Generalized coherent states in nuclear-physics problems

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The properties of generalized coherent states, which generate a basis of many-particle oscillator functions, are discussed. Depending on the choice of the generalized coherent states, it is possible to construct different oscillator basis functions convenient for describing different modes of nucleon systems—collective, internal, cluster, etc. It is shown that such states open up essentially new possibilities for investigating on a microscopic basis a large number of nuclear processes and nuclear properties such as collective excitations, scattering, nuclear reactions, and resonance states.

INTRODUCTION

One of the main problems in the microscopic theory of the nucleus is to separate the dominant correlations in the nuclear process under consideration and to take them into account correctly. In this direction of theoretical nuclear physics, much experience has been accumulated, and several methods have been proposed for separating the collective and internal degrees of freedom; approximations designed for the systematic solution of the Schrödinger equation of nuclear systems have been formulated. However, because of the absence of simple and reliable algorithms for taking into account the correlations included in these methods, many interesting ideas and approaches remain unrealized or are realized for a very restricted class of nuclear processes or nuclei (as a rule, the lightest).

On the other hand, the use of generalized coherent states (GCS)—also called irreducible coherent states, generating invariants, and generating functions—gives essentially new possibilities for taking into account the modes of nucleon motion that are dominant in a particular nuclear process and for solving the many-particle Schrödinger equation to a given accuracy.

A system of coherent states was used for the first time by Schrödinger in 1926 to describe nonspreading oscillator wave packets. Later, Glauber, studying the coherence of a beam of photons,[1] introduced the concept of coherent states. However, it was only in 1972 that Perelomov gave a clear definition of GCSs for different Lie groups.[2] This immediately stimulated the investigation of the properties of different physical systems by means of the GCS technique, since this was found to be very convenient for studying systems whose Hamiltonians can be represented in the form of a linear combination of the generators of the corresponding symmetry group.

However, in numerous problems, particularly in nuclear physics, there are many-particle Hamiltonians that do not reduce even to polynomials of finite degree in the generators of the corresponding dynamical symmetry group. Therefore, the use of GCSs in such problems proved to be ineffective.

Recently, however, a technique has been developed that makes it possible to extend significantly the applicability of GCSs and to use them to study systems whose investigation by means of the traditional algebraic methods had not given positive results.

In the present review, we present the basic ideas of the construction and the use of GCSs for different problems in nuclear physics.

In Sec. 1 we classify the problems of microscopic nuclear theory in accordance with the nature of the investigated nucleon correlations. We discuss the various forms of the Schrödinger equation (differential, integral, integro-differential, and algebraic) suitable for the investigation of correlation problems. We define generalized coherent states. In Sec. 2 we consider dynamical symmetry groups of nuclear systems. In Secs. 3–5 we present the ideas behind the construction of GCSs for the investigation of collective excitations, reactions, and resonances, respectively.

1. PROBLEMS IN MICROSCOPIC NUCLEAR THEORY. GENERALIZED COHERENT STATES

It is well known that when nuclear structure is investigated in the framework of microscopic approaches it is, as a rule, necessary to take into account correlations in the motion of the nucleons that arise as a result of the internucleon interaction. This is necessary even when the investigation is based on the concept of independent motion of nucleons in an average field of the nucleus. The clearest example of manifestation of nucleon correlations is the various effects associated with collective quadrupole motions of the nucleons, and also the effects due to clustering in nuclei.

For the theoretical study of nucleon correlations, it is customary to separate (explicitly or implicitly) from the complete set of $3A - 3$ independent variables $\{q\}$ of the nuclear wave function Ψ certain variables that directly reflect the dominant nucleon modes in the considered process, these distinguished variables being selected after a rearrangement of the variables. We denote these distinguished variables by $\{q_D\}$ and call them the dynamical variables. We denote all the remaining independent variables by $\{q_K\}$ and call them the kinematic variables. This division of the independent variables into two groups, $\{q\} = \{q_D, q_K\}$, is help-

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ful in clarifying the significance of the approximations employed in the theory. The number of dynamical variables (which, like q_K, are functions of the single-particle variables) depends on the nature of the investigated process, and also on the choice of the approximation used to describe it. After the dynamical and kinematic variables have been defined, the wave function of the many-particle system can be conveniently represented in the form

$$\Psi = A\{\sum_{\nu} \Phi_\nu(q_D)\,\chi_\nu(q_K)\}, \tag{1}$$

where $\hat{A}$ is the antisymmetrization operator. If the dynamical variables are symmetric with respect to permutation of the nucleon coordinates, the functions χ_ν will ensure the antisymmetry of the total wave function. The dependence of the wave function Ψ on q_K or, in other words, the form of the functions χ_ν is determined by the corresponding physical and kinematic conditions of the problem, for example, the Pauli principle. At the same time, the components $\Phi_\nu(q_D)$ of the many-particle function are uniquely determined by solving the dynamical equations which they satisfy.

Such a representation of the many-particle function Ψ is a generalization of the Born–Oppenheimer approximation[3] in the theory of molecules. In this case, the slow adiabatic motion of the nuclei corresponds to the modes of the nucleon systems that are due to the variation of the dynamical variables, while the rapid motion of the electrons of the molecule correspond to the modes associated with the kinematic variables.

Before we consider the construction of equations for the dynamical components $\Phi_\nu(q_D)$, we discuss the traditional problems of microscopic nuclear theory.

The problems solved by microscopic theory can be divided into three groups, depending on the nature of the correlations that play the dominant role in the processes. In the first group, we have problems involving the investigation of collective excitations of nuclei. It is obvious that in this case the dynamical variables are the collective variables, and the kinematic variables are the internal variables. The functions χ_ν will describe the internal state of the nucleus. Two methods have been proposed for correct separation of the internal and collective degrees of freedom—the method of hyperspherical functions (or the method of K harmonics)[5–8] and the method of generalized hyperspherical functions.[9–12] In the first method, the collective motion is associated with volume, monopole vibrations of the nucleus. The dynamical variable that reproduces the monopole motions is the hyperradius ρ:

$$\rho^2 = \sum_{i=1}^{A} \mathbf{r}_i^2 - \mathbf{R}_A^2,$$

where $\mathbf{r}_i$ is the coordinate of nucleon i, and $\mathbf{R}_A = (1/\sqrt{A})(\mathbf{r}_1 + \mathbf{r}_2 + \ldots + \mathbf{r}_A)$ is the normalized center-of-mass coordinate. In this method, the kinematic variables are the $3A-4$ internal angular variables $\{\theta_i\}$, $i = 1,2,\ldots, 3A-4$.

In the second method, monopole and quadrupole degrees of freedom, $a, b, c, \varphi, \theta, \psi$, are used simultaneously to separate the collective motions. Here, a, b, c are the principal values of the inertia tensor

$$Q_{\alpha\beta} = \sum_{i=1}^{A-1} x_{i\alpha} x_{i\beta};\ \alpha, \beta = x, y, z$$

($x_{i\alpha}$ is component α of the ith Jacobi vector $\mathbf{q}_i$), and φ, θ, ψ are the Eulerian angles that determine the orientation of the principal axes of the ellipsoid of inertia with respect to the external coordinate system. Instead of the variables a, b, c, one frequently uses the variables ρ, β, γ, which are related to a, b, c by

$$\rho^2 = a^2 + b^2 + c^2;\ \rho^2\beta\cos\gamma = c^2 - \frac{a^2+b^2}{2};$$

$$\rho^2\beta\sin\gamma = \frac{\sqrt{3}}{2}(a^2 - b^2).$$

The variables β and γ are microscopic analogs of the well-known variables of the Bohr–Mottelson model.[13] For the internal (kinematic) variables in the method of generalized hyperspherical functions it is necessary to take the degrees of freedom that describe the motion of the nucleons in the system of the principal axes of the ellipsoid of inertia or, which is the same thing, in the system of the principal axes of the mass quadrupole.

The internal function χ_ν satisfies the Pauli principle and, in addition, has definite values of the parity, spin S, isospin T, and their projections S_z and T_z, and also a definite $O(A-1)$ symmetry $[f_1 f_2 f_3]$. The last requirement is the strongest, but it is the one that makes it possible to separate branches of collective excitations associated with strong quadrupole, $E2$, and monopole, $E0$, transitions. These collective branches are observed in virtually all nuclei for which the probabilities of electromagnetic transitions have been measured, and therefore the possibility of making a restriction in the first approximation to a definite $O(A-1)$ symmetry in the theoretical study of the spectra of collective excitations has experimental confirmation.

It is necessary to use the group $O(A-1)$ (the group of orthogonal transformations in $(A-1)$-dimensional space) both because the parameters of this group can be conveniently chosen as independent variables of the internal functions $\{\chi_\nu(qK)\}$ (Refs. 10 and 11) and because the translationally and rotationally invariant basis functions of the irreducible representations of the group $O(A-1)$ provide the simplest means for expressing the ideas behind the shell model (the model of independent particles) suited for studying monopole and quadrupole excitations of a system of nucleons. In contrast to the functions of the translationally invariant shell model, the internal functions $\{\chi_\nu\}$ remain unchanged not only in a motion of the center of mass but also in a motion of the mass quadrupole, ensuring fulfillment of all the additional kinematic conditions satisfied by the nucleon coordinates in the intrinsic coordinate system of the nucleus.

It is a remarkable fact that certain standard functions in the theory of irreducible representations of the group of $(A-1)$-dimensional rotations and the group of permutations of the spatial coordinates and spin–isospin variables of a system of A nucleons (the properties of these functions are well known[10]) are invariant functions $\{\chi_\nu\}$, and there is a simple connection between the quantum numbers of the shell-model wave function written in the form (1) and the

quantum numbers of the standard invariant functions.

The second group of problems relates to the study of the interaction of nuclear systems (i.e., scattering, reactions). The microscopic basis for the solution of these problems is the resonating-group method,[14–17] in which it is assumed that the internal structure of the interacting subsystems (clusters) does not change as they approach and move away from each other. If it is necessary to investigate the interaction of two clusters consisting of A_1 and A_2 nucleons, then the components of the Jacobi vector $\mathbf{q}_1$ determining the relative position of the two clusters are chosen as the dynamical variables. The remaining Jacobi vectors describing the internal structure of these clusters are kinematic variables. The wave function of the system is represented in the form

$$\Psi = \hat{A}\ \{\varphi_1(A_1)\ \varphi_2(A_2)\ f(\mathbf{q}_1)\}. \tag{2}$$

The internal functions $\varphi_1(A_1)$ and $\varphi_2(A_2)$ of the clusters are fixed ($\chi_\nu \equiv \varphi_1(A_1)\varphi_2(A_2)$), while the function of the relative motion $f(\mathbf{q}_1)$ ($\Phi_\nu(q_D) \equiv f(\mathbf{q}_1)$) depends on the choice of the parameters of the nucleon–nucleon potential.

The third group of problems arises in the investigation of collective resonances, i.e., collective excitations, which lie in the continuum and are characterized by not only an excitation energy but also a width. The approximation of a fixed internal function used to solve the problems of the first group makes it possible to determine only an excitation energy, since in such an approximation one takes into account only a single channel of decay of the system into all the nucleons that constitute it. On the other hand, the resonating-group method makes it possible to find the resonances due to the centrifugal barrier, or to the Coulomb interaction, or, finally, to bound states in closed cluster channels when a many-channel problem is studied. Therefore, to describe the various collective resonances it is necessary to use not only the collective excitations but also a set of internal functions that reproduce the motion of the subsystems in the open channels. Therefore, in this case the dynamical variables will be the collective variables and the internal variables associated with the clustering of the nucleus. The analysis of the cluster functions made in Ref. 18 showed that the decay of the system into two clusters is described by internal functions whose $O(A-1)$ symmetry is $[f_1 + 2k,\ f_2,\ f_3]$, where $K = 0,1,2,\ldots$.

In the third group, we can also include the problems of the investigation of collective excitations when the interaction of the internal and collective degrees of freedom is taken into account more correctly than in the approximation of fixed internal motion. Such allowance for the interaction of the collective and internal motions is particularly topical in the description of nuclei with well-defined cluster structure. We must also include in this group the problems of the investigation of the interaction of nuclear systems that change their internal structure in the interaction process. Thus, in the solution of problems of the third group one uses a more accurate approximation than in the problems of the first two.

We now turn to the nuclear wave function (1). We recall that for all the problems listed above the kinematic functions $\chi_\nu(q_K)$ are fixed, while the dynamical components $\Phi_\nu(q_D)$ must be found by solving a differential or integro-differential equation (more precisely, a system of equations). To derive it, we must substitute in the many-particle Schrödinger equation

$$(\hat{H} - E)\ \Psi = 0 \tag{3}$$

the Ψ function in the form (1), multiply the resulting relation from the left by the function $\chi_{\nu'}(q_K)$, and integrate over the kinematic variables:

$$\langle \chi_{\nu'}(q_K) \mid \hat{H} - E \mid \hat{A} \sum_\nu \Phi_\nu(q_D)\ \chi_\nu(q_K) \rangle = 0. \tag{4}$$

We then obtain a system of coupled integro-differential equations (if the dynamical variables q_D are not symmetric with respect to permutations of the single-particle coordinates) or a system of differential equations (if q_K are symmetric). We recall that the collective coordinates, in contrast to cluster coordinates, are invariant with respect to permutation of the particles.

The tremendous difficulties associated with writing down explicitly the system of equations (4) for the dynamical components and the numerical solution of this system stimulated searches for algorithms for solving the problem as adequate as the one given above but not so complicated in practical implementation. One such algorithm is the generator-coordinate method,[19–21] which used an integral representation for the unknown collective components:

$$\Phi_\nu(q_D) = \int g(\beta)\ \Phi^0_\nu(q_D,\ \beta)\ d\beta, \tag{5}$$

where $\Phi^0_\nu(q_D, \beta)$ are components of the generator function $\Psi^0(\beta, \{q\})$, having the same or different form, depending on which of the groups of problems listed above is being considered. The functions $\Psi^0_\nu(q_D, \beta)$ depend on both the dynamical variables q_D and the generator parameters β; at the same time, the number of generator parameters must be equal to the number of variables q_D. It is readily seen that the integral representation for the dynamical components is equivalent to an integral representation of the total wave function,

$$\Psi = \int d\beta g(\beta)\ \Psi^0(\beta,\ \{q\}), \tag{6}$$

where Ψ is defined by the relation (1), the generator function $\Psi^0(\beta, \{q\})$ has the form[1)]

$$\Psi^0(\beta,\ \{q\}) = \hat{A}\{\sum_\nu \Phi^0_\nu(\beta,\ q_D)\ \chi_\nu(q_K)\}, \tag{7}$$

and, finally, the unknown weight function $g(\beta)$ satisfies the Hill–Wheeler integral equation

$$\int d\tilde{\beta}\ [\mathcal{H}(\beta,\ \tilde{\beta}) - E\mathcal{N}(\beta,\ \tilde{\beta})]\ g(\tilde{\beta}) = 0. \tag{8}$$

Here

$$\mathcal{N}(\beta,\ \tilde{\beta}) = \langle \Psi^0(\beta,\ \{q\}) \mid \Psi^0(\tilde{\beta},\ \{q\}) \rangle \tag{9}$$

[1)]The function $\Psi^0(\beta, \{q\})$, which is used to calculate the properties of collective excitations of nuclei, is equal to the Slater determinant constructed from the single-particle oscillator functions of an anisotropic oscillator. The frequencies of such an oscillator or combinations of them are the generator parameters.[22–25] For scattering problems, $\Psi^0(\beta, \{q\})$ is constructed from cluster orbitals of the Brink model $\exp\{-\frac{1}{2}(\mathbf{r}_i - \mathbf{R}_k)^2\}$, where $\mathbf{R}_k$ are cluster parameters playing the part of generator coordinates (Refs. 14, 16, 17, 26, and 27).

is the overlap integral (the integration is over all the single-particle coordinates),

$$\mathscr{H}(\beta, \tilde{\beta}) = \langle \Psi^0(\beta, \{q\}) \mid \hat{H} \mid \Psi^0(\tilde{\beta}, \{q\}) \rangle, \quad (10)$$

where $\hat{H}$ is the many-particle Hamiltonian operator.

The use of the integral representation for the wave function (or, which is the same thing, for the dynamical components) simplifies to a large degree the derivation of the integral equation but not the problem of numerical solution of it. For the solution of this equation, one generally uses a discrete variant of the integral equation (8),

$$\sum [\mathscr{H}(\beta_i, \beta_j) - E\mathscr{N}(\beta_i, \beta_j)] g(\beta_j) = 0, \quad (11)$$

where the summation is over a discrete set of points $\{\beta_j\}$. The accuracy of solution of the system of algebraic equations (9) depends on how felicitously the set of discrete points is chosen (they are called node points). The problem of choosing these points and, therefore, the problem of the exact solution of the system (11) has been considered in many studies (see, for example, Refs. 28 and 29), but the algorithms proposed are excessively cumbersome and require extensive numerical calculations, whose accuracy, moreover, is not always sufficiently well verifiable, particulary in the case of multidimensional generator coordinates.

The simplest method of solving the many-particle Schrödinger equation was found to be the oscillator-basis method, in which the function Ψ is expanded with respect to a complete set of many-particle oscillator functions,

$$\Psi = \sum_n C_n \mid n \rangle \quad (12)$$

or, which is the same thing,

$$\Phi_\nu(q_D) = \sum_n C_n \Phi_\nu^n(q_I), \quad (13)$$

where Φ_ν^n is a dynamical component of the many-particle oscillator function

$$\mid n \rangle = \hat{A}\{\sum_\nu \Phi_\nu^n(q_D)\, \chi_\nu(q_K)\}. \quad (14)$$

The expansion reduces the system of differential or integro-differential equations for the functions $\Phi_\nu(q_D)$ to the system

$$\sum_{n'} \{\langle n \mid \hat{H} \mid n' \rangle - E\delta_{n,n'}\} C_{n'} = 0 \quad (15)$$

of algebraic equations for the unknown coefficients C_n (in other words, for the wave function of the system in the oscillator representation). The method can be used not only to calculate the properties of collective excitations when the system makes finite motions but also with great success (as was shown in Refs. 30–32) to solve continuum problems. In actual calculations, the infinite system of equations (15) must be replaced by a finite system, and a finite set must be chosen from the complete set of basis functions to solve the many-particle equation. The accuracy of the solution of the system (15) is usually checked by studying the dependence of the calculated nuclear parameters on the number of basis functions employed. The number of basis functions is taken to be sufficiently large to make this dependence disappear. Nevertheless, there remains the problem of the contribution made by the terms omitted in the system (15), and also the question of the extent to which allowance for these terms improves the accuracy of the solutions obtained in the truncated basis. The influence of these terms can be taken into account if, following Ref. 31, it is assumed that from a certain n_0 onward the coefficients C_n take the asymptotic form $C_n \cong a_0 C_n^{\text{as}}$ (α_0 is a constant coefficient determined from the condition of matching of the exact solution C_ν, $\nu < n_0$, and the asymptotic solution, C_n^{as}, $n \geqslant n_0$), which is characteristic of each of the problems listed above. The coefficients C_n^{as} are uniquely determined by the number n of the basis function, and E is the energy of the concrete state and the phase shift (if continuum states are being considered). Taking into account the expressions for C_n^{as}, we reduce the system of equations (15) to

$$\sum_{n'=1}^{n_0-1} \{\langle n \mid \hat{H} \mid n' \rangle - E\delta_{nn'}\} C_{n'} + a_0 \sum_{\nu=n_0}^{\infty} C_\nu^{\text{as}} \langle n \mid \hat{H} \mid \nu \rangle = 0. \quad (16)$$

As a result, the system of linear homogeneous equations (15) is transformed into a closed system of nonlinear equations for the bound states or into a system of inhomogeneous linear equations for the continuum states. These systems, and also the explicit form of C_n^{as}, are discussed in more detail in Sec. 4.

