Scattering and resonances in a system of four quantum-mechanical particles

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A study is made of a system of four nonrelativistic spinless particles with two-particle interaction potentials that are analytic with respect to complex scale transformations. Systems of integral equations are constructed for finding the scattering operator $T(z)$ of the system and the operator $T^{\theta}(z)$, where θ is the complex parameter of a scale transformation that realizes analytic continuation of $T(z)$ to part of the unphysical sheet. The properties of the kernels of these equations are studied. Conditions on the interaction potentials under which the integral equations have a unique solution are obtained. The asymptotic behavior of the wave function of the system is studied by the method of stationary phase. A connection is established between the solutions of the corresponding homogeneous equations and the spectra of the Hamiltonians H and $H(\theta)$, i.e., with the energies of the bound states and resonances in a system of a few nonrelativistic particles. The use of the method of complex scale transformations is illustrated by the construction of the envelopes of the regions in which resonances are situated in systems with various potentials of analytic form: Coulomb, Yukawa, Yamaguchi, and exponential. The changes in the parameters of the two-particle resonances in systems of several particles are interpreted theoretically.

INTRODUCTION

The present review is devoted to an exposition of a unified method for describing scattering states, discrete states, and resonances in a system of four nonrelativistic particles. The method is based on the use of integral equations with compact kernels for the resolvent of the system's Hamiltonian.[1,2] In such an approach, the scattering amplitudes are determined by solving an inhomogeneous system of equations, in contrast to the states of the discrete spectrum and the resonances, which can be found from the corresponding homogeneous system after a complex scale transformation has been applied to it.[2–7]

It should be mentioned that the system of four particles with finite masses differs strongly from a system of three particles with finite masses. This difference is due not only to the increase in the number of particles but also to the appearance of an unbound process of a new type, namely, independent interaction of the particles in pairs, for example, (12) and (34). In nonstationary scattering theory, as was noted in Refs. 8–10, this requires a special proof of the existence of an asymptotic condition for the wave operators. In particular, for the same reason, the problem of the interaction of two particles in an external field, i.e., when the mass of the third particle is infinite, must be separated as an independent case that cannot be reduced to the three-particle problem. A detailed discussion of the questions touched on here can be found in Ref. 2 (pp. 370–371 and 173–177 of the Russian translation) and Refs. 11–14.

In stationary scattering theory, kernels of a particular type appear when integral equations are formulated for a problem with $N \geqslant 3$.[11–14] These kernels are associated with the scattering of independent mutually interacting subsystems of particles and can be obtained in explicit form by representing the Green's function of the complete system as a convolution of the Green's functions of the unbound subsystems. The use of the convolution formalism in the integral equations for problems of $N \geqslant 4$ particles with finite masses and $N \geqslant 2$ particles in an external field makes it possible to avoid the unphysical separation of the asymptotic behaviors inherent in the approaches with classification of the amplitudes with respect to the distinguished first interaction. For the differences between the systems of integral equations of the N-body problem, see the reviews of Refs. 15–17 and 19. In addition, the use of the convolution formalism makes it possible to solve the problem, important from the practical point of view, of establishing the connection between three- and four-dimensional perturbation theory and, as a consequence, between the three-dimensional diagrams that represent iterations of the Lippmann-Schwinger equation and nonrelativistic Feynman diagrams.[14,18,19]

The structure of the present paper is as follows: In Sec. 1, we derive and investigate integral equations for systems of four particles with finite masses. We study the properties of the kernels of these equations, construct systems of integral equations for the components of the total T scattering operator, and establish the connection between the matrix elements of these components and the amplitudes of reactions in the four-body problem. We also formulate conditions under which the integral equations have a unique solution. In Sec. 2, we consider the modification of the equations for describing the resonance states in the framework of the theory of complex scale transformations. We show that the solutions of the corresponding homogeneous equations are the eigenvalues of the Hamiltonian of the system subjected to a scale transformation. The method is used to find the positions of resonances in systems with various analytic potentials.

To conclude Sec. 2, we give a theoretical interpretation of the changes in the parameters of two-particle resonances manifested in reactions with the formation of three or more particles.

1. INTEGRAL EQUATIONS OF THE FOUR-BODY PROBLEM

Formulation of the problem and definitions

In this section, we shall formulate a system of equations for the scattering operators in the problem of four pairwise

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interacting particles with finite masses. As will be clear from what follows, this system has an ancillary nature and will be subsequently transformed by separating the primary singularities into a system of integral equations for the components of the total T scattering operator. We need the same system in Sec. 2 for discussing the resonances.

The system of equations which we shall be discussing has the same structure as the system of integral equations of the four-body problem in the framework of the second-quantization formalism.[18-20] The two systems (obtained in different approaches) lead to the same result in the determination of the elements of the S matrix of the system.[14,21] Since the equations of Refs. 18–20 were obtained originally by the summation of nonrelativistic Feynman diagrams, using the results of Refs. 14 and 21 one can readily establish a connection between the iterations of the equations studied below and Feynman diagrams and, thus, between three- and four-dimensional perturbation theory in the four-body problem.

The Hamiltonian of the four-particle system has the form

$$H = H_0 + V, \quad H_0 = \sum_{i=1}^{4} H_{0i}, \quad V = \sum_{\alpha} V_\alpha,$$
$$\alpha = ij,\ i < j,\ i,\ j = 1,\ 2,\ 3,\ 4, \tag{1}$$

where H_{0i} is the kinetic-energy operator of particle i with mass m_i, and V_{ij} is the operator of the interaction between particles i and j.

With regard to the potentials V_α, we assume the following. For any α, $V_\alpha(z)$ is an analytic function in the sector $|\arg z| < \sigma$, and for any φ with $|\varphi| < \sigma$ it satisfies $V_\alpha(e^{i\varphi} r_\alpha) \in L^2(R^3)$; r and k are defined in R^3. In addition, we shall assume that $V_\alpha(r_\alpha)$ decreases at infinity as $r_\alpha^{-2-\epsilon}, \epsilon > 0$. These conditions mean that the potential V_α is analytic with respect to scale transformations and belongs to the class $\mathbf{C}_\sigma$ [the class $\mathbf{C}_\sigma$ corresponds to the operator version of Aguilar-Balslev-Combes analytic potentials and is defined in Ref. 2 (p. 403 of the Russian translation) and Refs. 4 and 5]. For example, the Yukawa potential and exponential potentials belong to the class $\mathbf{C}_{\pi/2}$, a potential of Gaussian type to the class $\mathbf{C}_{\pi/4}$, and the Woods-Saxon potential $V(r) = V_0(1 + \exp\{(r-a)/R_0\})^{-1}$ to the class $\mathbf{C}_{\cot^{-1}(a/\pi R_0)}$, whereas a rectangular-well potential is not analytic. Note that potentials analytic with respect to scale transformations are not necessarily local. In particular, the widely used separable Yamaguchi potential is analytic in the class $\mathbf{C}_{\pi/2}$. We shall require the conditions imposed on the potentials V_α when studying the resonances.

As is readily seen, the potentials $V_\alpha(r_\alpha)$ (real for $r_\alpha \in \mathbf{R}^+$) are potentials of short-range type, and therefore the total scattering operator $T(z)$ of the system is defined and satisfies the Lippmann-Schwinger equation[8,21]

$$T(z) = V + VG_0(z)\, T(z), \tag{2}$$

where $G_0(z) = (z - H_0)^{-1}$ is the free Green's function of the system. In (2), the parameter z satisfies the condition $z \notin \sigma(H)$ [$\sigma(H)$ is the spectrum of the operator H], although in what follows as usual, we shall go to the limit $z \to E + i0$, $E \in \sigma_c(H)$, where $\sigma_c(H)$ is the continuous spectrum of the operator H.

Rearrangement of the Lippmann-Schwinger equations into a system of equations

Following the ideas explained in the Introduction and in Refs. 1, 2 and 8–14 about the role of the commuting Hamiltonians $H_\alpha = \Sigma_{i\in\alpha} H_{0i} + V_\alpha$ and $H_{\alpha'}$, $(\alpha,\alpha') \in \{(12.34, (13.24), (14.23)\}$ of the independent subsystems (pairs) of particles α and α', we split the operator V into the following terms:

$$V = \sum_{(\alpha,\alpha')} V_{\alpha,\alpha'}, \quad V_{\alpha,\alpha'} = V_\alpha + V_{\alpha'}. \tag{3}$$

It follows from (3) that the operator $T(z)$ can be represented in the form

$$T(z) = \sum_{(\alpha,\alpha')} T_{\alpha,\alpha'}(z), \tag{4}$$

where

$$T_{\alpha,\alpha'}(z) = V_{\alpha,\alpha'} + V_{\alpha,\alpha'} G_0(z)\, T(z). \tag{5}$$

Generalizing the system of equations (4), (5), we obtain a system of operators $T_{\alpha,\alpha'}(z)$

$$T_{\alpha,\alpha'}(z) = N_{\alpha,\alpha'}(z) + N_{\alpha,\alpha'}(z)\, G_0(z) \sum_{\beta,\beta'} T_{\beta,\beta'}(z),$$
$$\beta\beta' \neq \alpha\alpha', \tag{6}$$

where

$$N_{\alpha,\alpha'}(z) = [1 - V_{\alpha,\alpha'} G_0(z)]^{-1} V_{\alpha,\alpha'}. \tag{7}$$

As is readily seen, these last operators admit the representation

$$N_{\alpha,\alpha'}(z) = V_{\alpha,\alpha'} + V_{\alpha,\alpha'} G_{\alpha,\alpha'}(z)\, V_{\alpha,\alpha'}, \tag{8}$$

where $G_{\alpha,\alpha'}(z)$ denotes the resolvent of the Hamiltonian $H_{\alpha,\alpha'} = H_0 + V_{\alpha,\alpha'}$ of the independent particle pairs α and α'. The properties of the operators $N_{\alpha,\alpha'}(z)$ and $G_{\alpha,\alpha'}(z)$ have already been studied in detail (see Refs. 11–14), and therefore we formulate only the final result.

In accordance with Refs. 11–14, the representation of the Green's function $G_{\alpha,\alpha'}(z)$ as a convolution of the Green's functions of the independent particle pairs α and α',

$$G_{\alpha,\alpha'}(E + i\tau) = \int_{-\infty}^{\infty} \frac{d\varepsilon}{-2\pi i}\, g_\alpha(\varepsilon + i\tau_1) \otimes g_{\alpha'}(E - \varepsilon + i\tau_2),$$
$$g_\alpha(z) = \Big(z - \sum_{i\in\alpha} H_{0i} - V_\alpha\Big)^{-1}, \quad \tau = \tau_1 + \tau_2, \quad \tau_1 > 0, \quad \tau_2 > 0, \tag{9}$$

leads to the following representation of the operator $N_{\alpha,\alpha'}(z)$ for $z = E + i\tau$:

$$N_{\alpha,\alpha'}(z) = t_\alpha(z_\alpha) + t_{\alpha'}(z_{\alpha'}) + \mathscr{F}_{\alpha\otimes\alpha'}(z). \tag{10}$$

The operator $\mathscr{F}_{\alpha,\alpha'}(z)$ in (10) has the form

$$\mathscr{F}_{\alpha,\alpha'}(z) = \int_{-\infty}^{\infty} \frac{d\varepsilon}{-2\pi i} [g_{0\alpha}(\varepsilon + i\tau_1) \otimes I_{\alpha'} + I_\alpha \otimes g_{0\alpha'}(E - \varepsilon + i\tau_2)]$$
$$\times \{t_\alpha(\varepsilon + i\tau_1) \otimes t_{\alpha'}(E - \varepsilon + i\tau_2)\}$$
$$\times [g_{0\alpha}(\varepsilon + i\tau_1) \otimes I_{\alpha'} + I_\alpha \otimes g_{0\alpha'}(E - \varepsilon + i\tau_2)]. \tag{11}$$

The symbol $\otimes$ in (9) and (11) denotes the tensor product of the operators; I_α is the identity operator in the Hilbert space $\mathcal{G}_\alpha = \mathbf{L}^2(R^6)$ of the coordinates of particles i and $j (\alpha = ij)$, and similarly for $I_{\alpha'}$. In addition, in (10) and (11) we have used the following notation: $g_{0\alpha}(z) = (z - \Sigma_{i\in\alpha} H_{0i})^{-1}$ is the free Green's function of pair α (motion of the center of mass of the pair not yet separated), and $t_\alpha(z) = V_\alpha + V_\alpha g_\alpha(z) V_\alpha$ is the scattering operator of pair α, which acts on the space $\mathcal{I}_\alpha$. The parameters z_α and $z_{\alpha'}$ in (10) are defined by $z_\alpha = z - \Sigma_{i\in\alpha'} p_i/2m_i$, and similarly for α'. This means that the kernel of the operator $t_\alpha(z_\alpha)$ in the momentum representation can be expressed as follows:

$$\langle p_1p_2p_3p_4 \mid t_\alpha(z) \mid p_1^0p_2^0p_3^0p_4^0\rangle$$
$$= \langle p_ip_j \mid t_\alpha\left(z - \sum_{k'\in\alpha'} \frac{p_{k'}^2}{2m_{k'}}\right) \mid p_i^0p_j^0\rangle \prod_{k'\in\alpha'} \delta(p_{k'} - p_{k'}^0)$$
$$= \langle p_\alpha \mid t_\alpha\left(z - \frac{\mathcal{P}_\alpha^2}{2M_\alpha} - \sum_{k'\in\alpha'} \frac{p_{k'}^2}{2m_{k'}}\right)$$
$$\times \mid p_\alpha^0\rangle\, \delta(\mathcal{P}_\alpha - \mathcal{P}_\alpha^0) \prod_{k'\in\alpha'} \delta(p_k - p_k^0). \quad (12)$$

Here, $\mathcal{P}_\alpha = \Sigma_{l\in\alpha} p_l (\alpha = ij)$ is the c.m.s. momentum of pair α in R^3, $M_\alpha = m_i + m_j$, and $p_\alpha = (p_i m_j - p_j m_j)/M_\alpha$ are the momenta of the relative motion of the particles in pair α in R^3. The kernel of the operator $\mathcal{F}_{\alpha\otimes\alpha'}(z)$ in the momentum representation can be expressed as

$$\mathcal{F}_{\alpha\otimes\alpha'}(p_1p_2p_3p_4;\; p_1^0p_2^0p_3^0p_4^0;\; z)$$
$$= \delta(\mathcal{P} - \mathcal{P}^0)\, \delta(\mathcal{P}_{\alpha,\alpha'} - \mathcal{P}_{\alpha,\alpha'}^0)$$
$$\times \mathcal{F}_{\alpha\otimes\alpha'}\left(p_\alpha p_{\alpha'};\; p_\alpha^0 p_{\alpha'}^0;\; z - \frac{\mathcal{P}^2}{2M} - \frac{\mathcal{P}_{\alpha\alpha'}^2}{2M_{\alpha\alpha'}}\right), \quad (13)$$

where $\mathcal{P} = \Sigma_{i=1}^4 p_i$ is the total momentum of the system in R^{12}, and the momentum p_i is defined in R^3; $\mathcal{P}_{\alpha,\alpha'} = (\mathcal{P}_\alpha M_{\alpha'} - \mathcal{P}_{\alpha'} M_\alpha)/M$, where, $M = M_\alpha + M_{\alpha'}$, is the total mass of the system, and $M_{\alpha,\alpha'} = M_\alpha M_{\alpha'}/M$. In accordance with (11), the kernel $\widetilde{\mathcal{F}}_{\alpha\otimes\alpha'}(p_\alpha p_{\alpha''};\, p_\alpha^0 p_{\alpha'}^0; z)$ has the form

$$\widetilde{\mathcal{F}}_{\alpha\otimes\alpha'}(p_\alpha p_{\alpha'};\; p_\alpha^0 p_{\alpha'}^0;\; z)$$
$$= \int_{-\infty}^{\infty} \frac{d\varepsilon}{-2\pi i} \left(\frac{1}{\varepsilon - \widetilde{p}_\alpha^2 + i\tau_1} + \frac{1}{E - \varepsilon - \widetilde{p}_{\alpha'}^2 + i\tau_2}\right)$$
$$\times t_\alpha(p_\alpha, p_\alpha^0, \varepsilon + i\tau_1)\, t_{\alpha'}(p_{\alpha'}, p_{\alpha'}^0, E - \varepsilon + i\tau_2)$$
$$\times \left(\frac{1}{\varepsilon - \widetilde{p}_\alpha^{02} + i\tau_1} + \frac{1}{E - \varepsilon - \widetilde{p}_{\alpha'}^{02} + i\tau_2}\right). \quad (14)$$

On the basis of Eqs. (10) and (11) and the system (6), we can obtain a system of equations for the operators describing transitions to states in which the particles are distributed in independent pairs. In accordance with (6) and (10),

$$T_{\alpha,\alpha'}(z) = T_\alpha(z) + T_{\alpha'}(z) + T_{\alpha\otimes\alpha'}(z), \quad (15)$$

where $T_\alpha(z)$, $T_{\alpha\otimes\alpha'}(z)$ satisfy the system of equations

$$\left.\begin{aligned} T_\alpha(z) &= t_\alpha(z_\alpha) + t_\alpha(z_\alpha) G_0(z) \sum_{\alpha\neq\beta\neq\alpha'} T_\beta(z) \\ &\quad + t_\alpha(z) G_0(z) \sum_{\alpha\alpha'\neq\beta\beta'} T_{\beta\otimes\beta'}(z); \\ T_{\alpha\otimes\alpha'}(z) &= \mathcal{F}_{\alpha\otimes\alpha'}(z) + \mathcal{F}_{\alpha\otimes\alpha'}(z) G_0 \sum_{\alpha\neq\beta\neq\alpha'} T_\beta(z) \\ &\quad + \mathcal{F}_{\alpha\otimes\alpha'}(z) G_0(z) \sum_{(\alpha,\alpha')\neq(\beta,\beta')} T_{\beta\otimes\beta'}(z). \end{aligned}\right\} \quad (16)$$

The equation for the operator $T_\alpha(z)$ must be subjected to a further rearrangement in order to separate the grouping channels corresponding to three-particle interaction. To this end, we represent the operator $T_\alpha(z)$ in the form

$$T_\alpha(z) = T_\alpha^{(1)}(z) + \sum_{\alpha\in\eta} T_\alpha^\eta(z), \quad (17)$$

where η is a subsystem of three particles. The operators $T_\alpha^{(1)}(z)$ and $T_\alpha^\eta(z)$ in (17) are defined as follows:

$$\left.\begin{aligned} T_\alpha^{(1)}(z) &= t_\alpha(z_\alpha) + t_\alpha(z_\alpha) G_0(z) \sum_{(\alpha,\alpha')\neq(\beta,\beta')} T_{\beta\otimes\beta'(\beta,\beta')}(z); \\ T_\alpha^\eta(z) &= t_\alpha(z_\alpha) G_0(z) \sum_{\beta\in\eta} T_\beta(z). \end{aligned}\right\} \quad (18)$$

The second equation in (18) can be inverted by using the equations for the three-particle scattering operators in the subsystem η. As a result, we arrive at a system of equations for the auxiliary operators $T_\alpha^{(1)}(z)$, $T_\alpha^\eta(z)$, $T_{\alpha\otimes\alpha'}(z)$ in which all possible unbound processes are separated. This system has the form[12–14]

$$T_{\alpha\otimes\alpha'}(z) = \mathcal{F}_{\alpha\otimes\alpha'}(z) + \mathcal{F}_{\alpha\otimes\alpha'}(z) G_0(z)$$
$$\times \Big\{ \sum_{(\alpha,\alpha')\neq(\beta,\beta')} [T_\beta^{(1)}(z) + \sum_{\mu,\,\beta\in\mu} T_\beta^\mu(z)]$$
$$+ \sum_{(\alpha,\alpha')\neq(\beta,\beta')} T_{\beta\otimes\beta'}(z)\Big\};$$
$$T_\alpha^\eta(z) = \sum_{\gamma\in\eta} M_{\alpha,\gamma}^\eta(z_\eta) + \sum_{\gamma\in\eta} M_{\alpha,\gamma}^\eta(z_\eta)\, G_0(z)$$
$$\times \Big\{ \sum_{\substack{(\gamma,\gamma')\neq(\beta,\beta') \\ \beta\notin\eta}} T_\beta^{(1)}(z)$$
$$+ \sum_{\substack{(\gamma,\gamma')\neq(\beta,\beta') \\ \beta\in\eta,\,\eta\neq\mu}} T_\beta^\mu(z) + \sum_{(\gamma,\gamma')\neq(\beta,\beta')} T_{\beta\otimes\beta'}(z)\Big\};$$
$$T_\alpha^{(1)}(z) = t_\alpha(z) + t_\alpha(z_\alpha) G_0(z) \sum_{(\alpha,\alpha')\neq(\beta,\beta')} T_{\beta\otimes\beta'}(z). \quad (19)$$

The operators $M_{\alpha,\gamma}^\eta(z)$ are defined as the connected part of the three-particle amplitude $T_{\alpha\gamma}^\eta(z)$:

$$M_{\alpha,\gamma}^\eta(z) = T_{\alpha,\gamma}^\eta(z) - t_\alpha(z_\alpha)\, \delta(\alpha,\gamma),\;\; T_{\alpha,\gamma_1}^\eta(z)$$
$$= t_\alpha(z_\alpha)\, \delta(\alpha,\gamma) + t_\alpha(z_\alpha)\; G_0(z) \sum_{\beta\neq\alpha} T_{\beta,\gamma}^\eta(z).$$

The parameter z_η in (18) is defined in the same way as the parameter z_α: $z_\eta = z - \Sigma_{i\notin\eta} p_i^2/2m_i$. The kernel of the operator $M_{\alpha,\gamma}^\eta(z_\eta)$ in the momentum representation is expressed as follows:

$$\langle p_1p_2p_3p_4 \mid M_{\alpha,\gamma}^\eta(z_\eta) \mid p_1^0p_2^0p_3^0p_4^0\rangle = \delta(\mathcal{P} - \mathcal{P}^0)\, \delta(\mathcal{P}_\eta - \mathcal{P}_\eta^0)$$
$$\times M_{\alpha,\gamma}^\eta\left(p_\alpha p_{\alpha\eta},\;\; p_\alpha^0 p_{\alpha\eta}^0,\;\; z - \frac{\mathcal{P}^2}{2M} - \frac{\mathcal{P}_\eta^2}{2n_\eta}\right). \quad (20)$$

In (20), $p_{\alpha\eta}$ is the momentum of the relative motion of the center of mass of pair α and the third particle in the subsystem $p_{\alpha\eta} = [\mathcal{P}_\alpha(M_\eta - M_\alpha) - M_\alpha(\mathcal{P}_\eta - \mathcal{P}_\alpha)]/M_\eta$, p_η is the momentum of the relative motion of the center of mass of the three-particle subsystem η and the fourth particle, $p_\eta = [\mathscr{P}_\eta(M - M_\eta) - M_\eta(\mathscr{P} - \mathscr{P}_\eta)]/M$, and finally n_η is the reduced mass of the subsystem η and the fourth particle. In what follows, we shall work in the center-of-mass system, i.e., we shall set $\mathscr{P} = 0$. Then $\mathscr{P}_{\alpha,\alpha'} = \mathscr{P}_\alpha$ and $p_\eta = \mathscr{P}_\eta$. In addition, to simplify the expressions we shall use the notation $\tilde{\mathscr{P}}^2_\alpha = \mathscr{P}^2_\alpha/2M_{\alpha\alpha'}$, $\tilde{p}^2_\alpha = p^2_\alpha/2\mu_\alpha$, $\tilde{\mathscr{P}}^2_\eta \mathscr{P}^2_\eta/2n_\eta$, $\tilde{\mathscr{P}}^2_{\alpha\eta} = \mathscr{P}^2_{\alpha\eta}/2n_\alpha$, where $n_\alpha = M_\alpha(M_\eta - M_\alpha)/M_\eta$, $n_\eta = M_\eta(M - M_\eta)/M$. In this notation, the kinetic-energy operator is

$$H_0 = \tilde{\mathscr{P}}^2_\alpha + \tilde{p}^2_\alpha + \tilde{p}^2_{\alpha'},$$

or

$$H_0 = \tilde{\mathscr{P}}^2_\eta + \tilde{p}^2_\alpha + \tilde{p}^2_{\alpha\eta}.$$

Properties of the kernels of the system of equations for the auxiliary operators in the four-body problem

Since our main aim is to obtain a uniquely solvable system of integral equations with a compact kernel, we must subject the system of equations for the auxiliary operators (19) to a further rearrangement. The first step is to subtract the scattering operators corresponding to the unbound processes. Introducing the operators $\mathscr{K}_\alpha(z)$, $\mathscr{K}^\eta_\alpha(z)$, $\mathscr{K}_{\alpha\otimes\alpha'}(z)$ by

$$\mathscr{K}_\alpha(z) = T^{(1)}_\alpha(z) - t_\alpha(z);$$

$$\mathscr{K}^\eta_\alpha(z) = T^\eta_\alpha(z) - \sum_{\gamma\in\eta} M^\eta_{\alpha\gamma}(z);$$

$$\mathscr{K}_{\alpha\otimes\alpha'}(z) = T_{\alpha\otimes\alpha'}(z)$$

we arrive at a system of equations for these operators. We write the system in the form

$$\mathscr{K}(z) = \mathscr{K}_0(z) + A(z)\mathscr{K}(z), \tag{21}$$

where $\mathscr{K}(z)$ is a vector function whose elements are six operators of the type $K_\alpha(z)$, 12 operators of the type $\mathscr{K}^\eta_\alpha(z)$, and three operators $K_{\alpha\otimes\alpha'}(z)$; $A(z)$ is a 21×21 operator matrix. The explicit form of the elements of the vector function $\mathscr{K}_\alpha(z)$ and $A(z)$ can be readily established by means of the system of equations (19) and the definition of an operator of the type $\mathscr{K}$.

