Quark configurations in the lightest nuclei

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The *NN* interaction in the region of the usually discussed repulsive core is treated from the point of view of the quark model as a many-particle phenomenon that cannot be reduced to an *NN* collision. Various approximations used in quark calculations of *NN* forces are analyzed by means of a generalization of an algebraic technique developed earlier to study nucleon associations (clusters) in nuclei (Casimir operators, coefficients of fractional parentage, formalism of the group of permutations). It is shown that knowledge of the operator structure of the quark Hamiltonian makes it possible to draw important physical conclusions about the nature of the *NN* interaction. Under the assumption that the color magnetic interaction in quark systems is fairly strong (for example, capable of explaining the N–Δ splitting), a number of interesting consequences are obtained for the few-nucleon correlations in the lightest nuclei (d, t, α) namely, there is a color magnetic attraction of nucleons in the region of the core, the unusual quark configurations (s^4p^2, s^5p^4, s^6p^6, etc.) are distinguished in energy from the symmetric states (s^6, s^9, s^{12}), and the correlations are rigid because of the node structure of the wave functions (and not an infinite repulsive core).

INTRODUCTION

The properties of few-nucleon systems and the *NN* interaction in the region of comparatively short (on nuclear scales) distances of the order of the phenomenological core radius $r_c \approx 0.5$–0.7 F and the hadron radius r_h, which has a similar value, are currently being actively reconsidered from the point of view of the quark model. The idea of a short-range repulsion[1] in the *NN* system and the concept of the one-boson exchange potential (OBEP)[2] are no longer regarded as indisputable, although these concepts did guide the first very direct attempts to justify *NN* repulsion on the basis of quark–quark forces.[3–9]

The difficulty here is that an exact realization of a quark approach to the *NN* interaction is as yet impossible, since in the region $r \approx r_c, r_h$ the gluon and quark condensate makes a decisive contribution,[10] and QCD perturbation theory is invalid. Nevertheless, we already know enough to attempt to solve a number of problems associated, for example, with the symmetry properties of multiquark systems, i.e., associated with an operator structure of the qq interaction of the form $\Sigma_{\alpha=1}^{8} \lambda_i^{\alpha} \lambda_j^{\alpha} \sigma_i \sigma_j$, $\Sigma_{\alpha=1}^{8} \lambda_i^{\alpha} \lambda_j^{\alpha}$, etc. These specific operators determine nontrivial qualitative features of the interaction of nucleons when regarded as composite particles—"quark clusters." The study of these problems in connection with few-nucleon correlations[11] in the lightest nuclei is the subject of the present review. To a considerable extent, we use the algebraic formalism[12] of the model of nucleon associations (clusters) in nuclei. However, it is significantly generalized[13,14] in connection with the fact that quarks have colors.

The task is to establish the extent to which the idea of many-quark configurations in nuclei can assist in the creation of a unified basis for describing the *NN* interaction in a wide range of energies from zero to several giga-electron-volts, overcome the well-known difficulties in the theory of the lightest nuclei, and so forth. For example, the contribution of six-quark configurations to *NN* scattering may be effectively manifested through nodes of the wave function in the two-particle (NN) channel. This leads us to a new, "node" phenomenology, which can be regarded as an alternative to the unphysical infinite core in *NN* forces. The two types of phenomenology lead to phase-equivalent descriptions of *NN* scattering, but they differ in the off-shell behavior, and this must lead to observable effects in few-nucleon systems.

A few words about the development of these ideas. About ten years ago, we noted,[15] on the basis of our experience of investigating composite nuclear particles ($\alpha\alpha$, αt, $t\tau$)[10] and some other ideas,[17] that the phenomenology of *NN* repulsion is not so obvious as it appears. For example, there has long been discussion of short-distance repulsion[18] associated with the Pauli principle in the investigation of $\alpha\alpha$ scattering. But it was found that there is no repulsion; the Pauli principle does not hinder α particles from "penetrating each other"—it acts in a quite different way. Namely, there is a strong attraction, which appears to mask the presence of Pauli-forbidden states[19] (the configurations s^8 and s^6p^2), and this leads to a radical reformulation[20] of Levinson's theorem.[21] As a result, the S-state wave function in the region of overlapping of the α particles has two nodes whose positions are comparatively stable in a wide range of scattering energies; the phenomenological core in the $\alpha\alpha$ potential merely imitates the stability of the outer node.[16]

Our approach[22] to phase-shift analysis of singlet *NN* scattering was similar; this combined into a single picture the E dependence of the phase shifts of the lowest partial waves (in the S waves, the phases are shifted upward by π) in the region of low ($T_{lab} < 1$ GeV) and high ($T_{lab} \gg 1$ GeV) energies (Fig. 1). Instead of a repulsive core we used a deep short-

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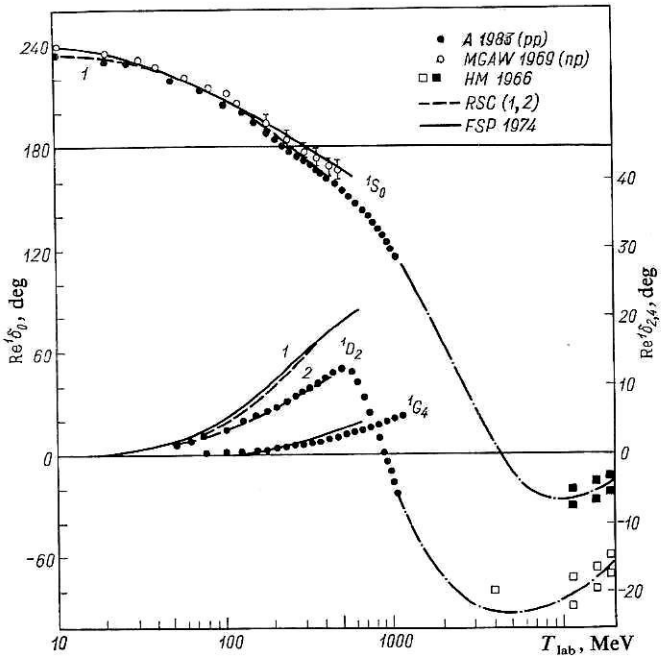

FIG. 1. Phase shifts in the singlet NN scattering channel: 1) without allowance for L dependence in the NN potential; 2) with allowance for the L dependence. RSC for the Reid potential[43]; FSP for the FSP model.[22] The chain curves are the extrapolation to the Born region in the FSP model with allowance for the generalized Levinson theorem.[22] The data of the phase-shift analyses are as follows: A and MGAW in accordance with Ref. 30, and HM in accordance with the results of the analysis of the differential cross section for $T_{\mathrm{lab}} \geqslant 10$ GeV in Ref. 31.

range attractive potential (the same for all partial waves) with one forbidden state in the S wave (abbreviated to FSP) plus a one-pion exchange potential in the peripheral region. In the FSP, the S-state wave function has one node at the point $r \approx r_c$.[22] This leads to a large positive value of the logarithmic derivative of the S wave in the region $r \approx r_c$. This derivative depends weakly on the energy, and this establishes a similarity between the FSP and the Feshbach–Lomon[23] boundary-condition model. Note that we are speaking of the logarithmic derivative at short distances $r \approx r_c$ and not the distances $r \approx 1.5$ F discussed in a number of later papers,[24–28] where this derivative has a pole dependence on the energy.

