Hydrodynamics of superfluid systems and the method of quasiaverages

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Fiz. Elem. Chastits At. Yadra **16**, 875–926 (July–August 1985)

The thermodynamics and hydrodynamics of macroscopic superfluid systems with broken phase invariance are constructed on the basis of Bogolyubov's concept of quasiaverages and the reduced-description method. The case of broken Galilean invariance is considered. A method for finding the low-frequency behavior of the Green's functions for arbitrary quasilocal operators is developed. An extension of the approach to systems in which translational invariance is also broken is discussed.

I. INTRODUCTION

At the present time, the theoretical foundation of the description of both equilibrium and nonequilibrium states of systems with spontaneously broken symmetry in statistical mechanics is the method of quasiaverages[1] and the reduced-description method.[2] This approach makes it possible to obtain the thermodynamics and the equations of motion for such macroscopic systems.

However, most of the physical results for systems with spontaneously broken symmetry have been obtained on the basis of a phenomenological approach. This applies in the first place to the superfluidity of He II, for the description of which the Tisza–Landau two-fluid model has proved to be effective. Indeed, in the framework of the model Landau constructed the equations of the ideal hydrodynamics of superfluid He II, while in the work of Landau and Khalatnikov dissipation processes were taken into account (see Ref. 3). Later, the phenomenological approach was applied to other systems close in their properties to superfluid systems, namely, systems for which either the particle-number operator or one of the components of the total spin is the generator of a broken symmetry. Such systems include, in particular, superconducing Fermi systems and planar ferromagnets and antiferromagnets.[4]

In recent years, there have been intensive investigations of superfluid ^{3}He. In contrast to He II, in which invariance of the state with respect to phase transformations is broken, the symmetry breaking in the case of ^{3}He is more complicated—besides the breaking of the symmetry with respect to phase transformations, there is also breaking of the symmetry with respect to three-dimensional rotations in both the coordinate and the spin space. Consequences of this are the more complicated structure of the order parameter, the vortex nature of the superfluid flow, and the appearance of diverse magnetic properties.[5–7]

In the present paper, which is based on the method of quasiaverages and the reduced-description method, we give a microscopic construction of the thermodynamics and hydrodynamics of macroscopic systems in which only phase invariance is broken. We do not assume that the Hamiltonian of the system possesses Galilean invariance. This makes it possible, in particular, to obtain results that apply to both Galileo–invariant and relativistic systems. In addition, our equations of superfluid hydrodynamics in the absence of Galilean invariance can be used to investigate the superfluidity of electrons in a metal or the phenomenon of quasiparticle superfluidity.

Further, we find the sources in the equations of the ideal hydrodynamics of a superfluid due to a weak external field, the operator structure of the Hamiltonian of the interaction of the system with the external field being arbitrary. The densities of the additive integrals of the motion and their corresponding flux densities, both of which occur in the hydrodynamic equations, are expressed in terms of the thermodynamic potential, while the sources are determined not only by the thermodynamic potential but also by the equilibrium mean value of the quasilocal operator that determines the interaction of the system with the external field.

The equations of hydrodynamics in the presence of arbitrary external sources make it possible to find the low-frequency behavior of the Green's functions $G_{\hat{a}\hat{b}}$ for arbitrary quasilocal operators $\hat{a}$ and $\hat{b}$. The allowance made for the anisotropy of the system associated with the fact that in the state of statistical equilibrium the superfluid momentum $\mathbf{p}$ and the normal velocity $\mathbf{v}_n$ are nonzero is important.

The approach considered here made it possible to study helical magnets, in which the symmetry with respect to spin rotations is broken.[8] In this case, the presence in equilibrium of a paramerer $\mathbf{p}\neq 0$ analogous to the superfluid momentum leads to the appearance of magnetic structure with helical pitch $2\pi/|\mathbf{p}|$; the case of planar magnetic structures ($\mathbf{p}=0$) was investigated in Ref. 4. In addition, this approach makes it possible to consider not only systems with spontaneously broken phase invariance but also one in which translational invariance is broken.[9]

It is also of interest to apply the approach to superfluid crystals,[10,11] in which the translational and phase invariance

 0090-4759/85/040389-21$04.40

are broken simultaneously, and also to the study of the superfluidity of ^{3}He, in which there is simultaneous breaking of phase and rotational invariance.

1. THE STATE OF STATISTICAL EQUILIBRIUM FOR SUPERFLUID BOSE SYSTEMS

For systems with spontaneously broken sysmmetry, the state of statistical equilibrium has a lower symmetry than the Hamiltonian. A convenient concept making it possible to describe systems with spontaneously broken symmetry is that of quasiaverages.

According to Bogolyubov,[1,12] the average (expectation) values in a state of statistical equilibrium (with broken symmetry) are determined by

$$\langle \ldots \rangle = \lim_{\nu\to 0} \lim_{V\to\infty} \operatorname{Sp} w_\nu \ \ldots, \quad w_\nu = \exp(\Omega_\nu - Y_\alpha \hat{\gamma}_\alpha - \nu \hat{f}). \quad (1)$$

Here, V is the volume of the system, $\hat{\gamma}_\alpha$ are the operators of the additive integrals of the motion with respect to the Hamiltonian $\mathscr{H} = \gamma_0$, Y_α are the thermodynamic forces conjugate to them, and Ω_ν is the thermodynamic potential, determined from the condition $\operatorname{Sp} w_\nu = 1$. The operator $\hat{f}$ possesses the symmetry of the investigated phase and lifts the degeneracy of the state of statistical equilibrium. The limit

$$\lim_{\nu\to 0} \lim_{V\to\infty} \frac{\Omega_\nu}{V} = \omega$$

determines the density of the thermodynamic potential. For quasiaverages (in contrast to ordinary averages) the principle of spatial correlation weakening is always valid, i.e.,

$$\langle \hat{a}(\mathbf{x})\, \hat{b}(\mathbf{y}) \rangle \xrightarrow[|\mathbf{x}-\mathbf{y}|\to\infty]{} \langle \hat{a}(\mathbf{x}) \rangle \langle \hat{b}(\mathbf{y}) \rangle, \quad (2)$$

where $\hat{a}(\mathbf{x})$ and $\hat{b}(\mathbf{y})$ are arbitrary quasilocal operators.

Suppose that the state of a superfluid is described by the statistical operator ρ, which satisfies the principle of spatial correlation weakening. The operator ρ evolves in time in accordance with von Neumann's equation

$$i\,\frac{\partial \rho(t)}{\partial t} = [\mathscr{H}, \rho(t)]. \quad (3)$$

Introducing the operator $\hat{\mathscr{P}}$ of the total momentum vector of the superfluid system, we determine the conditions of spatial homogeneity by

$$[\rho, \hat{\mathscr{P}} - \mathbf{p}\hat{N}] = 0. \quad (4)$$

This condition means that a macroscopically large number of particles can be in the state with momentum $\mathbf{p}$ (form a condensate). Indeed, for the operator $a_{\mathbf{p}'} \equiv V^{-1/2} \int d^3x\, e^{-\mathbf{p}'\mathbf{x}} \psi(\mathbf{x})$ we have

$$\operatorname{Sp} \rho\, [\hat{\mathscr{P}} - \mathbf{p}\hat{N}, a_{\mathbf{p}'}] = (\mathbf{p} - \mathbf{p}') \operatorname{Sp} \rho a_{\mathbf{p}'} = 0.$$

Thus, only for a state with momentum $\mathbf{p}' = \mathbf{p}$ can the mean value of the operator $a_{\mathbf{p}}$ be nonzero. Since $[\hat{N}, \rho] \neq 0$ for superfluid systems, for spatially homogeneous superfluid systems $[\rho, \mathscr{P}] \neq 0$, i.e., the spatially homogeneous state is not translationally invariant. However, let $\hat{a}(\mathbf{x})$ be an arbitrary quasilocal translationally invariant operator. Then

$$i\,\frac{\partial \hat{a}(\mathbf{x})}{\partial x_k} = [\hat{\mathscr{P}}_k, \hat{a}(\mathbf{x})]. \quad (5)$$

If $[\hat{N}, \hat{a}] = 0$, then in accordance with (4) $\operatorname{Sp} \rho [\hat{\mathscr{P}}_k\, \hat{a}(\mathbf{x})] = 0$, and , therefore, $\operatorname{Sp} \rho \hat{a}(\mathbf{x})$ does not depend on $\mathbf{x}$ in accordance with (5). Thus, the average value of an arbitrary translationally invariant operator that commutes with the particle-number operator $\hat{N}$ does not depend on $\mathbf{x}$ for a spatially homogeneous state [see (4)].

If the initial statistical operator ρ satifies the condition (4) of spatial homogeneity, then the system goes over rapidly, after the relaxation time τ_r, to the state of statistical equilibrium. This means that the following relation must hold:

$$\rho(t) \xrightarrow[t \gg \tau_r]{} w(t) \equiv \lim_{\nu\to 0} \lim_{V\to\infty} w_\nu(t),$$

$$w_\nu(t) = \exp\Big\{ \Omega_\nu - Y_\alpha \hat{\gamma}_\alpha - \nu Y_0 \int d^3x \left(\psi(\mathbf{x})\, e^{-i\varphi(\mathbf{x},t)} + \text{h.c.} \right) \Big\}. \quad (6)$$

Here $\hat{\gamma}_0 = \mathscr{H}$ is the Hamiltonian, $\hat{\gamma}_k \equiv \hat{\mathscr{P}}_k$ is the momentum operator, $\hat{\gamma}_4 \equiv \hat{N}$ is the particle-number operator, $\psi(\mathbf{x})$ is the operator of particle annihilation at the point $\mathbf{x}$, and $\varphi(\mathbf{x},t)$, as we shall see below, is the asymptotic phase of $\operatorname{Sp} \rho(t) \psi(\mathbf{x})$. Thus, the phenomenon of superfluidity is associated with breaking of the symmetry with respect to phase transformations.

The limit $V \to \infty$, $\nu \to 0$ is to be understood in the sense of mean values, i.e.,

$$\operatorname{Sp} w\hat{a} \equiv \lim_{\nu\to 0} \lim_{V\to\infty} \operatorname{Sp} w_\nu \hat{a}. \quad (7)$$

Thus, the statistical operator w must be regarded as a generalized operator, i.e., like generalized functions it is determined by the mean values of arbitrary quasilocal operators $\hat{a}(\mathbf{x})$ with kernels that decrease sufficiently rapidly.

The limit $t \gg \tau_r$ is also to be understood in the sense of mean values.

To find the asymptotic phase $\varphi(\mathbf{x},t)$, we note that the condition(4) of spatial homogeneity is satisfied for all times t with the same momentum vector $\mathbf{p}$ if it is satisfied at the initial time. Therefore, in accordance with (4) and (6)

$$[w(t), \hat{\mathscr{P}}_k - p_k \hat{N}] = 0 \quad (8)$$

and, therefore,

$$\varphi(\mathbf{x}, t) = \mathbf{p}\mathbf{x} + \varphi(0, t). \quad (9)$$

We have taken into account the fact that

$$e^{-i\hat{\mathscr{P}}\mathbf{y}} \psi(\mathbf{x})\, e^{i\hat{\mathscr{P}}\mathbf{y}} = \psi(\mathbf{x}+\mathbf{y}), \quad e^{i\hat{N}\alpha} \psi(\mathbf{x})\, e^{-i\hat{N}\alpha} = e^{-i\alpha} \psi(\mathbf{x}). \quad (10)$$

The method of quasiaverages presupposes that for any (with kernel that decreases sufficiently rapidly) quasilocal operator $\hat{a}(\mathbf{x})$ there exist quasiaverages, i.e,

$$\lim_{\nu\to 0} \lim_{V\to\infty} \operatorname{Sp} w_\nu\, \hat{a}(\mathbf{x}) \equiv \operatorname{Sp} w\hat{a}(\mathbf{x}) < \infty. \quad (11)$$

We note further that

$$[w_\nu(t), Y_0 \mathscr{H} + \mathbf{Y}\hat{\mathscr{P}} + Y_4 \hat{N} + \nu \hat{f}] = 0,$$

$$\hat{f} = Y_0 \int d^3x\, \psi(\mathbf{x})\, e^{-i\varphi(\mathbf{x},t)} + \text{h.c.}$$

Since the operator $[\hat{f}, \hat{a}(\mathbf{x})]$ is also quasilocal by virtue of the canonical commutation relations,

$$\lim_{v\to 0}\lim_{V\to\infty} v\,\mathrm{Sp}\, w_v(t)\,[\hat{f},\ \hat{a}(\mathbf{x})]=0.$$

Therefore

$$\lim_{v\to 0}\lim_{V\to\infty}\mathrm{Sp}\,[w_v(t),\ Y_0\mathscr{H}+\mathbf{Y}\hat{\mathscr{P}}+Y_4\hat{N}]\,\hat{a}(\mathbf{x})=0$$

and, therefore,

$$[w(t),\ Y_0\mathscr{H}+\mathbf{Y}\mathscr{P}+Y_4\hat{N}]=0.$$

Thus, taking into account (8), we have

$$[w(t),\ \mathscr{H}+p_0\hat{N}]=0,\quad p_0=\frac{Y_4+\mathbf{Yp}}{Y_0}. \tag{12}$$

We shall call this relation the stationarity condition. The statistical operator $w(t)$ must satisfy Eq. (3). Therefore

$$w(t+\tau)=e^{-i\mathscr{H}\tau}w(t)\,e^{i\mathscr{H}\tau}. \tag{13}$$

Hence, using (12), we obtain

$$w(t+\tau)=e^{ip_0\hat{N}\tau}w(t)\,e^{-ip_0\hat{N}\tau} \tag{14}$$

and, therefore, in accordance with (6), (10), and (14)

$$\varphi(\mathbf{x},\ t+\tau)=\varphi(\mathbf{x},\ t)+p_0\tau. \tag{15}$$

Thus, taking into account (9) and (15), we obtain

$$\varphi(\mathbf{x},\ t)=\mathbf{px}+p_0t+\chi,\ \chi\equiv\varphi(0,\ 0). \tag{16}$$

From Eqs. (6) and (16) we see that the state of statistical equilibrium is characterized by the thermodynamic parameters $Y_\alpha(\alpha=0,1,2,3,4)$, p_k,χ(Y_0 is the reciprocal temperature, $-Y_k/Y_0\equiv v_{nk}$ is the velocity of the normal component, $-Y_4/Y_0$ is the chemical potential, p_k is the momentum of the condensate particles, and χ is the phase). We now find the dependence of these thermodynamic parameters on the initial state $\rho(0)$.

Let $\zeta_\alpha(\mathbf{x})\equiv\{\hat{\varepsilon}(\mathbf{x}),\hat{\pi}_i(\mathbf{x}),\hat{n}(\mathbf{x})\}$ be the operators of the densities of the additive integrals of the motion, i.e., $\hat{\gamma}_\alpha=\int d^3x\hat{\zeta}_\alpha(\mathbf{x})$. Then the following differential conservation laws hold:

$$i\,[\mathscr{H},\ \hat{\zeta}_\alpha(\mathbf{x})]=-\frac{\partial\hat{\zeta}_{\alpha k}(\mathbf{x})}{\partial x_k}, \tag{17}$$

where $\hat{\zeta}_{\alpha k}(\mathbf{x})\equiv\{\hat{q}_k(\mathbf{x}),\hat{t}_{ik}(\mathbf{x}),\hat{j}_k(\mathbf{x})\}$ are the corresponding operators of the flux densities, which in accordance with Ref. 13 can be represented in the form

$$\hat{q}_k(\mathbf{x})=\frac{i}{2}\int d^3x'x'_k\int_0^1 d\xi\,[\hat{\varepsilon}(\mathbf{x}-(1-\xi)\mathbf{x}'),\ \hat{\varepsilon}(\mathbf{x}+\xi\mathbf{x}')];$$

$$\hat{t}_{ik}(\mathbf{x})=-\hat{\varepsilon}(\mathbf{x})\,\delta_{ik}+i\int d^3x'x'_k\int_0^1 d\xi\,[\hat{\varepsilon}(\mathbf{x}-(1-\xi)\mathbf{x}'),\ \hat{\pi}_i(\mathbf{x}+\xi\mathbf{x}')]; \tag{18}$$

$$\hat{j}_k(\mathbf{x})=i\int d^3x'x'_k\int_0^1 d\xi\,[\hat{\varepsilon}(\mathbf{x}-(1-\xi)\mathbf{x}'),\ \hat{n}(\mathbf{x}+\xi\mathbf{x}')].$$

Since $[\hat{N},\hat{\zeta}_{\alpha k}(\mathbf{x})]=0$ and the statistical operator $\rho(t)$ in (6) satisfies the condition (4), it follows from (17) that $\mathrm{Sp}\,\rho(t)\hat{\zeta}_\alpha(\mathbf{x})$ does not depend on t (this mean value is also independent of $\mathbf{x}$). Therefore

$$\mathrm{Sp}\,w(t)\,\hat{\zeta}_\alpha(0)=\mathrm{Sp}\,\rho(0)\,\hat{\zeta}_\alpha(0). \tag{19}$$

This relation determines the dependence of the thermodynamic parameters Y_α on the initial statistical operator $\rho(0)$.

We introduce the unitary operator

$$U_\varphi(t)=\exp\left\{-i\int d^3x\varphi(\mathbf{x},\ t)\,\hat{n}(\mathbf{x})\right\}. \tag{20}$$

Since $\hat{n}(\mathbf{x})=\psi^+(\mathbf{x})\psi(\mathbf{x})$,

$$U_\varphi(t)\,\psi(\mathbf{x})\,U_\varphi^+(t)=e^{i\varphi(\mathbf{x},\ t)}\psi(\mathbf{x}). \tag{21}$$

Taking the function (16) as $\varphi(\mathbf{x},t)$, we readily see that

$$\begin{aligned}U_\varphi(t)\,w_v(t)\,U_\varphi^+(t)&\equiv w'_v\\&=\exp\left\{\Omega_v-Y_0\mathscr{H}_\mathbf{p}-Y_k(\hat{\mathscr{P}}_k+p_k\hat{N})\right.\\&\left.\quad-Y_4\hat{N}-vY_0\int d^3x\,(\psi(\mathbf{x})+\psi^+(\mathbf{x}))\right\},\end{aligned} \tag{22}$$

where

$$\mathscr{H}_\mathbf{p}=U_\mathbf{p}\mathscr{H}U_\mathbf{p}^+,\ U_\mathbf{p}=\exp\left\{-i\mathbf{p}\int d^3x\mathbf{x}\hat{n}(\mathbf{x})\right\}. \tag{23}$$

We have also borne in mind that $U_\mathbf{p}\,\hat{\mathscr{P}}_k\,U_p^+=\hat{\mathscr{P}}_k+p_k\hat{N}$, since $[\hat{\mathscr{P}}_k\,\hat{n}(\mathbf{x})]=i\partial\hat{n}(x)/\partial x_k$. Since in accordance with (21) and (22)

$$\mathrm{Sp}\,w_v(t)\,\psi(\mathbf{x})=e^{i\varphi(\mathbf{x},\ t)}\,\mathrm{Sp}\,w'_v\psi(\mathbf{x}) \tag{24}$$

and $\mathrm{Sp}\ w'_v\psi(\mathbf{x})=\mathrm{Sp}\ w'_v\psi(\mathbf{x}))^*$, is follows that $\varphi(\mathbf{x},t)$ is the phase of $\mathrm{Sp}\,w_v(t)\psi(\mathbf{x})$.

Introducing the exact phase of $\psi(\mathbf{x})$,

$$\bar{\varphi}(\mathbf{x},\ t)=\mathrm{Im}\ln\mathrm{Sp}\,\rho(t)\,\psi(x)\equiv\mathbf{px}+\bar{\varphi}(t),$$

$$\bar{\varphi}(t)=\mathrm{Im}\ln\mathrm{Sp}\,\rho(t)\,\psi(0)$$

[we have used the fact that $\rho(t)$ is a spatially homogeneous state: see (4)], we can, using (24), readily show that χ in (16) is determined by

$$\chi=\bar{\varphi}(0)+\int_0^\infty d\tau\,(\dot{\bar{\varphi}}(\tau)-p_0) \tag{25}$$

(since $\bar{\varphi}(\tau)\underset{\tau\to\infty}{\longrightarrow}\rho_0\tau+\chi$, the integral converges as $\tau\to\infty$).

This formula also solves the problem of finding χ for the functional $\rho(0)$. Equation (6), which we rewrite in the form

$$\begin{aligned}\rho(t)&\xrightarrow[t\gg\tau_r]{}w(Y_\alpha,\ \mathbf{p},\ p_0t+\chi)\\&=\lim_{v\to 0}\lim_{V\to\infty}w_v(Y_\alpha,\ \mathbf{p},\ p_0t+\chi),\end{aligned} \tag{26}$$

where

$$\begin{aligned}&w_v(Y_\alpha,\ \mathbf{p},\ \chi)\\&=\exp\left\{\Omega_v-Y_\alpha\hat{\gamma}_\alpha-vY_0\int d^3x\,(\psi(\mathbf{x})\,e^{-i(\mathbf{px}+\chi)}+\text{h.c.})\right\},\end{aligned} \tag{27}$$

determines in conjuction with Eqs. (16), (19), and (25) the ergodic relation for superfluid Bose systems.

Summarizing, we can say that in these superfluid systems (which we shall call generalized superfluid systems, since we have nowhere used the requirement of Galilean invariance) the state of thermodynamic equilibrium is characterized by the thermodynamic forces Y_α associated with the additive integrals of the motion and also by the superfluid momentum **p** and the phase $\chi \equiv \varphi$ (0,0), the presence of which is due to the breaking of the symmetry of the state of statistical equilibrium. We emphasize that the dependence on the thermodynamic variables **p** and χ is introduced through infinitesimally small sources, and in the state with broken symmetry this dependence persists in the limit $\nu \to 0$.

2. THERMODYNAMICS OF SUPERFLUID SYSTEMS

We introduce the density of the thermodynamic potential ω:

$$\omega = \lim_{\nu\to 0}\lim_{V\to\infty}\frac{\Omega_\nu}{V} = \omega(Y_0,\ \mathbf{Y}^2,\ Y_4,\ \mathbf{p}^2,\ \mathbf{Yp})$$

$$= -\lim_{\nu\to 0}\lim_{V\to\infty}\frac{1}{V}\ln \operatorname{Sp}\exp\Big\{-Y_0\hat{\mathscr{H}}_\mathbf{p} - Y_k(\hat{\mathscr{P}}_k + p_k\hat{N})$$

$$-Y_4\hat{N} - \nu Y_0\int d^3x\,(\psi(\mathbf{x}) + \psi^+(\mathbf{x}))\Big\}, \qquad (28)$$

this being a function of the thermodynamic parameters Y_0, $\mathbf{Y}^2$, Y_4, $\mathbf{p}^2$, $\mathbf{Y}_\mathbf{p}$. Differentiating ω with respect to the thermodynamic forces Y_α and the superfulid momentum **p**, we obtain

$$\frac{\partial\omega}{\partial Y_0} = \operatorname{Sp} w'\hat{\varepsilon}_\mathbf{p}(0) = \operatorname{Sp} w\hat{\varepsilon}(0) \equiv \varepsilon,$$

$$\frac{\partial\omega}{\partial Y_l} = \operatorname{Sp} w'(\hat{\pi}_l(0) + p_l\hat{n}(0)) = \operatorname{Sp} w\hat{\pi}_l(0) \equiv \pi_l,$$

$$\frac{\partial\omega}{\partial Y_4} = \operatorname{Sp} w\hat{n}(0) \equiv n,\quad \frac{\partial\omega}{\partial p_l} \qquad (29)$$

$$= \lim_{\nu\to 0}\lim_{V\to\infty}\Big\{\frac{Y_0}{V}\operatorname{Sp} w'_\nu\frac{\partial\hat{\mathscr{H}}_\mathbf{p}}{\partial p_l} + \frac{Y_l}{V}\operatorname{Sp} w'_\nu\hat{N}\Big\}.$$

Here $w \equiv w(Y, \mathbf{p}, \chi)$ (since $[\hat{N}, \hat{\xi}_\alpha] = 0$, the traces in these expressions do not depend on χ). Noting further that $\partial\hat{\mathscr{H}}_\mathbf{p}\partial p_l = -i\int d^3x x_l[\hat{n}(\mathbf{x}), \hat{\mathscr{H}}_\mathbf{p}]$, we find

$$\lim_{\nu\to 0}\lim_{V\to\infty}\frac{1}{V}\operatorname{Sp} w'_\nu\frac{\partial\hat{\mathscr{H}}_\mathbf{p}}{\partial p_l}$$

$$= -i\int d^3x x_l \operatorname{Sp} w'[\hat{\varepsilon}_\mathbf{p}(\mathbf{x}),\ \hat{n}(0)],\quad \hat{\varepsilon}_\mathbf{p} = U_\mathbf{p}\hat{\varepsilon}U_\mathbf{p}^+.$$

On the other hand, averaging the expression (18) for the operator $\hat{j}_l(\mathbf{x})$ of the particle-number flux density with the statistical operator w and bearing in mind that $[U_\mathbf{p}, \hat{n}(0)] = 0, [\hat{\varepsilon}(\mathbf{x}), \hat{N}] = 0$, we obtain

$$j_l = \operatorname{Sp} w\hat{j}_l(0) = -i\int d^3x x_l \operatorname{Sp} w'[\hat{\varepsilon}_\mathbf{p}(\mathbf{x}),\ \hat{n}(0)] \qquad (30)$$

(we have noted that $[w', \hat{\mathscr{P}}] = 0$). Therefore,

$$j_l = \frac{1}{Y_0}\frac{\partial\omega}{\partial p_l} - \frac{Y_l}{Y_0}\frac{\partial\omega}{\partial Y_4}. \qquad (30')$$

Thus, we have found an expression for the particle-number flux density in terms of the density ω of the thermodynamic potential. Using Eqs. (29) and (30), we find the following fundamental thermodynamic equation:

$$d\omega = \varepsilon\, dY_0 + \pi_k\, dY_k + n\, dY_4 + (Y_0 j_l + Y_l n)\, dp_l, \qquad (31)$$

which represents the second law of thermodynamics for reversible processes in the superfluid.

