To the memory of J. A. Swieca

Axiomatic approach to local gauge quantum field theory and spontaneous symmetry breaking

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The covariant formulation of quantum electrodynamics (with indefinite metric) is considered. The picture of spontaneous symmetry breaking in (Abelian and non-Abelian) gauge theory is compared with Goldstone's theorem in standard quantum field theory (with positive metric). It is shown that the photon can be identified with the Goldstone particle responsible for breaking local gauge invariance, the photon belonging to the physical Hilbert space because the corresponding global $U(1)$ symmetry is not broken. A Lorentz and locally gauge-invariant nonlocal charge field is constructed. It is verified in the framework of perturbation theory that the corresponding renormalized composite field has on-shell matrix elements free of infrared divergences. It is asserted that such a field can generate physical electron states without the need to introduce a Lorentz-noninvariant cloud of soft photons.

INTRODUCTION

The dominant position of gauge theories in quantum field theory during the last decade is due, in the first place, to the possibility of renormalizing them (in the non-Abelian case). A systematic exposition of gauge perturbation theory (together with a proof of the renormalizability) is contained in Ref. 56 (see also Refs. 67 and 68). Practically all the existing results on the electroweak interactions have been obtained in the framework of perturbation theory (see, for example, Ref. 66). The principal success of quantum chromodynamics (QCD)—the realization and application of asymptotic freedom—is also associated with its renormalizability.

The systematic refinement of the mathematical structure of gauge theories proceeds in at least three directions.

The first direction includes studies devoted to the geometrical approach to gauge fields. The progress achieved in this direction relates above all to the solution of classical gauge equations (see Refs. 2 and 46). However, the differential-geometrical understanding of the quantum theory of gauge fields is still in an elementary stage (see Ref. 54).

Here, we shall also not be concerned with studies in the second direction—gauge theories on a lattice and the investigation of the possible transition to the continuous limit. This approach has yielded interesting results only in the case of space-time of low dimensions (2 and 3) (see Ref. 26).

The axiomatic approach to the local and covariant formulation of the quantum theory of gauge fields (in a space with indefinite metric) is in some sense close to the real applications of gauge theories, beginning with the simplest and best confirmed of them—quantum electrodynamics. In the present paper, we consider the fundamentals of this approach and some of its applications—above all, to spontaneously broken symmetries. In contrast to a number of studies (see, for example, Refs. 9, 28, and 29), it is here assumed that Lorentz invariance is not broken in the charge sector. The basis of this assumption is the construction of a Poincaré-covariant classical nonlocal composite field $\psi(x, A)$, which depends exponentially on the electromagnetic potential A_μ and is invariant with respect to local gauge transformations (see Sec. 1.B), and the analysis of the infrared behavior of the matrix elements of this field in the quantum case made recently in Ref. 15 on the basis of summation of the perturbation-theory series. The majority of the results which we consider were obtained by R. Ferrari, F. Strocchi, and the late S. A. Swieca (Refs. 21–24, 30, 57–61, 63, and 64). We do not consider here two fundamental achievements of the axiomatic approach to gauge theories—the proof of the superselection rule with respect to the electric charge in quantum electrodynamics[61] and the reconstruction theorem in a theory with indefinite metric.[48,50] The prehistory of the axiomatic approach to quantum electrodynamics can be traced in Refs. 61 and 73. (As the standard source on axiomatic quantum field theory in Hilbert space with positive Poincaré-invariant metric, we use the monograph of Ref. 6.)

CONVENTIONS AND NOTATION

We use the Pauli metric (with spacelike signature):

$$p^2=\eta^{\mu\nu}p_\mu p_\nu=\mathbf{p}^2-p_0^2,\quad (\eta^{\mu\nu})=(\eta_{\mu\nu})=\begin{bmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}. \tag{1}$$

The Fourier transformation of functions defined in Minkowski space is expressed in the form

$$\psi(x)=\int e^{ipx}\psi(p)\,d_4p,\quad \psi(p)=\int e^{-ipx}\psi(x)\,d^4x, \tag{2}$$

$$d_4p=\frac{d^4p}{(2\pi)^4},\quad px=p_\mu x^\mu.$$

The energy is identified with $-p_0\ (=p^0)$. (In the case of a particle in a stationary gravitational field, only $E=-p_0$ is in general a conserved quantity, whereas $p^0=g^{0\mu}p_\mu$ may depend on the time.) The covariant derivative for the wave function of a particle with charge e (or for a field that annihilates the charge e) has the form

$$\mathscr{D}_\mu = \partial_\mu - ieA_\mu \quad (\overline{\mathscr{D}}_\mu = \partial_\mu + ieA_\mu), \tag{3}$$

where $\partial_\mu = \partial/\partial x^\mu$.

It corresponds to the local gauge transformations

$$\psi(x) \mapsto \exp[ie\lambda(x)]\,\psi(x), \quad A_\mu(x) \mapsto A_\mu(x) + \partial_\mu\lambda(x). \tag{4}$$

The Dirac matrices are determined by

$$[\gamma_\mu, \gamma_\nu]_+ = 2\eta_{\mu\nu}, \tag{5a}$$

so that for all vectors p and q of Minkowski space $\mathrm{tr}\,\hat{p}\hat{q} = 4pq$, where

$$\hat{p} = p_\mu\gamma^\mu. \tag{5b}$$

The Dirac adjoint is defined by

$$\widetilde{\psi} = \psi^*\beta, \quad \beta\gamma_\mu + \gamma_\mu^*\beta = 0, \quad \beta = \beta^*; \tag{6a}$$

in the standard basis, in which

$$\gamma_\mu^* = \eta_{\mu\mu}\gamma_\mu,$$

we take

$$\beta = i\gamma^0 = -\gamma_4. \tag{6b}$$

The pseudoscalar matrix $\gamma_5\,(=\gamma^5)$ is chosen in the form $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$, so that

$$\gamma_5^2 = 1. \tag{7}$$

The canonical Poisson bracket is the (anti)commutator divided by i. In particular, if $\pi^\mu(x)$ are the canonically conjugate momenta corresponding to the field components $A_\mu(x)$, then

$$\delta(x^0 - y^0)\,\{A_\nu(x), \pi^\mu(y)\} = \delta(x-y)\,\delta_\nu^\mu. \tag{8}$$

Note that transition from a timelike to a spacelike metric is realized by changing the sign of the covariant components of the vectors (of the type p_μ, A_μ) and the substitution $\gamma^\mu \to i\gamma^\mu$.

1. CLASSICAL THEORY OF GAUGE FIELDS

A. Maxwell-Dirac electrodynamics in a covariant gauge

To be definite, we begin our exposition with the simplest example of the theory of the electron-positron field $\psi(x)$ interacting with the electromagnetic field $A_\mu(x)$. We choose the Lagrangian of the theory in nondegenerate form with a term $\mathscr{L}_{GF}$ specifying the class of covariant gauges and containing the so-called Hertz prepotential[1)]:

$$\mathscr{L} = \mathscr{L}_{\text{inv}} + \mathscr{L}_{GF}, \quad \mathscr{L}_{\text{inv}} = \mathscr{L}_D + \mathscr{L}_{em}, \quad \mathscr{L}_{GF} = \mathscr{L}_B^{(\xi)} + \mathscr{L}_{g,h}; \tag{9a}$$

here, the Dirac and Maxwell Lagrangians

$$\mathscr{L}_D = \widetilde{\psi}(m + \hat{\mathscr{D}})\,\psi, \quad \hat{\mathscr{D}} = \hat{\partial} - ie\hat{A}$$

and

$$\mathscr{L}_{em} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\,F^{\mu\nu} \tag{9b}$$

determine the gauge-invariant part $\mathscr{L}_{\text{inv}}$, while the "gauge-fixing" terms have the form

$$\mathscr{L}_B^{(\xi)} = \frac{1}{2}\xi B^2 - B\,\partial A, \tag{9c}$$

$$\mathscr{L}_{g,h} = \frac{1}{2}\,[G_{\mu\nu}H^{\mu\nu} - G_{\mu\nu}(\partial^\mu h^\nu - \partial^\nu h^\mu) - \partial_\mu g_\nu - \partial_\nu g_\mu)\,H^{\mu\nu}] - \partial g\,\partial h - A_\mu g^\mu. \tag{9d}$$

By variation with respect to $F_{\mu\nu}$, $G_{\mu\nu}$, and $H_{\mu\nu}$ we obtain the relations

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{10a}$$

$$G_{\mu\nu} = \partial_\mu g_\nu - \partial_\nu g_\mu, \quad H_{\mu\nu} = \partial_\mu h_\nu - \partial_\nu h_\mu, \tag{10b}$$

which include the standard connection between the Maxwell tensor $F_{\mu\nu}$ and the electromagnetic potential A_μ. The equations of motion

$$(\hat{\mathscr{D}} + m)\,\psi(x) = 0$$

$$= \widetilde{\psi}(x)\,(m - \overleftarrow{\hat{\mathscr{D}}})\,(\equiv -\partial_\mu\widetilde{\psi}(x)\,\gamma^\mu + \psi(x)\,(m - ie\hat{A}(x))) \tag{11}$$

for the spinor field ensure conservation of the current:

$$\partial_\mu j^\mu = 0, \tag{12a}$$

where

$$j^\mu(x) = ie\widetilde{\psi}(x)\,\gamma^\mu\psi(x). \tag{12b}$$

The gauge-dependent part of the Lagrangian, $\mathscr{L}_{GF}$, makes a contribution to the expression for the current through the potentials; from the equation of motion for the field A_μ we find

$$j^\mu(x) = \partial_\nu F^{\mu\nu}(x) + \mathfrak{A}^\mu(x), \tag{13a}$$

where the additional term in the divergence of the Maxwell field,

$$\mathfrak{A}^\mu(x) = g^\mu(x) - \partial^\mu B(x), \tag{13b}$$

is also a conserved current [as a consequence of (12)]:

$$\partial_\mu\mathfrak{A}^\mu(x) = 0, \quad \text{so that } \partial g(x) = \square\, B(x). \tag{13c}$$

The fields B and g^μ are (noncanonical) free fields. Indeed, from variation of the Lagrangian with respect to h_μ (and $\partial_\mu h_\nu$) we find

$$\partial_\nu G^{\mu\nu} - \partial^\mu\partial g \equiv -\square\, g^\mu = 0, \tag{14a}$$

so that [by virtue of (13c)]

$$\square^2 B = 0 = \square^2\mathfrak{A}^\mu. \tag{14b}$$

Finally, by variation with respect to B and with respect to g_μ (and $\partial_\mu g_\nu$) we obtain the constraints

$$\partial A(x) = \xi B(x), \tag{15a}$$

$$A_\mu(x) = \square\, h_\mu(x). \tag{15b}$$

It can be seen from (14b) and (15) that the divergence $\partial A = \square\partial h$ also satisfies the (noncanonical) free equation

$$\square^2\,\partial A = 0 = \square^3\,\partial h. \tag{15c}$$

These equations are invariant with respect to special gauge transformations of the second kind of the form (4) with

$$\lambda = \lambda(x, \xi) = \partial_\mu l^\mu(x, \xi) = \lambda_0(x) + \lambda_1(x)\,\xi, \tag{16a}$$

where l^μ and λ satisfy the free equations

$$\partial_\mu(\partial^\mu l^\nu - \partial^\nu l^\mu) = \square\, l^\nu - \partial^\nu\,\partial l = 0, \quad \square\,\lambda_0 = 0 = \square^2\,\lambda_1; \tag{16b}$$

the auxiliary fields B, h, and g transform as

[1)]The introduction of prepotentials of such form is convenient for the analysis of infrared singularities of Green's functions.[14,15,77,79]

$$h_\mu \mapsto h_\mu + l_\mu, \quad B \mapsto B + \frac{1}{\xi} \Box \partial l = B + \Box \lambda_1,$$

$$g_\mu \mapsto g_\mu + \partial_\mu \Box \lambda_1. \tag{16c}$$

For what follows, it is important that among these special gauge transformations there are "local" transformations, for which $\lambda(x)$ decreases at infinity. To see this, it is sufficient to note that any λ_0 of the form

$$\lambda_0(x) = \frac{f(t+r) - f(t-r)}{2r}, \quad t = x^0, \quad r = |\mathbf{x}|,$$

where f is a function that decreases smoothly at infinity on the real axis, is an everywhere regular solution of the d'Alembert equation that tends to zero as $r + |t| \to \infty$. Similarly, we can set $l_\mu = \partial_\mu \rho$, where

$$\rho(x) = t\frac{F(t-r) - F(t+r)}{4r} \quad (\Box \rho = \partial l = \lambda_0 \text{ for } F' = f)$$

gives (for corresponding F) a smooth decreasing solution of the equation $\Box^2 \rho = 0$, etc. [More general solutions can be obtained from this by the substitution $t \to t - \tau$, $r \to |\mathbf{x} - \mathbf{y}|$ and taking a (continuous) superposition with respect to τ and $\mathbf{y}$.]

The Lagrangian (9) is a generalization of the frequently employed Lagrangian

$$\mathcal{L}_{g_\mu = 0} = \mathcal{L}_{\text{inv}} + \mathcal{L}_B^{(\xi)} \tag{9e}$$

(see, for example, Ref. 65), which interpolates between the Landau gauge obtained for $\xi = 0$ [which in accordance with (16a) gives $\partial A = 0$] and the Gupta-Bleuler gauge for $\xi = 1$ (in it $\Box A^\mu = j^\mu - g^\mu = j^\mu$). Among the solutions of Eqs. (10)–(15) there are, for $g_\mu = 0$, solutions of the equations of motion for the Lagrangian (9e). In the quantum case, the theory with the Lagrangian (9e) can be realized in a subspace of the state space of the theory determined by the Lagrangian (9).

