$$

this being a manifestation of the Adler-Bell-Jackiw anomaly in the model.

Thus, the group (10) is a symmetry group of the Lagrangian, but the quantum system does not possess this symmetry, and therefore Q_5 is not a conserved quantum number.

We now turn to the construction of the physical state space.

The operator equation (7) must be understood in the sense of the vanishing in the physical space of all the matrix elements of the operator L_μ, i.e., $\langle\Phi|L_\mu|\Phi'\rangle = 0$ for all states $|\Phi\rangle$ and $|\Phi'\rangle$ in the physical space $\mathscr{H}_{\text{ph}}$.

Thus, in the space $\mathscr{H}$ this equation is a constraint which imposes restrictions on the state vectors.

They must be imposed in the form

$$L^-_\mu \mid \Phi\rangle = 0. \tag{16}$$

Here, L^-_μ is the negative-frequency part of the operator L_μ.

For all vectors $|\Phi\rangle$ satisfying the condition (16) the fulfillment of Eq. (7) is guaranteed:

$$\langle\Phi \mid L_\mu \mid \Phi'\rangle \equiv 0. \tag{17}$$

We define the vacuum in the space $\mathscr{H}$ by means of the formula

$$\Sigma^- \mid 0\rangle = \eta^- \mid 0\rangle = \psi_0^- \mid 0\rangle = 0.$$

Then $|0\rangle$ is one of the solutions of Eq. (16). The remaining vectors $|\Phi\rangle$ can be obtained from $|0\rangle$ by applying to it operators $\hat{0}$ that do not carry $|0\rangle$ out of the physical space. For this, they must commute with $L^\pm_\mu$:

$$[\hat{0},\ L^\pm_\mu] = 0. \tag{18}$$

One can show that these operators can be constructed from the operators $\Sigma(x)$, $\sigma(x)$, Q_F, $\widetilde{Q}_F$.

Here

$$Q_F = -\int j_0\, dx^1;\ \widetilde{Q}_F = \int j_1\, dx^1 \tag{19}$$

are the operators of the "fermion" number and "chirality" of the free fermions, respectively (see the Appendix), and $\sigma(x)$ is defined as[3]

$$\sigma_\pm(x) = \left(\frac{2\pi}{\mu}\right)^{1/2} \left(\frac{1 \pm \gamma_5}{2}\right) : \exp\{-i\sqrt{\pi}\gamma_5\eta(x) - i\sqrt{\pi}\widetilde{\eta}(x)\} : \psi_0(x) \exp\left\{i\frac{\pi}{2}\widetilde{Q}_F\right\}. \tag{20}$$

By direct calculation it can be verified[3] that these operators have the properties

$$\left.\begin{aligned} &[\sigma_\pm(x),\ \sigma_\pm(y)] = [\sigma_\pm(x),\ \sigma_\mp(y)] = 0;\\ &[\sigma_\pm(x),\ \sigma^+_\pm(y)] = [\sigma_\pm(x),\ \sigma^+_\mp(y)] = 0;\\ &\sigma_\pm(x)\, \sigma^+_\pm(x) = 1. \end{aligned}\right\} \tag{21}$$

The commutation relations of the unitary operators $\sigma_\pm$ with the charge operators have the form

$$[\sigma_\pm Q_F] = -\sigma_\pm,\ [\sigma_\pm \widetilde{Q}_F] = \mp\sigma_\pm. \tag{22}$$

In addition, $\sigma_\pm(x)$ commute in the physical space with the generators of translations and are therefore constant unitary operators.

Indeed, defining P_μ by

$$P_\mu = \int T_{\mu 0}(x)\, dx^1,$$

$$T_{\mu\nu}(x) = T_{\mu\nu}(\Sigma) + T_{\mu\nu}(\psi_0) - T_{\mu\nu}(\eta),$$

we obtain

$$[P_\mu \sigma_\pm (x)] = -\pi \, (L_\mu (x) \pm \varepsilon_{\mu\nu} \, L^\nu (x)) \, \sigma_\pm (x). \tag{23}$$

It follows from the condition $L_\mu = 0$ that $\sigma_\pm$ do not depend on x.

Since the field Σ comnutes with $\sigma_\pm$, Q_F, $\widetilde{Q}_F$, the physical space is the product of the Fock space of the field Σ and the space of the operators $\sigma_\pm$, Q_F, $\widetilde{Q}_F$ with the commutation relations (21) and (22):

$$\mathscr{H}_{\text{ph}} = \mathscr{H} (\Sigma) \otimes \mathscr{H} (\sigma, \; Q_F, \; \widetilde{Q}_F). \tag{24}$$

Since the Hamiltonian commutes with Q_F, $\widetilde{Q}_F$, these operators and the Hamiltonian have a common set of eigenvectors.

One of these vectors is $|0\rangle$:

$$Q_F \; |0\rangle = \widetilde{Q}_F \; | \; 0 \rangle = 0. \tag{25}$$

It follows from the commutation relations (22) that the operators $\sigma_\pm$ change the fermion and chiral numbers (Q_F and $\widetilde{Q}_F$) by ± 1. Thus, the basis in the space $\mathscr{H}(\sigma, Q_F, \widetilde{Q}_F)$ has the form

$$| \; n^+, \; n^- \rangle = (\sigma_+)^{n+} \, (\sigma_-)^{n-} \; | \; 0 \, \rangle. \tag{26}$$

Here, $n^\pm$ are arbitrary integers.

The vectors $|n^+, n^-\rangle$ have a definite fermion number and chirality:

$$\left.\begin{aligned} Q_F \, |n^+, \; n^- \rangle &= (n^+ + n^-) \, n^+, \; n^- \rangle \, ; \\ \widetilde{Q}_F \, |n^+, \; n^- \rangle &= (n^+ - n^-) \; |n^+, \; n^- \rangle \, . \end{aligned}\right\} \tag{27}$$

Since the state $|0, 0\rangle$ has zero energy, and the operators $\sigma_\pm$ commute with the Hamiltonian, all the vectors $|n^+, n^-\rangle$ (26) have zero energy. Thus, the ground state of the system is degenerate with respect to the fermion number and chirality.

As the ground state, one can choose any linear superposition of the states $|n^+, n^-\rangle$.

In particular, it can be chosen in such a way as to diagonalize the Hamiltonian and the operators $\sigma_\pm$ simultaneously. This is possible, since the Hamiltonian commutes with $\sigma_\pm$.

Since the operators $\sigma_\pm$ are unitary, the equation for the eigenvalues has the form

$$\sigma_\pm \; | \; \theta_+, \; \theta_- \rangle = \exp \, (-\mathrm{i}\theta_\pm) \; | \; \theta_+, \; \theta_- \rangle, \quad 0 \leqslant \theta_\pm < 2\pi. \tag{28}$$

This equation is solved by the vector

$$|\theta_+, \; \theta_- \rangle = \sum_{n^+, \, n^-} \exp \, \{\mathrm{i}n^+\theta_+ + \mathrm{i}n^-\theta_-\} \; | \; n^+, \; n^- \rangle, \tag{29}$$

the so-called θ vacuum.

It follows from the relations (22) that the different θ vacua are related by the unitary operators

$$U_F \, (\alpha) = \exp \, (\mathrm{i}\alpha Q_F); \; \widetilde{U}_F \, (\alpha') = \exp \, (\mathrm{i}\alpha' \widetilde{Q}_F) \tag{30}$$

in accordance with

$$\begin{aligned} &| \; Q_+ + \; \alpha + \; \alpha', \; \theta_- + \; \alpha - \alpha' \, \rangle \\ &= U_F \, (\alpha) \; \widetilde{U}_F \, (\alpha') \; | \; \theta_+, \; \theta_- \rangle. \end{aligned} \tag{31}$$

It follows from this that all the θ vacua are equivalent, i.e., by means of unitary operators they can be obtained from a single vacuum, for example, from $|\theta_+, \theta_-\rangle|_{\theta\pm=0}$.

Since there are no transitions between the different θ vacua, one can take as the vacuum of the system one of them with a fixed value of the parameters $\theta_\pm$.

The existence of the unitary operators (30) is due to the symmetry of the Lagrangian (1) under phase transformations of the form

$$\psi \, (x) \rightarrow \exp \, (\mathrm{i}\alpha) \, \psi \, (x), \;\; \psi \, (x) \rightarrow \exp \, (\mathrm{i}\alpha'\gamma_5) \, \psi \, (x). \tag{32}$$

Indeed, in the space $\mathscr{H}$ the operators (30) realize precisely these transformations:

$$U_F \psi \, (x) \; U_F^+ = \exp \, (\mathrm{i}\alpha) \; \psi \, (x);$$

$$\widetilde{U}_F \psi \, (x) \; \widetilde{U}_F^+ = \exp \, (\mathrm{i}\alpha'\gamma_5) \; \psi \, (x).$$

Note that the vacuum (29) is not invariant with respect to these transformations. Indeed, it follows from (31) that the vacuum is changed under these transformations. This is due to the fact that the vacuum does not have definite Q_F and $\widetilde{Q}_F$ numbers:

$$Q_F \; | \; \theta_+, \; \theta_- \rangle \neq c_1 \; | \; \theta_+, \; \theta_- \rangle;$$

$$\widetilde{Q}_F \; | \; \theta_+, \; \theta_- \rangle \neq c_2 \; | \; \theta_+, \; \theta_- \rangle. \tag{33}$$

Thus, the choice of the physical vacuum with a given value of $\theta_\pm$ leads to breaking of the invariance with respect to the transformations (32).