The next step is to go over to the components of the operators $\mathscr{K}_\alpha(z)$, $\mathscr{K}^\eta_\alpha(z)$, $\mathscr{K}_{\alpha\otimes\alpha'}(z)$ by separating the primary singularities in the kernels of these operators in the momentum representation. We recall that the primary singularities arise because of the existence of bound states in the subsystems (Refs. 12, 13, 22, and 23). In the case under consideration, these are subsystems of two and three particles and two independent pairs of particles. In accordance with the quoted papers, the primary singularities can be expressed in the form $[z - \varepsilon - f(p)]^{-1}$, where ε is the energy of a bound state in one of the subsystems, and $f(p)$ is a quadratic form in the momenta of the particles. If the number of bound states in the subsystems is finite and at the same time all $\varepsilon < 0$, then the primary singularities do not intersect the singularities of the free Green's function $G_0(z)$ and can be explicitly separated.[12,22] The corresponding procedure is what leads to the appearance of the components, the number of which is finite. The procedure for separating the components can be applied not only to the total T scattering operator of the system but also to certain auxiliary operators, into which the operator $T(z)$ is decomposed and for which a system of integral equations is formulated. We shall work mainly with the components of the auxiliary operators.

To solve the problem posed—the decomposition of the operators of type $\mathscr{K}$ into components—it is necessary to know the properties of the kernels of the system of equations (21). We now go over to the study of these properties.

The properties of the scattering operator of a two-particle system are well known.[22,24] However, before we formulate them, let us discuss the properties of the two-particle bound states. It follows from the results of Ref. 2, Chap. 13, and Ref. 25 that in the employed class of potentials the discrete spectrum $\sigma_d(h_\alpha)$ of the Hamiltonian h_α of the internal motion in pair α, $h_\alpha = \tilde{p}^2_\alpha + V_\alpha$, consists of not more than a finite number of eigenvalues of finite multiplicity, and among them there are no positive eigenvalues. As usual in such cases, we shall assume that there is precisely one eigenvalue of unit multiplicity: $-\kappa^2_\alpha \in \sigma_d(h_\alpha)$ with wave function $|\psi_\alpha\rangle$. In addition, we shall assume that none of the two-particle subsystems have virtual states with zero energy (for a discussion of these, see Ref. 24). Then we represent the kernel $t_\alpha(p_\alpha, p^0_\alpha, z)$ in the form

$$t(p, p^0, z) = \frac{\varphi(p)\varphi^*(p^0)}{z+\varkappa^2} + \hat{t}(p, p^0, z), \tag{22}$$

where $\varphi(p) = \langle p|V|\psi\rangle$, and $\hat{t}(p, p^0, z)$ is the nonsingular part of the two-particle amplitude. (If the two-particle system has a virtual state with zero energy, this state this leads to the appearance in $t(z)$ of a singularity of the form $z^{-1/2}$.[24]) The pole singularity in (22) of the form $(z+\kappa^2)^{-1}$ leads to the appearance in Eq. (21) of primary singularities of the form $(z + \kappa^2_\alpha - \tilde{\mathscr{P}}^2_\alpha - \tilde{p}^2_\alpha)^{-1}$.

The properties of the kernels $\mathscr{F}_{\alpha\otimes\alpha'}(z)$ given by (13) and (14) were studied in detail in Refs. 12, 13, 20, 26, and 27. The results of these studies are as follows. The operator $\mathscr{F}_{\alpha\otimes\alpha'}(z)$ can be represented in the form

$$\begin{aligned}\mathscr{F}_{\alpha\otimes\alpha'}(p_\alpha p_{\alpha'}, \cdot, z) &= \frac{\zeta_{\alpha,\alpha'}(p_\alpha p_{\alpha'})\zeta^*_{\alpha\alpha'}(\cdot)}{z+\varkappa^2_\alpha+\varkappa^2_{\alpha'}} + L_{\alpha\alpha'}(p_\alpha p'_{\alpha'}, \cdot, z)\\ &+ \frac{\varphi(p_\alpha)L_{\alpha'}(p_{\alpha'}, \cdot, z)}{z+\varkappa^2_\alpha-\tilde{p}^2_{\alpha'}} + \frac{\varphi(p_{\alpha'})L_\alpha(p_\alpha, \cdot, z)}{z+\varkappa^2_{\alpha'}-\tilde{p}^2_\alpha},\end{aligned} \tag{23}$$

where the operators $L_\beta(z)$, $\beta = \alpha, \alpha', (\alpha, \alpha')$ are the "out" components of the operator $\mathscr{F}_{\alpha\otimes\alpha'}(z)$. Each of these operators, in turn, can be decomposed into "in" components as follows:

$$\begin{aligned}L_\beta(\cdot, p^0_\alpha p^0_{\alpha'}, z) &= \frac{L^\alpha_\beta(\cdot, p^0_\alpha, z)\varphi^*_{\alpha'}(p^0_{\alpha'})}{z+\varkappa^2_{\alpha'}-\tilde{p}^{02}_\alpha}\\ &+ \frac{L^{\alpha'}_\beta(\cdot, p^0_{\alpha'}, z)\varphi^*_\alpha(p^0_\alpha)}{z+\varkappa^2_\alpha-\tilde{p}^{02}_{\alpha'}} + L^{\alpha,\alpha'}_\beta(\cdot, p^0_\alpha p^0_{\alpha'}, z).\end{aligned} \tag{24}$$

The function $\zeta_{\alpha,\alpha'}(P_\alpha, P_{\alpha'})$ has the form

$$\zeta_{\alpha,\alpha'}(p_\alpha, p_{\alpha'}) = \langle p_\alpha p_{\alpha'} | V_{\alpha,\alpha'} | \psi_\alpha \psi_{\alpha'}\rangle.$$

The expressions for the operators of type L that occur in the decomposition (23) are given in Appendix A. It follows from (13) and (23) that the kernel $\mathscr{F}_{\alpha\otimes\alpha'}(z)$ leads not only to primary singularities associated with the two-particle bound states but also to a primary singularity of the form $(z+\kappa_\alpha^2+\kappa_{\alpha'}^2-\tilde{\mathscr{P}}_\alpha^2)^{-1}$.

We now turn to the study of the properties of the kernel $M^{\eta}_{\alpha,\gamma}(z)$. As in the two-particle case, we shall be interested above all in the questions of the finiteness of the number of bound states in the three-particle subsystems and the absence of bound states at positive energies. We denote by h_η the Hamiltonian of the internal motion in the three-particle subsystem η, and by $\sigma_{\rm d}(h_\eta)$ its discrete spectrum. In the class of potentials employed (and in the absence of virtual states with zero energy in the two-particle subsystems) the negative discrete spectrum of the operator h_η [i.e., $\sigma_{\rm d}(h_\eta)\cap(-\infty,0)$] is finite, and each eigenvalue has finite multiplicity (Refs. 25, 28, and 30–32). The absence of bound states at positive energies can be proved for a fairly narrow class of potentials; among the most widely used nuclear potentials, only the Yukawa potential (and its linear combinations) are included in this class (Ref. 2, Chap. 13, §13). We shall assume that h_η does not have eigenvalues on the half-line $[0,\infty)$; moreover, we shall assume that $\sigma_{\rm d}(h_\eta)$ consists of a unique eigenvalue $-\kappa_\eta^2$ of unit multiplicity, and $-\kappa_\eta^2<\Sigma_\eta$, where Σ_η is the starting point of the continuous spectrum of $h_\eta:\Sigma_\eta=\min\limits_{\alpha\in\eta}(-\kappa_\alpha^2)$.

Under the assumptions that we have made, the operators $M^{\eta}_{\alpha,\gamma}(z)$, have a decomposition of the form[9,12]

$$\begin{aligned}M^{\eta}_{\alpha,\gamma}(p_\alpha,\ p_{\alpha\eta},\ p^0_\gamma,\ p^0_{\gamma\eta},\ z)&=\frac{\langle p_\alpha,\ p_{\alpha\eta}\mid V_\alpha P_\alpha V_\gamma\mid p^0_\gamma p^0_{\gamma\eta}\rangle}{z+\varkappa_\eta^2}\\&+\frac{\varphi_\alpha(p_\alpha)\mathscr{I}^{(1)}_{\alpha,\gamma}(p_{\alpha\eta},\ p^0_\gamma,\ p^0_{\gamma\eta}z)}{z+\varkappa_\alpha^2-\tilde{p}^2_{\alpha\eta}}+\frac{\mathscr{I}^{(2)}_{\alpha,\gamma}(p_\alpha p_{\alpha\eta},\ p^0_{\gamma\eta},\ z)\varphi^*_\gamma(p^0_\gamma)}{z+\varkappa_\gamma^2-\tilde{p}^{0\,2}_{\gamma\pi}}\\&+\frac{\varphi_\alpha(p_\alpha)\mathscr{I}^{(3)}_{\alpha,\gamma}(p_{\alpha\eta},\ p^0_{\gamma\eta}z)\varphi^*_\gamma(p^0_\gamma)}{(z+\varkappa_\alpha^2-\tilde{p}^2_{\alpha\eta})(z+\varkappa_\gamma^2-\tilde{p}^{0\,2}_{\gamma\eta})}+\mathscr{I}^0_{\alpha,\gamma}(p_\alpha p_{\alpha\eta},\ p^0_\gamma p_{\gamma\eta},\ z).\end{aligned}\tag{25}$$

In (25), $P_{\rm d}$ is the projection operator onto the discrete spectrum $\sigma_{\rm d}(h_\eta)$, which in our case can be represented in the form $P_{\rm d}=|\psi_\eta\rangle\langle\psi_\eta|,\ h_\eta|\psi_\eta\rangle=-\kappa_\eta^2|\psi_\eta\rangle$. The components $\mathscr{I}^{i}_{\alpha,\gamma}(z)$, $i=0,1,2,3$, can be expressed in terms of the solutions of the Faddeev equations for physical values of the energy, i.e., when $z=E\pm i0$ and $E\in(\Sigma_\eta,\infty)$. The corresponding treatment is given in Ref. 12; its main features are reproduced in Appendix B. In accordance with (20) and (25), the three-particle bound state in the subsystem η leads to the appearance of a primary singularity of the form $(z+\kappa_\eta^2-\tilde{\mathscr{P}}_\eta^2)^{-1}$.

Decomposition of auxiliary operators of the type $\mathscr{K}$ into components

On the basis of the properties of the kernels of the system of equations (21), it is possible to separate in this system all the primary singularities, and this brings with it a decomposition of the auxiliary operators of the type $\mathscr{K}$ (20) into components.

In accordance with (22) and the equation for the operator $\mathscr{K}_\alpha(z)$, this operator decomposes into components as follows:

$$\mathscr{K}_\alpha(\mathscr{P}_\alpha p_\alpha p_{\alpha'},\ \cdot,\ z)=u_\alpha(\mathscr{P}_\alpha p_\alpha p_{\alpha'},\ \cdot z)+\frac{\varphi_\alpha(p_\alpha)v_{\alpha'}(\mathscr{P}_\alpha p_{\alpha'},\ \cdot,\ z)}{z+\varkappa_\alpha^2-\tilde{\mathscr{P}}_\alpha^2-\tilde{p}^2_{\alpha'}},\tag{26}$$

where

$$\begin{aligned}&u_\alpha(\mathscr{P}_\alpha p_\alpha p_{\alpha'},\ \cdot,\ z)\\&=\langle\mathscr{P}_\alpha p_\alpha p_{\alpha'}\mid\hat{t}_\alpha(z_\alpha)G_0(z)\sum_{[(\alpha,\ \alpha')\neq\beta,\ \beta'}T_{\beta\otimes\beta'}(z)\mid\cdot\rangle,\\&v_{\alpha'}(\mathscr{P}_\alpha p_{\alpha'},\ \cdot,\ z)=\langle\varphi_\alpha\mathscr{P}_\alpha p_{\alpha'}\mid G_0(z)\sum_{\alpha,\ \alpha'\neq\beta\beta'}T_{\beta\otimes\beta'}(z)\mid\cdot\rangle\end{aligned}$$

$$.\tag{27}$$

and $|\cdot\rangle$ denotes a certain set of relative momenta of the particles in the initial state. The decomposition (23) of the kernel $\tilde{\mathscr{F}}_{\alpha\otimes\alpha'}(z)$ generates a decomposition of the operator $\mathscr{K}_{\alpha\otimes\alpha'}(z)$ into components of the form

$$\begin{aligned}\mathscr{K}_{\alpha\otimes\alpha'}(\mathscr{P}_\alpha p_\alpha p_{\alpha'},\ \cdot,\ z)&=u_{\alpha,\ \alpha'}(\mathscr{P}_\alpha p_\alpha p_{\alpha'},\ \cdot,\ z)\\&+\frac{\varphi_\alpha(p_\alpha)v^{\alpha'}_{\alpha\alpha'}(\mathscr{P}_\alpha p_{\alpha'},\ \cdot,\ z)}{z+\varkappa_\alpha^2-\tilde{\mathscr{P}}^2_\alpha-\tilde{p}^2_{\alpha'}}+\frac{\varphi_{\alpha'}(p_{\alpha'})v^{\alpha}_{\alpha\alpha'}(\mathscr{P}_\alpha p_\alpha,\ \cdot,\ z)}{z+\varkappa^2_{\alpha'}-\tilde{\mathscr{P}}^2_\alpha-\tilde{p}^2_\alpha}\\&+\frac{\zeta_{\alpha,\ \alpha'}(p_\alpha p_{\alpha'})v_{\alpha\alpha'}(\mathscr{P}_\alpha,\ \cdot,\ z)}{z+\varkappa^2_\alpha+\varkappa^2_{\alpha'}-\tilde{\mathscr{P}}^2_\alpha}.\end{aligned}\tag{28}$$

The components of the operator $\mathscr{K}_{\alpha\otimes\alpha'}(z)$ are given by

$$\begin{aligned}u_{\alpha,\ \alpha'}(\mathscr{P}_\alpha p_\alpha p_{\alpha'},\ \cdot,\ z)&=\langle\mathscr{P}_\alpha p_\alpha p_{\alpha'}\mid L_{\alpha,\ \alpha'}(z)G_0(z)\\&\times[\sum_{(\alpha\alpha')\neq(\beta,\ \beta')}T^{(1)}_\beta(z)+\sum_{\substack{(\alpha,\alpha')\neq(\beta,\ \beta')\\ \beta\in\mu}}T^{\mu}_\beta(z)\\&+\sum_{(\alpha,\ \alpha')\neq(\beta,\ \beta')}T_{\beta\otimes\beta'}(z)]\mid\cdot\rangle;\\v_{\alpha,\ \alpha'}(\mathscr{P}_\alpha,\ \cdot,\ z)&=\langle\zeta_{\alpha,\ \alpha'}\mathscr{P}_\alpha\mid G_0(z)[\ldots]\mid\cdot\rangle;\\v^{\alpha}_{\alpha,\ \alpha'}(\mathscr{P}_\alpha p_\alpha,\ \cdot,\ z)&=\langle\varphi_\alpha\mathscr{P}_\alpha p_\alpha\mid L_\alpha(z)G_0(z)[\ldots]\mid\cdot\rangle.\end{aligned}\tag{29}$$

The square brackets in the last two expressions of (29) are the same as in the definition for $u_{\alpha,\alpha'}(z)$. The operator $\mathscr{K}^{\eta}_{\alpha}(z)$ decomposes into components as follows:

$$\begin{aligned}\mathscr{K}^{\eta}_{\alpha}(\mathscr{P}_\eta p_\alpha p_{\alpha\eta},\ \cdot,\ z)&=u^{\eta}_{\alpha}(\mathscr{P}_\eta p_\alpha p_{\alpha\eta},\ \cdot,\ z)\\&+\frac{\Phi_{\alpha\eta}(p_\alpha p_{\alpha\eta})v_\eta(\mathscr{P}_\eta,\ \cdot,\ z)}{z+\varkappa^2_\eta-\tilde{\mathscr{P}}^2_\eta}\\&+\frac{\varphi_\alpha(p_\alpha)v^{\eta}_{\alpha}(\mathscr{P}_\eta p_{\alpha\eta},\ \cdot,\ z)}{z+\varkappa^2_\alpha-\tilde{\mathscr{P}}^2_\eta-\tilde{p}^2_{\alpha\eta}},\end{aligned}\tag{30}$$

where

$$\left.\begin{aligned}&v_\eta(\mathscr{P}_\eta,\ \cdot,\ z)=\langle\psi_\eta\mathscr{P}_\eta\mid\sum_{\gamma\neq\eta}V_\gamma G_0(z)\\&\times[\sum_{(\gamma,\ \gamma')\neq(\beta,\ \beta')}T^{(1)}_\beta(z)+\sum_{\substack{(\gamma,\ \gamma)\neq(\beta,\ \beta')\\ \beta\notin\eta,\ \beta\in\mu}}T^{\mu}_\beta(z)\\&+\sum_{(\gamma,\ \gamma')\neq(\beta,\ \beta')}T_{\beta\otimes\beta'}(z)]\mid\cdot\rangle;\\&u^{\eta}_{\alpha}(\mathscr{P}_\eta p_\alpha p_{\alpha\eta},\ \cdot,\ z)\\&=\langle\mathscr{P}_\eta p_\alpha p_{\alpha\eta}\mid\sum_{\gamma\in\eta}\left\{\mathscr{I}^0_{\alpha,\gamma}(z_\eta)+\mathscr{I}^{(2)}_{\alpha,\gamma}(z_\eta)\frac{\langle\varphi_\gamma\mid}{z_\eta+\varkappa^2_\gamma-h_{0\gamma\eta}}\right\}\\&\times G_0(z)[\ldots]\mid\cdot\rangle;\\&v^{\eta}_{\alpha}(\mathscr{P}_\eta p_{\alpha\eta},\ \cdot,\ z)=\langle\mathscr{P}_\eta p_{\alpha\eta}\mid\\&\times\sum_{\gamma\in\eta}\left\{\mathscr{I}^{(1)}_{\alpha,\gamma}(z_\eta)+\mathscr{I}^{(3)}_{\alpha,\gamma}(z_\eta)\frac{\langle\varphi_\gamma\mid}{z_\eta+\varkappa^2_\gamma-h_{0\gamma\eta}}\right\}\\&\times G_0(z)[\ldots]\mid\cdot\rangle.\end{aligned}\right\}\tag{31}$$

The square-bracket convention in (31) is the same as in (29). The operator $h_{0\gamma\eta}$ in (31) is the operator of the kinetic energy of the relative motion of the center of mass of pair γ and the third particle in the subsystem η, i.e., $\tilde{p}^2_{\gamma\eta}$.

In (26), (28), and (30), the primary singularities are separated only in the final state; the primary singularities can be separated similarly in the initial state. As a result of the procedure for separating the primary singularities described above, the 21 auxiliary operators of the type $\mathscr{K}$ are replaced by 60 components of the types u and v. For these components, using (19), (21), (26), (28), and (30) and the definitions of operators of the type $\mathscr{K}$, we can readily obtain a system of integral equations of the form

$$U(z) = U_0(z) + W(z)\,U(z), \tag{32}$$

which is to be considered in the auxiliary space of 52 functions of the types u and v. For comparison, we note that in the case of the Yakubovskiĭ equations[33] the system of 18 equations for the auxiliary operators yields a system of 43 integral equations for the components [under the same conditions on $\sigma_d(h_\alpha)$ and $\sigma_d(h_\eta)$]. In the general case, the number of components obtained from the system of equations (21) is

$$N_1 = 21 + 4\sum_\alpha n_\alpha + \sum_{\alpha\alpha'} n_\alpha n_{\alpha'} + \sum_\eta n_\eta, \tag{33}$$

where the number of states in $\sigma_d\ (n_\alpha)$ is denoted by n_α, and the number of states in $\sigma_d(n_\eta)$ by n_η, whereas the equations in Ref. 33 decompose into

$$N_2 = 18 + 3\sum_\alpha n_\alpha + \sum_{\alpha\alpha'} n_\alpha n_{\alpha'} + \sum_\eta n_\eta \tag{34}$$

components. The expression (34) is obtained under the assumption that the kernels of the types $N^{12,34}_{12,12}$ and $N^{12,34}_{12,34}$ in Ref. 33 are constructed by convolution of the Green's functions $g_{12}(z)$ and $g_{34}(z)$ [see the expression (9)]. In this case, the components of these operators can be readily obtained from the expressions given in Appendix A. In the original variant, it was proposed in Ref. 33 that the kernels $N^{12,34}_{(\cdot)}(z)$ should be sought as the solutions of a system of integral equations, as a result of which the number of components is increased to $N'_2 = N_2 + 6 + \Sigma_\alpha n_\alpha$. It is readily seen that $N_2 < N_1 < N'_2$. We continue the comparison of Eqs. (21) in Ref. 33 in Appendix C, in which we show that in the equations from Ref. 33 there is an unphysical separation of the asymptotic behaviors.

The system of integral equations (32) for components of the types u and v obtained by separating in the "in" state a primary singularity of the form $(z + \varkappa^2_\alpha + \varkappa^2_{\alpha'} - \tilde{\mathscr{P}}^{02}_{\alpha\alpha'})^{-1}$, is given in Appendix D.

We shall discuss the properties of the system (32), (C.1)–(C.3) below; we here establish the connection between the components u and v and the elements of the S matrix of the system. To be specific, we shall assume that these components satisfy the system of integral equations (D.1)–(D.3). In this system we set $z = -\varkappa^2_\alpha - \varkappa^2_{\alpha'} + \tilde{\mathscr{P}}^0_\alpha + i0$. Then for the elements of the S matrix of the system we can write down expressions of the form

$$\begin{aligned}
&S_{(\beta+\beta')\leftarrow(\alpha+\alpha')}(\mathscr{P}_\beta,\ \mathscr{P}^0_\alpha) = \delta(\mathscr{P}_\beta - \mathscr{P}^0_\alpha)\,\delta[(\beta\beta'),\ (\alpha,\ \alpha')]\\
&\qquad - 2\pi i\delta(\tilde{\mathscr{P}}^2_\beta - \tilde{\mathscr{P}}^{02}_\alpha)\,v_{\beta\beta'}(\mathscr{P}_\beta,\ \mathscr{P}^0_\alpha,\ E + i0);\\
&S_{\eta\leftarrow(\alpha+\alpha')}(\tilde{\mathscr{P}}_\eta,\ \tilde{\mathscr{P}}^0_\alpha) = -2\pi i\delta(\tilde{\mathscr{P}}_\eta - \varkappa^2_\eta - E)\\
&\qquad \times v_\eta(\mathscr{P}_\eta,\ \mathscr{P}^0_\alpha,\ E + i0);\\
&S_{\beta\leftarrow(\alpha+\alpha')}(\mathscr{P}_\beta p'_\beta \mathscr{P}^0_\alpha,\ E + i0) = 2\pi i\delta(\tilde{\mathscr{P}}^2_\beta + \tilde{p}^2_{\beta'} - \varkappa^2_\beta - E)\\
&\quad \times [v_{\beta'}(\mathscr{P}_\beta p_{\beta'},\ \mathscr{P}^2_\alpha,\ E + i0) + v^{\beta'}_{\beta\beta'}(\mathscr{P}_\beta p_{\beta'},\ \mathscr{P}^0_\alpha,\ E + i0)\\
&\qquad + \sum_{\beta\in\eta} v^\eta_\beta(\mathscr{P}_\eta p_{\beta\eta},\ \mathscr{P}^0_\alpha,\ E + i0)];\\
&S_{0\leftarrow(\alpha+\alpha')}(\mathscr{P}_\beta p_\beta p_{\beta'},\ \mathscr{P}^0_\alpha,\ E + i0)\\
&\quad = -2\pi i\delta(\tilde{\mathscr{P}}^2_\beta + \tilde{p}^2_{\beta'} + \tilde{p}^2_\beta - E)\\
&\qquad \times \{\sum [u_\beta(\mathscr{P}_\beta p_\beta p_{\beta'},\ \mathscr{P}^0_\alpha,\ E + i0)\\
&\qquad - \psi_\beta(p_\beta)\,v_{\beta'}(\mathscr{P}_\beta p_{\beta'} \mathscr{P}^0_\alpha,\ E + i0)]\\
&\qquad + \sum_{\beta\beta'} [u_{\beta\beta'}(\mathscr{P}_\beta p_\beta p_{\beta'},\ \mathscr{P}^0_\alpha,\ E + i0)\\
&\qquad - \psi_\beta(p_\beta)\,v^{\beta'}_{\beta\beta'}(\mathscr{P}_\beta p_{\beta'},\ \mathscr{P}^0_\alpha,\ \mathscr{P}^0_\alpha,\ E + i0)\\
&\qquad - \psi_{\beta'}(p_{\beta'})\,v^{\beta}_{\beta\beta'}(\mathscr{P}_\beta p_\beta,\ \mathscr{P}^0_\alpha,\ E + i0)\\
&\qquad - \psi_\beta(p_\beta)\,\psi_{\beta'}(p_{\beta'})\,v_{\beta\beta'}(\mathscr{P}_\beta,\ \mathscr{P}^0_\alpha,\ E + i0)]\\
&\qquad + \sum_{\beta\in\eta} \Big[u^\eta_\beta(\mathscr{P}_\eta p_\beta p_{\beta\eta},\ \mathscr{P}^0_\alpha,\ E + i0)\\
&\qquad - \psi_\beta(p_\beta)\,v^\eta_\beta(\mathscr{P}_\eta p_{\beta\eta},\ \mathscr{P}^0_\alpha,\ E + i0)\\
&\qquad + \frac{\langle p_\beta p_{\beta\eta} \mid V_\beta \mid \Phi_\eta\rangle\, v_\eta(\mathscr{P}_\eta,\ \mathscr{P}^0_\alpha,\ E + i0)}{E + \varkappa^2_\eta - \mathscr{P}^2_\eta - \tilde{p}^2_{\beta\eta}}\Big]\Big\}.
\end{aligned} \tag{35}$$