A gauge-invariant formulation of the theory is obtained formally if in the Lagrangian (9e) (or in the corresponding equations of motion) we go to the limit $B(x) \to 0$, $\xi \to \infty$, $\xi B(x) = \partial A(x)$ finite. In contrast to the Lagrangians (9a) and (9e), the gauge-invariant Lagrangian $\mathcal{L}_{\text{inv}}$ is degenerate—it does not depend on $\dot{A}_0 = \partial_0 A_0$ and leads to a Hamiltonian theory with constraints (see, for example, Refs. 20, 35, and 69 and the references given there to the earlier work of Dirac and others).

We now turn to the specification of the symplectic structure on the manifold of fields that occur in the Lagrangian (9). Regarding the fields A_μ, h_μ, g_μ, and ψ [which occur in (9) together with their derivatives] as generalized coordinates, we introduce for this purpose momenta canonically conjugate to them:

$$\pi_A^\mu(x) = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu(x)} = -F^{0\mu}(x) - \eta^{0\mu} B(x) \quad (\eta^{0\mu} = -\delta_0^\mu), \tag{17a}$$

$$\pi_g^\mu(x) = \frac{\partial \mathcal{L}}{\partial \dot{g}_\mu(x)} = -H^{0\mu} - \eta^{0\mu} \partial h,$$

$$\pi_h^\mu(x) = \frac{\partial \mathcal{L}}{\partial \dot{h}_\mu(x)} = -G^{0\mu} - \eta^{0\mu} \partial g, \tag{17b}$$

$$\pi_\psi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\psi}(x)} = -\tilde{\psi}(x)\gamma^0. \tag{17c}$$

The canonical Poisson brackets for the Bose fields are standard; the nontrivial equal-time bracket satisfies (8). But the properties of the "classical" Dirac field are not so usual and require some clarification. We determine them formally, letting the Planck constant $\hbar$ tend to zero in the canonical anticommutation relations of the quantum fields ψ and $\tilde{\psi}$. This leads to identification of the fields $\psi(x)$ and $\tilde{\psi}(x)$ with the elements of an infinite Grassmann algebra (i.e., to the assumption that they anticommute rigorously) and to the postulation of symmetric equal-time Poisson brackets:

$$\{\psi(t, \mathbf{x}), \pi_\psi(t, \mathbf{y})\} = \{\pi_\psi(t, \mathbf{y}), \psi(t, \mathbf{x})\}$$
$$= \lim_{\hbar \to 0} \frac{1}{i\hbar} [\psi(t, \mathbf{x}), \pi_\psi(t, \mathbf{y})]_+ = \delta(\mathbf{x} - \mathbf{y}), \tag{18a}$$

or, in the basis (6b),

$$\{\psi(t, \mathbf{x}), \psi^*(t, \mathbf{y})\} = -i\delta(\mathbf{x} - \mathbf{y}) \tag{18b}$$

(the remaining equal-time brackets vanish).

It follows from these relations in particular that the zeroth component of the Noether current (12) behaves as the density of electric charge:

$$\left.\begin{aligned} &\{j^0(t, \mathbf{x}), \psi(t, \mathbf{y})\} = ie\delta(\mathbf{x} - \mathbf{y})\psi(t, \mathbf{x}), \\ &\{j^0(t, \mathbf{x}), \tilde{\psi}(t, \mathbf{y})\} = -ie\delta(\mathbf{x} - \mathbf{y})\tilde{\psi}(t, \mathbf{x}), \\ &\{j^\mu(t, \mathbf{x}), A_\nu(t, \mathbf{y})\} = \{j^\mu(t, \mathbf{x}), B(t, \mathbf{y})\} = 0. \end{aligned}\right\} \tag{19}$$

Thus, the three-dimensional (spatial) integral of $j^0(x)$ can be identified with the generator of gauge transformations of the first kind. The expression for the conserved current, which generates the local gauge transformations (4) and (16), can be written in the form

$$J_\mu(x; l(x, \xi)) = G_{\mu\nu} l^\nu - g_\nu(\partial_\mu l^\nu - \partial^\nu l_\mu)$$
$$+ l_\mu \partial g - \partial l \overleftrightarrow{\partial_\mu} B - \frac{1}{\xi} \Box \partial l \overleftrightarrow{\partial_\mu} \partial h \tag{20}$$
$$(\lambda \overleftrightarrow{\partial_\mu} B = \lambda \partial_\mu B - B \partial_\mu \lambda).$$

Its conservation follows from Eqs. (14)–(16), whereas the determining Poisson brackets

$$\left.\begin{aligned} &\{Q_l, h_\mu(x)\} = l_\mu(x), \quad \{Q_l, B(x)\} = \frac{1}{\xi} \Box \lambda(x), \\ &\{Q_l, \psi(x)\} = ie\lambda(x)\psi(x), \quad \{Q_l, A_\mu(x)\} = \partial_\mu \lambda(x), \\ &\{Q_l, g_\mu(x)\} = \frac{1}{\xi} \partial_\mu \Box \lambda(x), \quad \lambda(x) = \partial l(x) \end{aligned}\right\} \tag{21}$$

for the generator

$$Q_l = \int J^0(x; l(x, \xi)) \, d^3x \tag{22}$$

of the local gauge transformations (4) and (16) are derived from (8) and (17), and from the relation

$$\{\dot{B}(t, \mathbf{x}), \psi(t, \mathbf{y})\} = ie\delta(\mathbf{x} - \mathbf{y})\psi(t, \mathbf{x}), \tag{23}$$

which, in its turn, follows from (19) and from the equation of motion (13).

The circumstance that the fields $g_\mu(x)$ and $B(x)$ satisfy the linear ("free") equations (14) makes it possible to calculate their Poisson brackets with the basic fields at different times:

$$\begin{rcases}\{g_\mu(x), h_\nu(y)\} = -\eta_{\mu\nu} D(x-y), \\ \{A_\mu(x), B(y)\} = \partial_\mu D(x-y), \\ \{B(x), h_\nu(y)\} = \frac{1}{2}(y_\nu - x_\nu) D(x-y), \end{rcases} \tag{24a}$$

$$\begin{rcases}\{B(x), \psi(y)\} = ieD(x-y)\psi(y), \\ \{B(x), \tilde{\psi}(y)\} = -ieD(x-y)\tilde{\psi}(y), \end{rcases} \tag{24b}$$

where $D(x)$ is the Pauli-Jordan commutator function of the massless scalar field:

$$D(x) = 2\pi i \int \varepsilon(p^0)\delta(p^2) e^{ipx} d_4p = \frac{1}{2\pi}\varepsilon(x^0)\delta(x^2), \tag{25a}$$

which satisfies

$$\Box D(x) = 0,\ D(0, \mathbf{x}) = 0,\ \partial_0 D(0, \mathbf{x}) = \delta(\mathbf{x}). \tag{25b}$$

B. Physical charged fields and gauge-invariant bilocal operators

We can now introduce charged fields invariant with respect to local gauge transformations (although, of course, they take on a nontrivial phase factor under global transformations). We introduce the nonlocal field

$$\psi(x; f, A) = \exp\left[-ie \int A_\mu(y) f^\mu(x-y) d^4y\right]\psi(x), \tag{26a}$$

where $f(x)$ is a real generalized function satisfying the equation

$$\partial^\mu f_\mu(x) = \delta(x) \tag{26b}$$

and the boundary condition

$$\int_{S^3(R)} \lambda(x) f^\mu(x) dS_\mu \xrightarrow[R\to\infty]{} 0 \tag{26c}$$

for any choice of the functions $\lambda(x)$ that decrease smoothly at infinity [$S^3(R)$ is the Euclidean sphere of radius R]. It is readily verified that the field (26) does not change under the local gauge transformations (4) [for which $\lambda(x)\to 0$ as $x\to\infty$, but the function $\lambda(x)$ is not required to satisfy Eq. (16b)]. Special cases of such "physical charged fields" were already considered by Dirac[16] (see also Refs. 3 and 4). In Ref. 15, this field is given a different realization, which is also expressed in terms of an exponential of a linear operator:

$$\psi(x, n; A) = E(x; n; A)\psi(x), \tag{27a}$$

where $n = (n^\mu)$ is an arbitrary nonzero vector, and

$$E(x, n; A) = \exp\left[\frac{-ie}{2}\int_{-\infty}^{+\infty} d\alpha\,\varepsilon(\alpha) n^\mu A_\mu(x-\alpha n)\right] \tag{27b}$$

[$\varepsilon(\alpha) = \operatorname{sign}\alpha$ is the sign function]. We leave the reader to prove that the representation (27) is obtained as a special case of (26) for

$$f_\mu(x) = -in_\mu \int \frac{e^{iqx}}{nq} d_4q = \frac{1}{2} n_\mu \int_{-\infty}^{\infty} \varepsilon(\alpha)\delta(x-\alpha n) d\alpha,$$

where the integral in the first equation must be understood in the principal-value sense. It is readily verified that for such a choice of f_μ local gauge invariance of the field (26a) [or (27)] holds for any smooth function $\lambda(x)$ satisfying $\lim \lambda(x-\alpha n) = 0$ for $n\neq 0$.

In terms of the field (27) we can, following Ref. 15, define a Poincaré-invariant nonlocal physical charged field

$$\psi(x; A) = \int d_4p \int d_4y e^{ip(x-y)}\psi(y; p, A) \tag{28}$$

(which is also invariant with respect to the local gauge transformations).

It follows from (11) that the Fourier transform of the field (27) satisfies the integral equation

$$(i\hat{p} + m)\psi(p; n, A) = ie\int d_4q\left[\hat{A}(p-q) - (\hat{p}-\hat{q})\frac{nA(p-q)}{n(p-q)}\right]\psi(q; n, A). \tag{29}$$

Note that (29) does not make it possible to obtain for $n = p$ a simple (closed) equation for the physical field

$$\psi(p; A) = \psi(p; p, A), \tag{30}$$

since the right-hand side of (29) contains an integral over q of $\psi(q; p, A)$ (for $q\neq p$).

The fields (27) can be used to construct gauge- and Poincaré-invariant bilocal fields corresponding to *a* pair of separated opposite charges. Setting $n = y - x$ in the product $\psi(x; n, A)\tilde{\psi}(y; n, A)$, we obtain the dipole field

$$\mathscr{D}(x, y) = \mathscr{D}(x, y; \psi, \tilde{\psi}, A) = \psi(x; y-x, A)\tilde{\psi}(y; y-x, A)$$
$$= \psi(x)\exp\left\{ie\int_0^1 (y^\mu - x^\mu) A_\mu(\alpha x + (1-\alpha)y) d\alpha\right\}\tilde{\psi}(y). \tag{31}$$

It is easy to show that the field $\mathscr{D}$ is invariant with respect to both local and global gauge transformations; in addition, it is Poincaré-covariant and satisfies the Hermiticity condition

$$\tilde{\mathscr{D}}(x, y) = \mathscr{D}(y, x)^* \beta\otimes\beta^{-1} = \mathscr{D}^\dagger(x, y), \tag{32}$$

where * denotes complex conjugation and transposition of the factors. (We recall that the fermion fields ψ and $\tilde{\psi}$ anticommute classically.)

C. Higgs model and non-Abelian gauge theories

The Abelian Higgs model of spontaneous symmetry breaking (Refs. 19, 33, 36, 39, 49, and 72) is the electrodynamics of a scalar charged field φ that interacts not only with the electromagnetic field but also with itself through the Higgs potential V_H, which has a minimum on a circle of nonzero radius in the complex φ plane. The Lagrangian of this model can be written in the form

$$\mathscr{L} = \mathscr{L}_A + \mathscr{L}_\varphi - V_H, \tag{33}$$

where $\mathscr{L}_A = \mathscr{L}_{em} + \mathscr{L}_B^\xi$ [see (9)] is the Lagrangian of the free electromagnetic field in the covariant gauge, and

$$\mathscr{L}_\varphi = -\overline{\mathscr{D}}^\mu\varphi^*\mathscr{D}_\mu\varphi, \quad (\mathscr{D}_\mu = \partial_\mu + ieA_\mu); \tag{34}$$

$$V_H = \frac{1}{2}\varphi^*\varphi(\lambda^2\varphi^*\varphi - m_H^2). \tag{35}$$

The minimum of the potential (35) is attained for

$$(\varphi^*\varphi)_{\min} = \frac{1}{2\lambda^2} m_H^2. \tag{36}$$

Choosing some point $\varphi_\alpha = (1/\lambda\sqrt{2}) m_H e^{i\alpha}$ on the circle (36) and making in the neighborhood of this minimum the change of variables

$$\varphi = \frac{1}{\sqrt{2}}\left(\frac{m_H}{\lambda} + \rho + i\chi\right)e^{i\alpha},\ \varphi^* = \frac{1}{\sqrt{2}}\left(\frac{m_H}{\lambda} + \rho - i\chi\right)e^{-i\alpha}, \tag{37}$$

we can express the Lagrangian (33) in terms of the fields ρ and χ, zero values of which correspond to the minimal energy. As a result, we obtain

$$\mathscr{L}=\mathscr{L}_0+\mathscr{L}'-m_H^4/2\lambda^2, \tag{38a}$$

where

$$\mathscr{L}_0=\mathscr{L}_A-\frac{m^2}{2}A^2-\frac{1}{2}[(\partial\chi)^2+(\partial\rho)^2]-\frac{m_H^2}{2}\rho^2-mA^\mu\partial_\mu\chi,$$
$$m=e\frac{m_H}{\lambda} \tag{38b}$$

plays the part of the free Lagrangian of the asymptotic particles, and

$$\mathscr{L}'=-eA^2\left(m\rho+e\frac{\chi^2+\rho^2}{2}\right)+eA_\mu(\rho\partial^\mu\chi-\chi\partial^\mu\rho)$$
$$-\frac{\lambda^2}{8}(\rho^2+\chi^2)-\frac{\lambda m_H}{2}(\rho^2+\chi^2)\rho \tag{38c}$$

the interaction Lagrangian. Variation of $\mathscr{L}_0$ leads to the linear equations of motion

$$(m^2-\Box)A_\mu+\partial_\mu(m\chi+\partial A-B)=0; \tag{39a}$$
$$\xi B=\partial A; \tag{39b}$$
$$\Box\chi+m\,\partial A=0; \tag{39c}$$
$$(m_H^2-\Box)\rho=0. \tag{39d}$$

Taking the divergence of (39a) and using (39c), we see that the field $B(x)$ satisfies in this case the equation $\Box B=0$. Applying the d'Alembert operator to both sides of (39b) and (39c), we find

$$\Box\partial A=0=\Box^2\chi. \tag{40}$$

The circumstance that the (asymptotic) field χ satisfies the fourth-order equation (40), whereas $\Box\chi$ does not in general (for $\xi\neq 0$) vanish, plays a part in the analysis of spontaneously broken symmetry (see Sec. 3.C).