The symmetry breaking is manifested in the fact that the vacuum expectation value of the operator $\sigma_\pm$, which is noninvariant with respect to phase transformations,

$$U_F \sigma_\pm U_F^+ = \exp \;\; (\mathrm{i}\alpha) \; \sigma_\pm; \;\; \widetilde{U}_F \sigma_\pm \widetilde{U}_F^+ = \exp \; (\pm \; \mathrm{i}\alpha') \; \sigma_\pm \, ,$$

is nonzero:

$$\langle \theta_+, \; \theta_- \; | \; \sigma_\pm \; | \; \theta_+, \; \theta_- \rangle = \exp \, (-\mathrm{i}\theta_\pm). \tag{34}$$

In Sec. 3, we shall show that the symmetry breaking is also responsible for the appearance of a quark condensate: $\langle \overline{\psi}\psi \rangle \neq 0$.

Thus, the symmetry (32) is broken, but Goldstone particles corresponding to the breaking (33) of this invariance are not present in the physical spectrum. This is the case because the conditions of Goldstone's theorem are not satisfied in the present case. Indeed, if the theorem is to hold, conserved currents must correspond to the transformations (32). These currents are j_μ and j_μ^5, the currents of the free fermions, but they do not commute with L_μ and therefore do not belong to the physical space. It is at this point that the conditions of Goldstone's theorem are not satisfied. Note that instead of the currents j_μ and j_μ^5 we cannot choose J_μ and J_μ^5, which correspond through Noether's theorem to the transformations (32), since the current J_μ^5 is not conserved, and the color symmetry generated by the current J_μ is not broken (in the physical sector $Q = 0$).

2. GAUGE TRANSFORMATIONS

In this section, we trace the connection between the constraint equation $L_\mu = 0$ and the requirement of gauge invariance of the theory.

We consider the gauge-transformation operator[4,3]

$$U \, [\alpha_\pm] = \exp \, \{-\mathrm{i} \int dx^1 \alpha_\pm \, (x) [\, L^0 \, (x) \pm L_1 \, (x)]\}, \tag{35}$$

which depends on functions $\alpha_\pm \, (x^\pm)$ arbitrary apart from satisfying the equation $\Box\alpha_\pm = 0$.

It follows from the representation (6), (8) that A_μ and ψ transform under U as

$$\left.\begin{aligned} U\psi(x)U^{+} &= \exp(i\alpha_{\pm}(x))\psi(x); \\ UA_{\mu}(x)U^{+} &= A_{\mu}(x)+\frac{1}{e}\partial_{\mu}\alpha_{\pm}(x). \end{aligned}\right\} \quad (36)$$

These transformations leave the Lorentz condition (5) invariant.

By means of the operator $U[\alpha_{\pm}]$, the condition (18) can be rewritten in the form

$$U\hat{0}U^{+}=\hat{0}. \quad (37)$$

This means that the physical operators are invariant with respect to the gauge transformations, and Eq. (16) ensures invariance of the physical state vectors with respect to these transformations.

In deriving the equation $Q=0$ (14), we made essential use of the constraint $L_\mu=0$, which guarantees the gauge invariance of the theory. Therefore, in the model the local invariance of the operators and the state vectors is intimately related to the confinement phenomenon.

All gauge transformations with a gauge function $\alpha(x)$ satisfying the condition $\Box\alpha=0$ are amenable to classification in terms of the behavior of the function $\alpha(x)$ at infinity.

The functions $\alpha(x)$ can be divided into classes, each characterized by two numbers[3]:

$$n_{\pm}[\alpha]=-\frac{1}{2\pi}\int_{-\infty}^{\infty}dx^{1}\,[\partial_{0}\alpha\mp\partial_{1}\alpha]. \quad (38)$$

The significance of these numbers can be clarified by noting that the function $\alpha(x)$ can always be represented in the form

$$\alpha(x)=\alpha_{-}(x_{-})+\alpha_{+}(x_{+}).$$

Then $n_{\pm}(\alpha)$ are determined by the asymptotic behavior of this function at infinity,

$$n_{\pm}(\alpha)=-\frac{1}{\pi}[\alpha_{\mp}(\infty)-\alpha_{\mp}(-\infty)].$$

It follows from the gauge invariance of the current $J_\mu(x)$ (12) that under gauge transformations the current j_μ has the transformation law

$$Uj_{\mu}U^{+}=j_{\mu}-\frac{1}{\pi}\partial_{\mu}\alpha,$$

whence

$$UQ_{\pm}U^{+}=Q_{\pm}-n_{\pm}(\alpha), \quad (39)$$

where $Q_{\pm}=\frac{1}{2}(Q_{F}\pm\widetilde{Q}_{F})$

It follows from the fact that the spectrum of the operators $Q_\pm$ is integrable and from the relation (39) that $n_\pm(\alpha)$ are integers.

Thus, at the quantun level there exist only operators $U(\alpha)$ for which $n_\pm(\alpha)$ are integers.

It can also be seen from (39) that the operators U carry a charge $Q_\pm$ equal to $n_\pm(\alpha)$.

The operator $U(\alpha)$ (35) has zero topological number $n_\pm=0$, being defined only for $\alpha(x)$ that decrease at infinity.

Indeed, calculating the commutator

$$\left[\int dx^{1}\,\alpha_{\pm}(x)\,(L^{0}+L_{1}),\ \eta^{+}(y)\right]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dx^{+}\alpha_{+}(x_{+})}{x_{+}-y_{+}-i\varepsilon}, \quad (40)$$

we see that it is defined only if $\alpha_{+}(x_{+})\to 0$ as $x_{+}\to\infty$. We shall clarify the meaning of the unitary (at one point) operators $\sigma(x)$ (20) in the space $\mathscr{H}$.

The action of these operators on the fields A_μ and ψ is expressed[3] by

$$\sigma_{\pm}(y)A_{\mu}(x)\sigma_{\pm}^{+}(y)=A_{\mu}(x)+\frac{1}{e}\frac{\partial}{\partial x^{\mu}}F_{\pm}(x\,|\,y),$$

$$\sigma_{\pm}(y)\psi(x)\sigma_{\pm}^{+}(y)=\exp\{iF_{\pm}(x\mid y)\}\psi(x), \text{ where } F_{\pm}(x\mid y)=-\pi\theta(x_{\mp}-y_{\mp}).$$

Thus, the unitary operators $\sigma_\pm(y)$ are operators of singular gauge transformations with gauge functions $\alpha(x)=F_\pm$.

Generally speaking, the gauge functions $F_\pm$ do not decrease at infinity and therefore have nonzero topological numbers:

$$n_{+}[F_{+}]=1,\ n_{-}[F_{+}]=0,\ n_{+}[F_{-}]=0,\ n_{-}[F_{-}]=1.$$

We note that the θ vacuum is an eigenstate of the operators $\sigma_\pm$ in the physical space; this guarantees stability of the θ vacuum with respect to such gauge transformations.

Since σ commutes with L_μ, the physical vectors (16) will be invariant with respect to all possible gauge transformations. The physical operators $\Sigma, \sigma, Q_F, \widetilde{Q}_F$ commute with L_μ by construction and are therefore invariant with respect to topologically trivial gauge transformations. With regard to topologically nontrivial gauge transformations, the operators σ and Σ are invariant, while the operators $Q_F, \widetilde{Q}_F$ acquire integral *c*-number corrections (39) and are therefore not invariant.

Since Q_F and $\widetilde{Q}_F$ connect different θ vacua in accordance with (31), fixing of the parameters $\theta_\pm$ essentially eliminates these operators from our study.

3. "LOW-ENERGY" RELATIONS

It follows from the foregoing that the Schwinger model, formulated as a gauge model of quantum field theory in the language of gauge, A_μ, and quark, ψ, fields, is ultimately equivalent to the quantum theory of the free massive pseudoscalar field Σ.

The Fock space of this field is a space of colorless states, and this means that the model possesses the confinement property expressed by the operator identity

$$Q=0, \quad (41)$$

where Q is the color operator.

We emphasize once more that the absence of charged states in the physical spectrum is intimately related to the restoration of local gauge invariance of the state vectors. This invariance follows directly from the equations of motion in the form $L_\mu=0$, where L_μ are the generators of the residual gauge transformations. It follows from this equation that the only operators with physical meaning are the ones that commute with L_μ, i.e., the gauge-invariant operators. It therefore is in particular meaningless to consider the operator of the quark field $\psi(x)$ in the physical space. In addition, from the condition $L_\mu = 0$ there follows the expression (13) for the current operator of the color charge:

$$J_\mu(x) = \frac{1}{\sqrt{\pi}} \varepsilon_{\mu\nu}\partial^\nu \Sigma, \tag{42}$$

from which (41) follows directly.