In constructing the hydrodynamics of the ideal superfluid, and also to investigate the low-frequency behavior of the Green's functions, it is necessary to express the mean values in the state w of the operators of the momentum, $\hat{t}_{ik}$, and energy, $\hat{q}_k$, flux densities in terms of the density ω of the thermodynamic potential. Turning to the determination of these quantities, we note that in accordance with (18)

$$\left.\begin{aligned} t_{ik} &= \operatorname{Sp} w\hat{t}_{ik}(0) = -\langle\hat{\varepsilon}_\mathbf{p}(0)\rangle\delta_{ik} \\ &-i\int d^3x x_k\langle[\hat{\varepsilon}_\mathbf{p}(\mathbf{x}),\ \hat{\pi}_i(0)]\rangle + p_i j_k, \\ q_k &= \operatorname{Sp} w\hat{q}_k(0) = -\frac{i}{2}\int d^3x x_k\langle[\hat{\varepsilon}_\mathbf{p}(\mathbf{x}),\ \hat{\varepsilon}_\mathbf{p}(0)]\rangle, \\ \langle\ldots\rangle &\equiv \operatorname{Sp} w'_\nu \ldots \end{aligned}\right\} \qquad (32)$$

(we have used the fact that $[w'_\nu, \hat{\mathscr{P}}_k] = 0$; we emphasize that the averaging $\langle\cdots\rangle$ is over the state w'_ν). For convenience, we first calculate the quantities

$$\left.\begin{aligned} t'_{ik} &= -\langle\hat{\varepsilon}'_\mathbf{p}(0)\rangle\delta_{ik} - i\langle[\Gamma_{ki},\ \hat{\varepsilon}'_\mathbf{p}(0)]\rangle + p_i j_k, \\ q'_k &= -\frac{i}{2}\int d^3x x_k\langle[\hat{\varepsilon}'_\mathbf{p}(\mathbf{x}),\ \hat{\varepsilon}'_\mathbf{p}(0)]\rangle, \end{aligned}\right\} \qquad (33)$$

where

$$\hat{\varepsilon}'_\mathbf{p}(\mathbf{x}) \equiv \hat{\varepsilon}_\mathbf{p}(\mathbf{x}) + \hat{n}(\mathbf{x})\frac{Y_4 + \mathbf{Yp}}{Y_0} + \nu(\psi^+(\mathbf{x}) + \psi(\mathbf{x}))$$

and

$$\Gamma_{ki} = \int d^3x x_k\hat{\pi}_i(\mathbf{x})$$

is the generator of the group of arbitrary linear transformations $x_i \to x'_i = a_{ik}x_k$. The last assertion follows from the fact that the operators $\psi(\mathbf{x})$ and $\psi'(\mathbf{x}) = \psi(a(\mathbf{x})|\det a|^{1/2}$ satisfy the same commutation relations and, therefore, are connected by a unitary transformation U_a:

$$U_a\psi(\mathbf{x})U_a^+ = \psi'(\mathbf{x}) = |\det a|^{1/2}\psi(a\mathbf{x}).$$

Considering infinitesimally small transformations $a_{ik} = \delta_{ik} + \xi_{ik}, |\xi| \ll 1$, we readily find that $U_a = 1 - i\xi_{kl}\Gamma_{lk}$. From the condition $\operatorname{Sp} w'_\nu = 1$ we obtain [see (22)]

$$e^{-\Omega} = \operatorname{Sp}\exp\Big\{-\int_V d^3x\hat{h}(\mathbf{x})\Big\},\quad \hat{h}(\mathbf{x}) = Y_0\hat{\varepsilon}'_\mathbf{p}(\mathbf{x}) + Y_l\hat{\pi}_l(\mathbf{x}).$$

Then, determining the operator $\hat{h}_a(\mathbf{x})$ by

$$U_a\hat{h}(\mathbf{x})U_a^+ \equiv \hat{h}_a(a\mathbf{x})|\det a|, \qquad (34)$$

we find that

$$e^{-\Omega} = \operatorname{Sp} U_a\exp\Big\{-\int_V d^3x\hat{h}(\mathbf{x})\Big\}U_a^+$$

$$= \operatorname{Sp}\exp\Big\{-\int_{V_a} d^3x\hat{h}_a(\mathbf{x})\Big\},$$

where $V_a \equiv V|\det a|$. Since the potential Ω is proportional to V,

$$\exp(-\Omega/|\det a|) = \operatorname{Sp}\exp\Big\{-\int_V d^3x\hat{h}_a(\mathbf{x})\Big\}.$$

Hence, bearing in mind that for infinitesimally small transformations $\det a = 1 + \delta_{kl}\xi_{kl}$, and noting that

$$\operatorname{Sp} w\Big(\frac{\partial\hat{h}_a(0)}{\partial\xi_{kl}}\Big)_{\xi=0} = -\overset{\nu}{\omega}\delta_{kl},\quad \overset{\nu}{\omega} = \lim_{V\to\infty}\Omega_\nu/V.$$

we obtain

$$\hat{h}(\mathbf{x}) = e^{-i\hat{\mathscr{P}}\mathbf{x}}\hat{h}(0)\,e^{i\hat{\mathscr{P}}\mathbf{x}},\quad [w'_\nu,\ \hat{\mathscr{P}}] = 0,$$

However, for small ξ_{kl}

$$U_a = 1 - i\xi_{kl}\Gamma_{lk},$$

and it therefore follows from (34) that

$$-i[\Gamma_{lk},\ \hat{h}(0)] = \left(\frac{\partial h_a(0)}{\partial \xi_{kl}}\right)_{\xi=0} + \delta_{kl}\hat{h}(0),$$

and therefore

$$i\langle[\Gamma_{lk},\ \hat{h}(0)]\rangle + \delta_{kl}\langle\hat{h}(0)\rangle = \overset{\nu}{\omega}\delta_{kl}. \tag{35}$$

Substituting here the expression for $\hat{h}(\mathbf{x})$ and using the readily verified equality

$$i[\Gamma_{ki},\ \hat{\pi}_l(0)] = -\delta_{ik}\hat{\pi}_l(0) - \delta_{lk}\hat{\pi}_i(0),$$

we have in accordance with (30), (31), (33), and (35)

$$t'_{ik} = p_i j_k - \delta_{ki}\frac{\overset{\nu}{\omega}}{Y_0} - \frac{Y_k}{Y_0}\langle\hat{\pi}_i(0)\rangle = \frac{p_i}{Y_0}\frac{\partial\omega}{\partial p_k} - \frac{\partial}{\partial Y_i}\frac{\omega Y_k}{Y_0}. \tag{36}$$

Recalling the definition of $\hat{\varepsilon}'_{\mathbf{p}}(0)$ and noting that

$$i[\Gamma_{ki},\ \hat{n}(0)] = -\delta_{ik}\hat{n}(0),$$

we find from (32), (33), and (36) the following expression for the momentum flux density in the state w:

$$t_{ik} = t'_{ik} = \frac{p_i}{Y_0}\frac{\partial\omega}{\partial p_k} - \frac{\partial}{\partial Y_i}\frac{\omega Y_k}{Y_0}. \tag{37}$$

We now turn to the calculation of the energy flux density q_k, which in accordance with (32) and (33) is related to q'_k by

$$q'_k = q_k - ip_0\langle[\Gamma_k,\ \hat{\varepsilon}_{\mathbf{p}}(0)]\rangle - \frac{i}{2}p_0^2\langle[\Gamma_k,\ \hat{n}(0)]\rangle,$$

$$p_0 = \frac{Y_4 + Y\mathbf{p}}{Y_0},$$

where $\Gamma_k \equiv \int d^3x x_k\hat{n}(\mathbf{x})$ is a generator of the group of Galileo transformations. (We recall that we do not assume invariance of the Hamiltonian with respect to such transformtions.) In deriving this formula, we have used the face that in accordance with the method of quasiaverages, as already noted, the limit of the averages of quasilocal operators as $\nu \to 0$ is finite.

Noting that

$$[\Gamma_k,\ \hat{n}(0)] = 0,\quad i\langle[\Gamma_k,\ \hat{\pi}_i(0)]\rangle = -\delta_{ik}n,\quad i\langle[\Gamma_k,\ \hat{\varepsilon}_{\mathbf{p}}(0)]\rangle = -j_k,$$

we obtain

$$q'_k = q_k + p_0 j_k. \tag{38}$$

To find q'_k, we use the obvious formula

$$[w'_\nu,\ \hat{\varepsilon}'_{\mathbf{p}}(\mathbf{x})] = w'_\nu\int_0^1 d\lambda\{-Y_0[\mathscr{H}'_{\mathbf{p}},\ \hat{\varepsilon}'_{\mathbf{p}}(\mathbf{x};\ \lambda)] - Y_k[\hat{\mathscr{P}}_k,\ \hat{\varepsilon}'_{\mathbf{p}}(\mathbf{x};\ \lambda)]\}, \tag{39}$$

where

$$\mathscr{H}'_{\mathbf{p}} \equiv \int d^3x\hat{\varepsilon}'_{\mathbf{p}}(\mathbf{x}),\quad \hat{\varepsilon}'_{\mathbf{p}}(\mathbf{x};\ \lambda) \equiv w^{-\lambda}\hat{\varepsilon}'_{\mathbf{p}}(\mathbf{x})\,w^{\lambda},$$

and the operator $\hat{q}'_k(\mathbf{x})$ of the energy flux density corresponding to the Hamiltonian $\mathscr{H}'_{\mathbf{p}}$,

$$i[\mathscr{H}'_{\mathbf{p}},\ \hat{\varepsilon}'_{\mathbf{p}}(\mathbf{x})] = -\frac{\partial\hat{q}'_k(\mathbf{x})}{\partial x_k},$$

has the form [see (18)]

$$\hat{q}'_k(\mathbf{x}) = \frac{i}{2}\int d^3x' x'_k\int_0^1 d\xi[\hat{\varepsilon}'_{\mathbf{p}}(\mathbf{x} - (1-\xi)\mathbf{x}'),\ \hat{\varepsilon}'_{\mathbf{p}}(\mathbf{x} + \xi\mathbf{x}')].$$

Since $[w'_\nu, \hat{\mathscr{P}}] = 0$, is follows from (33) that $q'_k = \mathrm{Sp}\, w'_\nu\hat{q}'_k(\mathbf{x})$. Bearing in mind that

$$[\hat{\mathscr{P}}_k,\ \hat{\varepsilon}'_{\mathbf{p}}(\mathbf{x})] = i\frac{\partial\hat{\varepsilon}'_{\mathbf{p}}(\mathbf{x})}{\partial x_k},$$

we find from (39) that

$$[w'_\nu,\ \hat{\varepsilon}'_{\mathbf{p}}(\mathbf{x})] = -i\frac{\partial}{\partial x_k}w'_\nu\int_0^1 d\lambda\{Y_0(\hat{q}'_k(\mathbf{x};\ \lambda) - \langle\hat{q}'_k\rangle) + Y_k(\hat{\varepsilon}'_{\mathbf{p}}(\mathbf{x};\ \lambda) - \langle\hat{\varepsilon}'_{\mathbf{p}}\rangle)\}.$$

Using this expression and also the principle of spatial correlation weakening, we readily find that

$$i\int d^3x x_k\,\mathrm{Sp}\,w'_\nu[\hat{\varepsilon}'_{\mathbf{p}}(\mathbf{x}),\ \hat{\zeta}'_\alpha(0)]$$
$$= Y_0\,\mathrm{Sp}\frac{\partial w'_\nu}{\partial Y_\alpha}\hat{q}'_k(0) + Y_k\,\mathrm{Sp}\frac{\partial w'_\nu}{\partial Y_\alpha}\hat{\varepsilon}'_{\mathbf{p}}(0)$$
$$= Y_0\frac{\partial q'_k}{\partial Y_\alpha} + Y_k\frac{\partial\varepsilon'}{\partial Y_\alpha} - (Y_0 j_k + Y_k n)\frac{\partial}{\partial Y_\alpha}\frac{Y_4 + Y\mathbf{p}}{Y_0},$$
$$\varepsilon' \equiv \langle\hat{\varepsilon}'_{\mathbf{p}}(0)\rangle, \tag{40}$$

where $\hat{\zeta}'_\alpha \equiv (\hat{\varepsilon}_{\mathbf{p}}\,\hat{\pi}_k + p_k\hat{n}, \hat{n})$. We have also used the fact that

$$\frac{\partial w'_\nu}{\partial Y_\alpha} = -w'_\nu\int_0^1 d\lambda\int d^3x(\hat{\zeta}'_\alpha(\mathbf{x};\ \lambda) - \langle\hat{\zeta}'_\alpha\rangle).$$

We mention that in deriving (40) we integrated by parts and ignored the integrals over the infinitely distant surface, which can always be done for $\nu \neq 0$ (for $\nu \neq 0$, the equilibrium correlations decrease sufficiently rapidly, something that, by virtue of Bogolyubov's theorem on singularities of the type $1/q^2$, cannot be said of the correlations when $\nu = 0$; at the end of the calculations, we must go to the limit $\nu \to 0$).

For $\alpha = 0$, it follows from (40) that

$$-2q'_k = Y_0\frac{\partial q'_k}{\partial Y_0} + Y_k\left(\frac{\partial\varepsilon}{\partial Y_0} + p_0\frac{\partial n}{\partial Y_0}\right). \tag{41}$$

For $\alpha = i$, taking into account (37) and (30), we have

$$Y_0\frac{\partial q'_k}{\partial Y_i} = \frac{\partial}{\partial Y_i}\frac{\omega Y_k}{Y_0} - \frac{\partial}{\partial Y_i}\left\{Y_k\left(\varepsilon + \frac{Y_4 + Y\mathbf{p}}{Y_0}n\right)\right\}. \tag{42}$$

For $\alpha = 4$, we obtain from (40)

$$Y_0\frac{\partial q'_k}{\partial Y_4} = -Y_k\left(\frac{\partial\varepsilon}{\partial Y_4} + p_0\frac{\partial n}{\partial Y_4}\right). \tag{43}$$

It follows from the second equation that

$$q'_k = -\frac{Y_k}{Y_0}\left(\frac{\partial\omega}{\partial Y_0} - \frac{\omega}{Y_0} + p_0\frac{\partial\omega}{\partial Y_4}\right) + C_k(Y_0,\ Y_4,\ \mathbf{p}), \quad (44)$$

where the constant of integration C_k does not depend on $\mathbf{Y}$. Substitution of this result in the third equation shows that $\partial C_k/\partial Y_4 = 0$. In turn, from the first equation we find that $C_k = C_k(\mathbf{p})/Y_2^0$.

Thus, if follows from (30), (38), and (44) that the energy flux density q_k in the state w of thermodynamic equilibrium is determined by

$$q_k = \mathrm{Sp}\, w\hat{q}_k(\mathbf{x}) = -\frac{\partial}{\partial Y_0}\frac{\omega Y_k}{Y_0} - \frac{p_0}{Y_0}\frac{\partial\omega}{\partial p_k} + \frac{C_k(\mathbf{p})}{Y_0^2}, \quad (45)$$

where $C_k(\mathbf{p})$ is an unknown constant that depends only on the superfluid momentum $\mathbf{p}$ and not on Y_0, $\mathbf{Y}$, Y_4.

The expressions (30), (37), and (45) for the densities and flux densities ζ_{ak} can obviously be written in the form

$$\zeta_{\alpha k} = -\frac{\partial}{\partial Y_\alpha}\frac{\omega Y_k}{Y_0} + \frac{\partial\omega}{\partial p_k}\frac{\partial}{\partial Y_\alpha}\frac{Y_4 + \mathbf{Y}\mathbf{p}}{Y_0} + \frac{\delta_{\alpha 0}}{Y_0^2}C_k(\mathbf{p}), \quad (46)$$
$$\zeta_\alpha = \partial\omega/\partial Y_\alpha.$$

We now represent the expressions for the flux densities in the form corresponding to two-fluid hydrodynamics. The thermodynamic potential ω is a function of $Y_0\mathbf{Y}^2$, Y_4, $\mathbf{p}^2$, $\mathbf{Y}\mathbf{p}$. We introduce the quantities ρ_n, ρ_s $\overset{*}{m}$, which are functions of these thermodynamic variables:

$$\rho_n \equiv -2Y_0\frac{\partial\omega}{\partial\mathbf{Y}^2}, \quad \rho_s \equiv \frac{2}{Y_0}\frac{\partial\omega}{\partial\mathbf{p}^2}\overset{*}{m}^2, \quad \frac{\rho_n}{\overset{*}{m}} = n - \frac{\partial\omega}{\partial(\mathbf{Y}\mathbf{p})}. \quad (47)$$

Then with allowance for (47) the fluxes j_k, t_{ik}, q_k take the form

$$\left.\begin{aligned} j_k &= \frac{\rho_n}{\overset{*}{m}}v_{nk} + \frac{\rho_s p_k}{\overset{*}{m}^2}, \quad t_{ik} = -\frac{\omega}{Y_0}\delta_{ik} + \rho_n v_{ni}v_{nk} + \rho_s\frac{p_i p_k}{\overset{*}{m}^2},\\ q_k &= v_{nk}\left[-\frac{\omega}{Y_0} + \varepsilon + \left(n - \frac{\rho_n}{\overset{*}{m}}\right)p_0\right] - \frac{\rho_s p_k}{\overset{*}{m}^2}p_0 + \frac{C_k(\mathbf{p})}{Y_0^2}. \end{aligned}\right\} \quad (48)$$

From this it can be seen that ρ_n represents the "mass" density of the normal component and ρ_s the "mass" density of the superfluid component. If $\overset{*}{m}$ is interpreted as an effective "particle mass," then $\mathbf{p}/\overset{*}{m}$ must be interpreted as the superfluid velocity. Note that the total density $\rho = \overset{*}{m}n = \overset{*}{m}\partial\omega/\partial Y_4$ is not in general equal to the sum of the normal, ρ_n, and the superfluid, ρ_s, densities: $\rho \neq \rho_n + \rho_s$. One can therefore introduce a certain density ρ_c in accordance with

$$\rho_c \equiv \rho - \rho_n - \rho_s, \quad (49)$$

whose appearance is due, as we shall see later, to the absence of Galilean (or relativistic) invariance of the theory.

3. HYDRODYNAMICS OF AN IDEAL SUPERFLUID

We now derive the hydrodynamic equations of an ideal superfluid. According to (17), the average values of the densities of the additive integrals of the motion $\xi_\alpha(\mathbf{x},t) = \mathrm{Sp}\,\rho(t)\hat{\zeta}_\alpha(\mathbf{x})$ in the state described by the statistical operator $\rho(t)$ satisfy the equations

$$\frac{\partial\zeta_\alpha}{\partial t} = -\frac{\partial\zeta_{\alpha k}}{\partial x_k}, \quad (50)$$

where $\zeta_{ak}(\mathbf{x},t) = \mathrm{Sp}\,\rho(t)\hat{\zeta}_{ak}(\mathbf{x})$. At times $t = \tau_r$ (τ_r is the relaxation time) the statistical operator $\rho(t)$ becomes a functional of the parameters $\zeta_\alpha(\mathbf{x},t)$, $\varphi(\mathbf{x},t)$ of the reduced description:

$$\rho(t) \xrightarrow[t \gg \tau_r]{} \sigma\{\zeta_\alpha(\mathbf{x}',\ t),\ \varphi(\mathbf{x}',\ t)\} \quad (51)$$

(this assertion is known as the functional hypothesis). The functional arguments ζ_α and φ are the asymptotic ($t \gg \tau_r$) values of $\mathrm{Sp}\,\rho(t)\hat{\zeta}_\alpha(\mathbf{x})$ and $\mathrm{Im}\ln\mathrm{Sp}\,\rho(t)\psi(\mathbf{x})$.

In accordance with the definition, the functional σ must satisfy the conditions

$$\mathrm{Sp}\,\sigma\hat{\zeta}_\alpha = \zeta_\alpha, \quad \mathrm{Im}\ln\mathrm{Sp}\,\sigma\psi = \varphi. \quad (52)$$

Since $[\mathscr{H}, \hat{\mathscr{P}}] = \mathscr{H}, \hat{N}] = 0$, consequences of equations (51) and (52) are the expressions

$$\left.\begin{aligned} &e^{-i\mathscr{H}\tau}\sigma\{\zeta_\alpha(\mathbf{x}',\ t),\ \varphi(\mathbf{x}',\ t)\}e^{i\mathscr{H}\tau}\\ &= \sigma\{\zeta_\alpha(\mathbf{x}',\ t+\tau),\ \varphi(\mathbf{x}',\ t+\tau)\},\\ &e^{i\hat{\mathscr{P}}\mathbf{x}}\sigma\{\zeta_\alpha(\mathbf{x}',\ t),\ \varphi(\mathbf{x}',\ t)\}e^{-i\hat{\mathscr{P}}\mathbf{x}}\\ &= \sigma\{\zeta_\alpha(\mathbf{x}+\mathbf{x}',\ t),\ \varphi(\mathbf{x}+\mathbf{x}',\ t)\},\\ &e^{i\hat{N}\varphi'}\sigma\{\zeta_\alpha(\mathbf{x}',\ t),\ \varphi(\mathbf{x}',\ t)\}e^{-i\hat{N}\varphi'}\\ &= \sigma\{\zeta_\alpha(\mathbf{x}',\ t),\ \varphi(\mathbf{x}',\ t)+\varphi'\}. \end{aligned}\right\} \quad (53)$$

Let the initial statistical operator $\rho(0)$ correspond to a spatially homogeneous state, i.e., satisfy conditions for which the ergodic relation (26) holds. Then the asymptotic values of the reduced-description parameters in (51) have the form $\zeta_\alpha(\mathbf{x},t) = \zeta_\alpha$, $\varphi(\mathbf{x},t) = \mathbf{p}\mathbf{x} + p_0 t + \chi$, where ζ_α, $\mathbf{p}$, χ are constant that do not depend on $\mathbf{x}$ or t. Therefore, comparing (26) and (51), we have

$$\sigma(\zeta_\alpha,\ \mathbf{p}\mathbf{x}' + p_0 t + \chi) = w(Y_\alpha,\ \mathbf{p},\ p_0 t + \chi) \quad (54)$$

or

$$\sigma(\zeta_\alpha,\ \mathbf{p}\mathbf{x}' + \chi) = w(Y_\alpha,\ \mathbf{p},\ \chi), \quad (55)$$

and the parameters Y_α are determined as functions of ζ and $\mathbf{p}$ from the relation

$$\mathrm{Sp}\,w(Y_\beta,\ \mathbf{p},\ p_0 t + \chi)\hat{\zeta}_\alpha = \zeta_\alpha. \quad (56)$$

The mean value $\mathrm{Sp}\,\sigma(\zeta,\varphi)\hat{a}(\mathbf{x})$ ($\hat{a}(\mathbf{x})$ is some quasilocal operator) must obviously be determined by the values of the functions $\zeta_\alpha(\mathbf{x}')$ and $\varphi(\mathbf{x}')$ in the neighborhood of the point $\mathbf{x}$. Therefore, it follows from (55) that in the leading approximation in the gradients of the parameters $\zeta_\alpha(\mathbf{x})$ and $\varphi(\mathbf{x})$

$$\mathrm{Sp}\,\sigma\{\zeta_\alpha(\mathbf{x}'),\ \varphi(\mathbf{x}')\}\hat{a}(\mathbf{x})$$
$$\approx \mathrm{Sp}\,\sigma\left\{\zeta_\alpha(\mathbf{x}),\ \varphi(\mathbf{x}) + (\mathbf{x}'-\mathbf{x})\frac{\partial\varphi(\mathbf{x})}{\partial\mathbf{x}}\right\}\hat{a}(\mathbf{x})$$
$$= \mathrm{Sp}\,w\left\{Y_\alpha(\mathbf{x}),\ \frac{\partial\varphi(\mathbf{x})}{\partial\mathbf{x}},\ \varphi(\mathbf{x}) - \mathbf{x}\frac{\partial\varphi(\mathbf{x})}{\partial\mathbf{x}}\right\}\hat{a}(\mathbf{x}),$$

where the quantities $Y_\alpha(\mathbf{x})$ are determined from the equations

$$\mathrm{Sp}\, w\{Y_\beta,\ \mathbf{p},\ \chi\}\,\hat{\zeta}_\alpha(\mathbf{x})=\zeta_\alpha(\mathbf{x}). \tag{57}$$

Thus, it is readily seen that

$$\begin{aligned}\mathrm{Sp}\,\sigma\{\zeta_\alpha(\mathbf{x}'),\ \varphi(\mathbf{x}')\}\,\hat{a}(\mathbf{x}) &\approx \mathrm{Sp}\, w\left\{Y_\alpha(\mathbf{x}),\ \frac{\partial\varphi(\mathbf{x})}{\partial\mathbf{x}},\ \varphi(\mathbf{x})\right\}\hat{a}(0)\\ &=\mathrm{Sp}\, w\{Y_\alpha(\mathbf{x}),\ \mathbf{p}(\mathbf{x}),\ \varphi(\mathbf{x})\}\,\hat{a}(0),\end{aligned} \tag{58}$$

where the superfluid momentum $\mathbf{p}(\mathbf{x})$ of the spatially inhomogeneous system is determined by

$$\mathbf{p}(\mathbf{x}) = \partial\varphi(\mathbf{x})/\partial\mathbf{x}. \tag{59}$$

When the gradients of the parameters of the reduced description are ignored, the fluxes $\zeta_{\alpha k}$ can be calculated in accordance with the expressions $\zeta_{\alpha k}(\mathbf{x},t) = \mathrm{Sp}\,\omega(Y_\beta\,\mathbf{p},\varphi)\hat{\zeta}_{\alpha k}(\mathbf{x})$. Thus, for the fluxes $\zeta_{\alpha k}(\mathbf{x},t)$ in the leading approximation in the gradients the expressions (46) are valid, and Y_α and ζ_α are related by Eq. (46): $\zeta_\alpha=\partial\omega/\partial Y_\alpha$. In addition, it must be borne in mind that Y_α, p_k and ζ_α, p_k are now slowly varying functions of the coordinates $\mathbf{x}$ and the time t.