The relations (35) are a consequence of the general properties of the components of the total T scattering operator of the N-particle system.[12,22,23,27] As an example, we shall prove the first of the relations (35); the validity of the remainder can be established similarly. It is readily seen that the component $v_{\beta\beta'}(\mathscr{P}_\beta \mathscr{P}^0_\alpha, z)$ arises on the separation of primary singularities of the form $(z + \varkappa^2_\beta + \varkappa^2_{\beta'} - \tilde{\mathscr{P}}^2_\beta)^{-1}$ in the "out" state and of the form $(z + \varkappa^2_\alpha + \varkappa^2_{\alpha'} - \tilde{\mathscr{P}}^{02}_\alpha)^{-1}$ in the "in" state in the operator $T_{\beta\otimes\beta'/\alpha\otimes\alpha'}(z)$, where the auxiliary operators of the type $T_{(\cdot)/\alpha\otimes\alpha'}$ satisfy the system of equations (19) under the condition that all the free terms in this system, excluding $\mathscr{F}_{\alpha\otimes\alpha'}(z)$, are zero. Further, we express the operators of the type $T_{\beta\otimes\beta/\gamma\otimes\gamma}(z)$ in terms of the operators $T_{\beta\beta'/\gamma,\gamma'}(z)$. The latter satisfy the system of equations (6) under the condition that all the free terms in this system, excluding $N_{\gamma,\gamma'}(z)$, are zero. Since

$$T_{\beta,\,\beta'/\gamma,\,\gamma'}(z) = V_{\beta,\,\beta}\delta[(\beta\beta'),\ (\gamma\gamma']] + V_{\beta\beta'}G(z)\,V_{\gamma\gamma'},$$

where $G(z) = (z - H)^{-1}$ is the total Green's function of the system, we obtain for the operator $v_{\beta,\beta'}(z)$ as a result of these operations a representation of the form

$$\begin{aligned}
&v_{\beta\beta'}(\mathscr{P}_\beta \mathscr{P}^0_\alpha z) = \langle \zeta_{\beta\beta'} \mathscr{P}_\beta \mid G_0(z) \mid \mathscr{P}^0_\alpha \zeta_{\alpha\alpha'}\rangle\\
&\quad \times \left[\begin{Bmatrix} 1, & (\alpha\alpha') \neq (\beta\beta') \\ 0, & (\alpha\alpha') = (\beta\beta') \end{Bmatrix}\right.\\
&\quad + \langle \zeta_{\beta\beta'} \mathscr{P}_\beta \mid G_0(z) \Big[\sum_{\substack{\beta\beta' \neq \gamma\gamma' \\ \alpha,\,\alpha' \neq \gamma\gamma'}} V_{\gamma\gamma'} + V^{\beta\beta'} G(z)\, V^{\alpha\alpha'}\Big]\\
&\qquad \times G_0(z) \mid \zeta_{\alpha\alpha'} \mathscr{P}^0_\alpha\rangle,
\end{aligned}$$

where $V^{\beta\beta'} = V - V_{\beta\beta'}$. Ultimately, for $v_{\beta\beta'}(\mathscr{P}_\beta \mathscr{P}^0_\alpha, E + i0)$ when $\mathscr{P}^2_\beta - \kappa^2_\beta - \kappa^2_{\beta'} \widetilde{\mathscr{P}}^{02}_\alpha - \kappa^2_\alpha - \kappa^2_{\alpha'}$ we obtain

$$\begin{aligned} v_{\beta\beta'}(\mathscr{P}_\beta \mathscr{P}^0_\alpha, E + i0) &= \langle \psi_\beta \psi_{\beta'} \mathscr{P}^0_\beta \mid (E - H_0) \mid \mathscr{P}^0_\alpha \psi_\alpha \psi_{\alpha'} \rangle \\ &\times \begin{bmatrix} 1, (\alpha, \alpha') \neq (\beta, \beta') \\ 0, (\alpha, \alpha') = (\beta, \beta') \end{bmatrix} + \langle \psi_\beta \psi_\beta \mathscr{P}_\beta \mid V^{\beta, \beta'} \\ &+ V^{\beta\beta'} G(E + i0) V^{\alpha\alpha'} \mid \mathscr{P}^0_\alpha \psi_\alpha \varphi_{\alpha'} \rangle \\ &= \langle \psi_\beta \psi_{\beta'} \mathscr{P}_\beta \mid V^{\beta\beta'} + V^{\beta\beta'} G(E + i0) V^{\alpha\alpha'} \mid \mathscr{P}^0_\alpha \psi_\alpha \psi_{\alpha'} \rangle, \end{aligned}$$

in agreement with the expression for the amplitude of the process $\alpha + \alpha' \rightarrow \beta + \beta'$ in Refs. 8 and 21.

Properties of the systems of integral equations (21) and (32)

We begin the discussion of the properties of the systems of integral equations for the auxiliary operators, (21), and for the components, (32), with the case $\operatorname{Im} z \neq 0$. The following result can be proved by the method of Refs. 3, 22, and 38.

The kernel of the system of integral equations (21) is a Hilbert-Schmidt operator in the space $\mathbf{L}^2_{21}(R^9)$, where $\mathbf{L}^2_r = \underbrace{\mathbf{L}^2 \otimes \cdots \otimes \mathbf{L}^2}_{r}$ if $\operatorname{Im} z \neq 0$ or $z - \Sigma \in \mathbf{R}^-$, where Σ is the point at which the continuous spectrum of the system begins.

Further, the homogeneous systems of equations corresponding to the systems (21) and (32) cannot have nontrivial solutions for $\operatorname{Im} z \neq 0$.

Indeed, it is easy to show that the two homogeneous systems can have nontrivial solutions only simultaneously, the connection between these solutions being given by relations analogous to (27), (29), and (31). In Refs. 14 and 20, it was shown that the function $\psi(z)$ constructed from the solution of the homogeneous system of equations

$$\widetilde{\mathscr{K}}(z) = A(z)\, \widetilde{\mathscr{K}}(z)$$

in the manner

$$\psi(z) = G_0(z) \Big[\sum_\alpha \widetilde{\mathscr{K}}_\alpha(z) + \sum_{\eta, \alpha \in \eta} \widetilde{\mathscr{K}}^\eta_\alpha(z) + \sum_{\alpha, \alpha'} \widetilde{\mathscr{K}}_{\alpha \otimes \alpha'}(z) \Big]$$

satisfies the Schrödinger equation

$$(z - H)\, \psi(z) = 0. \tag{36}$$

At the same time, it can be shown that if $\operatorname{Im} z \neq 0$ then $(1 + H_0)\psi(z) \in \mathbf{L}^2(R^9)$ and, therefore, $\psi(z)$ belongs to the region in which the Hamiltonian $H(1)$ is self-adjoint. From (36), we then obtain $\psi(z) = 0$, from which it is readily deduced that all the $\widetilde{\mathscr{K}}(z)$ vanish.[14] Thus, the systems of integral equations (21) and (32) do not have spurious solutions (with regard to the latter, see Refs. 39–41).

Now suppose $z = E + i0$ and $E \in \mathbf{R}^-$. In this case, the system of integral equations for the components (32) belongs to the class of integral equations with a fixed singularity and, therefore, is a Fredholm system.[42] In particular, in the given interval of energies the system (32) can be transformed into a system of integral equations whose kernels do not contain singularities at all. Transformations of such type are described in Refs. 43–46. The authors of Refs. 43 and 44 proceed from a transformation of the kernels, while in Refs. 45 and 46, which are based on Kantorovich's method,[47] the required components are transformed. Note that the method of Refs. 45 and 46 (after expansion with respect to partial waves) leads to a convenient separation of on- and off-shell effects.

The question of the uniqueness of the solution of the system of equations (32) for $z = E + i0$ and $E \in \mathbf{R}^-$ can be solved by the technique described in Ref. 22 and used in Ref. 20 to study the system of integral equations for the scattering amplitudes of two particles in an external field. For $E < \Sigma$, the situation is completely analogous to the case of complex z. Namely, if the homogeneous system of equations

$$U(z) = W(z)\, U(z) \tag{37}$$

has a nontrivial solution, then $z \in \sigma_{\text{disc}}(H)$ since the wave function $\psi(z)$ of the system constructed from $U(z)$,

$$\begin{aligned} \psi(\mathscr{P}_\alpha p_\alpha p_{\alpha'}, z) &= (z - \widetilde{\mathscr{P}}^2_\alpha - p^2_{\alpha'} - \widetilde{p}^2_\alpha)^{-1} \\ \times \Big\{ \sum_{\alpha\alpha'} \Big[& u_\alpha(\mathscr{P}_\alpha p_\alpha p_{\alpha'}, z) + \frac{\varphi_\alpha(p_\alpha)\, v_{\alpha'}(\mathscr{P}_\alpha p_{\alpha'}, z)}{z + \varkappa^2_\alpha - \widetilde{\mathscr{P}}^2_\alpha - \widetilde{p}^2_{\alpha'}} \\ & + u_{\alpha'}(\mathscr{P}_\alpha p_\alpha p_{\alpha'}, z) \\ & + \frac{\zeta_{\alpha\alpha'}(p_\alpha p_{\alpha'})\, v^{\alpha\alpha'}_{\alpha\alpha'}(\mathscr{P}_\alpha, z)}{z + \varkappa^2_\alpha + \varkappa^2_{\alpha'} - \widetilde{\mathscr{P}}^2_\alpha} + \frac{\varphi_{\alpha'}(p'_\alpha)\, v_\alpha(\mathscr{P}_\alpha p_\alpha z)}{z + \varkappa^2_{\alpha'} - \widetilde{\mathscr{P}}^2_\alpha - \widetilde{p}^2_\alpha} \\ & + u_{\alpha\alpha'}(\mathscr{P}_\alpha p_\alpha p_{\alpha'}, z) \\ & + \frac{\varphi_\alpha(p_\alpha)\, v^{\alpha'}_{\alpha\alpha'}(\mathscr{P}_\alpha p_{\alpha'}, z)}{z + \varkappa^2_\alpha - \widetilde{\mathscr{P}}^2_\alpha - p^2_\alpha} + \frac{\varphi_{\alpha'}(p_{\alpha'})\, v^\alpha_{\alpha\alpha'}(\mathscr{P}_\alpha p_\alpha, z)}{z + \varkappa^2_{\alpha'} - \widetilde{\mathscr{P}}^2_\alpha - \widetilde{p}^2_\alpha} \Big] \\ + \sum_{\substack{\alpha, \eta \\ \alpha \in \eta}} \Big[& u^\eta_\alpha(\mathscr{P}_\eta p_\alpha p_{\alpha\eta}, z) + \frac{\varphi_\alpha(p_\alpha)\, v^\eta_\alpha(\mathscr{P}_\eta, p_{\alpha\eta}, z)}{z + \varkappa^2_\alpha - \widetilde{\mathscr{P}}^2_\eta - \widetilde{p}^2_{\alpha\eta}} \\ & + \frac{\langle p_\alpha p_{\alpha\eta} \mid V_\alpha \mid \Phi_\eta \rangle\, v_\eta(\mathscr{P}_\eta, z)}{z + \varkappa^2_\eta - \widetilde{\mathscr{P}}^2_\eta} \Big] \Big\}, \end{aligned} \tag{38}$$

satisfies the conditions $(1 + H_0)\psi(z) \in L^2(R^9)$ and $H\psi(z) = z\psi(z)$. We consider the case $E \in (\Sigma, 0)$ with $E \neq -\varkappa^2_\alpha$, $E \neq -\varkappa^2_\alpha - \varkappa^2_{\alpha'}$, $E \neq -\varkappa^2_\eta$ for all α, (α, α'), and η. As we shall see in what follows, these points, and only these, can be points of accumulation for the energies of the bound states and resonances in the system.

Following Refs. 20 and 22, we proceed as follows. From the solution of the homogeneous system (37), we construct functions $\Psi_{\alpha,\alpha'}(\mathscr{P}_\alpha\, p_\alpha\, p_{\alpha'}, E + i\tau)$ and $\Phi_{\alpha\alpha'}(\mathscr{P}_\alpha\, p_\alpha\, p_{\alpha'}, E + i\tau)$ such that

$$\begin{aligned} & \mid \Psi_{\alpha\alpha'}(E + i\tau) \rangle \\ & = N_{\alpha\alpha'}(E + i\tau)\, G_0(E + i\tau) \sum_{\beta\beta' \neq \alpha\alpha'} \mid \Phi_{\beta\beta'}(E + i\tau) \rangle, \end{aligned} \tag{39}$$

and the function $\Phi_{\alpha\alpha'}(E + i\tau)$ has the form

$$\begin{aligned} & \Phi_{\alpha\alpha'}(\mathscr{P}_\alpha p_\alpha p_{\alpha'}, E + i\tau) \\ & = A + \sum_\eta B_\alpha + \sum_{\eta'} B_{\alpha'}, (\alpha \in \eta, \alpha' \in \eta). \end{aligned} \tag{40}$$

In (40), A denotes the first square bracket in (38), and B_α the second. The energy argument in the functions of type u and v is $E + i0$, and in all denominators $z = E + i\tau$. It follows from the properties of the operator $\mathbf{N}_{\alpha\alpha'}(z)$ that the function $|\psi_{\alpha\alpha'}(E + i\tau)\rangle$ (39) satisfies the equation

$$|\psi_{\alpha\alpha'}(E+i\tau)\rangle = V_{\alpha\alpha'}G(E+i\tau)$$
$$\times|\psi_{\alpha\alpha'}(E+i\tau)\rangle + V_{\alpha\alpha'}G_0(E+i\tau)\sum_{(\beta\beta')\neq(\alpha\alpha')}|\Phi_{\beta\beta'}(E+i\tau)\rangle. \quad (41)$$

We multiply Eq. (41) scalarly first by $\langle\psi_{\alpha\alpha'}(E+i\tau)|G_0(E-i\tau)$ and then by $\Sigma_{\beta\beta'\neq\alpha\alpha'}\langle\phi_{\beta\beta'}(E+i\tau)|G_0(E-i\tau)$. We transform the second of the resulting relations by means of (41) and add the two results. We obtain

$$\langle\psi_{\alpha\alpha'}(E+i\tau)|G_0(E-i\tau)-G_0(E+i\tau)|\psi_{\alpha\alpha'}(E+i\tau)\rangle$$
$$+\sum_{\beta\beta'\neq\alpha\alpha'}\langle\Phi_{\beta\beta'}(E+i\tau)|G_0(E-i\tau)$$
$$\times|\psi_{\alpha\alpha'}(E+i\tau)\rangle - \langle\psi_{\alpha\alpha'}(E+i\tau)|$$
$$\times G_0(E+i\tau)|\sum_{\beta\beta'}\Phi_{\beta\beta'}(E+i\tau)\rangle = 0. \quad (42)$$

We note further that the difference between the components of the vectors $|\psi_{\alpha\alpha'}(E+i\tau)\rangle$ and $|\Phi_{\alpha\alpha'}(E+i\tau)\rangle$ of the same kind is a function that tends to zero as $\tau\to 0$. Since in the last two terms in (42) the primary singularities of $\psi_{\alpha\alpha'}$ and $\Phi_{\alpha\alpha'}$ intersect only in the presence of three-particle bound states, when such are absent we can replace $\psi_{\alpha\alpha'}$ by $\Phi_{\alpha\alpha'}$. Thus, if the subsidiary condition $\sigma_d(h_\eta)=\varnothing$ is satisfied for any η, then (42) can be rewritten in the form

$$\langle\psi_{\alpha\alpha'}(E+i\tau)|G_0(E-i\tau)-G_0(E+i\tau)|\psi_{\alpha\alpha'}(E+i\tau)\rangle$$
$$+\sum_{\beta\beta'\neq\alpha\alpha'}\langle\Phi_{\beta\beta'}(E+i\tau)|G_0(E-i\tau)|\Phi_{\alpha\alpha'}(E+i\tau)\rangle$$
$$-\langle\Phi_{\alpha\alpha'}(E+i\tau)G_0(E+i\tau)|\sum_{\beta\beta'\neq\alpha\alpha'}\Phi_{\beta\beta'}(E+i\tau)\rangle = O(\tau), \quad (43)$$

where $O(\tau)\to 0$ as $\tau\to 0$. We sum the expressions (43) over (α,α'). Then

$$\sum_{\alpha\alpha'}\langle\psi_{\alpha\alpha'}(E+i\tau)|G_0(E-i\tau)-G_0(E+i\tau)|\psi_{\alpha\alpha'}(E+i\tau)\rangle$$
$$+\sum_{\alpha\alpha'\neq\beta\beta'}\langle\Phi_{\alpha\alpha'}(E+i\tau)|G_0(E-i\tau)-G_0(E+i\tau)|$$
$$\times\Phi_{\beta\beta'}(E+i\tau)\rangle = O(\tau). \quad (44)$$

In (44), we go to the limit $\tau\to 0$. We then note that by virtue of the conditions imposed on $\sigma_d(h_\eta)$ the second term is 0. The first of the terms in (44) is equal to the sum of nine positive terms, multiplied by $2\pi i$, each of which, as can be seen from foregoing, is equal to zero. Thus, we have

$$\left.\begin{aligned}&\int d\mathcal{P}_\alpha\,|v_{\alpha\alpha'}^{\alpha\alpha'}(\mathcal{P}_\alpha, E+i0)|^2\,\delta(E+\varkappa_\alpha^2+\varkappa_{\alpha'}^2-\tilde{\mathcal{P}}_\alpha^2)=0,\\ &\int d\mathcal{P}_\alpha\int dp_\alpha\,|v_{\alpha'}(\mathcal{P}_\alpha p_\alpha, E+i0)+v_{\alpha\alpha'}^{\alpha'}(\mathcal{P}_\alpha p_\alpha, E+i0)\\ &+\sum_{\eta,\,\alpha\in\eta}v_\alpha^\eta(\mathcal{P}_\eta p_{\alpha\eta}, E+i0)|^2\,\delta(E+\varkappa_\alpha^2-\tilde{\mathcal{P}}_\alpha^2-\tilde{p}_{\alpha\eta}^2)=0.\end{aligned}\right\} \quad (45)$$

By means of the relations (45) it is readily established that the wave function

$$|\psi(E+i\tau)\rangle = G_0(E+i\tau)\sum_{\alpha\alpha'}|\Phi_{\alpha\alpha'}(E+i\tau)\rangle$$

of the system is square-integrable for all $\tau\geqslant 0$. Indeed, to study the scalar product $\langle\psi(E+i\tau)|\psi(E+i\tau)\rangle$ it is sufficient to estimate the contribution from the terms in which there are intersecting primary singularities. The typical integral of this type has the form

$$\int d\mathcal{P}_\alpha\left|\frac{v_{\alpha\alpha'}^{\alpha\alpha'}(\mathcal{P}_\alpha, E+i0)}{E+i\tau+\varkappa_\alpha^2+\varkappa_{\alpha'}^2-\tilde{\mathcal{P}}_\alpha^2}\right|^2 f(\mathcal{P}_\alpha), \quad (46)$$
$$f(\mathcal{P}_\alpha)=\int dp_\alpha\int dp_{\alpha'}\left|\frac{\zeta_{\alpha\alpha'}(p_\alpha p_{\alpha'})}{E+i\tau-\tilde{\mathcal{P}}_\alpha^2-\tilde{p}_\alpha^2-\tilde{p}_{\alpha'}^2}\right|^2.$$

Using (45), we write $v_{\alpha\alpha'}^{\alpha\alpha'}(\mathcal{P}_\alpha, E+i0)$ in the form

$$v_{\alpha\alpha'}^{\alpha\alpha'}(\mathcal{P}_\alpha, E+i0) - v_{\alpha\alpha'}^{\alpha\alpha'}\left(\frac{\vec{\mathcal{P}}_\alpha}{\mathcal{P}_\alpha}\sqrt{2\mu_{\alpha\alpha'}(E+\varkappa_\alpha^2+\varkappa_{\alpha'}^2)}\right).$$

Since in the employed class of potentials the function $v_{\alpha\alpha'}^{\alpha\alpha'}(\mathcal{P}_\alpha, E+i0)$ is differentiable with respect to $\mathcal{P}_\alpha$, it is readily seen that no singularities arise at any $\tau\geqslant 0$ in the integrand in (46). In addition, it can be shown that $(1+H_0)\psi(E+i\tau)\in\mathbf{L}^2(R^9)$ for all $\tau\geqslant 0$.

Thus, we have here shown that in the absence of three-particle bound states and for $E\neq-\varkappa_\alpha^2$, $E\neq-\varkappa_\alpha^2-\varkappa_{\alpha'}^2$, $E\in(\Sigma,0)$ the homogeneous system of equations (37) has nontrivial solutions only at the points $\sigma_d(H)\cap(\Sigma,0)$.

Note that if $\Sigma_\eta=\min(-\varkappa_\eta^2)$ satisfies the inequality $\Sigma_\eta>\Sigma$, the result can be extended to the interval (Σ,Σ_η).

We now turn to the case $z=E+i0$, $E\in\mathbf{R}^+$. In the iterations of the system (32) and the powers of its kernel there appear not only primary singularities but also so-called secondary singularities, which are generated by the intersection of the singularities of the free Green's functions $G_0(E+i0)$. The main task here is to show that the secondary singularities become weaker and then disappear altogether as the order of the iteration is raised.[22,48]

We write the Nth power of the kernel $W(z)$ in the form

$$W^N(z) = X^{(N)}(z)\,G_0(z).$$

It is readily seen that the elements of the matrix of the kernel $X^{(N)}(z)$ have primary singularities of the momentum in the "in" state. Therefore, one can speak of components of the operators $X_{mn}^{(N)}(z)$. As an example, we write down one of the possible ways of decomposing $X_{mn}^{(N)}(z)$. into components:

$$X_{mn}^{(N)}(\cdot,\ \mathcal{P}_\alpha^0 p_\alpha^0 p_{\alpha'}^0,\ z) = {}_{mn}^{N}\mathcal{X}_\alpha(\cdot,\ \mathcal{P}_\alpha^0 p_\alpha^0 p_{\alpha'}^0,\ z)$$
$$+{}_{mn}^{N}\mathcal{Y}_\alpha(\cdot,\ \mathcal{P}_\alpha^0,\ p_\alpha^0,\ z)$$
$$\times\frac{\varphi_{\alpha'}^*(p_{\alpha'}^0)}{z+\varkappa_{\alpha'}^2-\tilde{\mathcal{P}}_\alpha^{02}-\tilde{p}_\alpha^{02}}, \quad (47)$$

where $(\cdot)$ denotes a certain set of momenta in the "out" state, for example, $(\mathcal{P}_\eta, P_{\alpha\eta})$. It was established in Ref. 9 that for $N\geqslant N_0$, where N_0 is fixed number, secondary singularities are absent. Although in Ref. 9 this result was proved for chains of three-dimensional diagrams, it is not particularly difficult to extend it to the system of equations (32).

As a result, on the basis of Refs. 9 and 48, it can be asserted that the operator $X^{(N)}(E+i0)$ obtained from $X_{mn}^{(N)}(E+i\tau)$ by going to the limit $\tau\to+0$ is a compact operator for $N\geqslant N_0$ in some auxiliary Banach space. The structure of this space is described in Ref. 9 and 48. It was shown in Ref. 48 that the Fredholm alternative applies to such kernels.

Whereas the Fredholm alternative applies to the system of integral equations (32) irrespective of the existence or not of three-particle bound states, in studying the question of the uniqueness of a solution for $z = E + i0$, $E \in R^+$, we must, as before, restrict ourselves to the case when $\sigma_d(h_\eta) = \varnothing$ for all η. At the same time, the solutions of the homogeneous system (37) also satisfy the relations (45), but to them one more is added. This relation has the form

$$\int d\mathscr{P}_\alpha \int dp_\alpha \int dp_{\alpha'} \Big| \sum_{\alpha\alpha'} \Big[u_\alpha(\mathscr{P}_\alpha p_\alpha p'_\alpha, E+i0) + u_{\alpha'}(\mathscr{P}_\alpha p_\alpha p_{\alpha'}, E+i0)$$
$$+ u_{\alpha\alpha'}(\mathscr{P}_\alpha p_\alpha p_{\alpha'}, E+i0) + \frac{\varphi_\alpha(p_\alpha)\{v_{\alpha'}(\mathscr{P}_\alpha p_{\alpha'}, E+i0) + v^{\alpha'}_{\alpha\alpha'}(\mathscr{P}_\alpha p_{\alpha'}, E+i0)\}}{\varkappa^2_{\alpha'} + \tilde{p}^2_{\alpha'}}$$
$$+ \frac{\varphi_{\alpha'}(P_{\alpha'})\{v_\alpha(\mathscr{P}_\alpha p_\alpha, E+i0) + v^{\alpha}_{\alpha\alpha'}(\mathscr{P}_\alpha p_\alpha, E+i0)\}}{\varkappa^2_{\alpha'} + \tilde{p}^2_{\alpha'}}$$
$$+ \frac{\zeta_{\alpha\alpha'}(p_\alpha p_{\alpha'})\, v^{\alpha\alpha'}_{\alpha\alpha'}(\mathscr{P}_\alpha, E+i0)}{\varkappa^2_\alpha + \varkappa^2_{\alpha'} + \tilde{p}^2_\alpha + \tilde{p}^2_{\alpha'}} \Big]$$
$$+ \sum_{\substack{\alpha,\eta \\ \alpha\in\eta}} \Big[u^\eta_\alpha(\mathscr{P}_\eta p_{\alpha\eta} p_\alpha, E+i0) + \frac{\varphi_\alpha(p_\alpha)\, v^\eta_\alpha(\mathscr{P}_\eta p_{\alpha\eta}, E+i0)}{\varkappa^2_\alpha + \tilde{p}^2_\alpha}$$
$$+ \frac{\langle p_\alpha p_{\alpha\eta} | V_\alpha | \Phi_\eta \rangle\, v_\eta(\mathscr{P}_\eta, E+i0)}{\varkappa^2_\eta + \tilde{\mathscr{P}}^2_\eta} \Big] \Big|^2$$
$$\times \delta(E - \tilde{\mathscr{P}}^2_\alpha - \tilde{p}^2_\alpha - \tilde{p}^2_{\alpha'}) = 0. \tag{48}$$

From (45) and (48), as for $E \in \mathbf{R}^-$, it can be concluded that the wave function $\psi(E + i0)$ (38) is square integrable. The technique needed for this was described above (in this connection, see also Refs. 20, 22, and 49). Compared with Ref. 49, the proof is greatly facilitated by the restriction of the class of potentials.