The operator of the color-charge density is nonzero, but the charge density in the Σ boson is identically zero:

$$\langle \Sigma \mid J_\mu(x) \mid \Sigma \rangle = 0.$$

We now consider briefly the physical meaning of the Σ boson. The properties of the Σ boson are intimately related to the chiral properties of the model. The original Lagrangian of the model is invariant with respect to chiral transformations of the form

$$\psi \to \exp(-i\alpha\gamma_5)\psi. \tag{43}$$

The operator of the chiral current in the physical space has the form

$$J^5_\mu = \frac{1}{\sqrt{\pi}} \partial_\mu \Sigma \tag{44}$$

and, accordingly, the operator of the chiral charge is

$$Q_5(t) = \frac{1}{\sqrt{\pi}} \int \partial_0 \Sigma(x^1, t)\, dx^1. \tag{45}$$

The equal-time commutator of this charge with the field Σ is

$$[Q_5(t)\,\Sigma(x^1, t)] = \frac{1}{i\sqrt{\pi}}. \tag{46}$$

It follows from this that the operator

$$U_5(t) = \exp(i\alpha Q_5(t)) \tag{47}$$

shifts Σ by a constant amount:

$$U_5(t)\,\Sigma(x^1, t)\,U_5^+(t) = \Sigma(x^1, t) + \frac{\alpha}{\sqrt{\pi}}. \tag{48}$$

In the extended space $\mathscr{H}$, the operator $U_5(t)$ realizes a γ_5 rotation of the quark fields,

$$U_5(t)\psi(x^1, t)\,U_5^+(t) = \exp(-i\alpha\gamma_5)\,\psi(x^1, t). \tag{49}$$

This follows directly from the operator solution (6) and the form (11), (12) of the operator $Q_5(t)$ in the extended space. Since the charge Q_5 depends on the time, the Hamiltonian and Q_5 do not commute and therefore do not have a common system of eigenvectors.

It follows from this that the vacuum and the Σ boson do not have a definite Q_5 number. For example,

$$Q_5(t)\,|\text{vac}\rangle = i\sqrt{m}\exp(i\,m\,t)\,a_0^+\,|\text{vac}\rangle, \tag{50}$$

where a_0^+ is the operator of creation of a Σ boson with zero momentum.

In addition, the symmetry with respect to the shifts (48) of the field Σ is manifestly broken, since this field is massive, and the mass term in the Lagrangian breaks the symmetry (48).

The appearance of the nonvanishing mass of the Σ boson is what solves the $U(1)$ problem in this model.

Note that the operator $\tilde{U}_F$ (30) also realizes the transformation (49), but in this case the field Σ is not affected.

In Sec. 1 we saw that the symmetry generated by the operator $\tilde{U}_F$ is broken spontaneously.

We now turn to a discussion of the possibility of treating the Σ boson as a quark–antiquark bound state.

From the operator solution (6), we can obtain a representation for the currents formed from the quark fields in terms of the operator $\Sigma(x)$,[20]

$$\left.\begin{aligned} J(x) &= \bar{\psi}(x)\,\psi(x) = -A{:}\cos(\Sigma/f_\Sigma + \theta); \\ J_5(x) &= -i\,\bar{\psi}(x)\,\gamma_5\psi(x) = A{:}\sin(\Sigma/f_\Sigma + \theta). \end{aligned}\right\} \tag{51}$$

We have here introduced the new notation

$$A = \gamma m/2\pi;\quad f_\Sigma = 1/2\sqrt{\pi};\quad \theta = \pi + \theta_- - \theta_+; \tag{52}$$

$\theta_\pm$ are arbitrarily fixed numbers.

It follows from (51) that the vacuum expectation values of the operators J and J_5 are nonzero:

$$\left.\begin{aligned} \langle \theta_+, \theta_- | J | \theta_+, \theta_- \rangle &= -A\cos\theta; \\ \langle \theta_+, \theta_- | J_5 | \theta_+, \theta_- \rangle &= A\sin\theta. \end{aligned}\right\} \tag{52a}$$

The sum of the squares of these quantities does not depend on the parameter θ:

$$(\langle \theta \mid J \mid \theta \rangle)^2 + \langle \theta \mid J_5 \mid \theta \rangle^2 = A^2. \tag{53}$$

This relation reflects the existence of a circle of equivalent vacua, the properties of which were considered in detail in Sec. 1.

The parameter θ can be eliminated by a redefinition of the fermion fields $\psi(x)$ by means of a γ rotation.

For the operators J and J_5, such a rotation reduces formally to the addition to Σ in (51) of a constant quantity:

$$\Sigma/f_\Sigma \to \Sigma/f_\Sigma - 2\alpha. \tag{54}$$

By the choice $\alpha = -\theta/2$ one can always make the effective angle θ zero. In this case,

$$\langle J \rangle = -A;\quad \langle J_5 \rangle = 0. \tag{55}$$

Such a choice corresponds to a nonzero value of the quark condensate $\langle\bar{\psi}\psi\rangle$ and a zero value of $\langle\bar{\psi}\gamma_5\psi\rangle$.

Thus, the choice of the ground state with a fixed value of the parameter θ leads to the existence of a quark condensate.

We now consider the expression (51) for $\theta = 0$ in the limit of large distances, at which the field $\Sigma(x)$ is exponentially small. Expanding (51) in a series in Σ, we obtain

$$\left.\begin{aligned} J(x) &\approx \langle J \rangle = \Psi(0)\,f_\Sigma; \\ J_5(x) &\simeq \Psi(0)\,\Sigma(x). \end{aligned}\right\} \tag{56}$$

Here, we have introduced the notation $\Psi(0) = -A/f_\Sigma$, and it follows from (56) that $\Psi(0)$ can be represented in the form

$$\Psi(0) = \langle \text{vac} | J_5(0) \mid \Sigma \rangle. \tag{57}$$

It can be seen from the relations (56) and (57) that the Σ boson can be regarded as a quark–antiquark state with wave function at the origin equal to $\Psi(0)$.

From (44) there follows an identity which establishes a direct proportionality between the divergence of the axial current and the field Σ:

$$\partial_\mu J^5_\mu = -2f_\Sigma m^2 \Sigma(x). \tag{58}$$

The form of this relation recalls the identity of partial conservation of the axial current,[25] which appears in the theory of strong interactions at the level of a hypothesis.

The Σ boson can also be regarded as "gluonium." Indeed, from the operator solution we have the identity

$$\frac{e}{4\pi}\varepsilon_{\mu\nu}F^{\mu\nu}(x) = -f_\Sigma m^2\Sigma(x), \qquad (59)$$

whence

$$\langle \text{vac} \mid -\frac{e}{4\pi}\varepsilon_{\mu\nu}F^{\mu\nu} \mid \Sigma\rangle = f_\Sigma m^2, \qquad (60)$$

which is equal to the residue of the Σ boson in the channel of pseudoscalar gluonium.

We now elucidate the meaning of the operator expansion at large distances. We take, for example, the expansion (56). It means that if we calculate the Green's function of two currents,

$$\Pi(x) = \langle \text{vac} \mid T(J_5(x) J_5(0)) \mid \text{vac}\rangle, \qquad (61)$$

then in the limit $x \to \infty$ a good approximation for $\Pi(x)$ is [see (56)]

$$\Pi(x) \approx |\psi(0)|^2 \langle \text{vac} \mid T(\Sigma(x)\Sigma(0)) \mid \text{vac}\rangle = |\psi(0)|^2 \frac{1}{i}\Delta(x, m^2), \qquad (62)$$

where $\Delta(x, m^2)$ is the Green's function of the massive field,

$$\Delta(x, m^2) = \frac{1}{(2\pi)^2}\int \frac{\exp(ipx)\,d^2p}{m^2-p^2-i\varepsilon} = \frac{i}{2\pi}K_0(m\sqrt{-x^2+i\varepsilon}), \qquad (63)$$

and K_0 is the Macdonald function.

For the two-point function $\Pi(x)$ we have the Källén-Lehmann representation

$$\Pi(x) = \frac{1}{2\pi i}\int_0^\infty \rho(s)\Delta(x, s)\,ds, \qquad (64)$$

where $\rho(s)$ is the spectral density function. The expansion (62) means that in the low-energy limit $\rho(s)$ can be well approximated by a delta function:

$$\rho(s) \approx F\delta(s-m^2), \qquad (65)$$

with F related to $\Psi(0)$ (6) by

$$F = 2\pi |\Psi(0)|^2 = 2\gamma^2 m^2. \qquad (66)$$

We shall show that in the limit of large distances the exact function $\Pi(x)$ is well described by the expression (62), while the exact $\rho(s)$ can be approximated by the resonance (65).

Using the representation (51), we can readily obtain an exact expression for $\Pi(x)$:.

$$\Pi(x) = A^2 \operatorname{sh}(-4\pi i\Delta(x, m^2)). \qquad (67)$$

A function $\rho(s)$ satisfying (64) and (67) is given by[26,27]

$$\rho(s) = F\delta(s-m^2) + \sum_{k=1}^{\infty} F\frac{2^{2k}}{(2k+1)!}\Omega_{2k+1}(s). \qquad (68)$$

Here, $\Omega_n(s)$ is the n-particle phase space, given by

$$\Omega_n(q^2) = \int \frac{dp_1^1\dots dp_n^1}{2p_1^0\dots 2p_n^0}\delta(p_1+\dots+p_n-q), \qquad (69)$$

$$p_i^0 = \sqrt{m^2+(p_i^1)^2}.$$

It can be seen from (68) that in the pseudoscalar channel there is a resonance with mass m and coupling constant F, in complete agreement with (65). And it can be seen from (67) that $\Pi(x)$ is indeed well approximated by the expression (62) when $x \gg 1/m$.