Finally, we give as an example of a non-Abelian gauge theory the Lagrangian of the Higgs field $\varphi=\binom{\varphi_1}{\varphi_2}$ interacting with the Yang-Mills field:

$$A_\mu(x)=g\frac{\tau_a}{2i}A_\mu^a,\quad (A_\mu^*(x)+A_\mu(x)=0), \tag{41}$$

where τ_a are the Pauli matrices (summation from 1 to 3 over the repeated index a is understood). Defining

$$\mathscr{D}_\mu=\partial_\mu+A_\mu,\ \mathscr{D}_\mu^*=\partial_\mu-A_\mu, \tag{42}$$

we set

$$\mathscr{L}=\mathscr{L}_{YM}-\mathscr{D}_\mu^*\varphi^*\mathscr{D}^\mu\varphi-V_H, \tag{43}$$

where V_H is the potential of the two-component Higgs field, which again is given by Eq. (35) (with $\varphi^*\varphi=\varphi_1^*\varphi_1+\varphi_2^*\varphi_2$), and the gauge-invariant part of the Yang-Mills Lagrangian has the form

$$\mathscr{L}_{YM}^{\rm inv}=\frac{1}{g^2}\,{\rm tr}\left([\mathscr{D}_\mu,\mathscr{D}_\nu]G^{\mu\nu}-\frac{1}{2}G_{\mu\nu}G^{\mu\nu}\right). \tag{44}$$

The Lagrangian of the doublet of spinor fields $\psi=\binom{\psi_1}{\psi_2}$, $\psi_{1,2}=(\psi_{1,2}^\alpha,\ \alpha=1,\ldots,4)$, interacting with the Yang Mills field is identical to $\mathscr{L}_{\mathscr{D}}$ (9b) but with $\widehat{\mathscr{D}}=\widehat{\partial}+\widehat{A}$, where $A_\mu\in{\rm su}(2)$ [see (41)]. Variation with respect to $G_{\mu\nu}$ reproduces the standard connection between $G_{\mu\nu}$ and the Yang-Mills field A_μ:

$$G_{\mu\nu}=g\frac{\tau_a}{2i}G_{\mu\nu}^a,$$
$$G_{\mu\nu}^a=\partial_\mu A_\nu^a-\partial_\nu A_\mu^a+g\varepsilon_{abc}A_\mu^bA_\nu^c. \tag{45}$$

The gauge transformations of the second kind for the fields φ, A_μ, and $G_{\mu\nu}$ have the form

$$\varphi\to u\varphi,\quad u=e^{-\chi},\quad \chi=g\frac{\tau_a}{2i}\chi^a; \tag{46a}$$
$$A_\mu\to u\mathscr{D}_\mu u^{-1}=uA_\mu u^{-1}+\partial_\mu\chi; \tag{46b}$$
$$G_{\mu\nu}\to uG_{\mu\nu}u^{-1} \tag{46c}$$

or

$$A_\mu^a\to R_b^aA_\mu^b+\partial_\mu\chi^a,\ \text{where}\ u\tau_au^{-1}=\tau_bR_a^b. \tag{47}$$

A direct consequence of the invariance of the Lagrangian (43) with respect to local gauge transformations is the expression of the Noether current as the divergence of the antisymmetric field-strength tensor:

$$j_a^\mu=\frac{\partial\mathscr{L}}{\partial A_\mu^a}=\partial_\nu G_a^{\mu\nu} \tag{48}$$

(see, for example, Ref. 69). As in Abelian theory, this equation plays the part of a constraint. When a gauge-fixing term is introduced into the Lagrangian, Eq. (48) acquires an extra term on the right-hand side [cf. (13)]. As we shall see below (in Sec. 2.C), the equation holds only in the weak sense in the local formulation of the quantum theory of gauge fields.

2. POSTULATES OF QUANTUM ELECTRODYNAMICS

The necessity of a modification of the basic principles of quantum field theory (see, for example, Ref. 6) when an attempt is made to include at least quantum electrodynamics is due to two features of gauge theories: 1) the indefiniteness of the Poincaré-invariant metric and the need to use "unphysical" fields (and states) in any local gauge; 2) the infrared behavior and the associated problem of defining asymptotic states of charged particles (see Refs. 27, 34, 40, 41, 44, 75, 76, 78, and 80). The modifications of the theory due to these features are very significant.

A. Space of state vectors in a local covariant gauge. Indefinite metric with Hilbert majorant (Refs. 5, 7, 47, 48, 50, and 61)

The need to use an indefinite metric in local quantum electrodynamics appears already in the simplest example of the theory of the free electromagnetic field $A_\mu(x)$. The two-particle function of the field $A_\mu(x)$, which satisfies the equation

$$(\xi-1)\,\partial_\mu(\partial A)=\xi\,\Box A_\mu, \tag{49}$$

corresponding to the Lagrangian (9e) for $e=0\ (=j_\mu)$,

$$\langle A_\mu(x)\,A_\nu(y)\rangle_0$$
$$=2\pi\int e^{ik(x-y)}\theta(k^0)\,[\eta_{\mu\nu}\delta(k^2)+(1-\xi)\,k_\mu k_\nu\delta'(k^2)]\,d_4k, \tag{50}$$

is obviously not positive definite for any value of the gauge parameter ξ. The nonpositivity of the scalar square of the vector

$$A(f)|0\rangle=\int A_\mu(x)\,f^\mu(x)\,d^4x|0\rangle$$

for a nontransverse test function f^μ is a general property of the free theory in any covariant gauge and generalizes to interacting fields (as will be shown in the following brief review). Moreover, it is not by chance that the vacuum Maxwell equation $\partial_\nu F^{\mu\nu}(x)=0$ is replaced in our case by the equation

$$\partial_\nu F^{\mu\nu}=\partial^\mu B. \tag{51}$$

According to Strocchi's analysis,[57] the free Maxwell equation cannot hold in the operator sense in any local or covariant gauge [if the theory is formulated in terms of the potential $A_\mu(x)$].

We assume that the smeared fields $A(f)$, $\psi(u)$, and $\tilde{\psi}(u)$ are realized as linear operators in the vector space D with the nondegenerate Hermitian form

$$\langle\Phi,\ \Psi\rangle=\overline{\langle\Psi,\ \Phi\rangle},\ (\langle\Phi,\ \alpha_1\Psi_1+\alpha_2\Psi_2\rangle \\ =\alpha_1\langle\Phi,\ \Psi_1\rangle+\alpha_2\langle\Phi,\ \Psi_2\rangle) \tag{52}$$

(they carry D into D). We assume further that for real test functions f and u the operator $A(f)$ is symmetric and that the operators $\psi(u)$ and $\psi^*(u)=\tilde{\psi}(u)\beta^{-1}$ are conjugate with respect to this form:

$$\langle\Phi,\ A(f)\Psi\rangle=\langle A(f)\Phi,\ \Psi\rangle,$$
$$\langle\Phi,\ \psi(u)\Psi\rangle=\langle\psi^*(u)\Phi,\ \Psi\rangle.$$

(We recall that a form $\langle,\rangle$ is said to be nondegenerate if from the equation $\langle\Phi,\Psi\rangle=0$ for all Ψ in D it follows that $\Phi=0$.) We require further that in the space D there be realized a pseudounitary representation of the Lie algebra of the Poincaré group (the infinitesimal condition of pseudounitarity with respect to a given form is the condition of anti-Hermiticity $\langle\Phi,X\Psi\rangle+\langle X\Phi,\Psi\rangle=0$ of the mathematical generators X of the Lie algebra).

In studying the topological properties of the theory, it is convenient to make the assumption that the form (52) has a Hilbert majorant. In other words, we assume that in D there is defined a positive scalar product $(,)$ that majorizes the form (52):

$$|\langle\Phi,\ \Psi\rangle|^2\leqslant c\,(\Phi,\ \Phi)\,(\Psi,\ \Psi),\quad c>0. \tag{53}$$

We shall not be interested in an individual scalar product $(,)$ but only in the class of majorizing scalar products that define equivalent topologies. Two scalar products $(,)_1$ and $(,)_2$ define equivalent topologies if and only if they majorize each other:

$$(\Phi,\ \Phi)_1\leqslant a_1\,(\Phi,\ \Phi)_2,\ (\Phi,\ \Phi)_2\leqslant a_2\,(\Phi,\ \Phi)_1,\ a_{1,\,2}>0. \tag{54}$$

The completion $\mathscr{H}$ of the space D with respect to the strong topology determined by convergence in the norm

$$\|\Phi-\Psi\|=(\Phi-\Psi,\ \Phi-\Psi)^{1/2} \tag{55}$$

does not depend on the particular choice of the scalar product within the given equivalence class.

As we have already said, in the framework of the covariant local formulation of quantum electrodynamics there is no positive scalar product invariant with respect to Poincaré transformations. However, assuming the existence of global pseudounitary transformations $U(a,\underset{\sim}{\Lambda})$, we can require the transformed scalar product

$$(\Phi,\ \Psi)_{a,\,\underset{\sim}{\Lambda}}=(U(a,\ \underset{\sim}{\Lambda})\,\Phi,\ U(a,\ \underset{\sim}{\Lambda})\,\Psi)$$

to determine the same topology as the original one. For this, it is necessary and sufficient if the operator $U(a,\underset{\sim}{\Lambda})$ is bounded with respect to the Hilbert norm:

$$\|U(a,\ \underset{\sim}{\Lambda})\|\leqslant B<\infty, \tag{56}$$

where $B=B(a,\underset{\sim}{\Lambda})$ is a continuous function of the group parameters. Here, the complex unimodular matrix $\underset{\sim}{\Lambda}\in SL(2,C)$ is related to the proper Lorentz transformation Λ by $\underset{\sim}{\Lambda}\sigma_\mu\underset{\sim}{\Lambda}^*=\sigma_\nu(\Lambda)^\nu_\mu$, $\underset{\sim}{\Lambda}^{*-1}\tilde{\sigma}_\mu\underset{\sim}{\Lambda}^{-1}=\tilde{\sigma}_\nu(\Lambda)^\nu_\mu$, where $\sigma_k=(-\tilde{\sigma}_k)$; $k=1,2,3$, are the Pauli matrices, and $\sigma_0=\tilde{\sigma}_0$ is the 2×2 unit matrix. The Poincaré invariance of the Hilbert topology follows from (56) and from the group law [which ensures that $U^{-1}(a\,\underset{\sim}{\Lambda})$ is bounded].

We consider as an example the single-particle space of free photons in the Gupta-Bleuler gauge. In accordance with (50) (for $\xi=1$) the state vectors in this space can be specified as functions $\Phi^\mu(p)$ on the light cone $p^0=|\mathbf{p}|$. The Poincaré-invariant indefinite form is

$$\langle\Phi,\ \Psi\rangle=\int\overline{\Phi^\mu}\,(p)\,\eta_{\mu\nu}\Psi^\nu\,(p)\,(dp)_0, \tag{57}$$
$$(dp)_m\frac{d^3p}{(2\pi)^3 2\sqrt{m^2+\mathbf{p}^2}}.$$

As a positive scalar product, it is natural to take

$$(\Phi,\ \Psi)=\int\overline{\Phi^\mu}\,(p)\,\delta_{\mu\nu}\Psi^\nu\,(p)\,(dp)_0. \tag{58}$$

It is readily verified that in this case Eq. (56) is satisfied if B^2 is identified with the largest eigenvalue of the positive matrix ${}^t\Lambda\Lambda$.

Note that in other gauges (for $\xi\neq1$), when the space of single-photon states can be realized in terms of 8-component vector functions on the cone (and both the invariant form $\langle,\rangle$ and the scalar product $(,)$ are given by more complicated expressions), the translation operators $U(a)$ are also nonunitary with respect to the positive scalar product, but the condition (56) remains valid.