Support for the conclusion that the effective NN potential in the region $r \approx r_c$ does not change the sign of its derivative and keeps a large gradient is also provided by the strong spin–orbit forces in this region, which lead to a splitting of the ^3P_J phase shifts by tens of degrees at $E_{\mathrm{lab}} \gtrsim 200$ MeV. Following Ref. 29, we form the spin–orbit combination $P_{LS} = -(1/12)(2^3P_0 + 3^3P_1 - 5^3P_2)$ of these phases (Fig. 2). In the region $E < 500$ MeV, in which phase-shift analysis is reliable, P_{LS} increases monotonically, while for $E \gtrsim 500$ MeV, where the data are less reliable, the tendency for P_{LS} to increase persists. Moreover, the sign of the spin–orbit interaction is the same as that of the corresponding term in the nucleon–nucleus optical potential, i.e., attraction if $LS > 0$ $(J = L + 1)$ and repulsion if $LS < 0$ $(J = L, L - 1)$.

The papers of Ref. 22 brought forth some responses,[32] but it is only now that we appear to have a microscopic basis for distinguishing the quark configuration s^4p^2 corresponding to the node in the NN system, i.e., the apparent presence of a "forbidden state." This basis is not directly related to the Pauli exclusion principle, since the s shell has 12 "free places"[34] but is determined by the nature of the qq interaction. Namely, the main part is played by the color magnetic interaction term $H_{\mathrm{CM}} \sim \alpha_s \Sigma_{i<j} \lambda_i \lambda_j \sigma_i \sigma_j$ (see Refs. 35 and 36), which determines a color magnetic attraction in qq pairs symmetric in the CS space and repulsion in antisymmetric ones. The upshot is that in the NN system the configuration $s^4p^2[42]_X[42]_{CS}$ (see Ref. 44) predominates over the previously discussed configuration $s^6[6]_X \times [2^3]_{CS}$, and this effect is enhanced still further on the transition to the $3N$ and $4N$ systems.[37,38] Below, we discuss these questions and consider also the necessary formal methods.

1. QUARK APPROACHES TO THE NN INTERACTION. NONRELATIVISTIC POTENTIAL MODELS

The combined quark configuration s^6 in the deuteron was considered for the first time by Dyson.[39] This is the simplest possibility, since in the nucleon one takes the configuration s^3 and the symmetric orbital Young tableau $[3]_X$. In accordance with the rules of outer multiplication of Young tableaux,[40] it is easy to obtain for the two-nucleon system

$$[3]_X \times [3]_X = \begin{cases} [6]_X + [42]_X & \text{for even } L = 0,\ 2\ \ldots; \\ [51]_X + [3^2]_X & \text{for odd } L = 1,\ 3\ \ldots \end{cases} \quad (1)$$

If the overlapping of the two $3q$ clusters takes place adiabatically, then at zero intercluster distance ($r_{3q-3q} = 0$) only the single Young tableau $[6]_X$ among those listed in (1) "survives," and the common configuration s^6 is realized (when $r_{3q-3q} = 0$, the quark orbitals of the different nucleons are indistinguishable).

Once the general features of the operator structure of the qq interaction had been clarified,[41,42] it was not difficult to establish[3–9] that the s^6 configuration corresponds to NN repulsion (a "soft" core of height 300–500 MeV; see below). This fact gave rise to general satisfaction, since it agreed well with the known phenomenology of NN scattering.[43] The papers of Refs. 3–5 used the standard adiabatic approach of atomic physics—the Born–Oppenheimer method. As we shall see below, this approximation did not make it possible to take into account the nucleon color-magnetic attraction

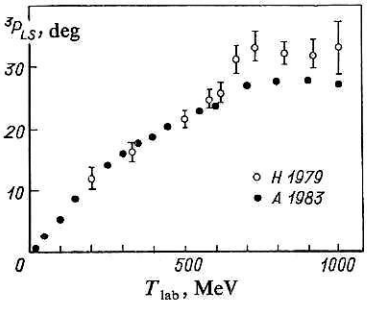

FIG. 2. Spin–orbit combination P_{LS} of phase shifts of the ^3P_J waves in accordance with the data of Ref. 30.

channels corresponding to the nontrivial symmetry $[42]_X$. The importance of these channels was noted in Refs. 15 and 44, which used quark shells and simple general algebraic arguments that follow from (1).

All these questions were then intensively discussed in Refs. 8 and 45–49, which used the resonating-group method.[50] This method, which formally is very universal is at the same time a specifically computer method and not an analytic method and is not sufficiently clear from the physical point of view. It was therefore worth comparing[45,48] the resonating-group method with our algebraic (essentially shell) approach,[13–15,44] and this made it possible to eliminate some original errors in the use of the resonating-group method (neglect of the important $6q$ states). The adiabatic variant of the resonating-group method proposed by Harvey,[6] which has the advantages of being more developed both analytically and algebraically, will undoubtedly play a constructive role. We shall therefore base our discussion on Harvey's approach. We shall consider all these questions successively, going over from simple to more complicated methods. At the same time, simple estimates independent of the form of the qq interaction can be made by means of the technique of Casimir invariants.

