Dynamical chaos of non-Abelian gauge fields

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The review studies a special class of Yang–Mills fields—spatially homogeneous fields (classical Yang–Mills mechanics), which have no analog in linear Abelian electrodynamics. Computer and analytic approaches show that such fields possess dynamical stochasticity, on the basis of which it may be asserted that the classical Yang–Mills equations without external sources constitute a nonintegrable system. The Higgs mechanism eliminates this stochasticity, and at a certain value of the vacuum expectation of the scalar field there is a phase transition of the disorder-order (confinement-deconfinement) type. The system with external sources apparently behaves similarly. The connection between this stochasticity and the mechanism of dimensional reduction in macroscopic systems and with the color-confinement phenomenon is considered. It is shown that the presence in the vacuum of random (Gaussian) currents leads to confinement of the fields generated by these currents. Attention is drawn to the possible manifestation of the stochasticity of the classical fields in multiparticle hadron-production processes. Such manifestation reflects universal stochastic features characteristic of systems of very different natures (statistics of the counting of thermoelectrons from random sources and photoelectrons from laser radiation that passes through a liquid in the critical state, developed turbulence in hydrodynamics, stellar systems, and KNO scaling in multiparticle production).

INTRODUCTION

Until comparatively recently, random (chaotic) behavior of dynamical systems was attributed to random initial conditions, or to random external disturbances (as, for example, in the case of Brownian motion), or, finally, to the excitation of a very large number of degrees of freedom. Of course, any one of these is sufficient for the occurrence of chaos in systems, but it was found that none of them plays the role of a necessary condition.[1–6]

It is now well established that numerous simple completely deterministic dynamical systems of classical mechanics possessing a small number ($n \geqslant 2$) of degrees of freedom are characterized by extremely irregular, exceptionally complicated, and effectively unpredictable motion—all this determined by the internal dynamics of the system.

This random (stochastic) behavior, which is in no way associated with the sufficient conditions for the occurrence of chaos listed above, can be naturally called dynamical stochasticity.

The mechanism of occurrence of such dynamical chaos consists in strong local instability of the motion.[3,4,7] The characteristic random properties of the system are manifested even in an individual trajectory of the system.

Dynamical chaos is characteristic of many nonlinear classical systems from different branches of physics and also other sciences (chemistry, hydrodynamics, biology, meteorology, ecology, etc.).

We shall see that it is also manifested in a specific manner in the classical theory of non-Abelian gauge fields.

The question of the complete integrability (or, rather, nonintegrability) of the classical Yang-Mills equations, which is directly related to the problem of their stochasticity, already has a history of its own and to a certain degree is related to the great popularity of classical solutions of, for example, instanton type,[8,9] on which great hopes were placed for the construction of the ground state of QCD.

However, all attempts to find additional integrals of the motion of the classical Yang-Mills equations proved unsuccessful. This gave rise to a program for seeking conservation laws expressed, not in terms of the potentials and fields, but in terms of contour variables.[10,11]

This circumstance was a stimulus to the investigation of the classical Yang-Mills equations, which are nonlinear by nature, to see if they contain a stochastic component. Besides its great independent interest, the investigation of non-Abelian gauge fields from the point of view of stochasticity is topical in view of the phenomenon (initially identified in solid-state physics) of the reduction in the dimension of quantum spin systems that interact with a random magnetic field.[12] In 1982, Olesen[13] conjectured that, by analogy with this phenomenon, random fields reduce the four-dimensional Yang-Mills theory to an effective two-dimensional theory possessing the confinement property. He showed that in the limit of an infinitely large number of colors ($N \to \infty$) a necessary and sufficient condition for confinement is the presence in the vacuum of random fields. The preprint of Ref. 14 showed for the example of the calculation of the Wilson expectation $W(C)$ in the limit $N \to \infty$ and under the restriction to planar contours that this reduction can be followed concretely. Calculations in SU(2) gauge theory on lattices[15] provide support for Olesen's hypothesis.

These considerations show that the confinement problem can be solved if it can be shown that random vacuum fields arise naturally in four-dimensional QCD and are an inseparable element of it.

In this review, we shall be concerned with studies that have discovered and proved stochasticity of free classical non-Abelian gauge fields.[16–18]

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As a result, it can be asserted that, in contrast to quantum electrodynamics, QCD is a theory that in the classical limit has clearly defined stochastic features.

The question of what happens to the stochasticity when we pass to the quantum systems is rather complicated and far from a definitive solution.

In general features,[19,20] it is to be expected that dynamical stochasticity in closed quantum systems with bounded phase space cannot occur, since the wave function (or density matrix) of such systems is always almost periodic, i.e., its frequency spectrum is discrete. In such systems, one can observe, at best, transient stochasticity. One can say that until, in semiclassical terms, the wave packet of such a system has spread, it will possess a classical and, therefore, stochastic trajectory, but then the strongest features of stochasticity must at least disappear.

However, in the quantum case for conservative systems one can hardly speak of trajectories even in the semiclassical limit. More appropriate here are the concepts of the spectrum and wave functions of the system. It is entirely justified and very important to consider what are the properties of the spectrum of a quantum system that exhibits stochastic motion in the classical limit.

It seems entirely likely[21] that in the semiclassical limit the quantum energy spectrum of a dynamical system consists of a regular and an irregular part. In the general case, the regular part of the spectrum (which varies weakly with the parameters of the Hamiltonian) corresponds for $\hbar = 0$ to regular classical trajectories wound around an invariant torus.

The irregular part of the spectrum (which depends strongly on the parameters of the Hamiltonian) corresponds in this limit to dynamical chaos.

Numerical simulation shows[22,23] that there is a correspondence between the fraction of the classical random motion and the fraction of the irregular part of the set of energy eigenvalues of the system. Such a part of the spectrum arises above the critical energy at which the classical regular motion begins to become random.

We note that the irregular part of the spectrum is, as a rule and when allowance is made for the symmetry properties of the Hamiltonian, associated with nonintersecting (repelling) levels, in accordance with the well-known theorems.[24,25]

We shall return (in Sec. 4) to the question of the nature of the spectrum of quantum systems that in the classical limit have a stochastic component.

Another criterion that distinguishes regular and irregular (i.e., in the classical limit corresponding to chaos) quantum states is associated with the behavior of the wave functions[26]: To the former there corresponds a regular interference pattern and large fluctuations in the intensity; to the latter, randomly distributed interference maxima and minima with suppressed intensity fluctuations.

The situation is different when a *quantum system* is open, i.e., is in a random external field. In this case, the arguments given above are in general invalid, and the question requires special investigation. The study of simple examples of such systems made in Ref. 27 shows that the quantum-mechanical properties of these systems, stochastic in the classical limit, do not impose strong restrictions on the manifestation of stochasticity. In other words, these quantum systems in a random field exhibit continuous spectral properties, also as the corresponding classical model.

These considerations suggest that the dynamical stochasticity found in free classical non-Abelian gauge fields leaves its traces in the real world of QCD, and there is hope that they are responsible for color confinement.

1. SPATIALLY HOMOGENEOUS YANG–MILLS FIELDS. EXACT SOLUTION. CLASSICAL YANG–MILLS MECHANICS

There are now numerous indications that the perturbation-theory vacuum of Yang–Mills theory is not the true vacuum. Arguments for such an assertion have both a classical[8,9] and a quantum[28,29] basis.

Qualitatively, gluon interaction ("pairing") gives rise to a gluon condensate, which is manifested in a nonvanishing vacuum expectation value of the square of the gluon field intensity and reduces the QCD ground-state energy calculated without allowance for this phenomenon.

From the classical point of view, it is important in the light of this fact to look for and analyze classical solutions of the Yang-Mills equations without external sources in Minkowski space; these may then serve as the basis for constructing and investigating the structure of the quantum QCD vacuum and the problem of asymptotic states.

We begin by considering free Yang-Mills fields corresponding to the group SU(2) in ordinary space–time.

The equations of motion are

$$\partial_\mu G^a_{\mu\nu} + g\varepsilon^{abc} A^b_\mu G^c_{\mu\nu} = 0, \tag{1}$$

where

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon^{abc} A^b_\mu A^c_\nu$$

(here and below, the Latin indices take the values 1, 2, 3, and the Greek indices take the values 0, 1, 2, 3).