The main difficulties in the implementation of the oscillator-basis method (they are associated with the calculation of the matrix elements of the Hamiltonian operator $\hat{H}$ and the other operators on the many-particle oscillator functions $|n\rangle$) can be overcome by means of the generating invariants, or generalized coherent staes (GCSs).[2,3] The GCSs $\Phi(\beta)$ depend on the coordinates (spatial, spin, and isospin) of all the nucleons without exception and also on a definite number of generator parameters $\beta = \{\beta_1, \beta_2, \ldots, \beta_k\}$, the important property of which is that they are generating functions of a complete set of basis functions $\{|n\rangle\}$ with definite properties:

$$\Phi(\beta) = \sum_n a_n \beta^n \mid n \rangle, \quad (17)$$

the degree $n = \{n_1, n_2, \ldots, n_k\}$ of the generator parameter β determining uniquely the quantum numbers of the basis function $|n\rangle$ (a_n is a structure constant). If we calculate the matrix element of an arbitrary operator $\hat{F}$ on the GCSs $\Phi(\beta)$ and $\Phi(\tilde{\beta})$ (we shall call such a matrix element a generating matrix element), then because

$$\langle \Phi(\beta) \mid \hat{F} \mid \Phi(\tilde{\beta}) \rangle = \sum_{n,n'} a_n a_n \beta^n \tilde{\beta}^{n'} \langle n \mid \hat{F} \mid n' \rangle \quad (18)$$

the matrix element of this operator on the basis functions $|n\rangle$ can be readily obtained by differentiating (18) with respect to the generator parameters:

$$\langle n \mid \hat{F} \mid n' \rangle = (a_n n! a_{n'} n'!)^{-1} \times \times \frac{\partial^n}{\partial \beta^n} \frac{\partial^{n'}}{\partial \tilde{\beta}^{n'}} \langle \Phi(\beta) \mid \hat{F} \mid \Phi(\tilde{\beta}) \rangle \mid_{\beta = \tilde{\beta} = 0}. \quad (19)$$

The structure constants α_n can be extracted by the same method from the matrix element of the identity operator or, in other words, from the overlap integral

$$\langle \Phi(\beta) \mid \Phi(\tilde{\beta}) \rangle. \quad (20)$$

One of the possible ways of constructing GCSs of a definite irreducible representation of a group G is to use the equation

$$\Phi(\beta) = e^{\beta \hat{J}^+} | 0\rangle, \tag{21}$$

where $|0\rangle$ is the lowest vector of the irreducible representation, and $\hat{J}^+$ is the raising generator of the Lie algebra of G. The result is unchanged if instead of the lowest one takes the highest vector, and instead of $\hat{J}^+$ one takes the lowering generator $\hat{J}^-$. When the group G (or rather its Lie algebra) has not one but several raising generators $\hat{J}^+ = \{\hat{J}_1^+, \hat{J}_2^+, ..., \hat{J}_k^+\}$, there will be as many generator parameters $\beta = \{\beta_1, \beta_2, ..., \beta_k\}$, and therefore the product $\beta \hat{J}^+$ in (21) becomes the contraction

$$\sum_i \beta_i \hat{J}_i^+.$$

If the irreducible representation of G is finite-dimensional, the sum over n in (17) is finite; otherwise, it is infinite. As an example, we give the GCSs of the irreducible representations of the rotation group $SO(3)$ in three-dimensional space[3]:

$$\Phi(\beta) = \sum_{n=0}^{2j} \sqrt{\frac{(2j)!}{n!\,(2j-n)!}}\, \beta^n | j, -j+n\rangle, \tag{22}$$

and also the GCSs of the irreducible representations of the noncompact group $SO(2,1)$ or $SU(1,1)$, which is locally isomorphic to it[3]:

$$\Phi(\beta) = \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(n+2j)}{n!\,\Gamma(2j)}}\, \beta^n | j, j+n\rangle. \tag{23}$$

The product $a_n \beta^n$ [it is contained in the expansion of the GCS $\Phi(\beta)$ (17)] is to be regarded as an oscillator function $|n\rangle$ defined in a complex space[2)]—the generalized Bargmann space.[33] This space has dimensions equal in number to the number of dynamical variables q_D and, therefore, the number of generator parameters β. The GCSs are then the kernel of the integral transformation (6) from the coordinate to the Bargmann representation and from the Bargmann to the coordinate representation. The oscillator functions

$$\langle \beta | n \rangle \equiv a_n \beta^n \tag{24}$$

in the generalized Bargmann representation form a complete orthonormal set of functions and have a simpler form than the same functions in the $(3A-3)$-dimensional coordinate space. It is expedient to use this basis $\{\langle \beta | n\rangle\}$ to expand $g(\beta)$, the eigenfunctions of the Hamiltonian in the Bargmann representation:

$$g(\beta) = \sum_n C_n \langle \beta | n \rangle. \tag{25}$$

Such an expansion is equivalent to an expansion of Ψ with respect to the basis functions of the coordinate representation $\{\langle x|n\rangle\}$:

$$\Psi = \sum_n C_n \langle x | n \rangle.$$

The many-particle Hamiltonian $\hat{H}$ can be associated with an effective Hamiltonian $\hat{H}_{\text{eff}}(\beta)$, which acts only on the variables β and whose eigenfunctions are $g(\beta)$. By means of the effective Hamiltonian one can very conveniently formulate various approximate methods for investigating many-particle systems and establishing qualitative features of the solutions that are obtained. Such a nuclear Hamiltonian with a realistic nucleon–nucleon interaction was constructed for the first time in Ref. 34 for Elliott's $SU(3)$ model.[35]

2. DYNAMICAL SYMMETRY GROUP OF THE NUCLEAR HAMILTONIAN

To construct the GCSs, it is first necessary to find the dynamical symmetry group[3)] of the investigated nuclear system or model Hamiltonian whose eigenfunctions are used as the basis for diagonalizing the microscopic Hamiltonian with a realistic nuclear interaction. We noted above that the multidimensional oscillator Hamiltonian $\hat{H}_{\text{osc}}$ is such a model Hamiltonian, and therefore the dynamical symmetry group of the nuclear Hamiltonian will be identical to that of $\hat{H}_{\text{osc}}$. As was shown by Moshinsky *et al.*,[37] this group is formed by skew-symmetric matrices that realize linear canonical transformations in the $(6A-6)$-dimensional phase space (i.e., in the space of the $3A-3$ momenta and $3A-3$ coordinates). Therefore, the noncompact symplectic group $Sp(6A-6,R)$ is the dynamical symmetry group of the nuclear Hamiltonian and $\hat{H}_{\text{osc}}$. The group $Sp(6A-6,R)$ contains as a subgroup the symmetry group $U(3A-3)$ of $\hat{H}_{\text{osc}}$.

All the oscillator functions of positive parity form a basis of the irreducible representation $[\frac{1}{4},\frac{1}{4},...,\frac{1}{4}] \equiv [(\frac{1}{4})^{3A-3}]$ of the group $Sp(6A-6,R)$, while the negative-parity functions form the irreducible representation $[\frac{3}{4},\frac{1}{4},...,\frac{1}{4}] \equiv [\frac{3}{4},(\frac{1}{4})^{3A-4}]$ of this group. Thus, the basis functions of the irreducible representations of $Sp(6A-6,R)$ describe all possible types of motions that exist in the nuclear system: internal, collective, cluster, etc. However, when solving specific problems one must make a restriction to only some of them, the modes of motion that are most important in the investigated processes. In turn, this means that instead of the GCSs of $Sp(6A-6,R)$ one must use GCSs of groups of lower dimension. We list below the reductions of the group $Sp(6A-6,R)$ that must be considered in the solution of each of the three groups of problems in microscopic nuclear theory, and we also identify the groups whose GCSs will be used to solve the corresponding problems.

To investigate collective nuclear excitations, one most often uses two chains of subgroups of $Sp(6A-6,R)$:

$$Sp(6A-6, R) \supset Sp(2, R) \otimes O(3A-3) \tag{26}$$

and

[2)] In accordance with the general requirement of the theory of GCSs,[2,3] the generator parameters β must be complex quantities. However, in the cases when $\Phi(\beta)$ are used as generating functions, the parameters β can be real.

[3)] It is well known (see, for example, Ref. 36) that the symmetry group G_s makes it possible to obtain from one eigenfunction Ψ_1 of the Hamiltonian $\hat{H}$ a set of eigenfunctions of $\hat{H}$ with the same energy as Ψ_1. The dynamical symmetry group $G_{\text{d.s.}}$, which includes G_s as a subgroup, makes it possible to construct from the function Ψ_1 the complete set of eigenfunctions of $\hat{H}$ associated with the different eigenvalues of $\hat{H}$. For this reason, the group $G_{\text{d.s.}}$ is sometimes called the group that generates the spectrum. The symmetry group couples into one irreducible representation all eigenfunctions of $\hat{H}$ with given energy, while the dynamical symmetry group combines the complete set of eigenfunctions of this Hamiltonian into one irreducible representation.

$$Sp\ (6A-6,\ R) \supset Sp\ (6,\ R) \otimes O\ (A-1), \qquad (27)$$

in which the groups $Sp(2,R)$ and $Sp(6,R)$ characterize the collective motion, while $O(3A-3)$ and $O(A-1)$ determine the internal motion. The first of these chains corresponds to the basis of the method of K harmonics,[5–8] and the second to the basis of the method of generalized hyperspherical functions.[9–12] The indices j and K of the irreducible representations of the groups $Sp(2,R)$ and $O(3A-3)$ are related by virtue of the complementarity of these groups[37] (see also Ref. 38) by the relation[39]

$$j=\frac{1}{2}K+\frac{3}{4}(A-1)-1. \qquad (28)$$

For the same reason, the indices $[\sigma_1\sigma_2\sigma_3]$ of the irreducible representation of $Sp(6,R)$ are related by[39]

$$\sigma_i=\frac{1}{2}f_i+\frac{1}{4}(A-1), \quad i=1,\ 2,\ 3 \qquad (29)$$

to the indices $[f_1f_2f_3]$ of the irreducible representation of $O(A-1)$. These relations reflect the fact that the internal motion uniquely determines the nature and variety of the collective modes.

In the minimal approximation of the K-harmonic method (i.e., in the approximation of a fixed internal motion), the collective motion is represented solely by monopole (volume) vibrations, whereas in the minimal approximation of the method of generalized hyperspherical functions quadrupole degrees of freedom are taken into account as well. Therefore, for more complete allowance of all modes of the collective motion and their coupling to the internal motion, it is necessary to construct GCSs of the group $Sp(6,R)$:

$$\Phi(\{b\})=\exp\{\sum_{r,s} b_{rs}\hat{A}^{+}_{rs}\}\mid 0\rangle, \qquad (30)$$

$$\hat{A}^{+}_{rs}=\sum_{i=1}^{A-1} a^{+}_{ir}a^{+}_{is}, \qquad (31)$$

where $\hat{A}^{+}_{rs}$ is the operator of production of the collective quanta, and a^{+}_{ir} is the operator of creation of an oscillator quantum for the degree of freedom ir. The vacuum function $|0\rangle$ is an $SU(3)$ multiplet,

$$\mid 0\rangle = \mid N_{\min}\ (\lambda_0\mu_0)\ \alpha LM\ [f_1f_2f_3]\rangle, \qquad (32)$$

whose symmetry indices $(\lambda_0\mu_0)$ and principal quantum number $N_{\min}$ are related to f_1,f_2,f_3 by

$$N_{\min}=f_1+f_2+f_3,\ \lambda_0=f_1-f_2,\ \mu_0=f_2-f_3. \qquad (33)$$

In Eq. (32), αLM are the quantum numbers of the reduction $SU(3)\supset SO(3)$.

The important part played by the group $Sp(6,R)$ in describing collective excitations of nuclei was noted for the first time in Ref. 39, in which a connection was also established between the indices of the irreducible representations of the group $O(A-1)$, which determine the internal motion of the system, and the indices of the irreducible representations of the symplectic group $Sp(6,R)$, which characterize the collective motion. In the later paper of Ref. 40, the problems of using the basis of the group $Sp(6,R)$ to calculate the spectrum of collective excitations were discussed. Since then, many studies have been devoted to this problem.[41–66] A new term has been introduced as a result—the symplectic shell model.[43,46] Since the $Sp(6,R)$ basis is fairly extensive and includes several branches of collective excitations, it became customary to use only part of the basis in concrete calculations and to consider subgroups of this group: $Sp(2,R)$ (Refs. 52–64) and $Sp(4,R)$.[51] The microscopic models that use the bases of these groups came to be known as the $Sp(2,R)$ model and the $Sp(4,R)$ model.

Each $Sp(6,R)$ vector must, if all the three indices σ_1, σ_2, σ_3 are different, be labeled by a set of nine quantum numbers. To determine these numbers, one uses the reduction

$$Sp\ (6,\ R) \supset U\ (3) \supset SU\ (3) \supset SO\ (3), \qquad (34)$$

which gives only six quantum numbers: N, the principal quantum number; $(\lambda\mu)$, the indices of the irreducible representation of $SU(3)$; L and M, the orbital angular momentum and the projection onto the z axis of the laboratory coordinate system; and the number α, the multiplicity of occurrence of the given L in the $SU(3)$ multiplet $(\lambda\mu)$. The three further unknown quantum numbers will determine the multiplicity of the $SU(3)$ representation in the fixed irreducible representation of $Sp(6,R)$. The problem of finding these quantum numbers and the operators corresponding to them is still unsolved.

The realization of the resonating-group method for a two-cluster system ($A=A_1+A_2$) leads to the following reduction of the group $Sp(6A-6,R)$:

$$Sp\ (6A-6,R) \supset Sp\ (6A_1-6,\ R) \otimes Sp\ (6A_2-6,\ R) \otimes Sp\ (6,\ R), \qquad (35)$$

where the basis functions of the irreducible representations of $Sp(6A_1-6,R)$ and $Sp(6A_2-6,R)$ will describe the internal state of the first and second clusters, respectively, while the $Sp(6,R)$ basis functions describe the relative motion of these clusters. Since the internal state of the interacting clusters is fixed in the resonating-group method, and the function of the relative motion agrees with the chosen nucleon–nucleon potential, to solve problems of the second group it is necessary to construct the GCSs of the group $Sp(6,R)$ in the space of cluster functions. But since the groups $Sp(2,R)$ and $O(3)$ are complementary in this space, to investigate two-cluster systems a restriction can be made to the GCSs of $Sp(2,R)$:

$$\Phi\ (\mathbf{R}) = \hat{A}\{\varphi_1\ (A_1)\varphi_2(A_2)\ e^{\mathbf{Ra}_1^{+}}\mid 0>\}, \qquad (36)$$

where the vector $\mathbf{R}$ is a cluster parameter and $\mathbf{a}_1^{+}$ is the operator of creation of an oscillator quantum of the relative motion of the clusters,

$$\mid 0\rangle = e^{-\frac{1}{2}\eta_i^2}.$$

Such GCSs will generate a basis of oscillator functions of the cluster model of both positive and negative parity, provided $A_+\neq A_2$. Otherwise ($A_1=A_2$), the GCSs (36) generate states of only positive or only negative parity.

As we have already mentioned, to describe collective resonances one must use a state basis that reproduces not only the excitation of the collective degrees of freedom but also the clustering of the nucleus. This means that the GCSs of the resonance states must contain as generator parameters

both b_{rs} and the cluster parameters **R**. The latter will generate the cluster generalized hyperspherical functions, and the b_{rs} will generate the collective excitations over such functions.

As a rule, GCSs are successfully used for systems whose Hamiltonians can be represented in the form of a linear combination of the generators of the dynamical symmetry group (see, for example, Refs. 3, 36, and 67). However, neither the nuclear Hamiltonian nor the operators of many other physical quantities can be represented in such a form. To extend the region of applicability of GCSs to such a case, the coordinate form of these objects was proposed in place of the operator form of the GCSs (21) in a number of studies (Refs. 16, 59, 64, 65, and 68). In the coordinate form, the GCSs $\Phi(\beta)$ for all the problems listed in the previous section have the form of Slater determinants

$$\Phi(\beta) \equiv \det \| \varphi_i(\mathbf{r}_j) \|,$$

which are constructed from suitably defined single-particle orbitals $\varphi_i(\mathbf{r}_j)$. These orbitals will be considered in the following sections of the review for each of the three groups of problems. We should also discuss the calculation of the generating matrix elements of various operators on the determinant functions. We here mention only that it is not too difficult to work with determinant functions. Löwdin[69] developed a fairly simple algorithm for calculating the matrix elements of single- and two-particle operators that eliminates the problem of obtaining the generating matrix elements

$$\langle \Phi(\beta) \mid \hat{F} \mid \Phi(\tilde{\beta}) \rangle.$$

Moreover, the differentiation of such matrix elements with respect to the generator parameters β and $\tilde{\beta}$ can always be done in general form, and expressions can be obtained in analytic form for the matrix elements

$$\langle n \mid \hat{F} \mid n' \rangle \tag{37}$$

on the basis functions $|n\rangle$ and $|n'\rangle$ with arbitrarily large values of the quantum numbers n and n' that characterize these functions. Knowledge of the matrix elements (37) for large n and n' makes it possible, on the one hand, to obtain asymptotic expressions for them, simplifying numerical calculations, and, on the other, to investigate the asymptotic form of the solutions $\{C_n\}$ for $n \gg 1$, this being needed to check the accuracy of the solutions obtained. Since, as already mentioned, the GCSs are generating functions, recursion relations for the matrix elements (37) can be established by means of the devices adopted in the theory of special functions, and this greatly simplifies the programming of these matrix elements on a computer.