Technique of Casimir invariants

In the general case, the Hamiltonian of the two-body qq interaction has the form[42] (we omit the summation with respect to the color index)

$$H_{\text{int}} = \sum_{i<j}^{N} \lambda_i \lambda_j V^c(r_{ij}) - \sum_{i<j}^{N} \lambda_i \lambda_j \sigma_i \sigma_j V(r_{ij}), \tag{2}$$

where $V^c(r)$ and $V(r)$ cannot yet be deduced from first principles, and the form of these potentials is to a large degree arbitrary. The operators $\Sigma_{i<j}\lambda_i\lambda_j$ and $\Sigma_{i<j}\lambda_i\lambda_j\sigma_i\sigma_j$ are invariants of the following groups: the color group $SU(3)_C$, the spin group $SU(2)_S$, and their extension—the CS group introduced by Jaffe[35]:

$$SU(6)_{CS} \supset SU(3)_C \times SU(2)_S. \tag{3}$$

Therefore, the eigenvalues of these operators can be expressed in terms of the corresponding Casimir invariants of the groups (3). The second-order Casimir invariant $C_2^{(n)}$ of the group $SU(n)$ is a generalization of the square of the total spin,

$$C_2^{(2)} \equiv S^2 = \left(\sum_{i=1}^{N} \frac{1}{2}\sigma_i^x\right)^2 + \left(\sum_{i=1}^{N} \frac{1}{2}\sigma_i^y\right)^2 + \left(\sum_{i=1}^{N} \frac{1}{2}\sigma_i^z\right)^2$$

to the case of a group of arbitrary rank n:

$$C_2^{(n)} = \sum_{a=1}^{n^2-1} \left(\frac{1}{2}\Lambda^a \frac{1}{2}\Lambda^a\right),$$

where $\Lambda^a = \Sigma_{i=1}^{N} \Lambda_i^a$ are the generators of the group $SU(n)$, normalized by the relation $\mathrm{Tr}\,\Lambda_i^a \Lambda_i^a = 2$ (like the σ matrices). For example, for the group $SU(3)_C$ in the direct product

$$\underbrace{q \times q \times \ldots \times q}_{N}$$

of the spinors

$$q = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

of this group we have

$$C_2^{(3)} = \sum_{a=1}^{8} \left(\sum_{i=1}^{N} \frac{1}{2}\lambda_i^a\right)\left(\sum_{j=1}^{N} \frac{1}{2}\lambda_j^a\right), \tag{4}$$

from which we obtain

$$\sum_{i<j}^{N}\sum_{a=1}^{8} \lambda_i^a \lambda_j^a = 2C_2^{(3)} - 2N\frac{4}{3}, \tag{5}$$

where the factor 4/3 is calculated on the basis of the normalization of the λ matrices: $\Sigma_{a=1}^{8} \frac{1}{2}\lambda_i^a \frac{1}{2}\lambda_i^a = \frac{8}{3}\,\mathrm{Tr}\,\frac{1}{2}\lambda_i^a \frac{1}{2}\lambda_i^a = \frac{4}{3}$. Such calculations lead to Jaffe's formula[35]

$$-\sum_{i<j}^{N} \lambda_i \lambda_j \sigma_i \sigma_j = -4C_2^{(6)} + 2C_2^{(3)} + \frac{4}{3}C_2^{(2)} + 8N. \tag{6}$$

The eigenvalues of the Casimir invariants $C_2^{(n)}([f])$ for an irreducible representation of $SU(n)$ can be expressed by means of the standard group-theoretical technique[53] in terms of lengths (f_i) of the rows of the Young tableau $[f_1] \equiv [f_1, f_2 \ldots, f_n]$, which is also an invariant of the given irreducible representation,[54]

$$C_2^{(n)}([f]) \equiv \langle [f] | C_2^{(n)} | [f] \rangle$$

$$= \frac{1}{2}[f_1'+n-1) + f_2'(f_2'+n-3) + \ldots + f_{n-1}'(f_{n-1}'-n+3)]$$

$$-\frac{1}{2n}\left(\sum_{i=1}^{n-1} f_i'\right)^2, \; f_i' = f_i - f_n. \tag{7}$$

Estimate of the color magnetic contribution

We first compare the contributions of the interaction (2) to the masses of the nucleon and Δ isobar. In shell notation, the wave functions of these baryons have the form

$$\left.\begin{aligned} |N\rangle &= |s^3\,[3]_X,\; [1^3]_C\,[21]_S\,([21]_{CS})\,[21]_I : [1^3]_{XCSI}\rangle; \\ |\Delta\rangle &= |s^3\,[3]_X,\; [1^3]_C\,[3]_S\,[1^3]_{CS}\,[3]_I : [1^3]_{XCSI}\rangle. \end{aligned}\right\} \tag{8}$$

Here, the Young tableau in the CS space, $[f]_{CS}$, is determined uniquely,

$$[1^3]_C \circ [21]_S = [21]_{CS},\; [1^3]_C \circ [3]_S = [1^3]_{CS} \tag{9}$$

in accordance with the inner-product rules.[40] Using (2) and (5)–(7), we obtain

$$\left.\begin{aligned} \langle N|H_{\text{int}}|N\rangle &= C + \langle V\rangle\,\langle [21]_{CS}| \\ &\quad -\sum_{i<j}^{N=3} \lambda_i\lambda_j\sigma_i\sigma_j |[21]_{CS}\rangle = C - 8\langle V\rangle; \\ \langle \Delta|H_{\text{int}}|\Delta\rangle &= C + 8\langle V\rangle, \end{aligned}\right\} \tag{10}$$

where

TABLE I. Mean values of the color magnetic operator in the channels $S, I = 1, 0$ and $0, 1$.

S, I	1, 0					0, 1			
$[f_{CS}]$	$[42]$	$[321]$	$[2^3]$	$[31^3]$	$[21^4]$	$[3^2]$	$[41^2]$	$[2^21^2]$	$[1^6]$
$\Delta_{CS} \equiv \Delta([f_{CS}]) \equiv$ $\equiv \langle [f_{CS}] \vert - \sum_{i<j}^{6} \lambda_i\lambda_j\sigma_i\sigma_j \vert [f_{CS}] \rangle$	$\frac{-88}{3}$	$\frac{-28}{3}$	$\frac{8}{3}$	$\frac{8}{3}$	$\frac{80}{3}$	-24	-24	8	48
$E_{CM} = \langle V \rangle \Delta_{CS}$, MeV	-550	-175	50	50	500	-450	-450	150	900
Orbital Young tableau $[f_X]$ compatible with $[f_{CS}]$ in the NN channel.	[42]	[42]	([6], [42])	[42]	[42]	[42]	[42]	([6] [42])	[42]

$$C = \langle V^c \rangle \langle [1^3]_C \vert \sum_{i<j}^{N=3} \lambda_i\lambda_j \vert [1^3]_C \rangle = -8 \langle V^c \rangle.$$

In (10), we have taken in front of the summation sign the orbital matrix element $\langle V \rangle$ of the two-body potential $V(r_{ij})$, this being the same for any pair of quarks in the s^3 configuration. The numerical value of $\langle V \rangle$ can be estimated from the known mass difference $m_\Delta - m_N \approx 300$ MeV:

$$\langle \Delta \vert H_{\text{int}} \vert \Delta \rangle - \langle N \vert H_{\text{int}} \vert N \rangle = 16 \langle V \rangle \approx 300 \text{ MeV}. \tag{11}$$