We shall seek a class of solutions of the system (1) for which the Poynting vector vanishes in a certain system[30]:

$$T_{0j} = G^a_{0i} G^a_{ji} = 0 \tag{2}$$

($T_{\mu\nu} = -G^a_{\mu\lambda} G^a_{\nu\lambda} + \frac{1}{4} g_{\mu\nu} G^{a^2}_{\lambda\rho}$ is the energy-momentum tensor of the field.)

In the gauge $A^a_0 = 0$, Eq. (1) and the condition (2) can be rewritten in the form

$$\ddot{A}^a_i - G^a_{ji,\,j} + g\varepsilon^{abc} A^b_j G^c_{ji} = 0; \tag{1a}$$

$$N^a \equiv \varepsilon^{abc} A^b_i \dot{A}^c_i = 0; \tag{1b}$$

$$\dot{A}^a_i G^a_{ij} = 0 \tag{2a}$$

(the dot above A^a_i denotes differentiation with respect to the time; $G_{ij,k} \equiv \partial_k G_{ij}$), where Eq. (1b) plays the part of a constraint. The quantity N^a is zero for free solutions. When an external current is present, $-N^a$ is the density of the external color charge j_0^a.

The constraint equations (1b) and (2a) lead to the relation

$$\dot{A}_i^a (A_{j,\,i}^a - A_{i,\,j}^a) = 0. \qquad (2b)$$

A sufficient condition for this relation to hold is fulfillment of the relations

a) $A_{i,\,j}^a = 0$; b) $\dot{A}_i^a = 0$; c) $A_{j,\,i}^a - A_{i,\,j}^a = 0$.

As will be seen from what follows, we shall study the most interesting case (a) of spatially homogeneous Yang-Mills fields, the fields in the given coordinate system then depending only on the time:

$$A_i^a = A_i^a(t).$$

The region of applicability of the equations for the homogeneous fields is determined by the condition that the variations in time are dominant in the system. In other words, this corresponds to the long-wavelength region of the spectrum (or strong fields):

$$|A_{i,\,j}^a| \ll A_i^{a^2}; \quad \lambda\, |\dot{A}_i^a| \gg 1.$$

It is to be hoped that the study of such fields will be helpful for obtaining information about the infrared regime of QCD—its most obscure point.

The equations of motion for homogeneous fields take the form

$$\ddot{A}_i^a - g^2 A_j^a A_j^b A_i^b + g^2 A_i^a A_j^b A_j^b = 0 \qquad (3)$$

with the constraint (1b).

Thus, for spatially homogeneous fields the Yang-Mills field equations (1) reduce to a discrete nonlinear mechanical system with Hamiltonian

$$H_{\rm YM} = \sum_{i,\,a=1}^{3} \frac{1}{2} (\dot{A}_i^a)^2 + \frac{g^2}{4} [(A_i^a A_i^a)^2 - (A_i^a A_j^a)^2]. \qquad (4)$$

It is readily seen that this Hamiltonian is symmetric with respect to the operation of transposing the matrix A_i^a, i.e., with respect to the internal and "external" (three-dimensional) spaces, which are both isotropic [SU(2) ⊗ O(3) symmetry]; therefore, as is readily seen, two "angular momenta" are conserved: the ordinary three-dimensional angular momentum

$$M_i = \varepsilon_{ijk} A_j^a \dot{A}_k^a = \text{const}$$

and the internal three-dimensional quantity N^a (1b), which is nonzero only for fields with sources and is equal to $-j_0^a$.

What has been said is a justification for calling the system of homogeneous Yang-Mills fields classical Yang-Mills mechanics, which, as will be shown below, exhibits fully the phenomenon of dynamical stochasticity.

The system (3) has nine degrees of freedom ($i, a = 1, 2, 3$) and four obvious conserved integrals: $H_{\rm YM}$ and M_i.

Before we analyze the situation with number $n \geqslant 2$ of degrees of freedom, we consider a simple case.[30] We seek a solution of the system (3) in the form

$$A_i^a = \frac{O_i^a}{g} f^{(a)}(t) \qquad (5)$$

[no summation over a in (5)], where O_i^a is a constant orthogonal matrix,

$$O_i^a O_i^b = \delta^{ab}. \qquad (6)$$

For $f^{(a)}(t)$ we obtain the system

$$\ddot{f}^{(a)} + f^{(a)}(\mathrm{f}^2 - f^{(a)^2}) = 0, \qquad (7)$$

where

$$\mathrm{f}^2 = \sum_{a=1}^{3} f^{(a)^2}.$$

It is well known that all conservative systems with one degree of freedom are integrable, and a particular solution of the system (7) for $f^{(1)} = f^{(2)} = f^{(3)} = f(t)$ can be readily found by using the energy integral:

$$f(t) = \left(\frac{2g^2}{3}\right)^{1/4} \mu\, \mathrm{cn}\left[\left(\frac{8g^2}{3}\right)^{1/4} \mu t;\ 1/\sqrt{2}\right], \qquad (8)$$

where cn(x; k) is the Jacobi elliptic cosine of argument x and modulus k; μ^4 is the Hamiltonian density T_{00} in the considered coordinate system.

The solution given by Eqs. (5), (6), and (8) is time-periodic with period $T = (3/8g^2)^{1/4}(4/\mu)K(1/\sqrt{2})$, where $K(x)$ is the complete elliptic integral of the first kind.

We note some interesting features of the solution obtained, although they do not directly bear on the stochasticity of the Yang-Mills equations.

The field intensities corresponding to the solution,

$$E_i^a = \frac{O_i^a}{g}\dot{f}; \quad H_i^a = \varepsilon_{ijk}\varepsilon^{abc}\frac{O_j^b O_k^c}{2g} f^2$$

$$\left(H_i^a = g\left(\frac{f}{\dot{f}}\right)^2 \varepsilon_{ijk}\varepsilon^{abc} E_j^b E_k^c\right),$$

are such that $\mathbf{E}^a$ ($\mathbf{H}^a$) are mutually orthogonal in the "rest" frame, and $\mathbf{H}^a$ is parallel to $\mathbf{E}^a$ ($a = 1, 2, 3$).

Further, it is readily seen that the argument of the periodic solution (8) in an arbitrary coordinate system obtained from our system, which "moves" with the wave, by an appropriate Lorentz transformation becomes $\xi \equiv kx = k_\mu x_\mu$, where $k_0 = \mu\gamma$, $k_i = \mu\gamma v_i$ [$\gamma = (1 - v^2)^{-1/2}$], i.e., $k^2 = \mu^2$. In linear massless electrodynamics such a solution cannot exist, since it is impossible to choose a coordinate system in which the Poynting vector of a wave is zero but not the energy density. It is this that distinguishes the solution (8) from the corresponding solution of Coleman.[31] By virtue of this, μ formally plays the role of a mass in the nonlinear wave (8).

Of course, one could from the very start seek solutions of the system (3) in the form $A_i^a(x) = A_i^a(\xi)$ with $k^2 = \mu^2$, but in this case the analogy with the classical dynamical system to which we have reduced the gauge field would be less clear.

2. TWO DEGREES OF FREEDOM. QUALITATIVE ANALYSIS OF COLOR OSCILLATIONS[16]

A nonlinear system with $n = 2$ can already have all the characteristic features of dynamical stochasticity even in the conservative case.

For the corresponding Hamiltonian of the system (3), introducing the notation $A_1^1 = x(t)/g$ and $A_2^2 = y(t)/g$ and setting $A_2^a = A_1^2 = 0$, we arrive at a nonlinear mechanical system on a plane with Hamiltonian

$$H = \frac{x^2 + y^2}{2} + \frac{x^2 y^2}{2} \quad (9)$$

and corresponding highly symmetric and seemingly simple coupled equations of motion, which we shall investigate:

$$\ddot{x} + xy^2 = 0; \quad \ddot{y} + yx^2 = 0. \quad (10)$$

The analysis of these equations is undoubtedly much simpler than that of the more general system (3) and, *a fortiori*, (1). However, if we shall have proved stochasticity of the system (10), it will be hard to imagine that the stochastic component vanishes completely in the more complicated system with $n > 2$ and, *a fortiori*, in the general case of spatially inhomogeneous Yang-Mills fields. In the following sections, we consider homogeneous Yang-Mills fields with $n > 2$.