It is well known that in the determinant function Ψ, which depends on the single-particle coordinates of the nucleons, separation to Jacobi coordinates makes it possible to separate the factor ψ_R describing the center-of-mass motion:

$$\Psi = \Phi\psi_R. \tag{38}$$

At the same time, the function Φ satisfies the condition of translational invariance, and if, in addition, a corresponding set of generator coordinates is defined for Ψ, then Φ is actually a GCS for the system described by the function Ψ.

Thus, when averaging different operators on the functions Ψ, it is necessary to take into account each time the elimination of the center-of-mass motion. This can be readily done by using, on the one hand, the property (38) and, on the other, the circumstance that practically all operators $\hat{F}$ used to investigate nuclear characteristics can be divided into three groups. The first contains translationally invariant operators, i.e., $\hat{F} = \hat{F}_0$. The second has the operators that contain the center-of-mass motion in the form of an additivie term, i.e., $\hat{F} = \hat{F}_0 + \hat{F}_R$. Finally, the third group consists of operators of the type $\hat{F} = \hat{F}_0 \hat{F}_R$. The potential-energy operator is an example of the operators of the first group. The kinetic-energy operator is an example of the second group, while the operator that determines the form factor of electron scattering is an example of the third:

$$\hat{F} = \frac{1}{2} \sum_{i=1}^{A} (1 + \hat{\tau}_{iz}) \, e^{i\mathbf{k}\mathbf{r}_i}.$$

Thus, calculating the overlap integral $\langle \Psi \mid \hat{F} \mid \tilde{\Psi} \rangle$ and using (38), it is possible to obtain the corresponding generating matrix elements in accordance with one of the following formulas:

$$\left.\begin{aligned}
&1.\ \langle \Phi \mid \hat{F}_0 \mid \tilde{\Phi} \rangle = \langle \Psi \mid \hat{F} \mid \tilde{\Psi} \rangle / \langle \psi_R \mid \tilde{\psi}_R \rangle; \\
&2.\ \langle \Phi \mid \hat{F}_0 \mid \tilde{\Phi} \rangle = \{ \langle \Psi \mid \hat{F} \mid \tilde{\Psi} \rangle \\
&\qquad - \langle \Phi \mid \tilde{\Phi} \rangle \langle \psi_R \mid \hat{F}_R \mid \tilde{\psi}_R \rangle \} / \langle \psi_R \mid \tilde{\psi}_R \rangle; \\
&3.\ \langle \Phi \mid \hat{F}_0 \mid \tilde{\Phi} \rangle = \langle \Psi \mid \hat{F} \mid \tilde{\Psi} \rangle / \langle \psi_R \mid \hat{F}_R \mid \tilde{\psi}_R \rangle.
\end{aligned}\right\} \tag{39}$$

The values of $\langle \psi_R \mid \tilde{\psi}_R \rangle$ and $\langle \psi_R \mid \hat{F}_R \mid \tilde{\psi}_R \rangle$ can be readily calculated from the single-particle overlap integrals to which the integral $\langle \Psi \mid \hat{F} \mid \Psi \rangle$ reduces.

3. GENERALIZED COHERENT STATES OF COLLECTIVE EXCITATIONS

We recall that a GCS of the group $Sp(6,R)$ in operator form has the form

$$\Phi(\{b_{rs}\}) = \exp\left\{ \sum_{r,s} b_{rs} \bar{A}^{+}_{rs} \right\} \mid 0\rangle, \tag{40}$$

and the vacuum vector $|0\rangle$ is an $SU(3)$ multiplet $(\lambda_0 \mu_0)$ such that

$$\lambda_0 = f_1 - f_2, \ \mu_0 = f_2 - f_3. \tag{41}$$

To establish the connection between the coordinate and operator forms of the GCSs, we consider the simplest case when $f_1 = f_2 = f_3$ and, therefore, $(\lambda_0 \mu_0) = (00)$. Such a vacuum vector is realized in the minimal approximation of the method of generalized hyperspherical functions for magic nuclei.

We consider the tensor b_{rs}. It can always be reduced to diagonal form. Let the three mutually orthogonal unit vectors $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ specify the orientation of the coordinate system in which the tensor b_{rs} is diagonal, and let $\beta_1, \beta_2, \beta_3$ be its principal values.

Then

$$b_{rs} = \sum_{\alpha} \beta_\alpha \ p_{\alpha r} \ p_{\alpha s}. \tag{42}$$

Here, $p_{\alpha r}$ is the rth component of the vector $\mathbf{p}_\alpha$. We direct the axes of the laboratory coordinate system along the principal axes of b_{rs}. In such a coordinate system, the components $\{\xi_i, \eta_i, \zeta_i\}$ of the Jacobi vector $\mathbf{q}_i$ $(i = 1,2,..., A-1)$ are

$$\xi_i = (\mathbf{p}_1\mathbf{q}_i),\ \eta_i = (\mathbf{p}_2\mathbf{q}_i),\ \zeta_i = (\mathbf{p}_3\mathbf{q}_i), \tag{43}$$

and the exponential in (40) has diagonal form:

$$\exp\{\beta_1\hat{A}^+_{\xi\xi} + \beta_2\hat{A}^+_{\eta\eta} + \beta_3\hat{A}^+_{\zeta\zeta}\}. \tag{44}$$

Here,

$$\bar{A}^+_{\xi\xi} = \sum p_{1r}\bar{A}^+_{rs}p_{1s},\ \bar{A}^+_{\eta\eta} = \sum p_{2r}\bar{A}^+_{rs}p_{2s}, \bar{A}^+_{\zeta\zeta} = \sum p_{3r}\bar{A}^+_{rs}p_{3s}$$

are the operators of creation of collective oscillator quanta, reduced to the coordinate system whose axes coincide with the direction of the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$. The vacuum vector $|0\rangle$ of the $SU(3)$ representation (00) is a scalar and therefore does not change on the transition from one coordinate system to another. In this case, the GCS $\Phi(\{\beta_\alpha\})$, where

$$\Phi(\{\beta_\alpha\}) = \exp\{\beta_1\hat{A}^+_{\xi\xi} + \beta_2\hat{A}^+_{\eta\eta} + \beta_3\hat{A}^+_{\zeta\zeta}\}\,|\,0\rangle,$$

generates a basis of the oscillator functions $|n_\xi, n_\eta, n_\zeta\rangle$,

$$\Phi(\{\beta_\alpha\}) = \sum_{n_\xi, n_\eta, n_\zeta} \frac{\beta_1^{n_\xi}\beta_2^{n_\eta}\beta_3^{n_\zeta}}{n_\xi!\, n_\eta!\, n_\zeta!} (\hat{A}^+_{\xi\xi})^{n_\xi} (\hat{A}^+_{\eta\eta})^{n_\eta} (\hat{A}^+_{\zeta\zeta})^{n_\zeta}\,|\,0\rangle$$

$$= \sum_{n_\xi, n_\eta, n_\zeta} \sqrt{\frac{\Gamma(n_\xi+J)\,\Gamma(n_\eta+J)\,\Gamma(n_\zeta+J)}{n_\xi!\,\Gamma(J)\,n_\eta!\,\Gamma(J)\,n_\zeta!\,\Gamma(J)}}\, \beta_1^{n_\xi}\beta_2^{n_\eta}\beta_3^{n_\zeta}\,|\,n_\xi, n_\eta, n_\zeta\rangle, \tag{45}$$

$(J = f + (A-1)/2)$, which is characterized by the quantum numbers n_ξ, n_η, n_ζ of the collective excitations with respect to each of the axes ξ, η, ζ.

We now show that the function

$$\Phi(\{\beta_\alpha\}) = [(1-\beta_1)(1-\beta_2)(1-\beta_3)]^{-J}$$
$$\times \exp\left\{-\sum_{i=1}^{A-1}\left(\frac{\beta_1}{1-\beta_1}\xi_i^2 + \frac{\beta_2}{1-\beta_2}\eta_i^2 + \frac{\beta_3}{1-\beta_3}\zeta_i^2\right)\right\}\,|\,0\rangle \tag{46}$$

is also a generating function of the oscillator basis $\{|n_\xi, n_\eta, n_\zeta\rangle\}$, i.e., the expansion (45) holds. For this, we introduce instead of the set of Cartesian components $\{\xi_i\}$ the hyperspherical radius $\rho_\xi = [\xi_1^2 + \xi_2^2 + ... + \xi_{A-1}^2]^{1/2}$ and the corresponding hyperspherical angles $\{\Theta_\xi^k\}$, $k = 1,2,..., A-2$. We introduce analogous hyperspherical coordinates $\{\Theta_\eta^k\}$ and ρ_ζ, $\{\Theta_\zeta^k\}$ in place of the sets $\{\eta_i\}$ and $\{\zeta_i\}$. Then the function $|n_\xi, n_\eta, n_\zeta\rangle$ can be written in the form

$$|\,n_\xi, n_\eta, n_\zeta\rangle = L^{J-1}_{n_\xi}(\rho_\xi^2)\, L^{J-1}_{n_\eta}(\rho_\eta^2)\, L^{J-1}_{n_\zeta}(\rho_\zeta^2)$$
$$\times \exp\left\{-\frac{1}{2}(\rho_\xi^2 + \rho_\eta^2 + \rho_\zeta^2)\right\} (\rho_\xi\rho_\eta\rho_\zeta)^f\, Q_{[fff]}(\{\Theta_\xi^k\}, \{\Theta_\eta^k\}, \{\Theta_\zeta^k\}), \tag{47}$$

where $L_n^\alpha(x)$ is a Laguerre polynomial. To obtain the function $\Phi(\{\beta_\alpha\})$ in the form (46), it is sufficient to use the factorization of the basis functions (47) in the variables ρ_ξ, ρ_η, ρ_ζ and the explicit form of the generating function $\mathcal{L}_\gamma$ for the Laguerre polynomials:

$$\mathcal{L}_\gamma(\rho, \beta) = (1-\beta)^{-\gamma}\exp\left\{-\frac{\beta}{1-\beta}\rho^2\right\} = \sum_n \beta^n L_n^{\gamma-1}(\rho^2). \tag{48}$$

We note further that the product

$$(\rho_\xi\rho_\eta\rho_\zeta)^f\, Q_{[fff]}(\{\Theta_\xi^k\}, \{\Theta_\eta^k\}, \{\Theta_\zeta^k\})$$

is invariant with respect to rotation of the coordinate axes and, when multiplied by the exponential $\exp\{-\frac{1}{2}(\rho_\xi^2 + \rho_\eta^2 + \rho_\zeta^2)\}$, is equal to the vacuum function $|0\rangle$.

The GCS $\Phi(\{\beta_\alpha\})$ (46) is written down in the coordinate system in which the tensor b_{rs} is diagonal. To obtain the expression for $\Phi(\{\beta_\alpha\})$ in the original laboratory coordinate system, we must replace the components ξ_i, η_i, and ζ_i by the scalar products $\mathbf{p}_1 \cdot \mathbf{q}_i$, $\mathbf{p}_2 \cdot \mathbf{q}_i$, and $\mathbf{p}_3 \cdot \mathbf{q}_i$ (43):

$$\Phi(\{\beta_\alpha\}) = [(1-\beta_1)(1-\beta_2)(1-\beta_3)]^{-J}$$
$$\times \exp\left\{-\sum_{i=1}^{A-1}\sum_{\alpha=1}^{3}\frac{\beta_\alpha}{1-\beta_\alpha}(\mathbf{p}_\alpha\mathbf{q}_i)^2\right\}\,|\,0\rangle. \tag{49}$$

This expression is a GCS of the group $Sp(6,R)$ in coordinate form. It is readily seen that is differs only by the factor

$$\exp\left\{-\frac{1}{2}q_A^2 - \sum_{\alpha=1}^{3}\frac{\beta_\alpha}{1-\beta_\alpha}(\mathbf{p}_\alpha\mathbf{q}_A)^2\right\}$$

$$\mathbf{q}_A = \frac{1}{\sqrt{A}}(\mathbf{r}_1 + \mathbf{r}_2 + \ldots + \mathbf{r}_A),$$

which describes the center-of-mass motion, from a Slater determinant constructed from single-particle orbitals of the form

$$\psi(\mathbf{r}_i) = \exp\left\{-\frac{1}{2}r_i^2 - \sum_\alpha\frac{\beta_\alpha}{1-\beta_\alpha}(\mathbf{p}_\alpha\mathbf{r}_i)^2\right\}$$
$$\times H_{n_x}(x_i)\, H_{n_y}(y_i)\, H_{n_z}(z_i),$$

where $H_n(x)$ is a Hermite polynomial, and multiplied by

$$\prod_{\nu=1}^{3}(1-\beta_\nu)^{-J_\nu}, \quad J_\nu = f_\nu + (A-1)/2.$$

Further, we shall consider GCSs of the group $Sp(6,R)$ for arbitrary irreducible representations of this group restricted only by some conditions that are natural from the physical point of view.

We restrict ourselves to irreducible representations of the group $Sp(6,R)$ whose $SU(3)$ multiplets can be realized by means of Slater determinants (more precisely, to the irreducible representations $(\lambda_0\mu_0)$ each of whose highest vectors can be expressed in the form of one Slater determinant). Thus, we make the indices f_1, f_2, f_3 satisfy certain conditions. However, these conditions identify the most interesting internal functions, namely, those whose construction can be realized without additional alternation with respect to the states of the nucleons of the open shells beyond the alternation needed to satisfy the requirements of the Pauli principle (in other words, there is no need to make an additional alternation with respect to the states of nucleons that have different spin–isospin quantum numbers).

We introduce three orthogonal unit vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ and, with them, single-particle states:

$$\psi^0(\mathbf{n};\ \nu\,|\,\mathbf{r})$$
$$= (\pi^{3/2}2^n\mathbf{n}!)^{-1/2}\exp\left(-\frac{1}{2}r^2\right) H_{n_1}(\mathbf{u}_1\mathbf{r})\, H_{n_2}(\mathbf{u}_2\mathbf{r})\, H_{n_3}(\mathbf{u}_3\mathbf{r})\,\zeta_\nu, \tag{50}$$

where

$$\mathbf{n} = \{n_1, n_2, n_3\},\ n = n_1 + n_2 + n_3,\ \mathbf{n}! = n_1! n_2! n_3!, \tag{51}$$

and ζ_ν is a spin–isospin function with quantum numbers ν. From the orbitals (50), we shall construct Slater determinants of a system of A nucleons, adopting the following two rules with regard to the order of filling of the orbitals of the open shells. First, the orbital $\psi^0(\mathbf{n};\nu|\mathbf{r})$ is filled not earlier than all the orbitals for which the number of excitation quanta with respect to at least one of the three directions $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is less than for the orbital $\psi^0(\mathbf{n};\nu|\mathbf{r})$ in the same direction and the spin–isospin function ζ_ν is the same, and, second, not earlier than all the orbitals obtained from $\psi^0(\mathbf{n};\nu|\mathbf{r})$ by transferring the excitation quanta (preserving their total number) from the direction $\mathbf{u}_2$ to the direction $\mathbf{u}_1$ and from the direction $\mathbf{u}_3$ to the directions $\mathbf{u}_2$ and $\mathbf{u}_1$. The first rule ensures the possibility of filling at once several open shells while preserving the factorization property of the Slater determinant (i.e., the separation from it in the form of a factor of the center-of-mass wave function when the Slater determinant is expressed in terms of Jacobi coordinates). The second rule has the consequence that in the coordinate system whose axes are directed along the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ the Slater determinant is the highest vector of the irreducible representation $(\lambda_0 \mu_0) = (f_1 - f_2, f_2 - f_3)$ of the group $SU(3)$, the indices f_1, f_2, f_3 being the total numbers of excitation quanta of the Slater determinant along the directions $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, respectively:

$$f_1 = \sum n_1,\ f_2 = \sum n_2,\ f_3 = \sum n_3.$$

In an arbitrary coordinate system, this same determinant is a generating invariant of the irreducible representation $(\lambda_0 \mu_0)$ of $SU(3)$, and the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ or the three Eulerian angles that specify the orientation of these vectors play the part of the generating parameters of the generating invariant.

Following the rules formulated above for each nucleus, we obtain in general several Slater determinants with different $SU(3)$ and $O(A-1)$ symmetry. We denote them by $\Psi^0([f_1, f_2, f_3])$, and their translationally invariant part by $\Phi^0([f_1 f_2 f_3])$. We call the latter the vacuum function of the corresponding irreducible representation $[\sigma_1\, \sigma_2\, \sigma_3]$ of the group $Sp(6,R)$. To establish which irreducible representation of $Sp(6,R)$ can be ascribed the dominant role in the ground state and in the spectrum of quadrupole–monopole excitations over the ground state, it is necessary to calculate the ground-state energy with some semirealistic nucleon–nucleon interaction or to analyze the $O(A-1)$ symmetry of the Nilsson diagrams associated (in accordance with experimental data) with the ground states.

The Slater determinant constructed from the single-particle orbitals

$$\psi(\mathbf{n};\ \nu \mid \mathbf{r}) = \exp\Big\{ - \sum_\alpha \frac{\beta_\alpha}{1-\beta_\alpha} (\mathbf{p}_\alpha \mathbf{r})^2 \Big\} \psi^0(\mathbf{n};\ \nu \mid \mathbf{r}), \tag{52}$$

forms together with the factor $\Pi_{\nu=1}^3 (1-\beta_\nu)^{J_\nu}$ a GCS of the irreducible representation $[\sigma_1\, \sigma_2\, \sigma_3]$ of $Sp(6,R)$. The generator parameters are the angles of orientation of the frame $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ and the angles that specify the frame $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ (i.e., the angles of orientation of the principal axes of the tensor b_{rs}), and also $\beta_1, \beta_2, \beta_3$—the principal values of the tensor b_{rs}.

We introduce now the second Slater determinant $\tilde{\Psi}([f_1 f_2 f_3])$, which is obtained from $\Psi([f_1 f_2 f_3])$ by replacing the frame $\{\mathbf{p}_\alpha\}$ by the frame $\{\mathbf{q}_\alpha\}$ and the principal values $\beta_1, \beta_2, \beta_3$ by $\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3$ (or, in other words, the tensor b_{rs} by the tensor $\tilde{b}_{rs}$) and, in addition, by replacing the frame $\{\mathbf{u}_\nu\}$ by the frame $\{\mathbf{v}_\nu\}$.