It follows from the inequality (53) that for given $\langle,\rangle$ and $(,)$ there exists a bounded Hermitian (with respect to the positive scalar product) operator β such that

$$\langle\Phi,\ \Psi\rangle=(\Phi,\ \beta\Psi). \tag{59}$$

We assume (with Strocchi *et al.*[58–61]) that β also has a bounded inverse operator. Then by redefining the scalar product within the given equivalence class one can achieve

$$\beta^2=1 \tag{60}$$

[at the same time, the inequality (53) will hold for $c=1$].

The covariance of the considered class of gauges can be expressed by the standard law of transformation of the basic fields

$$A_\nu(x)\to A_\mu(\Lambda x+a)\,\Lambda^\mu_\nu, \tag{61a}$$
$$\Psi(x)\to S^{-1}(\underset{\sim}{\Lambda})\,\psi(\Lambda x+a), \tag{61b}$$

where $S(\underset{\sim}{\Lambda})$ is a bispinor representation of the group SL(2, C). In the realization of the γ matrices in which $\gamma_5=i\gamma^0\gamma^1\gamma^2\gamma^3$ is diagonal ($\gamma_5=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$), the representation $S(\underset{\sim}{\Lambda})$ reduces to the form $S(\underset{\sim}{\Lambda})=\begin{pmatrix}\underset{\sim}{\Lambda}&0\\0&\underset{\sim}{\Lambda}^{*-1}\end{pmatrix}$.

B. Irreducibility and cyclicity. Composite fields. The concept of the electric charge[51,53]

In the axiomatic presentation of quantum electrodynamics, we write out explicitly the same equations of classical electrodynamics in which products of fields at one point, whose meaning requires a special investigation in the quantum case because of ultraviolet divergences,[18] are not encountered. We shall give, for example, Eq. (13), since it is linear with respect to the fields and currents in it, but we shall not use the explicit expression (12) for the current j^μ in terms of the charged fields ψ and $\tilde{\psi}$. Instead of this, we introduce a somewhat more abstract concept of a composite field in the quantum case.

We assume first (in accordance with the posed problem of the axiomatic description of local covariant gauges) that the basic fields[2)] A_μ, ψ, $\tilde{\psi}$, and B satisfy the usual locality conditions, namely, that the fields A_μ and B commute with each other and with the spinor fields in the case of spatial separation of their arguments, whereas ψ and $\tilde{\psi}$ anticommute locally with each other. In addition, we require that they satisfy the following condition of irreducibility.

With the algebra *FP* of polynomials of the smeared basic fields there is associated an algebra $\mathscr{F} \subset \mathscr{B}(\mathscr{H})$ of bounded operators in the Hilbert space $\mathscr{H}$; this algebra can be defined as the double commutant, $(FP)''$, of the algebra *FP* in $\mathscr{B}(\mathscr{H})$. [We recall that the commutant F' of some algebra F of operators in a Hilbert space $\mathscr{H}$ is, by definition, the algebra of all bounded operators in $\mathscr{H}$ that commute with F. One can think of $\mathscr{F} = (FP)''$ as the algebra generated by bounded functions of the diagonalizable operators in *FP*.] Similarly, we can define the algebras $\mathscr{F}(\mathscr{O}) = FP(\mathscr{O})''$, where $FP(\mathscr{O})$ is the algebra of polynomials of fields smeared by test functions with support in $\mathscr{O}$. The algebras $\mathscr{F}(\mathscr{O}) \subset \mathscr{F}$ define a net of *field algebras* as considered by Doplicher, Haag, and Roberts.[17] Irreducibility of the algebra $\mathscr{F}$ (or *FP*) means that any bounded (with respect to the Hilbert topology) operator that commutes with all elements of $\mathscr{F}$ is a multiple of the identity operator in $\mathscr{H}$.

We shall show that if the algebra F generated by the basic fields is irreducible, then any nonzero vector Ψ of the space $\mathscr{H}$ is cyclic with respect to this algebra. Indeed, if the closure of the linear manifold $\mathscr{F}\Psi$ with respect to the Hilbert norm is a true subspace of $\mathscr{H}$, then the operator of projection onto this subspace will commute with the algebra $\mathscr{F}$, contradicting the irreducibility hypothesis.

We now assume that the space D contains a Poincaré-invariant *vacuum vector* $|0\rangle$, for which

$$\langle 0 \mid 0 \rangle = 1. \tag{62}$$

(We do not assume uniqueness of this vector.) In accordance with what we have proved, the vector $|0\rangle$ is cyclic with respect to the algebra $\mathscr{F}$.

Suppose further that the current $j^\mu(x)$ is local with respect to the basic fields. Then in accordance with Borchers's well-known theorem (see Ref. 6, Theorem 5.2.7, p. 355), $j^\mu(x)$ is also a local field and, hence, belongs to the *Borchers's class* of basic fields. Fields that belong to this class but are not linear combinations of the basic fields and their derivatives are called *composite fields*. (The basic fields satisfy the condition of irreducibility already without the addition to them of composite fields acting on the same space D.)

From the spectral condition and the requirement of Lorentz invariance, and from the current conservation law there follows the Källén-Lehmann representation

$$w^{\mu\nu}(x-y) = \langle j^\mu(x)\, j^\nu(y)\rangle_0$$
$$= \int d_4 p e^{ip(x-y)}\theta(p^0)\, p^2 (p\eta_{\mu\nu} - p_\mu p_\nu) \int_0^\infty ds \rho(s)\, \delta(s+p^2). \tag{63}$$

Perturbation-theory analysis[25] indicates that in the case of a massive charged field $\rho(s)$ is integrable in the neighborhood of $s=0$. In particular, in the second order of perturbation theory, in which

$$\langle j^\mu(x)\, j^\nu(y)\rangle_0 = -e^2 \langle \tilde{\psi}_-(x)\, \gamma^\mu \psi_+(x)\, \tilde{\psi}_+(y)\, \gamma^\nu \psi_-(y)\rangle_0,$$

where $\psi_\pm(x)$ are terms of the free Dirac field with given signs of the frequency satisfying

$$\psi_+(x) \mid 0\rangle = \tilde{\psi}_-(x) \mid 0\rangle = 0 = \langle 0 \mid \tilde{\psi}_+(y) = \langle 0 \mid \psi_-(y),$$

the spectral function

$$\rho(s) = \frac{\theta(s-4m^2)}{6\pi s}\sqrt{1-\frac{4m^2}{s}}\left(1+\frac{2m^2}{s}\right)$$

vanishes for $s < 4m^2$.

We define an approximation of the electric charge,

$$Q_R = \int f_R(x)\, j^0(x)\, d^4x, \tag{64a}$$

where f_R is a test function that depends on the parameter R in such a way that

$$f_R(x) \to \delta(x^0) \quad \text{as} \quad R \to \infty. \tag{64b}$$

(The existence of the operator Q_R follows from the axioms which we have adopted, but this cannot be said *a priori* of its limit $Q = \lim_{R\to\infty} Q_R$.) To justify the necessity of such an approximation, we investigate the behavior of the vacuum expectation value of Q_R^2, using the representation (63) (see, for example, Ref. 51). To be specific, suppose

$$f_R(x) = \sqrt{\frac{2\pi}{R}}\exp\left(-\frac{r^2}{2R} - \frac{Rt^2}{2}\right)$$
$$= (2\pi R)^{3/2}\int d_4 p \exp\left\{ipx - \frac{R}{2}\mathbf{p}^2 - \frac{1}{2R}p_0^2\right\}, \tag{64c}$$

where $r = |\mathbf{x}|$, $t = x^0$. We note first that smearing with respect to the time components is necesary—replacement of f_R by $\exp(-r^2/2R)$ would lead to a divergent expression for $\langle Q_R^2\rangle_0$. However, integrating $\bar{f}_R(x) w^{00}(x-y) f_R(y)$ over x and y, we find [using (63) and (64) and replacing the variable of integration $\mathbf{p}$ by $\mathbf{R}^{-1/2}\mathbf{k}$]

$$\langle Q_R^2\rangle_0 = \frac{\sqrt{R}}{4\pi}\int_0^\infty ds \int d^3k \left(s+\frac{\mathbf{k}^2}{R}\right)^{-1/2} s\mathbf{k}^2\rho(s)$$
$$\times \exp\left[-\frac{1}{2}\left(\mathbf{k}^2 + \frac{s}{R} + \frac{\mathbf{k}^2}{R^2}\right)\right] \xrightarrow[R\to\infty]{} \infty.$$

Assuming the existence of a (self-adjoint) operator Q of the

[2)]In a gauge in which the parameter ξ in Eq. (15a) is nonzero, the field B is the derivative of the fields A_μ: $B = (1/\xi)\partial A$. For unity and compactness of the exposition, it is, however, always convenient to introduce it separately.

(conserved) electric charge in $\mathscr{H}$ (defined on state vectors with finite charge, including a vacuum vector, for which $Q|0\rangle = 0$), we see that Q_R cannot tend to Q on D with respect to the strong operator topology, since $\|(Q_R - Q)|0\rangle\|^2 = \|Q_R|0\rangle\|^2 = \langle Q_R^2\rangle_0 \to \infty$. The question arises of the conditions under which an operator Q does indeed exist and of the sense in which it is approximated by the smeared operators Q_R (64).

We note first that for any operator φ of the polynomial algebra FP with support in a bounded region $\mathscr{O}$ of space-time the commutator limit

$$\lim_{R\to\infty} [Q_R, \varphi] \tag{65}$$

exists. To prove this, it is simplest to assume that the functions f_R can be factorized in the form

$$f_R(x) = e_R(\mathbf{x})\, \delta_\varepsilon(x^0), \tag{66a}$$

where e_R and δ_ε are infinitely smooth non-negative functions with compact support such that

$$e_R(\mathbf{x}) = 1 \quad \text{for} \quad |\mathbf{x}| \leqslant R; \tag{66b}$$

$$\int dt\delta_\varepsilon(t) = 1, \quad \delta_\varepsilon(t) = 0 \quad \text{for} \quad |t| \geqslant \varepsilon = 0\left(\frac{1}{R}\right). \tag{66c}$$

Under these assumptions, it follows from the locality postulate that $[Q_R \varphi]$ does not depend on R for sufficiently large R (exceeding the maximal distance of the points of $\mathscr{O}$ from the origin). Thus, an infinitesimal transformation (47) (generated by some local Noether current) of the local fields always exists. This does not yet ensure the existence of a (self-adjoint) operator Q in the Hilbert space $\mathscr{H}$ that generates these transformations; Q exists if and only if the transformation (47) gives an (unbroken) symmetry.[3)] (Nonexistence of the operator Q or of the corresponding representation $e^{iQ\alpha}$ of the global gauge transformations is an example of a spontaneously broken symmetry.) In the following section, we give a simple criterion for the existence (or nonexistence) of the operator Q.

We conclude this subsection by adding a further requirement on the domain of definition D of the fields.

In the analysis of classical electrodynamics, we see that charged fields invariant with respect to local gauge transformations are necessarily nonlocal. We require the existence of the quantum analog of the nonlocal composite field $\psi(x; A)$ (28).

Suppose that for sufficiently good test functions f there exists the operator

$$\psi(f; A) = \int f(x)\, \psi(x; A)\, d^4x$$

from D to D, invariant with respect to local gauge transformations and such that

$$\lim_{R\to\infty} [Q_R, \psi(f; A)] = -\lim_{R\to\infty} [F^{0\nu}(\partial_\nu f_R),$$

$$\psi(f; A)] = -e\psi(f; A).$$

We summarize here some indications from perturbation theory (obtained in Ref. 15) that support such an assumption.

In the interaction representation, the quantized field $\psi(x; A)$ can be expressed in the form

$$\psi(x; A) = Z_\psi \int d_4p \int d^4y : \exp\left\{\frac{-ie}{2}\int_{-\infty}^{\infty} d\alpha\varepsilon(\alpha)\, p^\mu A_\mu(y - \alpha p)\right\} : \psi(y)\, e^{ip(x-y)}, \tag{67}$$

where :: denotes the normal product (of the "free" fields A_μ), and Z_ψ is a renormalization constant that includes all the dependence on the gauge (i.e., on the parameter ξ). The renormalized causal two-point function of the field can be expressed as follows in the neighborhood of the mass shell $m^2 + p^2 \approx 0$:

$$S^c(p, A) = i\int \langle T\psi(x, A)\, \tilde{\psi}(0; A)\rangle_0 e^{-ipx}\, d^4x$$

$$\approx \frac{m - i\hat{p}}{m^2 + p^2 - i0} \exp\left(\frac{\alpha}{2\pi}\, \frac{m^2 + p^2}{p^2} \ln \frac{m^2 + p^2 - i0}{m_0^2}\right), \tag{68}$$

where

$$S = T\exp\left(i\int \mathscr{L}_I(x)\, d^4x\right)$$

$$= T\exp\left(-e\int :\tilde{\psi}\gamma^\nu\psi A_\nu:(x)\, d^4x\right)$$

is the scattering operator, and m_0 is determined by the renormalization point of the wave function (in Ref. 15, the choice $m_0 = m$ is made; the more general form (68) has the advantage that in it one can pass to the conformally invariant limit $m \to 0$). The representation (68) corresponds to a sum of diagrams with many-photon exchange (with the omission of the internal electron loops, which in the considered case of an electron with nonzero mass make a negligible contribution in the neighborhood of the mass shell; see Ref. 1). The "physical propagator" $S^c(p, A)$ has the property

$$\lim_{p^2 \to -m^2} (m + i\hat{p})\, S^c(p, A) = 1, \tag{69}$$

which does not hold for the gauge-invariant but Lorentz-noninvariant field $\psi(x, n, A)$ (which generates "coherent states"; cf. Refs. 11, 27, 40, 41, 75, 76, and 78) and is lost in the limit $m \to 0$. This property ensures (for $m > 0$) the existence of the asymptotic Lehmann-Symanzik-Zimmermann limit of the field $\psi(x; A)$, which is a free electron field of mass m (see, for example, Ref. 6, §§4.1 and 4.2). Moreover, it is shown in Ref. 15 (again in the framework of perturbation theory) that the physical electron field $\psi(x; A)$ and the photon field $F_{\mu\nu}$ have gauge-invariant Green's functions and a scattering matrix free of infrared divergences.