It follows from the form of H (9) that any conserved integral $F(x, y, \dot{x}, \dot{y})$ of the system (10) must satisfy the partial differential equation

$$\dot{x}\frac{\partial F}{\partial x} + \dot{y}\frac{\partial F}{\partial y} = xy\left[y\frac{\partial F}{\partial \dot{x}} + x\frac{\partial F}{\partial \dot{y}}\right],$$

from which it can be seen that F cannot depend only on two of the variables $x, y, \dot{x}, \dot{y}$ and also be a polynomial of finite degree in these variables.

It is obvious that the "material point" described by (10) cannot leave the region bounded by the exponential curves $xy = \pm\sqrt{2}\mu^2$, where μ^4 is the "total energy" of the point. It is obvious that if a "point" with total energy μ^4 described by (10) is at some instant of time on the exponential curve $xy = \pm\sqrt{2}\mu^2$, then it will leave it along the normal into the allowed region.

We consider whether the system (10) has periodic trajectories.

It follows from the symmetry of the problem that a trajectory of it will be periodic if at least one of the following events occurs twice: a) the trajectory passes through the origin; b) the trajectory is perpendicular to one of the symmetry axes of the problem; c) the trajectory arrives on the equipotential curve.

These (sufficient) conditions of periodicity are helpful for classifying and describing the trajectories (see below), but it could be that there are other weaker sufficient criteria for periodicity of the trajectories of the system (10).

It is obvious that along the symmetry axes $x = \pm y$ the system executes the periodic oscillations (8) [events (a) and (c)]. Along the axes $x = 0$ and $y = 0$, the point goes away, as in electrodynamics, to infinity ($\ddot{x} = 0$, $\dot{x} \neq 0$, $\ddot{y} = 0$, $\dot{y} \neq 0$). But if at a certain instant of time the velocity of the point is not directed along the x or y axis, then it will not travel to infinity, though it may sometimes travel away from the center at an arbitrarily large distance and return after a finite time to the region $x \sim y$, as is readily seen from the fact that $\ddot{x}/x$ and $\ddot{y}/y$ are negative.

One can say that the considered motion occupies an intermediate position, as it were, between the finite and infinite types of motion.

In polar coordinates ($x = \rho\cos\varphi$, $y = \rho\sin\varphi$) the system (10) can be written in the form

$$\ddot{\varphi} + \frac{2\dot{\rho}}{\rho}\dot{\varphi} + \frac{\rho^2}{4}\sin 4\varphi = 0; \quad (11a)$$

$$\ddot{\rho} - \rho\dot{\varphi}^2 + \frac{\rho^3}{2}\sin^2 2\varphi = 0. \quad (11b)$$

With increasing distance from the origin ($\rho \gg \mu$; note that in our problem x, y, ρ have the dimensions of mass and not length!), for example, along the channel $\varphi \ll \pi/4$, $\sin 4\varphi \approx 4\varphi$, $\dot{\rho} > 0$, it can be seen from (11) that the frequency of the oscillations with respect to the coordinate φ increases with increasing distance from the center, while the amplitude decreases until $\dot{\rho} = 0$ ("turning point"); this last condition occurs after a finite interval of time, since $\ddot{\rho} \simeq -a(t)\rho^3(t)$ ($a > 0$), after which the "damping" regime is replaced by an "excitation" regime. Figure 1 shows a characteristic example of such behavior, taken from a computer display.

The motion with respect to ρ, averaged over the rapid oscillations of φ, is a random walk with large amplitudes ($\ddot{\rho} + a\rho^3 \approx 0$) from channel to channel with complicated motion in the region $x \approx y$ [which can be followed by numerical integration of the system (10) on a computer]. In the language of the variation in time of the color amplitudes A_1^1 and A_2^2, this picture corresponds alternately to rapid oscillations and decay of one color amplitude and growth of the other.

It is obvious that the system with three degrees of freedom $A_1^1 = x/g$, $A_2^2 = y/g$, $A_3^3 = z/g$ qualitatively reproduces the basic features in the behavior of the system (10) with $n = 2$ (see below), as outlined above. There are here six channels along the coordinate axes, the motion in them being similar to that in the channels of the two-dimensional system; i.e., with increasing distance from the center, the frequency of the oscillations of the trajectories increases,

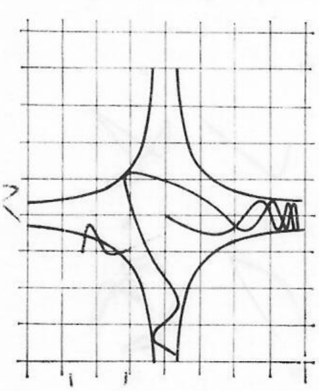

FIG. 1. Example of a random transition from channel to channel for the system (10), from a photograph of a computer display.

while the amplitude decreases until there is a stop, after which the regime of damping with respect to the spherical angle is replaced by the regime of excitation. The general picture of the variation in time of the color amplitudes in this three-dimensional case is characterized by alternating rapid oscillations and decrease of two color amplitudes and growth of the third. There are color "beats."

3. INSTABILITY OF PERIODIC TRAJECTORIES OF THE SYSTEM (10). STOCHASTICITY

Figure 2 gives examples of some periodic trajectories photographed from the display of a computer used to integrate the system (10).[16]

Figures 2a–2f show trajectories that pass through the origin and are perpendicular to either the equipotential line (Figs. 2a, 2b, 2d, and 2f) or the symmetry axis $y=0$ (Figs. 2c and 2e). The trajectories are arranged in order of decrease of their slope with respect to the x axis at the origin. The trajectory in Fig. 2a corresponds to oscillations in accordance with the elliptical cosine law (8).[30] A further decrease in the slope leads to an increase in the number of intersections with the x axis of the trajectories of the type in Figs. 2c, 2e and 2d, 2f.

We denote by α_n^0 these angles for trajectories of the type of Figs. 2c and 2e and by β_n^0 for trajectories of the type of Figs. 2d and 2f; n is the number of intersections of the trajectories with the x axis.

As $n\to\infty$, the angles α_n^0 and β_n^0 tend to zero.

From the figures shown, one can clearly see the tendency for the frequency to increase and the amplitude of the oscillations to decrease as the particle penetrates deeper into the channel along the x axis with decreasing angle to the x axis of the trajectories at the origin, in accordance with the qualitative analysis made in Sec. 2 for large ρ.

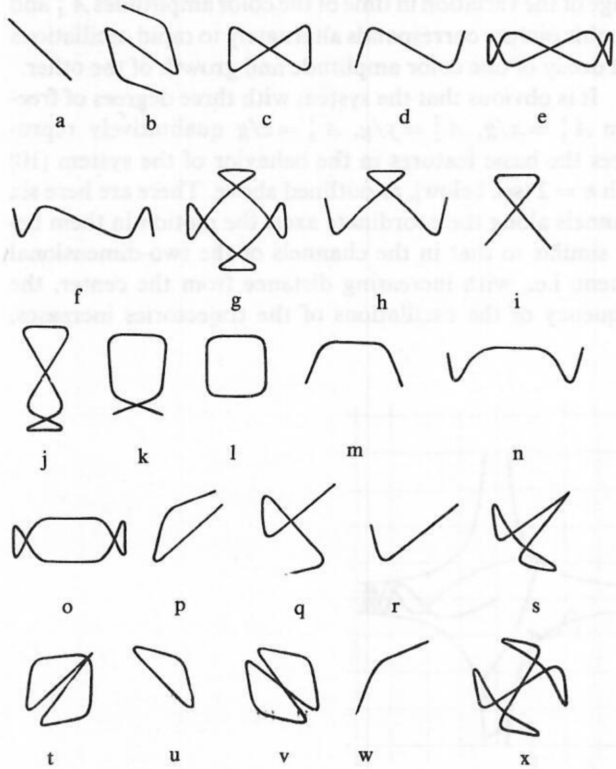

FIG. 2. Examples of periodic trajectories of the system (10) photographed from a computer display.

Figures 2g–2j are examples of trajectories that pass at right angles through the y axis at different distances from the center and at right angles either to the coordinate axes (Figs. 2g, 2j, 2l, 2m) or to the equipotential lines. With decreasing distance of these trajectories along the y axis from the center, they all penetrate further into the channel, and the picture considered in Sec. 2 ($\rho\gg\mu$) is again reproduced qualitatively. Figures 2p, 2q, and 2r show trajectories that are twice perpendicular to the equipotential lines.