4. PROPERTIES OF THE GREEN'S FUNCTIONS AT SHORT DISTANCES

The expression (68) gives a general representation for the spectral function in terms of the phase spaces $\Omega_n(s)$. However, in the general case Ω_n cannot be calculated. Nevertheless, it is possible to calculate the asymptotic behavior of $\rho(s)$ as $s \to \infty$. This is possible because of the asymptotic freedom of the model, which it possesses by virtue of the fact that the coupling constant has the dimensions of a mass squared, so that the perturbation-theory series can be made with respect to a dimensionless constant:

$$\alpha_s(p^2) = e^2/\pi p^2 = m^2/p^2. \qquad (70)$$

It can be seen from this expression that the effective coupling constant α_s decreases very rapidly (compared with α_s in QCD) in the region of high energies or short distances.

From the expression (67), we can readily find the asymptotic behavior of $\Pi(x)$ as $x \to 0$.

In the Euclidean space, the asymptotic behavior is[27]

$$\begin{aligned}\Pi(x) = \frac{1}{2\pi^2x^2}\Big\{&1+\alpha_s(x)(2-L_x) + \alpha_s^2(x)\left(\frac{11}{4}-\gamma^4-\frac{9}{4}L_x+\frac{1}{2}L_x^2\right)\\ &+\alpha_s^3(x)\left[\frac{317}{108}+2\gamma^4-\left(\frac{59}{18}+\gamma^4\right)L_x + \frac{5}{4}L_x^2-\frac{1}{6}L_x^3\right]+\frac{1}{4!}\alpha_s^4(x)\\ &\times\left[\frac{1021}{16}-30\gamma^4-\left(\frac{6359}{72}-42\gamma^4\right)L_x + \left(\frac{557}{12}-12\gamma^4\right)L_x^2-11L_x^3+L_x^4\right]\Big\}, \end{aligned} \qquad (71)$$

where

$$\alpha_s(x) = m^2x^2/4; \quad L_x = \ln[\gamma^2\alpha_s(x)].$$

We calculate the Fourier transform of the function $\Pi(x)$:

$$\Pi(p) = 2\pi\int \Pi(x)\exp(ipx)\,d^2x$$

in the limit of large momenta p.

If contact terms of the form $(\Box p)^n\delta(p)$, which are obtained from the terms of the form $(x^2)^n$ in the expansion (71), are ignored, we have[27]

$$\begin{aligned}\Pi(p) = &-\ln\frac{p^2}{\mu^2}+\alpha_2(p)+\alpha_s^2(p)\left[\ln\alpha_s(p)-\frac{1}{4}\right] +\alpha_s^3(p)\Big[2\ln^2\alpha_s(p)\\ &+2\ln\alpha_s(p)-\frac{35}{9}+4\gamma^4\Big]+\alpha_s^4(p)\Big[6\ln^3\alpha_s(p)-\frac{33}{2}\ln\alpha_s(p)\\ &-\left(\frac{123}{4}+36\gamma^4\right)\ln\alpha_s(p)-\frac{415}{48}-69\gamma^4-24\zeta(3)\Big];\\ &\zeta(3)\simeq 1{,}202.\end{aligned} \qquad (72)$$

In this expression, $\alpha_s(p)$ is given by (70).

We note that the first two terms in this expansion can be calculated by perturbation theory.[3)] They correspond to perturbation-theory diagrams of the form

The following terms of the series (72) cannot in general be obtained by perturbation theory, since they contain nonanalytic dependences on the coupling constant of the form $\ln^k \alpha_s$. The reason for this noncorrespondence is that these terms of the series are infrared-infinite. Therefore, if they are to be given a meaning, a procedure for eliminating these infinities is required.

It follows from (70) and (72) that the main contribution to $\Pi(p)$ as $p \to \infty$ is made by the first term in the expansion (72). Hence, taking into account (64), we obtain the asymptotic value of $\rho(s)$: $\rho(\infty) = 1$.

In the intermediate range of energies $9m^2 \leqslant s \leqslant 25m^2$ the function $\rho(s)$ is described by the three-particle phase space $\Omega_3(s)$, which can be calculated[27] exactly:

$$\Omega_3(s) = 2K(\omega)[(s + 2m\sqrt{s} - 3m^2) \times (s - 2m\sqrt{s} + m^2)]^{-\frac{1}{2}} \theta(s - 9m^2),$$

where $K(\omega)$ is the complete elliptic integral of the first kind, and

$$\omega^2 = \frac{(s - 2m\sqrt{s} - 3m^2)(s + 2m\sqrt{s} + m^2)}{(s + 2m\sqrt{s} - 3m^2)(s - 2m\sqrt{s} + m^2)}.$$

We note the slow arrival in the asymptotic region of the spectral density $\rho(s)$.

For example, at $s = 9m^2$, where the effective coupling constant is small ($\alpha_s = 0.11$), $\rho(s)$ differs by almost a factor of 2 from its asymptotic value:

$$\rho(9m^2) \simeq 1.92, \quad \rho(25m^2) \simeq 0.88$$

(the graph of the function $\rho(s)$ is shown in Fig. 1). At the same time, the expansion (72) works well at these scales, i.e., for the correlation function $\Pi(p)$ asymptotic freedom commences much more rapidly than for the spectral density $\rho(s)$.

It is possible that a similar situation is realized in QCD, in which the effective coupling constant α_s decreases at high energies much more slowly than in the model considered.

Recently, many studies have attempted to describe the properties of resonances in the framework of QCD on the basis of the use of sum rules.

Generally, one uses the Wilson expansion—a mathematical formalism which makes it possible to take into account semiphenomenologically the nonperturbative effects responsible for the properties of the resonances.

At the present time, the only method of obtaining the coefficient functions in the Wilson expansion is perturbation theory.

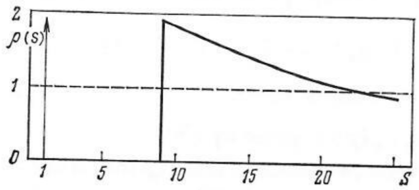


FIG. 1. Dependence of the spectral density ρ on s.

In the next section, we investigate the operator expansion in the example of the Schwinger model.

5. OPERATOR EXPANSION

We consider the operator expansion for the example of products of the currents $J(x)$ and $J_5(x)$. We introduce the notation

$$J^{\pm}(x) = \bar{\psi}\frac{1 \pm \gamma_5}{2}\psi = \frac{1}{2}(J(x) \pm iJ_5(x)). \tag{73}$$

From the operator representation (6) we can obtain a representation for $J^{\pm}$ in terms of the single dynamical operator Σ,[20]

$$J^{\pm}(x) = -\frac{A}{2} N \exp(\mp 2i\sqrt{\pi}\Sigma(x)). \tag{74}$$

Here, N denotes normal ordering.

Regular operators are separated from the chronological products of these currents by means of Wick's theorem in accordance with

$$T(J^+(x)J^-(0)) = \frac{A^2}{4}\exp(-4\pi i\Delta(x)) \times N\exp\{-2\sqrt{\pi}i(\Sigma(x) - \Sigma(0))\};$$
$$T(J^+(x)J^+(0)) = \frac{A^2}{4}\exp(4\pi i\Delta(x)) \times N\exp\{-2\sqrt{\pi}i(\Sigma(x) + \Sigma(0))\}. \tag{75}$$

The operator in (75) under the symbol of the normal ordering is regular as $x \to 0$, and it can therefore be expanded in a series in x_μ.