We now find an equation for the asymptotic phase $\varphi(\mathbf{x},t)$. Using the definition of the exact phase $\hat{\varphi}(\mathbf{x},t) = \mathrm{Im}\ln \mathrm{Sp}\,\pi(t)\psi(\mathbf{x})$ of $\psi(\mathbf{x})$, we have

$$\frac{\partial\bar{\varphi}(\mathbf{x},t)}{\partial t}=\mathrm{Re}\,\frac{\mathrm{Sp}\,\rho(t)\,[\mathscr{H},\ \psi(\mathbf{x})]}{\mathrm{Sp}\,\rho(t)\,\psi(\mathbf{x})},$$

whence, going over to the asymptotic region $t\gg\tau_r$ [see (51)] and replacing $\hat{\varphi}(\mathbf{x},t)$ by $\varphi(\mathbf{x},t)$, we obtain

$$\frac{\partial\varphi(\mathbf{x},t)}{\partial t}=\mathrm{Re}\,\frac{\mathrm{Sp}\,\sigma\,[\mathscr{H},\ \psi(\mathbf{x})]}{\mathrm{Sp}\,\sigma\psi(\mathbf{x})}.$$

Therefore, in the leading approximation in the gradients of $\zeta_\alpha(\mathbf{x})$ and $\partial\varphi(\mathbf{x},t)\partial\mathbf{x}$ we have

$$\frac{\partial\varphi(\mathbf{x},t)}{\partial t}=\mathrm{Re}\,\frac{1}{\mathrm{Sp}\,w\psi(0)}\,\mathrm{Sp}\,w\left\{Y(\mathbf{x}),\ \frac{\partial\varphi(\mathbf{x})}{\partial\mathbf{x}},\ \varphi(\mathbf{x})\right\}[\mathscr{H},\ \psi(0)]$$

or, noting that $[\omega,\mathscr{H}+p_0\hat{N}]=0$,

$$\frac{\partial\varphi(\mathbf{x},t)}{\partial t}=p_0(\mathbf{x},t)\equiv\frac{Y_4(\mathbf{x},t)+\mathbf{Y}(\mathbf{x},t)\,\mathbf{p}(\mathbf{x},t)}{Y_0(\mathbf{x},t)}. \tag{60}$$

At the same time, the superfluid momentum $\mathbf{p}(\mathbf{x},t)$ is related to the phase $\varphi(\mathbf{x},t)$ Eq. (59).

Equations (50) and (60), and also the expression (46), show that the phase $\varphi(\mathbf{x},t)$ enters the right-hand sides of these equations only through $\mathbf{p}(\mathbf{x},t)$ but not explicitly. Therefore, Eq. (60) is usually written in the form

$$\dot{\mathbf{p}}(\mathbf{x},t)=\nabla\left(\frac{Y_4(\mathbf{x},t)+\mathbf{Y}(\mathbf{x},t)\,\mathbf{p}(\mathbf{x},t)}{Y_0(\mathbf{x},t)}\right),\quad \mathrm{curl}\,\mathbf{p}(\mathbf{x},t)=0. \tag{61}$$

We now obtain an equation for the entropy density. The entropy density is determined by

$$s=-\frac{1}{V}\,\mathrm{Sp}\,w(t)\ln w(t)=-\omega+Y_\alpha\zeta_\alpha. \tag{62}$$

Therefore, noting that $\zeta_\alpha=\partial\omega/\partial Y_\alpha$, we have

$$\dot{s}=Y_\alpha\dot{\zeta}_\alpha-\frac{\partial\omega}{\partial p_k}\dot{p}_k$$

or, using Eqs. (50), (61), and (46), we obtain

$$\begin{aligned}\dot{s}=-\frac{\partial\omega}{\partial p_k}\frac{\partial}{\partial x_k}\frac{Y_4+\mathbf{Y}\mathbf{p}}{Y_0}-Y_\alpha\frac{\partial}{\partial x_k}\Big\{&-\frac{\partial}{\partial Y_\alpha}\frac{\omega Y_k}{Y_0}\\ &+\frac{\partial\omega}{\partial p_k}\frac{\partial}{\partial Y_\alpha}\frac{Y_4+\mathbf{Y}\mathbf{p}}{Y_0}+\frac{\delta_{\alpha 0}}{Y_0^2}C_k(\mathbf{p})\Big\}.\end{aligned} \tag{63}$$

Noting that $Y_\alpha(\partial/\partial Y_\alpha)((Y_4+\mathbf{Y}\mathbf{p})Y_0)=0$, we have

$$\dot{s}=\frac{\partial}{\partial x_k}\left\{\frac{Y_k}{Y_0}(Y_\alpha\zeta_\alpha-\omega)\right\}-Y_0\frac{\partial}{\partial x_k}\frac{C_k(\mathbf{p})}{Y_0^2}.$$

Thus,

$$\dot{s}=\frac{\partial}{\partial x_k}\left(\frac{Y_k}{Y_0}s\right)-Y_0\frac{\partial}{\partial x_k}\frac{C_k(\mathbf{p})}{Y_0^2}. \tag{64}$$

If we use the phenomenological principle of adiabaticity of the flows of an ideal superfluid, we must set the function $C_k(\mathbf{p})$ equal to zero. It can then be concluded from Eq. (64) that only the normal component of the fluid, which moves with velocity $-\mathbf{Y}/Y_0-\mathbf{v}_n$, possesses entropy.

4. GALILEO-INVARIANT AND RELATIVISTICALLY INVARIANT SYSTEMS

Let the quantum-mechanical equation be invariant with respect to the Galileo transformation, which is given by the unitary operator

$$U_\mathbf{p}=\exp\{-i\mathbf{p}\int d^3x\mathbf{x}\hat{n}(\mathbf{x})\},$$

where $\mathbf{p}=m\mathbf{v}$ (m is the particle mass). This means that the density operators $\hat{\zeta}_\alpha(\mathbf{x})$ have the following transformation properties under the unitary transformation $U_\mathbf{p}$:

$$\begin{aligned}&U_\mathbf{p}\hat{\varepsilon}(\mathbf{x})U_\mathbf{p}^+=\hat{\varepsilon}(\mathbf{x})+\mathbf{v}\hat{\pi}(\mathbf{x})+\hat{\mathbf{n}}(\mathbf{x})\frac{mv^2}{2}.\\ &U_\mathbf{p}\hat{\pi}_i(\mathbf{x})U_\mathbf{p}^+=\hat{\pi}_i(\mathbf{x})+mv_i\hat{n}(\mathbf{x}),\quad U_\mathbf{p}\hat{n}(\mathbf{x})U_\mathbf{p}^+=\hat{n}(\mathbf{x}).\end{aligned} \tag{65}$$

Using the first of these formulas, we have [see (23)]

$$\mathscr{H}_\mathbf{p}=\mathscr{H}+\mathbf{v}\hat{\mathscr{P}}+\frac{mv^2}{2}\hat{N}. \tag{66}$$

Therefore, in accordance with (28)

$$\omega(Y_\alpha,\ \mathbf{p})=\omega(Y'_\alpha,\ 0)\equiv\omega(Y'_\alpha), \tag{67}$$

where

$$Y'_0=Y_0,\quad Y'_k=Y_k+Y_0v_k,\quad Y'_4=Y_4+Y_kmv_k+Y_0\frac{mv^2}{2}. \tag{68}$$

We see thus that with allowance for rotational invariance the thermodynamic potential ω of Galileo-invariant systems is a function of the three independent Y'_0, $\mathbf{Y}'^2$, Y'_4. The transiformation (22) of the statistical operator corresponds to transition to a frame in which the condensate is at rest and the parameter $\mathbf{v}=\mathbf{p}/m\equiv\mathbf{v}_s$ has the meaning of the superfluid velocity. By virtue of (31) and (68), the second law of thermodynamics for Galileo-invariant superfluid systems has the form

$$d\omega=\zeta'_\alpha\,dY'_\alpha, \tag{69}$$

where ζ'_α are the densities of the additive integrals of the motion in the frame in which $\mathbf{v}_s = 0$, and

$$\varepsilon' = \varepsilon - \frac{\mathbf{p}\boldsymbol{\pi}}{m} + \frac{p^2}{2m} n, \quad \pi'_k = \pi_k - p_k n, \quad n' = n. \tag{70}$$

Equation (67) shows that for Galileo-invariant systems we have in accordance with (47) and (49) the relations

$$\overset{*}{m} = m, \ \rho_s = \rho - \rho_n,$$

and therefore the expressions (48) become

$$\left.\begin{aligned} j_k &= \frac{\rho_n}{m} v_{nk} + \frac{\rho_s}{m} v_{sk}, \ t_{ik} = -\frac{\omega}{Y_0}\delta_{ik} + \rho_s v_{si} v_{sh} \\ &\quad + \rho_n v_{ni} v_{nk}, \\ q_k &= v_{nk}\left(-\frac{\omega}{Y_0} + \varepsilon + \frac{\rho_s}{m}\frac{Y_4 + Y\mathbf{p}}{Y_0}\right) \\ &\quad - v_{sh}\frac{\rho_s}{m}\frac{Y_4 + Y\mathbf{p}}{Y_0} + \frac{C_k(\mathbf{p})}{Y_0^2}. \end{aligned}\right\} \tag{71}$$

We shall show that the requirement of Galilean invariance leads to vanishing of the function $C_k(\mathbf{p})$. To this end, we note that by virtue of (22) and (27)

$$\left(\frac{\partial w}{\partial p_k}\right)_Y = \mathrm{i}\,[\Gamma_k, w] + U^+_{\mathbf{p}}\left(\frac{\partial w'}{\partial p_k}\right)_Y U_{\mathbf{p}}. \tag{72}$$

It follows from the definition (22) of the statistical operator w' and Eq. (66) that

$$\left(\frac{\partial w'}{\partial p_l}\right)_Y = -w' \int_0^1 d\lambda \left\{\frac{Y_0}{m}(\hat{\mathscr{P}}_l(\lambda) - \langle\hat{\mathscr{P}}_l\rangle) + \left(Y_l + \frac{Y_0}{m} p_l\right)(\hat{N}(\lambda) - \langle\hat{N}\rangle)\right\},$$

$$\hat{N}(\lambda) \equiv w'^{-\lambda}\hat{N}w'^{\lambda}, \quad \langle\ldots\rangle \equiv \mathrm{Sp}\, w' \ldots$$

Therefore

$$\left(\frac{\partial w}{\partial p_l}\right)_Y = \mathrm{i}\,[\Gamma_l, w] + \frac{Y_0}{m}\left(\frac{\partial w}{\partial Y_l}\right)_{\mathbf{p}} + Y_l\left(\frac{\partial w}{\partial Y_4}\right)_{\mathbf{p}}$$

and, therefore,

$$\begin{aligned} \frac{\partial q_k}{\partial p_l} &= \frac{\partial}{\partial p_l}\mathrm{Sp}\, w\hat{q}_k(0) \\ &= -\mathrm{i}\,\mathrm{Sp}\, w\,[\Gamma_l, \hat{q}_k(0)] + \frac{Y_0}{m}\frac{\partial q_k}{\partial Y_l} + Y_l\frac{\partial q_k}{\partial Y_4}. \end{aligned} \tag{73}$$

Using the transformation properties (65) of the operators of the densities $\hat{\zeta}_a(\mathbf{x})$ and the relations (18), we find the transformation properties of the operators of the flux densities under a Galileo transformation:

$$\left.\begin{aligned} &U_{\mathbf{p}}\hat{\pi}_k(\mathbf{x})U^+_{\mathbf{p}} = \hat{\pi}_k(\mathbf{x}) + mv_k\hat{n}(\mathbf{x}), \\ &U_{\mathbf{p}}\hat{t}_{kl}(\mathbf{x})U^+_{\mathbf{p}} = \hat{t}_{kl}(\mathbf{x}) + mv_k\hat{\pi}_l(\mathbf{x}) + mv_l\hat{\pi}_k(\mathbf{x}) + v_kv_lm\hat{n}(\mathbf{x}), \\ &U_{\mathbf{p}}\hat{q}_k(\mathbf{x})U^+_{\mathbf{p}} = \hat{q}_k(\mathbf{x}) + v_l\hat{t}_{lk}(\mathbf{x}) + v_k\hat{\varepsilon}(\mathbf{x}) + \frac{v^2}{2}(\hat{\pi}_k(\mathbf{x}) \\ &+ v_km\hat{n}(\mathbf{x})) + v_kv_l\hat{\pi}_l(\mathbf{x}), \quad \mathbf{p} = m\mathbf{v}. \end{aligned}\right\} \tag{74}$$

Differentiating the last relation with respect to $\mathbf{v}$ and then setting $\mathbf{v} = 0$, we find

$$-\mathrm{i}\,[\Gamma_l, \hat{q}_k(\mathbf{x})] = \frac{1}{m}\{\hat{t}_{lk}(\mathbf{x}) + \delta_{lk}\hat{\varepsilon}(\mathbf{x})\}.$$

Substituting this expression in (73), we obtain

$$\frac{\partial q_k}{\partial p_l} = \frac{1}{m}\left(t_{lk} + \delta_{lk}\varepsilon + Y_0\frac{\partial q_k}{\partial Y_l}\right) + Y_l\frac{\partial q_k}{\partial Y_4}. \tag{75}$$

Since by (68)

$$\left.\begin{aligned} &\frac{\partial}{\partial Y_0}\Big|_{\mathbf{p}} = \frac{\partial}{\partial Y'_0} + \frac{p_k}{m}\frac{\partial}{\partial Y'_k} + \frac{p^2}{2m}\frac{\partial}{\partial Y'_4}, \ \frac{\partial}{\partial Y_k}\Big|_{\mathbf{p}} = \frac{\partial}{\partial Y'_k} + p_k\frac{\partial}{\partial Y'_4}, \\ &\frac{\partial}{\partial Y_4}\Big|_{\mathbf{p}} = \frac{\partial}{\partial Y'_4}, \ \frac{\partial}{\partial p_k}\Big|_{Y_\alpha} = \frac{\partial}{\partial p_k}\Big|_{Y'_\alpha} + \frac{Y'_0}{m}\frac{\partial}{\partial Y'_k} + Y'_k\frac{\partial}{\partial Y'_4}, \end{aligned}\right\} \tag{76}$$

we obtain, substituting the expressions (46) for ε, t_{lk}, and q_k in (75) and using (76),

$$\frac{\partial C_k(\mathbf{p})}{\partial p_l} = 0.$$

We have also used the face that the potential ω depends only on Y'_α and not on $\mathbf{p}$ [see (67)]. Hence and from the requirement of rotational invariance it follows that $C_k(\mathbf{p}) = 0$. Using Eqs. (71) and (67), we can readily show that the equations which we have obtained go over into the equations of Landau's two-fluid hydrodynamics.

We now consider the case when the system is invariant with respect to Lorentz transformations. In this case, the operators of the momentum, $\hat{\mathscr{P}}_k$ and the energy, $\mathscr{H}$, form a 4-vector $\hat{\mathscr{P}}^\mu \equiv (\hat{\mathscr{P}}_k \mathscr{H})$, which under the Lorentz tranformation

$$x^\mu \to x'^\mu = a^\mu_\nu x^\nu, \ (x^h \equiv x_h, \ x^0 \equiv t) \tag{77}$$

transforms in accordance with

$$\hat{\mathscr{P}}^\mu \to \hat{\mathscr{P}}'^\mu \equiv U_a\hat{\mathscr{P}}^\mu U^+_a = a^\mu_\nu\hat{\mathscr{P}}^\nu, \tag{78}$$

where U_a is a unitary transformation in the Hilbert space corresponding to the transformation (77), the explicit form of which we here omit.

The role of the particle-number operator $\hat{N}$ is played by the charge operator $\hat{Q}$, which is an invarant:

$$\hat{Q} \to \hat{Q}' \equiv U_a\hat{Q}U^+_a = \hat{Q}. \tag{79}$$

Thus, we can represent the equilibrium statistical operator of the relativistic superfluid in the form

$$\begin{aligned} w(Y_\mu, p_\mu) = \exp\Big\{&V\omega - Y_\mu\hat{\mathscr{P}}^\mu - Y_4\hat{Q} \\ &- vp_\mu \int_\sigma d\sigma^\mu(\hat{\varphi}(x)\,\mathrm{e}^{-\mathrm{i}p_\nu x^\nu} + \mathrm{h.c.})\Big\}, \end{aligned} \tag{80}$$

where

$$Y_\mu \equiv (Y_k, \ Y_0), \ p_\mu = (p_k, \ p_0), \ p_0 \equiv (Y_4 + Y\mathbf{p})/Y_0$$

(the hyperplane σ over which the integration in (80) is performed is orthogonal to the 4-vector p_μ; for simplicity, we consider the superfluidity of scalar particles described by the scalar field $\hat{\varphi}(x) \equiv e^{i\hat{\mathscr{P}}_\mu x^\mu}\hat{\varphi}(0)e^{-i\hat{\mathscr{P}}_\nu x^\nu}$). It follows from Eqs. (78)–(80) that

$$U_a w(Y_\mu, p_\mu) U^+_a = w(Y'_\mu, p'_\mu), \tag{81}$$

where

$$Y'_\mu = Y_\nu a^\nu_\mu, \quad p'_\mu = p_\nu a^\nu_\mu. \tag{82}$$

Thus, the quantities Y_μ and p_μ form two 4-vectors, and

$$Y_4 = -Y_\mu p^\mu \tag{83}$$

is an invariant (the connection between the covariant and contravariant components of 4-vectors is established by means of the diagonal metric tensor $g_{\mu\nu}$ with components $g_{00} = -1, g_{11} = g_{22} = g_{33} = 1$).

The condition (8) of spatial homogeneity and the stationarity condition (12) are combined into the single relativistically invariant relation

$$[w, \hat{\mathscr{P}}^{\mu} - p^{\mu}\hat{Q}] = 0. \tag{84}$$

Since the volume V is not a relativistic invariant, it is expedient to replace the density of the thermodynamic potential ω by $\omega_1 = \omega/Y_0 = \Omega/VY_0$, which is a relativistic invariant and has the physical meaning of the pressure [it follows from (81) that $\Omega = \omega V$ is an invariant; $Y_0 V$ is obviously one too]. Thus, $\omega' = \omega/Y_0$ is a function of the invariants Y^2, p^2, $Y_\mu p^\mu$,

$$\omega' = \omega'(Y^2, p^2, Y_\mu p^\mu). \tag{85}$$

In accordance with (46), the densities ζ_α and the fluxes $\zeta_{\alpha k}$ in the state of thermodynamic equilibrium were expressed in terms of the density ω of the thermodynamic potential, which is a function of the variables Y_0, Y_k, Y_4, p_k. This form of the expressions was, in particular, convenient for Galileo-invariant systems. For relativistically invariant systems, it is expedient to express the densities ζ_α and the fluxes $\zeta_{\alpha k}$ in terms of the potential ω', choosing as indenpendent variables the four-dimensional vectors Y_μ and p_μ ($Y_4 = -p_\mu Y^\mu$). It is readily seen in accordance with (46) that the charge density $n \equiv j^0$ and the charge flux density j^k form a 4-vector, and

$$j^{\mu} = \partial\omega'/\partial p_{\mu}. \tag{86}$$

Similarly, the energy density ε and the momentum density $\pi^k \equiv \pi_k$, and also the energy flux density $q^k \equiv q_k$ and momentum flux density $t^{kl} \equiv t_{kl}$, can be represented in accordance with (46) in the form

$$\left.\begin{aligned} &\varepsilon = \frac{\partial\omega' Y_0}{\partial Y_0} - p_0\frac{\partial\omega'}{\partial p_0}, \quad q^k = -\frac{\partial\omega' Y_k}{\partial Y_0} - p_0\frac{\partial\omega'}{\partial p_k} + \frac{C_k}{Y_0^2},\\ &\pi^k = Y_0\frac{\partial\omega'}{\partial Y_k} + p_k\frac{\partial\omega'}{\partial p_0}, \quad t^{kl} = -\frac{\partial\omega' Y_l}{\partial Y_k} + p_k\frac{\partial\omega'}{\partial p_l}. \end{aligned}\right\} \tag{87}$$

From this it can be seen that the quantities $t^{00} = \varepsilon, t^{k0} = \pi^k, t^{kl} = t^{kl}, t^{0k} = q^k$ from the second-rank tensor $t^{\mu\nu}$ (the energy-momentum tensor):

$$t^{\mu\nu} = -\frac{\partial\omega' Y^{\nu}}{\partial Y_{\mu}} + p^{\mu}\frac{\partial\omega'}{\partial p_{\nu}} \tag{88}$$

(for relativistically invariant systems, the constant C_k is also zero, since the quantities $t^{\mu\nu}$ must form a second-rank tensor, and the right-hand side of (88) is also a second-rank tensor). Note that by virtue of (85) the tensor $t^{\mu\nu}$ is symmetric, $t^{\mu\nu} = t^{\nu\mu}$.

We emphasize in conclusion that Eqs. (86) and (88) are formally valid for general superfluid systems [if the function $C_k(\mathbf{p})$ is ignored]; for in deriving Eqs. (86) and (88) from Eqs. (46) we merely went over from the independent variables Y_α ($\alpha = 0,1,2,3,4$), p_k to the independent variables Y_μ ($\mu = 0,1,2,3$), p_μ, where $p_0 = (Y_4 + \mathbf{Y}\mathbf{p})/Y_0$.

The introduction of the tensor $t^{\mu\nu}$ and the current vector j^μ makes it possible to express the hydrodynamic equations of the superfluid in the form

$$\frac{\partial t^{\mu\nu}}{\partial x^{\nu}} = 0, \quad \frac{\partial j^{\nu}}{\partial x^{\nu}} = 0. \tag{89}$$

The equation of motion for the superfluid momentum $\mathbf{p}$ (61) with the condition that the superfluid flow is irrotational are combined into the equation

$$\frac{\partial p^{\mu}}{\partial x_{\nu}} - \frac{\partial p^{\nu}}{\partial x_{\mu}} = 0, \tag{90}$$

where the 4-momentum p_ν is related to the phase φ by $p_\varphi = \partial\varphi/\partial x^\nu$. The entropy density $s = -(1/V)\mathrm{Sp}\, w \ln w$, equal in accordance with (87) to

$$s^0 = -\omega + Y_\mu \pi^\mu + Y_4 j^0 = Y_0 Y_\mu \frac{\partial\omega'}{\partial Y_\mu}, \tag{91}$$

and the entropy flux density s^k,

$$s^k = -\frac{Y_k}{Y_0} s^0, \tag{92}$$

are combined into the four-dimensional vector

$$s^{\mu} = -Y^{\mu} Y_{\nu} \frac{\partial\omega'}{\partial Y_{\nu}}, \tag{93}$$

which is the entropy 4-current.

It is a consequence of the conservation laws (89) and the equation of motion for the superfluid momentum p_μ (90) that the flow of the superfluid satisfies the adiabaticity condition

$$\partial s^{\mu}/\partial x^{\mu} = 0. \tag{94}$$

The system of equations (89), (90), and (94) obtained in our microscopic approach is completely equivalent to the equations of Ref. 14, the basis of which is a phenomenological approach to superfluid hydrodynamics. (The equations of the hydrodynamics of a relativistic superfluid were obtained earlier in Ref. 15 in a microscopic approach.)

To conclude this section, we note that Eqs. (89), (90), and (94) hold not only for relativistic systems but also for generalized superfluid systems whoose Hamiltonians are neither Galileo nor relativistically invariant. In contrast to relativistic systems, for which the pressure ω' is a function of the invariants p^2, Y^2, $p_\mu Y^\mu$ [see (85)], in the general case ω' will be an arbitrary function of Y_μ and p_μ. In the following sections, we shall find it convenient to use Eqs. (89) and (90) to study generalized superfluid systems and employ accordingly relativistic notation.

5. GREEN'S FUNCTIONS OF SUPERFLUID SYSTEMS

In the theory of normal systems, one introduces retarded (+) and advanced (−) Green's functions for a pair of translationally invariant quasilocal operators $\hat{a}(\mathbf{x})$ and $\hat{b}(\mathbf{x}')$:

$$G_{ab}^{\pm}(\mathbf{x}, t; \mathbf{x}', t') = \mp i\theta(\pm(t-t'))\,\mathrm{Sp}\, w\,[\hat{a}(\mathbf{x}, t), \hat{b}(\mathbf{x}', t')]. \tag{95}$$

Here, w is the equilibrium statistical operator, and

$$\hat{a}(\mathbf{x}, t) = \exp[i(\mathscr{H}t - \hat{\mathscr{P}}\mathbf{x})]\,\hat{a}(0)\exp[-i(\mathscr{H}t - \hat{\mathscr{P}}\mathbf{x})].$$

Since for normal systems $[w, \mathscr{H}] = [w, \hat{\mathscr{P}}] = 0$, the Green's functions are translationally invariant with respect to the coordinates and the time, i.e., they depend on $\mathbf{x}-\mathbf{x}'$, $t-t'$. In the case of superfluid systems, the Green's functions defined in this manner will not be translationally invariant, since the equilibrium statistical operator does not commute with $\mathscr{H}$ and $\hat{\mathscr{P}}$. However, if in Eq. (95) we understand by $\hat{a}(\mathbf{x},t)$ the operator

$$\hat{a}(\mathbf{x},\ t)=\exp\{\mathrm{i}\,(Ht-\hat{P}_k x_k)\}\,\hat{a}(0)\exp\{-\mathrm{i}\,(Ht-\hat{P}_k x_k)\}, \tag{96}$$

where

$$H\equiv\mathscr{H}+p_0\hat{N},\quad \hat{P}_k\equiv\hat{\mathscr{P}}_k-p_k\hat{N},$$

then the Green's functions determined by (95) will be translationally invariant, since $[w,H]=[w,\mathbf{P}]=0$. (In this connection, it is convenient to interpret the operators $\hat{\mathbf{P}}$ and H for superfluid systems as the operators of spatial and time translation, respectively.) Therefore, we arrive at the following definition of the two-time retarded $(+)$ and advanced $(-)$ Green's functions of superfluid systems:

$$G_{ab}^{\pm}(\mathbf{x},\ t)=\mp\mathrm{i}\theta(\pm t)\,\mathrm{Sp}\,w\,[\hat{a}(\mathbf{x},\ t),\ \hat{b}(0)], \tag{97}$$

where

$$w=\lim_{\nu\to\infty}\lim_{V\to\infty}\exp\left\{\Omega_\nu-Y_\alpha\hat{\gamma}_\alpha-\nu Y_0\int d^3x\,(\psi(\mathbf{x})\,e^{-\mathrm{i}(\mathbf{p}\mathbf{x}+\chi)}+\mathrm{h.c.})\right\}$$

is the equilibrium statistical operator [see (27)] and $\hat{a}(\mathbf{x},t)$ is determined by Eq. (96).