Color magnetic contribution to the *NN* interaction

Using (10) and (11), we now consider the even NN channel, taking, to be specific, the deuteron quantum numbers S, $I = 1, 0$. In contrast to the simple relations (9), the product of the color, $[2^3]_C$, and spin, $[42]_S$, Young tableaux in the channel $S, I = 1, 0$ determines a complete set of CS states:

$$[2^3]_C \circ [42]_S = [42]_{CS} + [321]_{CS} + [2^3]_{CS} + [31^3]_{CS} + [21^4]_{CS}. \tag{12}$$

In the channel $S, I = 0, 1$ we have the analogous relation

$$[2^3]_C \circ [3^2]_S = [3^2]_{CS} + [41^2]_{CS} + [2^21^2]_{CS} + [1^6]_{CS}. \tag{13}$$

For each CS state, the color magnetic interaction differs in sign and magnitude (Table I).

The most symmetric CS states, $[42]_{CS}$, $[321]_{CS}$, $[3^2]_{CS}$, $[41^2]_{CS}$, determine the channels of color magnetic attraction. In the remaining states, color magnetic repulsion is dominant. For orbital symmetry $[6]_X$ (configuration s^6) only one CS state satisfies the Pauli principle: $[f_{CS}] = [2^3]$ in the channel $S, I = 1, 0$ and $[f_{CS}] = [2^21^2]$ in the channel $S, I = 0, 1$. This is a direct consequence of the relation

$$[f_{CS}] = [\tilde{f}_{XI}], \tag{14}$$

which follows from the requirement of permutational antisymmetry (Pauli principle), and the Young tableau $[f_{XI}]$ for $[f_X] = [6]$ is uniquely determined by the value of the isospin: $[f_{XI}] = [3^2]$ for $I = 0$ and $[f_{XI}] = [42]$ for $I = 1$. In contrast, the orbital Young tableau $[42]_X$ (configuration s^4p^2) is compatible with any CS state of Table I. Therefore, the channels of the color magnetic attraction act in the configuration $s^4p^2[42]_X$ and are responsible for its "survival" even when there is complete overlapping of the $3q$ clusters (see below). The gain in the energy of the color magnetic interaction compared with $s^6[6]_X$ is $E_{CM}([42]_{CS}) - E_{CM}[2^3]_{CS}) \approx -600$ MeV.

As in the case of the N–Δ splitting, we have here not used any specific form of the potential $V(r_{ij})$, but we have ignored in this case the difference between the orbital matrix elements in the states s^2, sp, and p^2. Although the difference could be taken into account, this would in fact be excessive accuracy, since our treatment is rather approximate.

Born–Oppenheimer approximation

In this approximation, the adiabatic NN potential can be written in the form[3]

$$U_{\text{BO}}^{SI}(r) = N_A^{-1} (\hat{A} \{\langle N_+ \vert \langle N_- \vert\}_{SI} H_q \hat{A} \{\vert N_+ \rangle \vert N_- \rangle\}_{SI}) - 2 \langle N_+ \vert H_q \vert N_+ \rangle,$$
$$N_A^2 = \langle \{\langle N_+ \vert \langle N_- \vert\}_{SI} \hat{A} \{\vert N_+ \rangle \vert N_- \rangle\}_{SI} \rangle, \tag{15}$$

where $H_q = \Sigma_i (P^i/2m_q) + H_{\text{int}}$; $\vert N_\pm \rangle$ are the shell functions (8) constructed from orbitals $s_\pm$ "centered" at the points $\pm \frac{1}{2} r_{3q-3q}$:

$$\vert s_\pm^3 \rangle = \left(\frac{\Omega}{\pi}\right)^{3/4} \exp\left[-\frac{1}{2}\Omega \sum_{i=1}^{3} \left(r_i - \frac{1}{2} r_{3q-3q}\right)^2\right]; \tag{16}$$

r_{3q-3q} is a parameter of the basis and not a dynamical variable (r_{3q-3q} must not be confused with the relative coordinate r):

$$r_{3q-3q} \neq r = \frac{1}{3}(r_1 + r_2 + r_3) - \frac{1}{3}(r_4 + r_5 + r_6); \tag{17}$$

$\hat{A}$ antisymmetrizes with respect to permutations (P_{ij}) of quarks in different nucleons:

$$\hat{A} = \frac{1}{10}\left(I - \sum_{i=1}^{3}\sum_{j=4}^{6} P_{ij}\right), \quad \hat{A}^2 = \hat{A}. \tag{18}$$

The parameters of the potentials (2) and the variational pa-

rameter Ω are fitted to the N–A spectrum (in Refs. 3 and 52, $\Omega^{-1/2} = 0.66$–1.1 F). As $r_{3q-3q} \to 0$, the function $\hat{A}\{|N_+\rangle|N_-\rangle\}_{SI}$ automatically goes over into the s^6 configuration, since $s_+ \to s_-$, and the value of the potential (15) at the point $r_{3q-3q} = 0$ can be readily calculated by means of (10) and (11) and the values of Δ_{CS} in Table I ($[f_X] = [6]$):

$$U_{\rm BG}^{SI}(r_{3q-3q}=0) = \langle V\rangle \left\{\left\langle [f_{CS}]\left| -\sum_{i<j}^{N=6}\lambda_i\lambda_j\sigma_i\sigma_j\right|[f_{CS}]\right\rangle - 2\left\langle [21]_{CS}\left| -\sum_{i<j}^{N=3}\lambda_i\lambda_j\sigma_i\sigma_j\right|[21]_{CS}\right\rangle\right\} = \begin{cases} \frac{54}{3}\langle V\rangle \approx 350\ \text{MeV}, & S,\ I=1,\ 0; \\ 24\,\langle V\rangle \approx 450\ \text{MeV}, & S,\ I=0,\ 1. \end{cases} \tag{19}$$

The part played by the confinement potential $V^c(r_{ij})$ in the NN interaction is unimportant in the given approximation, as can be seen, for example, from the fact that at the point $r_{3q-3q} = 0$ the V^c contribution vanishes by virtue of the relation

$$\langle [2^3]_C|\sum_{i<j}^{6}\lambda_i\lambda_j|[2^3]_C\rangle - 2\langle [1^3]_C|\sum_{i<j}^{3}\lambda_i\lambda_j|[1^3]_C\rangle = 0. \tag{20}$$

The values given in (19) and (20) agree well with the microscopic calculation of Liberman (Fig. 3),[3] who used a harmonic-oscillator potential. However, the adiabatic approximation is not good for a system of quark clusters, not so much because the adiabaticity condition is not satisfied ($m_q \approx (1/3)m_N$, and therefore the ratio of the N and q kinetic energies is not small) but rather because of the automatic exclusion of the channels of color magnetic attraction [the symmetry $[42]_X$ does not survive as $r_{3q-3q} \to 0$; see Eq. (26) below]. A generalization of the adiabatic approach making it possible to take into account the channels of color magnetic attraction was proposed by Harvey.[6] Harvey also concluded that the confinement potential V^c plays an important part in the NN interaction.