We retain in this expansion the first few terms:

$$T(J^+(x)J^-(0)) = \frac{A^2}{4}\exp(-4\pi i\Delta(x)) \times N\Big\{1 - \frac{2\sqrt{\pi}i}{1!}\Big(x_\mu\partial_\mu\Sigma(0) + \frac{x_\mu x_\nu}{2!}\partial_\mu\partial_\nu\Sigma(0) + \frac{x_\mu x_\nu x_k}{3!}\partial_\mu\partial_\nu\partial_k\Sigma(0) + \frac{1}{4!}x_\mu x_\nu x_k x_n\partial_\mu\partial_\nu\partial_k\partial_n\Sigma(0)\Big)$$
$$+ \frac{1}{2!}(2\sqrt{\pi}i)^2((x_\mu\partial_\mu\Sigma(0))^2 + x_\mu x_\nu x_k\partial_\nu\partial_k\Sigma(0)\,\partial_\mu\Sigma(0) + \frac{1}{4}(x_\mu x_\nu\partial_\mu\partial_\nu\Sigma(0))^2 + \frac{1}{3}\Big(x_\mu\partial_\mu\Sigma(0)\,x_\nu x_k x_n\partial_\nu\partial_k\partial_n\Sigma(0)\Big)$$
$$+ \frac{1}{3!}(-2\sqrt{\pi}i)^3\Big[(x_\mu\partial_\mu\Sigma(0))^3 + \frac{3}{2!}(x_\mu\partial_\mu\Sigma)^2 x_\nu x_k\partial_\nu\partial_k\Sigma(0)\Big] + \frac{1}{4!}(2\sqrt{\pi}i)^4(x_\mu\partial_\mu\Sigma)^4 + 0(x^4)\Big\}. \tag{76}$$

In this case, the coefficient function

$$C_0(x) = \frac{A^2}{4}\exp(-4\pi i\Delta(x)) \tag{77}$$

must also be expanded in a series as $x \to 0$ to the requisite number of terms:

$$C_0(x) = \frac{1}{4\pi^2x^2}\Big\{1 + \alpha_s(x)(2 - \ln(\gamma^2\alpha_s)) + \alpha_s^2\Big(\frac{1}{2}\ln^2\gamma^2\alpha_s - \frac{9}{4}\ln\gamma^2\alpha_s + \frac{11}{4}\Big)\Big\}, \tag{78}$$

where

$$\alpha_s = e^2x^2/4\pi.$$

The expansion (76), (78) is the operator expansion of the chronological product of two currents at short distances.

This expansion is with respect to well-defined normal products of operators taken at one point. However, the coefficient functions, for example, $C_0(x)$, depend nonanalytically on the coupling constant e^2.

We shall show that these nonanalyticities can be eliminated by a redefinition of the operators.

Instead of the operator $N[\partial_\mu \Sigma(0)\partial_\mu \Sigma(0)]$ we define the operator $T[\partial_\mu \Sigma(0)\partial_\nu \Sigma(0)]$. For this, we consider the symmetric limit

$$\lim_{\varepsilon\to 0} T(\partial_\mu\Sigma(0)\partial_\nu\Sigma(0)) = N(\partial_\mu\Sigma(0)\,\partial_\nu\Sigma(0)) - \lim_{\varepsilon\to 0}\frac{1}{i}\,\partial_\mu\partial_\nu\Delta(\varepsilon).$$

Using the explicit expression for $\Delta(\varepsilon)$, we obtain

$$-\lim_{\varepsilon\to 0}\frac{1}{i}\,\partial_\mu\partial_\nu\Delta(\varepsilon) = -\frac{m^2}{8\pi}\,g_{\mu\nu}\ln(-\gamma^2\varepsilon^2 m^2/4).$$

We define the R operation for eliminating the ultraviolet infinities in the form

$$RT(\partial_\mu\Sigma(0)\,\partial_\mu\,\Sigma(0)) = \lim_{\varepsilon\to 0} T(\partial_\mu\Sigma(\varepsilon)\,\partial_\nu\Sigma(0)) + \frac{m^2}{8\pi}\,g_{\mu\nu}\ln(-\gamma^2\varepsilon^2\mu^2/4), \quad (79)$$

where μ is an arbitrary subtraction point with the dimensions of mass. Then the N product is related to the corresponding T product by

$$RT(\partial_\mu\Sigma(0)\,\partial_\mu\Sigma(0)) = N(\partial_\mu\Sigma(0)\,\partial_\nu\Sigma(0)) - \frac{m^2}{8\pi}\ln\frac{m^2}{\mu^2}\,g_{\mu\nu}. \quad (80)$$

With such a definition, the operator depends nonanalytically on the coupling constant e^2; for example, its vacuum expectation value is

$$\langle RT(\partial_\mu\Sigma(0)\,\partial_\nu\Sigma(0))\rangle = -\frac{m^2}{8\pi}\ln\frac{m^2}{\mu^2}\,g_{\mu\nu}. \quad (81)$$

From (80), we readily obtain

$$RT(x_\mu\partial_\mu\Sigma)^2 = N(x_\mu\partial_\mu\Sigma)^2 - \frac{(mx)^2}{8\pi}\ln\frac{m^2}{\mu^2}. \quad (82)$$

Similarly, we can redefine all the remaining operators, namely,

$$\left.\begin{aligned}
&RT(x_\mu x_\nu x_k\partial_\mu\Sigma\partial_\nu\partial_k\Sigma) = N(x_\mu\partial_\mu\Sigma x_k x_\nu\partial_k\partial_\nu\Sigma);\\
&RT(x_\mu x_\nu\partial_\mu\partial_\nu\Sigma)^2 = N(x_\mu x_\nu\partial_\mu\partial_\nu\Sigma)^2 - \frac{3}{8}\frac{(mx)^4}{4\pi}\ln\frac{m^2}{\mu^2};\\
&RT(x_\mu\partial_\mu\Sigma x_\nu x_k x_n\partial_\nu\partial_k\partial_n\Sigma) = N(x_\mu\partial_\mu\Sigma x_\nu x_k x_n\partial_\nu\partial_k\partial_n\Sigma)\\
&\quad + \frac{3}{8}\frac{(mx)^4}{4\pi}\ln\frac{m^2}{\mu^2};\\
&RT(x_\mu\partial_\mu\Sigma)^3 = N(x_\mu\partial_\mu\Sigma)^3\frac{3}{2}(mx)^2\ln\frac{m^2}{\mu^2}\,x_\mu\partial_\mu\Sigma;\\
&RT(x_\mu\partial_\mu\Sigma)^4 = N(x_\mu\partial_\mu\Sigma)^4 - \frac{3(mx)^2}{4\pi}\ln\frac{m^2}{\mu^2}N(x_\mu\partial_\mu\Sigma)^2\\
&\quad + \frac{3}{4}(mx^4)\frac{1}{(4\pi)^2}\ln^2\frac{m^2}{\mu^2}.
\end{aligned}\right\} \quad (83)$$

The formulas (83) make it possible to express the N products in the expansion (76) in terms of the corresponding T products. If this is done, the expansion (76) takes the form

$$\begin{aligned}
T(J^+(x)\,J^-(0)) &= C_0^R(x) - 2\sqrt{\pi}\,iC_1^R(x)\,x_\mu\partial_\mu\Sigma(0)\\
&- 2\sqrt{\pi}\,iC_1^R(x)\left\{\frac{x_\mu x_\nu}{2}\,\partial_\mu\partial_\nu\Sigma\right\}\\
&- 2\sqrt{\pi}\,iC_2^R(x)\left\{\frac{1}{3!}\,x_\mu x_\nu x_k\partial_\mu\partial_\nu\partial_k\Sigma\right.\\
&\left. + \frac{1}{4!}\,x_\mu x_\nu x_k x_n\partial_\mu\partial_\nu\partial_k\partial_n\Sigma\right\} + \frac{1}{2!}(2\sqrt{\pi}i)^2 C_1^R(x)\,RT(x_\mu\partial_\mu\Sigma)^2\\
&+ (2\sqrt{\pi}\,i)^2\frac{1}{2!}C_2^R(x)\left\{RT(x_\mu x_\nu x_k\partial_\mu\Sigma\partial_\nu\partial_k\Sigma)\right.\\
&+ \frac{1}{4}RT(x_\mu x_\nu\partial_\mu\partial_\nu\Sigma)^2\\
&\left. + \frac{1}{3}RT(x_\mu x_\nu x_k x_n\partial_\mu\Sigma\partial_\nu\partial_k\partial_n\Sigma)\right\} + \frac{1}{3}(-2\sqrt{\pi}\,i)^3 C_2^R(x)\\
&\times\left\{RT(x_\mu\partial_\mu\Sigma)^3 + \frac{3}{2}RT((x_\mu\partial_\mu\Sigma)^2 x_\nu x_k\partial_\nu\partial_k\Sigma)\right\}\\
&+ \frac{1}{4!}(2\sqrt{\pi}\,i)^4 C_2^R(x)\,RT(x_\mu\partial_\mu\Sigma)^4 + O(x)^2.
\end{aligned} \quad (84)$$

Here

$$\left.\begin{aligned}
C_0^R(x) &= \frac{1}{4\pi^2x^2}\left\{1 + \alpha_s(x)\left(2 - \ln\frac{\gamma^2x^2\mu^2}{4}\right)\right.\\
&\left. + \alpha_s^2\left(\frac{1}{2}\ln^2\frac{\gamma^2x^2\mu^2}{4} - \frac{9}{4}\ln\frac{\gamma^2x^2\mu^2}{4} + \frac{11}{4}\right)\right\};\\
C_1^R(x) &= \frac{1}{4\pi^2x^2}\left\{1 + \alpha_s(x)\left(2 - \ln\frac{\gamma^2x^2\mu^2}{4}\right)\right\};\\
C_2^R(x) &= \frac{1}{4\pi^2x^2}.
\end{aligned}\right\} \quad (85)$$