Finally, Figs. 2s–2x show trajectories perpendicular to the symmetry axes $x=\pm y$.

On the basis of this analysis of the trajectories in Fig. 2, it can be seen that the number of periodic trajectories of the type in Figs. 2c–2f, and also of the type in Figs. 2n and 2o, is countable; it can therefore be asserted that the set of periodic trajectories of the system (10) is at the most countable.

It is obvious that since no trajectory of the system (10) can lie entirely in one quadrant of Fig. 1, this fact and the symmetry of the problem means that all possible trajectories of the system can be obtained by specifying initial conditions in the form

$$y=0;\quad x=x_0>0;$$
$$\dot{x}=\sqrt{2}\,\mu^2\cos\alpha;\quad \dot{y}=\sqrt{2}\,\mu^2\sin\alpha\quad (0\leqslant\alpha\leqslant\pi).$$

The most important thing for us from this analysis is that the resulting periodic trajectories of the system are, as we have seen, very unstable with respect to small changes in the initial conditions (x_0,α); this is one of the manifestations of the stochasticity of the system (10).

On the basis of these arguments, it is to be expected that the trajectories of the system possess local instability, this being the cause of the strong dependence of the motion on not only the initial conditions but also various small disturbances.

The stochasticity of the system (10) can be seen explicitly not only by an investigation of the stability of the periodic trajectories with respect to the initial conditions. Another method, which clearly demonstrates this stochasticity, is associated with computer experiments using the method of Poincaré plots.[32] Experiments of this kind were made for the first time in the investigation of the motion of stars in the field of the Galaxy (Hénon-Heiles, Contopoulos, Ford, *et al.*[33–35]).

We program the computer to solve (10) by specifying the points of intersection of the phase trajectory of the system in the space $(x,\dot{x},y,\dot{y})$ with the plane $(y,\dot{y})$ for $\dot{x}>0$.[18] If the motion is periodic, the intersection occurs at a finite number of points; if the system is integrable, i.e., the trajectory winds around a torus, the points form a regular closed curve in the plane $(y,\dot{y})$. Finally, in the case of stochastic behavior of the system the point of intersection wanders randomly in the plane $(y,\dot{y})$, covering a finite area densely. It is such a behavior of the trajectories of the system (10) in the plane $(y,\dot{y})$ for $\dot{x}>0$ that is revealed by the computer,[18] this being a proof of the stochasticity of the system (10) [see Fig. 3, in which all the points intersecting the plane $(y,\dot{y})$ belong to one and the same trajectory]. The most characteristic and important property of random motion is the rapid, exponential instability of the trajectories of the system, i.e., the exponential separation of initially close phase trajectories: $R \sim e^{ht}$, where $h>0$.

This criterion of dynamical chaos is particularly helpful in numerical simulation.

The quantity h, which determines the exponential rate of separation of such trajectories, is the metric entropy of the random component of the motion and is sometimes called the KS entropy (Krillov-Kolmogorov-Sinai entropy).

If $h>0$, the motion has a stochastic component. Moreover, in accordance with the modern theory of dynamical systems,[7] the condition $h>0$ is the necessary and sufficient condition for stochasticity of almost all trajectories.[36]

The KS entropy is determined by the Lyapunov exponents $\Lambda_i\,(\Lambda_i>0)$,

$$h=\sum_{i=1}^{n-1}\Lambda_i \geqslant \Lambda_m, \tag{12}$$

where Λ_m is the maximal exponent (n in the majority of cases is the number of degrees of freedom of the system).

The quantity Λ_m is determined by the "distance" between neighboring trajectories in the phase space:

$$\rho^2=(\Delta x)^2+(\Delta y)^2+(\Delta z)^2+(\Delta\dot{x})^2+(\Delta\dot{y})^2+(\Delta\dot{z})^2$$

(to be specific, we consider the case with $n=3$, $A_1^1=x$, $A_2^2=y$, $A_3^3=z$, which was studied for the first time by means

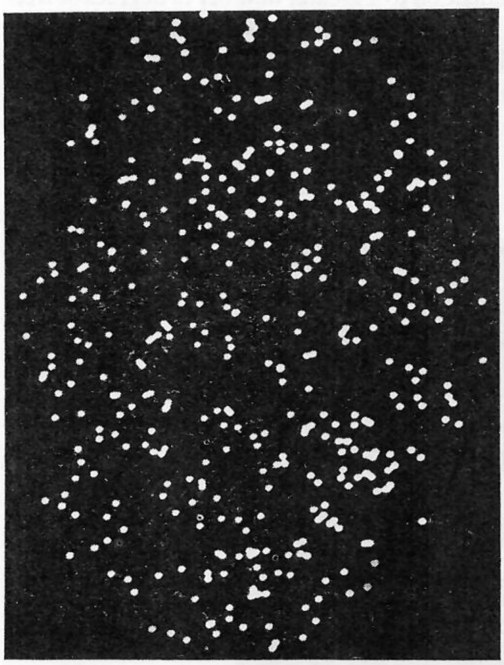

FIG. 3. Computer-display photograph of a plot in the $(y,\dot{y})$ plane showing the stochasticity of the system (10).

of the concept of KS entropy in Ref. 17):

$$\Lambda_m=\lim_{t\to\infty}\frac{\ln\rho(t)}{t}. \tag{13}$$

The behavior of trajectories that are near each other can be studied in the linear approximation; since we are interested in the strictly local behavior of such trajectories, the linear approximation is perfectly correct.

If $\Lambda_m>0$, then it follows from (12) that $h>0$ and such trajectories diverge exponentially. For an integrable system (quasiperiodic motion) $\rho(t)\sim t$ (power-law local instability) and $\Lambda_m=h=0$.

In Ref. 17, which uses the concept of KS entropy, exponential local instability of a system of the type (10) with $n=3$ was proved and the conclusion obtained above[16] to the effect that the system with $n=2$ is stochastic was confirmed.

For more details of the analysis of (10) from the point of view of the stochastic component which these equations contain, see Ref. 36, in which it is shown that the random component covers in both cases almost the complete energy surface.

One further criterion for stochasticity of the motion is sometimes taken to be the phenomenon of intersection (splitting) of the separatrices of the trajectories. Such an approach, used in Ref. 37, also confirms the conclusion drawn above to the effect that the system (10) is stochastic, i.e., it is nonintegrable (see also Ref. 38).[1)]

4. HIGGS MECHANISM AND STOCHASTICITY. DISORDER-ORDER PHASE TRANSITION IN THE CLASSICAL SYSTEM[18]

In recent years, much importance has become attached to the question of the realization of the phases in gauge theories[39–41]: the confinement phase, with which the disorder state is associated, and the Higgs phase—the state of order. By analogy, one can say that the absence of a complete set of nontrivial (so-called isolating) integrals in a classical system corresponds to the disorder phase found above in the system (10) and its generalizations to three degrees of freedom, while the order phase corresponds to systems with a complete set of isolating integrals (when their number is equal to the number of degrees of freedom).

In connection with what we have said above, it is of great interest to investigate classical gauge systems with spontaneous breaking in the gauge $A_0^a=0$.

The Hamiltonian corresponding to (4) has the form

$$H=H_{\rm YM}+\frac{1}{2}(\dot{B}_a^2+\dot{\sigma}^2)+\frac{g^2}{4}(A_i^aA_i^a)\Big[B_a^2/2+\Big(\frac{\sigma}{\sqrt{2}}+\eta\Big)^2\Big]$$
$$+\lambda^2\Big[B_a^2/2+\Big(\frac{\sigma}{\sqrt{2}}+\eta\Big)^2-\eta^2\Big]^2, \tag{14}$$

and the constraint equations are

$$\varepsilon^{abc}A_i^b\dot{A}_i^c-\frac{\eta}{\sqrt{2}}\dot{B}_a+\frac{1}{2}[\sigma\dot{B}_a-B_a\dot{\sigma}-\varepsilon^{abc}B_b\dot{B}_c]=0, \tag{15}$$

where η is the vacuum expectation value of the scalar field φ,

$$\varphi=\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}iB_1+B_2\\ \sqrt{2}\,\eta+\sigma-iB_3\end{pmatrix};$$

λ is the coupling constant of the self-interaction of the scalar field φ.