Since the averaging in (97) is over the state of statistical equilibrium determined by the operator w, the Green's functions $G_{ab}^{\pm}(\mathbf{x},t)$ are also functions of the thermodynamic forces Y_α and the superfluid momentum $\mathbf{p}$:

$$G_{ab}^{\pm}(\mathbf{x},\ t)=G_{ab}^{\pm}(\mathbf{x},\ t;\ Y_\alpha,\ \mathbf{p}).$$

The fourier components of the Green's functions $G_{ab}^{\pm}(\mathbf{k},\omega)$ are determined by the equation

$$G_{ab}^{\pm}(\mathbf{k},\ \omega)=\int d^3x\int_{-\infty}^{\infty}dt\,\mathrm{e}^{\mathrm{i}(\omega t-\mathbf{k}\mathbf{x})}G_{ab}^{\pm}(\mathbf{x},\ t). \tag{98}$$

These Green's functions determine the response of the system to an external perturbation. Indeed, suppose that in the limit $t\to-\infty$ the system is in the state of statistical equilibrium described by the statistical operator (26). At a certain time $t\rangle$ an external field is switched on, so that the Hamiltonian of the system becomes the operator $\mathscr{H}(t)=\mathscr{H}+V(t)$ and, accordingly, von Neumann's equation (3) takes the form

$$\mathrm{i}\,\frac{\partial\rho(t)}{\partial t}=[\mathscr{H}+V(t),\ \rho(t)].$$

Introducing instead of $\rho(t)$ the operator

$$\tilde{\rho}(t)=\mathrm{e}^{-\mathrm{i}p_0\hat{N}t}\rho(t)\,\mathrm{e}^{\mathrm{i}p_0\hat{N}t}, \tag{99}$$

we obtain for it the equation

$$\mathrm{i}\,\frac{\partial\tilde{\rho}(t)}{\partial t}=[\mathscr{H}+\tilde{V}(t),\ \tilde{\rho}(t)], \tag{100}$$

where

$$\tilde{V}(t)=\mathrm{e}^{-\mathrm{i}p_0\hat{N}t}V(t)\,\mathrm{e}^{\mathrm{i}p_0\hat{N}t}\equiv\int d^3x\,\xi(\mathbf{x},\ t)\,\hat{b}(\mathbf{x}) \tag{101}$$

[$\xi(\mathbf{x},t)$ is a c-number external field and $\hat{b}(\mathbf{x})$ is the quasilocal operator which determines the interaction of the particles with the external field]. Since in the limit $t\to-\infty$ there was no external field and the system was in equilibrium,

$$\tilde{\rho}(-\infty)=\mathrm{e}^{-\mathrm{i}p_0\hat{N}t}w(t)\,\mathrm{e}^{\mathrm{i}p_0\hat{N}t}\big|_{t\to-\infty}=w. \tag{102}$$

Assuming that the interaction of the system with the external field is weak, we can expand $\tilde{\rho}(t)$ in power of $\tilde{V}(t)$:

$$\tilde{\rho}(t)=w+\rho'(t)+\ldots, \tag{103}$$

where $[w,H]=0$ and $\rho'(t)\sim\xi(t)$. In the approximation linear in ξ, the equation for the statistical operator $\rho'(t)$ has the form

$$\mathrm{i}\,\frac{\partial\rho'(t)}{\partial t}=[H,\ \rho'(t)]+\int d^3x\,\xi(\mathbf{x},\ t)\,[\hat{b}(\mathbf{x}),\ w].$$

The solution of this equation with allowance for the initial condition $\rho'(-\infty)=0$ has the form

$$\rho'(t)=-\mathrm{i}\int_{-\infty}^{t}dt'\int d^3\mathbf{x}'\xi(\mathbf{x}',\ t')\,[\hat{b}(\mathbf{x}',\ t'-t),\ w].$$

Up to terms linear in the interaction $\tilde{V}(t)$, the mean value of the operator $\hat{a}(\mathbf{x})$ is determined by

$$a(\mathbf{x},\ t)=\mathrm{Sp}\,\tilde{\rho}(t)\,\hat{a}(\mathbf{x})=\mathrm{Sp}\,w\hat{a}(\mathbf{x})+a_\xi(\mathbf{x},\ t)+\ldots, \tag{104}$$

where

$$a_\xi(\mathbf{x},\ t)=\int_{-\infty}^{\infty}dt'\int d^3x'\xi(\mathbf{x}',\ t')\,G_{ab}^{+}(\mathbf{x}-\mathbf{x}',\ t-t') \tag{105}$$

and the Green's function $G_{ab}^{+}(\mathbf{x}-\mathbf{x}',t-t')$ is determined by Eq. (97).

The invariance of the equations of quantum mechanics with respect to continuous transformations leads to certain restrictions on the structure of the Green's functions.

We consider first the case when the quantum mechanical equations are invariant with respect to Galileo transformations. In this case, we have in accordance with (26)

$$U_{\mathbf{p}}w(Y_\alpha,\ \mathbf{p})\,U_{\mathbf{p}}^{+}=w(Y_\alpha',\ 0),$$

where the thermodynamic forces Y_α' are related to the thermodynamic forces Y_α by (68). Further, noting that

$$U_{\mathbf{p}}(\mathscr{H}+p_0\hat{N})\,U_{\mathbf{p}}^{+}=\mathscr{H}+\frac{\mathbf{p}}{m}\hat{\mathscr{P}}+\frac{Y_4'}{Y'}\hat{N},$$

$$U_{\mathbf{p}}(\hat{\mathscr{P}}_k-p_k\hat{N})\,U_{\mathbf{p}}^{+}=\hat{\mathscr{P}}_k,$$

and making a unitary transformation under the trace symbol Sp in Eq. (97), we obtain

$$G_{ab}^{\pm}(\mathbf{x},\ t;\ Y_\alpha,\ \mathbf{p})=G_{\tilde{a}\tilde{b}}^{\pm}\left(\mathbf{x}+\frac{\mathbf{p}}{m}t,\ t;\ Y_\alpha',\ 0\right), \tag{106}$$

where $\tilde{a}(0)\equiv U_a\hat{a}(0)U_a^{+}$, $\tilde{b}(0)\equiv u_a\hat{b}(0)U_a^{+}$.

Of course, such a restriction on the structure of the

Green's functions cannot be obtained for systems that are not Galileo invariant.

We now consider the case when the system is relativistically invariant. The Green's function is still determined by (97),

$$G^{\pm}_{ab}(x) = \mp i\theta(\pm x_0)\,\mathrm{Sp}\,w(Y_\mu,\ p_\mu)\,[\hat{a}(x),\ \hat{b}(0)],\quad (107)$$

where

$$\hat{a}(x) = \exp\{i(\hat{\mathcal{P}}^\mu - p^\mu\hat{Q})\,x_\mu\}\,\hat{a}(0)\exp\{-i(\hat{\mathcal{P}}^\mu - p^\mu\hat{Q})\,x_\mu\}$$

and the statistical operator $w(Y_\mu, p_\mu)$ is determined by Eq. (80):

$$w(Y_\mu,\ p_\mu) = \exp\Big\{V\omega - Y_\mu(\hat{\mathcal{P}}^\mu - p^\mu\hat{Q}) - \nu p_\mu\int_\sigma d\sigma^\mu(\hat{\varphi}(x)\,e^{-ip_\nu x^\nu} + \text{h.c.})\Big\}.$$

Since $\hat{\varphi}$ is a scalar field in accordance with (80),

$$U_a\hat{\varphi}(0)\,U_a^+ = \hat{\varphi}(0).\quad (108)$$

Therefore, bearing in mind that

$$U_a\hat{\mathcal{P}}^\mu U_a^+ = a^\mu_\nu\hat{\mathcal{P}}^\nu,\quad U_a Q U_a^+ = \hat{Q},$$

we find in accordance with (77), (82), and (105)

$$G^{\pm}_{ab}(x_\mu,\ Y_\mu,\ p_\mu) = G^{\pm}_{\tilde{a}\tilde{b}}(x'_\mu,\ Y'_\mu,\ p'_\mu),\quad (109)$$

where $\hat{a}(0) \equiv U_a\hat{a}(0)\,U_a^+$, $\hat{b}(0) \equiv U_a\hat{b}(0)\,U_a^+$ and the primed quantities are related to the unprimed ones by $x'_\mu = a^\nu_\mu x_\nu$, $Y'_\mu = Y_\nu a^\nu_\mu$, $p'_\mu = p_\nu a^\nu_\mu$. If for the operators $\hat{a}$ and $\hat{b}$ we choose, for example, the current operator $\hat{j}_\mu$, we readily see that

$$G^{\pm}_{j_\mu,\,j_\nu}(x,\ Y,\ p) = a^{\mu'}_\mu a^{\nu'}_\nu G^{\pm}_{j_{\mu'},\,j_{\nu'}}(x',\ Y',\ p').\quad (110)$$

We have also used the fact that

$$U_a\hat{j}_\mu(0)\,U_a^+ = a^{\mu'}_\mu\hat{j}_{\mu'}(0).\quad (111)$$

Similarly, using (108), we find

$$G^{\pm}_{\varphi,\,\varphi}(x,\ Y,\ p) = G^{\pm}_{\varphi,\,\varphi}(x',\ Y'\ \ p').$$

6. STRUCTURE OF THE LINEARIZED STATISTICAL OPERATOR AND EQUATIONS OF LINEARIZED HYDRODYNAMICS

In this section, we consider states of generalized superfluid systems near the state of statistical equilibrium, obtain the equations of linearized superfluid hydrodynamics, and find normal modes in the system.

Thus, we consider the equation of motion for the statistical operator $\rho(t)$:

$$i\frac{\partial\rho(t)}{\partial t} = [\mathcal{H},\ \rho(t)].\quad (112)$$

In accordance with (26), the equilibrium state is described by the statistical operator

$$w(t) = \exp\Big\{\Omega - Y_\alpha\hat{\gamma}_\alpha - \nu Y_0\int d^3x\,(\psi(\mathbf{x})\,e^{-i(\mathbf{px}+p_0 t)} + \text{h.c.})\Big\},\quad p_0 = (Y_4 + \mathbf{Yp})/Y_0.\quad (113)$$

(For simplicity, we also consider equilibrium states for which $\chi = 0$; we recall that $[w(t),\ \hat{\mathcal{P}} - \mathbf{p}\hat{N}] = 0$.) However, linearization around an equilibrium state that depends on the time t is inconvenient. We go over to the statistical operator

$$\tilde{\rho}(t) = e^{-ip_0\hat{N}t}\rho(t)\,e^{ip_0\hat{N}t}.\quad (114)$$

In this case, the equilibrium statistical operator around which we linearize is determined by

$$w \equiv w(0) = e^{-ip_0\hat{N}t}w(t)\,e^{ip_0\hat{N}t} = \exp\Big\{\Omega - Y_\alpha\hat{\gamma}_\alpha - \nu Y_0\int d^3x\,(\psi(\mathbf{x})\,e^{-i\mathbf{px}} + \text{h.c.})\Big\}.\quad (115)$$

In accordance with (112) and (114), the equation of motion for $\tilde{\rho}(t)$ takes the form

$$i\frac{\partial\tilde{\rho}(t)}{\partial t} = [H,\ \tilde{\rho}(t)],\quad H = \mathcal{H} + p_0\hat{N}.\quad (116)$$

We now set $\tilde{\rho}(t) = w + \tilde{\rho}'(t)$, where $\tilde{\rho}'(t)$ is the statistical operator that describes the deviation of the state of the system from equilibrium. At times $t \gg \tau_r$, the state of the system is, as we have already said, described by the parameters $\zeta_\alpha(\mathbf{x}, t)$ and $\varphi(\mathbf{x}, t)$. Therefore

$$\tilde{\rho}'(t) \xrightarrow[t\gg\tau_r]{} \sigma'(\zeta'(t),\ \varphi'(t)),\quad (117)$$

where

$$\sigma'(\zeta'(t),\ \varphi'(t)) = \int d^3x\,\{\hat{\sigma}_\alpha(\mathbf{x})\,\zeta'_\alpha(\mathbf{x},\ t) + \hat{\sigma}_\varphi(\mathbf{x})\,\varphi'(\mathbf{x},\ t)\},\quad (118)$$

$$\hat{\sigma}_\alpha(\mathbf{x}) \equiv \frac{\delta\sigma(\zeta,\ \varphi)}{\delta\zeta_\alpha(\mathbf{x})}\Big|_{\zeta=\bar{\zeta},\ \varphi=\mathbf{px}},\quad \hat{\sigma}_\varphi(\mathbf{x}) \equiv \frac{\delta\sigma(\zeta,\ \varphi)}{\delta\varphi(\mathbf{x})}\Big|_{\zeta=\bar{\zeta},\ \varphi=\mathbf{px}}$$

and $\zeta'_\alpha(\mathbf{x},t)$ and $\varphi'(\mathbf{x},t)$ are the deviations of the parameters $\zeta_\alpha(\mathbf{x},t)$ and $\varphi(\mathbf{x},t)$ from the equilibirum values, i.e., from $\bar{\zeta}_\alpha = \mathrm{Sp}\,w\hat{\zeta}_\alpha$ and $\bar{\varphi} = \mathrm{Im}\ln\mathrm{Sp}\,w\psi(\mathbf{x}) = \mathbf{p}\cdot\mathbf{x}$, respectively ($\zeta'_\alpha(\mathbf{x},t) = \zeta_\alpha(\mathbf{x},t) - \bar{\zeta}_\alpha$, $\varphi'(\mathbf{x},t) = \varphi(\mathbf{x},t) - \bar{\varphi}$). Noting that in accordance with (53)

$$e^{i\hat{\mathbf{P}}\mathbf{y}}\sigma(\zeta(\mathbf{x}'),\ \varphi(\mathbf{x}'))\,e^{-i\hat{\mathbf{P}}\mathbf{y}} = \sigma(\zeta(\mathbf{x}'+\mathbf{y}),\ \varphi(\mathbf{x}'+\mathbf{y}) - \mathbf{py}),\quad \hat{\mathbf{P}} = \hat{\mathcal{P}} - \mathbf{p}\hat{N}$$

we have

$$e^{i\hat{\mathbf{P}}\mathbf{y}}\hat{\sigma}_\alpha(\mathbf{x})\,e^{-i\hat{\mathbf{P}}\mathbf{y}} = \hat{\sigma}_\alpha(\mathbf{x}-\mathbf{y}),\quad e^{i\hat{\mathbf{P}}\mathbf{y}}\hat{\sigma}_\varphi(\mathbf{x})\,e^{-i\hat{\mathbf{P}}\mathbf{y}} = \hat{\sigma}_\varphi(\mathbf{x}-\mathbf{y}).$$

These expressions confirm once more that when $\mathbf{p} \neq 0$ it is expedient to interpret the operator $\hat{\mathbf{P}}$ (and not the operator $\hat{\mathcal{P}}$ as the operator of translations.

Since the phase $\bar{\varphi}$ in the state w is equal to $\mathbf{p}\cdot\mathbf{x}$, the phase in the state $w + \sigma'(\zeta'(t), \varphi'(t))$ can in accordance with (117) be represented in the frm

$$\varphi(\mathbf{x},\ t) = \mathrm{Sp}\,\sigma'(\zeta'(t),\ \varphi'(t))\,\hat{\varphi}(\mathbf{x}) + \mathbf{px},\quad (119)$$

where

$$\hat{\varphi}(\mathbf{x}) = \frac{i}{2\eta}(\psi^+(\mathbf{x})\,e^{i\mathbf{px}} - \psi(\mathbf{x})\,e^{-i\mathbf{px}}),\quad \eta = |\,\mathrm{Sp}\,w\psi\,|.\quad (120)$$

We shall call $\hat{\varphi}(\mathbf{x})$ the phase operator. We note that the phase operator can be written in the form [see (10)]

$$\hat{\varphi}(\mathbf{x}) = e^{-i\hat{\mathbf{P}}\mathbf{x}}\hat{\varphi}(0)\,e^{i\hat{\mathbf{P}}\mathbf{x}}.$$

Note that in accordance with the definition of $\sigma'(\zeta',\varphi')$

$$\zeta'_\alpha(\mathbf{x},\ t) = \mathrm{Sp}\,\sigma'(\zeta'(t),\varphi'(t))\,\hat{\zeta}_\alpha(\mathbf{x}),\quad \varphi'(\mathbf{x},\ t) = \mathrm{Sp}\,\sigma'(\zeta'(t),\ \varphi'(t))\,\hat{\varphi}(x). \tag{121}$$

From Eq. (116), taking into account (117), we obtain

$$\mathrm{i}\int d^3x\,\{\hat{\sigma}_\alpha(\mathbf{x})\,L_\alpha(\mathbf{x};\ t) + \hat{\sigma}_\varphi(\mathbf{x})\,L_\varphi(\mathbf{x};\ t)\} = [H,\ \sigma'(\zeta'(t),\ \varphi'(t))], \tag{122}$$

where

$$\left.\begin{array}{l} L_\alpha(\mathbf{x};\ t) = \mathrm{i}\,\mathrm{Sp}\,\sigma'(\zeta'(t),\ \varphi'(t))\,[H,\ \hat{\zeta}_\alpha(\mathbf{x})], \\ L_\varphi(\mathbf{x};\ t) = \mathrm{i}\,\mathrm{Sp}\,\sigma'(\zeta'(t),\ \varphi'(t))\,[H,\ \hat{\varphi}(\mathbf{x})] \end{array}\right\} \tag{123}$$

determine the linearized hydrodynamic equations of the superfluid:

$$\dot{\zeta}'_\alpha(\mathbf{x};\ t) = L_\alpha(\mathbf{x};\ t),\quad \dot{\varphi}'(\mathbf{x};\ t) = L_\varphi(\mathbf{x};\ t). \tag{124}$$

In principle, it is not difficult to develop an iterative procedure for finding the statistical operator $\sigma'(\xi'(t),\varphi'(t))$ in all orders of perturbation theory in the gradients of the hydrodynamic parameters.[13] However, we restrict ourselves to the leading approximation in the gradients of the parameters $\zeta'_\alpha(\mathbf{x},t)$ and $\varphi'(\mathbf{x},t)$. For this, it is sufficient to invoke the ergodic relation (26), which for convenience we represent in the form

$$e^{-\mathrm{i}Ht}\rho e^{\mathrm{i}Ht} \underset{t\gg\tau_r}{\rightarrow} w\,(Y_\alpha,\ \mathbf{p}+\mathbf{p}',\ (p_0+p'_0)\,t+\chi), \tag{125}$$

where the initial statistical operator ρ satisfies the equation $[\rho,\hat{\mathscr{P}}_k - (p_k + p'_k)\hat{N}] = 0$. The quantities $\mathbf{p}, p_0, \chi$ are functionals of the initial statistical operator ρ. Suppose that the statistical operator ρ differs little from the equilibrium statistical operator $w, \rho = w + \rho'$. Then, since $[w,\hat{P}_k] = 0$,

$$[\rho',\ \hat{P}_k] = p'_k\,[w,\ \hat{N}]. \tag{126}$$

In the zeroth approximation, we obviously have $\chi = 0$ ($\mathbf{p}'$ is of first order in the deviation from the state of equilibrium), and, therefore, the relation (125) takes in the linear approximation the form

$$e^{-\mathrm{i}Ht}\rho' e^{\mathrm{i}Ht} \xrightarrow[t\gg\tau_r]{} \left.\frac{\partial w}{\partial\zeta_\alpha}\right|_{\mathbf{p}} \zeta'_\alpha + \left.\frac{\partial w}{\partial p_k}\right|_\zeta p'_k + \left.\frac{\partial w}{\partial\varphi}\right|_{\mathbf{p},\,\zeta} (\chi' + p'_0 t), \tag{127}$$

where

$$\left.\begin{array}{l} \zeta'_\alpha = \mathrm{Sp}\,\rho'\hat{\zeta}_\alpha,\quad p'_0 = \zeta'_\alpha \left.\dfrac{\partial p_0}{\partial\zeta_\alpha}\right|_{\mathbf{p}} + \mathbf{p}' \left.\dfrac{\partial p_0}{\partial \mathbf{p}}\right|_\zeta, \\ \chi' = \mathrm{Sp}\,\rho'\hat{\varphi}(0) + \displaystyle\int_0^\infty d\tau\,\{\mathrm{Sp}\,\dot{\rho}'(\tau)\,\hat{\varphi}(0) - p'_0\}. \end{array}\right\} \tag{128}$$

For variation of the relation (125) we have chosen as independent variables, not the quantities Y_α, $\mathbf{p}$, χ, which are functional of ρ, but the quantities $\zeta_\alpha = \mathrm{Sp}\,w\hat{\zeta}_\alpha \equiv \zeta_\alpha(Y_\beta,\mathbf{p})$, $\mathbf{p}$, χ, since $\mathrm{Sp}\,\rho\hat{\zeta}_\alpha = \mathrm{Sp}\,w_\alpha$ because ζ_α are the densities of the additive integrals of the motion. In addition, we have used the fact that $p_0 = (Y_4 + \mathbf{Y}\cdot\mathbf{p})/Y_0$ and also the expression (25) for the phase χ. We emphasize that the relation (127) is valid for initial statistical operators that satisfy the condition (126).

We now trun to Eq. (117). Taking the initial statistical operator $\tilde{\rho}'(0)$ equal to the statistical operator ρ' in (127) and noting that in this case $\zeta'_\alpha(\mathbf{x},t) = \zeta'_\alpha$ do not depend on $\mathbf{x}$ or t by virtue of the conservation laws, while the phase $\varphi'(\mathbf{x},t)$ is in accordance with (125) equal to $\mathbf{p}'\mathbf{x} + p'_0 t + \chi'$, we obtain

$$e^{-\mathrm{i}Ht}\rho' e^{\mathrm{i}Ht} \xrightarrow[t\gg\tau_r]{} \int d^3x\,\{\hat{\sigma}_\alpha(\mathbf{x})\,\zeta'_\alpha + \hat{\sigma}_\varphi(\mathbf{x})\,(\mathbf{p}'\mathbf{x} + p_0 t + \chi')\}.$$

Comparing this expression with (127), we find that

$$\left.\frac{\partial w}{\partial\zeta_\alpha}\right|_{\mathbf{p}} = \int d^3x\hat{\sigma}_\alpha(\mathbf{x}),\quad \left.\frac{\partial w}{\partial\varphi}\right|_{\zeta,\,\mathbf{p}} = \int d^3x\hat{\sigma}_\varphi(\mathbf{x}),\quad \left.\frac{\partial w}{\partial p_k}\right|_\zeta = \int d^3x x_k\hat{\sigma}_\varphi(\mathbf{x}). \tag{129}$$

It is readily seen that these relations are in agreement with (96), since

$$e^{\mathrm{i}\hat{\mathbf{P}}\mathbf{y}} \left.\frac{\partial w}{\partial\zeta_\alpha}\right|_{\mathbf{p}} e^{-\mathrm{i}\hat{\mathbf{P}}\mathbf{y}} = \left.\frac{\partial w}{\partial\zeta_\alpha}\right|_{\mathbf{p}},\quad e^{\mathrm{i}\hat{\mathbf{P}}\mathbf{y}} \frac{\partial w}{\partial\varphi} e^{-\mathrm{i}\hat{\mathbf{P}}\mathbf{y}} = \frac{\partial w}{\partial\varphi},$$

$$e^{\mathrm{i}\hat{\mathbf{P}}\mathbf{y}} \left.\frac{\partial w}{\partial p_k}\right|_\zeta e^{-\mathrm{i}\hat{\mathbf{P}}\mathbf{y}} = \left.\frac{\partial w}{\partial p_k}\right|_\zeta + y_k \left.\frac{\partial w}{\partial\varphi}\right|_{\zeta,\,\mathbf{p}}.$$

Now let ρ' be a fairly arbitrary statistical operator. Then ignoring the gradients of $\zeta'_\alpha(\mathbf{x},t)$ but taking into account the gradients of the phase $\varphi'(\mathbf{x},t)$, we can represent the mean value $\mathrm{Sp}\,e^{\mathrm{i}Ht}\rho' e^{\mathrm{i}Ht}\hat{a}^-(\mathbf{x})$ for $t\gg\tau_r$ in the form

$$\mathrm{Sp}\,e^{-\mathrm{i}Ht}\rho' e^{\mathrm{i}Ht}\hat{a}(\mathbf{x}) \xrightarrow[t\gg\tau_r]{} \mathrm{Sp}\,\sigma'(\zeta'(t),\ \varphi'(t))\,\hat{a}(\mathbf{x})$$

$$\approx \zeta'_\alpha(\mathbf{x},\ t)\int d^3x'\,\mathrm{Sp}\,\hat{\sigma}_\alpha(\mathbf{x}')\,\hat{a}(\mathbf{x}) + \varphi'(\mathbf{x},\ t)\int d^3x'\,\mathrm{Sp}\,\sigma_\varphi(\mathbf{x}') \times \hat{a}(\mathbf{x}) + \frac{\partial\varphi'(\mathbf{x},\ t)}{\partial x_k}\int d^3x'\,(x'_k - x_k)\,\mathrm{Sp}\,\hat{\sigma}_\varphi(\mathbf{x}')\,\hat{a}(\mathbf{x})$$