Thus, the coefficient functions $C_i^R(x)$ have been freed of the logarithmic dependence on e^2. Averaging of (16) with respect to the vacuum gives the following representation for the Green's function:

$$\begin{aligned}
\langle T(J^+(x)\,J^-(0))\rangle &= C_0(x) = C_0^R(x) + \pi x^2 C_1^R(x)\langle RT(\partial_\mu\Sigma)^2\rangle\\
&+ \frac{\pi x^4}{2^4}C_2^R(x)\langle RT(\partial^2\Sigma)^2\rangle\\
&+ \frac{\pi^2x^4}{4}C_2^R(x)\langle RT(\partial_\mu\Sigma)^2(\partial_\nu\Sigma)^2\rangle + O(x^2),
\end{aligned} \quad (86)$$

where

$$\left.\begin{aligned}
\langle RT(\partial_\mu\Sigma)^2\rangle &= -\frac{m^2}{4\pi}\ln\frac{m^2}{\mu^2};\\
\langle RT(\partial^2\Sigma)^2\rangle &= -\frac{m^4}{4\pi}\ln\frac{m^2}{\mu^2};\\
\langle RT(\partial_\mu\Sigma)^2(\partial_\nu\Sigma)^2\rangle &= \frac{2m^4}{(4\pi)^2}\ln^2\frac{m^2}{\mu^2}.
\end{aligned}\right\} \quad (87)$$

In the momentum space, (86) takes the form

$$F_1(p) = 2\pi\int\exp(ipx)\langle TJ^+(x)\,J^-(0)\rangle\,d^2x;$$

$$\begin{aligned}
2F_1(p) &= -\ln\frac{p^2}{\bar{\mu}^2} + \alpha_s(p) + \alpha_s^2(p)\left[\ln\frac{\mu^2}{p^2} - \frac{1}{4}\right]\\
&- \frac{4\alpha_s(p)\,\pi}{p^2}\langle RT(\partial_\mu\Sigma)^2\rangle + O\left(\frac{1}{p^6}\ln^2 p^2\right).
\end{aligned} \quad (88)$$

In calculating the Fourier transform, we ignored contact terms of the form $\Box_p^n\delta(p)$. The parameter $\bar{\mu}$ arose after elimination of the ultraviolet infinities from the function $C_0^R(x)$. The parameter μ corresponds to separation of the nonanalyticities with respect to e^2, and $F_1(p)$ does not depend on it at all.

In order to be able to compare the expansion (88) with the standard Wilson expansion obtained in the framework of perturbation theory, we must express the field Σ in (88) in terms of the original quark or gluon fields. This can be done in several ways. If we use the relation

$$\Sigma(x) = -\frac{\sqrt{\pi}}{2e}\varepsilon_{\mu\nu}F^{\mu\nu}(x),$$

then the coefficient function of the corresponding operator in (88) will not depend on e^2, and in the following terms of the expansion (88) there will appear[28] a nonanalytic dependence of the coefficient functions of the form $1/e^n$, which cannot be obtained in the framework of perturbation theory.

A second way is to use the relation

$$\partial_\mu \Sigma = \sqrt{\pi}\,\overline{\psi}\gamma_\mu\gamma_5\psi.$$

In this case, the operator has the form

$$\langle RT\,(\partial_\mu\Sigma)^2\rangle = \pi\,\langle RT\,(\overline{\psi}\gamma_\mu\gamma_5\psi)^2\rangle. \quad (89)$$

The standard scheme for obtaining the Wilson expansion by perturbation theory leads to the result

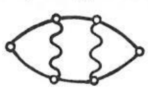
$$2F_1(p) = 2F_1^0(p) + \frac{2\pi e^2}{p^4}\langle(\overline{\psi}\gamma_\mu\psi)^2\rangle - \frac{e^2}{p^4}\langle F_{\mu\nu}^2\rangle - \frac{4e^2\pi}{p^4}\langle(\overline{\psi}\gamma_\mu\gamma_5\psi)^2\rangle, \quad (90)$$

where

$$2F_1^0(p) = -\ln\frac{p^2}{\overline{\mu}^2} + \alpha_s(p) + \alpha_s^2(p)\left[\ln\frac{\overline{\mu}^2}{p^2} + B\right]. \quad (91)$$

The first two terms in $F_1^0(p)$ are identical to the exact result (72) and, as already noted in Sec. 4, can be obtained perturbatively.

To the third term in F_1^0 there correspond perturbation-theory diagrams of the form



These diagrams are infrared-infinite, the integral with respect to the gluon line diverging at small momenta as $\int d^2k/k^2$.

In dimensional ε regularization, the diagrams become finite, and after the subtraction of the poles with respect to ε we obtain a parameter $\overline{\mu}$, corresponding to elimination of the infinities at small momenta.[4)]

If we identify the operators that arise in the framework of perturbation theory in the expansion (90) with the exact operators defined as (89),

$$\langle RTF_{\mu\nu}^2\rangle = 2\pi\,\langle RT\,(\overline{\psi}\gamma_\mu\psi)^2\rangle = -2\pi\,\langle RT\,(\overline{\psi}\gamma_\mu\gamma_5\psi)^2\rangle = -2\,\langle RT\,(\partial_\mu\Sigma)^2\rangle = \frac{m^2}{2\pi}\ln\frac{m^2}{\mu^2},$$

then the expansion (90) will be identical to the exact expansion (88), (89), the constant B in (91) determining the connection between the subtraction points $\overline{\mu}$ and μ:

$$\mu^2 = \overline{\mu}^2\exp\left(B + \frac{1}{4}\right).$$

We consider the chronologically ordered product $T[J^+(x)J^+(0)]$, which is remarkable in that its vacuum expectation value is zero in any order of perturbation theory. We shall show that in this case too the nonanalyticities with respect to e^2 can also be separated, i.e., they will all be contained in the operators.

The second of the expressions (75) can be rewritten in the form

$$T\,(J^+(x)\,J^+(0)) = \exp\,(4\pi i\Delta(x))\,N\,(J^+(x)\,J^+(0)).$$

At small x there exists the expansion

$$T\,(J^+(x)\,J^+(0)) = f_0(x)\,N\,(J^+(0))^2 + f_1(x)\,x_\mu N\,(\partial_\mu J^+(0)\,J^+(0)) + f_1(x)\,\frac{x_\mu x_\nu}{2}\,N\,(\partial_\mu\partial_\nu J^+(0)\,J^+(0)) + \ldots, \quad (92)$$

where

$$f_0(x) = \frac{m^2\gamma^2x^2}{4}\,(1 - \alpha_s(x)\,(2 - \ln\alpha_s));$$

$$f_1(x) = \frac{m^2\gamma^2x^2}{4},$$

i.e., $f_0(x)$ depends nonanalytically on e^2.

We determine the operators

$$RT\,(J^+(0))^2 = \lim_{\varepsilon\to 0} T\,(J^+(\varepsilon)\,J^+(0)) \equiv 0. \quad (93)$$

Similarly, from (92) we obtain

$$\left.\begin{aligned}
&RT\,(\partial_\mu\partial_\nu J^+(0)\,J^+(0)) = \lim_{\varepsilon\to 0} T\,\Box\,J^+(0)\,J^+(0)\\
&= \frac{1}{2}\,m^2\gamma^2 N\,(J^+(0))^2\,g_{\mu\nu};\\
&RT\,(x_\mu x_\nu x_i\partial_\mu\partial_\nu\partial_i J^+J^+) = \frac{3}{2}\,m^2\gamma^2 x_\mu N\,(\partial_\mu J^+J^+);\\
&RT(\partial_\mu\partial_\nu\partial_i\partial_j J^+(0)\,J^+(0)) = \frac{m^4\gamma^2}{2}N\,(J^+(0))^2\,(\delta_{\mu\nu}\delta_{ij} + \delta_{\mu i}\delta_{\nu j}\\
&+ \delta_{\mu j}\delta_{\nu i})\left(\ln\frac{m^2}{\mu^2} + 1\right) + \frac{m^2\gamma^2}{2}\,[N\,(\partial_\mu\partial_\nu J^+J^+)\,\delta_{ij}\\
&+ \text{symmetrization}].
\end{aligned}\right\} \quad (94)$$

With allowance for (94), the expansion (92) takes the form

$$T\,(J^+(x)\,J^+(0)) = \frac{1}{2!}\,x_\mu x_\nu RT\,(\partial_\mu\partial_\nu J^+J^+)\,f_0^R(x) + \frac{1}{3!}\,x_\mu x_\nu x_k RT\,(\partial_\mu\partial_\nu\partial_k J^+J^+) + \frac{1}{4!}\,x_\mu x_\nu x_k x_n RT\,(\partial_\mu\partial_\nu\partial_k\partial_n J^+J^+) + \ldots, \quad (95)$$

where $f_0^R(x) = 1 + \alpha_s(x)\,[\ln(\gamma^2x^2\mu^2/4) - 3]$ depends analytically on e^2, the nonanalyticity being contained in the operators

$$\langle RT\,\Box\,J^+J^+\rangle = m^2\gamma^2A^2/4;$$

$$\langle RT\,\Box^2 J^+J^+\rangle = m^4\gamma^2A^2\left(\ln\frac{m^2}{\mu^2} + 1\right).$$

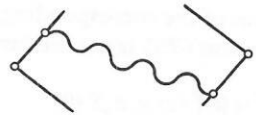
We introduce the Green's function

$$2F_2(p) = 2\pi\int\langle TJ^+(x)\,J^+(0)\rangle\exp\,(ipx)\,d^2x.$$

From (95), we obtain

$$2F_2(p) = -\frac{32\pi e^2}{p^6}\langle RT\,\Box\,J^+J^+\rangle + \ldots \quad (96)$$

The Wilson expansion by perturbation theory for the function F_2 reduces to calculation of a diagram of the form



and leads to the result

$$2F_2(p)=\frac{8\pi e^2}{p^4}\langle (J^+)^2\rangle-\frac{32\pi e^2}{p^6}\langle \Box J^+J^+\rangle+\dots$$

If we identify the resulting operators with the exact operators defined by (93) and (94), we obtain complete agreement with the exact expression (96).