Harvey's method

Harvey's approach[6] can be regarded as a "quark variant" of the Heitler–London method of valence bonds. We recall that in this model there is a splitting of the terms of the H_2 moleucle. Let $s_\pm$ be the $1s$ electron orbitals of two hydrogen atoms at the points $\pm\frac{1}{2}r_{\rm HH}$. From the orbital $s_\pm = \Psi_0(r_e \mp \frac{1}{2}r_{\rm HH})$ it is possible to construct normalized symmetric and antisymmetric states (we omit the indices e and HH):

$$\left.\begin{matrix} |s_+s_-\,[2]_X\rangle \\ |s_+s_-\,[1^2]_X\rangle \end{matrix}\right\} = \frac{\Psi_0\left(r_1+\frac{1}{2}r\right)\Psi_0\left(r_2-\frac{1}{2}r\right) \pm \Psi_0\left(r_1-\frac{1}{2}r\right)\Psi_0\left(r_2+\frac{1}{2}r\right)}{(1\pm\langle s_+|s_-\rangle)^{1/2}}. \tag{21}$$

The difference from the ordinary adiabatic approach is that the normalization is fixed, and therefore the antisymmetric state $[1^2]_X$ does not vanish as $r_{\rm HH} \to 0$. The indeterminate form 0/0 obtained in the limit $r_{\rm HH} \to 0$ can be evaluated by l'Hôpital's rule (by going over to the derivatives):

$$|s_+s_-\,[2]_X\rangle \xrightarrow[r_{\rm HH}\to 0]{} |(1s)^2\,[2]_X\rangle; \qquad |s_+s_-\,[1^2]_X\rangle \xrightarrow[r_{\rm HH}\to 0]{} |1s2p\,[1^2]_X\rangle. \tag{22}$$

In their dependence on the parameter $r_{\rm HH}$, the functions (21) describe two different "paths" in the space of electron configurations (at the point $r_{\rm HH} = 0$ they pass through orthogonal shell states of the He atom). Accordingly, in the H_2 molecule there are two different adiabatic potentials—for the symmetric and antisymmetric terms $^1\Sigma_s$ and $^3\Sigma_a$:

$$\left.\begin{matrix} U_{HL}^{S=0}(r) = \langle s_+s_-\,[2]_X|H_e|s_+s_-\,[2]_X\rangle - 2\langle s_+|H_e|s_+\rangle\,(^1\Sigma_s); \\ U_{HL}^{S=1}(r_{\rm HH}) = \langle s_+s_-\,[1^2]_X|H_e|s_+s_-\,[1^2]_X\rangle - 2\langle s_+|H_e|s_+\rangle\,(^3\Sigma_a). \end{matrix}\right\} \tag{23}$$

The terms $^1\Sigma_s$ and $^3\Sigma_a$ are uniquely determined by the value of the total spin of the channel: $s = 0$ or 1 (a consequence of the Pauli principle). It was this circumstance that justified the retention of the common normalization of the functions (21)—we have two independent channels.

When the Heitler—London method is generalized to the case of two $3q$ clusters, a qualitatively new situation arises, namely, for fixed "nuclear" quantum numbers of the channel ($S, I = 1, 0$ or $0, 1$) there is a complete set of quark CS states (12) or (13) and two types of orbital symmetry: $[6]_X$ and $[42]_X$. Therefore, if we apply the Heitler–London method directly to the $3q$ clusters, we obtain, for example, in the channel S, $I = 1$, 0 five different potentials for the orbital Young tableau $[42]_X$ and one for $[6]_X$ (the latter is identical to the adiabatic potential of the Born–Oppenheimer method). However, these potentials do not correspond to independent channels, since the interaction (2) generates the nondiagonal matrix elements $\langle [f_{CS}]|H_{\rm int}|[f'_{CS}]\rangle$ (f_{CS} ceases to be a good quantum number, since the CS symmetry is broken by the difference between the orbital matrix elements $\langle V\rangle$ in the pairs s^2, sp, p^2). It must also be borne in mind that in the $\Delta\Delta$ and CC channels (C is a color $3q$ cluster) one can obtain in exactly the same way corresponding Heitler–London potentials, and the matrix elements coupling the NN and $\Delta\Delta$ channels ($\langle N_+N_-|H_{\rm int}|\Delta_+\Delta_-\rangle$, etc.) will also be nonzero.

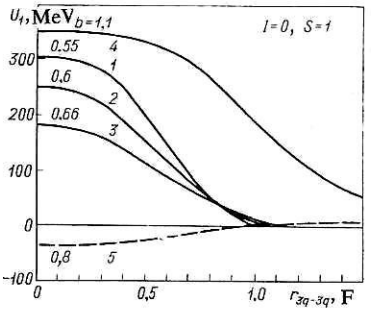

FIG. 3. Adiabatic NN potentials of the quark model: 1)–3) calculation in Harvey's approach[59] for the qq interactions from Ref. 52 for different values of the parameter b, in F; 4) calculation in accordance with the Born–Oppenheimer model from Ref. 3; 5) result of Harvey's calculation[6] for cluster functions not minimized with respect to the parameter b.

A "correct" generalization of the Heitler–London method to the quark case was proposed by Harvey in Ref. 6, in which the method was in fact augmented by a variational principle. The path in the space of $6q$ configurations $[f_X][f_{CS}]$, which depends on r_{3q-3q} as on a parameter, is determined from the principle of a minimum of the total energy H_q, which is equivalent to solving the probem of diagonalizing the energy matrix:

$$\mathrm{Det}\,\langle s_+^3 s_-^3\,[f_X]\,[f_{CS}]\,|H_q - E|s_+^3 s_-^3\,[\bar{f}_X]\,[\bar{f}_{CS}]\rangle = 0 \qquad (24)$$

for each fixed value of r_{3q-3q}. For example, in the channel S, $I=1$, 0 there is a 6×6 matrix, and it has six eigenvalues $E_n(r_{3q-3q})$, $n=1, 2, \ldots, 6$. As $r_{3q-3q}\to\infty$, the lowest eigenvalues E_1 and E_2 tend to the NN and $\Delta\Delta$ thresholds, respectively, while the remaining E_n, $n=3, 4, 5, 6$, increase unboundedly, since they correspond to closed CC channels.