Thus, we have established the following.

1. The Fredholm alternative applies to the system of integral equations (32) for all z, including the real axis $z = E + i0$, $E \in \mathbf{R}$.

2. The homogeneous system of equations (37) does not have nontrivial solutions for complex z (there are no spurious solutions).

3. If $\sigma_d(H_\eta) = \varnothing$ for all η and the homogeneous system has a nontrivial solution $v(E + i0)$, where $E \neq -\varkappa^2_\alpha$, $E \neq -\varkappa^2_\alpha - \varkappa^2_{\alpha'}$, then $E \in \sigma_d(H)$.

We shall return to a discussion of the questions touched on in this section when we study the eigenvalues of the kernel $A^\theta(z)$ obtained from the kernel of the system of integral equations (21) by means of a complex scale transformation. Completing this section, we note that in the literature there are other methods of studying the equations of the N-body problem. References to some of them can be found in the notes to the third and fourth volumes of the monograph by Reed and Simon.[1,2]

2. RESONANCE STATES IN SYSTEMS OF SEVERAL NONRELATIVISTIC PARTICLES

Resonances in systems of N particles with potentials analytic with respect to scale transformations

Here and in what follows, we shall discuss the question of resonances in many-particle systems and various methods of describing such resonances. Since resonances are usually associated with the poles of the analytic continuation of the matrix elements of the Green's functions of the system to the second (unphysical) sheet, it is necessary to possess a mathematical formalism making it possible to realize these analytic continuations. In recent years, the method of canonical transformations of the Hamiltonian of the system[2,4–8] has been developed strongly for these purposes; in it, the resonances are the eigenvalues of a non-self-adjoint Hamiltonian $U(a)HU^{-1}(a)$, where $U(a)$ is the operator of the canonical transformation, and a is the complex parameter of this transformation. We consider only one special case of such transformations—complex scale transformations.

The operator $U(\theta)$ of scale transformations on $L^2(\mathbf{R}^{3N})$ is defined as follows:

$$U(\theta) f(r_1 \ldots r_N) = \exp\left\{\tfrac{3}{2} N\theta\right\} f(e^\theta r_1, \ldots e^\theta r_N). \tag{49}$$

It is readily seen that $U(\theta)$ is unitary for real θ, and $U(\theta_1 + \theta_2) = U(\theta_1)U(\theta_2)$. The kinetic-energy operator H_0 transforms as follows:

$$H_0(\theta) = U(\theta) H_0 U^{-1}(\theta) = e^{-2\theta} H_0. \tag{50}$$

The relation (50), defined for real θ, admits analytic continuation to complex θ. Since the spectrum of the operator H_0 is the half-line $[0, \infty)$, it follows from (50) that $\sigma(H_0(\theta)) = \{z | \arg z = -2\operatorname{Im}\theta\}$, i.e., for $\operatorname{Im}\theta \neq 0$ the spectrum of H_0 can be determined from the real axis. The operator of the interaction V under the action of $U(\theta)$ takes the form

$$V(\theta) = U(\theta) V U^{-1}(\theta). \tag{51}$$

The corresponding class of potentials was described in Sec. 1. Then (see Ref. 2, Chap. XIII, ∮10) one can introduce the operator

$$\left.\begin{aligned} &H(\theta) = U(\theta) H U^{-1}(\theta), \quad |\operatorname{Im}\theta| < \sigma, \\ &H(\theta) = e^{-2\theta} H_0 + V(\theta) \end{aligned}\right\} \tag{52}$$

and the resolvent $G^\theta(z) = [z - H(\theta)]^{-1}$. If the potentials V_α are analytic in the class C_σ, the matrix elements of the resolvent $G(z)$, $0 \leqslant \arg z \leqslant 2\pi$, admit analytic continuation to the part D_σ of the unphysical sheet of energies in accordance with the law

$$\langle f | G(z) | f \rangle = \langle f(\theta^*) | G^\theta(z) | f(\theta) \rangle, \tag{53}$$

where $D_\sigma = \{z | -2\sigma < \arg(z - \Sigma) < 0\}$, $f(\theta) = U(\theta) f$. The functions f in (53) belong to the set, dense in $\mathbf{L}^2(\mathbf{R}^{3N})$, of analytic vectors for the generator of the group $U(\theta)$ (for more detail on this question, see Ref. 2).

We describe the spectrum of the operator $H(\theta)$[2,4–8]

1) $\sigma(H(\theta))$ depends only on $\operatorname{Im}\theta$ because $H(\theta_1)$ and $H(\theta_2)$ are unitarily equivalent for real $\theta_1 - \theta_2$, $\sigma(H(\theta)) = \sigma_{ess}(H(\theta)) \cup \sigma_d(H(\theta))$.

2) $\sigma_{\text{ess}}(H(\theta)) = \{e^{-2\theta}\lambda\,|\lambda \in (0,\infty)\} \cup \{\mu_i + e^{-2\theta}\lambda\,|\lambda \in 0,\infty)), \mu_i \in \sigma_{\text{d}}((h_i)\theta)\}$. Here, we have denoted by $h_i(\theta)$ the operator $U(\theta)h_i U^{-1}(\theta)$, where h_i is the Hamiltonian of the motion in the ith channel of the system. (In the four-body problem, there are three types of Hamiltonians of internal motion in the channels h_α, h_η, and $h_{\alpha,\alpha'} = h_\alpha + h_{\alpha'}$.)

3) $\sigma_{\text{d}}(H(\theta))$ is at most a countable set, the limit points for which can be only 0 and the points in $\sigma_{\text{d}}(h_i(\theta))$. For $|\text{IM}\,\theta| < \min(\sigma, \pi/2)$, we have $\sigma_{\text{d}}(H(\theta)) \cap \mathbf{R} = \sigma_{\text{d}}(H)$. If $0 < \text{Im}\,\theta_1 < \text{IM}\,\theta_2 < \pi/2$, then $\sigma_{\text{d}}(H(\theta_1)) \subset \sigma_{\text{d}}(H(\theta_2))$.

Property 2 means that the Hamiltonian $H(\theta)$ has for $|\text{IM}\,\theta| < \pi/2$ not only ordinary thresholds $-\kappa_i^2 \in \sigma_{\text{d}}$ but also complex thresholds $\mu_i \in \sigma_{\text{d}}(h_i(\theta))$, corresponding to resonances in the channels.

For the four-body problem, the complex thresholds in the channels α and (α, α') satisfy the conditions

$$2\,\text{Im}\,\theta < \arg \mu_\alpha < 0, \quad \mu_\alpha \in \sigma_d(h_\alpha),$$
$$\mu_{\alpha\alpha'} = \mu_\alpha + \mu_{\alpha'},$$

and the three-particle complex thresholds satisfy the conditions

$$2\,\text{Im}\,\theta < \arg(\mu_\eta - \Sigma_\eta) < 0,$$
$$\mu_\eta \in \sigma_d(h_\eta),$$

and Σ_η, as usual, is the point at which the continuum of the Hamiltonian h_η begins. The spectrum of the Hamiltonian $H(\theta)$ for the four-body system is shown in Fig. 1 [the crosses in Fig. 1 represent points in $\sigma(H(\theta))$].

To understand better the meaning of the complex scale transformation, we consider the formal theory of resonances[50] used traditionally in nuclear and atomic physics. This formal theory is based on the following expansion of the resolvent:

$$G(z) = R(z) + [1 + R(z)H] \times \sum_{ij} |\psi_i\rangle \frac{A_{ij}(z)}{\omega(z)} \langle\psi_j|[HR(z) + 1], \tag{54}$$

where the operator $P = \Sigma_i |\psi_i\rangle\langle\psi_j|$ is an orthogonal projector, $R(z)$ is the resolvent of the operator QHQ, $Q = 1 - P$, $R(z) = (zQ\text{-}QHQ)^{-1}$, $\omega(z)$ in (54) is the determinant with elements $\omega_{ij}(z) = z\delta_{ij} - \langle\psi_i|H + HR(z)H|\psi_j\rangle$, and $A_{ij}(z)$ is the corresponding minor. In the formal theory of resonances, the problem of finding the resonances reduces to finding the zeros of the analytic continuation of the determinant $\omega(z)$ to the unphysical sheet.

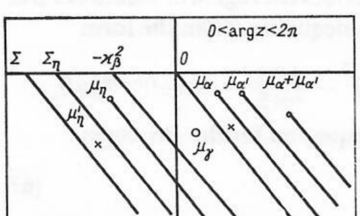

FIG. 1. Spectrum of the Hamiltonian $H(\theta)$ in the four-body problem. The continuous spectrum of the operator H begins at Σ; $\varkappa_\beta^2$ and Σ_η are real thresholds for H; $\mu_\alpha, \mu_\eta, \mu_\alpha + \mu_{\alpha'}$ are complex thresholds for H (thresholds of resonances[2]).

If the interaction potentials are analytic with respect to scale transformations, then an analytic continuation of the determinant $\omega(z)$ (under the condition that $|\psi_{i.}\rangle$ are analytic vectors) can be constructed by means of (53). Note that the determinant $\omega(z)$ satisfies the condition

$$\omega(z)\,\omega_G(z) = 1, \tag{55}$$

where $\omega_G(z)$ is the determinant with the elements $\langle\psi_i|G(z)|\psi_j\rangle$. Since the analytic continuation of $\omega_G(z)$ is realized by the determinant $\omega_G^\theta(z)$ with elements $\langle\psi_i(\theta)G(z)|\psi_j(\theta)\rangle$, it follows from (55) that the determinant $\omega(z)$ has an analytic continuation to the region $D(\theta) = \{z|-2\,\text{Im}\,\theta < \arg(z - \Sigma) < 0, |\text{Im}\,\theta| < \sigma\}$. At the same time, $\omega_G(z)$ and $\omega(z)$ are meromorphic functions in $D_\theta \setminus \sigma_{\text{ess}}[H(\theta)]$. Since $\omega(z)$ and $\omega_G(z)$ do not depend on θ in $D_{\theta_1} \cap D_{\theta_2}$, the domain of meromorphicity of these determinants can be extended to $D_\theta \setminus U_i \sigma_{\text{d}}(h_i(\theta))$. It follows from this that the zeros of $\omega(z)$, which are the bound states and resonances of the Hamiltonian H, and the poles of the analytic continuation of $\omega_G(z)$ can accumulate only at the thresholds (both real and complex) of the operator $H(\theta)$. This has already been noted. Further, the determinant $\omega_G(z)$ is a Weinstein-Aronszajn determinant of the second kind, and the expansion itself is a special case of the theory of degenerate perturbations.[51] As a consequence, the connection between the zeros of the analytic continuation of $\omega(z)$ and the multiplicity of z as an eigenvalue of $H(\theta)$ can be established by means of the second (W–A) formula. Let z_0 be a zero of multiplicity k of the determinant $\omega(z)$. Then ν_1, the multiplicity of z_0 as an eigenvalue of $H(\theta)$, is determined by

$$\nu_1 = \nu_2 + k,$$

where ν_2 is the multiplicity of z_0 as an eigenvalue of $(QHQ)(\theta)$, $(\nu_2 = 0$, if $z_0 \notin \sigma_{\text{d}}\{[Q \text{ and } \theta](\theta)\}$. But if z_0 is a pole of order k for $\omega(z)$, then $\nu_1 = \nu_2 - k$.

Some restrictions on the positions of the virtual poles of the many-particle Green's function[1)]

One of the important problems of the theory of many-particle resonances is that of finding regions of the unphysical sheet free of virtual poles (resonances). In the case of the two-particle problem, this problem has been studied by many authors (see Ref. 50, Chap. 12, §4, and also Refs. 52–54). For a certain class of analytic potentials, the method of Refs. 52–54 can be generalized to many-particle systems.

Thus, suppose $V_{ij} \in C_\sigma$ for each pair, $1 \leqslant i < j \leqslant N$. Then the positions of the poles of the resolvent on the second sheet can be found as the eigenvalues of the operator $H(\theta) = e^{-2\theta}H_0 + V(\theta)$. Setting $\theta = i\rho$, $0 < \rho < \sigma$, we write the equation for the resonance energies in the form

$$(H_0 + e^{2i\rho}V(\rho))\,|\psi\rangle = ze^{2i\rho}\,|\psi\rangle. \tag{56}$$

The function $|\psi\rangle$ in (56) can be assumed to be normalized to unity. From (56), we obtain

$$\langle\psi\,|\,H_0 + e^{2i\rho}V(\rho)\,|\,\psi\rangle = ze^{2i\rho}$$

[1)]The results of this section were obtained by the authors in collaboration with V. G. Aĭrapetyan.

or

$$\langle\psi \mid H_0 \mid \psi\rangle + \langle\psi \mid \text{Re } e^{2i\rho} V(\rho) \mid \psi\rangle = \text{Re } z e^{2i\rho};$$
$$\langle\psi \mid \text{Im } e^{2i\rho} V(\rho) \mid \psi\rangle = \text{Im } z e^{-2i\rho}. \quad (57)$$

Further, following Refs. 52–54, we multiply the first of the equations by $\cos\beta$ and the second by $\sin\beta$, where β is some arbitrary real parameter, and add the resulting relations. This gives us

$$\cos\beta \langle\psi \mid H_0 \mid \psi\rangle + \langle\psi \mid \text{Re } z\, e^{2i\rho \mp i\beta} V(\rho) \mid \psi\rangle = \text{Re } z\, e^{2i\rho \mp i\beta}. \quad (58)$$

Since the operator H_0 is positive definite, for $\cos\beta \geqslant 0$ the resonance energies must satisfy the inequality

$$-\langle\psi \mid \text{Re } e^{2i\rho \mp i\beta} V(\rho) \mid \psi\rangle \geqslant -\text{Re } z\, e^{2i\rho \mp i\beta}.$$

Now suppose the potential $V(\rho)$ satisfies the estimate $|V(\rho)| = |\sum_{ij} V_{ij}(e^{i\rho} r_{ij})| \leqslant A_\rho$ for $0 < \rho < \sigma$. Then we must have

$$A_\rho \geqslant -\text{Re } z\, e^{2i\rho \mp i\beta} \quad (\cos\beta \geqslant 0). \quad (59)$$

Similarly, for $\cos\beta \leqslant 0$

$$A_\rho \geqslant \text{Re } z\, e^{2i\rho \mp i\beta},$$

which, as is readily seen, is equivalent to (59). If $\min A_\rho$ as $\rho \to \sigma$ exists and is equal to A, then in (59) we can set $\rho = \sigma$.

As a result, for the resonance energies we obtain restrictions of the form

$$A \geqslant -\text{Re } z\, e^{2i\rho + i\beta}, \quad (60)$$

where β is an arbitrary parameter on the interval $(-\pi/2, \pi/2)$. Therefore, all the resonances must be in the region bounded by the envelope of the family of curves (60).

To illustrate this result, let us consider the case $\sigma = \pi/2$, which corresponds, for example, to a superposition of exponential potentials $V(r) = -V_0 e^{-\mu r}$. Since the resonance energies can be situated only in the lower half-plane, it is sufficient to study the single inequality

$$A \geqslant R\cos(\varphi + \beta), \quad (61)$$

where $\beta \in (0, \pi/2)$, $R = |z|$, $\varphi = \arg z$, $-\pi < \varphi < 0$.

As is well known (see, for example, Ref. 55), the envelope of a single-parameter family of curves $u(x,y,\beta) = 0$ can be obtained by eliminating the parameter β from the system of equations

$$\begin{cases} u(x, y, \beta) = 0 \\ \dfrac{\partial\varphi(x, y, \beta)}{\partial\beta} = 0. \end{cases}$$

From (61) we obtain $\sin(\beta + \varphi) = 0$, which (with allowance for the restrictions on the region of the parameters φ and β) is equivalent to the condition $\varphi + \beta = 0$ with $-\pi/2 < \varphi < 0$. Therefore, in the region $-\pi/2 < \varphi < 0$ the envelope is the arc of a circle of radius A. We note in passing that this result agrees with Ref. 53. In the region $-\pi < \varphi < \pi/2$, the condition $\sin(\beta + \varphi) = 0$, $\beta \in (0, \pi/2)$, cannot be satisfied, so that to solve the inequality (61) we proceed as follows. We note that the function $\cos(\varphi + \beta)$ increases monotonically with respect to β for $-\pi < \varphi < \pi/2$. This means that in the region $-\pi < \varphi < \pi/2$ the resonances lie above the curve $A = R\cos(\varphi + \pi/2) = -R\sin\varphi$, and this is equivalent to the condition $|\text{Im } z| \leqslant A$. Our result is shown in Fig. 2.

We can construct similarly the envelopes for any arbitrary $\sigma < \pi/2$. These envelopes are shown in Fig. 3. All that has been said above applies to the case when the potentials satisfy $\lim_{\rho \to \sigma} |V(\rho)| = A < \infty$. This estimate and, therefore, restrictions of the form (60) are not satisfied for potentials of the type of the Coulomb and Yukawa potentials. To construct the envelope of the region in which the resonance poles are situated, Regge's method can be used (Refs. 52 and 53; see also Ref. 50, Chap. 12, §4). This method is as follows.

Suppose that interaction potential $V(r)$ in a two-particle system belongs to the class σ, and that for all r the estimate $|V(e^{i\sigma} r)| \leqslant V_0/r$ holds. For the Coulomb potential[2)] and the Yukawa potential, $\sigma = \pi/2$. Further, for any function $\psi(r)$ in the region in which the Hamiltonian $H_0 = -(1/2\mu)\nabla^2$ is self-adjoint,

$$\int dr \frac{|\psi(r)|^2}{r^2} \leqslant 4 \int dr\, |\nabla\psi(r)|^2. \quad (62)$$

[In connection with the inequality (62), see Ref. 56, Chap. 11, p. 279 of the Russian translation, and also Ref. 57, pp. 192 and 352 of the Russian translation.] Note that the expression on the right-hand side of (62) is equal to $8\mu\langle\psi|H_0|\psi$, as a result of which the inequality (62) can be rewritten in the form

$$\int dr \frac{|\psi(r)|^2}{r^2} \leqslant 8\mu \langle\psi \mid H_0 \mid \psi\rangle. \quad (63)$$

Since the resonance energies satisfy the reltion (58), which at the given stage can be conveniently rewritten as

$$\cos\beta \langle\psi \mid H_0 \mid \psi\rangle + \cos\beta \langle\psi \mid \text{Re } e^{2i\rho} \times V(e^{i\rho} r) \mid \psi\rangle - \sin\beta \langle\psi \mid \text{Im } e^{2i\rho} V(e^{i\rho} r) \mid \psi\rangle = \text{Re } z\, e^{2i\rho + i\beta}, \quad (64)$$

it follows by virtue of (63) and the condition on the behavior of the modulus of $V(e^{i\rho} r)$ that the inequality

$$\langle\psi \mid \frac{\cos\beta}{8\mu} \frac{1}{r^2} - \frac{\sqrt{2}\, V_0}{r} - \text{Re } z\, e^{2i\rho + i\beta} \mid \psi\rangle \leqslant 0 \quad (65)$$

must be satisfied. This inequality can be satisfied only in the case when there exist values of r for which the integrand in (65) is nonpositive. If at the same time for any r (and any $\beta \in [0, \pi/2]$)

$$\frac{\cos\beta}{8\mu} \frac{1}{r^2} - \frac{\sqrt{2}\, V_0}{r} - \text{Re } z\, e^{2i\rho + i\beta} \geqslant 0, \quad (66)$$

then the inequality (66) describes the region of values of z free of resonances. Writing the inequality (66)in the form

$$\frac{\cos\beta}{8\mu} \left(\frac{1}{r} - \frac{4\sqrt{2}\,\mu V_0}{\cos\beta}\right)^2 - \frac{4\mu V_0^2}{\cos\beta} - \text{Re } z\, e^{2i\rho + i\beta} \geqslant 0,$$

we arrive at the following equation for the envelope:

$$-\text{Re } z\, e^{2i\rho + i\beta} = \frac{4\mu V_0^2}{\cos\beta}. \quad (67)$$

In (67), we can set $\rho = \sigma$. To be specific, we shall assume that

2)Note that in the earlier treatment we did not consider the long-range Coulomb potential. However, in the study of the problem of resonances this potential can be included in the scheme of the method of complex scale transformations.[2,4–8]

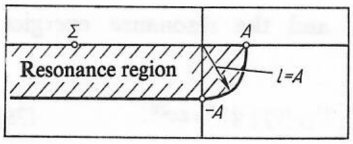

FIG. 2. Region containing the virtual poles of the Green's function for systems with potentials of exponential type.

$\sigma = \pi/2$ (the case of the Coulomb and Yukawa potentials, in which we are interested). Writing z in the form $z = x - iy$, $y \geqslant 0$, we obtain from (67)

$$x \cos\beta + y \sin\beta = \frac{4\mu V_0^2}{\cos\beta} = \frac{a_0}{\cos\beta},$$

or, equivalently,

$$x (\cos 2\beta + 1) + y \sin 2\beta = 2a_0. \quad (68)$$

Differentiating (68) with respect to β, we obtain $\tan 2\beta = y/x$, and after substitution in (68) this gives an equation for the envelope in the form of the parabola

$$y^2 = 4a_0 (a_0 - x). \quad (69)$$

Therefore, in a system of two particles interacting through a Yukawa-type potential all the resonances are at energies that satisfy the condition

$$x = E_R < a_0 = 4\mu V_0^2. \quad (70)$$

We note that the estimates (67)–(70) can be somewhat improved. To this end, we transform the relation (64) into the inequality

$$\mathrm{Re}\, z\, e^{2i\rho+i\beta} \geqslant \left\langle \psi \left| \cos\beta h_0 - \frac{\sqrt{2}\, V_0}{r} \right| \psi \right\rangle. \quad (71)$$

Since the ground state of the Hamiltonian $(-h^2/2\mu)\Delta - (ze^2/r)$ has energy $-(ze^2)^2\mu/2\hbar^2$, the Hamiltonian $(\cos\beta H_0 - \sqrt{2}\, V_0/r)$ is bounded below by the constant $A = -\mu V_0^2/\cos\beta$.

Therefore,

$$\mathrm{Re}\, z\, e^{2i\rho+i\beta} \geqslant -\frac{\mu V_0^2}{\cos\beta}$$

and a_0 in all the relations (67)–(70) can be replaced by μV_0^2, i.e., reduced by four times.

These methods can be readily extended to many-channel problems. Indeed, in the case of a system of several particles the inequality (71) can be written in the form

$$\mathrm{Re}\, z\, e^{2i\rho+i\beta} \geqslant \left\langle \psi \left| \cos\beta H_0 - \sum_{i<j}^{N} \frac{\sqrt{2}\, V_{0ij}}{r_{ij}} \right| \psi \right\rangle. \quad (72)$$

Further, we split H_0 into $N(N-1)/2$ parts:

$$H_0 = \sum_{i<j}^{N} \rho_{ij} H_0 = \sum_{i<j}^{N} \rho_{ij} (h_{0ij} + H_{0ij}),$$

where h_{0ij} is the kinetic-energy operator of the internal motion in the pair (ij). Then from (72) we obtain the inequality

$$\mathrm{Re}\, z\, e^{2i\rho+i\beta} \geqslant \sum_{i<j}^{N} \left\langle \psi \left| \rho_{ij} \cos\beta h_{0ij} - \frac{\sqrt{2}\, V_{0ij}}{r_{ij}} \right| \psi \right\rangle$$

$$\geqslant -\sum_{i<j}^{N} \frac{\mu_{ij} V_{0ij}^2}{\cos\beta \rho_{ij}} = -\frac{A}{\cos\beta}, \quad (73)$$

subject to the condition $\Sigma_{i<j}^{N} \rho ij = 1$. The envelope of the family of curves for $\rho = \sigma = \pi/2$ of the form (73) is again the parabola (69), in which a_0 must be taken to be the smallest value of the sum $\Sigma_{i<j}^{N} (\mu_{ij} V_{0ij} / \rho_{ij})$.

If $\sigma < \pi/2$, which corresponds, for example, to pair potentials of the form

$$V(r) = -V_0 \frac{e^{-\mu r}}{r} (\cos\lambda_1 r + \gamma \sin\lambda_2 r), \quad (74)$$

then the envelope of the region containing the resonances is obtained from (69) by a rotation counterclockwise through the angle $(\pi - 2\sigma)$. In particular, for the potentials (74) the quantity σ is obtained from the condition $\arg e^{i\sigma} (\mu + i\lambda) = \pi/2$, where $\lambda = \max(|\lambda_1|, |\lambda_2|)$.

The results are shown in Fig. 4.