In the first place, we shall be interested in the matrix element of the identity operator on the generating functions $\Psi([f_1 f_2 f_3])$ and $\tilde{\Psi}([f_1 f_2 f_3])$:

$$\langle \Psi([f_1 f_2 f_3]) \mid \tilde{\Psi}([f_1 f_2 f_3]) \rangle = \det \| F_{i,\tilde{i}} \|, \tag{53}$$

where $\| F_{i,\tilde{i}} \|$ is a matrix formed from the overlap integrals of orbitals $\psi(\mathbf{n};\nu|\mathbf{r})$ and $\tilde{\psi}(\tilde{\mathbf{n}};\tilde{\nu}|\mathbf{r})$,[4] which form the determinants $\Psi([f f_2 f_3])$ and $\Psi([f_1 f_2 f_3])$:

$$F_{i,\tilde{i}} = f^\nu_{\mathbf{n},\tilde{\mathbf{n}}} \delta_{\nu\tilde{\nu}} = \langle \mathbf{n} \mid \tilde{\mathbf{n}} \rangle \delta_{\nu\tilde{\nu}}; \tag{54}$$

$$i = \{\mathbf{n}, \nu\},\quad \tilde{i} = \{\tilde{\mathbf{n}}, \tilde{\nu}\},\quad \mathbf{n} = \{n_1, n_2, n_3\},\quad \tilde{\mathbf{n}} = \{\tilde{n}_1, \tilde{n}_2, \tilde{n}_3\}.$$

Because the spin-isospin functions are orthogonal, the matrix $\| F_{i,\tilde{i}} \|$ is quasidiagonal and decomposes into four blocks $\| f^\nu_{\mathbf{n},\tilde{\mathbf{n}}} \|$ and its determinant has the form of a product:

$$\det \| F_{i,\tilde{i}} \| = \prod_\nu \det \| f^\nu_{\mathbf{n},\tilde{\mathbf{n}}} \|. \tag{55}$$

The index ν takes four different values in accordance with the number of possible spin–isospin states ζ_ν. In the matrix element $f^\nu_{\mathbf{n},\tilde{\mathbf{n}}}$, this index indicates to which of the four blocks of the matrix it belongs.

In each of the matrix elements $f^\nu_{\mathbf{n},\tilde{\mathbf{n}}}$, the integrand contains the exponential

$$\exp\ \{ - \sum_{r,s} B_{rs}\, x_r\, x_s \},$$

where x_1, x_2, x_3 are the components of the vector $\mathbf{r}$, and $\| B_{rs} \|$ is a symmetric 3×3 matrix whose elements can be regarded as the components of a third-rank tensor

$$B_{rs} = \delta_{rs} + \sum_\alpha C_\alpha p_{\alpha r} p_{\alpha s} + \sum_\alpha \tilde{C}_\alpha q_{\alpha r} q_{\alpha s}. \tag{56}$$

Here

$$C_\alpha = \frac{\beta_\alpha}{1-\beta_\alpha},\quad \tilde{C}_\alpha = \frac{\tilde{\beta}_\alpha}{1-\tilde{\beta}_\alpha}. \tag{57}$$

Let $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ be the components of the vector $\mathbf{r}$ in the coordinate system whose axes are directed along the principal axes of the tensor B_{rs}; then

$$\sum_{r,s} B_{rs} x_r x_s = \lambda_1 \tilde{x}_1^2 + \lambda_2 \tilde{x}_2^2 + \lambda_3 \tilde{x}_3^2, \tag{58}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the principal values of the tensor B_{rs}.

We go over to new variables $\xi = \{\xi_1, \xi_2, \xi_3\}$:

$$\xi_1 = \sqrt{\lambda_1}\, \tilde{x}_1,\quad \xi_2 = \sqrt{\lambda_2}\, \tilde{x}_2,\quad \xi_3 = \sqrt{\lambda_3}\, \tilde{x}_3, \tag{59}$$

$$d\tilde{x}_1\, d\tilde{x}_2\, d\tilde{x}_3 = (\det \| B_{rs} \|)^{-1/2}\, d\xi_1\, d\xi_2\, d\xi_3,$$

$$\det \| B_{rs} \| = \lambda_1 \lambda_2 \lambda_3.$$

[4] By definition $\tilde{\psi}(\tilde{\mathbf{n}},\tilde{\nu}|\mathbf{r}) = (\pi^{3/2} 2^{\tilde{n}} \tilde{\mathbf{n}}!)^{-1/2} \exp\{ -1/2\, r^2 - \Sigma_\alpha \tilde{\beta}_\alpha / 1 - \tilde{\beta}_\alpha \times (\mathbf{q}_\alpha \mathbf{r})^2 \} H_{n1}(\mathbf{v}_1 \mathbf{r}) H_{n2}(\mathbf{v}_2 \mathbf{r}) H_{n3}(\mathbf{v}_3 \mathbf{r}) \zeta_\nu$.

The first rule for filling the orbitals of the Slater determinant in conjunction with the known invariance of the determinant with respect to the addition to one of its rows of a linear superposition of the remaining rows makes it possible without changing Ψ, to retain instead of the product of Hermite polynomials $H_{n_1}(\mathbf{u}_1\mathbf{r})H_{n_2}(\mathbf{u}_2\mathbf{r})H_{n_3}(\mathbf{u}_3\mathbf{r})$ in the expression for the orbitals $\psi(\mathbf{n}|\mathbf{r})$ only the leading powers of the arguments of the Hermite polynomials, i.e.,

$$2^{n_1}(\mathbf{u}_1\mathbf{r})^{n_1}2^{n_2}(\mathbf{u}_2\mathbf{r})^{n_2}2^{n_3}(\mathbf{u}_3\mathbf{r})^{n_3}.$$

The same is true for the orbitals $\tilde{\psi}(\mathbf{n}|\mathbf{r})$.

We now transform to the new variables the scalar product of one of the vectors $\mathbf{u}_i$ and the vector $\mathbf{r}$:

$$(\mathbf{u}_i\mathbf{r}) = \sum_\nu u_{i\nu}x_\nu = \sum_\nu \tilde{u}_{i\nu}\tilde{x}_\nu = \sum_\alpha \frac{\tilde{u}_{i\alpha}}{\sqrt{\lambda_\alpha}}\xi_\alpha = (\mathbf{u}_i'\boldsymbol{\xi}), \tag{60}$$

$$\mathbf{u}_i' = \left\{\frac{\tilde{u}_{i1}}{\sqrt{\lambda_1}}, \frac{\tilde{u}_{i2}}{\sqrt{\lambda_2}}, \frac{\tilde{u}_{i3}}{\sqrt{\lambda_3}}\right\}; \quad |\mathbf{u}_i'| = u_i'.$$

A similar relation holds for $\mathbf{v}_j \cdot \mathbf{r}$:

$$(\mathbf{v}_j\mathbf{r}) = (\mathbf{v}_j'\boldsymbol{\xi}).$$

Instead of the vectors $\{\mathbf{u}_i'\}$, which, as is readily seen, are not orthogonal, we introduce three unit orthonormal vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$. The latter are related to $\mathbf{u}_1'$, $\mathbf{u}_2'$, $\mathbf{u}_3'$ by a triangular transformation matrix $\|\alpha_{ij}\|$:

$$\mathbf{u}_i' = \sum_j \alpha_{ij}\mathbf{a}_j, \quad \mathbf{a}_j = \sum_i \bar{\alpha}_{ij}\mathbf{u}_i', \tag{61}$$

where

$$\|\alpha_{ij}\| = \begin{Vmatrix} \alpha_{11} & & \\ \alpha_{21} & \alpha_{22} & \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{Vmatrix}; \quad \|\bar{\alpha}_{ij}\| = \begin{Vmatrix} \bar{\alpha}_{11} & \bar{\alpha}_{12} & \bar{\alpha}_{13} \\ & \bar{\alpha}_{22} & \bar{\alpha}_{23} \\ & & \bar{\alpha}_{33} \end{Vmatrix}. \tag{62}$$

The diagonal elements of the matrix $\|\alpha_{ij}\|$ and of its inverse $\|\bar{\alpha}_{ij}\|$ (it is only these elements that we shall need in the following calculations) are equal to

$$\left.\begin{aligned} &\alpha_{11} = |\mathbf{u}_1'|, \quad \alpha_{22} = \frac{|[\mathbf{u}_1'\mathbf{u}_2']|}{|\mathbf{u}_1'|}, \quad \alpha_{33} = \frac{|(\mathbf{u}_1'\mathbf{u}_2'\mathbf{u}_3')|}{|[\mathbf{u}_1'\mathbf{u}_2']|}; \\ &\bar{\alpha}_{11} = (\alpha_{11})^{-1}; \quad \bar{\alpha}_{22} = (\alpha_{22})^{-1}, \quad \bar{\alpha}_{33} = (\alpha_{33})^{-1}. \end{aligned}\right\} \tag{63}$$

The use of the triangular matrix means that the vector $\mathbf{a}_1$ is collinear with the vector $\mathbf{u}_1'$, while $\mathbf{a}_2$ lies in the plane passing through $\mathbf{u}_1'$ and $\mathbf{u}_2'$. Similarly, instead of the vectors $\mathbf{v}_1'$, $\mathbf{v}_2'$, $\mathbf{v}_3'$ we introduce the orthonormal frame $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$:

$$\mathbf{v}_i' = \sum_j \tilde{\alpha}_{ij}\mathbf{b}_j, \quad \mathbf{b}_j = \sum_i \bar{\tilde{\alpha}}_{ij}\mathbf{v}_i'. \tag{64}$$

By virtue of the second rule for filling the orbitals of the Slater determinant, we can, without changing the value of the determinant $\Psi([f_1f_2f_3])$, replace each of the products

$$(\mathbf{u}_2'\boldsymbol{\xi})^{n_1}(\mathbf{u}_2'\boldsymbol{\xi})^{n_2}(\mathbf{u}_3'\boldsymbol{\xi})^{n_3}$$

in the expression for the corresponding orbital ψ by

$$\alpha_{11}^{n_1}\alpha_{22}^{n_2}\alpha_{33}^{n_3}(\mathbf{a}_1\boldsymbol{\xi})^{n_1}(\mathbf{a}_2\boldsymbol{\xi})^{n_2}(\mathbf{a}_3\boldsymbol{\xi})^{n_3}.$$

A similar procedure can be carried out for the orbitals $\tilde{\psi}$ of the determinant $\tilde{\Psi}([f_1f_2f_3])$.

It is now convenient, in order to ensure that the orbitals ψ and $\tilde{\psi}$ with different $n = n_1+n_2+n_3$ and $\tilde{n} = \tilde{n}_1+\tilde{n}_2+\tilde{n}_3$ but $\nu = \tilde{\nu}$ are biorthogonal,[5] to return from the maximal degrees of the scalar products $\mathbf{a}_i \cdot \boldsymbol{\xi}$ and $\mathbf{b}_j \cdot \boldsymbol{\xi}$ to the Hermite polynomials, but now with the arguments $\mathbf{a}_i \cdot \boldsymbol{\xi}$ and $\mathbf{b}_j \cdot \boldsymbol{\xi}$.

It is then clear that the overlap integral of the determinants $\Psi([f_1f_2f_3])$ and $\tilde{\Psi}([f_1f_2f_3])$ can be expressed in terms of the overlap integral of $\Psi^a([f_1f_2f_3])$ and $\Psi^b([f_1f_2f_3])$, which differ from the vacuum determinant functions $\Psi^0([f_1f_2f_3])$ and $\tilde{\Psi}^0([f_1f_2f_3])$ only in that the unit orthogonal vectors $\mathbf{a}_i$ and $\mathbf{b}_j$ occur in them instead of the vectors $\mathbf{u}_i$ and $\mathbf{v}_j$, i.e.,

$$\begin{aligned} &\langle\Psi([f_1f_2f_3]) \mid \tilde{\Psi}([f_1f_2f_3])\rangle \\ &= \langle\Psi^a([f_1f_2f_3]) \mid \Psi^b([f_1f_2f_3])\rangle (\det\|B_{rs}\|)^{-\frac{A}{2}} \\ &\times (\alpha_{11}\tilde{\alpha}_{11})^{f_1}(\alpha_{22}\tilde{\alpha}_{22})^{f_2}(\alpha_{33}\tilde{\alpha}_{33})^{f_3}. \end{aligned} \tag{65}$$

The overlap integral of oscillator functions having definite $SU(3)$ symmetry was found by Elliott[35]:

$$\begin{aligned} &\langle\Psi^a([f_1f_2f_3]) \mid \Psi^b([f_1f_2f_3])\rangle \\ &= (\mathbf{a}_1\mathbf{b}_1)^{f_1-f_2}([\mathbf{a}_1\mathbf{a}_2][\mathbf{b}_1\mathbf{b}_2])^{f_2-f_3}\{(\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3)(\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3)\}^{f_3}. \end{aligned} \tag{66}$$

In solving below the problem of constructing the single-particle density matrix on the functions $\Psi^a([f_1f_2f_3])$ and $\Psi^b([f_1f_2f_3])$, we shall give a simple derivation of this expression. Here, we note that the GCSs of the group $Sp(6,R)$ can be reduced by means of rotation and dilatation operations to the GCSs of the corresponding irreducible representation of the group $SU(3)$.

Thus, the integration over all the single-particle coordinates needed to calculate the matrix element of the identity operator on the functions $\Psi([f_1f_2f_3])$ and $\tilde{\Psi}([f_1f_2f_3])$ has been performed. Taking into account the connection between the operators $\mathbf{u}_i$, $\mathbf{v}_j$ and $\mathbf{a}_i$, $\mathbf{b}_j$, the relations

$$\begin{aligned} &\mathbf{a}_1 = \alpha_{11}^{-1}\mathbf{u}_1', \quad \mathbf{b}_1 = \tilde{\alpha}_{11}^{-1}\mathbf{v}_1', \\ &[\mathbf{a}_1\mathbf{a}_2] = \alpha_{11}^{-1}\alpha_{22}^{-1}[\mathbf{u}_1'\mathbf{u}_2'], \quad [\mathbf{b}_1\mathbf{b}_2] = \tilde{\alpha}_{11}^{-1}\tilde{\alpha}_{22}^{-1}[\mathbf{v}_1'\mathbf{v}_2'], \\ &(\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3) = \alpha_{11}^{-1}\alpha_{22}^{-1}\alpha_{33}^{-1}(\mathbf{u}_1'\mathbf{u}_2'\mathbf{u}_3'); \quad (\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3) = \tilde{\alpha}_{11}^{-1}\tilde{\alpha}_{22}^{-1}\tilde{\alpha}_{33}^{-1}(\mathbf{v}_1'\mathbf{v}_2'\mathbf{v}_3'), \end{aligned}$$

and also (65) and (66), we obtain

$$\begin{aligned} &\langle\Psi([f_1f_2f_3]) \mid \tilde{\Psi}([f_1f_2f_3])\rangle \\ &= (\det\|B_{rs}\|)^{-A/2}(\mathbf{u}_1'\mathbf{v}_1')^{f_1-f_2} \\ &\times([\mathbf{u}_1'\mathbf{u}_2'][\mathbf{v}_1'\mathbf{v}_2'])^{f_2-f_3}\{(\mathbf{u}_1'\mathbf{u}_2'\mathbf{u}_3')(\mathbf{v}_1'\mathbf{v}_2'\mathbf{v}_3')\}^{f_3}. \end{aligned} \tag{67}$$

We must now return to the vectors $\mathbf{u}_i$ and $\mathbf{v}_j$. For this, we note that in the coordinate system in which the matrix B_{rs} is diagonal,

$$\begin{aligned} (\mathbf{u}_i'\mathbf{v}_j') &= \sum_\alpha \frac{u_{i\alpha}v_{j\alpha}}{\lambda_\alpha} \\ &= \frac{1}{\lambda_1\lambda_2\lambda_3}(u_{i1}v_{j1}\lambda_2\lambda_3 + u_{i2}v_{j2}\lambda_3\lambda_1 + u_{i3}v_{j3}\lambda_1\lambda_2). \end{aligned} \tag{68}$$

But the expression in the brackets on the right-hand side of the equation is the contraction of the matrix $\|C_{kl}\|$ formed from the cofactors of the elements of the matrix $\|B_{kl}\|$, with the vectors $\mathbf{u}_i$ and $\mathbf{v}_j$, and $\lambda_2\lambda_3$, $\lambda_3\lambda_1$, $\lambda_1\lambda_2$ are the principal

[5] We recall that two sets of functions $\{\psi_i\}$ and $\{\varphi_i\}$ are said to be biorthogonal if $\langle\psi_i|\varphi_j\rangle = 0$ when $i \neq j$.

values of the tensor C_{kl}. Therefore,

$$(\mathbf{u}_i' \mathbf{v}_j') = (\det \| B_{kl} \|)^{-1} \sum_{r,s} C_{rs} u_{ir} v_{js}. \quad (69)$$

The matrix $\|C_{kl}\|$ can be readily calculated. Its matrix elements can be expressed in terms of Kronecker deltas δ_{kl} and the components of the vectors $\mathbf{p}_\alpha$ and $\mathbf{q}_{\alpha'}$:

$$C_{kl} = \delta_{kl}\left(1+\sum_\alpha C_\alpha + \sum_\alpha \widetilde{C}_\alpha\right) - \sum_\alpha C_\alpha p_{\alpha k} p_{\alpha l} - \sum_\alpha \widetilde{C}_\alpha q_{\alpha k} q_{\alpha l} + \sum_{\nu<\mu} C_\nu C_\mu [\mathbf{p}_\nu \mathbf{p}_\mu]_k [\mathbf{p}_\nu \mathbf{p}_\mu]_l + \sum_{\nu<\mu} \widetilde{C}_\nu \widetilde{C}_\mu [\mathbf{q}_\nu \mathbf{q}_\mu]_k [\mathbf{q}_\nu \mathbf{q}_\mu]_l + \sum_{\nu,\mu} C_\nu \widetilde{C}_\mu [\mathbf{p}_\nu \mathbf{q}_\mu]_k [\mathbf{p}_\nu \mathbf{q}_\mu]_l. \quad (70)$$

The contraction of the tensor C_{rs} with the vectors $\mathbf{u}_1$ and $\mathbf{v}_1$ is

$$\sum_{r,s} C_{rs} u_{1r} v_{1s} = \prod_\nu [(1-\beta_\nu)(1-\widetilde{\beta}_\nu)]^{-1} \mathscr{A}, \quad (71)$$

where

$$\mathscr{A} = (\mathbf{R}^1\mathbf{S}^1) - \sum_{\lambda,\mu} \beta_\lambda \widetilde{\beta}_\mu (\mathbf{p}_\lambda \mathbf{q}_\mu)([\mathbf{p}_\lambda \mathbf{R}^1][\mathbf{q}_\mu \mathbf{S}^1)] + \sum_{\lambda,\mu} \alpha_\lambda \widetilde{\alpha}_\mu (\mathbf{p}_\lambda \mathbf{q}_\mu)(\mathbf{p}_\lambda \mathbf{R}^1)(\mathbf{q}_\mu \mathbf{S}^1), \quad (72)$$

$$\left.\begin{aligned} \alpha_1 &= \beta_2\beta_3, \quad \alpha_2 = \beta_3\beta_1, \quad \alpha_3 = \beta_1\beta_2, \\ \widetilde{\alpha}_1 &= \widetilde{\beta}_2\widetilde{\beta}_3, \quad \widetilde{\alpha}_2 = \widetilde{\beta}_3\widetilde{\beta}_1, \quad \widetilde{\alpha}_3 = \widetilde{\beta}_1\widetilde{\beta}_2. \end{aligned}\right\} \quad (73)$$

The vectors $\mathbf{R}^\nu$ are obtained from the vectors $\mathbf{u}_\nu$ by an appropriate stretching of their components in the coordinate system with axes along the unit vectors $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$. In this system,

$$\mathbf{R}^\nu = \{(\mathbf{R}^\nu \mathbf{p}_1), (\mathbf{R}^\nu \mathbf{p}_2), (\mathbf{R}^\nu \mathbf{p}_3)\}, \quad (\mathbf{R}^\nu \mathbf{p}_\lambda) = (1-\beta_\lambda)(\mathbf{u}^\nu \mathbf{p}_\lambda). \quad (74)$$

In turn, the components of the vectors $\mathbf{S}^\nu$ in the coordinate system associated with the frame $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$ are determined by the transformation

$$(\mathbf{S}^\nu \mathbf{q}_\mu) = (1-\beta_\mu)(\mathbf{v}_\nu \mathbf{q}_\mu). \quad (75)$$

To derive the expression (72), it is expedient to represent the vectors $\mathbf{u}_i$ and $\mathbf{v}_j$ in the form

$$\mathbf{u}_i = \sum_\nu (\mathbf{u}_i \mathbf{p}_\nu)\, \mathbf{p}_\nu, \quad \mathbf{v}_j = \sum_\nu (\mathbf{v}_j \mathbf{q}_\nu)\, \mathbf{q}_\nu \quad (76)$$

and to first find the contraction of the tensor C_{kl} with the vectors $\mathbf{p}_\lambda$ and $\mathbf{q}_\mu$ and only then form the contraction.