(we recall that $\hat{a}(\mathbf{x}) = e^{-\mathrm{i}\hat{P}\mathbf{x}}\hat{a}(0)e^{\mathrm{i}\hat{P}\mathbf{x}}$) or, taking into account (129), in the form

$$\mathrm{Sp}\,e^{-\mathrm{i}Ht}\rho' e^{\mathrm{i}Ht} a\,(\hat{\mathbf{x}}) \xrightarrow[t\gg\tau_r]{} \zeta'_\alpha(\mathbf{x},\ t) \left.\frac{\partial\langle a\rangle}{\partial\zeta_\alpha}\right|_{\mathbf{p}} + \varphi'(\mathbf{x},\ t) \left.\frac{\partial\langle a\rangle}{\partial\varphi}\right|_{\mathbf{p},\,\zeta} + \frac{\partial\varphi'(\mathbf{x},\ t)}{\partial x_k} \left.\frac{\partial\langle a\rangle}{\partial p_k}\right|_\zeta,$$

where $\langle\hat{a}\rangle \equiv \mathrm{Sp}\,w\hat{a}(0)$. As was shown in Sec. 4, the equations of superfluid hydrodynamics have the form

$$\frac{\partial t^{\mu\nu}}{\partial x^\nu} = 0,\quad \frac{\partial j^\nu}{\partial x^\nu} = 0,\quad \frac{\partial p^\mu}{\partial x_\nu} - \frac{\partial p^\nu}{\partial x_\mu} = 0, \tag{130}$$

and the phase $\varphi(\mathbf{x},t)$ is related to the superfluid 4-momentum by $p_\nu = \partial\varphi/\partial x^\nu$. We linearize Eq. (130) about the equilibrium state, choosing as parameters describing the deviation from the equilibrium state the quantities $\delta Y_\mu(\mathbf{x},t) = Y_\mu(\mathbf{x},t) - \overline{Y}_\mu$ and $\delta p_\mu(\mathbf{x},t) = p_\mu(\mathbf{x},t) - \bar{p}_\mu$ ($\overline{Y}_\mu$ and $\bar{p}_\mu$ are the equilibrium values of $Y_\mu, p_\mu, \mu = 0,1,2,3$). Then the hydrodynamic equations (130) when linearized about the equilibrium state take the form

$$\left.\begin{array}{l} k_\nu\left(\dfrac{\partial t^{\mu\nu}}{\partial Y_\lambda}\,\delta Y_\lambda(k) + \dfrac{\partial t^{\mu\nu}}{\partial p_\lambda}\,\delta p_\lambda(k)\right) = 0, \\ k_\nu\left(\dfrac{\partial j^\nu}{\partial Y_\lambda}\,\delta Y_\lambda(k) + \dfrac{\partial j^\nu}{\partial p_\lambda}\,\delta p_\lambda(k)\right) = 0, \\ k_\nu\delta p_\mu(k) - k_\mu\delta p_\nu(k) = 0, \end{array}\right\} \tag{131}$$

where $\delta Y_\nu(k)$, $\delta p_\nu(k)$ are the Fourier components of the corresponding quantities. Using Eq. (86) and (88), we readily see that

$$\left.\begin{aligned}\frac{\partial t^{\mu\nu}}{\partial Y_\lambda} &= -\frac{\partial^2 Y^\nu \omega'}{\partial Y_\mu \partial Y_\lambda} + p^\mu \frac{\partial j^\nu}{\partial Y_\lambda},\\ \frac{\partial t^{\mu\nu}}{\partial p_\lambda} &= g^{\mu\lambda} j^\nu - g^{\mu\nu} j^\lambda - Y^\nu \frac{\partial j^\lambda}{\partial Y_\mu} + p^\mu \frac{\partial j^\nu}{\partial p_\lambda}.\end{aligned}\right\} \quad (132)$$

Substituting these expressions in the first equation of (131) and noting that $\delta p_\lambda(k) = ik_\lambda \delta\varphi(k)$, we obtain

$$\delta Y_\lambda(k)(a^\lambda p^\mu - D^{\mu\lambda}) + i\delta\varphi(k)(bp^\mu - (kY)a^\mu) = 0, \quad (133)$$

where

$$D^{\mu\nu} \equiv \frac{\partial^2 (kY)\omega'}{\partial Y_\mu \partial Y_\nu}, \quad a^\lambda \equiv k_\nu \frac{\partial^2 \omega'}{\partial Y_\lambda \partial p_\nu}, \quad b \equiv k_\nu k_\mu \frac{\partial^2 \omega'}{\partial p_\nu \partial p_\mu}. \quad (134)$$

Using the notation (134), we express the second equation of (131) in the form

$$a^\lambda \delta Y_\lambda(k) + i\delta\varphi(k) b = 0. \quad (135)$$

Equations (133) and (135) have a nontrivial solution if

$$\Delta(k) \equiv b - a^\lambda D^{-1}_{\lambda\mu} a^\mu (kY) = 0. \quad (136)$$

We see that to determine the normal modes in the system we must find the matrix D^{-1}. In accordance with the definition (134), this matrix can be represented in the form

$$D^{\mu\nu} = (kY) B^{\mu\nu} + k^\nu A^\mu + k^\mu A^\nu, \quad (137)$$

where

$$B^{\mu\nu} \equiv \frac{\partial^2 \omega'}{\partial Y_\mu \partial Y_\nu}, \quad A^\nu \equiv \frac{\partial \omega'}{\partial Y_\nu}. \quad (138)$$

The structure of the matrix $B^{\mu\nu}$ is simpler (since it does not depend on the wave vector) than that of the matrix $D^{\mu\nu}$. Using (137), it is easy to show that

$$\begin{aligned}(kY) D^{-1}_{\nu\mu} &= B^{-1}_{\nu\mu} + \frac{(kB^{-1}k)}{D}(B^{-1}A)_\nu (B^{-1}A)_\mu \\ &+ \frac{(AB^{-1}A)}{D}(B^{-1}k)_\nu (B^{-1}k)_\mu - \frac{(kY)+(AB^{-1}k)}{D} \\ &\times [(B^{-1}A)_\nu (B^{-1}k)_\mu + (B^{-1}A)_\mu (B^{-1}k)_\nu], \end{aligned} \quad (139)$$

where

$$D = [(kY) + (AB^{-1}k)]^2 - (kB^{-1}k)(AB^{-1}A). \quad (140)$$

Note that since the right-hand side of Eq. (139) is finite for $(kY) = 0$, the matrix $D^{-1}_{\mu\nu}$ has a singularity at $(kT) = 0$.

It is easy to show that the dispersion relation (136) when $\mathbf{Y} = \mathbf{p} = 0$ takes with allowance for the definitions (46), (47), and (48) the form

$$\Delta(\mathbf{k}, \omega) = \omega^4 - \omega^2 k^2 (B + \rho_c C) - k^4 \frac{s\rho_s}{Y_0 \rho_n} \frac{A}{\overset{*}{m}{}^2} = 0, \quad (141)$$

where

$$\left.\begin{aligned} A &\equiv \frac{\partial P}{\partial \zeta_0}\frac{\partial}{\partial \zeta_4}\left(\frac{Y_4}{Y_0}\right) - \frac{\partial P}{\partial \zeta_4}\frac{\partial}{\partial \zeta_0}\left(\frac{Y_4}{Y_0}\right),\\ B &\equiv \frac{1}{\overset{*}{m}}\left(\frac{\partial P}{\partial \zeta_4} - \frac{Y_4}{Y_0}\frac{\partial P}{\partial \zeta_0}\right) + \frac{s}{Y_0\rho_n}\left[\frac{\partial P}{\partial \zeta_0} + \frac{\rho_s}{\overset{*}{m}}\frac{\partial}{\partial \zeta_0}\left(\frac{Y_4}{Y_0}\right)\right],\\ C &\equiv \frac{1}{\overset{*}{m}{}^2}\left[\frac{\partial}{\partial \zeta_4}\left(\frac{Y_4}{Y_0}\right) - \frac{Y_4}{Y_0}\frac{\partial}{\partial \zeta_0}\left(\frac{Y_4}{Y_0}\right)\right. \\ &\quad \left. + \overset{*}{m}\frac{s}{Y_0\rho_n}\frac{\partial}{\partial \zeta_0}\left(\frac{Y_4}{Y_0}\right)\right].\end{aligned}\right\} \quad (141')$$

For Galileo-invariant systems we have in accordance with (67), (47), and (49) $\rho_c = 0$, $\overset{*}{m} = m$, and A and B take the form

$$A = -m^2 \frac{\sigma}{\rho c_V}\left(\frac{\partial P}{\partial \rho}\right)_T, \quad B = \left(\frac{\partial P}{\partial \rho}\right)_\sigma + \frac{\rho_s T \sigma^2}{\rho_n c_V} \quad (142)$$

($P = -\omega/Y_0 \equiv -\omega'$ is the pressure, $\sigma = s/\rho = Y_0(P + \zeta_0 + \zeta_4 Y_4/Y_0)/\rho$ is the entropy of unit mass, c_V is the heat capacity of unit mass at constant volume, and $Y_0^{-1} \equiv T$ is the temperature). In this case, the dispersion relation $\Delta(\mathbf{k},\omega) = 0$ leads to the well-known expressions for the velocities $u_{1,2}$ of first and second sound:

$$\begin{aligned} u^2_{1,2} &= \frac{1}{2}\left[\left(\frac{\partial P}{\partial \rho}\right)_\sigma + \frac{T\sigma^2\rho_s}{c_V\rho_n}\right] \\ &\pm \sqrt{\frac{1}{4}\left[\left(\frac{\partial P}{\partial \rho}\right)_\sigma + \frac{T\sigma^2\rho_s}{c_V\rho_n}\right]^2 - \left(\frac{\partial P}{\partial \rho}\right)_T \frac{T\sigma^2\rho_s}{c_V\rho_n}} \end{aligned}$$

(to the indices 1 and 2 there correspond the upper and lower signs + and −). This formula applies to the case when $\mathbf{Y} = \mathbf{p} = 0$.

If we consider now a superfluid in which $\mathbf{p} = m\mathbf{v}_s = 0$, $\mathbf{Y} = -Y_0\mathbf{v}_n \neq 0$, then in this case the twofold dengeneracy is lifted and instead of two mode branches there are four. For small $\mathbf{Y}(v_n/u_{1,2} \ll 1)$, the phase velocities of the four branches are determined by

$$u_1^\pm = \pm u_1 + \delta u_1, \quad u_2^\pm = \pm u_2 + \delta u_2,$$

where

$$\begin{aligned} k\delta u_{1,2} &= -\left(\frac{\mathbf{kY}}{Y_0}\right)(1+Z) \mp \frac{1}{u_1^2 - u_2^2}\left(\frac{\mathbf{kY}}{Y_0}\right) \\ &\times \left\{\frac{\sigma}{Y_0}\left(\frac{\partial P}{\partial \zeta_0}\right)_{\zeta_4}\left(1 + \frac{2\rho_s}{\rho}\right) - \frac{\sigma}{\rho_n}\left[\frac{Y_0}{c_V}\left(\frac{\partial P}{\partial \rho}\right)_T\right.\right. \\ &\times \left.\left(\frac{\partial \rho_n}{\partial Y_0}\right)_{P,\,\mathbf{Y}=0} - \frac{\sigma\rho_s^2}{Y_0 c_V \rho}\right] - \frac{\rho_s}{\rho}\left(\frac{\partial P}{\partial \rho}\right)_\sigma \\ &+ \left.\left[\left(\frac{\partial P}{\partial \rho}\right)_\sigma + \frac{\sigma^2\rho_s}{Y_0 c_V \rho_n}\right] Z\right\}, \end{aligned} \quad (143)$$

$$Z \equiv \frac{\sigma}{2\rho_n}\left[\frac{Y_0}{c_V}\left(\frac{\partial \rho_n}{\partial Y_0}\right)_{P,\,\mathbf{Y}=0} + \left(\frac{\partial P}{\partial \zeta_0}\right)_{\zeta_4}\left(\frac{\partial \rho_n}{\partial Y_4}\right)_{Y_0,\,\mathbf{Y}=0}\right].$$

Note that under the assumption $c_p = c_V$ (this relation is satisfied with good accuracy in superfluid helium) these expressions go over into those of Khalatnikov,[16] which in our notation have the form

$$\begin{aligned} k\delta u_{1,2} &= -\left(\frac{\mathbf{kY}}{Y_0}\right)\left\{1 - \frac{T\sigma}{2c_V\rho_n}\left(\frac{\partial \rho_n}{\partial T}\right)_{P,\,\mathbf{Y}=0}\right. \\ &\mp \left.\left[\frac{\rho_s}{\rho} - \frac{T\sigma}{2c_V\rho_n}\left(\frac{\partial \rho_n}{\partial T}\right)_{P,\,\mathbf{Y}=0}\right]\right\}. \end{aligned}$$

7. HYDRODYNAMICS OF A SUPERFLUID IN EXTERNAL FIELDS

In the previous sections, we have constructed the thermodynamics of superfluid systems and found fluxes for the densities of the additive integrals of the motion in the state of statistical equilibrium. In the derivation of the equations of ideal hydrodynamics, an important role was played by the ergodic relation (6), which is valid for spatially homogeneous states.

In this section, we study the effect of fairly arbitrary weak slowly varying external fields on the evolution of the system. To solve this problem, we consider the equation of motion for the statistical operator $\rho(t)$:

$$\mathrm{i}\,\frac{\partial \rho(t)}{\partial t}=[\mathscr{H}+V(t),\,\rho(t)],\tag{144}$$

where $V(t)$ is the Hamiltonian of the interaction of the particles with the external field. Since we shall linearize this equation about the state w [see (115)], we shall find it convenient (as in Sec. 6) to go over to the new representation $\tilde{\rho}(t)=\mathrm{e}^{-\mathrm{i}p_0\hat{N}t}\rho(t)\mathrm{e}^{\mathrm{i}p_0\hat{N}t}$. In this representation, the equation of motion for the statistical operator $\tilde{\rho}(t)$ has the form

$$\mathrm{i}\,\frac{\partial \tilde{\rho}(t)}{\partial t}=[H+\tilde{V}(t),\,\tilde{\rho}(t)],\; H\equiv\mathscr{H}+p_0\hat{N},\tag{145}$$

where the Hamiltonian of the interaction of the particles with the external field $\xi(\mathbf{x},t)$ is given by

$$\tilde{V}(\xi(t))=\mathrm{e}^{-\mathrm{i}p_0\hat{N}t}\,V(t)\,\mathrm{e}^{\mathrm{i}p_0\hat{N}t}=\int d^3x\,\xi(\mathbf{x},\,t)\,\hat{b}(\mathbf{x})\tag{146}$$

[$\hat{b}(\mathbf{x})$ is some quasilocal operator].

When a weak external field is present, we shall also assume, provided the frequency of the field is low compared with τ_r^{-1}, that the state of the system can still be described by the parameters $\zeta_\alpha(\mathbf{x},t)$ and $\varphi(\mathbf{x},t)$. In this case, the statistical operator $\tilde{\rho}(t)$ when $t\gg\tau_r$ will depend on the time not only through $\zeta_\alpha(\mathbf{x},t)$ and $\varphi(\mathbf{x},t)$ but also through the external field $\xi(\mathbf{x},t)$ and all its time derivatives $\dot{\xi}(\mathbf{x},t)$, $\ddot{\xi}(\mathbf{x},t)$,...:

$$\begin{aligned}\tilde{\rho}(t)\xrightarrow[t\gg\tau_r]{}&\tilde{\rho}(\zeta_\alpha(t),\,\varphi(t);\,\xi(t),\,\dot{\xi}(t)\,\ldots)\\ \equiv&\tilde{\rho}(\zeta_\alpha(t),\,\varphi(t);\,t),\end{aligned}\tag{147}$$

and

$$\begin{aligned}&\mathrm{Sp}\,\tilde{\rho}(\zeta_\beta(t),\,\varphi(t);\,t)\,\hat{\zeta}_\alpha(\mathbf{x})=\zeta_\alpha(\mathbf{x},\,t),\\ &\mathrm{Im}\ln\mathrm{Sp}\,\tilde{\rho}(\zeta_\beta(t),\,\varphi(t);\,t)=\varphi(\mathbf{x},\,t)\end{aligned}\tag{148}$$

($\varphi(\mathbf{x},t)$ is the phase of $\psi(\mathbf{x})$ in the state $\tilde{\rho}(\zeta,\varphi;t)$). We emphasize that in this formula the functional arguments $\zeta_\beta,\varphi,\xi,\dot{\xi},\ldots$, regarding as functions of $\mathbf{x}$, must be assumed to be independent of each other.

Comparing Eq. (147) with (51), we obtain for $\xi=\dot{\xi}=\ldots=0$

$$\tilde{\rho}(\zeta_\alpha(t),\,\varphi(t);\,0,\,0\,\ldots)=\sigma(\zeta_\alpha(t),\,\varphi(t)).\tag{149}$$

We linearize the asymptotic relation (147) about the state (115), setting $\tilde{\rho}(t)=w+\tilde{\rho}'(t)$. Taking into account the relation (149), we obtain

$$\tilde{\rho}'(t)\xrightarrow[t\gg\tau_r]{}\sigma'(\zeta'_\alpha(t),\,\varphi'(t))+\rho(\xi(t))+\ldots,\tag{150}$$

where the statistical operator $\sigma'(\zeta'_\alpha(t),\varphi'(t))$ is determined by Eq. (118), and the parameters ζ'_α and φ' in (118) are unknown linear functionals of the field $\xi(\mathbf{x},t)$, $\dot{\xi}(\mathbf{x},t)$,... . In Eq. (150), $\rho(\xi(t))\equiv\rho(\xi(t),\dot{\xi}(t),\ldots)$ is the deviation linear in ξ of the statistical operator $\tilde{\rho}(\zeta_\alpha,\varphi;t)$ from w due to the explicit dependence of $\tilde{\rho}(\zeta_\alpha,\varphi;t)$ on the field $\xi(\mathbf{x},t)$.

Noting that the phase in the state $\tilde{\rho}'(\zeta',\varphi';t)$ has in accordance with (150) the form

$$\varphi'(\mathbf{x},\,t)=\mathrm{Sp}\,\{\sigma'(\zeta'(t),\,\varphi'(t))+\rho(\xi(t))\}\,\hat{\varphi}(\mathbf{x}),\tag{151}$$

and taking into account (121), we find that

$$\mathrm{Sp}\,\rho(\xi(t))\,\hat{\zeta}_\alpha(\mathbf{x})=0,\;\mathrm{Sp}\,\rho(\xi(t))\,\hat{\varphi}(\mathbf{x})=0.\tag{152}$$

We have here taken into account the fact that

$$\mathrm{Sp}\,\{\sigma'(\zeta'(t),\,\varphi'(t))+\rho(\xi(t))\}\,\hat{\zeta}_\alpha(\mathbf{x})=\zeta'_\alpha(\mathbf{x},\,t).$$

From Eq. (145), taking into account the expansion (150), we obtain

$$\begin{aligned}&\mathrm{i}\int d^3x\,\{\hat{\sigma}_\alpha\mathbf{x}\,(L_\alpha(\mathbf{x};\,t)+\eta_\alpha(\mathbf{x};\,t))+\hat{\sigma}_\varphi(\mathbf{x})\,(L_\varphi(\mathbf{x};\,t)\\ &+\eta_\varphi(\mathbf{x};\,t))\}+\mathrm{i}\,\frac{\partial\rho(\xi(t))}{\partial t}\\ &=[H,\,\sigma'(\zeta'(t),\,\varphi'(t))]+[H,\,\rho(\xi(t))]+[\tilde{V}(\xi(t)),\,w],\end{aligned}\tag{153}$$

where the quantities $L_\alpha(\mathbf{x};t)$, $L_\varphi(\mathbf{x},t)$ [see (123)] and

$$\left.\begin{aligned}\eta_\alpha(\mathbf{x};\,t)=&\,\mathrm{i}\,\mathrm{Sp}\,w\,[\tilde{V}(\xi(t)),\,\hat{\zeta}_\alpha(\mathbf{x})]\\ &+\mathrm{i}\,\mathrm{Sp}\,\rho(\xi(t))\,[H,\,\hat{\zeta}_\alpha(\mathbf{x})];\\ \eta_\varphi(\mathbf{x};\,t)=&\,\mathrm{i}\,\mathrm{Sp}\,w\,[\tilde{V}(\xi(t)),\,\hat{\varphi}(\mathbf{x})]\\ &+\mathrm{i}\,\mathrm{Sp}\,\rho(\xi(t))\,[H,\,\hat{\varphi}(\mathbf{x})];\end{aligned}\right\}\tag{154}$$

$$\begin{aligned}&\eta_\alpha(\mathbf{x};\,t)\equiv\eta_\alpha(\mathbf{x};\,\xi(t),\,\dot{\xi}(t)\,\ldots),\;\eta_\varphi(\mathbf{x};\,t)\\ &\quad\equiv\eta_\varphi(\mathbf{x};\,\xi(t),\,\dot{\xi}(t)\,\ldots)\end{aligned}$$

are unknown linear functionals of $\dot{\xi}(t)$, $\xi(t)$,... which determine the linearized hydrodynamic equations of the superfluid

$$\begin{aligned}&\dot{\zeta}_\alpha(\mathbf{x},\,t)-L_\alpha(\mathbf{x},\,t)=\eta_\alpha(\mathbf{x},\,t);\;\dot{\varphi}(\mathbf{x},\,t)\\ &\quad-L_\varphi(\mathbf{x};\,t)=\eta_\varphi(\mathbf{x};\,t)\end{aligned}\tag{155}$$

in the presence of the "sources" $\eta_\alpha(\mathbf{x};t)$ and $\eta_\varphi(\mathbf{x};t)$ associated with the external field $\xi(\mathbf{x},t)$. Therefore

$$\begin{aligned}&\mathrm{i}\int d^3x\,\{\hat{\sigma}_\alpha(\mathbf{x})\,\eta_\alpha(\mathbf{x};\,t)+\hat{\sigma}_\varphi(\mathbf{x})\,\eta_\varphi(\mathbf{x};\,t)\}\\ &+\mathrm{i}\,\frac{\partial\rho(\xi(t))}{\partial t}=[H,\,\rho(\xi(t))]+[\tilde{V}(\xi(t)),\,w].\end{aligned}\tag{156}$$

We now turn to the determination of the sources $\eta_\alpha(\mathbf{x};t)$ and $\eta_\varphi(\mathbf{x};t)$. To this end, we transform the system of equations (156) and (154) to a form convenient for finding $\eta_\alpha(\mathbf{x};t)$ and $\eta_\varphi(\mathbf{x};t)$ in perturbation theory with respect to the spatial and time derivatives of $\xi(\mathbf{x},t)$. It is readily seen that Eq. (156) is equivalent to the integral equation

$$\begin{aligned}&\mathrm{e}^{-\mathrm{i}H\tau}\rho(\xi(t))\,\mathrm{e}^{\mathrm{i}H\tau}=\rho(\xi(t))-\mathrm{i}\int_0^\tau d\tau'\mathrm{e}^{-\mathrm{i}H\tau'}\\ &\times\Big\{\mathrm{i}\,\frac{\partial\rho(\xi(t))}{\partial t}+\mathrm{i}\int d^3x\,(\hat{\sigma}_\alpha(\mathbf{x})\,\eta_\alpha(\mathbf{x};\,t)\\ &+\hat{\sigma}_\varphi(\mathbf{x})\,\eta_\varphi(\mathbf{x};\,t))-[\tilde{V}(\xi(t)),\,w]\Big\}\,\mathrm{e}^{\mathrm{i}H\tau'}.\end{aligned}$$

Since in the limit $\tau\to\infty$ we have in accordance with (51)

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}H\tau}\rho(\xi(t))\,\mathrm{e}^{\mathrm{i}H\tau}&\xrightarrow[\tau\to\infty]{}\sigma'(\zeta'(\tau;\,t),\,\varphi'(\tau;\,t))\\ &=\mathrm{e}^{-\mathrm{i}H\tau}\sigma'(\zeta'(0;\,t),\,\varphi'(0;\,t))\,\mathrm{e}^{\mathrm{i}H\tau},\end{aligned}$$

$$\begin{aligned}&\sigma'(\zeta'(0,\,t),\,\varphi'(0;\,t))\\ &\quad=\int d^3x\,\{\hat{\sigma}_\alpha(\mathbf{x})\,\zeta'_\alpha(\mathbf{x},\,0;\,t)+\hat{\sigma}_\varphi(\mathbf{x})\,\varphi'(\mathbf{x},\,0;\,t)\}\end{aligned}$$

[the parameters $\zeta'_\alpha(\mathbf{x},\tau;t), \varphi'(\mathbf{x},\tau;t)$ satisfy with respect to the variables $\mathbf{x}$ and τ the linearized hydrodynamic equations with the initial conditions $\zeta'_\alpha(\mathbf{x},0;t) \equiv \zeta'_\alpha(\mathbf{x};t)$, $\varphi'(\mathbf{x},0;t) \equiv \varphi'(\mathbf{x};t)$, which are linear functionals of $\xi(\mathbf{x},t)$, $\dot\xi(\mathbf{x},t)$, $\ddot\xi(\mathbf{x},t)$,..., etc., i.e, $\zeta'_\alpha(\mathbf{x},t) = \zeta'_\alpha(\mathbf{x};\xi(t),\dot\xi(t),...)$, $\varphi'(\mathbf{x};t) = \varphi'(\mathbf{x};\xi(t),\dot\xi(t),...)$, where τ is a parameter], we finally obtain with allowance for this limiting relation

$$\rho(\xi(t)) = \sigma'(\zeta'(\mathbf{x}';t), \varphi'(\mathbf{x}';t)) - \int_0^\infty d\tau e^{-iH\tau} \times\Big\{ i[H, \sigma'(\zeta'(\mathbf{x}';t), \varphi'(\mathbf{x}';t))] + i[\tilde V(\xi(t)), w] + \frac{\partial\rho(\xi(t))}{\partial t} + \int d^3x\,(\hat\sigma_\alpha(\mathbf{x})\,\eta_\alpha(\mathbf{x};t) + \hat\sigma_\varphi(\mathbf{x})\,\eta_\varphi(\mathbf{x};t))\Big\} e^{iH\tau}. \tag{157}$$

The parameters $\zeta'_\alpha(\mathbf{x},t)$ and $\varphi'(\mathbf{x},t)$ are determined in accordance with (152) from the equations

$$\mathrm{Sp}\,\rho(\xi(t))\,\hat\zeta_\alpha(\mathbf{x}) = 0,\quad \mathrm{Sp}\,\rho(\xi(t))\,\hat\varphi(\mathbf{x}) = 0.$$

To Eq. (157) we can apply the standard iterative procedure with respect to the spatial and time derivatives of the field $\xi(\mathbf{x},t)$. We are interested in the mean values $\mathrm{Sp}\,\rho(\xi(t))\dot a(\mathbf{x})$, which depend on the point $\mathbf{x}$. These are determined by the values of the c-number functions $\zeta'_\alpha(\mathbf{x},t)$ and $\varphi'(\mathbf{x}',t)$ in the neighborhood of the point $\mathbf{x}$. Having in mind the calculation of these mean values, we shall represent the statistical operator $\rho(\xi(t))$ in the form of an expansion in powers of the gradients of these functions at the point $\mathbf{x}$.