As an example, we consider the question of the resonances in the system H^- which consists of a proton and two electrons. The Hamiltonian of the H^- system has the form

$$H = -\frac{\hbar^2}{2m_e}\Delta_1 - \frac{\hbar^2}{2m_e}\Delta_2 - \frac{e^2}{r_1} - \frac{e^2}{r_2} + \frac{e^2}{r_{12}} = H_0 + V.$$

Letting the parameter of the scale transformation tend to $\pi/2$, we obtain in complete analogy with (57)

$$\left.\begin{aligned} &\lim_{\rho\to\pi/2} \langle\psi(\rho) \mid H_0 \mid \psi(\rho)\rangle = \mathrm{Re}\, z\, e^{2i\frac{\pi}{2}} = -\mathrm{Re}\, z \geqslant 0, \\ &\lim_{\rho\to\pi/2} \langle\psi(\rho) \mid V \mid \psi(\rho)\rangle = \mathrm{Im}\, z\, e^{2i\pi/2} = -\mathrm{Im}\, z. \end{aligned}\right\} \quad (75)$$

The first of the relations shows that in systems with a purely Coulomb interaction all resonances are at negative energies.[7] Since $\mathrm{Im}\, z \leqslant 0$, writing z in the form $z = -x - iy$, $x \geqslant 0$, $y \geqslant 0$ we reduce (75) to the form

$$\lim_{\rho\to\frac{\pi}{2}} \langle\psi(\rho) \mid H_0 - \beta V \mid \psi(\rho)\rangle = x - \beta y, \quad (76)$$

where β is some arbitrary positive parameter. From (76), we obtain the chain of inequalities

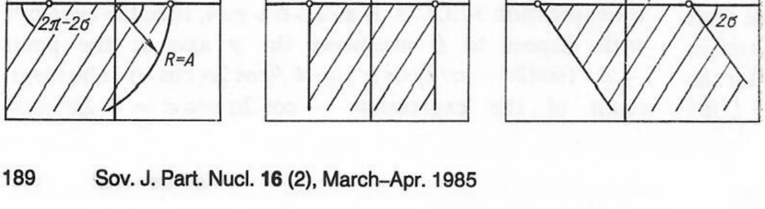

FIG. 3. Region containing the resonances for systems with potentials of the class C_σ for three values of σ.

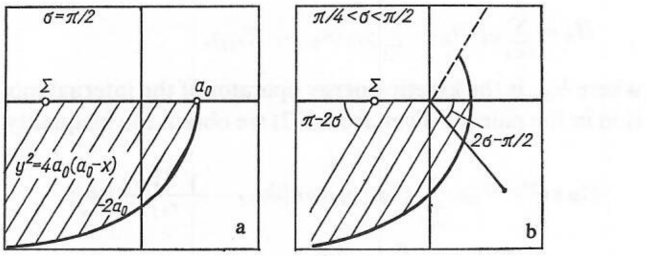

FIG. 4. Envelopes for the region containing the resonances for systems with potentials of Yukawa type (a) and the type $V(r) = -V_0(e^{-\mu r}/r(\cos\lambda_1 r + \gamma\sin\lambda_2 r)$ (b).

$$x-\beta y \geqslant \lim_{\rho\to\pi/2}\left\langle \psi(\rho)\left|H_0-\frac{\beta e^2}{r_{12}}\right|\psi(\rho)\right\rangle \geqslant$$
$$\geqslant \lim_{\rho\to\pi/2}\left\langle \psi(\rho)\left|h_{0\,12}-\frac{\beta e^2}{r_{12}}\right|\psi(\rho)\right\rangle \geqslant -\frac{R\beta^2}{4}, \quad (77)$$

where $\mathrm{Ry} = e^4 m_e/\hbar^2 = 27.12$ eV is the atomic unit of energy (rydberg). As is readily established the envelope of the family of curves

$$x-\beta y = -\frac{\mathrm{Ry}}{4}\beta^2$$

is the parabola

$$y^2 = \mathrm{Ry}\, x. \quad (78)$$

Since $\sigma_{\mathrm{ess}}(H) = [-R/2, \infty]$, the half-widths of all resonances in the system H^- satisfy the estimate $\Gamma/2 \leqslant \mathrm{Ry}/\sqrt{2}$. In deriving this estimate, we ignore the fact that the electrons are identical. It is easy to show that in the state with spin $S=0$ the half-widths satisfy as before the estimate $\Gamma_0/2 \leqslant R/\sqrt{2}$, and in the state with total spin $S=1$ the estimate is reduced by 2 times: $\Gamma_1/2 \leqslant R/2\sqrt{2}$.

Note that in the expressions (75)–(77) we cannot directly set $\rho = \pi/2$. This is due to the fact that for $\operatorname{Im}\theta = \pi/2$ virtual states of the negative Coulomb potential are opened, and these violate the conditions under which the theory of complex scale transformations works. Therefore, the resonances in the H^- system cannot be sought as eigenvalues of the Hamiltonian $H_0 + iV$. Indeed, it follows from the properties of the scale transformation that $\sigma_d(H(\theta^*)) = \sigma_d(H(\theta))$. If one could directly set $\theta = i\tau/2$, then, replacing V by $-V$, we would find that the spectrum of the operator $H_0 + iV$ can be situated only on the real axis, i.e., no resonances arise. This contradiction shows that the restriction $\rho < \pi/2$ is essential and that the relations (75)–(77) are satisfied only in the sense of limits.

We note further that the envelope (78) is determined exclusively by the strength of the interelectron interaction potential. If this potential were a purely attractive Coulomb potential, then in the system there would be only virtual and bound states.

The third class of potentials for which one can construct the envelope of the region containing the resonances is the class of potentials of finite rank (separable). To be specific, we restrict ourselves to potentials of unit rank, i.e., we shall assume that each of the pair potentials can be represented in the form $V = \lambda|g\rangle\langle g|$, where $|g\rangle$ is an analytic vector for the group of scale transformations on $L^2(R^3)$. Then $V(\theta) = U(\theta) VU(\theta)^{-1} = \lambda|g(\theta)\rangle\langle g(\theta)|$, and the resonance energies satisfy the relation

$$\langle\psi\mid H_0\mid\psi\rangle + \langle\psi\mid e^{2\theta}\sum_{i<j}^{N} V_{ij}(\theta)\mid\psi\rangle = ze^{2\theta}. \quad (79)$$

As before, we reduce (79) to the form

$$\cos\beta\langle\psi\mid H_0\mid\psi\rangle + \operatorname{Re}\langle\psi\mid e^{2i\rho\pm i\beta} \times \sum_{i<j}^{N} V_{ij}(e^{i\rho})\mid\psi\rangle = \operatorname{Re} z\, e^{2i\rho\pm i\beta}.$$

By virtue of the definition of the potentials V_{ij}, we have functions $g(\theta)\in L^2(R^3)$ for $0<\rho<\sigma$. Therefore, the operators $V_{ij}(e^{i\rho})$ are Hilbert-Schmidt operators with norm

$$\| V(e^{i\rho})\|_2 = |\lambda| \int dr\, |g(e^{i\rho}r)|^2$$

and all the resonance poles must satisfy the inequality

$$\operatorname{Re} z\, e^{2i\rho\pm i\beta} \geqslant -\sum_{i<j}^{N} A_{ij}^{\rho}, \quad (80)$$

where $A_{ij}^{\rho} = \|V_{ij}(e^{i\rho})\|$. For each fixed ρ, the envelopes of the one-parameter family (80) are shown in Fig. 3. Varying the parameter $\rho, \rho < \sigma$, we can construct a curve that bounds the region containing the resonances.

As an example, we consider a Yamaguchi potential of the form $g(r) = \sqrt{\pi/2}(e^{-\gamma r}/2)$. In this case

$$A^{\rho} = |\lambda|\frac{\pi}{2}\int dr\left|\frac{e^{-2\gamma r e^{i\rho}}}{r^2}\right|$$
$$= 2\pi^2|\lambda|\int_0^{\infty} dr \exp\{-2\gamma r\cos\rho\} = \frac{\pi^2|\lambda|}{\gamma\cos\rho}.$$

Thus, for the Yamaguchi potential the envelope of the region containing the resonances can be determined from the equation

$$\operatorname{Re} z\, e^{2i\rho + i\beta} \geqslant -\frac{A}{\cos\rho},$$
$$A = \pi^2\sum_{i<j}^{N}\frac{|\lambda_{ij}|}{\gamma_{ij}}, \quad \beta\in(0, \pi/2), \ \rho\in(0, \pi/2). \quad (81)$$

It is difficult to construct the envelope of the two-parameter family of curves (81) in a general form. Therefore, as an orientation we calculate two characteristic points situated on the intersection of this envelope with the coordinate axes. It can be seen in Fig. 3 that the intersection of the envelope with respect to β with the x axis is at the point $A/(\sin 2\alpha\cos\alpha)$ for $0<\alpha<\pi/2$ and at the point $A/\cos\alpha$ for $\pi/4 \leqslant \alpha < \pi/2$. The maximum of the expression $\sin 2\alpha\cos\alpha$ can be determined from the condition $\tan\alpha = 1/\sqrt{2}$ and is $4/3\sqrt{3}$. Further, we have $\min_{\alpha\in(0,\pi/4)}[1/(\sin 2\alpha\cos\alpha)] = 3\sqrt{3}/4 < \sqrt{2} = \min_{\alpha\in(\pi/4,\pi/2)}(\cos\alpha)^{-1}$. Thus, the point of intersection of the envelope with the x axis is determined by the expression $3\sqrt{3}A/4$. If $\pi/2>\alpha>\pi/4$, then the envelope with respect to β intersects the y axis at the point $-A/[\sin(2\alpha-\pi/2)\cos\alpha] = A/(\cos 2\alpha\cos\alpha)$. The maximum of the expression $-\cos 2\alpha\cos\alpha = -2x^3 + x$,

$x = \cos\alpha$, is attained at $x = 1/\sqrt{6}$ and is equal to $2/3\sqrt{6}$. Therefore, the point of intersection of the envelope with the y axis is $-A\,3\sqrt{6}/2$. The approximate form of the envelope of the region containing the resonances for the sum of Yamaguchi potentials is shown in Fig. 5.

Integral equations for the resolvent of the Hamiltonian $H(\theta)$ in the four-body problem

Suppose the parameter z in the resolvent set of $H(\theta)$ lies in the region

$$D_0 = \{z \mid -2\operatorname{Im}\theta < \arg z < 0,\quad 0 \leqslant \operatorname{Im}\theta < \sigma\}.$$

We represent the resolvent $G^\theta(z)$, $z \in D_0$, in the form

$$G^\theta(z) = e^{2\theta} R^\theta(z(\theta)), \tag{82}$$

where $R^\theta(\zeta)$ is the resolvent of the operator $\widetilde{H}(\theta) = H_0 + e^{2\theta}V(\theta)$, $R^\theta(\zeta) = (\zeta - \widetilde{H}(\theta))^{-1}$, and the parameter $z(\theta)$ is equal to $ze^{2\theta}$ and therefore, lies on the physical sheet. Taking into account this circumstance and using the transformation (82), it is possible to construct integral equations for the resolvents $G^\theta(z)$ and $R^\theta(z)$ analogous to the system (21). As is readily seen, the kernel $A^\theta(z)$ of the system of integral equations for finding the operator $T^\theta(z)$,

$$T^\theta(z) = V(\theta) + V(\theta)\,G^\theta(z)\,V(\theta),$$

for values of z lying on the physical sheet can be obtained from the kernel of the system (21) by means of the transformation $U(\theta)$:

$$A^\theta(z) = U(\theta)\,A(z)\,U^{-1}(\theta). \tag{83}$$

At the same time, the elements of the matrix $B^\theta(z)$,

$$A^\theta(z) = B^\theta(z)\,G_0^\theta(z)$$

are operators $t_\alpha^\theta(z)$, $(M_{\alpha,\beta}^\eta)^\theta(z)$ and $\mathscr{F}_{\alpha\otimes\alpha'}^\theta(z)$. We obviously have

$$\left.\begin{aligned} t_\alpha^\theta(z) &= V_\alpha'(\theta) + V_\alpha(\theta)\,g_\alpha^\theta(z)\,V_\alpha(\theta) = V_\alpha(\theta) + e^{2\theta}V_\alpha(\theta) \\ &\quad\times r_\alpha(z(\theta))\,V_\alpha(\theta), \\ (M_{\alpha,\beta}^\eta)^\theta(z) &= V_\alpha(\theta)\,\delta_{\alpha,\beta} + V_\alpha(\theta)\,g_\eta^\theta(z)\,V_\beta(z) \\ &= V_\alpha(\theta)\,\delta_{\alpha,\beta} + e^{2\theta}V_\alpha(\theta)\,r_\eta^\theta(z(\theta))\,V_\beta(\theta), \end{aligned}\right\} \tag{84}$$

where $r_\alpha(\zeta) = (\zeta - \tilde{h}_\alpha(\theta))^{-1}$, $r_\eta(\zeta) - (\zeta - \tilde{h}_\eta(\theta))^{-1}$. The operator $\widetilde{\mathscr{F}}_{\alpha\otimes\alpha'}^\theta(z)$, which is obtained from $\mathscr{F}_{\alpha\otimes\alpha'}^\theta(z)$ by separating the relative motion of the centers of mass of pairs α and α', can be expressed in terms of the resolvent $r_{\alpha,\alpha'}(z) = (z - \tilde{h}_\alpha^{(\theta)} - \tilde{h}_{\alpha'}^{(\theta)})^{-1}$.

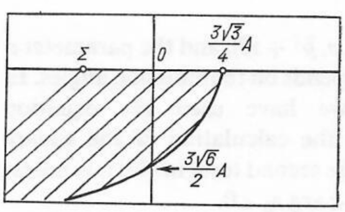

FIG. 5. Region containing the resonances for systems with two-body Yamaguchi potentials.

For the resolvent $r_{\alpha\alpha'}(t)$ we can write down a representation of the form

$$r_{\alpha,\alpha'}(z) = \int\limits_\Gamma \frac{d\varepsilon}{-2\pi i}\, r_\alpha(\varepsilon) \otimes r_{\alpha'}(z - \varepsilon), \tag{85}$$

where the contour Γ, which is traversed in the anticlockwise direction, has the following property. All points of Γ lie in the resolvent set of the Hamiltonian h_α, and the contour $\Gamma_1 = \{\zeta \mid z - \zeta_1, \zeta_1 \in \Gamma\}$ contains the spectrum of $h_{\alpha'}$ within it.

Equation (85) can be proved by direct calculation on the basis of the spectral properties of the operator $h_\alpha(\theta)$.

By means of (84) and (85), the kernel $A^\theta(z)$ can be analytically continued to the region D_0 of the unphysical sheet. At the same time, as was pointed out above, all elements of the matrix $B^\theta(z)$ will be calculated on the physical sheet of energies. In the case of the two- and three-particle kernels, this fact is obvious by virtue of (84), and for the kernel $\widetilde{\mathscr{F}}_{\alpha\otimes\alpha'}^\theta(z)$ it is sufficient to choose the contour Γ in the form shown in Fig. 6. The crosses in Fig. 6 are denoted from $\sigma_\alpha[\tilde{h}_\alpha(\theta)]$, and the points are the elements of the set $\{z(\theta)\} \setminus \sigma_d(\tilde{h}_{\alpha'}(\theta))$. It is readily seen that the necessary choice of the contour Γ is always possible except for the case when $z(\theta) \in \sigma_d(h_{\alpha,\alpha'}(\theta))$.

Thus, we have constructed a system of integral equations of the form

$$\mathscr{K}^\theta(z) = \mathscr{K}_0^\theta(z) + A^\theta(z)\,\mathscr{K}^\theta(z), \tag{86}$$

which has meaning for all z satisfying the condition $-2\operatorname{Im}\theta < \arg z < 2\pi - 2\operatorname{Im}\theta$. We now consider the properties of this system.

The elements of the matrix $B^\theta(z)$ are decomposed in the standard manner into components, and it is obvious that these components appear because of the points of the discrete spectra of the operators $\tilde{h}_\alpha(\theta)$, $\tilde{h}_\eta(\theta)$, and $\tilde{h}_{\alpha,\alpha'}(\theta)$. Since $G_0^\theta(z)$ in the region considered is nonsingular, the following proposition holds: The operator $A^\theta(z)$ is a Hilbert-Schmidt operator for $z \notin \sigma_{ess}(H(\theta))$, and the operator $U^\theta(z)$, the kernel of the system of integral equations for the components, is compact for all z, $-2\operatorname{Im}\theta < \arg z < 2\pi - 2\operatorname{Im}\theta$. Further, it is obvious that the nontrivial solutions of the homogeneous equation

$$\mathscr{K}^\theta(z) = A^\theta(z)\,\mathscr{K}^\theta(z),\quad z \notin \sigma_{ess}(H(\theta)),$$

describe the discrete spectrum of $H(\theta)$, i.e., the points from $\sigma_d(H)$ for $\operatorname{Im}\theta < \pi/2$ and the resonances that lie in D_0.

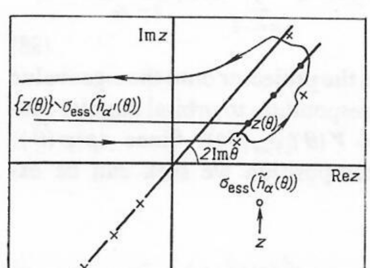

FIG. 6. Contour of integration in Eq. (85).

On the changes of the parameters of two-particle resonances in three-particle reactions

As an example of the use of the technique of complex scale transformations, we consider the question of the parameters of two-particle resonances produced in three-particle reactions under the condition that the energy of the system is equal to the energy of one of the three-particle resonances. In such a formulation, the problem corresponds to the conditions of the Migdal-Watson model (see, for example, Ref. 58). However, as will be shown below, analysis of the three-particle dynamics of the problem predicts, in contrast to the Migdal-Watson approximation, changes in the parameters of the two-particle resonances. In particular, it is found that the two-particle resonances whose background phase shifts satisfy the condition $\pi/2 \leqslant \delta_{bg} \leqslant \pi$ are shifted forward, while resonances with δ_{bg} between 0 and $\pi/2$ may be shifted either forward or backward. (We here adhere to the definition of the phase shift used in Ref. 21, p. 287 of the Russian translation, Fig. 13.3.) The magnitude of the shift can be predicted from the change in the shape of the resonance. This conclusion is confirmed by the analysis of the experimental data (see, for example, Ref. 59).

To separate a three-particle resonance, the formal theory of resonances is the most suitable. Assuming for simplicity that the three-particle resonance corresponds to total orbital angular momentum $L = 0$, we find that the decay vertex of the resonance $|\Phi(z)\rangle$ is proportional to $[1 + R(z)H]|\psi\rangle$, where $|\psi\rangle$ is an analytic vector for $U(\theta)$ on $\mathbf{L}^2(\mathbf{R}^6)$ and $\langle\Phi(\theta^*)|\psi(\theta)\rangle \neq 0$. Here, $|\Phi(\theta)\rangle$ denotes the wave function of the three-particle resonance. Then the amplitude of three-particle breakup (under the condition that the reaction proceeds through a sufficiently narrow isolated level of the compound nucleus) can be expressed approximately in the form

$$T(p_{12}, q, z) = C\,\langle p_{12}q_3\,[t_{12}(z)\,G_0(z) + 1] \times (V_{13} + V_{23})\,|\,\Phi(z)\rangle, \quad (87)$$

$$z = \tilde{p}_{12}^2 + \tilde{q}_3^2 + \mathrm{i}0.$$

In (87) we have assumed that in the subsystem 12 there is a fairly narrow resonance, and we have omitted the terms with $t_{13}(z)$ and $t_{23}(z)$ not containing resonances.

We separate the resonance part in $t_{12}(z)$. In accordance with the theory of complex scale transformations, the resonance contribution to $t^\theta(z)$ can be expressed by the term

$$\sum_{m=-l}^{l} \frac{V(\theta)\,|\,\psi_m(\theta)\rangle\langle\psi_m(\bar{\theta})\,|\,V(\theta)}{z-z_0} = \sum_{m=-l}^{l} \frac{|\,\varphi_m(\theta)\rangle\langle\varphi_m(\bar{\theta})\,|}{z-z_0}. \quad (88)$$

In (88), $\Sigma|\psi_m(\theta)\rangle\langle\psi_m(\overline{\theta})|$ is the projector onto the eigenvalue z_0 of the operator $h(\theta)$ corresponding to orbital angular momentum l and $|\varphi_m(\theta)\rangle = V(\theta)|\psi_m(\theta)\rangle$. Since $\langle p|\varphi(\overline{\theta})\rangle = \langle -p|\varphi(\theta)\rangle^*$, the decomposition we seek can be expressed in the form[3]

$$t^\theta(p, \bar{p}', z) = (-1)^l(2l+1)\,p_l(p\cdot p')\frac{\chi^\theta(p)\,\chi^\theta(p')}{z-z_0} + \tilde{t}^\theta(p, p', z), \quad (89)$$

where $\tilde{t}(z)$ is the nonresonance part of the amplitude. Setting $\theta = 0$, we obtain the contribution of the resonance to the amplitude $t(p, p', z)$ under the condition that $\chi^0(p)$ exists. In particular, this condition is satisfied for a superposition of Gaussian potentials if $-\pi/2 < \arg z_0$, and also for generalized Yukawa potentials of the type μ_0 (see Ref. 1):

$$V(r) = \frac{1}{r}\int^{\infty} d\zeta\sigma(\zeta)\,\mathrm{e}^{-\zeta r},$$

provided $\mu_0 + \mathrm{Im}(2\mu_{12}z_0)^{1/2} > 0$.

A result analogous to (89) can be obtained from Weinberg's quasiparticle method. Let α_j be a resonance eigenvalue of the kernel $VG_0(z)$, i.e., let α_j be fairly near unity. Then the contribution of this eigenvalue to the amplitude $t(p, p', z)$ can be described as follows[50]:

$$\frac{\langle p\,|\,\Phi_j(z)\rangle\langle -p'\,|\,\Phi_j(z)\rangle}{1-\alpha_j(z)}, \quad (90)$$

where $\alpha_j(z)\Phi_j(z) = VG_0(z)\Phi_j(z)$, and $\langle\Phi_j(\overline{z})\Phi_j(z)\rangle = 1$. Assuming that $\alpha_j(z_0) = 1$ and expanding $\Phi_j(z)$ and $\alpha_j(z)$ in powers of $z - z_0$, we obtain a result analogous to (89), and we can take $\chi^0(p)$ to be

$$\left(\left.\frac{d\alpha_j(z)}{dz}\right|_{z=z_0}\right)^{-1/2}\langle p\,|\,\Phi_j(\mathrm{Re}\,z_0 + \mathrm{i}0)\rangle.$$

We now transform the expression (27), for which we write $G_0(E + i0)$ in the form $\mathscr{P}[1/(E - H_0)] - \mathrm{i}\pi\delta(E - H_0)$ and note that $E = \tilde{p}_{12}^2 + \tilde{q}_3^2$. In addition, when calculating the integral with the principal value we do not consider nonresonance terms. As a result, we find that the amplitude $T(p_{12}, q_3, z)$ is proportional to

$$p_{12}t_{12}^l(p_{12})\,\Phi_1(p_{12}, q_3) + \frac{\chi^0(p_{12})}{\tilde{p}_{12}^2 - z_0}\,\Phi_2(p_{12}, q_3), \quad (91)$$

where

$$\left.\begin{aligned} \Phi_1(p_{12}, z_3) &= \mu_{12}\int d\Omega\, p_{12}'P_l(p_{12}, p_{12}') \\ &\times \langle p_{12}\frac{p_{12}'}{p_{12}'}q\,|\,(V_{13} + V_{23})\,|\,\Phi(E + \mathrm{i}0)\rangle; \\ \Phi_2(p_{12}, q_3) &= \mathscr{P}\int dp_{12}'\frac{\chi^0(p_{12}')\,P_l(p_{12}\cdot p_{12}')}{\tilde{p}_{12}^2 - \tilde{p}_{12}^{12}} \\ &\times \langle p_{12}'q_3\,|\,(V_{13} + V_{23})\,|\,\Phi(E + \mathrm{i}0)\rangle. \end{aligned}\right\} \quad (92)$$

Further, it can be assumed that the functions Φ_1 and Φ_2 depend weakly on p_{12} in the neighborhood of the resonance for fixed emission angles of the particles. This assumption is justified at least for a sufficiently narrow two-particle resonance. Then for $T(p_{12}, q_3, E + i\theta)$ we can obtain a parametrization of the form

$$p_{12}t_{12}^l(p_{12}) + \left[\frac{t_{12}^l(p_{12})}{\tilde{p}_{12}^2 - z_0}\right]^{1/2}\rho. \quad (93)$$

In (91) and (93), $t^l(p) = t^l(p, p, \tilde{p}^2 + \mathrm{i}0)$, and the parameter ρ in (93) is a constant which depends on the emission angles. In the derivation of (93), we have used the equation $t^l(p) = (\tilde{p}^2 - z_0)^{-1}\chi^{02}(p)$. In the calculation of the square root in the denominator in the second term in (93) it is necessary to remember that $-\pi < \arg z_0 < 0$.

It should be borne in mind that the parametrization (93) is possible only for the types of reaction which we are considering, i.e., ones that proceed through a three-particle resonance. Otherwise, the changes in the parameters of the two-particle resonance will depend to a strong degree on the off-shell effects in the three-particle amplitudes; in the case considered, the off-shell effects are to a large degree weakened.

We now turn to the analysis of the expression (93). The first term in (93) shifts the two-particle resonance along the energy axis independently of the background phase shift. But the second term depends on δ_{bg}, and for $0<\delta_{bg}<\pi/2$ this term shifts the resonance backward or forward. As a result, resonances with δ_{bg} can be shifted either forward or backward. The change in the shape of the resonance as a function of δ_{bg} can be studied similarly. The change in the position and shape of a resonance can be readily established for definite processes.