Without going into the details of this calculation, we note that the infrared divergences are eliminated by introducing a parameter μ with the dimensions of mass in the definition of the propagator of the Hertz potentials h_ν. The photon propagator is written in the form

$$D^c_{\rho\sigma}(x; \xi) = i\langle TA_\rho(x)\, A_\sigma(0)\rangle_0$$

$$= i\Box\langle Th_\rho(x)\, A_\sigma(0)\rangle_0 = -[\eta_{\rho\sigma}\Box - (1 - \xi)\,\partial_\rho\partial_\sigma]\, E^c(\mu x), \tag{70a}$$

where

[3)]Definitions of the concepts of broken and unbroken symmetries are given below in Sec. 3. In the case of exact symmetry, the operator Q is given by Eq. (102).

$$E^c(\mu x) = \int E^c(k;\ \mu)\, e^{ikx} d_4k = \frac{i}{(4\pi)^2} \ln \frac{4}{\beta(\mu x)^2 + i0}; \quad (70b)$$

$$\beta = e^{2\gamma - 1} (\approx 1.178), \quad \gamma = -\Gamma'(1) = 0.5772\ldots,$$

$$E^c(k;\ \mu) = \lim_{\varepsilon \downarrow 0} \frac{d}{d\varepsilon} \left[\varepsilon \frac{1}{(k^2 - i0)^2} \left(\frac{k^2 - i0}{\mu^2} \right)^\varepsilon \right]$$

$$= \lim_{\lambda \downarrow 0} \left[\frac{1}{(k^2 + \lambda^2 - i0)^2} + i\pi^2 \ln \frac{e\lambda^2}{\mu^2} \delta(k) \right]; \quad (70c)$$

$$-\Box E^c(\mu x) = D^c(x) = \frac{1}{(2\pi)^2} \frac{1}{x^2 + i0} = \int \frac{e^{ikx}}{k^2 - i0} d_4k, \quad (70d)$$

where the derivatives in (70a) are transferred to the electron propagators, with which $D^c_{\mu\nu}$ are integrated. This corresponds to working with an effective Lagrangian in which the interaction term $j^\mu A_\mu(x)$ is replaced by $h_\mu \Box j^\mu$ (or, equivalently, $-1/2H^\mu_\nu \partial_\mu j_\nu$).

Note that in accordance with the Bloch-Nordsieck theorem[4,52,62] the parameter μ in the expressions for cross sections between coherent states is effectively replaced by the instrument resolution ΔE.

As already noted, the field $\psi(x, A)$ is not local: The commutator $[\psi(x, A), \psi(y, A)]$ does not vanish in the case of a spatial separation of the arguments and it cannot be represented even classically as a functional that depends only on the values of the basic fields in a restricted region of space–time. However, we assume that the vectors in D generated by the action of these fields on the vacuum can be approximated by local states. More precisely, we assume that the vacuum is cyclic with respect to the algebra FP_0 of polynomials of the basic fields smeared by test functions of compact support. In other words, we assume that the set

$$D_0 = FP_0 \,|\, 0\rangle$$

(which contains only localized states) is dense in $\mathscr{H}$ with respect to the Hilbert topology.

C. Nonfulfillment of the Maxwell equation div $E = j^0$ in a local gauge

In this subsection, we shall analyze the need for the introduction of a field of the type B, which occurs in Eq. (13) in axiomatic quantum electrodynamics.

As already noted, the introduction of the term $\mathscr{L}_{GF} = \mathscr{L}^{(\xi)}_B + \mathscr{L}_{gh}$, which determines the class of covariant gauges, and the Lagrangian (9) of the electromagnetic field is dictated by the desire to work (in the framework of classical electrodynamics) with the standard canonical formalism. In the axiomatic quantum theory of interacting fields, one uses not the canonical commutation relations but only two consequences of them: local (anti)commutativity of the basic fields and the property of the "smeared charge" of generating global gauge transformations of the fields:

$$\lim_{R\to\infty} [Q_R,\ \psi(x)] = -e\psi(x), \quad \lim_{R\to\infty} [Q_R,\ \tilde{\psi}(x)] = e\tilde{\psi}(x). \quad (71)$$

The following question arises: If so little remains of the canonical formalism, cannot one get by without the introduction of fields of the type $B(x)$, $g_\mu(x)$, and $h_\mu(x)$ and postulate instead of (13) the usual gauge-invariant Maxwell equations? We shall show that this is not the case.

Proposition 2.1. In a local gauge in which $[\mathbf{E}(x), \psi(y)] = 0$ for $(x-y)^2 > 0 (E^k(x) = F^{0k}(x))$, the condition (71) with $e \neq 0$ is incompatible with Gauss's law

$$\operatorname{div} \mathbf{E} = j^0(x), \quad (72)$$

which relates the charge density to the divergence of the electric field.

Proof. Suppose that Q_R is given by the expression (64), where $f_R(x)$ has the form (66). For each fixed y, one can find a sufficiently large radius $R = R(y)$ such that every x in the support of f_R for which $|\mathbf{x}| \geqslant R$ is spacelike with respect to y. Using Gauss's theorem and the assumption (72), we find

$$[Q_R,\ \psi(y)] = \int dx^0 \delta_\varepsilon(x^0) \int_{|\mathbf{x}|=R} ds\, [\mathbf{E}(x),\ \psi(y)], \quad (73)$$

where ds is the element of area on the sphere $S^2(R) = \{\mathbf{x} \in \mathbf{R}^3;\ |\mathbf{x}| = R\}$ (the 3-vector ds is directed along the outer normal to the sphere). By virtue of the locality assumption (71), the commutator in the integrand vanishes, and this leads to vanishing of the charge e.

Equation (73), which plays the part of a constraint in the manifestly gauge-invariant formulation of classical electrodynamics, can be preserved in local quantum electrodynamics (in the weak sense).

We assume that the space D contains a subspace D' with the following properties:

a) D' is invariant with respect to the action of the operators $U(a, \Lambda)$ of the representation of the quantum-mechanical Poincaré group and contains a vacuum vector $|0\rangle$, for which $\langle 0|0\rangle = 1$;

b) D' is mapped into itself under the action of gauge-invariant elements of the polynomial algebra FP; in particular,

$$F_{\mu\nu}(f)\, D' \subset D', \quad j_\mu(f)\, D' \subset D'. \quad (74)$$

Moreover, if on the space D there is defined the field $\psi(f; A)$ (67), then it also leaves the subspace D' invariant:

$$\psi(f;\ A)\, D' \subset D'; \quad (75)$$

c) the constraints that are derived from the gauge-invariant singular Lagrangian are satisfied in the sense of the expectation values with respect to vectors in D'; more precisely,

$$\langle \Psi \,|\, j^\mu(f) + F^{\mu\nu}(\partial_\nu f) \,|\, \Phi \rangle = 0$$

$$\text{for any} \quad \Phi \in D',\ \Psi \in \mathscr{H}' = \overline{D}', \quad (76)$$

where $\overline{D}'(=\mathscr{H}')$ is the closure of D' with respect to the Hilbert topology;

d) the Poincaré-invariant sesquilinear form $\langle , \rangle$ is nonnegative definite on $\mathscr{H}'$:

$$\langle \Phi,\ \Phi \rangle \geqslant 0 \quad \text{for all} \quad \Phi \in \mathscr{H}'. \quad (77)$$

We shall show that from the listed properties and from the postulates of this section it follows that D contains nonzero vectors with vanishing scalar square (77). For this, we need above all the following generalization of the Reeh-Schlieder theorem.

Proposition 2.2. Suppose that in a theory with indefinite form of the type (59), where the self-adjoint operator β is bounded and has a bounded inverse, the vacuum is cyclic with respect to the polynomial algebra FP of basic local fields $\varphi_k(x)$. [In the case of quantum electrodynamics, the set of fields φ_k includes the charged fields $\psi(x)$, $\tilde{\psi}(x)$, the electromagnetic potential $A_\mu(x)$, and the field $B(x)$.] In such a case, the finite linear combinations of vectors of the form

$$\Phi(f_1, \ldots, f_j) = \varphi_1(f_1) \ldots \varphi_j(f_j) \mid 0\rangle, \ \operatorname{supp} f_k \subset \mathcal{O}, \tag{78}$$

where $\mathcal{O}$ is an arbitrary fixed open set in Minkowski space, are dense in $\mathcal{H}$.

Scheme of the proof. Suppose that the closure $\mathcal{H}(\mathcal{O})$ of the manifold of linear combinations of vectors of the form (78) is not identical to $\mathcal{H}$. Then $\mathcal{H}$ contains a vector, say, Ψ', orthogonal (with respect to the given positive scalar product) to the subspace $\mathcal{H}(\mathcal{O})$. By virtue of the assumption of the existence and boundedness of the operator β^{-1} there is a vector $\Psi \in \mathcal{H}$ such that $\Psi' = \beta\psi = \beta^*\psi$ and, therefore, in accordance with (59),

$$(\Psi', \Phi) = \langle\Psi, \Phi\rangle = 0 \quad \text{for all} \quad \Phi \in \mathcal{H}(\mathcal{O}). \tag{79}$$

In particular, all the generalized functions

$$F_\Psi = F_\Psi^{\varphi_1 \ldots \varphi_n}(x_1, \ldots, x_n) = \langle\Psi \mid \varphi_1(x_1) \ldots \varphi_n(x_n) | 0\rangle \tag{80}$$

vanish for $x_k \in \mathcal{O}(k = 1, \ldots, n)$. On the other hand, the functions F_ψ are the limiting values of analytic functions $F_\psi(z_1, \ldots, z_n)$ that are holomorphic in the tube domain $\operatorname{Im} z_n \in V_+$, $\operatorname{Im}(z_n - z_{n-1}) \in V_+, \ldots, \operatorname{Im}(z_2 - z_1) \in V_+$, where V_+ is the future cone: $V_+ = \{y \in \mathbf{R}^4 : y^0 > |\mathbf{y}|\}$. The locality of the fields φ_k makes it possible to prove analyticity of each funcion F_ψ in a larger domain that includes real spacelike separated points. It follows from this that the function (80) vanishes identically, and hence, since the vacuum is cyclic, $v = 0$ (for details, see Ref. 59).

Corollary. The equation $\varphi(x)|0\rangle = 0$, where φ is an arbitrary local field in the theory, implies the operator equation $\varphi(x) = 0$.

Indeed, taking the domain $\mathcal{O}$ spacelike with respect to the point x, we find by virtue of the local commutativity that $\varphi(x)\Phi = 0$ for any Φ of the form (78). Using Proposition 2.2, we conclude from this that $\varphi(x) = 0$.

We apply the result of the corollary to the field

$$\mathfrak{A}^\mu(x) = j^\mu(x) - \partial_\nu F^{\mu\nu}(x). \tag{81}$$

The operator $\mathfrak{A}^0(f_R)$ has a nontrivial commutation relation with the field $\psi(x)$ for $e \neq 0$ (by virtue of Proposition 2.1) and, hence, the vector

$$\Phi_R = \mathfrak{A}^0(f_R) | 0\rangle \tag{82}$$

does not vanish, whereas

$$\langle\Phi_R, \Phi_R\rangle = 0. \tag{83}$$

It follows from the existence of a nonzero vector with vanishing scalar square and from the nondegeneracy of the form $\langle,\rangle$ that in the space $\mathcal{H}$ there are necessarily vectors with negative $\langle,\rangle$ square. Thus, the indefiniteness of the metric in the space of the state vectors, which is clear from (50) in the case of the free field A_μ in the covariant gauge, necessarily also holds in the electrodynamics of interacting fields in any local gauge.

Let $\mathcal{H}''$ be the subspace of vectors of zero length of the space $\mathcal{H}'$. The space of physical states is identified with the factor space $\mathcal{H}'/\mathcal{H}''$ (or with its closure with respect to the topology of the scalar product $\langle,\rangle$).

D. Conclusions. Postulates of gauge theories

The exposition in the previous parts of this section was to some extent inductive, namely, we attempted to justify the choice of the basic requirements of gauge theory by using our experience in quantum electrodynamics.

It is appropriate to summarize the results of this discussion, listing the postulates of quantum electrodynamics in a local covariant gauge. We shall maintain a generality that permits inclusion in the treatment of non-Abelian gauge fields and theories with spontaneously broken symmetry.

The basic entity of the theory is the polynomial algebra FP of local fields $\varphi_k(f) = \int \varphi_k(x) f(x) d^4x$ and its representation by unbounded operators in the Hilbert space $\mathcal{H}$ with indefinite form $\langle,\rangle$, which is invariant with respect to the Hilbert topology. The basic fields φ_k include a certain system of Hermitian gauge fields $A_\mu^a(x)$, spinor fields $\psi(x)$ and their conjugates, and scalar fields, which we shall call Higgs fields, although fields of the type $B(x)$, in terms of which the class of covariant gauges is fixed, are also included here. Among the fields that generate the algebra FP, there are also composite local fields. These include the conserved currents $j_a^\mu(x)$ and the components of the energy-momentum tensor $T_{\mu\nu}(x)$, and, in the non-Abelian case, the field intensities $G_{\mu\nu}^a(x)$ as well [see (45)]. The space of test functions $J(\mathbf{R}^4)$ contains infinitely smooth functions of compact support in $\mathcal{D}(\mathbf{R}^4)$ and is contained in the Schwartz space $\mathcal{S}(\mathbf{R}^4)$ of rapidly decreasing smooth functions.