In the the zeroth approximation in the gradients, we have in accordance with (118), (119), (129), and (146)

$$\overset{(0)}{\sigma}{}'(\zeta'(t), \varphi'(t)) = \overset{(0)}{\zeta}{}'_\alpha(\mathbf{x},t)\int d^3x'\hat\sigma_\alpha(\mathbf{x}') + \overset{(0)}{\varphi}{}'(\mathbf{x},t)\int d^3x'\hat\sigma_\varphi(\mathbf{x}') = \overset{(0)}{\zeta}{}'_\alpha(\mathbf{x},t)\frac{\partial w}{\partial\zeta_\alpha}\Big|_{\mathbf{p}} + \overset{(0)}{\varphi}{}'(\mathbf{x},t)\frac{\partial w}{\partial\varphi}\Big|_{\zeta,\mathbf{p}}, \tag{158}$$

$$\overset{(0)}{V}(\xi(t)) = \xi(\mathbf{x},t)\int d^3x'\hat b(\mathbf{x}'),$$

where $\overset{(0)}{\zeta}{}'_\alpha(\mathbf{x},t) \sim \xi(\mathbf{x},t)$, $\varphi'(\mathbf{x},t) \sim \xi(\mathbf{x},t)$. Here and in what follows, we must bear in mind that the parameters $\zeta'_\alpha(\mathbf{x},t)$ and $\varphi'(\mathbf{x},t)$ are determined by the values of the external field $\xi(\mathbf{x}',t)$ in the neighborhood of the point $\mathbf{x}$.

Thus, if we take into account (158), the statistical operator in the zeroth approximation $\overset{(0)}{\rho}(\xi(t))$ (proportional to $\xi(\mathbf{x},t)$) has the structure

$$\overset{(0)}{\rho}(\xi(t)) = \overset{(0)}{\zeta}{}'_\alpha(\mathbf{x},t)\frac{\partial w}{\partial\zeta_\alpha} + \overset{(0)}{\varphi}{}'(\mathbf{x},t)\frac{\partial w}{\partial\varphi} - \int_0^\infty d\tau e^{-iH\tau}\Big\{ i[\overset{(0)}{V}(\xi(t)), w] + i\Big[H, \overset{(0)}{\zeta}{}'_\alpha(\mathbf{x},t)\frac{\partial w}{\partial\zeta_\alpha} + \overset{(0)}{\varphi}{}'(\mathbf{x},t)\frac{\partial w}{\partial\varphi}\Big] + \frac{\partial w}{\partial\zeta_\alpha}\overset{(0)}{\eta}_\alpha(\mathbf{x};t) + \frac{\partial w}{\partial\varphi}\overset{(0)}{\eta}_\varphi(\mathbf{x};t)\Big\} e^{iH\tau}. \tag{159}$$

It follows from this that $\Big[\overset{(0)}{\rho}(\xi(t)), P_k\Big] = 0$. Then from the relation (154), which determines $\overset{(0)}{\eta}_\alpha(\mathbf{x};t)\,\overset{(0)}{\eta}_\varphi(\mathbf{x};t)$ in the zeroth approximation in the field inhomogeneities,

$$\left.\begin{aligned} \overset{(0)}{\eta}_\alpha(\mathbf{x};t) &= i\,\mathrm{Sp}\,w\,[\overset{(0)}{\tilde V}(\xi(t)), \hat\zeta_\alpha(\mathbf{x})] + i\,\mathrm{Sp}\,\overset{(0)}{\rho}(\xi(t))\,[H, \hat\zeta_\alpha(\mathbf{x})], \\ \overset{(0)}{\eta}_\varphi(\mathbf{x};t) &= i\,\mathrm{Sp}\,w\,[\overset{(0)}{\tilde V}(\xi(t)), \hat\varphi(\mathbf{x})] + i\,\mathrm{Sp}\,\overset{(0)}{\rho}(\xi(t))\,[H, \hat\varphi(\mathbf{x})], \end{aligned}\right\} \tag{160}$$

and the fact that $[H,\hat\zeta_\alpha(\mathbf{x})] = [\hat P_k, \hat\zeta_{\alpha k}(\mathbf{x})]$, it follows that

$$\overset{(0)}{\eta}_\alpha(\mathbf{x};t) = i\,\mathrm{Sp}\,w\,[\overset{(0)}{\tilde V}(\xi(t)), \hat\zeta_\alpha(\mathbf{x})]. \tag{161}$$

Since $\hat\zeta_\alpha(\mathbf{x})$ are the operators of the densities of the additive integrals of the motion and $\mathrm{Sp}\,\overset{(0)}{\rho}(\xi(t))\hat\zeta(\mathbf{x}) = 0$, using (161) we obtain $\overset{(0)}{\zeta}{}'_\alpha(\mathbf{x},t) = \mathrm{Sp}\,\overset{(0)}{\sigma}{}'(\zeta'(t))\varphi'(t))\hat\zeta_\alpha(\mathbf{x}) = 0$.

Therefore the statistical operator $\overset{(0)}{\rho}(\xi(t))$ has the form the condition

$$\overset{(0)}{\rho}(\xi(t)) = \overset{(0)}{\varphi}{}'(\mathbf{x},t)\frac{\partial w}{\partial\varphi} - \int_0^\infty d\tau e^{-iH\tau} \times\Big\{ i[\overset{(0)}{\tilde V}(\xi(t)), w] + \frac{\partial w}{\partial\varphi}\overset{(0)}{\eta}_\varphi(\mathbf{x};t) + \frac{\partial w}{\partial\zeta_\alpha}\overset{(0)}{\eta}_\alpha(\mathbf{x};t)\Big\} e^{iH\tau} \tag{162}$$

(we have used the fact that $[H,\partial w/\partial\varphi] = 0$). The parameter $\hat\varphi'(\mathbf{x},t)$ is determined from the condition

$$\mathrm{Sp}\,\overset{(0)}{\rho}(\xi(t))\,\hat\varphi(\mathbf{x}) = 0.$$

Bearing in mind that $[w,\hat P_k] = [w,H] = 0$, we find in accordance with (161) and (146)

$$\overset{(0)}{\eta}_\alpha(\mathbf{x};t) = -i\xi(\mathbf{x},t)\,\mathrm{Sp}\,w\,[\hat N, \hat b(0)]\,Y_0\frac{\partial p_0}{\partial Y_\alpha}\Big|_{\mathbf{p}}. \tag{163}$$

To determine $\overset{(0)}{\eta}_\varphi(\mathbf{x};t)$, it is necessary to find $\Big[\overset{(0)}{\rho}(\xi(t)), H\Big]$. From (162), we obtain

$$[\overset{(0)}{\rho}(\xi(t)), H] = i\lim_{\tau\to\infty} e^{-iH\tau}\Big\{ i[\overset{(0)}{\tilde V}(\xi(t)), w] + \frac{\partial w}{\partial\varphi}\overset{(0)}{\eta}_\varphi(\mathbf{x};t) + \frac{\partial w}{\partial\zeta_\alpha}\overset{(0)}{\eta}_\alpha(\mathbf{x};t)\Big\} e^{iH\tau} - i\Big\{ i[\overset{(0)}{\tilde V}(\xi(t)), w] + \frac{\partial w}{\partial\varphi}\overset{(0)}{\eta}_\varphi(\mathbf{x};t) + \frac{\partial w}{\partial\zeta_\alpha}\overset{(0)}{\eta}_\alpha(\mathbf{x};t)\Big\}. \tag{164}$$

Since $H = \mathscr{H} + p_0\hat N$ depends on ζ_α, $\mathbf{p}$, and $[w,H] = 0$,

$$e^{-iH\tau}\frac{\partial w}{\partial\zeta_\alpha}e^{iH\tau} = \frac{\partial w}{\partial\zeta_\alpha} + \tau\frac{\partial p_0}{\partial\zeta_\alpha}\Big|_{\mathbf{p}}\frac{dw}{\partial\varphi}. \tag{165}$$

Therefore, noting that $[H,\partial w/\partial\varphi] = 0$, we rewrite the relation (164) in the form

$$[\overset{(0)}{\rho}(\xi(t)), H] = -\lim_{\tau\to\infty}\Big\{ e^{-iH\tau}[\overset{(0)}{\tilde V}(\xi(t)), w]e^{iH\tau} - i\tau\overset{(0)}{\eta}_\alpha(\mathbf{x};t)\frac{\partial p_0}{\partial\zeta_\alpha}\Big|_{\mathbf{p}}\frac{\partial w}{\partial\varphi}\Big\} + [\overset{(0)}{\tilde V}(\xi(t)), w].$$

It follows from this that the second equation of (160) takes the form

$$\overset{(0)}{\eta}_{\varphi}(\mathbf{x};\ t) = -\mathrm{i}\lim_{\tau\to\infty}\left\{\mathrm{Sp}\,\mathrm{e}^{-\mathrm{i}H\tau}\,[\overset{(0)}{\widetilde{V}}(\xi(t)),\ w]\right.$$
$$\left.\times\ \mathrm{e}^{\mathrm{i}H\tau}\hat{\varphi}(\mathbf{x}) - \mathrm{i}\tau\overset{(0)}{\eta}_{\alpha}(\mathbf{x};\ t)\,\frac{\partial p_0}{\partial\zeta_\alpha}\Big|_{\mathrm{p}}\right\}. \quad (166)$$

Using the ergodic relation (127), we readily see that

$$\lim_{\tau\to\infty}\left\{\mathrm{i}\mathrm{e}^{-\mathrm{i}H\tau}\,[\overset{(0)}{\widetilde{V}}(\xi(t)),\ w]\,\mathrm{e}^{\mathrm{i}H\tau} + \tau\,\frac{\partial w}{\partial\varphi}\,\overset{(0)}{\eta}_{\alpha}(\mathbf{x};\ t)\,\frac{\partial p_0}{\partial\zeta_\alpha}\Big|_{\mathrm{p}}\right\}$$
$$= -\frac{\partial w}{\partial\zeta_\alpha}\Big|_{\mathrm{p}}\overset{(0)}{\eta}_{\alpha}(\mathbf{x};\ t) - \frac{\partial w}{\partial\varphi}\,\chi',$$

where χ' is determined by (128), in which the operator $\mathrm{i}\left[w, \overset{(0)}{V}(\xi(t))\right]$. is taken as the initial operator ρ'. Thus, we have

$$\overset{(0)}{\eta}_{\varphi}(\mathbf{x};\ t) = -\chi'(\mathbf{x},\ t). \quad (167)$$

If $[N,\hat{b}] = 0$, then $\overset{(0)}{\eta}_{\alpha}(\mathbf{x};t) = 0$ [see (163)]. In this case, the expression (166) for $\overset{(0)}{\eta}_{\varphi}(\mathbf{x};t)$ can be simplified. To this

$$[w,\ \int d^3x\,(\psi(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}\mathbf{px}} + \psi^+(\mathbf{x})\,\mathrm{e}^{\mathrm{i}\mathbf{px}})].$$

end, we again use the ergodic relation, in which $[w,\int d^3x(\psi(\mathbf{x})\mathrm{e}^{\mathrm{i}\mathbf{px}} + \psi^+(\mathbf{x})\mathrm{e}^{\mathrm{i}\mathbf{px}})]$ is chosen as the initial statistical operator ρ'. Since this statistical operator commutes with $\hat{P}_k$, we obtain in accordance with (166), (128), and (127)

$$\overset{(0)}{\eta}_{\varphi}(\mathbf{x};\ t) = \frac{1}{2\eta}\,\xi(\mathbf{x},\ t)\,\frac{\partial b}{\partial\zeta_\alpha}\Big|_{\mathrm{p}}$$
$$\times\int d^3x'\,\mathrm{Sp}\,[w,\ \psi^+(\mathbf{x}')\,\mathrm{e}^{\mathrm{i}\mathbf{px}'} - \psi(\mathbf{x}')\,\mathrm{e}^{-\mathrm{i}\mathbf{px}'}]\,\hat{\zeta}_\alpha(0).$$

Noting that $[w,H] = 0$, $[w,\hat{P}_k] = 0$, we readily transform $\overset{(0)}{\eta}_{\varphi}(\mathbf{x};t)$ to the form

$$\overset{(0)}{\eta}_{\varphi}(\mathbf{x};\ t) = -\xi(\mathbf{x},\ t)\left(\frac{\partial b}{\partial\zeta^\mu}\,p^\mu + \frac{\partial b}{\partial\zeta_4}\right),\ [\hat{N},\ \hat{b}] = 0 \quad (168)$$

(we have here used relativistic notation).

Since in the considered case $[\hat{N},\hat{b}] = 0$) the leading approximation for $\eta_\alpha(\mathbf{x};t)$ vanishes, we must for what follows find the quantities $\eta_\alpha(\mathbf{x};t)$ in the first approximation in the gradients of the external field [see (154)]:

$$\overset{(1)}{\eta}_{\alpha}(\mathbf{x};\ t) = \mathrm{i}\,\mathrm{Sp}\,w\,[\overset{(1)}{\widetilde{V}}(\xi(t)),\ \hat{\zeta}_\alpha(\mathbf{x})] + \mathrm{i}\,\mathrm{Sp}\,\overset{(1)}{\rho}(\xi(t))\,[H,\ \hat{\zeta}_\alpha(\mathbf{x})]. \quad (169)$$

In accordance with (118), (129), and (146), the operators $\overset{(0)}{\sigma}'(\zeta'(t),\varphi'(t))$, and $\overset{(1)}{V}(\xi(t))$ have the form

$$\left.\begin{aligned}
&\overset{(1)}{\sigma}'(\zeta'(t),\ \varphi'(t)) = \overset{(1)}{\zeta}{}'_{\alpha}(\mathbf{x};\ t)\,\frac{\partial w}{\partial\zeta_\alpha} + \overset{(1)}{\varphi}{}'(\mathbf{x};\ t)\,\frac{\partial w}{\partial\varphi}\\
&+ \frac{\partial\overset{(0)}{\varphi}(\mathbf{x};\ t)}{\partial x_k}\left\{\frac{\partial w}{\partial p_k} - x_k\,\frac{\partial w}{\partial\varphi}\right\},\\
&\overset{(1)}{\widetilde{V}}(\xi(t)) = \frac{\partial\xi(\mathbf{x},\ t)}{\partial x_k}\int d^3x'\,(x'_k - x_k)\,\hat{b}(\mathbf{x}'),
\end{aligned}\right\} \quad (170)$$

where $\overset{(1)}{\zeta}{}'_{\alpha}(\mathbf{x};t) \sim \partial\xi(\mathbf{x},t)/\partial x_k;\ \overset{(1)}{\varphi}{}'(\mathbf{x};t) \sim \partial\xi(\mathbf{x},t)/\partial x_k$. Therefore, the operator $\overset{(1)}{\rho}(\xi(t))$ has the general structure [see (157)]

$$\overset{(1)}{\rho}(\xi(t)) = \overset{(1)}{\sigma}{}'(\zeta'(t),\ \varphi'(t))$$
$$-\int_0^\infty d\tau\,\mathrm{e}^{-\mathrm{i}H\tau}\left\{\mathrm{i}\,[H,\ \overset{(1)}{\sigma}{}'(\zeta'(t),\ \varphi'(t))]\right.$$
$$+\ \mathrm{i}\,[\overset{(1)}{\widetilde{V}}(\xi(t)),\ w] + \frac{\partial w}{\partial\varphi}\,\overset{(1)}{\eta}_{\varphi}(\mathbf{x};\ t)$$
$$\left.+\ \frac{\partial w}{\partial\zeta_\alpha}\,\overset{(1)}{\eta}_{\alpha}(\mathbf{x};\ t) + \frac{\partial\overset{(0)}{\eta}_{\varphi}(\mathbf{x};\ t)}{\partial x_k}\left(\frac{\partial w}{\partial p_k}\Big|_{\zeta} - x_k\,\frac{\partial w}{\partial\varphi}\Big|_{\zeta,\ \mathrm{p}}\right)\right\}\mathrm{e}^{\mathrm{i}H\tau}. \quad (171)$$

Noting that $H = \mathscr{H} + p_0\hat{N}$ satisfies the relation

$$\mathrm{i}\left[H,\ \frac{\partial w}{\partial\zeta_\alpha}\Big|_{\mathrm{p}}\right] = -\frac{\partial p_0}{\partial\zeta_\alpha}\Big|_{\mathrm{p}}\,\frac{\partial w}{\partial\varphi}\Big|_{\zeta,\ \mathrm{p}},$$

and taking into account the form of $\overset{(1)}{\sigma}{}'(\zeta'(t),\varphi'(t))$ (170) and

$$\left[H,\ \frac{\partial w}{\partial\varphi}\right] = 0,\ \left[\hat{P}_l,\ \frac{\partial w}{\partial p_k}\Big|_{\zeta}\right] = -\mathrm{i}\delta_{kl}\,\frac{\partial w}{\partial\varphi}\Big|_{\zeta,\ \mathrm{p}},$$

we have

$$\mathrm{i}\,\mathrm{Sp}\,\overset{(1)}{\rho}(\xi(t))\,[H,\ \hat{\zeta}_\alpha(\mathbf{x})] = -\mathrm{i}\lim_{\tau\to\infty}\mathrm{Sp}\,\mathrm{e}^{-\mathrm{i}H\tau}$$
$$\times\,[\overset{(1)}{\widetilde{V}}(\xi(t)),\ w]\,\mathrm{e}^{\mathrm{i}H\tau}\hat{\zeta}_\alpha(\mathbf{x}) + \mathrm{i}\,\mathrm{Sp}\,[\overset{(1)}{\widetilde{V}}(\xi(t)),\ w]\,\hat{\zeta}_\alpha(\mathbf{x}).$$

Therefore [see (169) and $\overset{(1)}{V}(\xi(t))$ from (170)]

$$\overset{(1)}{\eta}_{\alpha}(\mathbf{x};\ \ t) = \mathrm{i}\,\frac{\partial\xi(\mathbf{x},\ t)}{\partial x_k}\lim_{\tau\to\infty}\mathrm{Sp}\,\mathrm{e}^{\mathrm{i}H\tau}$$
$$\times\left[w,\ \int d^3x'\,x'_k\hat{\zeta}_\alpha(x')\right]\mathrm{e}^{-\mathrm{i}H\tau}\hat{b}(0). \quad (172)$$

To simplify this expression, we introduce

$$\overset{(1)}{\eta}{}'_{\alpha}(\mathbf{x};\ t) = \mathrm{i}\,\frac{\partial\xi(\mathbf{x},\ t)}{\partial x_k}\lim_{\tau\to\infty}\mathrm{Sp}\,\mathrm{e}^{\mathrm{i}H\tau}$$
$$\times\left[w,\ \int d^3x'\,x'_k\hat{\zeta}'_\alpha(\mathbf{x}')\right]\mathrm{e}^{-\mathrm{i}H\tau}\hat{b}(0), \quad (173)$$

where

$$\hat{\zeta}'_\alpha(\mathbf{x}) \equiv \hat{\zeta}_\alpha(\mathbf{x}) - \hat{\zeta}_4(\mathbf{x})\,Y_0\,\frac{\partial p_0}{\partial Y_\alpha}\Big|_{\mathrm{p}}.$$

The quantities $\overset{(1)}{\eta}_{\alpha}$ and $\overset{(1)}{\eta}{}'_{\alpha}$ are evidently related by the equations

$$\overset{(1)}{\eta}_{\alpha}(\mathbf{x};\ t) = \overset{(1)}{\eta}{}'_{\alpha}(\mathbf{x};\ t) + Y_0\,\frac{\partial p_0}{\partial Y_\alpha}\Big|_{\mathrm{p}}\,\overset{(1)}{\eta}_{4}(\mathbf{x};\ t). \quad (174)$$

Bearing in mind that

$$\left[\hat{P}_k,\ \left[w,\ \int d^3x'\,x'_l\hat{\zeta}'_\alpha(\mathbf{x}')\right]\right] = 0,$$

and using the ergodic relation (127), we find for $\alpha = 0, 1, 2, 3$

$$\overset{(1)}{\eta}{}'_{\alpha}(\mathbf{x};\ t) = \mathrm{i}\,\frac{\partial\xi(\mathbf{x},\ t)}{\partial x_l}\,\frac{\partial\langle\hat{b}(0)\rangle}{\partial\zeta_\beta}\,\mathrm{Sp}\,w\int d^3x'\,x'_l$$
$$\times\,[\hat{\zeta}'_\alpha(\mathbf{x}'),\ \hat{\zeta}_\beta(0)],\ [\hat{N},\ \hat{b}] = 0, \quad (175)$$

where $\langle\hat{b}(0)\rangle \equiv \mathrm{Sp}\,w\hat{b}(0)$.

To find $\overset{(1)}{\eta}_4(\mathbf{x};t)$, we must calculate the limit of the expression in (172) for $\alpha = 4$. The ergodic relation (127) leads to

$$\overset{(1)}{\eta}_4(\mathbf{x};\, t) = \frac{\partial \xi(\mathbf{x},\, t)}{\partial x_l} \left\{ \mathrm{i}\, \frac{\partial \langle \hat{b}(0)\rangle}{\partial \zeta_\alpha} \int d^3x' x'_l \times \mathrm{Sp}\, w \left[\hat{\zeta}_4(\mathbf{x}'),\, \hat{\zeta}_\alpha(0) \right] - \frac{\partial \langle \hat{b}(0)\rangle}{\partial p_k} \bigg|_{\zeta} \right\}. \tag{176}$$

Thus, in accordance with (174)–(176), the "sources" $\overset{(1)}{\eta}_\alpha(\mathbf{x};t)$ have the form

$$\overset{(1)}{\eta}_\alpha(\mathbf{x};\, t) = -\frac{\partial \xi(\mathbf{x},\, t)}{\partial x_l} \left\{ \frac{\partial \langle \hat{b}(0)\rangle}{\partial \zeta_\beta} K_{l;\,\alpha\beta} + Y_0 \frac{\partial p_0}{\partial Y_\alpha}\bigg|_{\mathbf{p}} \frac{\partial \langle \hat{b}(0)\rangle}{\partial p_l}\bigg|_{\zeta} \right\}, \tag{177}$$

where

$$K_{l,\,\alpha\beta} \equiv -\mathrm{i} \int d^3x\, x_l\, \mathrm{Sp}\, w\, [\hat{\zeta}_\alpha(\mathbf{x}),\, \hat{\zeta}_\beta(0)] = K_{l;\,\beta\alpha}.$$

Using (32), we find in explicit form the elements of the matrix $K_{l;\alpha\beta}$. For $\alpha = 4, \beta = 0, 1, 2, 3, 4$

$$K_{l;\,4\beta} = \delta_{\beta 0}\zeta_{4l} + \delta_{\beta l}\zeta_4. \tag{178}$$

For $\alpha = \mu = 0, 1, 2, 3$, and $\beta = \nu = 0, 1, 2, 3$

$$K_{l;\,\mu\nu} = \delta_{\mu 0}\delta_{\nu 0} 2\zeta_{0l} + \delta_{\mu 0}\delta_{\nu k}(\zeta_{kl} + \delta_{kl}\zeta_0) + \delta_{\mu i}\delta_{\nu k}(\delta_{li}\zeta_k + \delta_{lk}\zeta_i).$$

It is easy to show that from the last equation with allowance for (178) and (46) it is possible to represent the elements of $K_{l;\alpha\beta}$ in the compact form $(\alpha, \beta = 0, 1, 2, 3, 4)$

$$K_{l;\,\beta\alpha} = -Y_0 \frac{\partial \zeta_{\beta l}}{\partial Y_\alpha}\bigg|_{\mathbf{p}} - Y_l \frac{\partial \zeta_\beta}{\partial Y_\alpha}\bigg|_{\mathbf{p}} + p^\beta \frac{\partial \zeta_\alpha}{\partial p_l}\bigg|_{Y},\quad p^\beta = (p^\nu,\, 1),\quad (\nu = 0,\, 1,\, 2,\, 3). \tag{179}$$

We have found the sources $\eta_\alpha(\mathbf{x};t)$ in the superfluid hydrodynamic equations that are proportional to $\partial\xi(\mathbf{x},t)/\partial x_k$. We now find the sources $\eta_\alpha(\mathbf{x};t)$ that are proportional to the time derivatives $\partial\xi(\mathbf{x},t)/dt$ of the external field. In accordance with (154),

$$\overset{1}{\eta}_\alpha(\mathbf{x};\, t) = \mathrm{i}\, \mathrm{Sp}\, \overset{1}{\rho}(\xi(t))\, [H,\, \hat{\zeta}_\alpha(\mathbf{x})],$$

where $\overset{1}{\rho}(\xi(t))$ is the term in the expansion of the statistical operator $\rho(\xi(t))$ proportional to $\partial\xi(\mathbf{x},t)/\partial t$. In accordance with Eq. (157),

$$\overset{1}{\rho}(\xi(t)) = \overset{1}{\sigma}{}'(\zeta'(t),\, \varphi'(t)) - \int_0^\infty d\tau\, e^{-\mathrm{i}H\tau} \times \left\{ \mathrm{i}\, [H,\, \overset{1}{\sigma}{}'(\zeta'(t),\, \varphi'(t))] + \frac{\partial w}{\partial \varphi} \overset{1}{\eta}_\varphi(\mathbf{x};\, t) + \frac{\partial w}{\partial \zeta_\alpha} \overset{1}{\eta}_\alpha(\mathbf{x};\, t) + \frac{\partial \overset{(0)}{\rho}(\xi(t))}{\partial t} \right\} e^{\mathrm{i}H\tau}, \tag{180}$$

where is accordance with (118) and (129)

$$\overset{1}{\sigma}{}'(\zeta'(t),\, \varphi'(t)) = \frac{\partial w}{\partial \zeta_\alpha} \overset{1}{\zeta}{}'_\alpha(\mathbf{x};\, t) + \frac{\partial w}{\partial \varphi} \overset{1}{\varphi}{}'(\mathbf{x};\, t)$$

and $\overset{1}{\zeta}{}'_\alpha(\mathbf{x};t), \overset{1}{\varphi}{}'(\mathbf{x};t)$ are proportional to $\partial\xi(\mathbf{x},t)/\partial t$. Since $\left[H, \overset{1}{\sigma}{}'(\zeta'\varphi')\right] \sim (\partial w/\partial\varphi)|_{\zeta,\mathbf{p}}$, bearing in mind that $[H, \partial w/\partial\varphi] = 0$, we find from Eq. (180)

$$\overset{1}{\eta}_\alpha(\mathbf{x};\, t) = -\lim_{\tau\to\infty} \mathrm{Sp}\, e^{-\mathrm{i}H\tau} \dot{\rho}^{(0)}(\xi(t))\, e^{\mathrm{i}H\tau} \hat{\zeta}_\alpha(\mathbf{x}) + \mathrm{Sp}\, \dot{\rho}^{(0)}(\xi(t))\, \hat{\zeta}_\alpha(\mathbf{x}). \tag{181}$$

Noting that the statistical operator $\rho^{(0)}(\xi(t))$ corresponds to a spatially homogeneous state, $[\rho^{(0)}(\xi(t)), \hat{P}_k] = 0$, and since $[H, \hat{\zeta}_\alpha(\mathbf{x})] = [\hat{P}_k \hat{\zeta}_{\alpha\kappa}(\mathbf{x})]$, we have

$$\mathrm{Sp}\, e^{-\mathrm{i}H\tau} \dot{\rho}^{(0)}(\xi(t))\, e^{\mathrm{i}H\tau} \hat{\zeta}_\alpha(\mathbf{x}) = \mathrm{Sp}\, \dot{\rho}^{(0)}(\xi(t))\, \hat{\zeta}_\alpha(\mathbf{x}),$$

and, therefore,

$$\overset{1}{\eta}_\alpha(\mathbf{x};\, t) = 0. \tag{182}$$

Thus, Eqs. (163), (167), (168), (177), and (182) give in the leading approximation expressions for the sources $\eta_\alpha(\mathbf{x};t)$, $\eta_\varphi(\mathbf{x};t)$ in the superfluid hydrodynamic equations in the case of an external field $\xi(\mathbf{x};t)$ that varies slowly in space and time.