APPENDIX A. EXPRESSION FOR THE COMPONENTS OF THE OPERATOR $\mathscr{F}_{\alpha\otimes\alpha'}(z)$

On the basis of the results of Refs. 12, 13, and 27, the expressions for the components of the operator $\mathscr{F}_{\alpha\otimes\alpha'}(z)$ [for the operators $L^{\rho}_{\beta}(z), \beta,\rho=\alpha;\alpha';(\alpha,\alpha')$] can be reduced to the form

$$L^{\alpha}_{\alpha}(p_\alpha,\ p^0_\alpha,\ z)=\hat{t}_\alpha(p_\alpha p^0_\alpha;\ z+\varkappa^2_{\alpha'})+(z+\varkappa^2_\alpha+\varkappa^2_{\alpha'})^{-1}\psi_\alpha(p_\alpha)\psi^*_\alpha(p^0_\alpha),$$

$$L^{\alpha'}_{\alpha}(p_\alpha,\ p^0_{\alpha'},\ z)=\frac{\varphi_\alpha(p_\alpha)\varphi^*_{\alpha'}(p_{\alpha'})}{z-\tilde{p}^2_\alpha-\tilde{p}^{02}_{\alpha'}}+(z+\varkappa^2_\alpha+\varkappa^2_{\alpha'})^{-1}\psi_\alpha(p_\alpha)\psi^*_{\alpha'}(p^0_{\alpha'}),$$

$$L^{\alpha\alpha'}_{\alpha}(p_\alpha,\ p^0_\alpha p^0_{\alpha'},\ z)=[\tau_\alpha(p_\alpha p^0_\alpha,\ z+\varkappa^2_{\alpha'})-\tau_\alpha(p_\alpha p^0_\alpha,\ z-\tilde{p}^2_{\alpha'})]\psi^*_{\alpha'}(p^0_{\alpha'})$$

$$+\frac{\hat{t}_\alpha(p_\alpha p^0_\alpha,\ z-p^{02}_{\alpha'})\varphi^*_{\alpha'}(p^0_{\alpha'})}{z-\tilde{p}^2_\alpha-\tilde{p}^{02}_{\alpha'}}-\psi_\alpha(p_\alpha)\zeta^*_{\alpha\alpha'}(p^0_\alpha p^0_{\alpha'}). \tag{A1}$$

The function $\tau(p, p^0, z)$ in (A1) is defined as follows:

$$\tau(p,\ p^0,\ z)=\hat{t}(p,\ p^0,\ z)-V(p-p^0)$$

$$=\int dq\,\frac{t(p,\ q,\ \tilde{q}^2+i0)\,t(q,\ p^0,\ \tilde{q}^2-i0)}{z-\tilde{q}^2}. \tag{A2}$$

By virtue of the symmetry of the expression (14) for $\mathscr{F}_{\alpha\otimes\alpha'}(p_\alpha p_{\alpha'}, p^0_\alpha p^0_{\alpha'}, z)$ with respect to transposition of the "in" and "out" states, the expressions for $L^{\beta}_{\alpha}(z)$, $\beta=\alpha,\alpha',(\alpha,\alpha')$ can be obtained from (A1) by replacing the index α by α' and vice versa. The expressions for the operator $L^{\alpha}_{\alpha\alpha'}(z)$ can be obtained from $L^{\alpha\alpha'}_{\alpha}(z)$ by means of the relation

$$[L^{\alpha}_{\alpha\alpha'}(z)]^+=L^{\alpha\alpha'}_{\alpha}(z^*),$$

i.e.,

$$L^{\alpha}_{\alpha,\alpha'}(p_\alpha p_\alpha,\ p^0_\alpha,\ z)=[L^{\alpha,\alpha'}_{\alpha}(p^0_\alpha,\ p_\alpha p'_\alpha,\ z^*)]^*.$$

The expression for $L^{\alpha'}_{\alpha,\alpha'}(z)$ is obtained similarly.

The expression for $L^{\alpha\alpha}_{\alpha\alpha'}(z)$ has the form

$$L^{\alpha\alpha'}_{\alpha\alpha'}(z)=A_1(z)+A_2(z)+A_3(z). \tag{A3}$$

The first term in (A3) is

$$A_1(z)=\frac{\varphi_\alpha(p_\alpha)\varphi^*_\alpha(p^0_\alpha)}{\tilde{p}^2_\alpha-\tilde{p}^{02}_\alpha}[\Phi_{\alpha'}(p_{\alpha'}p^0_{\alpha'},\ \tilde{p}^2_\alpha)-\Phi_{\alpha'}(p_{\alpha'}p^0_{\alpha'},\ \tilde{p}^{02}_\alpha)], \tag{A4}$$

where

$$\Phi_{\alpha'}(p,\ p^0,\ a^2)=\frac{1}{-\varkappa^2_\alpha-a^2}[\tau_{\alpha'}(p,\ p^0,\ z+\varkappa^2_\alpha)-\tau_{\alpha'}(p,\ p^0,\ z-a^2)].$$

The expression for the second term in (A3) can be obtained from (A4) by interchanging the indices. The last term in (A3) is

$$A_3(z)=\sum_{i=1}^{4}B_i(z), \tag{A5}$$

where

$$B_1(z)=V_\alpha(p_\alpha-p^0_\alpha)V_{\alpha'}(p_{\alpha'}-p^0_{\alpha'})\times\left(\frac{1}{z-\tilde{p}^2_\alpha-\tilde{p}^{02}_\alpha}+\frac{1}{z-\tilde{p}^2_{\alpha'}-\tilde{p}^{02}_\alpha}\right),$$

$$B_2(z)=V_\alpha(p_\alpha-p^0_\alpha)\Bigg[\frac{\tau_{\alpha'}(p_{\alpha'}p^0_{\alpha'},\ z-\tilde{p}^2_\alpha)-\tau_{\alpha'}(p_{\alpha'}p^0_{\alpha'},\ z-\tilde{p}^{02}_\alpha)}{\tilde{p}^2_\alpha-p^{02}_\alpha}$$

$$+\frac{\tau_{\alpha'}(p_{\alpha'}p^0_{\alpha'},\ z-\tilde{p}^2_\alpha)}{z-\tilde{p}^2_\alpha-\tilde{p}^{02}_{\alpha'}}+\frac{\tau_{\alpha'}(p_{\alpha'},\ p^0_{\alpha'},\ z-\tilde{p}^{02}_\alpha)}{z-\tilde{p}^{02}_\alpha-\tilde{p}^2_{\alpha'}}\Bigg]. \tag{A6}$$

The expression for $B_3(z)$ is obtained from the expression for $B_2(z)$ by interchanging the indices. Finally, $B_4(z)$ is an integral of the form

$$B_4(z)=\int_{-\infty}^{\infty}\frac{d\varepsilon}{-2\pi i}\left(\frac{1}{\varepsilon-\tilde{p}^2_\alpha+i\tau_1}+\frac{1}{E-\varepsilon-\tilde{p}^2_{\alpha'}+i\tau_2}\right)\times\tau_\alpha(p_\alpha p^0_\alpha,\ \varepsilon+i\tau_1)\tau_{\alpha'}(p_{\alpha'}p^0_{\alpha'},\ E-\varepsilon+i\tau_2)\times\left(\frac{1}{\varepsilon-\tilde{p}^{02}_\alpha+i\tau_1}+\frac{1}{E-\varepsilon-p^{02}_{\alpha'}+i\tau_2}\right). \tag{A7}$$

By a transformation of the singular denominators, (A7) is reduced to the form

$$B_4(z)=\frac{1}{\tilde{p}^2_\alpha-\tilde{p}^{02}_\alpha}[\mathscr{D}(\tilde{p}^2_\alpha-i\tau_1)-\mathscr{D}(\tilde{p}^{02}_\alpha-i\tau_1)]$$

$$+\frac{1}{z-\tilde{p}^2_\alpha-\tilde{p}^2_{\alpha'}}[\mathscr{D}(\tilde{p}^2_\alpha-i\tau_2)-\mathscr{D}(E-\tilde{p}^{02}_{\alpha'}+i\tau_2)]$$

$$+\frac{1}{z-\tilde{p}^2_{\alpha'}-\tilde{p}^{02}_\alpha}[\mathscr{D}(\tilde{p}^{02}_\alpha-i\tau_1)-\mathscr{D}(E-\tilde{p}^2_{\alpha'}+i\tau_2)]$$

$$+\frac{1}{\tilde{p}^{02}_{\alpha'}-\tilde{p}^2_{\alpha'}}[\mathscr{D}(E-\tilde{p}^2_{\alpha'}+i\tau_2)-\mathscr{D}(E-\tilde{p}^{02}_{\alpha'}+i\tau_2)], \tag{A8}$$

where

$$\mathscr{D}(\zeta)=\int_{-\infty}^{\infty}\frac{d\varepsilon}{-2\pi i}\,\frac{\tau_\alpha(p_\alpha p^0_\alpha,\ \varepsilon+i\tau_1)\tau_{\alpha'}(p_{\alpha'}p^0_{\alpha'};\ E-\varepsilon+i\tau_2)}{\varepsilon-\zeta}. \tag{A9}$$

Note that in some cases, for example, for a separable Yamaguchi potential, the integral (A9) can be calculated explicitly. If in the decomposition of $\mathscr{F}_{\alpha\otimes\alpha'}(z)$ into components we ignore the term $B_4(z)$, then such an approximation [given by the explicit expressions (A1)–(A6)] will have the same meaning as the Amado-Lovelace approximation in the three-particle problem.[3,29] Note also that the connection between the properties of the operator $\mathscr{F}_{\alpha\otimes\alpha'}(z)$ and its components $L_\beta(z)$ and the cluster properties of the Møller operators and the S matrices in the system of two noninteracting subsystems of particles was studied in Refs. 11–13 and 27.

As is readily seen, terms of the form $[1/(\widetilde{p}_\alpha^2-\widetilde{p}_\alpha^{02})][\mathscr{D}(\widetilde{p}_\alpha^2-i\tau_1)-\mathscr{D}(\widetilde{p}_\alpha^{02}\widetilde{p}_\alpha^{02}-i\tau_1)]$ arise from transformation of an integral like

$$\int_{-\infty}^{\infty}\frac{d\varepsilon}{-2\pi i}\,\frac{\tau_\alpha(p_\alpha p_\alpha^0,\ \varepsilon+i\tau_1)\,\tau_2(E-\varepsilon+i\tau_2)}{(\varepsilon-\widetilde{p}_\alpha^2+i\tau_1)(\varepsilon-\widetilde{p}_\alpha^{02}+i\tau_1)}. \quad (A10)$$

Substituting in (A10) the expression for $\tau_{\alpha'},(z)$ in the form (A2) and noting that for $\tau_1\neq 0$ and $\tau_2\neq 0$ the integration can be performed in any order, we obtain for (A10) a representation of the form

$$\int dq_{\alpha'}\frac{\tau_\alpha(p_\alpha p_\alpha^0,\ z-\widetilde{q}_{\alpha'}^2)\,\tau_{\alpha'}(p_{\alpha'}q_{\alpha'},\ \widetilde{q}_{\alpha'}^2+i0)\,\tau_{\alpha'}(q_{\alpha'}p_{\alpha'}^0,\ \widetilde{q}_{\alpha'}^2-i0)}{(z-\widetilde{p}_\alpha^2-\widetilde{q}_{\alpha'}^2)(z-\widetilde{p}_\alpha^{02}-\widetilde{q}_{\alpha'}^2)}. \quad (A11)$$

It can be seen from (A11) that integrals of the type (A10) do not have any singularities for $z\in\mathbf{C}\setminus[0,\infty)$, since the denominators in (A11) are in this case nonsingular. From this result, the properties of the functions $\tau(p,p^0,z)$,[3,22] and the explicit expressions for the components $L_\alpha^\rho(z)$, we conclude that for $z\in\mathbf{C}\setminus[0,\infty)$ these components have no singularities with respect to either the energy or the momenta.

APPENDIX B. DECOMPOSITION OF THE THREE-PARTICLE AMPLITUDE $M_{\alpha,\beta}^\eta(z)$

Below, we shall obtain the decomposition (25) for the three-particle amplitude $M_{\alpha,\beta}^\eta(z)$. To shorten the expressions, we shall denote $p_{\alpha\eta}$ by q_α and omit everywhere the index η except in the notation for the energy of a three-particle bound state. The basis for the following treatment will be the relation proved in Ref. 22 for the Møller operators Ω_μ, $\mu=0,\alpha$:

$$\sum_\mu\overline{\Omega_\mu\Omega_\mu^+}=I-P_\alpha, \quad (B1)$$

where I is the identity operator in the Hilbert space of functions that depend on the variables p_α and q_α, and P_α is the projection operator onto the discrete spectrum of the Hamiltonian h_η. From (B1) we obtain the following representation for the operator $T_{\alpha\beta}(z)=V_\alpha\delta_{\alpha\beta}+V_\alpha(z-h_\eta)^{-1}V_\beta$:

$$T_{\alpha,\beta}(z)=V_\alpha\delta_{\alpha,\beta}+\frac{V_\alpha|\psi\rangle\langle\psi|V_\beta}{z+\varkappa_\eta^2}+\sum_\mu\int d(\cdot)_\mu\frac{V_\alpha\Omega_\mu|\cdot\rangle_\mu{}_\mu\langle\cdot|\Omega_\mu^+V_\beta}{z-E_\mu},$$

$$|\cdot\rangle_\alpha=|q_\alpha^0\rangle,\quad d(\cdot)_\alpha=dq_\alpha^0,\quad E_\alpha=-\varkappa_\alpha^2+q_\alpha^{02}/2n_\alpha=-\varkappa_\alpha^2+\widetilde{q}_\alpha^{02},$$

$$|\cdot\rangle_0+|p^0,\ q^0\rangle,\quad d(\cdot)_0=dp^0\,dq^0,\quad E_0=p_0^2/2\mu+q^{02}/2n=\widetilde{p}^{02}+\widetilde{q}^{02}. \quad (B2)$$

In accordance with the assumptions made above, $\langle p_\alpha q_\alpha|V_\alpha\Omega_\mu|p_\mu^0\rangle$ and $\langle p_\alpha q_\alpha|V_\alpha-\Omega_0|p_\alpha^0q_\alpha^0\rangle$ can be expressed uniquely in terms of the amplitudes $\mu_{\alpha\beta}(E+i0),E\in\sigma_c(h_\eta)$ and the components of these amplitudes. We have

$$\langle p_\alpha q_\alpha|V_\alpha\Omega_0|p_\alpha^0q_\alpha^0\rangle=t_\alpha(p_\alpha p_\alpha^0,\ \widetilde{p}_\alpha^{02})\,\delta(q_\alpha-q_\alpha^0)+\sum_\gamma M_{\alpha\gamma}(p_\alpha q_\alpha,\ p_\gamma^0q_\gamma^0,\ E_0+i0),$$

$$\langle p_\alpha q_\alpha|V_\alpha\Omega_\gamma|q_\gamma^0\rangle=\varphi_\alpha(p_\alpha)\,\delta(q_\alpha-q_\alpha^0)\,\delta_{\alpha,\gamma}+\zeta_{\alpha,\gamma}(p_\alpha q_\alpha,\ q_\gamma^0,\ E_\gamma+i0). \quad (B3)$$

In (B3), we have introduced "in" components of the operators $M_{\alpha,\gamma}(z)$:

$$M_{\alpha,\gamma}(p_\gamma q_\alpha,\ p_\gamma^0q_\gamma^0,\ z)=\tau_{\alpha\gamma}(p_\alpha q_\alpha,\ p_\gamma^0q_\gamma^0,\ z)+\frac{\zeta_{\alpha\gamma}(p_\alpha^0q_\alpha,\ q_\gamma^0,\ z)\,\varphi_\gamma^*(p_\gamma^0)}{z+\varkappa_\gamma^2-\widetilde{q}_\gamma^{02}}. \quad (B4)$$

Besides the decomposition (B4), in what follows we shall also need the "out" components of the operators $M_{\alpha\gamma}(z)$ and $\zeta_{\alpha\gamma}(z)$:

$$\left.\begin{aligned}M_{\alpha\gamma}(p_\alpha q_\alpha,\ p_\gamma^0q_\gamma^0,\ z)&=u_{\alpha\gamma}(p_\alpha q_\alpha,\ p_\gamma^0q_\gamma^0,\ z)+\frac{\varphi(p_\alpha)\,v_{\alpha\gamma}(q_\alpha,\ p_\gamma^0q_\gamma^0,\ z)}{z+\varkappa_\alpha^2-\widetilde{q}_\alpha^{12}};\\ \zeta_{\alpha,\gamma}(p_\alpha q_\alpha,\ q_\gamma^0,\ z)&=\rho_{\alpha\gamma}(p_\alpha q_\alpha,\ q_\gamma^0,\ z)+\frac{\varphi_\alpha(p_\alpha)\,\omega_{\alpha\gamma}(q_\alpha,\ q_\gamma^0,\ z)}{z+\varkappa_\alpha^2-\widetilde{q}_\alpha^2}.\end{aligned}\right\} \quad (B5)$$

We recall once more that for $z=E+i0$ and $E\in\sigma_c(h_\eta)$ the components of the operators $M_{\alpha\gamma}(z)$, i.e., functions of the type u, v, ζ, ρ, ω, etc., are uniquely determined by the solution of the three-particle integral equations.

Substituting (B3) in (B2), we obtain as a result of simple algebraic manipulations

$$\begin{aligned}&M_{\alpha,\beta}(p_\alpha q_\alpha,\ p_\beta'q_\beta',\ z)\\&=\int dp^0\,dq^0\frac{\langle p_\alpha q_\alpha|t_\alpha(E_0+i0)|p^0q^0\rangle}{z-E_0}\\&\times\langle p^0q^0|t_\beta(E_0-i0)|p_\beta'q_\beta'\rangle[1-\delta_{\alpha\beta}]\\&+\frac{\zeta_{\alpha\beta}(p_\alpha q_\alpha,\ p_\beta',\ E_\beta'+i0)\,\varphi_\beta^*(p_\beta')}{z-E_\beta'}+\frac{\varphi_\alpha(p_\alpha)\,\zeta_{\beta,\alpha}^*(p_\beta'q_\beta',\ q_\alpha,\ E_\alpha+i0)}{z-E_\alpha}\\&+\sum_{\gamma\neq0}\int dq_\gamma^0\frac{\zeta_{\alpha\gamma}(p_\alpha q_\alpha,\ q_\gamma^0,\ E_\gamma+i0)\,\zeta_{\beta\gamma}^*(p_\beta'q_\beta',\ q_\gamma^0,\ E_\gamma+i0)}{z-E_\gamma}\\&+\sum_{\gamma\neq0}\int dp^0\,dq^0\frac{M_{\alpha\gamma}(p_\alpha q_\alpha,\ p^0q^0,\ E_0+i0)\,t_\beta(p_\beta^0p_\beta,\ \widetilde{p}_\beta^{02}-i0)\,\delta(q_\alpha-q_\alpha^0)}{z-E_0}\\&+\sum_{\gamma\neq0}\int dp^0\,dq^0\frac{t_\alpha(p_\alpha p_\alpha^0,\ \widetilde{p}_\alpha^{02}+i0)\,M_{\beta\gamma}^*(p_\beta'q_\beta',\ p^0q^0,\ E_0+i0)\,\delta(q_\alpha-q_\alpha^0)}{z-E_0}\\&+\sum_{\mu,\ \gamma\neq0}\int dp^0\,dq^0\frac{M_{\alpha\gamma}(p_\alpha q_\alpha,\ p^0q^0,\ E_0+i0)\,M_{\beta,\mu}^*(p_\beta'q_\beta',\ p^0q^0,\ E_0+i0)}{z-E_0},\end{aligned}$$

$$E_\alpha=-\varkappa_\alpha^2+\widetilde{q}_\alpha^2,\quad E_\beta'=-\varkappa_\beta^2+\widetilde{q}_\beta'^2. \quad (B6)$$

The next step consists of decomposing (B6) into components. For this, it is necessary to substitute in (B6) the decompositions (B5) and transform the singular denominators. One of the possible ways of doing this is to use the identities

$$\frac{1}{a}\,\frac{1}{A}=\frac{1}{A+a}\left(\frac{1}{a}+\frac{1}{A}\right);\quad \frac{1}{B}\,\frac{1}{a}=\frac{1}{B+a}\left(\frac{1}{B}+\frac{1}{a}\right);$$
$$\frac{1}{A}\,\frac{1}{a}\,\frac{1}{B}=\frac{1}{A+a}\,\frac{1}{A}\,\frac{1}{B}+\frac{1}{A+a}\,\frac{1}{B+a}\left(\frac{1}{a}+\frac{1}{B}\right),$$

where $A^{-1}=(E+\mathrm{i}0-E_\alpha)^{-1}$, $B^{-1}=(E-\mathrm{i}0+E'_\beta)^{-1}$, $a^{-1}=(z-E)^{-1}$, so that $(A+a)^{-1}=(z-E_a)^{-1}$ and $(B+a)^{-1}=(z-E'_\beta)^{-1}$. Performing these operations, we obtain

$$M_{\alpha\beta}(p_\alpha q_\alpha,\ p'_\alpha q'_\alpha,\ z)=\Phi(p_\alpha q_\alpha,\ p'_\beta q'_\beta,\ z)$$
$$+\varphi_\alpha(p_\alpha)\,\varphi^*_\beta(p'_\beta)$$
$$\times\left[\frac{I_1(q_\alpha q'_\beta)}{z-E_\alpha}+\frac{I_2(q_\alpha q'_\beta)}{(E'_\beta+\mathrm{i}0-E_\alpha)(z-E'_\beta)}+\frac{I_3(q_\alpha,\ q'_\beta)}{(z-E_\alpha)(E_\alpha-\mathrm{i}0-E'_\beta)}\right], \tag{B7}$$

where

$$I_1(q_\alpha,\ q'_\beta)=\int dp^0\,dq^0\,\frac{\langle\varphi_\alpha q_\alpha\mid p^0q^0\rangle\langle p^0q^0\mid\varphi_\beta q'_\beta\rangle(1-\delta_{\alpha\beta})}{(E_0+\mathrm{i}0-E_\alpha)(E_0-\mathrm{i}0-E'_\beta)}$$
$$+\sum_{\gamma\neq0}\int dq^0_\gamma\,\frac{\omega_{\alpha\gamma}(q_\alpha q^0_\gamma;\ E_\gamma+\mathrm{i}0)\,\omega^*_{\beta\gamma}(q'_\beta q^0_\gamma,\ E_\gamma+\mathrm{i}0)}{(E_\gamma+\mathrm{i}0-E_\alpha)(E_\gamma-\mathrm{i}0-E'_\beta)}$$
$$+\sum_{\gamma,\,\mu\neq0}\int dp^0\,dq^0\,\frac{v_{\alpha\gamma}(q_\alpha,\ p^0q^0,\ E_0+\mathrm{i}0)\,v^*_{\beta,\,\mu}(q'_\beta,\ p^0q^0,\ E_0+\mathrm{i}0)}{(E_0+i0-E_\alpha)(E_\gamma-\mathrm{i}0-E'_\beta)};$$
$$I_2(q_\alpha q'_\beta)=\omega_{\alpha\beta}(q_\alpha q'_\beta,\ E'_\beta+\mathrm{i}0);$$
$$I_3(q_\alpha q'_\beta)=\omega^*_{\beta\alpha}(q'_\beta,\ q_\alpha,\ E_\alpha+\mathrm{i}0);$$

$$\Phi(q_\alpha p_\alpha,\ p'_\beta,\ q'_\beta,\ z)=\Phi_1(p_\alpha q_\alpha p'_\beta q'_\beta,\ z)+\frac{\varphi_\alpha(p_\alpha)\,\Phi_2(q_\alpha p'_\beta q'_\beta,\ z)}{z-E_\alpha}$$
$$+\frac{\Phi_3(p_\alpha q_\alpha,\ q'_\beta,\ z)\,\varphi^*_\beta(p'_\beta)}{z-E'_\beta}+\frac{\varphi_\alpha(p_\alpha)\,\Phi_4(q_\alpha,\ q'_\beta,\ z)\,\varphi^*_\beta(p'_\beta)}{(z-E_\alpha)(z-E'_\beta)}. \tag{B8}$$

In turn, the functions $\Phi_i(z)$, $i=1, 2, 3, 4$, have the form

$$\Phi_1(p_2q_\alpha,\ p'_\beta q'_\beta,\ z)=\int dp^0\,dq^0\,\frac{\langle p_\alpha q_\alpha\mid t_\alpha(E_0+\mathrm{i}0)\mid p^0q^0\rangle}{z-E_0}$$
$$\times\langle p^0q^0\mid t_\beta(E_0-\mathrm{i}0)\mid p'_\beta q'_\beta\rangle(1-\delta_{\alpha\beta})+$$
$$+\sum_{\gamma\neq0}\int dq^0_\gamma\,\frac{\rho_{\alpha,\,\gamma}(p_\alpha q_\alpha,\ q^0_\gamma,\ E_\gamma+\mathrm{i}0)\,\rho^*_{\beta,\,\gamma}(p'_\beta q'_\beta,\ q^0_\gamma,\ E_\gamma+\mathrm{i}0)}{z-E_\gamma}$$
$$+\int dp^0\,dq^0\Big[\sum_{\gamma\neq0}u_{\gamma\alpha}(p_\alpha q_\alpha,\ p^0q^0,\ E_0+\mathrm{i}0)\langle p^0q^0\mid t_\beta(E_0-\mathrm{i}0)\mid p'_\beta q'_\beta\rangle$$
$$+\sum_{\gamma\neq0}\langle p_\alpha q_\alpha\mid t_\alpha(E_0+\mathrm{i}0)\mid p^0q^0\rangle\,u^*_{\beta\gamma}(p'_\beta q'_\beta,\ p^0q^0,\ E_0+\mathrm{i}0)$$
$$+\sum_{\gamma,\,\mu\neq0}u_{\alpha,\,\gamma}(p_\alpha q_\alpha,\ p^0q^0,\ E_0+\mathrm{i}0)\,u^*_{\beta\mu}(p'_\beta q'_\beta,\ p^0q^0,\ E_0+\mathrm{i}0)\Big]\frac{1}{z-E_0};$$

$$\Phi_2(q_\alpha,\ p'_\beta q'_\beta,\ z)=\int dp^0\,dq^0\left(\frac{1}{z-E_0}+\frac{1}{E_0+\mathrm{i}0-E_\alpha}\right)$$
$$\times\langle\varphi_\alpha q_\alpha\mid p^0q^0\rangle\langle p^0q^0\mid t_\beta(E_0$$
$$+\mathrm{i}0)\mid p'_\beta q'_\beta\rangle(1-\delta_{\alpha\beta})+\rho^*_{\beta\alpha}(p'_\beta q'_\beta,\ q_\alpha,\ E_\alpha+\mathrm{i}0)$$
$$+\sum_{\gamma\neq0}\int dq^0_\gamma\left(\frac{1}{z-E_\gamma}+\frac{1}{E_\gamma+\mathrm{i}0-E_\alpha}\right)$$
$$\times\omega_{\alpha\gamma}(q_\alpha q^0_\gamma,\ E_\gamma+\mathrm{i}0)\,\rho^*_{\beta\gamma}(p'_\beta q'_\beta,\ q^0_\gamma,\ E_\gamma+\mathrm{i}0)$$
$$+\sum_{\gamma\neq0}\int dp^0\,dq^0\left(\frac{1}{z-E_0}+\frac{1}{E_0+\mathrm{i}0-E_\alpha}\right)$$
$$\times v_{\alpha\gamma}(q_\alpha,\ p^0q^0,\ E_0+\mathrm{i}0)\langle p^0q^0\mid t_\beta(E_0-\mathrm{i}0)\mid p'_\beta q'_\beta\rangle$$
$$+\sum_{\gamma,\,\mu\neq0}\int dp^0\,dq^0\left(\frac{1}{z-E_0}+\frac{1}{E_0+i0-E_\alpha}\right)v_{\alpha\gamma}(p_\alpha p^0q^0,\ E_0+\mathrm{i}0)$$
$$\times u^*_{\beta\mu}(p'_\beta q'_\beta,\ p^0q^0,\ E_0+\mathrm{i}0). \tag{B9}$$