In the coordinate system in which the matrix B_{rs} is diagonal, the scalar product of the vectors $[\mathbf{u}_1' \times \mathbf{u}_2']$ and $[\mathbf{v}_1' \times \mathbf{v}_2']$ also has a simple form:

$$[(\mathbf{u}_1'\mathbf{u}_2')[\mathbf{v}_1'\mathbf{v}_2']) = \frac{[\mathbf{u}_1\mathbf{u}_2]_1 [\mathbf{v}_1\mathbf{v}_2]_1}{\lambda_2\lambda_3} + \frac{[\mathbf{u}_1\mathbf{u}_2]_2 [\mathbf{v}_1\mathbf{v}_2]_2}{\lambda_3\lambda_1} + \frac{[\mathbf{u}_1\mathbf{u}_2]_3 [\mathbf{v}_1\mathbf{v}_2]_3}{\lambda_1\lambda_2} = \frac{1}{\lambda_1\lambda_2\lambda_3} \sum_\nu [\mathbf{u}_1\mathbf{u}_2]_\nu [\mathbf{v}_1\mathbf{v}_2]_\nu \lambda_\nu = (\det \| B_{kl} \|)^{-1} \sum_{r,s} B_{rs} [\mathbf{u}_1\mathbf{u}_2]_r [\mathbf{v}_1\mathbf{v}_2]_s. \quad (77)$$

As in the calculation of the preceding contraction, we again first find

$$\sum_{r,s} B_{rs} p_{\lambda r} q_{\mu s},$$

and then, using the relations (76), we obtain the required expression

$$\sum_{r,s} B_{rs} [\mathbf{u}_1\mathbf{u}_2]_r [\mathbf{v}_1\mathbf{v}_2]_s = \prod_\nu [(1-\beta_\nu)(1-\widetilde{\beta}_\nu)]^{-1} \mathscr{B}, \quad (78)$$

where

$$\mathscr{B} = ([\mathbf{R}^1\mathbf{R}^2][\mathbf{S}^1\mathbf{S}^2]) - \sum_{\lambda,\mu} \beta_\lambda \widetilde{\beta}_\mu (\mathbf{p}_\lambda \mathbf{p}_\mu)(\mathbf{p}_\lambda [\mathbf{R}^1\mathbf{R}^2])(\mathbf{q}_\mu [\mathbf{S}^1\mathbf{S}^2]), \quad (79)$$

Finally, it is easy to show that

$$(\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3)(\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3) = \frac{\hat{\pi}}{\lambda_1\lambda_2\lambda_3} = (\det \| B_{rs} \|)^{-1} \hat{\pi}. \quad (80)$$

Here, $\hat{\pi} = 1$ if both triplets of vectors—$\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ and $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$—are right- or left-handed simultaneously, and $\hat{\pi} = -1$ if one is right- and the other is left-handed.

The determinant of the matrix $\| B_{kl} \|$ was found in Ref. 64:

$$\det \| B_{kl} \| = \prod_\nu [(1-\beta_\nu)(1-\widetilde{\beta}_\nu)]^{-1} \Delta, \quad (81)$$

$$\Delta = 1 - \sum_{r,s} (\beta_r\widetilde{\beta}_s - \alpha_r\widetilde{\alpha}_s)(\mathbf{p}_r\mathbf{q}_s)^2 - D\widetilde{D}, \quad (82)$$

$$D = \beta_1\beta_2\beta_3, \quad \widetilde{D} = \widetilde{\beta}_1\widetilde{\beta}_2\widetilde{\beta}_3.$$

Thus, we have calculated all the elements that make up the overlap integral (67), and we can now write down an expression for it, taking into account first the factor due to the center-of-mass motion:

$$\langle \psi_R \mid \widetilde{\psi}_R \rangle = \Delta^{-1/2}.$$

As a result, we find that the overlap integral of two GCSs of the irreducible representation $[\sigma_1\sigma_2\sigma_3]$ of $Sp(6,R)$ is

$$\langle \Phi([f_1f_2f_3]) \mid \widetilde{\Phi}([f_1f_2f_3]) \rangle = \frac{\mathscr{A}^{f_1-f_2}\mathscr{B}^{f_2-f_3}}{\Delta^{f_1+(A-1)/2}}. \quad (83)$$

The expression (83) generalizes the well-known[35] Elliott overlap integral of functions of a definite $SU(3)$ symmetry to the case of functions of a definite $Sp(6,R)$ symmetry. Each of the functions for which the overlap is calculated depends on nine generator parameters (if $f_1 \neq f_2 \neq f_3$) and is a superposition of all the basis functions of the irreducible representation $[\sigma_1\sigma_2\sigma_3]$, and any of the basis functions of this irreducible representation can be separated from the corresponding GCS by a simple projection operation.

The overlap integral (83) is helpful for investigating the quantum numbers of the basis functions, and also for calculating the coefficients of the linear superposition with which the normalized basis functions occur in the generating function.

We defer for the moment the discussion of the overlap integrals (83) as well as the question of calculating the coefficients of the superposition, and now consider the calculation of the generating matrix elements of the kinetic- and potential-energy operators of a system of nucleons.

To find the matrix element of the kinetic-energy operator $\hat{T}$, we use the relation

$$\langle \alpha \mid \hat{T} \mid \alpha' \rangle = \frac{\hbar^2}{2mr_0^2} \begin{cases} N + \frac{3}{2}(A-1), & \alpha = \alpha' \\ -\langle \alpha \mid \rho^2 \mid \alpha' \rangle, & \alpha \neq \alpha', \end{cases} \quad (84)$$

where α is the set of quantum numbers that characterize the basis function, N is the principal quantum number of the function $|\alpha[f_1f_2f_3]\rangle$, ρ^2 is the sum of the squares of the Jacobi coordinates, and r_0 is the oscillator radius. Thus, the calculation of $\langle\alpha|\hat{T}|\alpha'\rangle$ reduces to the calculation of the matrix element of the operator ρ^2. For this, it is convenient to use the auxiliary operator

$$\hat{\Gamma}=\exp\left(-\gamma\sum_{i=1}^{A}r_i^2\right)=\prod_{i=1}^{A}\exp\left(-\gamma r_i^2\right). \tag{85}$$

Since this operator can be represented in the form of the product

$$\hat{\Gamma}=\exp\left(-\gamma\rho^2\right)\exp\left(-\gamma q_A^2\right),$$

where $\mathbf{q}_A=\Sigma_{i=1}^{A}\mathbf{r}_i/\sqrt{A}$ is the normalized center-of-mass coordinate, and each of the determinants $\Psi([f_1f_2f_3])$ and $\tilde{\Psi}([f_1f_2f_3])$ factorizes after transition from the single-particle coordinates $\{\mathbf{r}_i\}$ to the Jacobi vectors $\{\mathbf{q}_i\}$ and takes the form of a product of the wave function of the relative motion of the nucleons $\Phi([f_1f_2f_3])$ (or $\tilde{\Phi}([f_1f_2f_3])$) and the center-of-mass wave function ψ_R (or $\tilde{\psi}_R$), it follows that

$$\begin{aligned}&\langle\Psi([f_1f_2f_3])\mid\hat{\Gamma}\mid\tilde{\Psi}([f_1f_2f_3])\rangle\\&=\langle\Phi([f_1f_2f_3])\mid e^{-\gamma\rho^2}\mid\tilde{\Phi}([f_1f_2f_3])\rangle\langle\psi_R\mid e^{-\gamma q_A^2}\mid\tilde{\psi}_R\rangle,\end{aligned}$$

this relation expressing the matrix element of the operator $\tilde{\Gamma}$ in terms of a factor that is readily calculated—the overlap integral of the center-of-mass wave functions with the operator $\exp(-\gamma q_A^2)$—and the matrix element of the operator $\exp(-\gamma\rho^2)$ on the GCSs. It is the latter that is of interest, since, by differentiating it n times with respect to the parameter γ and then setting γ equal to zero, we obtain the matrix element of the operator ρ^{2n} and, in particular, of the operator ρ^2, which, as was noted above, gives exhaustive information about the matrix elements of the kinetic-energy operator $\hat{T}$.

The matrix element of the operator $\exp(-\gamma\rho^2)$ on GCSs of the group $Sp(6,R)$ has the form

$$\begin{aligned}&\langle\Phi([f_1f_2f_3])\mid e^{-\gamma\rho^2}\mid\tilde{\Phi}([f_1f_2f_3])\rangle\\&=(1+\gamma)^{-f_1-f_2-f_3-\frac{3}{2}(A-1)}\frac{\mathscr{A}_\gamma^{f_1-f_2}\mathscr{B}_\gamma^{f_2-f_3}}{\Delta_\gamma^{f_1+(A-1)/2}},\end{aligned} \tag{86}$$

where

$$\begin{aligned}\mathscr{A}_\gamma&=\left(1-\frac{\gamma}{1+\gamma}P\right)\left(1-\frac{\gamma}{1+\gamma}\tilde{P}\right)(\mathbf{R}^1\mathbf{S}^1)\\&+\sum_\lambda\left[\frac{\gamma}{1+\gamma}\left(1-\frac{\gamma}{1+\gamma}\tilde{P}\right)\beta_\lambda+\frac{\gamma^2}{(1+\gamma)^2}\left(1+\frac{1-\gamma}{\gamma}\tilde{P}\right)\alpha_\lambda\right]\\&\times(\mathbf{p}_\lambda\mathbf{R}^1)(\mathbf{p}_\lambda\mathbf{S}^1)+\sum_\mu\left[\frac{\gamma}{1+\gamma}\left(1-\frac{\gamma}{1+\gamma}P\right)\tilde{\beta}_\mu\right.\\&\left.+\frac{\gamma^2}{(1+\gamma)^2}\left(1+\frac{1-\gamma}{\gamma}P\right)\tilde{\alpha}_\mu\right](\mathbf{q}_\mu\mathbf{R}^1)(\mathbf{q}_\mu\mathbf{S}^1)\\&+\sum_{\lambda,\mu}\frac{\gamma^2}{(1+\gamma)^2}\left(\beta_\lambda-\frac{1-\gamma}{\gamma}\alpha_\lambda\right)\left(\tilde{\beta}_\mu-\frac{1-\gamma}{\gamma}\tilde{\alpha}_\mu\right)\\&\times(\mathbf{p}_\lambda\mathbf{q}_\mu)(\mathbf{p}_\lambda\mathbf{R}^1)(\mathbf{q}_\mu\mathbf{S}^1)\\&-\frac{1}{(1+\gamma)^2}\sum_{\lambda,\mu}\beta_\lambda\tilde{\beta}_\mu(\mathbf{p}_\lambda\mathbf{q}_\mu)([\mathbf{p}_\lambda\mathbf{R}^1][\mathbf{q}_\mu\mathbf{S}^1]);\end{aligned} \tag{87}$$

$$\begin{aligned}\mathscr{B}_\gamma&=([\mathbf{R}^1\mathbf{R}^2][\mathbf{S}^1\mathbf{S}^2])-\frac{\gamma}{1+\gamma}\sum_\lambda\beta_\lambda(\mathbf{p}_\lambda[\mathbf{R}^1\mathbf{R}^2])(\mathbf{p}_\lambda[\mathbf{S}^1\mathbf{S}^2])\\&-\frac{\gamma}{1+\gamma}\sum_\mu\tilde{\beta}_\mu(\mathbf{q}_\mu[\mathbf{R}^1\mathbf{R}^2])(\mathbf{q}_\mu[\mathbf{S}^1\mathbf{S}^2])\\&-\frac{1-\gamma}{1+\gamma}\sum_{\lambda,\mu}\beta_\lambda\tilde{\beta}_\mu(\mathbf{p}_\lambda\mathbf{q}_\mu)(\mathbf{p}_\lambda[\mathbf{R}^1\mathbf{R}^2])(\mathbf{q}_\mu[\mathbf{S}^1\mathbf{S}^2]);\end{aligned} \tag{88}$$

$$\begin{aligned}\Delta_\gamma&=1-\frac{\gamma}{1+\gamma}(P+\tilde{P})+\frac{\gamma^2}{(1+\gamma)^2}(Q+\tilde{Q}+P\tilde{P})\\&+\frac{\gamma(1-\gamma)}{(1+\gamma)^2}(P\tilde{Q}+Q\tilde{P})-\frac{\gamma^3}{(1+\gamma)^3}(D+\tilde{D})\\&-\frac{\gamma^2(1-\gamma)}{(1+\gamma)^3}(Q\tilde{Q}+D\tilde{P}+P\tilde{D})\\&-\frac{\gamma(1-\gamma)^2}{(1+\gamma)^2}(D\tilde{Q}+Q\tilde{D})-\frac{(1-\gamma)^3}{(1+\gamma)^3}D\tilde{D}\\&-\sum_{\lambda,\mu}\left[\frac{1}{(1+\gamma)^2}\beta_\lambda\tilde{\beta}_\mu+\frac{\gamma}{(1+\gamma)^3}(\alpha_\lambda\tilde{\beta}_\mu+\beta_\lambda\tilde{\alpha}_\mu)\right.\\&\left.-\frac{1-\gamma}{(1+\gamma)^3}\alpha_\lambda\tilde{\alpha}_\mu\right](\mathbf{p}_\lambda\mathbf{q}_\mu)^2;\end{aligned} \tag{89}$$

$$P=\sum_\nu\beta_\nu,\quad\tilde{P}=\sum_\nu\tilde{\beta}_\nu,\quad Q=\sum_\nu\alpha_\nu,\quad\tilde{Q}=\sum_\nu\tilde{\alpha}_\nu.$$

The generating matrix element of the operator ρ^2 can be expressed in terms of the logarithmic derivatives with respect to γ of $\mathscr{A}_\gamma$, $\mathscr{B}_\gamma$ and Δ_γ:

$$\begin{aligned}&\langle\Phi([f_1f_2f_3])\mid\rho^2\mid\tilde{\Phi}([f_1f_2f_3])\rangle=\left[f_1+f_2+f_3+\frac{3}{2}(A-1)\right.\\&+(f_1-f_2)\frac{\partial}{\partial\gamma}\ln\mathscr{A}_\gamma|_{\gamma=0}+(f_2-f_3)\frac{\partial}{\partial\gamma}\ln\mathscr{B}_\gamma|_{\gamma=0}\\&\left.-\left(f_1+\frac{A-1}{2}\right)\frac{\partial}{\partial\gamma}\ln\Delta_\gamma|_{\gamma=0}\right]\langle\Phi([f_1f_2f_3])\mid\tilde{\Phi}([f_1f_2f_3])\rangle.\end{aligned} \tag{90}$$

Our next task is to find the matrix elements of the potential-energy operator

$$\hat{V}=\sum_{i<j}V_0\exp\left[-\frac{(\mathbf{r}_i-\mathbf{r}_j)^2}{s_0^2}\right] \tag{91}$$

of the nucleon–nucleon interaction for a two-body potential in Gaussian form. Note that once we have calculated this matrix element we can readily go over from it when necessary to the matrix element for the potential energy of the two-body potential $\mathscr{V}(\mathbf{r})$, represented in the form of an integral over s of $\exp(-r^2/s^2)$ with kernel $K(s)$:

$$\mathscr{V}(\mathbf{r})=\int_0^\infty K(s)\exp\left(-r^2/s^2\right)ds.$$

A large class of potentials admits such a representation.

Since the GCSs of the group $Sp(6,R)$ can, as was noted above, be reduced to the GCS $\Phi^0([f_1f_2f_3])$ of $SU(3)$, we concentrate all attention on calculating the matrix element

$$\langle\Phi^0([f_1f_2f_3])\mid\hat{V}\mid\Phi^0([f_1f_2f_3])\rangle.$$

(Note that the matrices that realize the transition from the GCS of the group $Sp(6,R)$ to the corresponding GCS of $SU(3)$ form the group $GL(3,R)$. These matrices can be represented as products of an orthogonal matrix and a diagonal matrix. The elements of the diagonal matrix are $\sqrt{\lambda_1}$, $\sqrt{\lambda_2}$, $\sqrt{\lambda_3}$, where $\lambda_1,\lambda_2,\lambda_3$ are the principal values of the matrix $\|B_{rs}\|$.)

In calculating the overlap integral of the vacuum func-

tions, we used Elliott's well-known result, and there was no need to investigate the matrix associated with this overlap integral, namely, $\mathscr{I}_{i,\tilde{i}}$, whose determinant is the overlap integral of the vacuum functions:

$$\left.\begin{aligned}&\langle\Psi^0([f_1f_2f_3])\mid\widetilde{\Psi}^0([f_1f_2f_3])\rangle=\det\|\mathscr{I}_{i,\tilde{i}}\|,\\&\mathscr{I}_{i,\tilde{i}}=g^{\nu}_{\mathbf{n},\tilde{\mathbf{n}}}\delta_{\nu\tilde{\nu}}=\langle\mathbf{n}\mid\tilde{\mathbf{n}}\rangle\,\delta_{\nu\tilde{\nu}},\\&i=\{\mathbf{n},\ \nu\},\quad\tilde{i}=\{\tilde{\mathbf{n}},\ \tilde{\nu}\}.\end{aligned}\right\}\tag{92}$$

At the same time, to construct the single-particle density matrix (i.e., the integral of the product of $\Psi^0([f_1f_2f_3])$ and $\widetilde{\Psi}^0([f_1f_2f_3])$ over all the single-particle vectors $\{\mathbf{r}_i\}$ except one), and with it the matrix elements $\langle\Psi^0([f_1f_2f_3])|\hat{V}|\widetilde{\Psi}^0([f_1f_2f_3])\rangle$, it is necessary in accordance with the results of Löwdin[69] to find first the cofactors $A_{i,\tilde{i}}$ of the elements of the matrix $\mathscr{I}_{i,\tilde{i}}$ (or, which is the same thing, the cofactors $a^{\nu}_{\mathbf{n},\tilde{\mathbf{n}}}$ of the matrix $g^{\nu}_{\mathbf{n},\tilde{\mathbf{n}}}$), in terms of which the single-particle density matrix is expressed.