We investigate in detail the case with two degrees of freedom of a gauge field [see (9)] interacting with a Higgs vacuum ($B_a = \sigma = 0$):

$$H \equiv \mu^4 = \frac{1}{g^2}\left[\frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{x^2y^2}{2} + \frac{g^2\eta^2}{4}(x^2 + y^2)\right]. \quad (16)$$

The additional potential energy $x^2 + y^2$ is here also naturally spherically symmetric, and again $N^a = 0$.

It is clear that for large fields this addition, corresponding to a linear oscillator, is unimportant, and we must obtain random motion of the system (10). Conversely, in small fields one could think that the nonlinearity x^2y^2 can be ignored and that stable regular oscillations should be expected.

By means of the scale transformation $x \rightarrow \alpha x$, $y \rightarrow \alpha y$, $t \rightarrow \beta t$, it is easy to show that the motion of the system (16) is characterized by a single important dimensionless parameter: $\pi = g^2\eta^4/4\mu^4$.

For $\pi = 0$ we have, of course, the stochastic motion investigated in detail in Secs. 2 and 3. At large values of π, as already noted, we expect regular motion.

We now see that in fact the system (16) is stochastic not only for $\pi = 0$ but also for small but finite $\pi \ll \pi_c$.

Our task is to calculate on a computer the critical value π_c of the parameter at which a phase transition takes place in the following sense: At large values of π, the system is nearly integrable, and the trajectory in the phase space $(x, \dot{x}, y, \dot{y})$ represents winding around a torus[42] (the measure of the ergodic trajectories is zero [1,2,43]), i.e., the order phase is realized, while for small but finite $\pi(\pi < \pi_c)$ the motion is, as for $\pi = 0$, stochastic, i.e., the disorder phase is realized.

In Sec. 3 we have already described the part of the computational experiment referring to the solution of the system of equations (10) on a computer which gives the points of intersection of the phase trajectory of the system in the space $(x, \dot{x}, y, \dot{y})$ with the phase plane $(y, \dot{y})$ for $\dot{x} > 0$.

Figure 4, a photograph from the computer display, shows the plot in the $(y, \dot{y})$ plane for the value $\pi = 4.84$; it can be seen that the points of intersection of the trajectory with the plane form closed regular curves. The centers of the three small closed curves correspond to stable trajectories, and the two points of intersection of the closed lines correspond to unstable periodic trajectories (intersection of separatrices at nonzero angle, as already mentioned in Sec. 3).

It is in the neighborhood of these last points of intersection that there first appear "macroscopic" regions of ergodic motion of nonzero measure [Fig. 5, $\pi = 0.35$; cf. Refs. 33–35 (see also Ref. 36)].

With a further decrease in π, the area occupied by the stochastic component increases sharply and at the critical value $\pi = \pi_c \approx 0.15$ becomes almost equal to the entire allowed region of the motion in the $(y, \dot{y})$ plane. The picture becomes similar to the one in Fig. 3 and corresponds to developed stochasticity (we emphasize once more that all the points in this figure correspond to a single trajectory).

In Ref. 36, the motion-stabilizing role of the Higgs mechanism was analyzed in more detail. For the case with $n = 2$ at small π ($\mu \gg 1$, $\pi \ll \pi_c$) the motion is stochastic [$h \sim \mu/\ln\mu \sim \pi^{-1/4}/\ln(1/\pi) > 0$]. At large π ($\pi \gg \pi_c$), the stochastic component persists only in an exponentially narrow layer around the separatrix—the system is integrable in the sense of Kolmogorov-Arnol'd-Moser theory.[43,42]

Since h depends continuously on π, the "phase transition" found in Ref. 18 evidently has a transition region that is not sharply defined. [See also Ref. 44, in which μ_c (or π_c) is determined using an approach based on study of the topology of the energy surface $H = H(x, y, \dot{x}, \dot{y})$. For $\pi < \pi_c = 2/3$, gradual transition from regular to irregular trajectories is characteristic of our problem. For $\pi > \pi_c$, one can identify a compact invariant set filled with regular trajectories.]

On the transition to a greater number of degrees of freedom ($n = 3$),[36] the situation becomes more complicated. Even at small μ ($\pi \gg 1$) an appreciable stochastic component can be obtained under certain conditions. At large μ ($\pi \ll 1$), as for $n = 2$, the stochastic component covers almost the

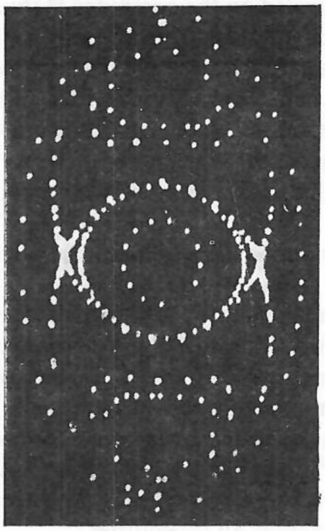

FIG. 4. Photograph of a computer display showing a plot in the $(y, \dot{y})$ plane for $\pi = 4.84$.

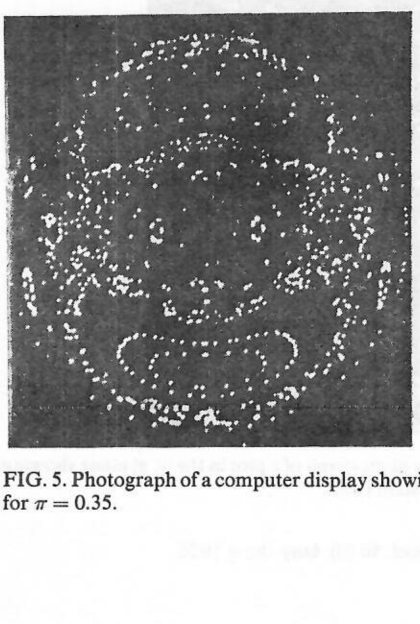

FIG. 5. Photograph of a computer display showing a plot in the $(y, \dot{y})$ plane for $\pi = 0.35$.

entire energy surface except for small regions along the coordinate axes.

Thus, it can be assumed that for $n > 2$ the Higgs mechanism does not completely eliminate the stochastic component even at large values of π.[36]

We make a remark associated with the study of the Yang-Mills-Higgs system (16) in the quantum-mechanical limit.

The corresponding problem with a nonlinearity parameter α was considered in Ref. 23:

$$H = -\frac{1}{2}\frac{\partial^2}{\partial x^2} - \frac{1}{2}\frac{\partial^2}{\partial y^2} + \frac{1}{2}(x^2 + y^2) + \alpha x^2 y^2. \quad (17)$$

In the classical problem, Pullen and Edmonds[23] also found a transition (at high energies, which, as is readily seen, correspond to small values of our parameter π) to chaos from the regular motion.

In the quantum problem (17), the term with α being treated, it is true, as a perturbation, it is found in accordance with the expectation of Ref. 21, as discussed in the Introduction, that there is an intimate correlation between the fraction of the classical stochastic motion and the fraction of the part of the energy spectrum of the Hamiltonian (17) in which the eigenvalues are strongly sensitive to small variations in the parameter of the perturbation (directly corresponding to the absence of level crossing).

The region of energies at which the transition from the one regime to the other is observed is the same for the two cases—classical and quantum.

In Ref. 23 it can be clearly seen how with increasing energy the fraction of nondegenerate energy eigenvalues with a strong dependence on a small change in the nonlinearity parameter α increases,

$$\Delta_i^2 = |[E_i(\alpha + \Delta\alpha) - E_i(\alpha)] - [E_i(\alpha) - E_i(\alpha - \Delta\alpha)] \approx |\partial^2 E_i/\partial\alpha^2|(\Delta\alpha)^2,$$

while the fraction of levels with $\Delta_i^2 \approx 0$ decreases.

It is of interest to study this problem without regarding the parameter α as a perturbation.

It must, however, be borne in mind that a quantum-mechanical system is not yet a quantum-field system with infinitely many degrees of freedom.