Thus, in Harvey's approach we obtain on the basis of the eigenvalues $E_n(r_{3q-3q})$, which depend on r_{3q-3q} as on a parameter, the following expressions for the potentials of the NN, $\Delta\Delta$, and CC channels, i.e., the "terms" Σ_{NN}, $\Sigma_{\Delta\Delta}$, and Σ_{CC^n} (we omit the index $3q$–$3q$):

$$\left.\begin{aligned}
&\Sigma_{NN}: U_{HL}^{SI}(r)_{NN} = E_1(r) - 2\langle N|H_q|N\rangle,\\
&\Sigma_{\Delta\Delta}: U_{HL}^{SI}(r)_{\Delta\Delta} = E_2(r) - 2\langle \Delta|H_q|\Delta\rangle,\\
&\Sigma_{CC_n}: U_{HL}^{SI}(r)_{CC_n} = E_n(r) - 2\langle C_n|H_q|C_n\rangle,\ n=3, 4, 5, 6,
\end{aligned}\right\} \qquad (25)$$

where C is a color eq cluster belonging to definite values of the color, spin, and isospin quantum numbers (we denote them by the single index n).

In Harvey's approach, as in the Heitler–London approach, normalized states $N_{NL}^{-1}|s_+^3 s_-^3\,[f_X][f_{CS}]\rangle$ are used. Here, $|s_+^3 s_-^3\,[f_X][f_{CS}]\rangle$ is a shell state constructed from the orbitals $S_\pm$ by means of the technique of coefficients of fractional parentage (see Sec. 4) and a normalization factor calculated by means of the same technique,[6]

$$N_{HL}^2(f_X,\ r_{3q-3q}) = \langle s_+^3 s_-^3\,[f_X]\,[f_{CS}]|s_+^3 s_-^3\,[f_X]\,[f_{CS}]\rangle$$
$$= \begin{cases} 1 + 9\langle s_+|s_-\rangle^2 + 9\langle s_+|s_-\rangle^4 + \langle s_+|s_-\rangle^6, & [f_X]=[6],\\ 1 - \langle s_+|s_-\rangle^2 - \langle s_+|s_-\rangle^4 + \langle s_+|s_-\rangle^6, & [f_X]=[42]. \end{cases} \qquad (26)$$

In the limit $r_{3q-3q}\to 0$, we obtain for the $[42]_X$ normalized state an indeterminate form 0/0 and, after differentiation, the configuration s^4p^2.

Results of adiabatic calculations

Unfortunately, in Ref. 6 Harvey did not use the CS scheme of classifying states and made the calculations for the SI scheme[15] on a restricted basis (a 3×3 matrix was diagonalized). Allowance was not made for the most important CS states (for example, $[42]_{CS}$ and $[321]_{CS}$ in the channel S, $I=1$, 0); there were some further omissions, which were criticized by a number of authors.[51,55,58] In particular, Harvey used the cluster functions (16), which do not satisfy the condition of a minimum of the energy with respect to the parameter $b=\Omega^{-1/2}$. Then, as follows from general theorems,[56] the nondiagonal matrix elements of the first term of the interaction (2), corresponding to an "average field," will be nonzero (and not small). For example, the matrix elements

$$\langle (1s)^3\,[3]_X\,[f_{CS}]\,|\sum_{i<j}^{3}\lambda_i\lambda_j V^c\,|\,1s\,(1p)^2 - (1s)^2\,2s\,[3]_X\,[f_{CS}]\rangle$$

do not vanish for the cluster functions (16), and in the six-quark basis these same terms mix the configurations s^6 and s^4p^2. This complicates the symmetry analysis, and, moreover, the incorrect basis does not make it possible to separate in "pure" form the effect of the color magnetic attraction contained in the terms diagonal with respect to the configuration s^4p^2. Complete clarity has now been achieved with regard to this question, and the correct solution on a complete Ω-minimized basis was obtained in Refs. 51, 55, and 57–59.

In Fig. 3 we summarize the results of different calculations of the NN potential in Harvey's approach[6,59] and in the ordinary adiabatic approach.[3] In all cases apart from Ref. 6 the nucleon functions (16) satisfy the condition of a minimum of the energy with respect to the parameter $b=\Omega^{-1/2}$:

$$d\,\langle s^3\,|\,H_q\,|\,s^3\rangle/db = 0. \qquad (27)$$

We consider the solution at the characteristic point $r_{3q-3q}=0$. The eigenfunction Ψ_1 of Eq. (24) at $r_{3q-3q}=0$ is a superposition of the configurations s^6 and s^4p^2:

$$\Psi_1(r_{3q-3q}=0) = C_0|s^6\,[6]_X\,[2^3]_{CS}\rangle + \sum_\nu C_\nu|s^4p^2\,[42]_X\,[f_{CS}^\nu]\rangle, \qquad (28)$$

where $[f_{CS}^\nu]$ takes all values $[f_{CS}]$ on the right-hand side of (12). In Fig. 4, the solutions (28) are given in the form of projections into the NN channel,

$$\Phi_{NN}(r) = \langle\Psi_1\,(r_{3q-3q}=0)\,\hat{A}$$
$$\times\,\{|\,s_+^3\,[3]_X\,[21]_{CS}\rangle\,|\,s_-^3\,[3]_X\,[21]_{CS}\rangle\}_{S=1,\,I=0}, \qquad (29)$$

and into the $\Delta\Delta$ and N^*N^* channels. The variants 1, 2, and 3 differ only in the form of the qq potentials V and V^c used in

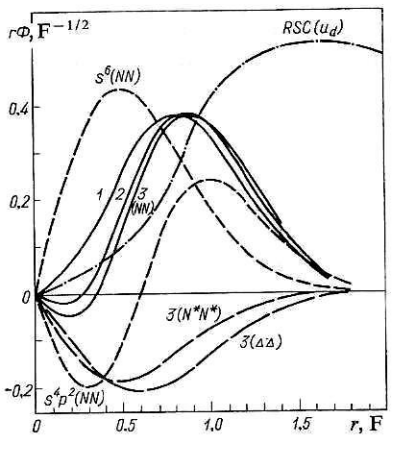

FIG. 4. Eigenfunctions (28) of Eq. (24) projected into the two-particle NN, $\Delta\Delta$, and N^*N^* channels at the point $r_{3q-3q}=0$. In accordance with the results of calculations using a complete basis: 1) from Ref. 51; 2) and 3) from Ref. 59; for variants 2 and 3 given in Fig. 3. Also given for comparison are the projections into the NN channel of the "pure" configurations s^6 and s^4p^2 and the s-wave deuteron function.[43]