The calculation of the cofactors is trivial for the elements of the diagonal matrix. But the matrix $g^{\nu}_{\mathbf{n},\tilde{\mathbf{n}}}$ is not, since the states $|\mathbf{n}\rangle$ and $|\tilde{\mathbf{n}}\rangle$ are not biorthogonal if they belong to one shell. The problem therefore arises of making them biorthogonal. The possibility of a solution of this problem by simple means follows from the multiplicative form of the determinant of the matrix $g^{\nu}_{\mathbf{n},\tilde{\mathbf{n}}}$. We now show how this problem is solved.

In accordance with the rules formulated above, the filling of the nth shell must begin with the orbital

$$|\,n,\ 0,\ 0\rangle=\frac{\exp(-r^2/2)}{\sqrt{2^n\mathbf{n}!\,\pi^{3/2}}}H_n(\mathbf{u}_1\mathbf{r})\equiv|\,(n,\ 0),\ n\rangle\,(\mathbf{u}_1\mathbf{v}_1)^{n/2}.$$

The next orbital among those taken into account has the form

$$|\,n-1,\ 1,\ 0\rangle=\frac{\exp(-r^2/2)}{\sqrt{2^n(n-1)!\,\pi^{3/2}}}H_{n-1}(\mathbf{u}_1\mathbf{r})\,H_1(\mathbf{u}_2\mathbf{r}).$$

It differs from the previous one in having one of its oscillator quanta directed along the vector $\mathbf{u}_2$. The states $|n-1,1,0\rangle$ and $|n,0,0\rangle$ are not orthogonal if $\mathbf{u}_1\neq\mathbf{v}_1$. However, the linear superposition

$$\begin{aligned}&(\mathbf{u}_1\mathbf{v}_1)\,|\,n-1,\ 1,\ 0\rangle-\sqrt{n}\,(\mathbf{u}_2\mathbf{v}_2)\,|\,n,\ 0,\ 0\rangle\\&\quad=(\mathbf{u}_1\mathbf{v}_1)^{\frac{n-2}{2}}(\mathbf{u}_3\mathbf{v}_3)^{1/2}\,|\,(n-2,\ 1),\ n\rangle\end{aligned}$$

is orthogonal with respect to the state $|\widetilde{(n,0),n}\rangle\sim|\widetilde{n,0,0}\rangle$, and the analogous linear superposition

$$\begin{aligned}&(\mathbf{u}_1\mathbf{v}_1)^{\frac{n-2}{2}}(\mathbf{u}_3v_3)^{\frac{1}{2}}\,|\,(n-2,\ 1),\ n\rangle\\&=(\mathbf{u}_1\mathbf{v}_1)\,|\,\widetilde{n-1,\ 1,\ 0}\rangle-\sqrt{n}(\mathbf{u}_2\mathbf{v}_1)\,|\,\widetilde{n,\ 0,\ 0}\rangle\end{aligned}$$

is orthogonal with respect to $|(n,0),n\rangle$, as is readily seen directly.

To make the expressions for the biorthogonal orbitals as simple as possible in the case when the oscillator quanta are distributed along the three mutually perpendicular directions $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ and to make the transformations made with them maximally simple, we write the orbital $|n_1,n_2,n_3\rangle$ in the form of the derivative of an exponential whose argument contains the vector parameter $\mathbf{t}=\{t_1,t_2,t_3\}$:

$$|\,n_1,\ n_2,\ n_3\rangle=D(\mathbf{n},\ \mathbf{t})\exp\Big\{-t^2+2\sum_{\alpha}t_\alpha(\mathbf{u}_\alpha\mathbf{r})-\frac{1}{2}r^2\Big\}\Big|_{t=0},\tag{93}$$

$$D(\mathbf{n},\ \mathbf{t})=(2^n\mathbf{n}!\,\pi^{3/2})^{-1/2}\frac{\partial^{n_1}}{\partial t_1^{n_1}}\frac{\partial^{n_2}}{\partial t_2^{n_2}}\frac{\partial^{n_3}}{\partial t_3^{n_3}}.$$

This expression is the well-known differential representation for the Hermite polynomials. We shall use a similar form of expression, but with the vector parameter $\mathbf{s}=\{s_1,s_2,s_3\}$, for the orbital $|\tilde{\mathbf{n}}\rangle$. We also introduce six new operators:

$$\begin{aligned}&\frac{\partial}{\partial\tau_1}=(\mathbf{u}_1\mathbf{v}_1)^{-1/2}\frac{\partial}{\partial t_1},\quad\frac{\partial}{\partial\sigma_1}=(\mathbf{u}_1\mathbf{v}_1)^{-\frac{1}{2}}\frac{\partial}{\partial s_1},\\&\frac{\partial}{\partial\tau_2}=[(\mathbf{u}_1\mathbf{v}_1)(\mathbf{u}_3\mathbf{v}_3)]^{-1/2}\begin{vmatrix}(\mathbf{u}_1\mathbf{v}_1)\frac{\partial}{\partial t_1}\\(\mathbf{u}_2\mathbf{v}_1)\frac{\partial}{\partial t_2}\end{vmatrix},\quad\frac{\partial}{\partial\sigma_2}\\&\qquad=[(\mathbf{u}_1\mathbf{v}_1)(\mathbf{u}_3\mathbf{v}_3)]^{1/2}\begin{vmatrix}(\mathbf{v}_1\mathbf{u}_1)\frac{\partial}{\partial s_1}\\(\mathbf{v}_2\mathbf{u}_1)\frac{\partial}{\partial s_2}\end{vmatrix},\\&\frac{\partial}{\partial\tau_3}=(\mathbf{u}_3\mathbf{v}_3)^{-1/2}\begin{vmatrix}(\mathbf{u}_1\mathbf{v}_1)(\mathbf{u}_1\mathbf{v}_2)\frac{\partial}{\partial t_1}\\(\mathbf{u}_2\mathbf{v}_1)(\mathbf{u}_2\mathbf{v}_2)\frac{\partial}{\partial t_2}\\(\mathbf{u}_3\mathbf{v}_1)(\mathbf{u}_3\mathbf{v}_2)\frac{\partial}{\partial t_3}\end{vmatrix},\quad\frac{\partial}{\partial\sigma_3}\\&\qquad=(\mathbf{u}_3\mathbf{v}_3)^{-1/2}\begin{vmatrix}(\mathbf{v}_1\mathbf{u}_1)(\mathbf{v}_1\mathbf{u}_2)\frac{\partial}{\partial s_1}\\(\mathbf{v}_2\mathbf{u}_1)(\mathbf{v}_2\mathbf{u}_2)\frac{\partial}{\partial s_2}\\(\mathbf{v}_3\mathbf{u}_1)(\mathbf{v}_3\mathbf{u}_2)\frac{\partial}{\partial s_3}\end{vmatrix}.\end{aligned}\tag{94}$$

The transition from the parameters $\{t_i\}$, $\{s_i\}$ to the parameters $\{\tau_i\}$, $\{\sigma_i\}$ is made by means of a simple triangular transformation:

$$\left.\begin{aligned}t_1&=(\mathbf{u}_1\mathbf{v}_1)^{-1/2}\tau_1-[(\mathbf{u}_1\mathbf{v}_1)(\mathbf{u}_3\mathbf{v}_3)]^{-1/2}(\mathbf{u}_2\mathbf{v}_1)\tau_2\\&\quad+(\mathbf{u}_3\mathbf{v}_3)^{-1/2}(\mathbf{u}_1\mathbf{v}_3)\tau_3,\\t_2&=[(\mathbf{u}_1\mathbf{v}_1)(\mathbf{u}_3\mathbf{v}_3)]^{-1/2}\tau_2+(\mathbf{u}_3\mathbf{v}_3)^{-1/2}(\mathbf{u}_2\mathbf{v}_3)\tau_3,\\t_3&=(\mathbf{u}_3\mathbf{v}_3)^{-1/2}(\mathbf{u}_3\mathbf{v}_3)\tau_3;\end{aligned}\right\}\tag{95a}$$

$$\left.\begin{aligned}s_1&=(\mathbf{u}_1\mathbf{v}_1)^{-1/2}\sigma_1-[(\mathbf{u}_1\mathbf{v}_1)(\mathbf{u}_3\mathbf{v}_3)]^{-1/2}(\mathbf{u}_1\mathbf{v}_2)\sigma_2\\&\quad+(\mathbf{u}_3\mathbf{v}_3)^{-1/2}(\mathbf{u}_3\mathbf{v}_1)\sigma_3;\\s_2&=[(\mathbf{u}_1\mathbf{v}_1)(\mathbf{u}_3\mathbf{v}_3)]^{-1/2}\sigma_2+(\mathbf{u}_3\mathbf{v}_3)^{-1/2}(\mathbf{u}_2\mathbf{v}_3)\sigma_3,\\s_3&=(\mathbf{u}_3\mathbf{v}_3)^{-1/2}(\mathbf{u}_3\mathbf{v}_3)\sigma_3.\end{aligned}\right\}\tag{95b}$$

We now have the possibility of reducing the expressions for the biorthogonal orbitals:

$$\begin{aligned}&|\,(n_1-n_2,\ n_2-n_3)\,n\rangle\\&=D(\mathbf{n},\ \tau)\exp\Big\{-t^2+2\sum_{\alpha}t_\alpha(\mathbf{u}_\alpha\mathbf{r})-\frac{1}{2}r^2\Big\}\Big|_{t=0},\end{aligned}\tag{96a}$$

$$\begin{aligned}&|\,(m_1-m_2,\ m_2-m_3)\,m\rangle\\&=D(\mathbf{m},\ \sigma)\exp\Big\{-s^2+2\sum_{\alpha}s_\alpha(\mathbf{v}_\alpha\mathbf{r})-\frac{1}{2}r^2\Big\}\Big|_{s=0}.\end{aligned}\tag{96b}$$

We shall show that the orbitals $|(n_1-n_2,\,n_2-n_3)n\rangle$ and $|m_1-m_2,\,m_2-m_3)m\rangle$ are orthogonal if $n_3\neq m_3$ or if

$n_3 = m_3$ but $n_2 \neq m_2$ or, finally, if $n_3 = m_3$, $n_2 = m_2$ but $n_1 \neq m_1$. We first note that

$$\int_{-\infty}^{\infty}\int\int d\mathbf{r}\exp\left\{-t^2-s^2+2\sum_\alpha t_\alpha(\mathbf{u}_\alpha\mathbf{r})+2\sum_\alpha s_\alpha(\mathbf{v}_\alpha\mathbf{r})-r^2\right\}$$
$$=\pi^{3/2}\exp\left\{2\sum_{\alpha,\beta}t_\alpha s_\beta(\mathbf{u}_\alpha\mathbf{v}_\beta)\right\}=\pi^{3/2}\exp\left\{2\sum_i\tau_i\sigma_i\right\}. \quad (97)$$

Substituting (97) in the overlap integral of the two orbitals, we find that

$$\langle(n_1-n_2,\ n_2-n_3)\ n\mid(m_1-m_2,\ m_2-m_3)\ m\rangle$$
$$=\delta_{n_1,m_1}\delta_{n_2,m_2}\delta_{n_3,m_3}.$$

The use of the biorthogonal orbitals (96a) and (96b) makes it possible to simplify greatly the expression for the single-particle density matrix $\rho(\lambda,\mu|1,2)$ constructed on the Slater determinants $\Phi^0([f_1f_2f_3])$ and $\tilde{\Phi}^0([f_1f_2f_3])$. It becomes additive not only with respect to the spin–isospin quantum numbers but also with respect to the quantum numbers $\mathbf{n}=\{n_1,n_2,n_3\}$ of the biorthogonal orbitals:

$$\rho(\lambda,\ \mu\mid 1,\ 2)=\left[\sum_\nu\rho_\nu(\mathbf{r}_1,\ \mathbf{r}_2)\,\zeta_\nu(1)\,\zeta_\nu(2)\right](\mathbf{u}_1\mathbf{v}_1)^\lambda(\mathbf{u}_3\mathbf{v}_3)^\mu, \quad (98)$$

$$\lambda=f_1-f_2,\quad \mu=f_2-f_3,$$

where

$$\rho_\nu(\mathbf{r}_1,\ \mathbf{r}_2)=\sum_{\mathbf{n}}D(\mathbf{n},\ \tau)\,D(\mathbf{n},\ \sigma)$$
$$\times\exp\left\{-t^2-s^2+2\sum_\alpha t_\alpha(\mathbf{u}_\alpha\mathbf{r}_1)\right.$$
$$\left.+2\sum_\alpha s_\alpha(\mathbf{v}_\alpha\mathbf{r}_2)-\frac{1}{2}(r_1^2+r_2^2)\right\}\Big|_{t=s=0}. \quad (99)$$

The generating matrix element of the potential-energy operator is determined by two main expressions—the direct and exchange integrals

$$\left.\begin{aligned}V^{\mathrm{dir}}_{\nu\tilde{\nu}}&=\int\int d\mathbf{r}_1\,d\mathbf{r}_2\rho_\nu(\mathbf{r}_1,\ \mathbf{r}_1)\,\rho_{\tilde{\nu}}(\mathbf{r}_2,\ \mathbf{r}_2)\,V_0\\&\times\exp\left[+\frac{(\mathbf{r}_1-\mathbf{r}_2)^2}{s_0^2}\right]\equiv V^+_{\nu\tilde{\nu}},\\V^{\mathrm{exch}}_{\nu\tilde{\nu}}&=\int\int d\mathbf{r}\,d\mathbf{r}_2\rho_\nu(\mathbf{r}_1,\ \mathbf{r}_2)\,\rho_{\tilde{\nu}}(\mathbf{r}_2,\ \mathbf{r}_1)\,V_0\\&\times\exp\left[-\frac{(\mathbf{r}_1-\mathbf{r}_2)^2}{s_0^2}\right]\equiv V^-_{\nu\tilde{\nu}}.\end{aligned}\right\} \quad (100)$$

It follows from (99) that both these expressions have a simple differential representation:

$$V^\pm_{\nu\tilde{\nu}}=z^{3/2}V_0\sum_{\mathbf{n},\mathbf{m}}D(\mathbf{n},\ \tau)\,D(\mathbf{n},\ \sigma)$$
$$\times D(\mathbf{m},\ \tilde{\tau})\,D(\mathbf{m},\ \tilde{\sigma})\,\pi^3\exp\left\{-(1-z)\left(\frac{\mathbf{t}-\tilde{\mathbf{t}}}{\sqrt{2}}\right)^2\right.$$
$$-(1-z)\left(\frac{s-\tilde{s}}{\sqrt{2}}\right)^2+\sum_{i,j}[(t_i+\tilde{t}_j)(s_i+\tilde{s}_j)$$
$$\left.\pm z(t_i-\tilde{t}_j)(s_i-\tilde{s}_j)](\mathbf{u}_i\mathbf{v}_j)\right\}\Big|_{\substack{t=s=0\\ \tilde{t}=\tilde{s}=0}}. \quad (101)$$

The parameters $\{\tilde{\tau}_i\tilde{\sigma}_i\}$ and $\{\tilde{t}_i,\tilde{s}_i\}$ are related by the same triangular relations (95a) and (95b) as the parameters $\{\tau_i,\sigma_i\}$ and $\{t_i,s_i\}$. In addition, $z=(1+2\,r_0^2/s_0^2)^{-1}$, where r_0 is the oscillator radius.

To make the transition from the matrix elements (100) of the potential-energy operator on the vacuum functions to the matrix elements on the generating functions $\Phi([f_1f_2f_3])$ and $\tilde{\Phi}([f_1f_2f_3])$ of the complete basis of the irreducible representation of $Sp(6,R)$, it is necessary in the expressions (100) to replace the vectors $\mathbf{u}_i$ and $\mathbf{v}_j$ by $\mathbf{a}_i$ and $\mathbf{b}_j$ [see the definitions (61)–(64), the comment on them, and Eq. (65)] and, in addition, to write the exponential $\exp(-r^2/s_0^2)$ of the potential-energy operator of the nucleon–nucleon interaction in accordance with the transformation (59) in the form

$$\exp\left\{-\frac{1}{s_0^2}\left(\frac{\xi_1^2}{\lambda_1}+\frac{\xi_2^2}{\lambda_2}+\frac{\xi_3^2}{\lambda_3}\right)\right\}.$$

Then, after calculation of the integral over the coordinates of the two nucleons, we must repeat in the reverse order all the operations described in detail above that ensure the return from the vectors $\mathbf{a}_i$ and $\mathbf{b}_i$ and the eigenvalues of the matrix $\|B_{kl}\|$ to the vectors $\mathbf{u}_i$ and $\mathbf{v}_j$ and the tensors b_{kl} and $\tilde{b}_{kl}$.

4. GENERALIZED COHERENT STATES OF CLUSTER SYSTEMS

During the last decade, a striking jump has occurred in the theory of nuclear reactions (at least those that involve light and even medium nuclei). It has become possible to calculate quite systematically (at the microscopic level with allowance for the many-particle kinematics and dynamics) processes such as the collision with one another of ^{16}O or ^{40}Ca nuclei or processes of inelastic scattering of nucleons, deuterons, and particles by p-shell nuclei with allowance for disintegration, excitation, or charge exchange of the latter. The calculations are made in the framework of the resonating-group method.

This method is essentially a microscopic variant of the coupled-channel method, and it is now successfully displacing not only the phenomenological variants of the coupled-channel method but also the optical model. Whereas earlier experiments on scattering of nucleons by nuclei or of nuclei by other nuclei were invariably interpreted by the optical model, the resonating-group method is used more and more frequently by those who have the possibility. And this transition from the optical model to the resonating-group model is necessary not only for basic reasons (instead of a phenomenological approach with parameters that must be fixed not only as functions of the scattered nuclei but also as functions of the energy at which the scattering takes place, one uses a microscopic approach that does not contain any arbitrary or free parameters) but also for practical reasons—using the resonating-group method, it is possible to explain the differential cross sections for scattering through large angles (which are associated with the effect of the Pauli principle) as well as the behavior of the scattering cross sections near different thresholds, and, finally, one can take into account resonances in various coupled channels, in particular closed ones.