8. LOW-FREQUENCY LIMITING BEHAVIOR OF THE GREEN'S FUNCTIONS

In this section, we find the actual structure of the Green's functions $G^{\pm}_{ab}(\mathbf{k},\omega)$ in the region of small wave vectors $\mathbf{k}$($kl \ll 1$, where 1 is the mean free path) and low frequencies ω($\omega\tau_r \ll 1$, where τ_r is the relaxation time). For this, we shall use the superfluid hydrodynamic equations in the form (89) and (90). In the presence of external sources $\eta_\alpha(\mathbf{x};t)$, $\eta_\varphi(\mathbf{x};t)$ [see (154)], these equations have the form

$$\frac{\partial t^{\mu\nu}(\mathbf{x},\, t)}{\partial x^\nu} = \eta^\mu(\mathbf{x};\, t);\quad \frac{\partial j^\nu(\mathbf{x},\, t)}{\partial x^\nu} = \eta^4(\mathbf{x};\, t);\quad \frac{\partial p^\mu(\mathbf{x},\, t)}{\partial x_\nu} - \frac{\partial p^\nu(\mathbf{x},\, t)}{\partial x_\mu} = \eta^{\mu\nu}(\mathbf{x};\, t). \tag{183}$$

The last equation in (183) is a consequence of the equations

$$\varphi(\mathbf{x},\, t) = p_0(\mathbf{x},\, t) + \eta_\varphi(\mathbf{x};\, t),\quad p_k(\mathbf{x},\, t) = \frac{\partial \varphi(\mathbf{x},\, t)}{\partial x_k},$$

from which it can be seen that

$$\eta_{\mu\nu}(\mathbf{x};\, t) = \frac{\partial \eta_\varphi(\mathbf{x};\, t)}{\partial x_l} (g_{\mu l} g_{\nu 0} - g_{\mu 0} g_{\nu l}). \tag{184}$$

Assuming that the "sources are small," we linearize Eq. (183) about the equilibrium state, choosing (as in Sec. 6) as the parameters that describe the deviation from the equilibrium state the quantities $\delta Y_\lambda(\mathbf{x};t) = Y_\lambda(\mathbf{x};t) - \bar{Y}_\lambda$ ($\lambda = 0, 1, 2, 3$) (the deviations of the thermodynamic forces $Y_\lambda(\mathbf{x};t)$ from the equilibrium values $\hat{Y}_\lambda$), the quantity $\delta p_0(\mathbf{x};t) = p_0(\mathbf{x};t) - \bar{p}_0$, and the phase $\delta\varphi(\mathbf{x};t)$ (in the equilibrium state, the phase $\bar{\varphi}$ is assumed to be equal to zero). Then the hydrodynamic equations (183) linearized about the equilibrium state with $\bar{\varphi} = 0$ have the form [cf. (131)]

$$\left.\begin{aligned} &ik_\nu\left\{\frac{\partial t^{\mu\nu}}{\partial Y_\lambda}\delta Y_\lambda(\mathbf{k},\ \omega)+\frac{\partial t^{\mu\nu}}{\partial p_\lambda}\delta p_\lambda(\mathbf{k},\ \omega)\right\}=\eta^\mu(\mathbf{k},\ \omega);\\ &ik_\nu\left\{\frac{\partial j^\nu}{\partial Y_\lambda}\delta Y_\lambda(\mathbf{k},\ \omega)+\frac{\partial j^\nu}{\partial p_\lambda}\delta p_\lambda(\mathbf{k},\ \omega)\right\}=\eta^4(\mathbf{k},\ \omega);\\ &k_\nu\delta p_\mu(\mathbf{k},\ \omega)-k_\mu\delta p_\nu(\mathbf{k},\ \omega)=\eta_\varphi(\mathbf{k},\ \omega)\,(k_\mu g_{\nu0}-g_{\mu0}k_\nu).\end{aligned}\right\}\qquad(185)$$

The quantities $\delta Y_\lambda(\mathbf{k},\omega), \delta p_\lambda(\mathbf{k},\omega), \eta^\mu(\mathbf{k},\omega), \eta^4(\mathbf{k},\omega), \eta_\varphi(\mathbf{k},\omega)$ that occur here are the Fourier components of the corresponding quantities. We write the solution of the last equation in the form

$$\delta p_\nu(\mathbf{k},\ \omega)=ik_\nu\delta\varphi(\mathbf{k},\ \omega)-g_{\nu0}\eta_\varphi(\mathbf{k},\ \omega).\qquad(186)$$

Substituting the expression obtained for $\delta p_\nu(\mathbf{k},\omega)$ in the remaining equations (185), we obtain

$$\left.\begin{aligned} &ik_\nu\left\{\frac{\partial t^{\mu\nu}}{\partial Y_\lambda}\delta Y_\lambda(\mathbf{k},\ \omega)+ik_\lambda\frac{\partial t^{\mu\nu}}{\partial p_\lambda}\delta\varphi(\mathbf{k},\ \omega)\right\}\\ &=\eta^\mu(\mathbf{k},\ \omega)+ik_\nu\frac{\partial t^{\mu\nu}}{\partial p_0}\eta_\varphi(\mathbf{k},\ \omega)\equiv\bar\eta^\mu(\mathbf{k},\ \omega),\\ &ik_\nu\left\{\frac{\partial j^\nu}{\partial Y_\lambda}\delta Y_\lambda(\mathbf{k},\ \omega)+ik_\lambda\frac{\partial j^\nu}{\partial p_\lambda}\delta\varphi(\mathbf{k},\ \omega)\right\}\\ &=\eta^4(\mathbf{k},\ \omega)+ik_\nu\frac{\partial j^\nu}{\partial p_0}\eta_\varphi(\mathbf{k},\ \omega)\equiv\bar\eta^4(\mathbf{k},\ \omega),\end{aligned}\right\}\qquad(187)$$

or, using the definitions (134),

$$\left.\begin{aligned} &(p^\mu a^\lambda-D^{\mu\lambda})\,\delta Y_\lambda(\mathbf{k},\ \omega)+i\,(bp^\mu-(kY)\,a^\mu)\\ &\times\delta\varphi(\mathbf{k},\ \omega)=-i\bar\eta^\mu(\mathbf{k},\ \omega),\\ &a^\lambda\delta Y_\lambda(\mathbf{k},\ \omega)+ib\delta\varphi(\mathbf{k},\ \omega)=-i\bar\eta^4(\mathbf{k},\ \omega).\end{aligned}\right\}\qquad(188)$$

Eliminating from the first equation $a^\lambda\,\delta Y_\lambda(\mathbf{k},\omega)$ by means of the second equation, we find

$$\delta Y_\nu(\mathbf{k},\ \omega)=iD^{-1}_{\nu\mu}\{\bar\eta^\mu(\mathbf{k},\ \omega)-p^\mu\bar\eta^4(\mathbf{k},\ \omega)-(kY)\,a^\mu\delta\varphi(\mathbf{k},\ \omega)\}\qquad(189)$$

and, therefore, taking into account the second equation of (188),

$$\delta\varphi(\mathbf{k},\omega)=-\frac{1}{\Delta}\{\bar\eta^4(\mathbf{k},\ \omega)\,(1-a^\lambda D^{-1}_{\lambda\mu}p^\mu)+a^\lambda D^{-1}_{\lambda\mu}\bar\eta^\mu(\mathbf{k},\ \omega)\}.\qquad(190)$$

We now consider the actual structure of the sources η^μ and η^4, $\eta^{\mu\nu}$ on the right-hand side of (183). As we have seen in Sec. 7, their explicit form depends strongly on whether or not the operator $\hat b(\mathbf{x})$ in (146) commutes with the particle-number operator $\hat N$. We consider first the case when $[\hat N,\hat b]\neq 0$. The Fourier components of the sources $\eta^\mu(\mathbf{k},\omega)$ in the continuity equation for the energy–momentum tensor are determined in accordance with (163) by

$$\overset{(0)}{\eta}{}^\mu(\mathbf{k},\ \omega)=\xi(\mathbf{k},\ \omega)\frac{\partial\langle\hat b(0)\rangle}{\partial\varphi}p^\mu.\qquad(191)$$

The Fourier component of the source $\eta^4(\mathbf{k},\omega)$ in the continuity equation for the current is determined by

$$\overset{(0)}{\eta}{}^4(\mathbf{k},\ \omega)=\xi(\mathbf{k},\ \omega)\frac{\partial\langle\hat b(0)\rangle}{\partial\varphi}.\qquad(192)$$

Since the quantity $\eta_\varphi(\mathbf{k},\omega)$ occurs in the sources $\bar\eta^\mu(\mathbf{k},\omega)$ and $\bar\eta^4(\mathbf{k},\omega)$ in conjuction with the factor k_ν, in the leading approximation in k

$$\overset{(0)}{\bar\eta}{}^\mu(\mathbf{k},\ \omega)=\overset{(0)}{\eta}{}^\mu(\mathbf{k},\ \omega),\ \overset{(0)}{\bar\eta}{}^4(\mathbf{k},\ \omega)=\overset{(0)}{\eta}{}^4(\mathbf{k},\ \omega).\qquad(193)$$

If $[\hat N,\hat b]=0$, then the sources η^μ and η^4 vanish in the leading approximation in k [see (163)], and we must therefore find the sources in the following approximation in the wave vector and frequency of the external field $\xi(\mathbf{k},\omega)$. In accordance with Eqs. (177) and (182), the source $\eta^\mu(\mathbf{k},\omega)$ is determined by

$$\overset{(1)}{\eta}{}^\mu(\mathbf{k},\omega)=-ik_l\xi(\mathbf{k},\omega)\left\{\frac{\partial\langle\hat b(0)\rangle}{\partial\zeta_\beta}\bigg|_{\mathbf{p}}K_{l;\,\mu\beta}+p^\mu\frac{\partial\langle b(0)\rangle}{\partial p_l}\bigg|_\zeta\right\}.$$

For the source $\overset{(1)}{\eta}{}^4(\mathbf{k},\omega)$, we have in accordance with (177)

$$\overset{(1)}{\eta}{}^4(\mathbf{k},\ \omega)=-ik_l\xi(\mathbf{k},\ \omega)\left\{\frac{\partial\langle\hat b(0)\rangle}{\partial\zeta_\beta}\bigg|_{\mathbf{p}}K_{l;\,4\beta}+\frac{\partial\langle\hat b(0)\rangle}{\partial p_l}\bigg|_\zeta\right\}.$$

Finally, the source $\overset{(0)}{\eta}_\varphi(\mathbf{k},\omega)$ is determined by [see (168)]

$$\begin{aligned}\overset{(0)}{\eta}_\varphi(\mathbf{k},\ \omega)&=-\xi(\mathbf{k},\ \omega)\left(\frac{\partial\langle\hat b(0)\rangle}{\partial\zeta^\nu}p^\nu+\frac{\partial\langle\hat b(0)\rangle}{\partial\zeta^4}\right)\\ &=-\xi(\mathbf{k},\ \omega)\frac{\partial\langle\hat b(0)\rangle}{\partial\zeta^\alpha}\bigg|_{\mathbf{p}}p^\alpha.\end{aligned}\qquad(194)$$

To keep the expressions compact, we have introduced the vector $p^\alpha\equiv(p^\mu,1)$, $\alpha=\mu$, 4. Therefore, using (187) and (194), we obtain for the sources $\overset{(1)}{\bar\eta}{}^\mu(\mathbf{k},\omega)$ and $\overset{(1)}{\bar\eta}{}^4(\mathbf{k},\omega)$ the expressions

$$\left.\begin{aligned}&\overset{(1)}{\bar\eta}{}^\mu(\mathbf{k},\ \omega)=-i\xi(\mathbf{k},\ \omega)\left\{k_l\left[\frac{\partial\langle\hat b(0)\rangle}{\partial\zeta_\beta}\bigg|_{\mathbf{p}}K_{l;\,\mu\beta}\right.\right.\\ &\left.\left.+p^\mu\frac{\partial\langle\hat b(0)\rangle}{\partial p_l}\bigg|_\zeta\right]+k_\nu\frac{\partial t^{\mu\nu}}{\partial p_0}\frac{\partial\langle\hat b(0)\rangle}{\partial\zeta_\beta}\bigg|_{\mathbf{p}}p^\beta\right\};\\ &\overset{(1)}{\bar\eta}{}^4(\mathbf{k},\ \omega)=-i\xi(\mathbf{k},\ \omega)\left\{k_l\left[\frac{\partial\langle\hat b(0)\rangle}{\partial\zeta_\beta}\bigg|_{\mathbf{p}}K_{l;\,4\beta}\right.\right.\\ &\left.\left.+\frac{\partial\langle\hat b(0)\rangle}{\partial p_l}\bigg|_\zeta\right]+k_\nu\frac{\partial j^\nu}{\partial p_0}\frac{\partial\langle\hat b(0)\rangle}{\partial\zeta_\beta}\bigg|_{\mathbf{p}}p^\beta\right\},\end{aligned}\right\}\qquad(195)$$

these occurring in the epressions for $\delta Y_\mu(\mathbf{k},\mu)$ and $\delta\varphi(\mathbf{k},\omega)$. Taking into account (178) and (179) and going over from the derivatives $(\partial\langle\hat b(0)\rangle/\partial\zeta_\beta)_{\mathbf{p}}$ $(\partial\langle\hat b(0)\rangle/\partial\mathbf{p})\zeta$ to the derivatives $(\partial\langle\hat b(0)\rangle/\partial Y_\mu)_{p\nu}$, $(\partial\langle\hat b(0)\rangle/\partial p_\mu)_Y$ in accordance with

$$\left.\begin{aligned}&\left(\frac{\partial\langle\hat b(0)\rangle}{\partial\zeta_\alpha}\right)_{\mathbf{p}}\\ &\quad=\frac{\partial\langle b(0)\rangle}{\partial Y_\mu}\left(\frac{\partial Y_\mu}{\partial\zeta_\alpha}\right)_{\mathbf{p}}+\frac{\partial\langle\hat b(0)\rangle}{\partial p_\mu}\frac{1}{Y_0}\delta_{\mu0}\left(\frac{\partial Y_\alpha}{\partial\zeta_\beta}\right)_{\mathbf{p}}p^\beta;\\ &\left(\frac{\partial\langle\hat b(0)\rangle}{\partial p_l}\right)_\zeta=-\frac{\partial\langle\hat b(0)\rangle}{\partial Y_\mu}\left(\frac{\partial\zeta_\alpha}{\partial p_l}\right)_Y\left(\frac{\partial Y_\mu}{\partial\zeta_\alpha}\right)_{\mathbf{p}}\\ &+\frac{\partial\langle\hat b(0)\rangle}{\partial p_\mu}\left[\delta_{\mu0}\left(\frac{Y_l}{Y}-\frac{p^\beta}{Y_0}\frac{\partial\zeta_\alpha}{\partial p_l}\bigg|_Y\frac{\partial Y_\beta}{\partial\zeta_\alpha}\bigg|_{\mathbf{p}}\right)+\delta_{\mu l}\right],\end{aligned}\right\}\qquad(196)$$

we obtain

$$\left.\begin{aligned}
&\overset{(1)}{\eta}{}^{\mu}(\mathbf{k},\ \omega)=\mathrm{i}\xi(\mathbf{k},\ \omega)\left\{\frac{\partial\langle\hat b(0)\rangle}{\partial Y_\nu}A^\mu_\nu+\frac{\partial\langle\hat b(0)\rangle}{\partial p_\nu}B^\mu_\nu\right\};\\
&\overset{(1)}{\eta}{}^{4}(\mathbf{k},\ \omega)=\mathrm{i}\xi(\mathbf{k},\ \omega)\left\{\frac{\partial\langle\hat b(0)\rangle}{\partial Y_\nu}A_\nu+\frac{\partial\langle\hat b(0)\rangle}{\partial p_\nu}B_\nu\right\};\\
&\overset{(0)}{\eta}_{\varphi}(\mathbf{k},\ \omega)=-\xi(\mathbf{k},\ \omega)\left\{\frac{\partial\langle\hat b(0)\rangle}{\partial Y_\nu}\left(\frac{\partial Y_\nu}{\partial\zeta_\beta}\right)_{\mathrm{p}}p^\beta+\right.\\
&\left.+\frac{\partial\langle\hat b(0)\rangle}{\partial p_\nu}\delta_{\nu0}\frac{p^\alpha}{Y_0}\left(\frac{\partial Y_\alpha}{\partial\zeta_\beta}\right)_{\mathrm{p}}p^\beta\right\},
\end{aligned}\right\}\quad(197)$$

where

$$\left.\begin{aligned}
&A_\mu=Y_0a^\lambda\left(\frac{\partial Y_\lambda}{\partial\zeta^\mu}\right)_{\mathrm{p}},\quad B_\mu=-k_\mu+\delta_{\mu0}a^\lambda\left(\frac{\partial Y_\lambda}{\partial\zeta^\alpha}\right)_{\mathrm{p}}p^\alpha,\\
&\qquad(\alpha=0,\ 1,\ 2,\ 3,\ 4),\\
&A^\mu_\nu=(kY)\,g^\mu_\nu+Y_0\left(\frac{\partial Y_\lambda}{\partial\zeta^\nu}\right)_{\mathrm{p}}(p^\mu a^\lambda-D^{\mu\lambda}),\\
&\qquad(\nu,\ \mu,\ \lambda=0,\ 1,\ 2,\ 3),\\
&B^\mu_\nu=-p^\mu k_\nu+\delta_{\nu0}\left(\frac{\partial Y_\lambda}{\partial\zeta^\alpha}\right)_{\mathrm{p}}p^\alpha(p^\mu a^\lambda-D^{\mu\lambda}).
\end{aligned}\right\}\quad(198)$$

In deriving (197), we have used the fact that in accordance with (132) and (134)

$$k_\lambda\frac{\partial t^{\mu\lambda}}{\partial\zeta_\beta}\bigg|_{\mathrm{p}}=\frac{\partial Y_\lambda}{\partial\zeta_\beta}\bigg|_{\mathrm{p}}(p^\mu a^\lambda-D^{\mu\lambda})+k_\lambda\frac{\partial t^{\mu\lambda}}{\partial p_0}\frac{\partial p_0}{\partial\zeta_\beta}\bigg|_{\mathrm{p}},$$

$$k_\lambda\frac{\partial j^\lambda}{\partial\zeta_\beta}\bigg|_{\mathrm{p}}=a^\lambda\frac{\partial Y_\lambda}{\partial\zeta_\beta}\bigg|_{\mathrm{p}}+k_\lambda\frac{\partial j^\lambda}{\partial p_0}\frac{\partial p_0}{\partial\zeta_\beta}\bigg|_{\mathrm{p}}.$$

We shall describe the scheme for finding the low-frequency limiting behaviors of the Green's functions $G^+_{ab}(\mathbf{k},\omega)$. In accordance with (105), the Fourier transform of $a_\xi(\mathbf{x},t)$ is related to the Green's function by

$$a_\xi(\mathbf{k},\ \omega)=\xi(\mathbf{k},\ \omega)\,G^+_{ab}(\mathbf{k},\ \omega).\qquad(199)$$

On the other hand, $a_\xi(\mathbf{x},t)$ in the region of large t ($t\gg\tau_r$) can in accordance with (104) and (150) be represented in the form

$$a_\xi(\mathbf{x},\ t)=\mathrm{Sp}\,\sigma'(\zeta'(t),\ \varphi'(t))\,\hat a(\mathbf{x})+\mathrm{Sp}\,\rho(\xi(t))\,\hat a(\mathbf{x}),\qquad(200)$$

where the operator $\sigma'(\xi'(t),\varphi'(t))$ is determined by (118),

$$\sigma'(\zeta'(t),\ \varphi'(t))=\int d^3x\,\{\hat\sigma_\alpha(\mathbf{x})\,\zeta'_\alpha(\mathbf{x},\ t)+\hat\sigma_\varphi(\mathbf{x})\,\varphi'(\mathbf{x},\ t)\}.$$

Since $\mathrm{Sp}\,\hat\sigma_\alpha(\mathbf{x}')\hat a(\mathbf{x})$, $\mathrm{Sp}\,\hat\sigma_\varphi(\mathbf{x}')\hat a(\mathbf{x})$ depend on the difference $\mathbf{x}-\mathbf{x}'$, the Fourier transform $a_\xi(\mathbf{k},\omega)$ of $a_\xi(\mathbf{x},t)$ contains terms that derive from the first term in (200) proportional to $\delta\zeta_\alpha(\mathbf{k},\omega)$ and $\delta\varphi(\mathbf{k},\omega)$. From the equations of superfluid hydrodynamics with sources $\eta(\mathbf{k},\omega)$ [which are proportional to $\xi(\mathbf{k},\omega)$] it follows that $\delta\zeta_\alpha(\mathbf{k},\omega)$ and $\delta\varphi(\mathbf{k},\omega)$ are singular in the region of small $\mathbf{k}$ and ω. In accordance with the previous section 7, the Fourier transform of the second term in Eq. (200) is regular in the region of small $\mathbf{k}$ and ω. Bearing in mind what we have just said, we can represent the Fourier transform of $a_\xi(\mathbf{x},t)$ in the region of small $\mathbf{k}$ and ω in accordance with (200) in the leading approximation in the form

$$a_\xi(\mathbf{k},\ \omega)=\frac{\partial\langle\hat a\rangle}{\partial\zeta_\alpha}\,\delta\zeta_\alpha(\mathbf{k},\ \omega)+\frac{\partial\langle\hat a\rangle}{\partial p_l}\,\mathrm{i}k_l\delta\varphi(\mathbf{k},\ \omega)$$
$$+\frac{\partial\langle\hat a\rangle}{\partial\varphi}\,\delta\varphi(\mathbf{k},\ \omega)\qquad(201)$$

or

$$a_\xi(\mathbf{k},\ \omega)=\frac{\partial\langle\hat a\rangle}{\partial Y_\mu}\,\delta Y_\mu(\mathbf{k},\ \omega)+\frac{\partial\langle\hat a\rangle}{\partial p_\mu}\,\delta p_\mu(\mathbf{k},\ \omega)$$
$$+\frac{\partial\langle\hat a\rangle}{\partial\varphi}\,\delta\varphi(\mathbf{k},\ \omega).\qquad(201')$$

Proceeding from the equations of linearized hydrodynamics in the presence of sources, we shall find $\delta Y_\mu(\mathbf{k},\omega)$, $\delta p_\mu(\mathbf{k},\omega)$, $\delta\varphi(\mathbf{k},\omega)$ [they will be proportional to $\xi(\mathbf{k},\omega)$] and, comparing (199) with (201'), we shall find the Fourier transforms of the Green's functions $G^+_{ab}(\mathbf{k},\omega)$ in the region of small $\mathbf{k}$ and ω.