And although, as we have noted, the homogeneous fields with which we have been concerned in this review correspond to the long-wave part of the spectrum of the classical Yang-Mills system, the corresponding formulation of the problem in quantum field theory must not ignore the fact that the infrared problem is a strong-coupling problem.

5. CLASSICAL YANG-MILLS MECHANICS WITH $n > 3$

If, as has been shown in the previous sections, dynamical stochasticity is already inherent in the classical Yang-Mills mechanics (i.e., in spatially homogeneous fields) for $n = 2$ and 3, an increase in the number of degrees of freedom of the system can, generally speaking, only strengthen the stochasticity of the motion.

At the same time, an increase in n introduces a new aspect, which we shall consider in this section.

We shall study the system (3) for $n = 4$: $A_i^3 = A_3^1 = A_3^2 = 0$.

In this case, the third component of the conserved angular momentum is nonzero,

$$M_3 = A_1^a \dot{A}_2^a - A_2^a \dot{A}_1^a, \quad (18)$$

and the constraint (the vanishing of N^a) has the form

$$A_i^1 \dot{A}_i^2 - A_i^2 \dot{A}_i^1 = 0. \quad (19)$$

The form of the potential $U = (g^2/4)(A_1^1 A_2^2 - A_1^2 A_2^1)^2$ suggests the ansatz

$$\left.\begin{aligned} gA_1^1 &= \xi_1 + \xi_2; \quad gA_1^2 = \xi_3 + \xi_4; \\ gA_2^1 &= \xi_3 - \xi_4; \quad gA_2^2 = \xi_1 - \xi_2, \end{aligned}\right\} \quad (20)$$

which "mixes" the components of the different isotopic vectors

$$\mathbf{A}^{(1)}(A_1^1,\ A_2^1,\ 0);\ \mathbf{A}^{(2)}(A_1^2,\ A_2^2,\ 0).$$

The Hamiltonian takes the form

$$g^2 H = \sum_{i=1}^{4} \dot{\xi}_i^2 - \frac{1}{4}(\xi_1^2 + \xi_4^2 - \xi_2^2 - \xi_3^2)^2. \quad (21)$$

Expressing M_3 from (18) in terms of the variables (20) and using the constraint (19), we obtain

$$g^2 \frac{M_3}{4} = \xi_4\dot{\xi}_1 - \xi_1\dot{\xi}_4 = \xi_2\dot{\xi}_3 - \xi_3\dot{\xi}_2. \quad (22)$$

It is also convenient to introduce the variables

$$\left.\begin{aligned} \xi_1 &= r_1 \sin\varphi; \quad \xi_2 = r_2 \sin\theta; \\ \xi_4 &= r_1 \cos\varphi; \quad \xi_3 = r_2 \cos\theta, \end{aligned}\right\} \quad (23)$$

and then (22) can be written very simply as

$$g^2 \frac{M_3}{4} = r_1^2 \dot{\varphi} = -r_2^2 \dot{\theta}. \quad (24)$$

Substituting (23) and (24) in (21), we finally obtain

$$g^2 H = \frac{\dot{r}_1^2 + \dot{r}_2^2}{2} + \frac{M_3^2}{32}\left[\frac{1}{r_1^2} + \frac{1}{r_2^2}\right] + \frac{1}{4}(r_1^2 - r_2^2)^2. \quad (25)$$

Of course, setting $M_3 = 0$ and making the substitution $r_1 = (x + y)/2$, $r_2 = (x - y)/2$, we arrive at the already investigated Hamiltonian (9) with $n = 2$.

Equation (25) was obtained for the first time in Ref. 45, which used not the Hamiltonian, as here, but the axial gauge $A_3^a = 0$.

Frøyland[46] recently arrived at Eq. (25), but he did not note the constraint condition which leads to the equality (in his notation) $L_1 = L_2$ of the two constants that play the role of our M_3 ($= L_1 = L_2$).

It is true that throughout Ref. 46 he considers (as an assumption) precisely the case $L_1 = L_2$.

The system with the Hamiltonian (25) does not in fact depend on the parameter M_3, since by means of the scale transformations

$$r_1 \rightarrow \left(\frac{M_3}{4}\right)^{1/3} r_1;\ r_2 \rightarrow \left(\frac{M_3}{4}\right)^{1/3} r_2;\ t \rightarrow \left(\frac{M_3}{4}\right)^{-1/3} t$$

it is possible to transform H to the form[45]

$$g^2 H = \left(\frac{M_3}{4}\right)^{4/3} \left[\frac{\dot{r}_1^2 + \dot{r}_2^2}{2} + \frac{1}{2}\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right) + \frac{1}{4}(r_1^2 - r_2^2)^2\right]. \tag{26}$$

The corresponding equations of motion are

$$\left.\begin{aligned} \ddot{r}_1 &= \frac{1}{r_1^3} + r_1(r_1^2 - r_2^2); \\ \ddot{r}_2 &= \frac{1}{r_2^3} - r_2(r_1^2 - r_2^2). \end{aligned}\right\} \tag{27}$$

The presence of the centrifugal barrier in (26) makes the region of $r_1 = r_2 = 0$ forbidden.

The trajectories of the system in each of the quadrants of the coordinate system r_1, r_2 lie within the region bounded by the exponential curve shown in Fig. 6.

We have here qualitatively the same picture of the random walk of a particle in the channel along the bisector of the quadrant as in the case of the system (10). Eliminating the trivial case when the particle velocity is directed exactly along the channel axis, analysis shows that the particle cannot in general reach infinity, since the width of the channel decreases with the time more rapidly (as t^{-2}) than does the amplitude of the oscillations about the channel axis ($\sim t^{-1/2}$).[45] This circumstance is ultimately responsible for the stochasticity of the system, which is particularly clearly manifested at values of $H = \mu^4$ that are not small.

We shall not dwell in detail on the analysis of this stochasticity or the question of the regular component, referring the reader to Refs. 45, 47, and 48. We make only one remark.

In Sec. 4 we saw the stabilizing effect of the Higgs mechanism on the system (10). Essentially, it arose from the introduction into the system (10) of the additional parameter π, by the variation of which it was possible to change the regime of the motion at a given μ. In the system (25), despite the nonvanishing angular momentum M_3 there is no such parameter. If, however, we introduce on the right-hand side of the classical Yang-Mills equations the charge density $J_\mu^a = (\rho^a, 0)$ (essentially of quantum origin and generated, for example, by heavy virtual quarks), $\rho^a = \rho \varepsilon^{abc} n_1^b n_2^c$, where n_i^a are unit vectors in the internal space, then the integrals of the motion M_3 and N have the form

$$M_3 = r_1^2 \dot{\varphi} - r_2^2 \dot{\theta};\quad \frac{\rho}{2} = r_1^2 \dot{\varphi} + r_2^2 \dot{\theta}.$$

Then instead of (26) we shall have[48] [after the scale transformation $r_1 \rightarrow (\mu + \rho)/4)^{1/3} r_1$, etc.]

$$g^2 H = \frac{\dot{r}_1^2 + \dot{r}_2^2}{2} + \frac{1}{2}\left(\frac{1}{r_2^2} + \frac{\lambda^2}{r_1^2}\right) + \frac{1}{4}(r_1^2 - r_2^2)^2, \tag{28}$$

where in the system there appears besides the energy the new parameter

$$\lambda = (\mu - \rho)/(\mu + \rho).$$

The numerical investigation made in Ref. 48 shows that for $\lambda \neq 1$ and given H there is a transition from stochastic to regular motion when λ increases. For $\lambda > \lambda_c(H)$, the system is nearly regular. Of course, the stabilizing effect associated with the nonvanishing charge density ρ^a ($\lambda \gg 1$) is of a quite different nature from the stabilization associated with the Higgs mechanism.

6. GENERAL CASE OF CLASSICAL YANG-MILLS MECHANICS

In Ref. 49, the most general case of classical Yang-Mills mechanics with nine degrees of freedom was considered.

The potential $A_i^a(t)$ of the Yang-Mills field can always be represented as

$$(O_1 E O_2^T)_i^a, \tag{29}$$

where E is diagonal,

$$E = \begin{pmatrix} x(t) & 0 & 0 \\ 0 & y(t) & 0 \\ 0 & 0 & z(t) \end{pmatrix}, \tag{30}$$

and O_1 and O_2 are orthogonal time-dependent matrices.