The GCSs of cluster systems needed to solve problems in the resonating-group method are the ones that have been

most investigated among the GCSs listed in Sec. 1. In Ref. 16 there is a detailed exposition of the technique for calculating the matrix elements of single- and two-particle operators on the basis of the results of Löwdin.[69] In various studies[68,70–75] an algorithm has been developed for constructing GCSs in invariant form and calculating in invariant form the matrix elements of various operators on such GCSs.

The generalized coherent states for cluster systems are constructed from the single-particle orbitals

$$\psi(\mathbf{n},\ k;\ \nu \mid \mathbf{r}) = (2^n \mathbf{n}!\, \pi^{3/2})^{-1/2} \times H_{n_1}(\mathbf{u}_1,\ \mathbf{r}-\mathbf{R}_k)\, H_{n_2}(\mathbf{u}_2,\ \mathbf{r}-\mathbf{R}_k) \times H_{n_3}(\mathbf{u}_3,\ \mathbf{r}-\mathbf{R}_k) \exp\left\{-\frac{1}{2}r^2 + 2\mathbf{R}_k\mathbf{r} - R_k^2\right\} \zeta_\nu(i). \quad (102)$$

Here, $\mathbf{R}_k$ is the cluster parameter, which determines the position of cluster k. The determinant function constructed from these orbitals, $\Psi_k(A_k)$, where $k = 1,2,...$ (it describes the internal state of cluster k), can be represented as a product of the center-of-mass wave function

$$\psi_R^{(k)} = \exp\left\{-\frac{1}{2}\eta_k^2 + 2\sqrt{A_k}\mathbf{R}_k\eta_k - A_k R_k^2\right\}, \quad (103)$$

$$\eta_k = \frac{1}{\sqrt{A_k}} \sum_{i=1}^{A_k} \mathbf{r}_i$$

and the function $\varphi_k(A_k)$. The latter is a GCS of the irreducible representation $(\lambda_k\ \mu_k)$ of the group $SU(3)$. The generator parameters of this GCS are three mutually orthogonal unit vectors $\mathbf{u}_1^k, \mathbf{u}_2^k, \mathbf{u}_3^k$. The determinant function of the two-cluster system has, in its turn, the form

$$\Psi = \hat{A}\left\{\varphi_1(A_1)\,\varphi_2(A_2) \exp\left[-\frac{1}{2}\sum_{i=1}^{A-1} q_i^2 + 2\sqrt{\frac{A_1A_2}{A}}(\mathbf{R}_1-\mathbf{R}_2)\,\mathbf{q}_1 - \frac{A_1A_2}{A}(\mathbf{R}_1-\mathbf{R}_2)^2\right]\right\}\psi_R, \quad (104)$$

where ψ_R is the center-of-mass function:

$$\psi_R = \exp\left\{-\frac{1}{2}q_A^2 + \frac{2}{\sqrt{A}}(A_1\mathbf{R}_1 + A_2\mathbf{R}_2)\,\mathbf{q}_A - \frac{1}{A}(A_1\mathbf{R}_1 + A_2\mathbf{R}_2)^2\right\}. \quad (105)$$

The Jacobi vector $\mathbf{q}_1$, which describes the relative motion of the two clusters, and the center-of-mass vector $\mathbf{q}_A$ of the system of A nucleons are related to η_1 and η_2, the normalized Jacobi vectors of the centers of mass of the first and second cluster, by

$$\begin{pmatrix}\mathbf{q}_1\\ \mathbf{q}_A\end{pmatrix} = \begin{pmatrix}\sqrt{\frac{A_2}{A}} & -\sqrt{\frac{A_1}{A}}\\ \sqrt{\frac{A_1}{A}} & \sqrt{\frac{A_2}{A}}\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}. \quad (106)$$

Note that the function $\exp\{2\mathbf{q}\cdot\mathbf{R} - R^2\}$ is a generating function for the oscillator functions $|Nlm\rangle$ (N is the principal quantum number, l is the orbital angular momentum, and m is its projection):

$$\exp\{2\mathbf{q}\mathbf{R} - R^2\} = \sum_{Nlm} A_{Nl} R^N |Nlm\rangle\, Y_{lm}^*(\hat{\mathbf{R}}), \quad (107)$$

where

$$A_{Nl} = \left[\frac{4\pi}{2^N (N-l)!!\,(N+l+1)!!}\right]^{1/2},$$

where $\hat{\mathbf{R}} = \mathbf{R}/|\mathbf{R}|$ is the unit vector that specifies the direction of the vector $\mathbf{R}$. The projection operator $\hat{\mathcal{P}}_{Nlm}$, which is used to separate from (107) the oscillator functions $|Nlm\rangle$,

$$\hat{\mathcal{P}}_{Nlm} \exp\{2\mathbf{q}\mathbf{R} - R^2\} = A_{Nlm} \mid Nlm\rangle,$$

contains two operations, namely, differentiation with respect to the modulus of the vector $\mathbf{R}$ and integration with respect to the angular variables of the unit vector $\hat{\mathbf{R}}$:

$$\hat{\mathcal{P}}_{Nlm} = \int d\hat{\mathbf{R}} Y_{lm}(\hat{\mathbf{R}})\, \frac{1}{N!}\, \frac{d^N}{dR^N}. \quad (108)$$

The order in which these operations is performed is arbitrary.

When the matrix elements are calculated on the determinant functions Ψ, it is customary to introduce instead of the vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ the new vector

$$\mathbf{R} = \mathbf{R}_1 - \mathbf{R}_2,$$

and, in addition, to make $\mathbf{R}_1$ and $\mathbf{R}_2$ satisfy the condition

$$A_1\mathbf{R}_1 + A_2\mathbf{R}_2 = 0.$$

By virtue of this condition, the center-of-mass wave function takes the simple form

$$\psi_R = \exp\left(-\frac{1}{2}q_A^2\right).$$

The overlap integral of such functions is equal to unity (if the normalizing factor is chosen appropriately). However, to simplify the calculations it is much more convenient to take one of the clusters at the origin, i.e., to set, for example, $\mathbf{R}_1 = 0$. Then $\mathbf{R} = -\mathbf{R}_2$, and

$$\psi_R = \exp\left\{-\frac{1}{2}q_A^2 - \frac{2A_2}{\sqrt{A}}\mathbf{R}\mathbf{q}_A - \frac{A_2^2}{A}q_A^2\right\}. \quad (109)$$

The overlap integral of these functions is

$$\langle\psi_R|\tilde{\psi}_R\rangle = \exp\left\{\frac{2A_2^2}{A}(\mathbf{R}\tilde{\mathbf{R}})\right\}. \quad (110)$$

To construct the single-particle density matrix by means of which the matrix elements of the operators in which we are interested will be calculated, it is convenient to use the differential representation (93) for the Hermite polynomials. Using this representation to calculate the single-particle overlap integrals, we can then readily construct biorthogonal sets of single-particle functions in the same way as was done in the previous section.

For simplicity, we shall consider the wave functions of two interacting s nuclei, i.e., nuclei with numbers of nucleons A_1, $A_2 \leqslant 4$. However, many of the results given below are also valid in the more general case. The overlap integral for the positive-parity states is

$$\langle\Phi|\tilde{\Phi}\rangle = \sum_{r=0}^{i_0} D_r \operatorname{ch}(\mathcal{E}_r\mathbf{R}\tilde{\mathbf{R}}), \quad (111)$$

where

$$\mathcal{E}_r = 2A_1A_2/A - 2r. \quad (112)$$

If the nucleons of the second cluster (under the condition

that $A_2 \leqslant A_1$) are in the same spin–isospin states as the nucleons of the first, then

$$D_r = \frac{(-1)^r i_0!}{r!\,(i_0-r)!}, \quad i_0 = A_2. \tag{113}$$

For the negative-parity states, the overlap integral is

$$\langle \Phi | \tilde{\Phi} \rangle = \sum_{r=0}^{i_0} D_r \operatorname{sh}(\epsilon_r \mathbf{R}\tilde{\mathbf{R}}). \tag{A}$$

We now expand the overlap integrals in powers of $\mathbf{R}$ and $\tilde{\mathbf{R}}$:

$$\langle \Phi | \tilde{\Phi} \rangle = \sum_{n=0}^{\infty} \frac{B_n}{(2n+f)!} (\mathbf{R}\tilde{\mathbf{R}})^{2n+f} =$$

$$= \sum_{n=0}^{\infty} \frac{B_n}{(2n+f)!} (R\tilde{R})^{2n+f} \sum_{L} J_L^{2n+f} (Y_L(\hat{\mathbf{R}})\, Y_L(\hat{\tilde{\mathbf{R}}})). \tag{114}$$

Here

$$B_n = \sum_{r=0}^{i_0} D_r \epsilon_r^{2n+f}, \quad J_L^{\lambda} = \frac{4\pi\lambda!}{(\lambda-L)!!\,(\lambda+L+1)!!}, \tag{115}$$

and $(\hat{Y}_L(\mathbf{R}) Y_L(\hat{\mathbf{R}}))$ is the scalar product of two spherical functions. The effect of the Pauli principle on the wave functions (104) has the consequence that the expansion of the overlap integrals begins with a power f of the generator parameter R that is fixed for the given nucleus and parity. The parameter f, and also n uniquely determine the $SU(3)$ symmetry of the oscillator functions generated by the two-cluster function (104). If A_1, $A_2 \leqslant 4$, then

$$\lambda = 2n+f, \ \mu = 0. \tag{B}$$

Oscillator functions with $\lambda < 2n+f$ are forbidden by the Pauli principle.

We now calculate the matrix element of the kinetic-energy operator. As in the previous section, for this purpose we use the operator

$$\hat{F} = \exp(-\gamma\rho^2) = \exp\Big\{-\gamma \sum_{i=1}^{A-1} q_i^2\Big\}. \tag{A}$$

The matrix element of this operator is equal to the integral

$$\langle \Phi | e^{-\gamma\rho^2} | \tilde{\Phi} \rangle = \int d\mathbf{q}_1\, d\mathbf{q}_2 \ldots d\mathbf{q}_{A-1} \varphi_1(A_1)\, \varphi_2(A_2)$$

$$\times \hat{A}\Big\{\varphi_1(A_1)\, \varphi_2(A_2)$$

$$\times \exp\Big[-(\gamma+1)\sum_{i=1}^{A-1} q_i^2 + 2\sqrt{\frac{A_1A_2}{A}}\,(\mathbf{R}\mathbf{q}_1 + \tilde{\mathbf{R}}\mathbf{q}_1)$$

$$- \frac{A_1A_2}{A}(R^2+\tilde{R}^2)\Big]\Big\}. \tag{116}$$

Instead of the vectors $\mathbf{q}_i$, we introduce new vectors ξ_i such that

$$\xi_i = (\gamma+1)^{1/2}\mathbf{q}_i, \tag{117a}$$

and also new generator parameters

$$\mathbf{R}' = (\gamma+1)^{-1/2}\mathbf{R}, \ \tilde{\mathbf{R}}' = (\gamma+1)^{-1/2}\tilde{\mathbf{R}}. \tag{117b}$$

As a result of this substitution, the matrix element of the operator $\exp(-\gamma\rho^2)$ is

$$\langle \Phi | e^{-\gamma\rho^2} | \tilde{\Phi} \rangle = (\gamma+1)^{-\frac{3}{2}(A-1)-N_1-N_2}$$

$$\times \exp\Big\{-\frac{\gamma}{\gamma+1}\,\frac{A_1A_2}{A}(R^2+\tilde{R}^2)\Big\}\langle \Phi(\mathbf{R}') | \Phi(\tilde{\mathbf{R}}')\rangle. \tag{118}$$

In deriving this expression, we used the fact that the functions $\varphi_1(A_1)$ and $\varphi_2(A_2)$ are homogeneous in the Jacobi coordinates, their degrees of homogeneity being equal to the principal quantum numbers N_1 and N_2 that characterize these functions. For s nuclei, $N_1 = N_2 = 0$.

From (118), differentiating with respect to γ, we find the generating matrix element of the operator $\frac{1}{2}\rho^2$,

$$\langle \Phi(\mathbf{R}) | \tfrac{1}{2}\rho^2 | \Phi(\tilde{\mathbf{R}})\rangle = \frac{1}{2}\Big[N_1+N_2+\frac{3}{2}(A-1)$$

$$+ \frac{A_1A_2}{A}(R^2+\tilde{R}^2)$$

$$- \frac{\partial}{\partial\gamma}\Big]\langle \Phi(\mathbf{R}') | \Phi(\tilde{\mathbf{R}}')\rangle|_{\gamma=0}, \tag{119}$$

and then the matrix elements of this operator on the oscillator functions. The nonvanishing matrix elements $\langle n|\frac{1}{2}\rho^2|m\rangle$ are the ones for which $m = n$ and $m = n \pm 1$:

$$\langle n | \tfrac{1}{2}\rho^2 | n\rangle = \frac{1}{2}[2n+f+N_1+N_2+\tfrac{3}{2}(A-1)], \tag{120}$$

$$\langle n | \tfrac{1}{2}\rho^2 | n+1\rangle = \langle n+1 | \tfrac{1}{2}\rho^2 | n\rangle =$$

$$= \frac{A_1A_2}{2A}\sqrt{\frac{B_n}{B_{n+1}}}\,(2n+f-L+2)(2n+f+L+3). \tag{121}$$

For very large values of n, for which the Pauli principle can be ignored, the nondiagonal matrix elements of the operator $\frac{1}{2}\rho^2$ are equal to the matrix elements of r^2 on the three-dimensional oscillator functions:

$$\langle n | \tfrac{1}{2}\rho^2 | n+1\rangle \simeq \frac{1}{4}\sqrt{(2n+f-L+2)(2n+f+L+3)}. \tag{122}$$

The generating matrix elements of the potential-energy operator are superpositions of terms of the form[6]

$$\exp\{-C_1R^2 - C_2\tilde{R}^2\}\operatorname{ch}(C_3\mathbf{R}\tilde{\mathbf{R}}) \text{ or}$$

$$\exp\{-C_1R^2 - C_2\tilde{R}^2\}\operatorname{sh}(C_3\mathbf{R}\tilde{\mathbf{R}}). \tag{123}$$

The coefficients of such a superposition, and also C_1, C_2, and C_3 also contain the parameters of the nucleon–nucleon potential. For example, for potentials with a Gaussian radial dependence

$$C_3 = 2\frac{A_1A_2}{A} - 2r - 1 \pm z, \quad z = \Big(\frac{2r_0^2}{s_0^2}+1\Big)^{-1}, \tag{124}$$

where r_0 is the oscillator radius and s_0 is the range of the NN potential. The parameters C_1 and C_2 take one of the following three sets of values:

$$C_1 = C_2 = -\frac{1}{2}(1-z); \quad C_1 = -\frac{1}{2}(1-z),$$

$$C_2 = 0; \quad C_1 = 0, \quad C_2 = -\frac{1}{2}(1-z). \tag{125}$$

The expansion of the expressions in powers of the generator parameters (it can be readily obtained by using Leibniz's

[6] *Translator's Note.* The Russian notation for the trigonometric, inverse trigonometric, hyperbolic trigonometric functions, etc., is retained here and throughout the article in the displayed equations.

well-known rule for the derivative of the product of two functions) has the form

$$\exp\{-C_1R^2-C_2\tilde{R}^2\}\operatorname{ch}(C_3\mathbf{R}\tilde{\mathbf{R}})$$
$$=\sum_{m_1,\,m_2}^{\infty}(-1)^{m_1+m_2}R^{2m_1}\tilde{R}^{2m_2}\times\sum_{k=0}^{\bar{m}}(\mathbf{R}\tilde{\mathbf{R}})^{2k}\frac{C_1^{m_1-k}C_2^{m_2-k}C_3^{2k}}{(m_1-k)!\,(m_2-k)!\,(2k)!};\qquad(126)$$

$$\exp\{-C_1R^2-C_2\tilde{R}^2\}\operatorname{sh}(C_3\mathbf{R}\tilde{\mathbf{R}})$$
$$=\sum_{m_1,\,m_2}^{\infty}(-1)^{m_1+m_2}R^{2m_1+1}\tilde{R}^{2m_2+1}\times\sum_{k=0}^{\bar{m}}(\mathbf{R}\tilde{\mathbf{R}})^{2k+1}\frac{C_1^{m_1-k}C_2^{m_2-k}C_3^{2k+1}}{(m_1-k)!\,(m_2-k)!\,(2k+1)!},\qquad(127)$$

where $\bar{m}=\min\{m_1,m_2\}$. From this we find that the matrix elements of the potential-energy operator are

$$\langle m_1L|\hat{V}|m_2L\rangle=\sum_{k=0}^{\bar{m}}A^L_{m_1m_2}(k),\qquad(128)$$

$$A^L_{m_1m_2}(k)=(-1)^{m_1+m_2}\sqrt{\frac{(2m_1+f)!\,(2m_2+f)!}{B_{m_1}B_{m_2}}}\times\frac{J_L^{2k+f}}{\sqrt{J_L^{2m_1+f}J_L^{2m_2+f}}}\frac{C_1^{m_1-k}C_2^{m_2-k}C_3^{2k+f}}{(m_1-k)!\,(m_2-k)!\,(2k+f)!}.\qquad(129)$$

In addition, the diagonal matrix elements contain constants that determine the threshold energy for disintegration of the system.

In the limit when $m_1, m_2\gg 1$ the sum over k in (128) can be calculated analytically.

Suppose $m_1+m_2=2\nu$, $m_1-m_2=n$. We restrict ourselves to values of n that satisfy the inequality $n\ll\nu$, i.e., we shall consider the matrix elements near the principal diagonal. For simplicity, we set $f=L=0$. The generalization to the case of nonvanishing L will be given below.

We assume that in the sum (128) there exists a maximal term $A_{m_1m_2}(k)$ with $k=\alpha\nu$, where $\alpha<1$, and that the main contribution to this sum is made by the terms for which k satisfies the inequality $|\alpha\nu-k|\ll\alpha\nu$. The following calculations confirm these assumptions.