We consider first the simpler case when $[\hat N,\hat b]\neq0$. It follows from Eqs. (189)–(192) that in the leading approximation

$$\left.\begin{aligned}
&\delta\varphi(\mathbf{k},\ \omega)=-\frac{1}{\Delta}\,\xi(\mathbf{k},\ \omega)\,\frac{\partial\langle\hat b\rangle}{\partial\varphi},\\
&\delta Y_\nu(\mathbf{k},\ \omega)=\frac{\mathrm{i}}{\Delta}\,\xi(\mathbf{k},\ \omega)\,(kY)\,D^{-1}_{\nu\mu}a^\mu\,\frac{\partial\langle\hat b\rangle}{\partial\varphi}.
\end{aligned}\right\}\quad(202)$$

Substitution these expressions in (201') and comparing the expression obtained for $a_\xi(\mathbf{k},\omega)$ with (199), we find

$$G^+_{ab}(\mathbf{k},\ \omega)=\frac{1}{\Delta}\,\frac{\partial\langle\hat b\rangle}{\partial\varphi}\left\{-\frac{\partial\langle\hat a\rangle}{\partial\varphi}-\frac{\partial\langle\hat a\rangle}{\partial p_\mu}\,\mathrm{i}k_\mu\right.$$
$$\left.+\frac{\partial\langle\hat a\rangle}{\partial Y_\mu}\,\mathrm{i}\,(kY)\,D^{-1}_{\mu\nu}a^\nu\right\}.\qquad(203)$$

We now consider the case when $[\hat N,\hat b]=0$. As can be seen from (201'), to determine $a_\xi(\mathbf{k},\omega)$ it is first necessary to find $\delta Y_\mu(\mathbf{k},\omega)$ and $\delta p_\mu(\mathbf{k},\omega)$, $\delta\varphi(\mathbf{k},\omega)$. From (190) and (197), taking into account (198), we obtain

$$\delta\varphi(\mathbf{k},\ \omega)=\frac{\mathrm{i}}{\Delta}\,\xi(\mathbf{k},\ \omega)\left\{\frac{\partial\langle\hat b\rangle}{\partial p_\nu}\,k_\nu-\frac{\partial\langle\hat b\rangle}{\partial Y_\nu}\,(kY)\,a^\lambda D^{-1}_{\lambda\nu}\right\},\qquad(204)$$

whence in accordance with (186), (189), and (197) we have

$$\left.\begin{aligned}
&\delta p_\mu(\mathbf{k},\ \omega)=\xi(\mathbf{k},\ \omega)\left\{\frac{\partial\langle\hat b\rangle}{\partial Y_\nu}\left[\frac{k_\mu}{\Delta}(kY)\,a^\lambda D^{-1}_{\lambda\nu}\right.\right.\\
&\left.\qquad+\delta_{\mu0}\,\frac{\partial Y_\nu}{\partial\zeta^\beta}\,p^\beta\right]\\
&\left.+\frac{\partial\langle\hat b\rangle}{\partial p_\nu}\left[-\frac{k_\nu k_\mu}{\Delta}+\delta_{\mu0}\delta_{\nu0}\,\frac{p^\alpha p^\beta}{Y_0}\,\frac{\partial Y_\alpha}{\partial\zeta^\beta}\right]\right\},\\
&\delta Y_\mu(\mathbf{k},\ \omega)=\xi(\mathbf{k},\ \omega)\left\{\frac{\partial\langle\hat b\rangle}{\partial Y_\nu}\left[-(kY)\,D^{-1}_{\nu\mu}\right.\right.\\
&\left.+Y_0\,\frac{\partial Y_\mu}{\partial\zeta^\nu}-\frac{(kY)^2}{\Delta}\,a^\lambda D^{-1}_{\lambda\mu}a^{\lambda'}D^{-1}_{\lambda'\nu}\right]+\\
&\left.+\frac{\partial\langle\hat b\rangle}{\partial p_\nu}\left[\delta_{\nu0}\,\frac{\partial Y_\mu}{\partial\zeta^\alpha}\,p^\alpha+\frac{(kY)}{\Delta}\,k_\nu a^\lambda D^{-1}_{\lambda\mu}\right]\right\}.
\end{aligned}\right\}\quad(205)$$

Substituting the resulting expressions (204) and (205) in (201') and taking into account (199), we find the limiting behavior of $G^+_{ab}(\mathbf{k},\omega)$ in the case $[\hat N,\hat b]=0$. However, we write down directly the low-frequency behavior of the Green's function for arbitrary quasilocal operators $\hat a$ and $\hat b$, i.e., we also take into account Eq. (203):

$$G^+_{ab}(\mathbf{k},\ \omega)=\frac{1}{\Delta}\Big\{\Big(-\frac{\partial\langle\hat{a}\rangle}{\partial\varphi}-\frac{\partial\langle\hat{a}\rangle}{\partial p_\mu}\,ik_\mu+i\,\frac{\partial\langle\hat{a}\rangle}{\partial Y_\mu}(kY)'a^\lambda D^{-1}_{\lambda\mu}\Big)\Big(\frac{\partial\langle\hat{b}\rangle}{\partial\varphi}-\frac{\partial\langle\hat{b}\rangle}{\partial p_\nu}\,ik_\nu+i\,\frac{\partial\langle\hat{b}\rangle}{\partial Y_\nu}(kY)\,a^{\lambda'}D^{-1}_{\lambda'\nu}\Big)-\frac{\partial\langle\hat{a}\rangle}{\partial Y_\mu}\,\frac{\partial\langle\hat{b}\rangle}{\partial Y_\nu}\,\Delta\,(kY)\,D^{-1}_{\mu\nu}\Big\}. \quad (206)$$

We have here ignored the contribution $G'_{ab}(\mathbf{k},\omega)$ of the nonpole term in $G'_{ab}(\mathbf{k},\omega)$, which has the form

$$G'_{ab}(\mathbf{k},\ \omega)=\frac{\partial\langle\hat{a}\rangle}{\partial p_\mu}\,\frac{\partial\langle\hat{b}\rangle}{\partial Y_\nu}\,\delta_{\mu0}\,\frac{\partial Y_\nu}{\partial\zeta^\beta}\,p^\beta+\frac{\partial\langle\hat{a}\rangle}{\partial Y_\mu}\,\frac{\partial\langle\hat{b}\rangle}{\partial p_\nu}\,\delta_{\nu0}\,\frac{\partial Y_\mu}{\partial\zeta^\beta}\,p^\beta+\frac{\partial\langle\hat{a}\rangle}{\partial Y_\mu}\,\frac{\partial\langle\hat{b}\rangle}{\partial Y_\nu}\,Y_0\,\frac{\partial Y_\mu}{\partial\zeta^\nu}+\frac{\partial\langle\hat{a}\rangle}{\partial p_\mu}\,\frac{\partial\langle\hat{b}\rangle}{\partial p_\nu}\,\delta_{\mu0}\delta_{\nu0}\,\frac{p^\alpha p^\beta}{Y_0}\,\frac{\partial Y_\alpha}{\partial\zeta^\beta},$$

since we have already ignored terms of this type by ignoring in $a_\xi(\mathbf{k},\omega)$ terms of the type $\mathrm{Sp}\,\rho(\xi)\hat{a}$. We recall that

$$\Delta=b-(kY)\,a^\lambda D^{-1}_{\lambda\mu}a^\mu$$

determines the poles of the Green's functions (the mode branches) of the superfluid. We emphasize that there are no poles associated with $\det D=0$. This is due to the fact that $1/\Delta$ near the singularity $\det D=0$ behaves as $\det D$, $1/\Delta\sim\det D$. Therefore, the structure of the limiting behavior of the Green's function (206) does not have poles associated with the singularities of the matrix D. One can show that the singularities associated with the vanishing of $\det D$ cancel.

If the potential ω corresponds to a relativistically invariant system, then p_ν and Y_ν are 4-vectors, and $D_{\mu\nu}$ is a 4-tensor. In this case, formula (206) gives a relativistically covariant representation of the low-frequency limiting behavior of the Green's function [see (107)].

In some cases, it is convenient to represent the low-frequency behavior of the Green's functions by means of a bilinear combination of the derivatives $\partial\langle\hat{a}\rangle/\partial\zeta_\alpha$, $\partial\langle\hat{a}\rangle/\partial p_l$, $\partial\langle\hat{a}\rangle/\partial\varphi$ and the derivatives $\partial\langle\hat{b}\rangle/\partial\zeta_\alpha$, $\partial\langle\hat{b}\rangle/\partial p_i$, $\partial\langle\hat{b}\rangle/\partial\varphi$. Such a representation is convenient for nonrelativistic and generalized superfluid systems.

Since

$$\frac{\partial\langle\hat{a}\rangle}{\partial p_\mu}\Big|_{Y_\nu}=\frac{\partial\langle\hat{a}\rangle}{\partial\zeta_\alpha}\Big(\frac{\partial\zeta_\alpha}{\partial p_\mu}\Big)_{Y_\nu}+\frac{\partial\langle\hat{a}\rangle}{\partial p_i}\,\delta_{\mu i},$$

$$\frac{\partial\langle\hat{a}\rangle}{\partial Y_\mu}\Big|_{p_\nu}=\frac{\partial\langle\hat{a}\rangle}{\partial\zeta_\alpha}\Big(\frac{\partial\zeta_\alpha}{\partial Y_\mu}\Big)_{p_\nu},$$

proceeding from Eq. (206), we represent the pole part of the low-frequency limiting behavior of $G^+_{ab}(\mathbf{k},\omega)$ in the form[17]

$$\begin{aligned}G^+_{ab}(\mathbf{k},\ \omega)&=\frac{\partial\langle\hat{a}\rangle}{\partial\zeta_\alpha}\,\frac{\partial\langle\hat{b}\rangle}{\partial\zeta_\beta}\,G^+_{\zeta_\alpha\zeta_\beta}(\mathbf{k},\ \omega)\\&-\frac{i}{\eta}\,\frac{\partial\langle\hat{a}\rangle}{\partial\zeta_\alpha}\Big(\frac{\partial\langle\hat{b}\rangle}{\partial\varphi}-ik_l\,\frac{\partial\langle\hat{b}\rangle}{\partial p_l}\Big)G^+_{\zeta_\alpha\psi}(\mathbf{k},\ \omega)\\&-\frac{i}{\eta}\,\frac{\partial\langle\hat{b}\rangle}{\partial\zeta_\beta}\Big(\frac{\partial\langle\hat{a}\rangle}{\partial\varphi}+ik_l\,\frac{\partial\langle\hat{a}\rangle}{\partial p_l}\Big)G^+_{\psi\zeta_\beta}(\mathbf{k},\ \omega)\\&-\frac{1}{\eta^2}\Big(\frac{\partial\langle\hat{a}\rangle}{\partial\varphi}+ik_l\,\frac{\partial\langle\hat{a}\rangle}{\partial p_l}\Big)\Big(\frac{\partial\langle\hat{b}\rangle}{\partial\varphi}-ik_i\,\frac{\partial\langle\hat{b}\rangle}{\partial p_i}\Big)G^+_{\psi\psi}(\mathbf{k},\ \omega).\end{aligned} \quad (207)$$

Here, the Green's functions $G^+_{\zeta_\alpha\zeta_\beta}(\mathbf{k},\omega)$, $G^1_{\zeta_\alpha\psi}(\mathbf{k},\omega)$, $G^+_{\psi\zeta_\alpha}(\mathbf{k},\omega)$, $G^+_{\psi\psi}(\mathbf{k},\omega)$ (we shall call them the basis functions) are determined by

$$\left.\begin{aligned}&G^+_{\zeta_\alpha\zeta_\beta}(\mathbf{k},\ \omega)=-\frac{Z_\alpha Z_\beta}{\Delta}-(kY)\,\frac{\partial\zeta_\alpha}{\partial Y_\mu}\,D^{-1}_{\mu\nu}\,\frac{\partial\zeta_\beta}{\partial Y_\nu}\,;\\&G^+_{\zeta_\alpha\psi}(\mathbf{k},\ \omega)=\frac{\eta}{\Delta}\,Z_\alpha,\quad G^+_{\psi\zeta_\alpha}(\mathbf{k},\ \omega)=-\frac{\eta}{\Delta}\,Z_\alpha;\\&G^+_{\psi\psi}(\mathbf{k},\ \omega)=\frac{\eta^2}{\Delta},\quad \eta=|\mathrm{Sp}\,w\psi|,\end{aligned}\right\} \quad (208)$$

where

$$Z_\alpha=\frac{\partial\zeta_\alpha}{\partial p_\lambda}\,k_\lambda-\frac{\partial\zeta_\alpha}{\partial Y_\lambda}\,(kY)\,a^\mu D^{-1}_{\mu\lambda}.$$

We now investigate in more detail the structure of Δ as a function of the 4-vector k_ν. We consider the case when k_ν satisfies the conditions

$$k_\nu Y^\nu=k_\nu p^\nu=0$$

(in particular, if $\mathbf{Y}=\mathbf{p}=0$, then $\omega=0$). From the definitions of the matrix $B_{\nu\mu}$ and A_ν (138) when $k_\nu Y^\nu=k_\nu p^\nu=0$ it follows that $A^\nu B^{-1}_{\nu\mu}k^\mu\equiv(AB^{-1}k)=0$. It can be seen from this that the expression (140) for D has the form

$$D=-(kB^{-1}k)\ (AB^{-1}A). \quad (209)$$

Further, in accordance with (139) we have in the considered situation

$$(kY)\,D^{-1}_{\nu\mu}=B^{-1}_{\nu\mu}-\frac{(B^{-1}A)_\nu\,(B^{-1}A)_\mu}{(AB^{-1}A)}-\frac{(B^{-1}k)_\nu\,(B^{-1}k)_\mu}{(kB^{-1}k)}. \quad (210)$$

It follows from (134), which determines a^μ, that $a^\mu\sim k^\mu$. Therefore

$$(kY)\,a^\nu D^{-1}_{\nu\mu}a^\mu=0,\quad (kY)=(kp)=0.$$

Thus, $\Delta(k)$ [see (136)] has the form

$$\Delta(k)=k^2e,\quad e=2\,\frac{\partial\omega'}{\partial p^2}. \quad (211)$$

Therefore, for the limiting behavior of $G^+_{\psi\psi}(k)$ we obtain [see (208)]

$$G^+_{\psi\psi}(k)=\frac{\eta^2}{k^2e}. \quad (212)$$

If $\mathbf{Y}=\mathbf{p}=0$ and $\omega=0$, then this expression for the case of a Galileo-invariant superfluid system is, when (47) is taken into account, identical to Bogolyubov's result of Ref. 18.

Now suppose

$$(kY)=0,\ (kp)\neq0. \quad (213)$$

In this case, if $Y = 0$, then $\omega = 0$ and $kp = \mathbf{k}\cdot\mathbf{p} \equiv |\mathbf{k}||\mathbf{p}|\cos\theta$, where θ is the angle betwen the vectors $\mathbf{k}$ and $\mathbf{p}$. It is readily seen from (139) that by virtue of the condition (213)

$$(kY)\, D^{-1}_{\nu\mu} A^{\mu} = (kY)\, D^{-1}_{\nu\mu} k^{\mu} = 0.$$

Therefore, the expression $(kY) a^{\nu} D^{-1}_{\nu\mu} a^{\mu}$ has the structure

$$(kY)\, a^{\nu} D^{-1}_{\nu\mu} a^{\mu} = (kp)^2 \frac{ak^2 + b\,(kp)^2}{k^2 + d\,(kp)^2},$$

where a, b, d can in accordance with (139) be expressed in terms of the thermodynamic potential ω' (the pressure). Using (208) and (136), we see that the general structure of the limiting behavior of $G^{+}_{\psi\psi}(k)$ has in the considered case the form

$$G^{+}_{\psi\psi}(k) = \frac{\eta^2\,[k^2 + (kp)^2 d]}{(kp)^4 (b - fd) + k^2 (kp)^2 (a - f - ed) - ek^4}, \qquad (214)$$

where in accordance with (211) the quantity f is determined from the equation

$$k_{\mu} k_{\nu} \frac{\partial^2 \omega'}{\partial p_{\mu}\, \partial p_{\nu}} = ek^2 + f\,(kp)^2.$$

[Equation (214) determines the explicit dependence of the Green's function $G^{+}_{\psi\psi}(k)$ on the angle θ.]

To conclude this section, we write down the limiting behaviors of the "basis" Green's functions $G^{\pm}_{\psi\psi}(\mathbf{k}, G^{\pm}_{\zeta_\alpha\psi}\omega)$, $(\mathbf{k},\omega)$, $G^{\pm}_{\zeta_\alpha\zeta_\beta}(\mathbf{k},\omega)$ for $\omega\tau_r \ll 1$, $kl \ll 1$ in the case when $\mathbf{Y} = 0$, $\mathbf{p} = 0$ (we do not assume that the superfluid system is either Galileo or relativistically invariant). Taking into account (208), we obtain

$$G^{\pm}_{\psi\psi}(\mathbf{k}, \omega) = -\frac{\eta^2}{\Delta(\mathbf{k}, \omega)} \Big\{ \omega^2 \Big[\frac{Y_4}{Y_0} \frac{\partial}{\partial \zeta_0} \Big(\frac{Y_4}{Y_0} \Big) - \frac{\partial}{\partial \zeta_4} \Big(\frac{Y_4}{Y_0} \Big) \Big] + k^2 \frac{s}{Y_0 \rho_n} A \Big\};$$

$$G^{\pm}_{\zeta_\alpha\psi}(\mathbf{k}, \omega) = \delta_{\alpha j} \frac{k_j \eta}{\Delta(\mathbf{k}, \omega)} \Big[\omega^2 \Big(\frac{Y_4}{Y_0} \frac{\partial P}{\partial \zeta_0} - \frac{\partial P}{\partial \zeta_4} \Big) - k^2 \frac{s}{Y_0 \rho_n} \frac{\rho_s + \rho_c}{\overset{*}{m}} A \Big] - \frac{Y_0 \eta \omega}{\Delta(\mathbf{k}, \omega)} \Big\{ \omega^2 \frac{\partial}{\partial Y_\alpha} \Big(\frac{Y_4}{Y_0} \Big) + k^2 \Big[\frac{1}{\overset{*}{m}} \Big(\frac{\partial \zeta_\alpha}{\partial Y_0} + \frac{Y_4}{Y_0} \frac{\partial \zeta_\alpha}{\partial Y_4} \Big) - \frac{s}{Y_0 \rho_n} \frac{\partial \zeta_\alpha}{\partial Y_4} \Big] A \Big\};$$

$$G^{+}_{\zeta_0\zeta_0}(\mathbf{k}, \omega) = \frac{\omega^2 k^2}{\Delta(\mathbf{k}, \omega)} \Big\{ \frac{\rho_c}{\overset{*}{m}{}^2} \Big[\Big(\frac{s}{Y_0} \Big)^2 \frac{\overset{*}{m}}{\zeta_4 \rho_n} - \Big(\frac{Y_4}{Y_0} \Big)^2 \Big] + \frac{1}{\overset{*}{m} \zeta_4} \Big[\frac{\rho_s}{\rho_n} \Big(\frac{s}{Y_0} \Big)^2 + (P + \zeta_0)^2 \Big] \Big\} - \frac{k^4}{\Delta(\mathbf{k}, \omega)} \frac{\rho_s s}{\rho_n \overset{*}{m}{}^2} \frac{\partial \zeta_0}{\partial Y_0} A;$$

$$G^{\pm}_{\zeta_0\zeta_j}(\mathbf{k}, \omega) = \frac{\omega k_j}{\Delta(\mathbf{k}, \omega)} \Big\{ \omega^2 (P + \zeta_0) + k^2 \Big[\frac{Y_0 \rho_c}{\overset{*}{m}{}^2} \Big(\frac{\partial \zeta_0}{\partial Y_0} + \frac{Y_4}{Y_0} \frac{\partial \zeta_0}{\partial Y_4} - \frac{s \overset{*}{m}}{Y_0 \rho_n} \frac{\partial \zeta_0}{\partial Y_4} \Big) - \frac{\rho_s s}{\rho_n \overset{*}{m}} \frac{\partial \zeta_0}{\partial Y_4} \Big] A \Big\};$$

$$G^{\pm}_{\zeta_0\zeta_4}(\mathbf{k}, \omega) = \frac{k^2}{\overset{*}{m} \Delta(\mathbf{k}, \omega)} \Big[\omega^2 \Big(P + \zeta_0 + \frac{\rho_c}{\overset{*}{m}} \frac{Y_4}{Y_0} \Big) - k^2 \frac{\rho_s s}{\rho_n \overset{*}{m}} \frac{\partial \zeta_0}{\partial Y_4} A \Big];$$

$$G^{\pm}_{\zeta_i\zeta_j}(\mathbf{k}, \omega) = \Big(\delta_{ij} - \frac{k_i k_j}{k^2} \Big) G^{t}(\mathbf{k}, \omega) + \frac{k_i k_j}{k^2} G^{l}(\mathbf{k}, \omega);$$

$$G^{l}(\mathbf{k}, \omega) = \frac{k^2}{\Delta(\mathbf{k}, \omega)} \Big\{ \omega^2 \Big[(P + \zeta_0) \frac{\partial P}{\partial \zeta_0} + \zeta_4 \frac{\partial P}{\partial \zeta_4} \Big] + k^2 \frac{s}{Y_0} \Big[\frac{(\rho_s + \rho_c)^2}{\rho_n} + \rho_s \Big] \frac{A}{\overset{*}{m}{}^2} \Big\};$$

$$G^{t}(\mathbf{k}, \omega) = \frac{k^2}{\Delta(\mathbf{k}, \omega)} \Big\{ \omega^2 \rho_n \Big[\rho_c C + B + \frac{\rho_s}{\overset{*}{m}{}^2} \Big(\frac{\partial}{\partial \zeta_4} \frac{Y_4}{Y_0} - \frac{Y_4}{Y_0} \frac{\partial}{\partial \zeta_0} \frac{Y_4}{Y_0} \Big) \Big] + k^2 \frac{\rho_s}{\overset{*}{m}{}^2} \frac{s}{Y_0} A \Big\};$$

$$G^{\pm}_{\zeta_i\zeta_4}(\mathbf{k}, \omega) = \frac{k_i \omega}{\overset{*}{m} \Delta(\mathbf{k}, \omega)} \Big\{ \omega^2 \overset{*}{m} \zeta_4 + k^2 \Big[\frac{Y_0 \rho_c}{\overset{*}{m}{}^2} \Big(\frac{\partial \zeta_4}{\partial Y_0} + \frac{Y_4}{Y_0} \frac{\partial \zeta_4}{\partial Y_4} \Big) - \frac{(\rho_s + \rho_c)}{\rho_n} s \frac{\partial \zeta_4}{\partial Y_4} \Big] A \Big\};$$

$$G^{\pm}_{\zeta_4\zeta_4}(\mathbf{k}, \omega) = \frac{k^2}{\overset{*}{m}{}^2 \Delta(\mathbf{k}, \omega)} \Big[\omega^2 (\overset{*}{m} \zeta_4 - \rho_c) - k^2 \frac{\rho_s}{\rho_n} s \frac{\partial \zeta_4}{\partial Y_4} A \Big]$$

[the quantities A, B, C, $\Delta(\mathbf{k},\omega)$ are determined by Eqs. (141′) and (141)].

If the superfluid system is Galileo-invariant, then with allowance for (142) the limiting behaviors which we have found for the Green's functions $G^{\pm}_{\psi\psi}(\mathbf{k},\omega)$, $G^{\pm}_{\zeta_\alpha\psi}(\mathbf{k},\omega)$ go over into the results of Bogolyubov[18] and Galasiewicz,[19] respectively,and those for $G^{\pm}_{\zeta_\alpha\zeta_\beta}(\mathbf{k},\omega)$ go over into the results of Hohenberg and Martin.[20]

[1]N. N. Bogolyubov, "Quasiaverages in problems of statistical mechanics," Preprint *D*-781 [in Russian], JINR, Dubna (1961).

[2]N. N. Bogolyubov, *Problemy dinamicheskoĭ teorii v statisticheskoĭ fizike*, Gostekhizdat, Moscow (1946); English translation: "*Problems of a dynamical theory in statistical physics*," *in*: *Studies in Statistical Mechanics*, Vol. 1 (eds. J. De Boer and G. E. Uhlenbeck), North-Holland Amsterdam (1962).

[3]I. M. Khalatnikov, *Teoriya sverkhtekuchesti* (*The Theory of Superfluidity*), Nauka, Moscow (1971).

[4]B. I. Halperin and P. C. Hohenberg, Phys. Rev. **188**, 898 (1969).

[5]V. P. Mineev, "Superfluid ^{3}He: an introduction to the subject" [in Russian], Usp. Fiz. Nauk **139**, 303 (1983) [Sov. Phys. Usp. **26**, 160 (1983)].

[6]G. E. Volovik, "Superfluid properties of the A phase of ^{3}He [in Russian],Usp. Fiz. Nauk **143**, 73 (1984) [Sov. Phys. Usp. **27**, 363 (1984)].

[7]Sverkhtekuchest' geliya-3 (Superfluidity of Helium-3; Collection of Russian translations), Mir, Moscow (1977).

[8]M. Yu. Kovalevskiĭ, S. V. Peletminskiĭ, and Yu. V. Slyusarenko, "Hydrodynamics of magnets with helical structure" [in Russian], in: Dokl. na III Mezhdunarodnom simpoziume po izbrannym problemam statisticheskoĭ mekhaniki (Proc. of the Third International Symposium on Selected Problems of Statistical Mechanics), Dubna (1984).

[9]N. M. Lavrinenko, S. V. Peletminskiĭ, and Yu. V. Slyusarenko, "Kinetics of systems with crystal structure" [in Russian], Teor. Mat. Fiz. **53**, 456 (1982); S. V. Peletminskiĭ and Yu. V. Slyusarenko, "Thermodynamics and kinetics of systems with crystal structure" [in Russian], Preprint 84-9R, Institute of Theoretical Physics, Kiev (1984).

[10]A. F. Andreev and I. M. Liftshits, "Quantum theory of defects in crystals" [in Russian], Zh. Eksp. Teor. Fiz. **56**, 2057 (1969) [Sov. Phys. JETP **29**, 1107 (1969)].

[11]W. M. Slaslow, Phys. Rev. B **15**, 173 (1977).

[12]N. N. Bogolyubov, Jr. and B. I. Sadovnikov, *Nekotorye voprosy statisticheskoĭ mekhaniki* (*Some Problems of Statistical Mechanics*), Vysshaya Shkola, Moscow (1975).

[13]A. I. Akhiezer and S. V. Peletminskiĭ, *Metody statisticheskoĭ fiziki*, Nauka, Moscow (1977), English translation: *Methods of Statistical Physics*, Pergamon Press, Oxford (1981).

[14]V. V. Lebedev and I. M. Khalatnikov, "Relativistic hydrodynamics of superfluids" [in Russian], Zh. Eksp. Teor. Fiz. **83**, 1601 (1982) [Sov. Phys. JETP **56**, 923 (1982)].

[15]M. Yu. Kovalevskiĭ, S. V. Peletminskiĭ, A. N. Tarasov, and P. I. Fomin, "The method of quasiaverages and relativistic hydrodynamics of superfluids" [in Russian], Preprint 83–151R, Institute of Theoretical Physics, Kiev (1983).

[16]I. M. Khalatnikov, "Propagation of sound in moving helium II and the effect of a heat flux on the propagation of second sound" [in Russian], Zh. Eksp. Teor. Fiz. **30**, 617 (1956) [Sov. Phys. JETP **3**, 649 (1956)].

[17]M. Yu. Kovalevskiĭ, S. V. Peletminskiĭ, and A. N. Tarasov, "Low-frequency limiting behavior of the Green's functions of superfluid Bose systems" [in Russian], Preprint 83-18, Khar'kov Physicotechnical Institute (1983).

[18]N. N. Bogolyubov, "On the hydrodynamics of superfluids" [in Russian], Preprint *R*-1395, JINR, Dubna (1963).

[19]Z. Galasiewicz, "Three-legs Green's functions for the superfluid Bose-systems," Bull. Acad. Pol. Sci. Cl. III, **15**, 191 (1967).

[20]P. C. Hohenberg and P. C. Martin, "Microscopic theory of superfluid helium," Ann. Phys. (N.Y.) **34**, 291 (1965).

Translated by Julian B. Barbour