Introducing the antisymmetric matrices ω and Ω,

$$\left.\begin{aligned} \omega &= O_1^T \dot{O}_1 = -\dot{O}_1^T O_1; \\ \Omega &= O_2^T \dot{O}_2 = -\dot{O}_2^T O_2, \end{aligned}\right\} \tag{31}$$

we obtain for the Hamiltonian

$$H_{\mathrm{YM}} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + T_{\mathrm{YM}} + \frac{g^2}{4}(x^2y^2 + y^2z^2 + z^2x^2), \tag{32}$$

where

$$T_{\mathrm{YM}} = \frac{1}{2}\sum_{a=1}^{3}\{I_a(\omega_a^2 + \Omega_a^2) - 2J_a \omega_a \Omega_a\}; \tag{32a}$$

$$\omega_a = \frac{1}{2}\varepsilon_{abc}\omega_{bc};\quad \Omega_i = -\frac{1}{2}\varepsilon_{ijk}\Omega_{jk}; \tag{33}$$

$$\left.\begin{aligned} I_1 &= y^2 + z^2;\ I_2 = x^2 + z^2;\ I_3 = x^2 + y^2; \\ J_1 &= 2yz;\ J_2 = 2xz;\ J_3 = 2xy. \end{aligned}\right\} \tag{34}$$

If the analogy with the classical mechanics of a point

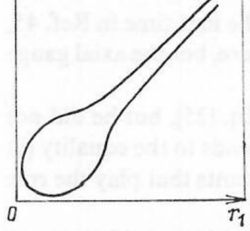

FIG. 6. Form of the equipotential curve of the Hamiltonian (26). Only the first quadrant is shown.

was helpful in the preceding sections, the analogy with the mechanics of a rigid body is here suggested. True, this "rigid body" has t-dependent moments of inertia J_i and I_i and "rotates" in the ordinary and the internal spaces, since if we project M_i and N^a onto a coordinate system that "moves" with the "rigid body,"

$$N^a = O_1^{ab} n^b; \quad M_i = O_{2_{ij}} m_j;$$
$$N^a = I_a \omega_a - J_a \Omega_a; \quad M_k = I_k \Omega_k - J_k \omega_k,$$

and differentiate them with respect to the time ($\dot{N}^a = \dot{M}_i = 0$), we obtain the "Euler equations" of classical mechanics

$$\frac{d\mathbf{n}}{dt} = [\mathbf{n}, \boldsymbol{\omega}]; \quad \frac{d\mathbf{m}}{dt} = [\mathbf{m}, \boldsymbol{\Omega}]. \qquad (35)$$

We shall not analyze in detail Eqs. (35) and (32), but refer the reader to the original paper of Ref. 49, where there is also a generalization to the case of an arbitrary gauge group SU(N).

7. STOCHASTICITY AND CONFINEMENT

The discovery of dynamical chaos of free classical non-Abelian gauge fields and the "phase transition" of the disorder—order type (confinement phase–Higgs phase) in them—this is the content of the previous sections—makes it very attractive to believe that the observed phenomena are to some extent preserved in the real (i.e., quantum) vacuum of QCD and that it is the presence in it of random color vacuum fields that is responsible for color confinement.

In the Introduction we have already argued that the disordered (stochastic) vacuum could be the cause of confinement [analogy with the reduction in the dimension of quantum spin systems in a random field, Olesen's hypothesis about the reduction of the four-dimensional Yang-Mills theory to the two-dimensional theory in the limit $N \to \infty$, and lattice calculations, which make this hypothesis plausible for SU(2) symmetry].

We consider the last argument based on Monte Carlo calculations on a lattice of the distribution $\rho_C(\alpha)$ of the eigenvalues of the Wilson loops, $\langle W(C)\rangle = \int_{-\pi}^{+\pi} d\alpha \langle \rho_C(\alpha)\rangle e^{i\alpha}$, since it shows that stochastic phenomena are to some extent present in QCD.[15]

As follows from the results of Ref. 15, for distances (loop radii) r less than the confinement radius r_C, the distribution of the loop eigenvalues (the spectral density) $\rho_C(\alpha)$ has a peak at $\alpha = 0$, indicating a strong correlation of the fields at short distances.

However, for loops with $r > r_C$ the distribution $\rho_C(\alpha)$ becomes virtually uniform, i.e., the fields are weakly correlated and the distribution of the eigenvalues $W(C)$ corresponds to disordered configurations.

This argument, if it is not an artifact of the Monte Carlo calculations on a lattice, shows that QCD does indeed contain in some form a stochastic component.

The question of the mechanism of this stochasticity—of whether it is a manifestation and "relic" of the classical stochasticity found and described in Secs. 1–6—remains, of course, open.

At the present time, there are numerous mechanisms that "ensure" color confinement, the most popular of which is the mechanism based on the condensation of vortices and magnetic monopoles.[40,50,51]

No less popular is the mechanism[52] based on the idea of a decrease in the vacuum energy through the formation of a gluon condensate.[28,29]

Monte Carlo calculations[53–55] show that quarks are confined in SU(2) and SU(3) lattice gauge theories.

Among the less orthodox confinement mechanisms, we mention Ref. 56 of Kirzhnits *et al.*, in which the possible stochasticity of QCD mentioned above is associated with a phenomenon like the localization phenomenon in disordered systems.

As is well known, the phenomenon of localization by means of a random potential leads to unusual properties of the spectrum of the corresponding quantum problem: It is quasicontinuous (like the set of rational numbers), but wave functions corresponding to energies nearly equal in magnitude are localized at a large distance from one another. Therefore, to such localized wave functions there corresponds a discrete spectrum, the levels of which are determined by the properties of the random potential that acts between the quarks (in particular, the localization length).

As a result, qualitative arguments suggest that in the quark-antiquark system in the one-dimensional approximation a linearly rising effective potential is established.

What was said above, in conjunction with the studies of Refs. 13 and 14 mentioned in the Introduction, in which confinement arises in the limit $N \to \infty$ as a consequence of stochasticity, makes it extremely important to show that in fact stochasticity is sufficient for confinement in quantum field theory.

We shall show[57] that if in the functional integral of the theory we take into account only fields generated by randomly distributed currents, then the corresponding two-particle Green's function corresponds to confinement. Of course, the contribution to the gluon propagator is not exhausted by such fields, since there is undoubtedly an important class of fields of a different, nonstochastic nature which are particularly important for distances that are short and intermediate compared with the confinement radius r_C or, in our problem, the correlation radius μ^{-1} of the random currents. It is these fields that must ensure the property of asymptotic freedom.

We cannot say whether they are present at $r \sim \mu^{-1} \sim r_C$ together with the stochastic component (the presence of this component is indicated by the Monte Carlo calculations mentioned above[15]), and, if so, what is their relative contribution compared with that of the random fields.

In other words, the question in which we are interested is that of the Green's function of the field quanta if they are generated by random color currents $J_\mu^a(x)$ with a Gaussian distribution (so-called white noise)

$$\langle J_\mu^a(x) J_\nu^b(y)\rangle = \mu^2 \delta_{\mu\nu} \delta^{ab} \delta^{(4)}(x-y). \qquad (36)$$

In accordance with what has been said, the dynamics of the field quanta (gluons) at distances of order $r_c \sim \mu^{-1}$ is determined by the stochastic equation of motion

$$\frac{\delta S}{\delta A^a_\mu} = J^a_\mu, \quad (37)$$

where S is the action of the theory in four-dimensional space-time (we shall use below the Euclidean formulation), and the quantum averaging is determined by the relation

$$\langle A^{a_1}_{\mu_1}(x_1) \ldots A^{a_n}_{\mu_n}(x_n)\rangle = \langle \tilde{A}^{a_1}_{\mu_1}(x_1) \ldots \tilde{A}^{a_n}_{\mu_n}(x_n)\rangle_J. \quad (38)$$

The fields $\tilde{A}^a_\mu$ are determined from Eq. (37), while on the right-hand side of (38) the averaging $\langle\ldots\rangle_J$ is over the Gaussian distribution of the currents

$$\exp\left\{-\frac{1}{2\mu^2}\int J^a_\mu(x) J^a_\mu(x)\, d^4x\right\},$$

which corresponds to (36).