The expressions for the functions $\Phi_2(q_\alpha,p'_\beta q'_\beta,z)$ and $\Phi_3(q_\alpha p_\alpha,q'_\beta,z)$ are related by

$$\Phi_2(q,\ p,\ k,\ z^*)=\Phi^*_3(p,\ k,\ q,\ z),$$

in which the function $\Phi_2(z)$ must be taken, not for the operator $M_{\alpha,\beta}(z)$ as in (B9), but for the operator $M_{\beta,\alpha}(z)$. Finally, we write down the expression for $\Phi_4(z)$:

$$\Phi_4(q_\alpha,\ q'_\beta,\ z)=\int dp^0\,dq^0\left(\frac{1}{z-E_0}+\frac{1}{E_0-\mathrm{i}0-E'_\beta}\right)\langle\varphi_\alpha q_\alpha\mid p^0q^0\rangle$$
$$\langle p^0q^0\mid\varphi_\beta q'_\beta\rangle(1-\delta_{\alpha\beta})+\sum_{\gamma\neq0}\int dq^0_\gamma\left(\frac{1}{z-E_\gamma}+\frac{1}{E_\gamma-\mathrm{i}0-E'_\beta}\right)$$
$$\times\omega_{\alpha\gamma}(q_\alpha q^0_\gamma,\ E_\gamma+\mathrm{i}0)\,\omega^*_{\beta\gamma}(q'_\beta,\ q^0_\beta,\ E_\gamma+\mathrm{i}0)$$
$$+\sum_{\gamma,\,\mu\neq0}\int dp^0\,dq^0\left(\frac{1}{z-E_0}+\frac{1}{E_0-\mathrm{i}0-E'_\beta}\right)$$
$$\times v_{\alpha\gamma}(p_\alpha q_\alpha,\ p^0q^0,\ E_0+\mathrm{i}0)$$
$$\times v^*_{\beta\mu}(p'_\beta q'_\beta,\ p^0q^0,\ E_0+\mathrm{i}0).$$

We now transform the last term in (B7), for which we prove an identity relating I_1, I_2, and I_3,

$$I_1(q_\alpha,\ q'_\beta)+\frac{I_2(q_\alpha,\ q'_\beta)}{E'_\beta+\mathrm{i}0-E_\alpha}+\frac{I_3(q_\alpha,\ q'_\beta)}{E_\alpha-\mathrm{i}0-E'_\beta}=\Phi_5(q_\alpha,\ q'_\beta), \tag{B10}$$

to the function $\Phi_5(q_\alpha,q'_\beta)$, which is continuous with respect to its variables. To this end, we consider the identity

$$\langle\psi_\alpha q_\alpha\mid\sum_\mu\Omega_\mu\Omega^+_\mu\mid\psi_\beta q'_\beta\rangle\langle\psi_\alpha q_\alpha\mid I-P_d\mid\psi_\beta q'_\beta\rangle. \tag{B11}$$

In accordance with the properties of the components $v_{\alpha\beta}(q_\alpha,q'_\beta,p'_\beta,z)$ and $\omega_{\alpha\beta}(q_\alpha,q'_\beta,z)$ we have [see Ref. 22, Eqs. (5.25), (5.26), (9.13), (9.14), etc.]

$$\left.\begin{aligned}\langle\psi_\alpha q_\alpha\mid\Omega_\gamma\mid\cdot\rangle_\gamma&=\langle\psi_\alpha q_\alpha\mid\cdot\rangle_\gamma+\frac{\omega_{\alpha\gamma}(q_\alpha,\ q^0_\gamma E^0_\gamma+\mathrm{i}0)}{E_\gamma+\mathrm{i}0-E_\alpha}\\&\quad+\lambda_{\alpha\gamma}(q_\alpha q^0_\gamma E_\gamma+\mathrm{i}0),\\ \langle\psi_\alpha q_\alpha\mid\Omega_0\mid\cdot\rangle_0&=\sum_\gamma\Big[\frac{v_{\alpha,\,\gamma}(q_\alpha,\ p^0q^0,\ E_0+\mathrm{i}0)}{E_0+\mathrm{i}0-E_\alpha}\\&\quad+\eta_{\alpha\gamma}(q_\alpha,\ p^0q^0,\ E_0+\mathrm{i}0).\end{aligned}\right\} \tag{B12}$$

The functions $\omega_{\alpha\gamma}(q_\alpha,q^0_\gamma,E_\gamma+\mathrm{i}0)$ and $\lambda_{\alpha,\gamma}(q_\alpha,q^0_\gamma,E_\gamma+\mathrm{i}0)$ in (B12) are continuous functions of their variables (see Lemma 9.1 from Ref. 22), and the functions $v_{\alpha,\gamma}(q_\alpha,p^0,q^0,E+\mathrm{i}0)$ and $\eta_{\alpha\gamma}(q_\alpha p^0q^0,E_0+\mathrm{i}0)$ can have at most only secondary singularities, of which there are none in the neighborhood of the point $\tilde{q}^2_\alpha=E_0+\varkappa^2_\alpha$. Substituting the decomposition (B12) in (B11), we arrive at the identity (B10) with a function Φ_5 of the form

$$\Phi_5(q_\alpha q'_\beta)=\langle\psi_\alpha q_\alpha\mid P_d\mid\psi_\beta q'_\beta\rangle+\langle\psi_\alpha q_\alpha\mid\psi_\beta q'_\beta\rangle(1-\delta_{\alpha\beta})$$
$$+\sum_{\alpha\neq\gamma\neq\beta}\int dq^0_\gamma\langle\psi_\alpha q_\alpha\mid\psi_\gamma q^0_\gamma\rangle\langle\psi_\gamma q^0_\gamma\mid\psi_\beta q'_\beta\rangle$$
$$+\sum_{\gamma\neq\beta}\int d\gamma^0_\gamma\,\frac{\omega_{\alpha\gamma}(q_\alpha,\ q^0_\gamma,\ E_\gamma+\mathrm{i}0)\langle\psi_\gamma q^0_\gamma\mid\psi_\beta q'_\beta\rangle}{E_\gamma+\mathrm{i}0-E_\alpha}$$
$$+\sum_{\gamma\neq\alpha}\int dq^0_\gamma\,\frac{\langle\psi_\alpha q_\alpha\mid\varphi_\gamma q^0_\gamma\rangle\,\omega^*_{\beta\gamma}(q'_\beta q^0_\gamma,\ E_\gamma+\mathrm{i}0)}{E_\gamma-\mathrm{i}0-E'_\beta}$$
$$+\sum_\gamma\int dq^0_\gamma\Big\{\lambda_{\alpha,\,\gamma}(q_\alpha,\ q^0_\gamma,\ E_\gamma+\mathrm{i}0)\Big[\langle\psi_\gamma q^0_\gamma\mid\psi_\beta q'_\beta\rangle,$$

$$
+\frac{\omega^*_{\beta\gamma}(q'_\beta q^0_\gamma,\ E_\gamma+\mathrm{i}0)}{E_\gamma-\mathrm{i}0-E'_\beta}
$$

$$
+\lambda^*_{\beta\gamma}(q'_\beta q^0_\gamma,\ E_\gamma+\mathrm{i}0)\Big]+\Big[\langle\psi_\alpha q_\alpha\mid\psi_\gamma q^0_\gamma\rangle
$$

$$
+\frac{\omega_{\alpha\gamma}(q_\alpha q^0_\gamma,\ E_\gamma+\mathrm{i}0)}{E_\gamma+\mathrm{i}0-E_\alpha}\Big]\lambda^*_{\beta\gamma}(q'_\beta,\ q^0_\gamma,\ E_\gamma+\mathrm{i}0)\Big\}
$$

$$
+\sum_{\gamma,\,\mu\neq 0}\int dp^0\,dq^0\Big[\frac{v_{\alpha\gamma}(q_\alpha,\ p^0q^0,\ E_0+\mathrm{i}0)}{E_0+\mathrm{i}0-E_\alpha}\eta^*_{\beta\mu}(q'_\beta,\ p^0q^0,\ E_0+\mathrm{i}0)
$$

$$
+\eta_{\alpha\gamma}(q_\alpha p^0q^0,\ E_0+\mathrm{i}0)\frac{v^*_{\beta\mu}(q'_\beta,\ p^0q^0,\ E_0+\mathrm{i}0)}{E_0-\mathrm{i}0-E'_\beta}
$$

$$
+\eta_{\alpha\gamma}(q_\alpha,\ p^0q^0,\ E_0+\mathrm{i}0)\,\eta^*_{\beta\mu}(p'_\beta,\ p^0q^0,\ E_0+\mathrm{i}0)\Big]. \tag{B13}
$$

One can show that $\Phi_5(q_\alpha, q'_\beta)$ is a continuous (Hölder) function of its variables. Combining (B7) and (B10), we obtain the result

$$
M_{\alpha\beta}(p_\alpha q_\alpha,\ p'_\beta q'_\beta,\ z)=\Phi(p_\alpha q_\alpha,\ p'_\beta q'_\beta,\ z)
$$

$$
+\frac{\varphi_\alpha(p_\alpha)\,\Phi_5(q_\alpha q'_\beta)\,\varphi^*_\beta(p'_\beta)}{z-E_\alpha}
$$

$$
+\frac{\varphi_\alpha(p_\alpha)\,\varphi^*_\beta(p'_\beta)}{E'_\beta+\mathrm{i}0-E_\alpha}\left(\frac{1}{z-E'_\beta}-\frac{1}{z-E_\alpha}\right)I_2(q_\alpha q'_\beta)
$$

$$
=\Phi(p_\alpha q_\alpha,\ p'_\beta q'_\beta,\ z)+\frac{\varphi_\alpha(p_\alpha)\,\Phi_5(q_\alpha q'_\beta)\,\varphi^*_\beta(p'_\beta)}{z-E_\alpha}
$$

$$
+\frac{\varphi_\alpha(p_\alpha)\,I_2(q_\alpha q'_\beta)\,\varphi^*_\beta(p'_\beta)}{(z-E_\alpha)(z-E_{\beta'})}. \tag{B14}
$$

The last equation is satisfied in the sense of generalized functions (for details, see Ref. 12). From (B14), we readily obtain the decomposition (25). At the same time,

$$
\mathcal{J}^{(1)}_{\alpha\beta}(q_\alpha p'_\beta q'_\beta,\ z)=\Phi_2(q_\alpha p'_\beta q'_\beta,\ z)+\Phi_5(q_\alpha q'_\beta)\,\varphi^*_\beta(p'_\beta);
$$

$$
\mathcal{J}^{(2)}_{\alpha,\beta}(p_\alpha q_\alpha,\ q'_\beta,\ z)=\Phi_3(p_\alpha q_\alpha,\ q'_\beta,\ z);
$$

$$
\mathcal{J}^{(3)}_{\alpha,\beta}(q_\alpha,\ q'_\beta,\ z)=\Phi_4(q_\alpha,\ q'_\beta,\ z)+\omega_{\alpha\beta}(q_\alpha,\ q'_\beta,\ E'_\beta+\mathrm{i}0);
$$

$$
\mathcal{J}^{0}_{\alpha,\beta}(p_\alpha q_\alpha,\ p'_\beta q_\beta,\ z)=\Phi_1(p_\alpha q_\alpha,\ p'_\beta q'_\beta,\ z). \tag{B15}
$$

It is readily seen that the asymmetry which arises in (B15) with respect to interchange of the "in" and "out" states is due to the nonsymmetric form of the transformation of the three singular denominators. In a sense, this asymmetry is unavoidable but it does not have fundamental significance. On the basis of the explicit expressions for functions of the type Φ, it can be shown that as $z\to E_\alpha+\mathrm{i}0$ or $z\to'_\beta+\mathrm{i}0$ the functions Φ_i, $i=2, 3, 4$, reduce to components of the type v, ζ, ω. In particular, it can be readily established from (B15) that

$$
\lim_{z\to E'_\beta+\mathrm{i}0}\mathcal{J}_{\alpha\beta}(q_\alpha q'_\beta,\ z)=\omega_{\alpha\beta}(q_\alpha q'_\beta\ E'_\beta+\mathrm{i}0).
$$

Let us draw some conclusions from the above. To separate in explicit form the singularities with respect to z in the neighborhood of the point $-\varkappa^2_\eta\in\sigma_\mathrm{d}(h_\eta)$ we have used the spectral expansion (B2) for the scattering amplitude, in which we have transformed the singular denominators and eliminated by means of the identity (B10) the singular denominators $(E_\alpha+\mathrm{i}0-E'_\beta)^{-1}$ that appear in such an approach. As a result, the components of the three-particle amplitude are expressed in terms of the solutions of the three-particle integral equations for $z=E+\mathrm{i}0$, $E\in\sigma_\mathrm{c}(h_\eta)=(\Sigma_\eta,\infty)$, and by virtue of the assumptions made these solutions are unique. We note in passing that this approach can be used to study the properties of components of the type u, v, ζ, ω and to obtain for these components approximations consistent with unitarity.

APPENDIX C. THE OPERATOR $\mathcal{F}_{\alpha\otimes\alpha'}(z)$ AND ASYMPTOTIC BEHAVIOR OF THE WAVE FUNCTION OF THE SYSTEM IN THE COORDINATE REPRESENTATION

This appendix is devoted to the elimination of the problem of the coordinate asymptotic behaviors of the wave functions of a system of two particles in an external field and four particles with finite masses in the part of the problem associated with kernels of the type $\mathcal{F}_{\alpha\otimes\alpha'}(z)$. This means that we shall study the asymptotic behavior of expressions of the form

$$
G_0(z)\,\mathcal{F}_{\alpha\otimes\alpha'}(z)\,G_0(z)\,S(z)|\psi_\mathrm{in}\rangle.
$$

We begin with the case of a system of two particles in an external field. We shall be interested in the behavior of the matrix elements $\langle r_1r_2|\psi_{1\otimes 2}(E+\mathrm{i}0)\rangle$ as $r_1\to\infty$, $r_2\to\infty$, where

$$
|\psi_{1\otimes 2}(z)\rangle=\int_{-\infty}^{\infty}\frac{d\varepsilon}{-2\pi\mathrm{i}}[g_{01}(\varepsilon+\mathrm{i}\tau_1)t_1(\varepsilon+\mathrm{i}\tau_1)g_{01}(\varepsilon+\mathrm{i}\tau_1)]
$$

$$
\otimes[g_{02}(E-\varepsilon+\mathrm{i}\tau_2)\,t_2(E-\varepsilon+\mathrm{i}\tau_2)\,g_{02}(E-\varepsilon+\mathrm{i}\tau_2)]\,S(z)\mid\varphi_\mathrm{in}\rangle. \tag{C1}
$$

Since

$$
\langle r\mid g_0(E+\mathrm{i}0)\mid r'\rangle=-\frac{\mu}{2\pi}\frac{\mathrm{e}^{\mathrm{i}\sqrt{2\mu E}\,|r-r'|}}{|r-r'|},
$$

the expression for $\langle r_1r_2|\psi_{1\otimes 2}(E+\mathrm{i}0)\rangle$ for r_1 and r_2 tending to infinity can be reduced to the form

$$
\langle r_1r_2\mid\psi_{1\otimes 2}(E+\mathrm{i}0)\mid_{r_1\to\infty,\ r_2\to\infty}\sim\frac{\mu_1\mu_2}{(2\pi)^2}\int_0^E\frac{d\varepsilon}{-2\pi\mathrm{i}}
$$

$$
\times\exp\{\mathrm{i}\sqrt{2\mu_1\varepsilon}\,r_1+\mathrm{i}\sqrt{2\mu_2(E-\varepsilon)}\,r_2\}(r_1r_2)^{-1}
$$

$$
\times\langle k_1(\varepsilon),\ k_2(\varepsilon)\mid R(\varepsilon)\,S(E+\mathrm{i}0)\mid\psi_\mathrm{in}\rangle. \tag{C2}
$$

In (C2), $\mathbf{k}_1(\varepsilon)=\sqrt{2\mu_1\varepsilon}(\mathbf{r}_1/r_1)$, $\mathbf{k}_2(\varepsilon)=\sqrt{2\mu_2(E-\varepsilon)}(\mathbf{r}_2/r_2)$ and

$$
R(\varepsilon)=[t_1(\varepsilon+\mathrm{i}0)\,g_{01}(\varepsilon+\mathrm{i}0)]
$$

$$
\otimes[t_2(E-\varepsilon+\mathrm{i}0)\,q_{02}(E+\mathrm{i}0-\varepsilon)].
$$

Further, we shall assume that the relation $r_1=r_2\alpha$ holds; this means that the ratio of the velocities of the particles after scattering is α. A similar device was used in Ref. 35 to calculate the coordinate asymptotic behaviors of the three-particle wave function. For convenience, we set $\alpha=v_1/v_2$, the relation $\mu_1v_1^2/2+\mu_2v_2^2/2=E$ holding. The asymptotic behavior of the integral (C2) under the condition that $\langle k_1(\varepsilon)\bar{k}_2(\varepsilon)|R(\varepsilon)S(E+\mathrm{i}0)|\psi_\mathrm{in}\rangle$ is a continuous function of ε on the interval of integration can be calculated by the method of stationary phase (for a discussion of this method, see, for example, Ref. 34). The function $\sqrt{2\mu_1\varepsilon}$

$+\sqrt{2\mu_2(E-\varepsilon)}/v_2/v_1$ on the interval $(0,E)$ has a unique—and nondegenerate—stationary point $\varepsilon=\mu_1 v_1^2/2$. As a result, we have

$$\langle r_1 r_2 \mid \psi_{1\otimes 2}(E+\mathrm{i}0) \mid \psi_{\mathrm{in}}\rangle\,|_{r_1\to\infty,\ r_2\to\infty,\ r_1/r_2=v_1/v_2} \sim A\exp\{\mathrm{i}k_1 r_1+\mathrm{i}k_2 r_2\}(r_1 r_2)^{-1}\,T(k_1 k_2), \qquad \text{(C3)}$$

where

$$k_1=\mu_1 v_1;\quad k_2=\mu_2 v_2;\quad T(k_1 k_2)=\langle k_1 k_2 \mid R(\widetilde{k}_1^2)\,S(E+\mathrm{i}0)\mid \psi_{\mathrm{in}}\rangle,$$

and the coefficient A is determined by the "range of action" of the point of stationary phase. Although in (C3) we separate independent scattering of the particle by the center, in what follows, this effect will be masked by the circumstance that the coefficient A depends on r_1:

$$A=\frac{(\mu_1\mu_2)^{3/2}\,\mathrm{e}^{\mathrm{i}\pi/4}\,v_1 v_2}{8\pi^2\sqrt{\pi}\sqrt{\dfrac{r_1}{v_1}E}}. \qquad \text{(C4)}$$

If we introduce the parameter ρ by means of the relation $\rho^2=2\mu_1 r_1^2+2\mu_2 r_2^2$, the expressions (C3) and (C4) can be reduced to the form

$$\langle r_1 r_2 \mid \psi_{1\otimes 2}(E+\mathrm{i}0)\rangle\,|_{r_1\to\infty,\ r_2\to\infty,\ r_1/r_2=v_1/v_2} \sim \frac{E^{3/4}(\mu_1\mu_2)^{3/2}\,\mathrm{e}^{\mathrm{i}\pi/4}}{\pi^2\sqrt{2\pi}}\,\frac{\mathrm{e}^{\mathrm{i}\sqrt{E}\rho}}{\rho^{5/2}}\,T(k_1 k_2). \qquad \text{(C5)}$$

The result (C5) agrees with the expression for the coordinate asymptotic behavior of the three-particle wave function given in Ref. 35.

The expressions (C3) and (C5) describe the asymptotic behavior of the wave function of the system of two particles in the external field in the coordinate representation, provided that $\langle k_1(\varepsilon)k_2(\varepsilon)|R(\varepsilon)S(E+\mathrm{i}0)|\psi_{\mathrm{in}}\rangle$ is a continuous function of ε for $\varepsilon\in(0,E)$. This is always the case if the initial channel in the system contains a bound state. But if the asymptotic Hamiltonian in the "in" state is H_0, this is not the case. As we shall now see, it is in fact for zero initial channel that the effect of simultaneous independent scattering of the particles by the force center can be separated in pure form.

We specify the initial state in the system in the form $|p_1^0 p_2^0\rangle$, and we assume that $E=\tilde{p}_1^{02}+\tilde{p}_2^{02}$ and $p_1^0+p_2^0=0$. We separate in $S(E+i0)$ the amplitude $t_{12}(E+i0)$ and consider the contribution of this amplitude to (C2). We obtain

$$I=\langle r_1 r_2 \mid G_0(E+\mathrm{i}0)\,\mathcal{F}_{1\otimes 2}(E+\mathrm{i}0)\,G_0(E+\mathrm{i}0)\,t_{12}(E+\mathrm{i}0)\mid p_1^0 p_2^0\rangle_{\substack{r_1\to\infty\\ r_2\to\infty}}$$
$$\sim\frac{\mu_1\mu_2}{(2\pi)^2}\int dq_1\int_0^E\frac{d\varepsilon}{-2\pi\mathrm{i}}\,\frac{\exp\{\mathrm{i}\sqrt{2\mu_1\varepsilon}\,r_1+\mathrm{i}\sqrt{2\mu_2(E-\varepsilon)}\,r_2\}}{r_1 r_2}$$
$$\times\langle k_1(\varepsilon)\mid t_1(\varepsilon+\mathrm{i}0)\mid q_1\rangle\langle k_2(\varepsilon)\mid t_2(E-\varepsilon+\mathrm{i}0\mid -q_1\rangle$$
$$\times\left(\frac{1}{\varepsilon-q_1^2+\mathrm{i}0}+\frac{1}{E-\varepsilon-q_1^2/2\mu_2+\mathrm{i}0}\right)\frac{t_{12}(q_1,\ p_{12}^0,\ \widetilde{p}_{12}^{02}+\mathrm{i}0)}{E-\widetilde{q}_1^2-\dfrac{q_1^2}{2\mu_2}+\mathrm{i}0}. \qquad \text{(C6)}$$

By virtue of the lemma on singular integrals,[22] the integral over ε in (C6) leads to a Hölder function with respect to the argument q_1. Then the free Green's function $G_0(E+i0)$ in (C6) can be expressed in the form of the Sochocki formula $G_0(E+\mathrm{i}0)=\mathrm{i}\pi\delta(E+H_0)+\mathcal{P}[1/(E-H_0)]$.

We write I in the form $I=I_1+I_2$, where I_1 is determined by the contribution to (C6) from $-\mathrm{i}\pi\delta(E-H_0)$. Below, we shall be interested in only I_1. The integrand for I_1 contains a factor of the form $1/(\varepsilon-\tilde{q}_1^2+\mathrm{i}0)+1/(\tilde{q}_1^2-\varepsilon+\mathrm{i}0)$, which is obviously equal to $-2\pi\mathrm{i}\delta(\varepsilon-\tilde{q}_1^2)$. The upshot is

$$I_1\,|_{\substack{r_1\to\infty\\ r_2\to\infty}}\sim\frac{\mathrm{e}^{\mathrm{i}p_{12}^0 r_1}}{r_1}\,\frac{\mathrm{e}^{\mathrm{i}p_{12}^0 r_2}}{r_2}\int d\Omega_{q_1}\,\langle p_{12}^0\mid t_1\left(\frac{p_{12}^{02}}{2\mu_1}+\mathrm{i}0\right)\mid p_{12}^0\rangle$$
$$\times\langle p_{12}^0\mid t_2\left(\frac{p_{12}^{02}}{2\mu_2}+\mathrm{i}0\right)\mid -p_{12}^0\rangle\langle p_{12}^0\mid t_{12}(\widetilde{p}_{12}^{02}+\mathrm{i}0)\mid p_{12}^0\rangle. \qquad \text{(C7)}$$

It is readily seen that (C7) describes the effect of simultaneous indepedent scattering of the particles by the force center when the interaction V_{12} is included. In the case of the equations of Refs. 22 and 33, this effect is not separated in explicit form because of the employed classification of the amplitudes with respect to the first interaction. Our treatment proves the assertion formulated in the Introduction concerning the existence of an unphysical separation of the asymptotic behaviors in equations with classification of the amplitudes in accordance with the distinguished first interaction. We note that the effect under discussion is associated with the scattering of real particles [all the amplitudes in (C7) are on the energy shell]. In this connection, we mention Refs. 36 and 37, in which similar situations were studied for the example of a system of three particles (with finite masses) in the case of a zero initial reaction channel.