We find the values of α from the condition of a maximum of $A_{m_1m_2}(k)$. For this purpose, we represent k in the form $k=\alpha\nu+p$, where $|p|\ll\alpha\nu$, and we then write down expressions for $\ln A_{m_1m_2}(k)$, restricting ourselves to the leading terms at large values of ν:

$$\ln A_{m_1m_2}(k)\approx 2\left[(1-\alpha)\ln\frac{2\sqrt{\bar{C}_1\bar{C}_2}}{1-\alpha}+\alpha\ln\frac{\bar{C}_3}{\alpha}\right]\nu-\ln[2\pi\alpha^{3/2}(1-\alpha)\,\nu]-2p\left[\ln\frac{2\alpha}{1-\alpha}+\frac{\sqrt{\bar{C}_1\bar{C}_2}}{\bar{C}_3}\right]-\frac{1}{\alpha(1-\alpha)}\frac{p^2}{\nu}-\frac{1}{4}\frac{1}{1-\alpha}\frac{n^2}{\nu}+\ldots,\qquad(130)$$

where

$$\bar{C}_i=C_iA/2A_1A_2.\qquad(131)$$

In (130), we have omitted the terms of order p/ν, n/ν, $1/\nu$ and the small terms of higher order in the reciprocal powers of ν. We obtain the equation for α by setting equal to zero the derivative with respect to α of the terms linear in ν on the right-hand side of Eq. (130):

$$\frac{2\sqrt{\bar{C}_1\bar{C}_2}}{\bar{C}_3}\frac{1-\alpha}{\alpha}=1.\qquad(132)$$

It follows from (132) that

$$\alpha=\frac{\bar{C}_3}{2\sqrt{\bar{C}_1\bar{C}_2}+\bar{C}_3},\quad 1-\alpha=\frac{2\sqrt{\bar{C}_1\bar{C}_2}}{2\sqrt{\bar{C}_1\bar{C}_2}+\bar{C}_3}.\qquad(133)$$

Returning to $A_{m_1m_2}(k)$, we write them in an asymptotic form valid at large values of m_1 and m_2:

$$A_{m_1m_2}(k)\simeq[\pi(1-\alpha)\,\alpha^{3/2}]^{-1}\frac{(2\sqrt{\bar{C}_1\bar{C}_2}+\bar{C}_3)^{m_1+m_2}}{m_1+m_2}\times\exp\left\{-\frac{2}{\alpha(1-\alpha)}\frac{p^2}{m_1+m_2}-\frac{(m_1-m_2)^2}{2(1-\alpha)(m_1+m_2)}\right\}.\qquad(134)$$

Thus, our assumptions have been confirmed—the coefficients $A_{m_1m_2}(k)$ do indeed have a maximum at $k=\alpha\nu$ and decrease exponentially with increasing $p^2=(k-\alpha\nu)^2$ for fixed $\nu=\frac{1}{2}(m_1+m_2)$. Replacing the sum over k in (128) by a Poisson integral, we readily obtain the result

$$\langle m_1,\ L\,|\hat{V}|\,m_2,\ L\rangle\cong\frac{(-1)^{m_1+m_2}}{\sqrt{2\pi(1-\alpha)\,\alpha^2}}\frac{(2\sqrt{\bar{C}_1\bar{C}_2}+\bar{C}_3)^{m_1+m_2}}{\sqrt{m_1+m_2}}\times\exp\left\{-\frac{1}{2(1-\alpha)}\frac{(m_1-m_2)^2}{m_1+m_2}\right\}.\qquad(135)$$

This expression is also valid when $L\ll m_1,m_2$.

In all cases $2\sqrt{\bar{C}_1\bar{C}_2}+\bar{C}_3$ does not exceed unity, and therefore the calculated sum (128) either decreases slowly as $1/\sqrt{m}$, if $m_1=m_2=m$, $m\to\infty$, and $2\sqrt{\bar{C}_1\bar{C}_2}+\bar{C}_3=1$, or decreases as

$$\exp\{-2m\,|\ln(2\sqrt{\bar{C}_1\bar{C}_2}+\bar{C}_3)|\},\qquad(136)$$

if $2\sqrt{\bar{C}_1\bar{C}_2}+\bar{C}_3<1$.

We investigate the asymptotic behavior of the amplitudes C_n. We recall that in the oscillator representation the Schrödinger equation has the form

$$\sum_{m_2}(\langle m_1|\hat{H}|m_2\rangle-E\delta_{m_1m_2})\,C_{m_2}=0,\ m_1=0,\ 1,\ 2\ldots,\qquad(137)$$

where

$$\langle m_1|\hat{H}|m_2\rangle=\langle m_1|\hat{T}|m_2\rangle+\langle m_1|\hat{V}|m_2\rangle.\qquad(138)$$

Usually, the solution of the Schrödinger equation in the coordinate representation is begun with an investigation of the asymptotic behavior, i.e., the behavior of the wave function at large values of the radius (and also in the neighborhood of other singular points, if there are any). Having established the asymptotic behavior of the wave function, we use it essentially in the choice and implementation of practically any scheme of calculation. However, as a rule such study of the asymptotic behavior of the amplitudes C_n of the expansion with respect to a discrete basis is not made by assuming that the method of gradual extension of the discrete basis automatically eliminates all the problems associated with the asymptotic behavior of the amplitudes C_n as $n\to\infty$. And

even in the cases when rapid convergence of the considered values to the limiting value corresponding to an infinitely large number of basis functions is not ensured, the poor convergence is, as a rule, attributed to the properties of the basis, though, as will be seen below, it may have a different origin. This is that in such a scheme the correct asymptotic behavior of the resulting solutions is not taken into account.

To investigate the asymptotic behavior of C_n, we consider Eq. (137) for $m_1 \gg 1$. In this case, the sum

$$\sum_{m_2} \langle m_1|\hat{V}|m_2\rangle C_{m_2} \tag{139}$$

takes the form of a linear superposition of the expressions

$$\sum (m_1) = \sum_{m_2} \frac{(-1)^{m_1+m_2}}{\sqrt{m_1+m_2}} a \exp\left\{-b\frac{(m_1-m_2)^2}{m_1+m_2}\right\} \tag{140}$$

[the constants a and b can be readily determined by comparing this expression with (135)]. The remaining terms in (139), being exponentially small, can be omitted. It can be shown that the expression (140) also gives a vanishing contribution to (139) and, therefore, to (137) to the accuracy with which the sum over m_2 in (139) can be replaced by an integral. For this, we write out separately in (140) the sums over the even and odd values of m_2, and we then go to the limit of an integral:

$$\begin{aligned}\sum (m_1) &\cong \frac{a}{\sqrt{2m_1}} \sum_p \left\{\exp\left[-b\frac{(2p)^2}{2m_1}\right] C_{m_1+2p} \right.\\ &\left. - \exp\left[-b\frac{(2p-1)^2}{2m_1}\right] C_{m_1+2p-1}\right\}\\ &\cong \frac{a}{\sqrt{2m_1}} \int_{-\infty}^{\infty} dp \frac{d}{dp}\left\{\exp\left[-b\frac{p^2}{m_1}\right] C_{m_1+2p}\right\} \cong 0. \end{aligned} \tag{141}$$

Thus, in the equations in (137) corresponding to large values of m_1 we can omit the terms involving the matrix elements of the potential-energy operator. Therefore, the equations of the system (137) in the limit of large values of m_1 take the simple form

$$\begin{aligned}&\langle m_1|\hat{T}|m_1-1\rangle C_{m_1-1} + (\langle m_1|\hat{T}|m_1\rangle - E) C_{m_1}\\ &\quad + \langle m_1|\hat{T}|m_1+1\rangle C_{m_1+1} = 0. \end{aligned} \tag{142}$$

These equations are equivalent to the differential equation

$$-\frac{d}{dx} x \frac{d}{dx} C + \left[\frac{1}{4}\frac{(L+1/2)^2}{x} - \mathscr{E}\right] C = 0, \tag{143}$$

where

$$C_n \to C(x), \quad x = 2n + L + 3/2, \quad \mathscr{E} = \frac{mr_0^2}{\hbar^2} E. \tag{144}$$

The solutions of this equation for bound states ($\mathscr{E} < 0$), can be expressed in terms of Macdonald functions[76]:

$$C_{nL} \cong \sqrt[4]{x} K_L(\varkappa\sqrt{x}), \quad \varkappa = \sqrt{2|\mathscr{E}|}, \tag{145}$$

which have a damped exponential asymptotic behavior $\exp\{-\varkappa\sqrt{x}\}$, and for the continuum states ($\mathscr{E} > 0$)

$$\begin{aligned}C_{nL} &\cong \sqrt[4]{x}\,[j_L(k\sqrt{2x}) - \operatorname{tg}\delta_L n_L(k\sqrt{2x})],\\ k &= \sqrt{2\mathscr{E}}. \end{aligned} \tag{146}$$

Here, $j_L(x)$ and $n_L(x)$ are spherical Bessel and Neumann functions, respectively. The last relation establishes the connection between the coefficients C_{nL} and the scattering data—the phase shifts δ_L. One can relax the conditions on the value of n and write the relationship between C_{nL} and the phase shifts δ_L in the form

$$C_{nL} = A_{nL}(k) - \operatorname{tg}\delta_L B_{nL}(k), \tag{147}$$

extending thereby the region of applicability of the limiting expression for C_{nL}. The functions A and B of the discrete variable are the expansion coefficients of the regular and irregular solutions of the Schrödinger equation, respectively, for the free motion, and they are also the regular and irregular solutions of the linear three-term recursion relations (142), in which $C_m = A_{mL}$ or $C_m = B_{mL}$. If $4n \gg (kr_0)^2$, then (147) goes over into (146).

The relation (146) is the basis of the algebraic version of the resonating-group method developed at the Institute of Theoretical Physics of the Ukrainian Academy of Sciences (see Refs. 30, 31, 68, 77, and 78).

5. GENERALIZED COHERENT STATES FOR RESONANCE STATES

To construct GCSs for resonance states, it is necessary to take the orbitals

$$\begin{aligned}\psi(\mathbf{n}, k; \mathbf{v}|\mathbf{r}) &= (2^n \mathbf{n}! \pi^{3/2})^{-1/2} H_{n_1}(z_1) H_{n_2}(z_2) H_{n_3}(z_3)\\ &\times \exp\left\{-\frac{1}{2}r^2 - \sum_\alpha \frac{\beta_\alpha}{1-\beta_\alpha}(\mathbf{p}_\alpha \mathbf{r})^2\right.\\ &\left. + 2\sum_\alpha \frac{(\mathbf{R}_k \mathbf{p}_\alpha)(\mathbf{p}_\alpha \mathbf{r})}{1-\beta_\alpha} - \sum_\alpha \frac{(\mathbf{R}_k \mathbf{p}_\alpha)^2}{1-\beta_\alpha}\right\} \zeta_{\mathbf{v}},\\ z_i &= (\mathbf{u}_i^k, \ \mathbf{r} - \mathbf{R}_k), \end{aligned} \tag{148}$$

which besides the cluster parameters $\mathbf{R}_k$ contain the deformation parameters β_1, β_2, β_3. In some limiting cases, the new orbitals (148) are equal to the orbitals (52) and (102) used earlier to construct the GCSs of cluster systems and of collective excitations. Thus, setting $\beta_\alpha = 0$ ($\alpha = 1,2,3$) in (148) and bearing in mind that

$$\sum_\alpha p_{\alpha r} p_{\alpha s} = \delta_{rs}, \tag{149}$$

we obtain the cluster orbitals (102). And if in (148) we let the cluster parameter $\mathbf{R}_k$ tend to zero, then we arrive at the orbitals (52). These limiting processes relate the resonance GCSs to the cluster and collective GCSs.

Besides the limit $\beta_\alpha = 0$, the resonance GCSs can be reduced to cluster GCSs by means of transformations of rotation and dilatation, which were described in detail in Sec. 3. To see this, we consider the overlap integral of the resonance GCSs $\Phi(R, \beta)$ and $\Phi(\tilde{R}, \tilde{\beta})$. By analogy with (104), the overlap integral of the resonance GCSs of a two-cluster system is

$$\begin{aligned}\langle \Phi(R, \beta)\, \Phi(\tilde{R}, \tilde{\beta})\rangle &= \int d\mathbf{r}_1 d\mathbf{r}_2 \ldots d\mathbf{r}_{A-1} \varphi_1(A_1) \varphi_2(A_2)\\ &\times \hat{A}\left\{\tilde{\varphi}_1(A_1)\tilde{\varphi}_2(A_2) \exp\left[-\sum_{i=1}^{A-1}\sum_{r,s} B_{rs} x_{ir} x_{is}\right.\right.\\ &+ 2\sum_\alpha \frac{(\mathbf{R}\mathbf{p}_\alpha)(\mathbf{p}_\alpha \mathbf{r}_1)}{1-\beta_\alpha}\\ &\left.\left. + 2\sum_\alpha \frac{(\mathbf{S}\mathbf{q}_\alpha)(\mathbf{q}_\alpha \mathbf{r}_1)}{1-\tilde{\beta}_\alpha} - \sum_\alpha \frac{(\mathbf{R}\mathbf{p}_\alpha)^2}{1-\beta_\alpha} - \sum_\alpha \frac{(\mathbf{S}\mathbf{q}_\alpha)^2}{1-\beta_\alpha}\right]\right\}, \end{aligned} \tag{150}$$

where $\mathbf{R} = \sqrt{A_1 A_2/A}\ (\mathbf{R}_1 - \mathbf{R}_2)$, the quadratic form B_{rs} is determined by Eq. (56), and each of the functions φ_k ($\widetilde{\varphi}_k$, $k = 1,2$) belongs to the $O(A_k - 1)$ symmetry $[f_1^k f_2^k f_3^k]$ and contains the set of generator parameters $\{\mathbf{u}_\alpha^k\}$ (respectively, $\{\mathbf{v}_\alpha^k\}$). As in Sec. 3, we first introduce new variables $\xi_{iv} = \Sigma_\mu g_{v\mu} x_{i\mu}$, in which the quadratic form $\Sigma_{r,s} B_{rs} x_{ir} x_{is}$ takes the simple form

$$\sum_{r,\,s} B_{rs} x_{ir} x_{is} = \xi_i^2,\ i = 1, 2, \ldots, A - 1.$$

Then, by means of the triangular transformation (61) we go over to the new generator parameters $\{\mathbf{a}_\alpha^k\}$ and $\{\mathbf{b}_\alpha^k\}$. In addition, we redefine the cluster parameters:

$$R'_\nu = \sum_{\alpha\mu} \frac{\mathbf{R}\mathbf{p}_\alpha}{1-\beta_\alpha}\, p_{\alpha\mu} g^-_{\mu\nu},$$
$$\widetilde{R}'_\nu = \sum_{\alpha\mu} \frac{\mathbf{S}\mathbf{q}_\alpha}{1-\widetilde{\beta}_\alpha}\, q_{\alpha\mu} g^-_{\mu\nu}. \tag{151}$$

Here, $g^-_{\nu\mu}$ are the elements of the matrix $\|g^-_{\nu\mu}\|$, the inverse of $\|g_{\nu\mu}\|$. After such transformations, the overlap integral becomes

$$\langle \Phi(R,\ \beta) | \Phi(\widetilde{R},\ \widetilde{\beta})\rangle = F(R,\ \beta;\ \widetilde{R},\ \widetilde{\beta})$$
$$\times \int d\xi_1\, d\xi_2 \ldots d\xi_{A-1} \varphi_1(A_1)\, \varphi_2(A_2)$$
$$\times \widetilde{A}\Big\{\widetilde{\varphi}_1(A_1)\, \widetilde{\varphi}_2(A_2) \exp\Big[-\sum_{i=1}^{A-1} \xi_i^2 + 2(\mathbf{R}' + \widetilde{\mathbf{R}}')\,\xi_1$$
$$- (R'^2 + \widetilde{R}'^2)\Big]\Big\}. \tag{152}$$

The $(3A - 3)$-dimensional integral in this relation is equal to the overlap integral of the cluster GCSs $\langle \Phi(R')|\Phi(\widetilde{\mathbf{R}}')\rangle$, and the function $F(R, \beta; \widetilde{R}, \widetilde{\beta})$ is equal to

$$F(R,\ \beta;\ \widetilde{R},\ \widetilde{\beta}) = (\det \|B_{rs}\|)^{-(A-1)/2}$$
$$\times \prod_{k=1}^{2} (\alpha_{11}^k \widetilde{\alpha}_{11}^k)^{j_1^k} (\alpha_{22}^k \widetilde{\alpha}_{22}^k)^{j_2^k} (\alpha_{33}^k \widetilde{\alpha}_{33}^k)^{j_3^k}$$
$$\times \exp\Big\{-\sum_\alpha \Big[\frac{(\mathbf{R}\mathbf{p}_\alpha)^2}{1-\beta_\alpha} - \frac{(\mathbf{S}\mathbf{q}_\alpha)^2}{1-\widetilde{\beta}_\alpha}\Big] + R'^2 + \widetilde{R}'^2\Big\}, \tag{153}$$

where $\mathbf{R}'$ and $\mathbf{R}'$ are determined by Eq. (151), and the parameters α_{11}^k, α_{22}^k, α_{33}^k are determined for each value $k = 1,2$ by Eq. (63). Thus, the rotation and dilatation transformations reduce the resonance GCSs to cluster GCSs. To go over from the matrix elements of the operator $\hat{F}$ calculated on the cluster GCSs to the matrix elements of this operator on the resonance GCSs, it is necessary to make the reverse transition from the parameters $\{\mathbf{a}_\alpha^k\}$ and $\{\mathbf{b}_\alpha^k\}$ to the parameters $\{\mathbf{u}_\alpha^k\}$ and $\{\mathbf{v}_\alpha^k\}$, replace the operators $\mathbf{R}'$ and $\widetilde{\mathbf{R}}'$ by the original cluster parameters $\mathbf{R}$ and $\widetilde{\mathbf{R}}$ (151), and then multiply the resulting expression by the function $F(R, \beta, \widetilde{R}, \widetilde{\beta})$.

Resonance GCSs for a system of two interacting clusters, one an α particle and the other formed by k nucleons ($k \leqslant 4$, $A = 4 + k$), were constructed for the first time in Ref. 79, in which a simplified variant of the orbitals (148) was used ($\beta_1 = \beta$, $\tilde{\beta}_1 = \tilde{\beta}$, $\beta_2 = \beta_3 = \tilde{\beta}_2 = \tilde{\beta}_3 = 0$; $\mathbf{p}_1 \| \mathbf{R}, \mathbf{q}_1 \| \mathbf{S}$):

$$\exp\Big\{-\frac{1}{2} r^2 - \frac{\beta}{1-\beta}(\mathbf{pr})^2 + 2\frac{(\mathbf{Rr})}{1-\beta} - \frac{R^2}{1-\beta}\Big\}.$$

By means of these GCSs a calculation was made in Refs. 59, 61, and 80 of the spectrum of collective excitations of the lightest p-shell nuclei, and in Refs. 30 and 78 a study was made of the bound and continuum states of the nuclei ^{7}Li and ^{8}Be in the $\alpha + t$ and $\alpha + \alpha$ channels, respectively.

CONCLUSIONS

The technique of generalized coherent states is thus a very powerful tool for investigating problems of nuclear structure. The technique is currently being developed strongly, and it is to be hoped that in the near future it will provide the basis for the solution of new and more complicated problems that cannot be attacked by other means.

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