We now verify the vacuum-dominance hypothesis.[14]

According to this hypothesis, the expectation value of the operator $\bar\psi\Gamma_1\psi\bar\psi\Gamma_2\psi$ can be approximately estimated by

$$\langle\bar\psi\Gamma_1\psi\bar\psi\Gamma_2\psi\rangle=\frac{1}{4}\langle\bar\psi\psi\rangle^2(\mathrm{tr}\,\Gamma_1\,\mathrm{tr}\,\Gamma_2-\mathrm{tr}\,(\Gamma_1\Gamma_2)).$$

Here, $\langle\bar\psi\psi\rangle$ is the quark condensate, which can be found exactly from the expression (74):

$$\langle\bar\psi\psi\rangle=-A.$$

Application to the case

$$\Gamma_{1,2}=(1\pm\gamma_5)/2$$

gives

$$\langle J^-(0)\,J^+(0)\rangle=\frac{1}{4}A^2,\ \langle J^+(0)\,J^+(0)\rangle=0.$$

The second of these expressions agrees with (93), whereas the first gives an incorrect result, since $\langle RTJ^-J^+\rangle$ depends on μ but the constant A does not. An analogous situation holds in the case of vector currents.

Summarizing, we can say that there exist different ways of defining regular composite operators.

They correspond to different representations of the operator expansion. In one of the definitions of the operators the exact expansion corresponds to the perturbation-theory expansion.

Complete identity of these expansions is possible only if one knows the connection between the ultraviolet subtraction point μ for the operators and the infrared subtraction point $\bar\mu$ for the coefficient functions. In the framework of perturbation theory, this connection cannot be established.

6. SUM RULES

One of the approximate methods for obtaining information on the behavior of QCD in the low-energy region is to use finite energy sum rules[21,22] and their modifications.[14,29]

In many cases, the use of sum rules has made it possible to describe the properties of resonances. It is therefore of interest to test this method for the example of the exactly solvable model that we have considered.[27]

We consider the pseudoscalar channel, which is described by the current J_5.

The problem is to recover on the basis of the asymptotic behavior (71) of the two-point Green's function J_5 (61) the parameters of the resonance $\Sigma(m^2=e^2/\pi, F=2m^2\gamma^2)$ by means of sum rules.

We write down sum rules with an exponential weight function:

$$\int\rho^{\exp}(s)\,e^{-s/M^2}\frac{ds}{M^2}=\hat B_M\Pi(p^2)=\Pi(M^2), \qquad (97)$$

where $\Pi(M^2)$ is the Borel transform of the function $\Pi(p^2)$, which is defined in accordance with

$$\Pi(M^2)=\hat B_M\Pi(p^2)=\lim_{\substack{p^2\to\infty\\ n\to\infty\\ p^2/n=M^2}}\frac{(-1)^n}{\Gamma(n)}(p^2)^n\frac{d^n}{(dp^2)^n}\Pi(p^2).$$

Using the expansion (72) for the function $\Pi(p^2)$ in the high-energy region, we obtain[27]

$$\Pi(M^2)=1+\frac{m^2}{M^2}-\left(\frac{m^2}{M^2}\right)^2\left(\ln\frac{M^2}{m^2}+0.673\right)$$
$$+\left(\frac{m^2}{M^2}\right)^3\left(17{,}7+0{,}846\ln\frac{M^2}{m^2}+\ln^2\frac{M^2}{m^2}\right)+O\left(\frac{m^8}{M^8}\ln^3M^2\right). \qquad (98)$$

This expansion is valid for $M^2\gg m^2$. In what follows, it is convenient to fix the scale by setting $m^2=1$.

As $\rho^{\exp}(s)$, for which there exists the exact expression (68), we choose the widely used approximation in the form

$$\rho^{\exp}(s)=F_R\delta(s-m_R^2)+\theta(s-9). \qquad (99)$$

This choice of the spectral density takes into account the existence of both the resonance and the continuum that begins at the three-particle threshold and has the asymptotic behavior $\rho(\infty)=1$.

With allowance for this approximation, the sum rules (97) take the form

$$\hat\Pi(M^2)\equiv\Pi(M^2)-\exp(-9/M^2)=\frac{F_R}{M^2}\exp\left(-\frac{m_R^2}{M^2}\right).$$

Differentiating this relation with respect to M^2, we obtain the second equation for determining the two parameters m_R^2 and F_R:

$$\frac{d}{dM^2}(\hat\Pi(M^2)\,M^2)=\frac{F_Rm_R^2}{M^4}\exp\left(-\frac{m_R^2}{M^2}\right).$$

The parameter m_R^2 is found from these equations in the form

$$m_R^2=M^2(1+M^2\hat\Pi'(M^2)/\hat\Pi(M^2)).$$

We shall estimate the accuracy of the determination of M_R^2. In the approximation linear with respect to the error we have

$$\frac{\Delta m_R^2}{m_R^2}=\left(1-\frac{M^2}{m_R^2}\right)\left\{\frac{\Delta\hat\Pi'}{\hat\Pi'}-\frac{\Delta\hat\Pi}{\hat\Pi}\right\}.$$

It follows from this that the error in the determination of m_R^2 increases linearly at large M^2. For this error to be of the order of the error in the determination of the function $\hat\Pi(M^2)$, it is necessary to work in the region $M^2\lesssim m_R^2$. However, even for the boundary value $M^2=m_R^2$ the expansion (98) is invalid:

$$\Pi(1)=1+1-0.673+17.7+\dots$$

Thus, in the present model it is not possible to calculate the parameters of the resonance by using standard sum rules, since the expansion (98) fails in the region of the resonance.

This is due to the fact that in the model which we are investigating the resonance lies outside the region of asymptotic freedom. Indeed, the effective coupling constant $\alpha_s(p^2)$ in the region of the resonance $p^2=m^2$ is equal to unity, this corresponding to the strong-coupling region.

We shall now show that nevertheless the resonance can be described by means of sum rules in the x space.

In the x space, the expansion (71) in the region $mx \simeq 1$ of the resonance converges well and, moreover, takes into account the contact terms.

We note that the Källén-Lehmann representation (64), written in the form

$$2\pi^2 x^2 \Pi(x) = \frac{x^2}{2} \int_0^\infty \rho(s) K_0(x\sqrt{s})\, ds, \qquad (100)$$

represents in the Euclidean domain a set of sum rules that depends on the parameter x. Since the function $K_0(x\sqrt{s})$ is exponentially small as $x \to \infty$, by choosing x fairly large we can ensure suppression of the background compared with the resonance in the expression (100).

Choosing the approximation for $\rho(s)$ in the form

$$\rho(s) = F_R \delta(s - m_R^2) + \theta(s - s_0),$$

where s_0 is the beginning of the effective continuum, and substituting it in (100), we obtain the equation[27]

$$\frac{1}{2} F_R x^2 K_0(m_R x) = 2\pi^2 x^2 \Pi(x) - \sqrt{s_0}\, x K_1(x\sqrt{s_0}) \equiv f(x). \qquad (101)$$

The equation for determining the parameter m_R can be obtained from (101) in the form

$$2 - E^{-1}(z) = xf'(x)/f(x), \quad z = m_R x, \qquad (102)$$

where $E(z) = K_0(z)/zK_1(z)$.

The graphs of the functions on the left- and right-hand sides of Eq. (102) are given in Fig. 2 for the parameter value $s_0 = 9$.

Curve 1 corresponds to the function $2 - E^{-1}(x)$, curve 2 to the function $xf'/f(x)$, where $f(x)$ is determined by the expression (101), and curves 3, 4, 5, 6 to the function $xf'/f(x)$, where $f(x)$ can be calculated in accordance with (101), and for $\Pi(x)$ the first, second, third, and fourth approximations with respect to $\alpha_s(x)$ in the expansion (71) have been taken, respectively.

We estimate the accuracy of the determination of the parameter m_R from Eq. (102):

$$\frac{\Delta m_R}{m_R} = \frac{E^2(z)}{E'(z)\, z} \Delta\left(\frac{xf'}{f}\right) = \frac{E(2E-1)}{zE'} \left\{\frac{\Delta f'}{f'} - \frac{\Delta f}{f}\right\}. \qquad (103)$$

To ensure that the accuracy in the determination of m_R is of the same order as the accuracy in the determination of the function $f(x)$ itself, we work in the region in which the coefficient

$$P(z) = E(z)(2E(z) - 1)/zE'(z)$$

is of order unity (or less). This condition makes it impossible to work in the region of very short distances, since in this region the coefficient P is very large:

$$P(z) \to 2 \ln^2 z, \quad z \to 0.$$

The most suitable region is the neighborhood of the resonance $m_R x \simeq 1$: here $P \lesssim 1$, and the expansion (71) still works.