We assume that the Hilbert space $\mathcal{H}$ with nondegenerate Hermitian form $\langle,\rangle$ and the operator-valued generalized functions $\varphi_k(f)$ satisfy the following postulates.

1. The indefinite form $\langle,\rangle$ has the form (59), where β is a bounded self-adjoint operator in $\mathcal{H}$ with a bounded inverse.

2. In $\mathcal{H}$ there is realized a representation $U(a, \underset{\sim}{\Lambda})$ of the universal covering $\mathcal{P}_Q = \mathrm{ISL}(2, \mathbf{C})$ of the Poincaré group ("the quantum Poincaré group"), this representation conserving the indefinite form

$$\langle U(a, \underset{\sim}{\Lambda})\Phi, U(a, \underset{\sim}{\Lambda})\Psi\rangle = \langle\Phi, \Psi\rangle \tag{84}$$

for all $\Phi, \Psi \in \mathcal{H}$. For fixed $(a, \underset{\sim}{\Lambda})$, the operator $U(a, \underset{\sim}{\Lambda})$ is bounded with respect to the Hilbert topology. For any choice of Φ and Ψ, the function $\langle\Phi, U(a, \underset{\sim}{\Lambda})\Psi\rangle$ is continuous with respect to the parameters of the group.

3. In $\mathcal{H}$ there exists a (possibly, nonunique) Poincaré-invariant vacuum vector $|0\rangle$ which satisfies

$$\langle 0 \mid 0\rangle = 1. \tag{85}$$

4. The operator-valued generalized functions $\varphi_k(f)$, where f ranges over a space of test functions,[4)] have a com-

[4)]Experience with two-dimensional models suggests that in a situation with confinement it is not in general possible to use test functions of the Schwartz space $\mathcal{S}$. In this case, one can evidently use the space $J(\mathbf{R}^4)$ of Fourier transforms of functions of the Jaffe class.[38] Note that $\mathcal{D}(\mathbf{R}^4) \subset J(\mathbf{R}^4) \subset \mathcal{S}(\mathbf{R}^4)$. In quantum electrodynamics and in models of Salam-Weinberg type one can get by with the space of rapidly decreasing test functions $\mathcal{S}(\mathbf{R}^4)$.

mon invariant domain of definition $D \subset \mathscr{H}$, which includes the vacuum vector and satisfies

$$U(a, \underset{\sim}{\Lambda}) D \subset D. \tag{86}$$

They are covariant with respect to the representation U of the group $\mathscr{P}_Q$ in $\mathscr{H}$. In particular, the transformation law of the current j^μ is

$$U(a, \underset{\sim}{\Lambda}) j^\mu(x) U(a, \underset{\sim}{\Lambda})^{-1} = (\Lambda^{-1})^\mu_\nu j^\nu(\Lambda x + a). \tag{87}$$

5. The usual spectral condition holds:

$$\operatorname{supp} \int \langle \Phi \mid U(a, 1) \Psi \rangle e^{ipa} d^4 a \subset \bar{V}_+ = \{p \in \mathbb{R}^4 : p^0 \geqslant |\mathbf{p}|\} \tag{88}$$

(where the existence of a Fourier transform is assumed in the sense of the theory of generalized functions[5])

6. There is the locality property of the fields with the normal connection between the spin and statistics.

7. The vacuum is cyclic with respect to the polynomial algebra of basic fields with compact supports.

In other words, the linear hull $D_0 = FP_0|0\rangle$ of vectors of the form (88), where φ_k are basic fields, is dense in $\mathscr{H}$ with respect to the Hilbert topology.

8. There is a compact gauge Lie group G that acts by automorphisms of the algebra FP and leaves the energy-momentum tensor invariant. There exists a set of conserved currents j^μ_a and corresponding tensors $G^{\mu\nu}_a$ ($a = 1, \ldots,$ dim G) of the gauge fields, which transform in accordance with the adjoint representation of G. The smeared charges

$$Q_{aR} = \int j^0_a(x) f_R(x) d^4 x, \tag{89}$$

where $f_R(x)$ is a sequence of test functions, generate the infinitesimal gauge transformations

$$\frac{\partial}{\partial \chi_a} U(\chi) \psi(x) \Big|_{\chi=0} = i g t_a \psi(x), \tag{90}$$

where t_a are Hermitian matrices satisfying the commutation relations

$$[t_a, t_b] = i f_{abc} t_c \tag{91}$$

of the Lie algebra of G. In other words,

$$\lim_{R\to\infty} [Q_{aR}, \psi(x)] = -g t_a \psi(x); \tag{92}$$

$$\lim_{R\to\infty} [Q_{aR}, j^\mu_b(x)] = i f_{abc} j^\mu_c(x). \tag{93}$$

[In the special case of quantum electrodynamics, the group G is the $U(1)$ matrix, t_a reduces to multiplication by 1, $f_{\rm abc} = 0$, and we use the notation e and $F^{\mu\nu}$ for the charge g and the tensor $G^{\mu\nu}$.]

9. The space $\mathscr{D}(\subset\mathscr{H})$ contains a subspace $D' \subset D$ (whose Hilbert closure is denoted by $\mathscr{H}'$) that satisfies conditions a–d in Sec. 2.C [at the same time, the conditions (74) and (75) remain valid only in quantum electrodynamics (see the requirement 10); in the case of non-Abelian gauge theory, j^μ and $F^{\mu\nu}$ must also be replaced in Eq. (76) by j^μ_a and $G^{\mu\nu}_a$]. In particular, in accordance with (76)

$$\langle \Psi | \mathfrak{A}^\mu_a(x) | \Phi \rangle = 0 \quad \text{for all} \quad \Phi \in D', \ \Psi \in \mathscr{H}' \tag{93a}$$

and

$$\mathfrak{A}^\mu_a(x) = j^\mu_a - \partial_\nu G^{\mu\nu}_a(x). \tag{93b}$$

(As was noted at the end of Sec. 1, this requirement is a reflection of the local gauge invariance of the theory.)

10. In the case of quantum electrodynamics [or, more generally, if G contains a central subgroup $U(1)$] the Maxwell tensor satisfies a linear equation [a "Bianchi identity," a necessary condition of representability in the form (10)]:

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \ \ [\text{ or } \ d(F_{\mu\nu} dx^\mu \Lambda dx^\nu) = 0].$$

In a covariant gauge in which Eqs. (13) and (14) hold, the subspace D' is determined by the linear condition

$$\mathfrak{A}^{(-)}_\mu(x) \Phi = 0, \tag{94}$$

where $\mathfrak{A}^{(-)}_\mu$ is the negative-frequency part of the (generalized) free field $\mathfrak{A}_\mu(x)$. In addition, there exists a nonlocal Poincaré-covariant field $\psi(x, A)$ that leaves D' invariant [see (75)] and satisfies

$$[\mathfrak{A}^\mu(x), \psi(y, A)] = [j^\mu(x) - \partial_\nu F^{\mu\nu}(x), \psi(y, A)] = 0; \tag{95}$$

$$\lim_{R\to\infty} [Q_R, \psi(x, A)] = -e\psi(x, A). \tag{96}$$

These postulates make it possible to derive the usual properties of the Wightman functions (see Ref. 6), except for the positive definiteness and the cluster-decomposition property. Conversely, from a set of Wightman functions with these properties for which the order of the generalized function

$$\int w_n(x_1, \ldots, x_j, x_{j+1}, \ldots, x_n) \times g(x_{j+1}, \ldots, x_n) d^4 x_{j+1} \ldots d^4 x_n$$

with respect to the variables $x_1, \ldots, x_j$ does not depend on n one can recover a Hilbert space $\mathscr{H}$ with bounded indefinite form, a pseudounitary representation U of the Poincaré group $\mathscr{P}_Q$, and field operators such that requirements 1–10 are satisfied, with the possible exception of the boundedness of the operators β^{-1} and $U(a, \underset{\sim}{A})$ (see Ref. 50).

Postulates 5, 8, and 9 ensure the existence of a nontrivial (closed) subspace $\mathscr{H}''$ of the space $\mathscr{H}'$ in which the Poincaré-invariant form $\langle , \rangle$ vanishes identically. This form generates a positive scalar product on the factor space $\mathscr{H}'/\mathscr{H}''$.

Our last requirement, which does not impose any new restrictions on the structure of the theory, gives its physical interpretation.

11. *The space of physical states* $\mathscr{H}_{\rm PHYS}$ is the completion with respect to the norm $\|\Phi\| = \sqrt{\langle \Phi, \Phi \rangle}$ of the factor space $\mathscr{H}'/\mathscr{H}''$. *Physical operations* are operators in $\mathscr{H}$ that together with their conjugates with respect to the indefinite form $\langle , \rangle$ leave $\mathscr{H}'$ (or D') invariant. (Then, as a consequence, they also leave the subspace $\mathscr{H}'$ invariant and, therefore, generate operators on the factor space $\mathscr{H}_{\rm PHYS}$.) Self-adjoint (with respect to $\langle , \rangle$) physical operations that are invariant with respect to global gauge transformations are called *observables*. The transition probabilities and expectation values of observables are defined in terms of the Poincaré-invar-

[5]For this purpose, it is sufficient for the Hilbert norm $\|U(al)\|$ to be a continuous polynomially bounded function of a.

iant scalar product in $\mathscr{H}_{\rm PHYS}$.

Concluding this section, it is appropriate to comment on the common assertion to the effect that there is spontaneous breaking of the Lorentz invariance in sectors with nonzero charge in quantum electrodynamics[6)] (Refs. 9, 28, 29, 31, and 55).

The contradiction between this assertion and our postulate 2 is eliminated if it is possible (in the framework of the approach presented here) to construct a model with a smaller Lorentz-noninvariant space of charged states. We shall explain this possibility in the language of scattering theory.

In the framework of perturbation theory,[7)] there is constructed in Ref. 15, the paper considered here, a Hilbert space of physical asymptotic states $\mathscr{H}^{\rm ex}$ and a scattering operator $S(\mu)$, where μ is the parameter (with the dimensions of mass) introduced by Eq. (70), and: 1) in $\mathscr{H}^{\rm ex}$ there is realized a unitary representation of the Poincaré group that commutes with the charge operator and leaves $S(\mu)$ invariant; 2) the cross sections of processes of scattering of charged particles with the participation of a finite number of photons are finite and depend on μ; 3) the sum of the integrals of these cross sections with an increasing number of soft photons (with total energy not exceeding the given resolution ΔE) does not depend on μ (but does depend on ΔE in accordance with the Bloch-Nordsieck theorem). The theory of Fröhlich *et al.*[28,29] and Buchholz[9] would correspond in this language to some space of coherent asymptotic states $\mathscr{H}^{\rm ex}_{\Delta/E} \subset \mathscr{H}^{\rm ex}$ (in which there would not be states, say, with one electron and a finite number of photons) and to an operator S such that the above-mentioned sums of integrals of the cross sections would be reproduced as the squares of the moduli of the matrix elements of the S operator between suitable coherent states. In the charged sector of the space $\mathscr{H}^{\rm ex}_{\Delta E}$, Lorentz invariance would be broken by construction.

However, we do not rule out the possibility that in a subspace with fixed charge of the physical Hilbert space the algebra of local observables is reducible.

3. SPONTANEOUS SYMMETRY BREAKING IN THE AXIOMATIC APPROACH

A. Infinitesimal characteristic of spontaneously broken symmetry. Generalized Goldstone theorem

To define the concept of "spontaneous symmetry breaking," it is necessary to distinguish algebraic symmetries from symmetries determined by unitary operators in the space of state vectors β.

By an *algebraic symmetry* we shall understand above all an automorphism[8)] $\varphi \to \tau(\varphi)$ of the polynomial algebra FP [or the corresponding algebra of bounded operators $\mathscr{F} = (FP)''$]. If the fields that generate the algebra FP are related by a system of equations, we shall assume that the symmetries preserve these equations. In particular, in the case of gauge theory, with the postulates formulated in the previous section, we assume that the symmetries τ leave the weak equations (93) and the current conservation law invariant. The symmetry τ is said to *exact* if there exists a pseudounitary operator U (or, more generally, an operator U satisfying $|\langle U\Phi, U\Psi\rangle| = |\langle \Phi, \Psi\rangle|$) such that

$$\tau(\varphi) = U\varphi U^{-1} \text{ and } U\,|\,0\rangle = |\,0\rangle. \tag{97}$$

Otherwise, the symmetry is said to be *spontaneously broken*. A one-parameter group $\tau_\alpha(\varphi)$ of algebraic symmetries [with an additive parameter α satisfying $\tau_0(\varphi) = \varphi$] is called a *gauge group* if:

1) $\tau_\alpha[FP(\mathscr{O})] = FP(\mathscr{O})$;

2) $\tau_\alpha(\varphi)^* = \tau_\alpha(\varphi^*)$ (where the asterisk denotes conjugation with respect to the indefinite scalar product $\langle,\rangle$);

3) there exists a conserved current $j^\mu(x)$ and a skew-symmetric (local) tensor field $F^{\mu\nu}(x)$ that satisfy (76) and are such that the smeared charge Q_R (64), (66) is the infinitesimal generator of the group τ_α:

$$i\,\frac{d}{d\alpha}\tau_\alpha(\varphi) = \lim_{R\to\infty}[Q_R,\ \tau_\alpha(\varphi)]. \tag{98}$$

(We use the notation of Abelian gauge theory, though these discussions are also suitable for a non-Abelian theory. In the general case, one can take it that j^μ and $F^{\mu\nu}$ are an abbreviated notation for j^μ_a and $G^{\mu\nu}_a$ for some a.)