(2). Variant 1 is in accordance with the calculation of Ref. 51, in which a one-gluon exchange potential in the Fermi–Breit approximation[42] was used. Variants 2 and 3 are in accordance with an analogous calculation[59] for the nonsingular potentials of Ref. 52. It can be seen that the wave function Φ_{NN} has a node in the region $r \leqslant r_h$, the position of the node being shifted to the left compared with the case of the "pure" configuration $s^4p^2[42]_X[42]_{CS}$. The probability of the s^4p^2 configuration in the solutions 1, 2, and 3 shown in Fig. 4 is 50–70%. Thus, in Harvey's approach the degrees of freedom corresponding to the symmetry $[42]_X$ are "unfrozen," but the dynamics of the $6q$ system is described in the approach in an oversimplified manner—on the basis of the adiabatic approximation, for which there is here no justification (there is no small expansion parameter).

2. Results of calculations by the resonating-group method

In recent years, quite a number of computer calculations of NN scattering have used the resonating-group method.[8,45–48] These took into account two ($NN + \Delta\Delta$) or three ($NN + \Delta\Delta + CC$) coupled channels and used the SI classification of states in accordance with the quantum numbers of the groups

$$SU(4)_{SI} \supset SU(2)_S \times SU(2)_I. \tag{30}$$

It is readily seen that even in the three-channel equations of the resonating-group method not more than half of the Pauli-allowed states were taken into account. The baryon states N, Δ, C which have the same Young tableau $[3]_{SI}$ in the SI space were used. In the outer product of these Young tableaux only a restricted number of $6q$ symmetries is contained:

$$[3]_{SI} \times [3]_{SI} = \begin{cases} [51]_{SI} + [3^2]_{SI}, & \text{symmetric with respect to permutations of } 3q \text{ clusters,} \\ [6]_{SI} + [42]_{SI}, & \text{antisymmetric with respect to permutations of } 3q \text{ clusters;} \end{cases} \tag{31}$$

in contrast, in the $S, I = 1, 0$ channel five SI states satisfy the Pauli principle in the orbital state $[42]_X$:

$$[42]_S \circ [3^2]_I = [51]_{SI} + [3^2]_{SI} + [41^2]_{SI} + [321]_{SI} + [2^21^2]_{SI}. \tag{32}$$

Therefore, in order to take into account all the degrees of freedom of the $6q$ system in the resonating-group approximation, it is necessary to include in the treatment coupling to other channels, for example, the N^*N^* channel. The resonance $N^*_{\frac{1}{2}^-}(1535)$ has the SI symmetry $[21]_{SI}$ and orbital Young tableau $[21]_X$. In this case, in the outer product of the two Young tableaux we have

$$[21] \times [21] = \begin{cases} [42] + [2^3] + [321] + [31^3], & \text{symmetric with respect to permutation of } 3q, \\ [3^21^3] + [41^2] + [321] + [2^21^2], & \text{antisymmetric with respect to permutation of } 3q. \end{cases} \tag{33}$$

In the second row (states symmetric with respect to $3q$–$3q$ permutation) all the Young tableaux from (32) lost in (31) are retained.

Thus, complete allowance for the channels of color magnetic attraction requires that the wave function in the resonating-group approximation[50] be written down for at least the following three coupled channels:

$$\Psi^{SI}_{RGM}(x_1, y_1, x_2, y_2, r) = \hat{A}\chi_{NN}(r)\{\varphi_N(x_1, y_1)\varphi_N(x_2, y_2)\}_{SI} + \hat{A}\chi_{\Delta\Delta}(r)\{\varphi_\Delta(x_1, y_1)\varphi_\Delta(x_2, y_2)\}_{SI} + \hat{A}\chi_{N^*N^*}(r)\{\varphi_{N^*}(x_1, y_1)\varphi_{N^*}(x_2, y_2)\}_{SI} \tag{34}$$

or that one takes into account polarization terms[50] of the form

$$\sum_f C_f\varphi_f(x_1, y_1, x_2, y_2, r). \tag{35}$$

We have here used Jacobi relative coordinates [see also (17)]

$$x_1 = r_1 - r_2,\ y_1 = \frac{1}{2}(r_1 + r_2) - r_3\ , \text{etc.}, \tag{36}$$

which distinguishes φ_N and φ_Δ from the functions (16); φ_f are the shell configurations $s^{N-n}p^n[f_X][f_{CS}]$, $n = 0, 2 \ldots$, $N = 6$.

Let us consider the available results. As usual, the variational principle (one varies the functions $\chi_{NN}, \chi_{\Delta\Delta}, \ldots$) leads to a system of coupled integro-differential equations for the projected wave functions:

$$\left.\begin{aligned} u_{NN}(r) &= (\Psi^{SI}_{RGM},\ \{\varphi_N\varphi_N\}_{S=1,\ I=0}), \\ u_{\Delta\Delta}(r) &= (\Psi^{SI}_{RGM},\ \{\varphi_\Delta\varphi_\Delta\}_{S=1,\ I=0})\ , \text{etc.} \end{aligned}\right\} \tag{37}$$

Below the $\Delta\Delta$ threshold, the functions (37) have the asymptotic behavior (S waves were considered) $u_{NN}(r) \sim \frac{1}{kr}\sin(kr + \delta_0)$, $u_{\Delta\Delta}(r) \sim \frac{1}{\varkappa r}\sinh \varkappa r$, where $\varkappa = [E - 2(m_\Delta - m_N)]^{1/2}m_N^{1/2}$, $E = k^2/m_N$. The S-wave phase shifts $\delta_0(E)$ obtained in the calculations of Refs. 8 and 48, which used potentials (2) of the form

$$\left.\begin{aligned} V^c(r_{ij}) &= \frac{\alpha_s}{4}\left(\frac{1}{r_{ij}} - \frac{\pi}{m_q^2}\delta(r_{ij})\right) - ar_{ij}; \\ V(r_{ij}) &= -\frac{\alpha_s}{4}\frac{\pi}{m_q^2}(\delta r_{ij}), \end{aligned}\right\} \tag{38}$$

have the correct slope $d\delta_0/dk = -kr_0$ ($E = k^2/m_N \gtrsim 50$ MeV), corresponding to a hard-core radius $r_0 = 0.4$–0.7 F (see Ref. 48) but do not agree even in sign with the results of the phase-shift analysis of Ref. 30. The reason for the discrepancy is readily found by noting that the qq forces (2) and (38) must not describe the peripheral part ($r \gtrsim b$) of the NN interaction, which is determined by pion exchange. Indeed, it was shown in Refs. 8 and 48 that addition of an OPEP contribution at distances $r \geqslant b$ completely corrects the behavior of the phase shifts.