We now consider the problem of finding the coordinate asymptotic behaviors of the four-particle wave function for the example of a wave function $\psi_{\alpha\otimes\alpha'}(E+\mathrm{i}0)$ of the form

$$|\psi_{\alpha\otimes\alpha'}(z)\rangle=G_0(z)\,\mathcal{F}_{\alpha\otimes\alpha'}(z)\,S(z)\mid\psi_{\mathrm{in}}\rangle.$$

As in the previous case, we shall study the asymptotic behavior of $r_\alpha r_{\alpha'}R_{\alpha\alpha'}|\psi_{\alpha\otimes\alpha'}(E+\mathrm{i}0)\rangle$ (r_α, $r_{\alpha'}$, and $R_{\alpha\alpha'}$ are the coordinates conjugate to the momenta p_α, $p_{\alpha'}$, and $\mathcal{P}_{\alpha\alpha'}$) under the condition that $r_\alpha/v_\alpha=r_{\alpha'}/v_{\alpha'}=R_{\alpha\alpha'}/v_{\alpha\alpha'}$ ($v_\alpha=p_\alpha/\mu_\alpha$, $v_{\alpha\alpha'}=\mathcal{P}_{\alpha\alpha'}/M_{\alpha\alpha'}$). Since the operator $\mathcal{F}_{\alpha\otimes\alpha'}$ actually depends on $z-\mathcal{P}_\alpha^2$ (13), it can be represented in the form

$$\mathcal{F}_{\alpha\otimes\alpha'}(z)=\int_{-\infty}^{\infty}\frac{d\varepsilon_1}{-2\pi\mathrm{i}}\,G_{0\alpha\alpha'}(\varepsilon+\mathrm{i}\tau_1)\,\widetilde{\mathcal{F}}_{\alpha\otimes\alpha'}(E-\varepsilon+\mathrm{i}\tau_2), \qquad \text{(C8)}$$

where

$$\langle\mathcal{P}_{\alpha\alpha'}\mid G_{0\alpha\alpha'}(z)\mid\mathcal{P}_{\alpha\alpha'}^0\rangle=\left(z-\frac{\mathcal{P}_{\alpha\alpha'}^2}{2M_{\alpha\alpha'}}\right)^{-1}\delta(\mathcal{P}_{\alpha\alpha'}-\mathcal{P}_{\alpha\alpha'}^0).$$

Substituting in (C8) the expression (14) for $\widetilde{\mathcal{F}}_{\alpha\otimes\alpha'}(z)$ and using the expressions for the free Green's functions in the coordinate representation, we obtain in complete analogy with (C2)

$$\langle r_\alpha r_{\alpha'}R_{\alpha\alpha'}\mid\psi_{\alpha\otimes\alpha'}(E+\mathrm{i}0)\rangle\Big|_{r_\alpha\to\infty,\ \frac{r_\alpha}{v_\alpha}=\frac{r_{\alpha'}}{v_{\alpha'}}=\frac{R_{\alpha\alpha'}}{v_{\alpha\alpha'}}}$$
$$\sim\frac{\mu_\alpha\mu_{\alpha'}M_{\alpha\alpha'}}{(2\pi)^3}\int_0^E\frac{d\varepsilon_1}{-2\pi\mathrm{i}}\int_0^E\frac{d\varepsilon_2}{-2\pi\mathrm{i}}\,\frac{\mathrm{e}^{\mathrm{i}\varphi(\varepsilon_1\varepsilon_2)r_\alpha}}{r_\alpha r_{\alpha'}R_{\alpha\alpha'}}\,\theta(E-\varepsilon_1-\varepsilon_2)$$
$$\times\langle k_\alpha(\varepsilon)\,k_{\alpha'}(\varepsilon)\,\mathcal{K}_{\alpha\alpha'}(\varepsilon)\mid R_{\alpha\otimes\alpha'}(\varepsilon)\,S(E+\mathrm{i}0)\mid\psi_{\mathrm{in}}\rangle,$$
$$E=\frac{\mu_\alpha v_\alpha^2}{2}+\frac{\mu_{\alpha'}v_{\alpha'}^2}{2}+\frac{M_{\alpha\alpha'}v_{\alpha\alpha'}^2}{2}. \qquad \text{(C9)}$$

In (C9), we have used the following notation: $\theta(x)$ is the Heaviside function,

$$\theta(x)=\begin{cases}1, & x>0,\\ 0, & x<0,\end{cases}$$

$$\varphi(\varepsilon_1\varepsilon_2)=\sqrt{2\mu_\alpha\varepsilon_2}+\sqrt{2\mu_{\alpha'}(E-\varepsilon_1-\varepsilon_2)}\frac{v'_\alpha}{v_\alpha}+\sqrt{2M_{\alpha\alpha'}\varepsilon_1}\frac{v_{\alpha\alpha'}}{v_\alpha}, \tag{C10}$$

$$\mathbf{k}_\alpha(\varepsilon)=\sqrt{2\mu_\alpha\varepsilon_2}\frac{\mathbf{r}_\alpha}{r_\alpha},\quad \mathbf{k}_{\alpha'}(\varepsilon)=\sqrt{2\mu_{\alpha'}(E-\varepsilon_1-\varepsilon_2)}\frac{\mathbf{r}_{\alpha'}}{r_{\alpha'}},$$

$$\mathscr{K}_{\alpha\alpha'}(\varepsilon)=\sqrt{2M_{\alpha\alpha'}\varepsilon_1}\frac{\mathbf{R}_{\alpha\alpha'}}{R_{\alpha\alpha'}}.$$

Finally, the operator $R_{\alpha\otimes\alpha'}(\varepsilon)$ has the form

$$R_{\alpha\otimes\alpha'}(\varepsilon)=[t_\alpha(\varepsilon_2+i0)\,g_{0\alpha}(\varepsilon_2+i0)]$$
$$\otimes[t_{\alpha'}(E-\varepsilon_1-\varepsilon_2+i0)\,g_{02}(E-\varepsilon_1-\varepsilon_2+i0)].$$

Assuming that the matrix element in (C9) is a continuous function of ε_1 and ε_2 [i.e., the initial reaction channel is either the channel (2 + 2) or (3 + 1)], to calculate the asymptotic behavior in (C9) we again use the method of stationary phase. In the region of integration, the function $\varphi(\varepsilon_1\varepsilon_2)$ has a unique—and nondegenerate—stationary point (i.e., a point at which $\partial\varphi/\partial\varepsilon_1=0$ and $\partial\varphi/\partial\varepsilon_2=0$). This point $(\varepsilon_1,\varepsilon_2)$ is determined by the natural relations $\varepsilon_1=(M_{\alpha\alpha'}v^2_{\alpha\alpha'})/2$, $\varepsilon_2=\mu_\alpha v^2_\alpha/2$.

As a result, the asymptotic behavior of the expression (C9) can be written as

$$\langle r_\alpha r_{\alpha'}R_{\alpha\alpha'}\mid\psi_{\alpha\otimes\alpha'}(E+i0)\rangle\Big|_{r_\alpha\to\infty,\ \frac{r_\alpha}{v_\alpha}=\frac{r_{\alpha'}}{v_{\alpha'}}=\frac{R_{\alpha\alpha'}}{v_{\alpha\alpha'}}}$$
$$\sim A_0\exp\{ip_\alpha r_\alpha+ip_{\alpha'}r_{\alpha'}+i\mathscr{P}_{\alpha\alpha'}R_{\alpha\alpha'}\}(r_\alpha r_{\alpha'}R_{\alpha\alpha'})^{-1}$$
$$\times\langle p_\alpha p'_\alpha\mathscr{P}_{\alpha\alpha'}\mid R_{\alpha\otimes\alpha'}(\tilde{p}^2_\alpha,\tilde{\mathscr{P}}^2_{\alpha\alpha'})\,S(E+i0)\mid\psi_{\text{in}}\rangle, \tag{C11}$$

where p_α, $p_{\alpha'}$, $\mathscr{P}_{\alpha\alpha'}$ are the momenta after scattering, and

$$A_0=\frac{\mu_\alpha\mu_{\alpha'}\mu_{\alpha\alpha'}}{(2\pi)^3(-2\pi i)^2}\times\exp\left\{i\frac{\pi}{4}\operatorname{sign}[\lambda_+(A)-\lambda_-(A)]\right\}$$
$$\times 2\pi/r_\alpha\sqrt{|\det A|}. \tag{C12}$$

In (C12) there is a matrix with elements $\partial^2\varphi/\partial\varepsilon_i\partial\varepsilon_j$; $\lambda_+(A)$ and $\lambda_-(A)$ are the numbers of its positive and negative eigenvalues. After simple calculations, we obtain

$$|\det A|=\frac{2E}{v^2_\alpha}(\mu_\alpha v^2_\alpha\mu_{\alpha'}v^2_{\alpha'}M_{\alpha\alpha'}v^2_{\alpha\alpha'})^{-1},$$

$$\lambda_+(A)=2,\quad \lambda_-(A)=0,$$

$$A_0=-\frac{e^{-i\frac{\pi}{4}}(\mu_\alpha\mu_{\alpha'}M_{\alpha\alpha'})^{3/2}v^2_\alpha v_{\alpha'}v_{\alpha\alpha'}}{(2\pi)^4 r_\alpha\sqrt{2E}}.$$

Further, we introduce the parameter ρ by $\rho^2=2\mu r^2_\alpha+2\mu_{\alpha'}r^2_{\alpha'}+2M_{\alpha\alpha'}R^2_{\alpha\alpha'}$. Then $r_\alpha/v_\alpha=r_{\alpha'}/v_{\alpha'}=R_{\alpha\alpha'}/v_{\alpha\alpha'}=\rho/2\sqrt{E}$, and the asymptotic behavior in which we are interested can be represented in the form

$$\langle r_\alpha r_{\alpha'}R_{\alpha\alpha'}\mid\psi_{\alpha\otimes\alpha'}(E+i0)\rangle\Big|_{r_\alpha\to\infty,\ \frac{r_\alpha}{v_\alpha}=\frac{r_{\alpha'}}{v_{\alpha'}}=\frac{R_{\alpha\alpha'}}{v_{\alpha\alpha'}}}$$
$$\sim-\frac{e^{i\frac{\pi}{4}}(\mu_\alpha\mu_{\alpha'}M_{\alpha\alpha'})^{3/2}E^{3/2}}{\pi^4\sqrt{2}}\frac{e^{i\sqrt{E}\rho}}{\rho^4}$$
$$\times\langle p_\alpha p_{\alpha'}\mathscr{P}_{\alpha\alpha'}\mid R_{\alpha\otimes\alpha'}(\tilde{p}^2_\alpha,\tilde{\mathscr{P}}^2_{\alpha\alpha'})\,S(E+i0)\mid\psi_{\text{in}}\rangle. \tag{C13}$$

If the initial channel of the system is determined by the collision of not two but three or four particles, then for $E>0$ the asymptotic behavior of $|\psi_{\alpha\otimes\alpha'}(E+i0)\rangle$ contains not only expressions of the type (C9) and (C13) but also expressions of the type (C7), i.e., the effect of the simultaneous scattering of the independent particle subsystems α and α' can, as before, be separated in a pure form.

APPENDIX D. SYSTEM OF INTEGRAL EQUATIONS FOR COMPONENTS CORRESPONDING TO THE INITIAL CHANNEL $\alpha+\alpha'$

The system of integral equations for the components of type u and v obtained from (27), (29), and (31) after separation in the in state of the primary singularity of the form $(z+\varkappa^2_\alpha+\varkappa^2_{\alpha'}-\tilde{\mathscr{P}}^{02}_{\alpha,\alpha'})^{-1}$, is

$$\begin{bmatrix}u_\beta(\mathscr{P}_\beta p_\beta p_{\beta'},\mathscr{P}^0_\alpha,z)\\ v_{\beta'}(\mathscr{P}_\beta p_{\beta'},\mathscr{P}^0_\alpha z)\end{bmatrix}=\int\prod_{i=1}^{4}dk_i\,\delta(\mathscr{K})\,\delta(\mathscr{P}_\beta-\mathscr{K}_\beta)\,\delta(p_\beta-k_\beta)$$
$$\times\begin{bmatrix}\hat{t}_\beta(p_\beta,k_\beta,z-\tilde{\mathscr{P}}^2_\beta-\tilde{p}^2_{\beta'})\\ \varphi^*_\beta(k_\beta)\end{bmatrix}\frac{1}{z-\tilde{\mathscr{P}}^2_\beta-\tilde{p}^2_{\beta'}-\tilde{k}^2_\beta}\Big\{\zeta_{\alpha,\alpha'}(k_\alpha k_{\alpha'})$$
$$\times\delta(\mathscr{K}_\alpha-\mathscr{P}^0_\alpha)[1-\delta(\alpha,\beta)][1-\delta(\alpha',\beta)]$$
$$+\sum_{(\gamma,\gamma')\neq(\beta\beta')}\Big[u_{\gamma,\gamma'}(\mathscr{K}_\gamma k_\gamma k_{\gamma'},\mathscr{P}^0_\alpha,z)$$
$$+\frac{\varphi_\gamma(k_\gamma)\,v^{\gamma'}_{\gamma,\gamma'}(\mathscr{K}_\gamma k_{\gamma'},\mathscr{P}^0_\alpha,z)}{z+\varkappa^2_\gamma-\tilde{\mathscr{K}}^2_\gamma-\tilde{k}^2_{\gamma'}}$$
$$+\frac{\varphi_{\gamma'}(k_{\gamma'})\,v^{\gamma}_{\gamma,\gamma'}(\mathscr{K}_\gamma k_\gamma,\mathscr{P}^0_\alpha,z)}{z+\varkappa^2_{\gamma'}-\tilde{\mathscr{K}}^2_\gamma-\tilde{\lambda}^2_\gamma}$$
$$+\frac{\zeta_{\gamma,\gamma'}(k_\gamma k_{\gamma'})\,v_{\gamma\gamma'}(\mathscr{K}_\gamma,\mathscr{P}^0_\alpha,z)}{z+\varkappa^2_\gamma+\varkappa^2_{\gamma'}-\tilde{\mathscr{K}}^2_\gamma}\Big]\Big\}; \tag{D1}$$

$$\begin{bmatrix}u^\eta_\beta(\mathscr{P}_\eta p_\beta p_{\beta\eta},\mathscr{P}^0_\alpha,z)\\ v^\eta_\beta(\mathscr{P}_\eta p_{\beta\eta},\mathscr{P}^0_\alpha,z)\\ v_\eta(\mathscr{P}_\eta,\mathscr{P}^0_\alpha,z)\end{bmatrix}=\int\prod_{i=1}^{4}dk_i\,\delta(\mathscr{K})\,\delta(\mathscr{P}_\eta-\mathscr{K}_\eta)$$

$$\times\begin{bmatrix}\sum_{\gamma\in\eta}\Big[\mathscr{T}^{\eta^0}_{\beta,\gamma}(p_\beta\tilde{p}_{\beta\eta},k_\gamma k_{\gamma r},z-\tilde{\mathscr{P}}^2_\eta)+\dfrac{\mathscr{T}^{\eta^2}_{\beta\gamma}(\bar{p}_\beta p_{\beta\eta},k_{\gamma\eta},z-\tilde{\mathscr{P}}^2_\eta)\varphi^*_\gamma(k_\gamma)}{z+\varkappa^2_\gamma-\mathscr{P}^2_\eta-\tilde{k}^2_{\gamma\eta}}\Big]\\ \sum_{\gamma\in\eta}\Big[\mathscr{T}^{\eta^1}_{\beta,\gamma}(p_{\beta\eta},k_\gamma k_{\gamma\eta},z-\mathscr{P}^2_\eta)+\dfrac{\mathscr{T}^{\eta^3}_{\beta\gamma}(\bar{p}_{\beta\eta},k_{v\eta},z-\tilde{\mathscr{P}}^2_\eta)\,\varphi^*_\gamma(k_\gamma)}{z+\varkappa^2_\gamma-\tilde{\mathscr{P}}^2_\eta-\tilde{k}^2_{\gamma\eta}}\Big]\\ \sum_{\gamma\in\eta}\langle\Phi_\eta\mid V_\gamma\mid k_\gamma k_{\gamma\eta}\rangle\end{bmatrix}$$

$$\times\frac{1}{z-\tilde{\mathscr{P}}^2_\eta-\tilde{k}^2_\gamma-\tilde{k}^2_{\gamma\eta}}$$

$$\times\Big\{\zeta_{\alpha,\alpha'}(k_\alpha k_{\alpha'})\,\delta(\tilde{\mathscr{K}}_\alpha-\tilde{\mathscr{P}}^0_\alpha)\,\delta[1-\delta(\alpha,\gamma)][1-\delta(\alpha'\gamma)]$$

$$+\sum_{\substack{(\gamma\gamma')\neq(\rho\rho')\\ \rho\notin\eta}}\left[u_{\rho}(\mathscr{K}_{\rho}k_{\rho}k_{\rho'},\ \widetilde{\mathscr{P}}_{\alpha}^{0},\ z)+\frac{\varphi_{\rho}(k_{\rho})\,v_{\rho'}(\mathscr{K}_{\rho}k_{\rho},\ \overline{\mathscr{P}}_{\alpha}^{0},\ z)}{z+\varkappa_{\rho}^{2}-\mathscr{K}_{\rho}^{2}-k_{\rho}^{2}}\right]$$

$$+\sum_{\substack{(\gamma\gamma')\neq(\rho\rho')\\ \rho\notin\eta,\ \rho\in\mu}}\left[u_{\rho\mu}(\mathscr{K}_{\mu}k_{\rho}k_{\rho\mu},\ \mathscr{P}_{\alpha}^{0},\ z)\ \frac{\varphi_{\rho}(k_{\rho})\,v_{\rho}^{\mu}(\mathscr{K}_{\mu}k_{\rho\mu},\ \overline{\mathscr{P}}_{\alpha}^{0},\ z)}{z+\varkappa_{\rho}^{2}-\widetilde{\mathscr{K}}^{2}-\widetilde{k}_{\rho\mu}^{2}}\right.$$

$$\left.+\frac{\langle k_{\rho}k_{\rho\mu}\mid V_{\rho}\mid\Phi_{\mu}\rangle(\mathscr{K}_{\mu},\ \mathscr{P}_{\alpha}^{0},\ z)\,v_{\mu}}{z+\varkappa_{\mu}^{2}-\widetilde{\mathscr{K}}_{\mu}^{2}}\right]$$

$$+\sum_{(\gamma\gamma')\neq(\rho\rho')}\left[u_{\rho\rho'}(\mathscr{K}_{\rho}k_{\rho}k_{\rho'},\ \mathscr{P}_{\alpha}^{0},\ z)\right.$$

$$+\frac{\varphi_{\rho}(k_{\rho})\,v_{\rho\rho'}^{\rho'}(\mathscr{K}_{\rho}k_{\rho'},\ \mathscr{P}_{\alpha}^{0},\ z)}{z+\varkappa_{\rho}^{2}-\widetilde{\mathscr{K}}_{\rho}^{2}-k_{\rho'}^{2}}+\frac{\varphi_{\rho'}(k_{\rho'})\,v_{\rho\rho'}^{\rho}(\mathscr{K}_{\rho}k_{\rho},\ \mathscr{P}_{\alpha}^{0},\ z)}{z+\varkappa_{\rho'}^{2}-\widetilde{\mathscr{K}}_{\rho}^{2}-\widetilde{k}_{\rho}^{2}}$$

$$\left.\left.+\frac{\zeta_{\rho\rho'}(k_{\rho}k_{\rho'})\,v_{\rho\rho'}(\mathscr{K}_{\rho},\ \mathscr{P}_{\alpha}^{0},\ z)}{z+\varkappa_{\rho}^{2}+\varkappa_{\rho'}^{2}-\widetilde{\mathscr{K}}_{\rho}^{2}}\right]\right\};\qquad\text{(D2)}$$

$$\begin{bmatrix}u_{\beta\beta'}(\mathscr{P}_{\beta}p_{\beta}p_{\beta'},\ \mathscr{P}_{\alpha}^{0},\ z)\\ v_{\beta\beta'}^{\beta}(\mathscr{P}_{\beta}p_{\beta},\ \mathscr{P}_{\alpha}^{0},\ z)\\ v_{\beta\beta'}(\mathscr{P}_{\beta},\ \mathscr{P}_{\alpha}^{0},\ z)\end{bmatrix}=\int\prod_{i=1}^{4}dk_{i}\,\delta(\mathscr{K})\,\delta(\mathscr{P}_{\beta}-\mathscr{K}_{\beta})$$

$$\times\begin{bmatrix}L_{\beta\beta'}^{\beta\beta'}(p_{\beta}p_{\beta'},\ k_{\beta}k_{\beta'},\ z-\widetilde{\mathscr{P}}_{\beta}^{2})\\ +\dfrac{L_{\beta\beta'}^{\beta}(p_{\beta}p_{\beta'},\ k_{\beta},\ z-\widetilde{\mathscr{P}}_{\beta}^{2})\,\varphi_{\beta'}^{*}(k_{\beta'})}{z-\varkappa_{\beta'}^{2}-\widetilde{\mathscr{P}}_{\beta}^{2}-\widetilde{k}_{\beta}^{2}}\\ +\dfrac{L_{\beta\beta'}^{\beta'}(p_{\beta}p_{\beta}',\ k_{\beta}',\ z-\widetilde{\mathscr{P}}_{\beta}^{2})\,\varphi_{\beta}^{*}(k_{\beta})}{z+\varkappa_{\beta}^{2}-\widetilde{\mathscr{P}}_{\beta}^{2}-\widetilde{k}_{\beta'}^{2}}\\ L_{\beta}^{\beta\beta'}(p_{\beta},\ k_{\beta}k_{\beta'},\ z-\widetilde{\mathscr{P}}_{\beta}^{2})+\dfrac{L_{\beta}^{\beta}(p_{\beta},\ k_{\beta},\ z-\widetilde{\mathscr{P}}_{\beta}^{2})\,\varphi_{\beta'}^{*}(k_{\beta'})}{z+\varkappa_{\beta'}^{2}-\widetilde{\mathscr{P}}_{\beta}^{2}-\widetilde{k}_{\beta}^{2}}\\ +\dfrac{L_{\beta}^{\beta}(p_{\beta},\ k_{\beta},\ z-\widetilde{\mathscr{P}}_{\beta}^{2})\,\varphi_{\beta}^{*}(k_{\beta})}{z+\varkappa_{\beta}^{2}-\widetilde{\mathscr{P}}_{\beta}^{2}-\widetilde{k}_{\beta'}^{2}}\\ \zeta_{\beta,\beta'}^{*}(k_{\beta}k_{\beta'})\end{bmatrix}$$

$$\times\frac{1}{z-\widetilde{\mathscr{P}}_{\beta}^{2}-\widetilde{k}_{\beta}^{2}-\widetilde{k}_{\beta'}^{2}}\left\{\zeta_{\alpha\alpha'}(k_{\alpha}k_{\alpha'})\,\delta(\mathscr{P}_{\alpha}^{0}-\mathscr{K}_{\alpha})\right.$$

$$\times[1-\delta(\alpha,\ \beta)]\,[1-\delta(\alpha'\beta)]$$

$$+\sum_{(\gamma\gamma')\neq(\beta\beta')}\left[u_{\gamma}(\mathscr{K}_{\gamma}k_{\gamma}k_{\gamma'},\ \mathscr{P}_{\alpha}^{0},\ z)+\frac{v_{\gamma'}(\mathscr{K}_{\gamma}k_{\gamma'},\ \mathscr{P}_{\alpha}^{0},\ z)\,\varphi_{\gamma}(k_{\gamma})}{z+\varkappa_{\gamma}^{2}-\widetilde{\mathscr{K}}_{\gamma}^{2}-\widetilde{k}_{\gamma'}^{2}}\right]$$

$$+\sum_{\substack{(\gamma\gamma')\neq(\beta\beta')\\ \gamma\notin\mu}}\left[u_{\gamma}^{\mu}(\mathscr{K}_{\mu}k_{\gamma}k_{\gamma\mu},\ \mathscr{P}_{\alpha}^{0},\ z)+\frac{\varphi_{\gamma}(k_{\gamma})\,v_{\gamma}^{\mu}(\mathscr{K}_{\mu}k_{\gamma\mu}\,\mathscr{P}_{\alpha}^{0},\ z)}{z+\varkappa_{\gamma}^{2}-\widetilde{\mathscr{K}}_{\mu}^{2}-\widetilde{k}_{\gamma\mu}^{2}}\right.$$

$$\left.+\frac{\langle k_{\gamma}k_{\gamma\mu}\mid V_{\gamma}\mid\Phi_{\mu}\rangle\,v_{\mu}(\mathscr{K}_{\mu},\ \mathscr{P}_{\alpha}^{0},\ z)}{z+\varkappa_{\mu}^{2}-\widetilde{\mathscr{K}}_{\mu}^{2}}\right]$$

$$+\sum_{(\gamma\gamma')\neq(\beta\beta')}\left[u_{\gamma\gamma'}(\mathscr{K}_{\gamma}k_{\gamma}k_{\gamma'},\ \mathscr{P}_{\alpha}^{0},\ z)\right.$$

$$+\frac{\varphi_{\gamma}(k_{\gamma})\,v_{\gamma\gamma'}^{\gamma'}(\mathscr{K}_{\gamma}k_{\gamma'},\ \mathscr{P}_{\alpha}^{0},\ z)}{z+\varkappa_{\gamma}^{2}-\widetilde{\mathscr{K}}_{\gamma}^{2}-k_{\gamma}^{2}}+\frac{\varphi_{\gamma'}(k_{\gamma'})\,v_{\gamma\gamma'}^{\gamma}(\mathscr{K}_{\gamma}k_{\gamma},\ \widetilde{\mathscr{P}}_{\alpha}^{0},\ z)}{z+\varkappa_{\gamma'}^{2}-\widetilde{\mathscr{K}}_{\gamma}^{2}-\widetilde{k}_{\gamma}^{2}}$$

$$\left.\left.+\frac{\zeta_{\gamma\gamma'}(k_{\gamma}k_{\gamma'})\,v_{\gamma\gamma'}(\mathscr{K}_{\gamma},\ \mathscr{P}_{\alpha}^{0},\ z)}{z+\varkappa_{\gamma}^{2}+\varkappa_{\gamma'}^{2}-\widetilde{\mathscr{K}}_{\gamma}^{2}}\right]\right\}.\qquad\text{(D3)}$$

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