Proposition 3.1. For spontaneous breaking of a gauge symmetry with generator Q_R, it is necessary and sufficient that there exists a smeared polynomial of the fields with compact support $\varphi \in FP_0$ and such that

$$c \equiv \lim_{R\to\infty}\langle[Q_R,\ \varphi]\rangle_0 \neq 0. \tag{99}$$

Remark. The existence of the limit on the left-hand side of (99) has already been proved [see (65)].

Proof. The *sufficiency* of the condition (99) is obvious: If the symmetry were exact, then by virtue of (97) the vacuum expectation value

$$\langle\tau_\alpha(\varphi)\rangle_0 = \langle U_\alpha\varphi U_\alpha^{-1}\rangle_0 = \langle\varphi\rangle_0 \tag{100}$$

would not depend on α, so that in accordance with (98) the limit on the left-hand side of (99) would be zero.

We shall prove the *necessity* of the condition (99) by

6)Without going into a detailed discussion of the arguments of these papers, we note that Gupta's criticism[32] of the equation $\partial_0\partial A = \partial_0\partial(A^{\rm in})$ applies to the interaction representation and evidently does not apply to Eq. (2.14) of Ref. 28.

7)Note that in constructive quantum field theory asymptotic completeness has not been proved even for two-dimensional models.

8)We recall that, by definition, the automorphism τ conserves the algebraic operations:

$$\tau(\lambda_1\varphi_1 + \lambda_2\varphi_2) = \lambda_1\tau(\varphi_1) + \lambda_2\tau(\varphi_2),$$
$$\tau(\varphi_1\varphi_2) = \tau(\varphi_1)\,\tau(\varphi_2),\quad (\lambda_k \in \mathbb{C}).$$

Note that discrete symmetry transformations, including time reversal, are antiautomorphisms of the considered algebra [for which $\tau(\lambda\varphi) = \bar\lambda\tau(\varphi)$, $\tau(\varphi_1\varphi_2) = \tau(\varphi_2)\tau(\varphi_1)$]. The continuous (connected) symmetry groups with which we shall be dealing in this section are given by continuous groups of automorphisms.

assuming the contrary. Suppose

$$\lim_{R\to\infty} \langle [Q_R,\ \varphi]\rangle_0 = 0 \quad \text{for all} \quad \varphi \in FP_0. \tag{101}$$

Then there exists a symmetric (with respect to the form $\langle,\rangle$) charge operator Q and a pseudounitary operator U_α defined by the equations

$$Q\varphi|0\rangle = \lim_{R\to\infty} [Q_R,\ \varphi]|0\rangle, \tag{102}$$

$$U_\alpha \varphi|0\rangle = \tau_\alpha(\varphi)|0\rangle \quad (\varphi \in FP_0). \tag{103}$$

We show first of all that this definition is correct, i.e., that from $\varphi_1|0\rangle = \varphi_2|0\rangle$ there follows $Q_R\varphi_1|0\rangle = Q_R\varphi_2|0\rangle$ and $U_\alpha\varphi_1|0\rangle = U_\alpha\varphi_2|0\rangle$. Indeed, for any φ in FP_0

$$\langle \varphi Q(\varphi_1 - \varphi_2)\rangle_0 = \lim_{R\to\infty} \langle [Q_R,\ \varphi(\varphi_1-\varphi_2)] - [Q_R,\ \varphi](\varphi_1-\varphi_2)\rangle_0 = 0$$

and the correctness of the definition (102) follows from the cyclicity requirement 7. However, from this and from (98) it follows that

$$\frac{d}{d\alpha}\langle \varphi U_\alpha(\varphi_1-\varphi_2)\rangle_0 = 0$$

and, therefore, $U_\alpha(\varphi_1 - \varphi_2)|0\rangle = (\varphi_1-\varphi_2)|0\rangle = 0$.

The pseudounitarity of the operator U_α follows from the equivalence of (101) and (100) and from the chain of equations

$$\langle \varphi_1^* U_\alpha^* U_\alpha \varphi_2\rangle = \langle \tau_\alpha(\varphi_1)^* \tau_\alpha(\varphi_2)\rangle_0 = \langle \tau_\alpha(\varphi_1^*)\tau_\alpha(\varphi_2)\rangle_0 = \langle \tau_\alpha(\varphi_1^*\varphi_2)\rangle_0 = \langle \varphi_1^*\varphi_2\rangle_0.$$

The condition of symmetry for the operator Q,

$$\langle \Phi_1,\ Q\Phi_2\rangle = \langle Q\Phi_1,\ \Phi_2\rangle \quad \text{for} \quad \Phi_k = \varphi_k\,|\,0\rangle,\ k = 1,\ 2,$$

is obtained from here by differentiation with respect to α, Thus, denying the condition (99) leads to exact gauge symmetry, i.e., this condition is indeed necessary for its spontaneous breaking.

In the analysis of spontaneous symmetry breaking in local gauge field theory with indefinite metric (i.e., in a theory in which the postulates 1–10 of Sec. 2.*D* are satisfied) a central part is played by the following generalization of Goldstone's theorem (see Ref. 49).

Proposition 3.2. Suppose that in a local quantum field theory there exists a φ in FP_0 for which the condition (99) of spontaneous breaking of a continuous symmetry is satisfied. Then the Fourier transform of the matrix element $\langle [j^0(x),\ \varphi]\rangle_0$ contains a singularity of the type $\delta(p^2)$.

Remark. We assume conservation of the current j^μ, through whose zeroth component the regularized charge Q_R (64) is defined, and admit indefiniteness of the Poincaré-invariant metric in the space of the state vectors; however, we do not require fulfillment of the weak Maxwell equation (76), which is characteristic of local gauge theory.

Scheme of the proof. Under our assumptions, the Jost-Lehmann-Dyson representation (see, for example Refs. 70, 71, and 74) holds:

$$\langle [j^0(x),\ \varphi]\rangle_0 = \int_0^\infty dm^2 \int d^3y\, [\rho_1(m^2,\ \mathbf{y})\, D_m(x^0,\ \mathbf{x}-\mathbf{y}) + \rho_2(m^2,\ \mathbf{y})\,\partial_0 D_m(x^0,\ \mathbf{x}-\mathbf{y})], \tag{104}$$

where D_m is the commutation function for a free scalar field of mass m,

$$D_m(x) = 2\pi i \int \varepsilon(p^0)\,\delta(m^2+p^2)\,e^{ipx} d_4 p, \tag{105}$$

and $\rho_k(m^2, \mathbf{y})$ $(k = 1, 2)$ are generalized functions of compact support with respect to the 3-vector $\mathbf{y}$. Following Swieca,[64] we can decompose each such generalized function of compact support into a sum

$$\rho_k(m^2,\ \mathbf{y}) = \rho_k(m^2)\,\delta(\mathbf{y}) + \nabla\sigma_k(m^2,\ \mathbf{y}), \tag{106a}$$

where $\boldsymbol{\sigma}_k$ is also a vector function of finite support, and

$$\rho_k(m^2) = \int \rho_k(m^2,\ \mathbf{y})\, d^3\mathbf{y}. \tag{106b}$$

We shall show that

$$\rho_1(m^2) = 0,\quad \rho_2(m^2) = c\,\delta(m^2), \tag{107}$$

where $c\ (\neq 0)$ is the constant (99).

We first of all free ourselves from the terms $\nabla\sigma_k$ by multiplying both sides of (104) by $e_R(\mathbf{x})$ [see (66b)] and integrating over $\mathbf{x}$; repeated integration by parts (with respect to both $\mathbf{y}$ and $\mathbf{x}$) gives

$$\int e_R(\mathbf{x})\,\langle [j^0(x),\ \varphi]\rangle_0\, d^3x$$

$$= \int_0^\infty dm^2 \int d^3x \Big\{ [\rho_1(m^2)\, D_m(x) + \rho_2(m^2)\,\partial_0 D_m(x)]\, e_R(\mathbf{x}) - \int d_3y\, [\sigma_1(m^2,\ \mathbf{y})\, D_m(x^0,\ \mathbf{x}-\mathbf{y}) + \sigma_2(m^2,\ \mathbf{y})\,\partial_0 D_m(x^0,\ \mathbf{x}-\mathbf{y})]\,\nabla e_R(\mathbf{x})\Big\} \equiv F_R(x^0).$$

For sufficiently large R, the gradient $\nabla e_R(\mathbf{x})$ vanishes on the support of the integral with respect to $\mathbf{y}$, so that

$$F(t) = \lim_{R\to\infty} F_R(t) = \int_0^\infty dm^2 \Big[\rho_1(m^2)\,\frac{\sin(mt)}{m} + \rho_2(m^2)\cos(mt)\Big]. \tag{108}$$

On the other hand, the current conservation law guarantees that $\mathbf{F}(t)$ does not depend on t. This is possible only for ρ_1 and ρ_2 specified by Eq. (107), and that is what we wanted to prove.

Corollary, If the Poincaré-invariant metric $\langle,\rangle$ is positive and the space of physical states is the complete Hilbert space $\mathscr{H}$, then from the proposition there follows the standard Goldstone theorem, according to which a (physical) particle of zero mass exists in a theory with a spontaneously broken continuous symmetry.

In the following subsection, we shall consider how the singularity of the type $\delta(p^2)$, whose existence is established by the generalized Goldstone theorem, affects the physical mass spectrum in a gauge theory.

B. The Higgs phenomenon. The photon as a Goldstone boson

Proposition 3.3. Suppose that under the assumptions of Proposition 3.2 the current j^μ satisfies postulate 9 of local gauge invariance [including the weak form (76) of the Maxwell equation]. Then the singularity $\delta(p^2)$ whose existence is proved by the previous proposition does not contribute to the matrix elements between vectors in D' and thus does not correspond to a physical particle with vanishing mass.

Proof. The field $\mathfrak{A}^{\mu}(x)$ (81) has all the properties of a conserved current. Moreover, by Proposition 2.1 $\mathfrak{A}^0(f_R)$ has the same commutation properties (as $R\to\infty$) with the elements of the algebra FP_0 as Q_R. Thus, to it we can apply Proposition 3.2, from which there follows the existence of a vector Ψ in D_0 such that the Fourier transform of the matrix element $\langle 0|\mathfrak{A}^0(x)|\Psi\rangle$ has a singularity of the type $\delta(p^2)$. By definition, vectors Ψ with this property contain a contribution from a state with a massless (Goldstone) particle. If it is assumed that among them there is a vector in D' (or $\mathscr{H}'$), we arrive at a contradiction with the assumption 9 that the matrix elements of $\mathfrak{A}^{\mu}$ between the physical states are zero.

Hitherto, we have studied the spontaneous breaking of global gauge invariance [although in Proposition 3.3 we have assumed the validity of Eq. (76), which reflects the local gauge invariance]. However, spontaneous breaking of local gauge invariance is a feature of the standard formalism of the quantum theory of gauge fields (although this is seldom noted). We give here a simple criterion for such breaking[23,37,58] and investigate its effect on the mass spectrum of the physical particles.

Proposition 3.4. If in the quantum theory of a charged field $\varphi(x)$ the two-point function $\langle\varphi(x)\varphi^*(y)\rangle_0$ is nonzero, then the local Abelian gauge symmetry $\varphi(x)\to e^{ie\lambda(x)}\varphi(x)$ is spontaneously broken.

Proof. If there existed a (pseudo)unitary operator $U(\lambda)$ such that

$$U(\lambda)\,\varphi(x)\,U(\lambda)^{-1}=e^{ie\lambda(x)}\,\varphi(X),$$
$$U(\lambda)\,|\,0\rangle\,\langle 0\,|\,U^{-1}(\lambda)=|\,0\rangle\,\langle 0\,|,$$

then all the vacuum expectation values of the fields φ would have to be invariant with respect to gauge transformations of the second kind; in particular, we should have

$$\langle\varphi(x)\,\varphi^*(y)\rangle_0=\langle\varphi(x)\,\varphi^*(y)\rangle_0\exp\{ie\,[\lambda(x)-\lambda(y)]\}.$$

Because $\lambda(x)$ is arbitrary, this is possible only when

$$\langle\varphi(x)\,\varphi^*(y)\rangle_0=c\delta(x-y).$$

The spectral condition leads to $c=0$, and this contradicts the assumption of the proposition.

Thus, quantum electrodynamics is an example of a spontaneously broken local gauge symmetry. In this case, the following generalization of Proposition 4.1 holds.

Proposition 3.5. Let $J^{\mu}[x;\lambda(x)]$ be a translationally noninvariant Noether current [determined by an expression of the type (20)] that generates local gauge transformations. If the two-point function is nonvanishing, $\langle\varphi(x)\varphi^*(y)\rangle_0\neq 0$ and $f(x,y)$ is a (nonzero) test function of compact support, then the Fourier transform of the matrix element

$$\langle [J^{\mu}(x;\ \lambda(x)),\ \iint\varphi(y)\,\varphi^*(z)\,f(y,\ z)\,d^4y\ d^4z]\rangle_0 \quad (109)$$

has a singularity of the type $\delta^{(k)}(p^2)$ for some $k=0,1,\ldots$ (see Refs. 22 and 58).