Analysis of curve 6 in accordance with (101) and (102) at the point $x = 1$ leads to the parameter values $m_R \simeq 1.1$ and $F_R \simeq 7.4$.

We recall that the exact values of the parameters are $m = 1$ and $F \simeq 6.3$.

If we take into account the following corrections in the expansion (71) and obtain a good approximation to the exact curve 2 at the point $x = 2$, better accuracy can be ensured. For example, doing this for curve 2 at the point $x = 2$ leads to $m_R \simeq 1.0$ and $F_R \simeq 6.3$.

This good agreement is due to the strong suppression of the background and the comparatively small value of the coefficient $P = 0.2$.

From the point of view of fulfillment of the sum rule (100), the parameter choice $s_0 = 9$ is not the best. For better fulfillment of the relation (100), it is necessary to choose s_0 somewhat less than 9, in order to take into account effectively the large positive excursion above unity of the exact function $\rho(s)$ (see Fig. 1).

In Fig. 3, we give graphs of the function $m_R(1/x^2)$ for different values of the function s_0 [the function $\Pi(x)$ is calculated up to the fourth order in α_s]. It can be seen from the figure that the most stable curve corresponds to $s_0 = 6$. Applying our analysis to this curve at the point $x = 1$, we obtain $m_R \simeq 1.0$ and $F_R \simeq 6.18$.

Thus, use of the sum rules in the x space makes it possible to calculate the parameters of the resonance fairly reliably, whereas in the p space this cannot be done because of the poor convergence of the series at scales of the order of the

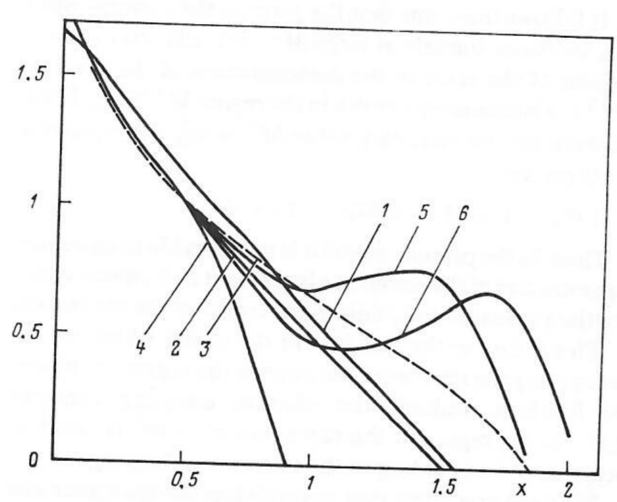


FIG. 2. Graphs of the functions used in Eq. (102).

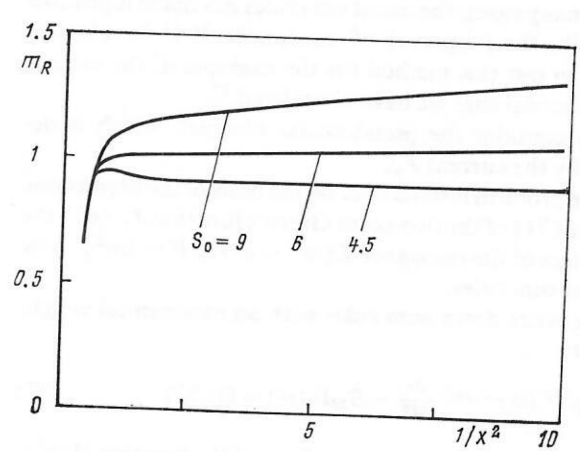


FIG. 3. Dependence of m_R on $1/x^2$ for different values of s_0.

mass of the resonance. However, our example does not preclude application of the sum rules to calculation of the parameters of a resonance in the p space in models in which the series converge well at the scales of the resonance (see, for example, Ref. 15).

In conclusion, we thank N. V. Krasnikov, V. A. Kuz'min, V. A. Matveev, A. A. Pivovarov, V. A. Rubakov, M. E. Shaposhnikov, and the participants of a seminar of the Theoretical Department of the Institute of Nuclear Research of the USSR Academy of Sciences for helpful discussions.

We thank F. V. Tkachev and K. G. Chetyrkin for clarifying some aspects of Wilson's perturbation-theory expansion.

APPENDIX

In this Appendix, we describe the basic properties of massless fermions in two-dimensional space.

The Dirac equation for the massless fermions has the form

$$i\gamma_\mu\partial_\mu\psi_0(x)=i\begin{pmatrix}0 & \partial_0-\partial_1\\ \partial_0+\partial_1 & 0\end{pmatrix}\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix}=0.$$

This equation decouples into two independent equations for the components $\psi_{1,2}$:

$$(\partial_0-\partial_1)\psi_2=0;\quad (\partial_0+\partial_1)\psi_1=0.$$

It follows from this that ψ_1 depends only on $x^-=x_0-x^1$, and ψ_2 only on $x^+=x^0+x^1$, i.e., ψ_1 is a wave that propagates to the right, ψ_2 a wave that propagates to the left.

The decomposition into positive- and negative-frequency parts has the form

$$\left.\begin{aligned}\psi_1(x)&=\int dp^1\,e^{-ipx}\theta(p^1)\frac{a_1^-(p^1)}{\sqrt{2\pi}}+\int dp^1\,e^{ipx}\theta(p^1)\,b_1^+(p^1)/\sqrt{2\pi};\\ \psi_2(x)&=\int dp^1\,e^{-ipx}\theta(-p^1)\frac{a_2^-(p^1)}{\sqrt{2\pi}}\\ &+\int dp^1\,e^{ipx}\theta(-p^1)\,b_2^+(p^1)/\sqrt{2\pi}.\end{aligned}\right\}\qquad (A1)$$

Here

$$px=|p^1|x^0-p^1x^1.$$

We introduce the notation

$$(a_i^-)^*=a_i^+,\ (b_i^+)^*=b_i^-;\ i=1,\ 2.$$

At the quantum level, the operators $a_i^\pm$ and $b_i^\pm$ satisfy the anticommutation relations

$$\left.\begin{aligned}&\{a_i^-(p^1),\ a_j^+(q^1)\}=\delta_{ij}\delta(p^1-q^1);\\ &\{b_i^-(p^1),\ b_j^+(q^1)\}=\delta_{ij}\delta(p^1-q^1);\\ &\{a_i^\pm(p^1),\ b_j^{\pm(\mp)}(q^1)\}=0.\end{aligned}\right\}\qquad (A2)$$

The operators $a_i^\pm$ (respectively, $b_i^\pm$) with index $i=1$ are the operators of creation and annihilation of particles (respectively, antiparticles) of the first species. These particles are characterized by the fact that they all propagate to the right. The operators $a_i^\pm$ (respectively, $b_i^\pm$) with $i=2$ correspond to particles of the second species, which propagate to the left.

The numbers of particles of the first and second species are conserved separately. Indeed, the conserved charges Q_F and $\tilde{Q}_F$ have the form

$$Q_F=\int dx^1 : \bar{\psi}_0\gamma_0\psi_0 : =\int dp^1\,\theta(p^1)\,[a_1^+(p^1)\,a_1^-(p^1)-b_1^+(p^1)\,b_1^-(p^1)]$$
$$+\int dp^1\,\theta(-p^1)\,[a_2^+(p^1)\,a_2^-(p^1)-b_2^+(p^1)\,b_2^-(p^1)];$$
$$\tilde{Q}_F=\int dx^1 : \bar{\psi}_0\gamma_0\gamma_5\psi_0 : =\int dp^1\,\theta(p^1)\,[a_1^+(p^1)\,a_1^-(p^1)-b_1^+(p^1)\,b_1^-(p^1)]$$
$$-\int dp^1\,\theta(-p^1)\,[a_2^+(p^1)\,a_2^-(p^1)-b_2^+(p^1)\,b_2^-(p^1)]$$

and are the sum and the difference of the particles of the first and second species, respectively. The assertion made above follows from the conservation of Q_F and $\tilde{Q}_F$.

From (A1) and the commutation relations (A2) we obtain the following expressions for the anticommutator and the Green's function of the free massless fields[23]:

$$\{\psi_{1,2}(x_0,\ x^1)\,\psi_{1,2}^+(y_0,\ y^1)\}=\delta(x^\mp-y^\mp);$$
$$\langle\psi_{1,2}(x_0,\ x^1)\,\psi_{1,2}^+(y_0,\ y^1)\rangle=\frac{1}{2\pi i}\,\frac{1}{x^\mp-y^\mp-i0}.$$

[1)]Methods for studying the structure of the ground state in quantum systems with degeneracy were developed by N. N. Bogolyubov in his pioneering studies on the theory of superfluidity and superconductivity.

[2)]It is possible that this result is also valid for four-dimensional theories.

[3)]We were assisted in establishing this fact by A. A. Pivovarov, F. V. Tkachev, and K. G. Chetyrkin.

[4)]The last term in $F_1^0(p)$ (91) was calculated by L. R. Surguladze and F. V. Tkachev.

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