The generating functional of our theory corresponding to (38) is given by the expression

$$Z(h^a_\mu) = \int DJ \exp\left\{-\int\left[\frac{1}{2\mu^2} J^a_\mu(x) J^a_\mu(x) - h^a_\mu(x)\tilde{A}^a_\mu(x)\right] d^4x\right\}, \quad (39)$$

the differentiation of which with respect to the quantum source $h^a_\mu(x)$ determines the Green's functions (38).

Going over in (39) from the variables J to the variables $\tilde{A}$ by the introduction of the δ function corresponding to (37) and using the standard device for expressing the determinant that arises in terms of anticommuting vector fields ψ^a_μ, $\bar{\psi}^a_\mu$, we obtain the expression (for simplicity, we omit in all that follows the Lorentz and internal indices)

$$Z(h) = \int D\bar{\psi} D\psi DA \exp\left\{-\int\left[-\frac{1}{2\mu^2}\left(\frac{\delta S}{\delta A}\right)^2 \times \delta(x-y) - \bar{\psi}(x)\frac{\delta^2 S}{\delta A(x)\,\delta A(y)}\psi(y) - hA(x)\,\delta(x-y)\right]\right\} d^4x\, d^4y. \quad (40)$$

Note that the introduction here of the stochastic equation (37) in the functional integral is different in that here an auxiliary time is not introduced as a fifth coordinate, and we work in real space-time. At the same time, there arises an intimate connection between stochastic differential equations of the type (37) and supersymmetry, which was noted recently.[58]

From the form of (40) one can actually already see that in the tree approximation a two-particle function of the type (38) has confinement properties [first term in the exponential (40)].

We shall show this in a different way associated with the introduction of the superfield $\Phi^a_\mu(x,\theta)$:

$$\Phi^a_\mu(x, \theta) = A^a_\mu(x) + \bar{\psi}^a_\mu(x)\theta + \bar{\theta}\psi^a_\mu(x) + C^a_\mu\bar{\theta}\theta, \quad (41)$$

where θ, $\bar{\theta}$ are anticommuting variables ($\theta^2 = \bar{\theta}^2 = \{\theta, \bar{\theta}\} = 0$).

It is easy to show that (40) can be written in the form

$$Z(h) = \int D\Phi(x, \theta) \exp\left\{-\int\left[\mathcal{L}(\Phi) - \frac{\mu^2}{2}\Phi^+\frac{\partial^2}{\partial\bar{\theta}\,\partial\theta}\Phi - H\Phi\right]\right\} d^4x\, d\bar{\theta}\, d\theta, \quad (42)$$

where $H = h(x)\bar{\theta}\theta$.

It can be shown from (42) that the Fourier transform of the propagator $\langle\Phi^a_\mu\Phi^b_\nu\rangle$ of the superfield has the structure ($p^2 \leqslant \mu^2$)

$$(p^2 + \mu^2\bar{\alpha}\alpha)^{-1}\,\delta^{ab}\delta_{\mu\nu},$$

where $\bar{\alpha}$, α are the Grassmann variables corresponding (after Fourier transformation) to the variables $\bar{\theta}$, θ.

Integrating with respect to them, we arrive at confinement, since as a result the Fourier transform of $\langle A^\alpha_\mu A^\alpha_\nu\rangle$ has the form $\delta_{\mu\nu}\delta^{ab}\mu^2/p^4$, i.e., the exchange of such quanta ensures a potential between static sources that rises linearly with r (for $r \gtrsim r_c$).

Our study shows that stochasticity of the sources that generate the corresponding fields (and, as the Monte Carlo calculations of Ref. 15 show, they can be near $r \gtrsim r_c$) is a sufficient condition for a potential that rises linearly with r.

It can be shown that stochasticity is also a necessary condition if the theory is based on local field theories. However, the attentive reader will no doubt see that such confinement is not something peculiar to gauge theories, since, as can be seen from our derivation, any quantum field theory characterized by the stochasticity condition (36) (it is obvious that the restriction to white noise as an example of stochasticity is not of decisive importance) will have a propagator with the behavior μ^2/p^4 for $p^2 \lesssim \mu^2$.

This can be clearly seen from the following chain of symbolic equations and relations:

$$\Box A = -J; \; A = -\Box^{-1}J;$$
$$\langle A(x)\,A(y)\rangle = \Box_x^{-1}\Box_y^{-1}\langle J(x)\,J(y)\rangle = \mu^2\Box_x^{-1}\Box_y^{-1}\delta^{(4)}(x-y)$$

etc., from which our assertion follows—and indeed by virtue of the correlation function (36).

But what then distinguishes the gauge theory of non-Abelian fields among other theories?

Obviously, the dynamical stochasticity inherent in it and which we considered in detail in this review for the classical case.

8. IN LIEU OF CONCLUSIONS

The question of whether (see the Introduction) traces of the dynamical stochasticity of the classical non-Abelian gauge fields described in this paper are preserved in the real hadron world is not simple.

The consequences mentioned in the Introduction for quantum systems possessing dynamical stochasticity in the classical limit (as we already know, QCD is such a system) associated with the structure of the spectrum and the properties of the wave functions are well known.[21,24–26]

This suggests that the real hadron spectrum must reflect the irregularity corresponding to the dynamical chaos of the classical Yang-Mills theory that we have established.

But how is this irregularity to be detected, and with what is the real spectrum to be compared?

In this direction, one can hardly expect any real advance. However, it is quite possible that one could find other characteristics of hadronic phenomena in which dynamical stochasticity of the non-Abelian gauge fields that control the world of hadrons is established.

In this direction, real progress is possible. Besides the above-mentioned studies based on Monte Carlo calculations on lattices,[15] we must here mention especially the series of papers by Carruthers *et al.*,[59–61] in which multiparticle production of hadrons was considered in the light of the stochastic behavior typical of developed turbulence and the distribution of galaxies in the Universe.

In these studies it was found that these phenomena from different fields of physics are characterized by general and common features that have a stochastic basis.

All these phenomena are described by a common distribution (generalized Bose-Einstein distribution):

$$P_n^{(k)} = \frac{(n+k-1)!}{n!\,(k-1)!} \frac{(\bar{n}/k)^n}{\left(1+\frac{\bar{n}}{k}\right)^{n+k}}.$$

This distribution corresponds to k independent random (Gaussian) sources. It can be shown that the distribution $P_n^{(k)}$ also describes the statistics of counting of electrons from k random (thermal) sources, and also the statistics of photoelectrons from coherent laser radiation that passes through a liquid in the critical state (the corresponding distribution for the case of a liquid in the normal state is described by a simple Poisson distribution). With regard to the multiple (nondiffractive) production of hadrons, one can say that the distribution $P_n^{(k)}$ reflects the fact that hadrons are apparently emitted by random (Gaussian) sources, and gluon matter strongly excited as a result of a collision (quark-gluon plasma?), giving soft hadronization, behaves as a turbulent fluid. The generality of these phenomena is particularly striking in the sense that they are characterized by a universal connection between $\bar{n}$ and k:

$$k = \bar{n}^{1/D}$$

for large k, with D almost the same for the distribution of galaxies ($D = 2.50$) and multiparticle production of hadrons ($D \approx 2.56$). For developed turbulence, the quantity corresponding to D characterizes the dissipative correlation function and is the fractal dimension[62] of the cascade model of turbulence.[63] Measurements give for D values 2.6–2.8.

This numerical coincidence, noted in Ref. 61, may have a common basis and apparently reflects a topological property common to these different phenomena [it is not impossible that D is related to the Lyapunov exponents (12) or the KS entropy (see the footnote at the end of Sec. 3)]. It is obvious that the bases of these phenomena have not a quantum but a decidedly classical aspect, so that one can, for example, say in the light of this observation that the well-known KNO scaling in multiparticle production, which has been well described by means of a distribution $P_n^{(k)}$,[59–61] is not a consequence of various dynamical and geometrical models but reflects detail-independent general stochastic features common to phenomena from the most varied branches of science. The dynamical chaos of non-Abelian gauge fields belongs to this group of phenomena.

[1)]Savvidy[64] has recently shown that the classical Yang-Mills mechanics considered here is a Kolmogorov K system, i.e., it has the strongest statistical properties. We note that elliptical star systems are also systems of this type[65] (cf. Sec. 8).

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