Only a term containing a singularity of the type $\delta(p^2)$ can in principle make a contribution in the calculation of the matrix elements between the vectors of the subspace $D'\subset D$ [defined by the conditions a–d of Sec. 2] and, thus, between the vectors of the physical Hilbert space $\mathscr{H}_{\rm PHYS}$ [defined by requirement 11 of Sec. 2.D]. It can be shown that the presence of a singularity of the type $\delta(p^2)$ in $\mathscr{H}_{\rm PHYS}$ (i.e., the existence of a physical Goldstone particle) is controlled by the behavior of the theory under global gauge transformations. The following theorem of Swieca[63] holds.

Proposition 3.6. Suppose that in an Abelian gauge theory the postulates of Sec. 2.D and the condition (101) (which ensures the existence of the charge Q) are satisfied. Suppose further that there exist (generalized) physical asymptotic states $|p\rangle(=|p;\ m,\ e\rangle)$ with momentum p, mass m, and charge $e\neq 0$, normalized by the condition

$$\langle p|q\rangle=(2\pi)^3\,(p^0+q^0)\,\delta(\mathbf{p}-\mathbf{q}),\quad p^{(0)}=\sqrt{m^2+\mathbf{p}^2},$$
$$q^0=\sqrt{m^2+\mathbf{q}^2}. \quad (110)$$

Then the operator of the square of the mass has a continuous spectrum with lower bound m^2.

Scheme of the proof. It follows from the Poincaré covariance of the theory that the matrix element of $F_{\mu\nu}$ between the states $|p\rangle$ and $|q\rangle$ has the form

$$\langle p|F_{\mu\nu}(x)|q\rangle=2i\,(q_\mu p_\nu-q_\nu p_\mu)\,\rho(t)\,e^{i(q-p)x}, \quad (111a)$$

where

$$t=-(p-q)^2=2\,(pq+m^2). \quad (111b)$$

Hence, using the weak form of the Maxwell equation (76), we find

$$\langle p|j^{\mu}(x)|q\rangle=(p^{\mu}+q^{\mu})\,t\rho(t)\,e^{i(q-p)x}. \quad (112)$$

We assume that $|p\rangle$ and $|q\rangle$ are states with charge e and together with the normalization condition (110) give

$$\langle p\mid Q\mid q\rangle=(p^0+q^0)\,(2\pi)^3\delta\,(\mathbf{p}-\mathbf{q})\;t\rho\,(t),$$

i.e.,

$$\lim_{t\to 0}t\rho\,(t)=e\neq 0. \quad (113)$$

Thus, the "form factor" $\rho(t)$ behaves as $1/t$ as $t\to 0$. However, the analysis in §2 of Ref. 10 shows that if there were a gap between the single-particle hyperboloid $-p^2=m^2$ and the beginning of the continuum in the singly charged sector, the function $\rho(t)$ would be smooth at $t=0$. Thus, a mass gap indeed does not exist in the singly charged sector.

Remark. This result remains valid in space–time of any dimension not lower than three. In the two-dimensional case it is not valid, since then there is still an invariant skew-symmetric tensor $\varepsilon_{\mu\nu}$, which can contribute to the right-hand side of (111a). We note also the important part played by the Maxwell equation (76) in the validity of the result. If (76) does not hold, nor does Proposition 3.6. Indeed, if we add to the Lagrangian of spinor electrodynamics a term of the type

$$-\frac{1}{2}\,m^2\,(A_\mu-\partial_\mu\chi)\,(A^\mu-\partial^\mu\chi),$$

where $\chi(x)$ is a scalar gauge field that transforms as $\chi(x)\to\chi(x)+\lambda(x)$ under the transformations (97), then the "photon" acquires a mass in the physical space. However, in this case Eq. (76) is replaced by

TABLE I. Table of quantum numbers of the basic fields in the Salam-Weinberg model.

Field	t	t_3	y	$Q=t_3+\frac{y}{2}$	Spin	Defining relations
$\psi_{eL}=\begin{pmatrix}\psi_{e1}\\ \psi_{e2}\end{pmatrix}$, $\psi_{\mu L}=\begin{pmatrix}\psi_{\mu 1}\\ \psi_{\mu 2}\end{pmatrix}$	1/2	$\begin{pmatrix}1/2\\ -1/2\end{pmatrix}$	-1	$\begin{pmatrix}0\\ -1\end{pmatrix}$	1/2	$\psi_L=\frac{1}{2}(1+\gamma_5)\psi_L$
ψ_{eR}, $\psi_{\mu R}$	0	0	-2	-1	1/2	$\psi_R=\frac{1}{2}(1-\gamma_5)\psi_R$
$\varphi=\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$ (Higgs field)	1/2	$\begin{pmatrix}1/2\\ -1/2\end{pmatrix}$	1	$\begin{pmatrix}1\\ 0\end{pmatrix}$	0	—
$\mathbf{W}_\mu=\begin{pmatrix}W_\mu^+\\ W_\mu^0\\ W_\mu^-\end{pmatrix}$, $\mathbf{G}_{\mu\nu}$	1	$\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}$	0	$\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}$	1	$G^i_{\mu\nu}=\partial_\mu W^i_\nu-\partial_\nu W^i_\mu+g\varepsilon_{ijk}W^j_\mu W^k_\nu$
B_μ, $B_{\mu\nu}=\partial_\mu B_\nu-\partial_\nu B_\mu$	0	0	0	0	1	—

$$\langle\Psi\,|\,j^\mu(x)-\partial_\nu F^{\mu\nu}(x)-m^2(A^\mu-\partial^\mu\chi)\,|\,\Phi\rangle=0$$

for $\Psi\in\mathscr{H}'$, $\Phi\in D'$, i.e., the current will not be the divergence of an antisymmetric tensor even in the sense of expectation values with respect to physical states.

The results of Propositions 3.3–3.6 can be summarized as follows. In the space with indefinite metric $D\,(\subset\mathscr{H})$ in a local Abelian gauge theory there is always a singularity of the type $\delta^{(k)}(p^2)$ due to the spontaneously broken local gauge invariance. If at the same time the corresponding global gauge symmetry is not broken, then there is a physical particle with zero mass. But if the global gauge symmetry is spontaneously broken, then in both Abelian and non-Abelian gauge theory a singularity of the type $\delta(p^2)$ appears only in the calculation of the matrix elements between the states of the unphysical part of the Hilbert space $\mathscr{H}$.

These results are very remarkable and can be regarded as a weighty qualitative argument in support of gauge theories. Indeed, the unique zero-mass boson observed in nature—the photon—can be regarded as a Goldstone boson corresponding to spontaneously broken local gauge invariance in quantum electrodynamics in which there is exact global $U(1)$ symmetry (responsible for conservation of the electric charge). (This fact was already noted by Ferrari and Picasso[23] in 1977; see also the more general treatment in Ref. 37.) We recall that the standard Goldstone theorem, which holds in local quantum field theory in a space with positive metric, predicts the existence of a scalar massless particle accompanying spontaneous symmetry breaking, but such particles are not observed in nature.

C. Gauge-invariant composite fields corresponding to physical particles in the Salam-Weinberg model

It is generally assumed that the theory of the electroweak interactions gives an example of spontaneous breaking of a global gauge symmetry. In fact, this interpretation is open to serious objections (see Refs. 12, 18, 30, and 43). There is a more attractive (although less developed and less popular) point of view, according to which the global symmetry remains exact and at the same time the physical results of the Salam-Weinberg theory such as the spontaneous generation of the mass of the vector bosons of the weak interaction remain valid. This is achieved by the fact that the particles are associated, not with the original basic fields of the theory, but with suitable gauge-invariant composite fields.

To present this point of view, we must ignore the quantum numbers of the basic fields in the Salam-Weinberg model (see, for example, Chap. 8 of Ref. 56).

The electromagnetic potential A_μ and the field of the neutral vector boson Z_μ are given by the linear combinations

$$A_\mu=\cos\theta B_\mu+\sin\theta W^0_\mu,\quad Z_\mu=-\sin\theta B_\mu+\cos\theta W^0_\mu,\qquad(114)$$

where θ ($=\theta_W$ is the angle of the weak interactions ("Weinberg angle").

Suppose the classical Higgs potential V_H is given by the expression (35). Then we assume that the vacuum expectation value of the gauge-invariant composite field[9)] $\varphi^*\varphi$ is equal to the value (36) at which V_H attains a minimum:

$$\langle\varphi^*(x)\,\varphi(x)\rangle_0=\frac{1}{2\lambda^2}\,m_H^2.\qquad(115)$$

We require that the physical particles correspond to the following gauge-invariant composite fields:

left lepton ($l=e^-$ or μ^-):

$$C_l\,(\varphi^*\psi_{lL})\,(x,\,A),\qquad(116a)$$

where the dependence on A_μ of the product $\varphi^*\psi_{lL}$ is determined (classically) by analogy with (28);

l-neutrino ν_l:

$$C_\nu\det[\varphi(x)\otimes\psi_{lL}(x)]=C_\nu[\varphi_1(x)\,\psi_{l2}(x)-\varphi_2(x)\,\psi_{l1}]_L;\qquad(116b)$$

[9)]Here and in what follows, we do not attempt to give a precise meaning to the product of operator-valued generalized functions at one point but regard them as new fields (in the class of Borchers's fields $\psi(*)$, $\varphi(*)$, W_μ, and B_μ) with transformation properties suggested by the properties of the corresponding classical fields.

neutral scalar Higgs meson:

$$C_H\varphi^*(x)\,\varphi(x); \quad (117)$$

$W_\pm$ mesons:

$$W_+ = C_W(\varphi^*\tau\varepsilon\varphi^*G_{\mu\nu})(x, A), \ W_- = W_+^*; \quad (118a)$$

Z meson:

$$C_Z(\cos\theta\varphi^*\tau\varphi G_{\mu\nu} + \sin\theta\varphi^*\varphi B_{\mu\nu}); \quad (118b)$$

photon:

$$C_\gamma(-\sin\theta\varphi^*\tau\varphi G_{\mu\nu} + \cos\theta\varphi^*\varphi B_{\mu\nu}). \quad (118c)$$

We note that these expressions go over into the well-known expressions [see Ref. 56] if we replace everywhere φ by

$$\varphi_{cl}(\alpha) = \frac{1}{\lambda\sqrt{2}} m_H e^{i\alpha}\begin{pmatrix}0\\1\end{pmatrix}, \quad (119)$$

and in (118) also replace φ^* by the complex-conjugate quantity and set

$$C_l = \frac{\lambda\sqrt{2}}{m_H}e^{i\alpha}, \quad C_\nu = -\frac{\lambda\sqrt{2}}{m_H}e^{-i\alpha},$$

$$C_W = \frac{\lambda^2\sqrt{2}}{m_H^2}e^{2i\lambda}, \quad C_\gamma = -C_Z = \frac{2\lambda^2}{m_H^2}.$$

A small modification of the standard description of the Higgs mechanism, according to which it is not $\varphi(x)$ but the composite field $\varphi^*(x)\varphi(x)$ that is translated by the amount $m_H^2/2\lambda^2$, reproduces the well-known expressions for the masses of the $W_\pm$ and Z bosons:

$$m_{W_\pm} = \frac{em_H}{2\lambda\sin\theta} = m_Z\cos\theta, \quad (120)$$

where e is the charge of the positron ($e^2/4\pi = 1/137$). For the definition (116) of the physical fields of the leptons, the mechanism of mass generation of the electron and muon also works.

The situation outlined in the Salam-Weinberg model can be extended to more general theories with spontaneous symmetry breaking (for example, to grand-unification models). In the general case, we have the following proposition.

Proposition 3.7. Let $G_{\{\varphi_{cl}\}}$ be the little group of the orbit $\{\varphi_{cl}\}$ of the minima of the Higgs potential (i.e., the subgroup of G that leaves invariant some point of the orbit) and let T_F be an (irreducible) fermion representation of the gauge group G.

A. If the subgroup $G_{\{\varphi_{cl}\}}$ is trivial, then there exists a linear correspondence between the fields ψ that transform in accordance with the representation T_F and the space of G-invariant composite fields of the form $P(\varphi)\psi[=P_\alpha(\varphi)\psi^\alpha]$. This correspondence is determined up to invariant fields of the same form that vanish on the orbit $\{\varphi_{cl}\}$.

B. If the little group $G_{\{\varphi_{cl}\}}$ is nontrivial, then every $G_{\{\varphi_{cl}\}}$-invariant irreducible projection Π_ψ in the space of the representation T_F corresponds to a G-covariant operator $P(\varphi)$ that is polynomial in φ and φ^* and is identical to $\Pi_{\{\varphi_{cl}\}}$ on the orbit $\{\varphi_{cl}\}$.

See the proof of this proposition in Ref. 30.

This approach to the Higgs mechanism requires changes in standard perturbation theory. If one begins, as usual, with some fixed point φ_{cl} on the orbit of the minima of the potential V_H, then it is necessary to average the obtained result over the orbit $\{\varphi_{cl}\}$. It is easy to see that such a modification of the rules of calculation do not affect the form of the gauge-invariant matrix elements.

The arguments which indicate that the vacuum expectation value of the gauge-noninvariant Higgs field φ vanishes (in contrast to the current conviction) are nonperturbative in nature. They are based on study of Abelian gauge theory on a lattice[13] and on consideration of the contribution of a rarefied instanton gas in $SU(2)$ Yang-Mills theory[30] (see also Ref. 49).

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