Figure 5 shows the functions u_{NN} and $u_{\Delta\Delta}$ from the complete series of resonating-group calculations[8,48] at different energies. All the u_{NN} have a node at $r \approx r_0 \lesssim b$, and this leads to a constant slope of the phase shifts, which was previously interpreted as the manifestation of a hard core in the NN forces. It can be directly seen that the hard core in the phenomenological NN potentials[43] (or the hard boundary

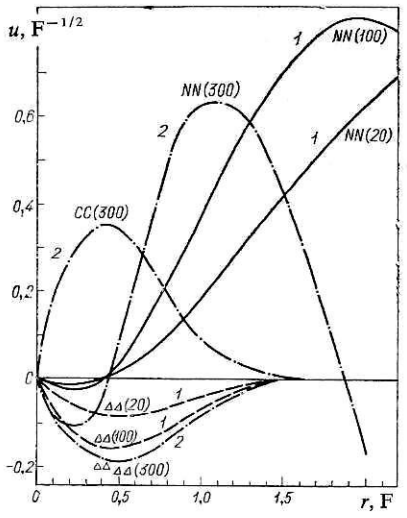

FIG. 5. Wave functions (37) obtained in resonating-group calculations: 1) in the $NN + \Delta\Delta$ approximation[8]; 2) in the $NN + \Delta\Delta + CC$ approximation.[48] The numbers in the brackets are the energies in MeV, measured from the NN threshold.

condition in the model of Ref. 23) actually imitates the stable boundary conditions that follow from quark microscopy in the region $r \lesssim b$. If we compare the resonating-group functions with what is given by the "pure" configuration s^4p^2 (see Fig. 4), we see that there is a strong difference in the amplitudes to the left of the node. However, it is to be expected that when complete allowance is made for the symmetry $[42]_X$ in (34) [or (35)] the suppression of u_{NN} at short distances will be much less, and u_{NN} will approach the wave function in the FSP model.[22]

2. RELATIVISTIC MODELS (BAGS)

In a number of studies (Refs. 4, 24, 25, and 44), the MIT bag model[36] has been used to describe the NN interaction in the region $r \lesssim 1.3$–1.5 F. The role of the color magnetic forces in the NN interaction does not depend on what model is used in the quark sector, i.e., relativistic or nonrelativistic. In the relativistic model, the orbitally excited configurations $s^4_{1/2}/p^2_{1/2}$, $s^4_{1/2}p^2_{3/2}$, and $s^4_{1/2}p_{1/2}p_{3/2}$ will be as important in the description of the NN interaction as $s^4 - p^2$ is in the nonrelativistic treatment. Here, we shall show this by means of an appropriate algebraic technique. The MIT model[36] has been discussed fairly fully in the literature,[36,60,61] and we therefore restrict ourselves to the most general comments concerning quark dynamics in a bag.

Free quarks in a bag

The confinement of the quarks in the bag is not due to any two-body interaction, as in (2), but corresponds to a vacuum phase transition at the boundary of the bag.[62] The gluon condensate, which is regarded as a model of the physical vacuum outside the bag, is characterized by a single constant $-B$, which has the dimensions of a volume energy. Within the bag we have the ordinary perturbation-theory QCD vacuum, which, therefore, has excess volume energy $+B$ above the physical vacuum. The action of the free quark fields Ψ_α (the subscript α represents color and flavor) is expressed in the form of an integral over the finite volume V of the bag[61] [if the bag is spherical, it has finite radius R and $V = (4/3)\pi R^3$]:

$$W = -\int d^4x \left[\left(-\frac{1}{2}\overline{\Psi}_\alpha \, \mathrm{i}\, \partial_\mu \gamma^\mu \Psi_\alpha + m_q \overline{\Psi}_\alpha \Psi_\alpha + B \right) \vartheta_B(x) + \frac{1}{2}(\overline{\Psi}_\alpha \Psi_\alpha)\, \delta_S(x) \right],$$

where

$$\vartheta_B(x) = \begin{cases} 1, & r < R, \quad -\infty < t < \infty; \\ 0, & r > R, \end{cases} \tag{39}$$

and $\delta_S(x)$ is a "surface" Δ function, $\partial_\mu \vartheta_B(x) = n_\mu \delta_S(x)$, $n_\mu n^\mu = -1$. Stationarity of the action (39) with respect to the variations $\delta\Psi$ and $\delta\vartheta_B$ leads to relations that are linear and nonlinear in the fields Ψ and $\overline{\Psi}$ (we omit the indices α):

$$\frac{\delta W}{\delta \overline{\Psi}} = 0, \quad \vartheta_B(x)\,(\mathrm{i}\,\partial_\mu \gamma^\mu - m_q)\,\Psi + \frac{1}{2}\delta_S(x)\,(\mathrm{i}\; n_\mu \gamma^\mu - 1)\,\Psi = 0; \tag{40}$$

$$\frac{\delta W}{\delta \vartheta_B} = 0,$$

$$\delta_S(x)\left[\frac{1}{2}\overline{\Psi}\; \mathrm{i}\; \overleftrightarrow{\partial_\mu} \gamma^\mu \Psi - m_q \overline{\Psi}\Psi - B - \frac{1}{2} n^\mu \partial_\mu (\overline{\Psi}\Psi) \right] = 0. \tag{41}$$

It follows from (40) that the fields Ψ and $\overline{\Psi}$ satisfy the free Dirac equation within the bag and the boundary conditions

$$\mathrm{i}\; n_\mu \gamma^\mu \Psi_\alpha = \Psi_\alpha, \quad \overline{\Psi}_\alpha \; \mathrm{i}\; n_\mu \gamma^\mu = -\overline{\Psi}_\alpha, \quad r = R \tag{42}$$

on its surface. The energy–momentum tensor of the system satisfies the continuity equation $\partial_\mu T^{\mu\nu} = 0$,

$$T^{\mu\nu}(x) = \vartheta_B(x)\left(\frac{1}{2}\overline{\Psi}_\alpha \gamma^\mu \; \mathrm{i}\; \overleftrightarrow{\partial_\mu} \Psi_\alpha + g^{\mu\nu} B \right), \tag{43}$$

and, therefore, we have the conservation laws $P^\mu = \int_V T^{0\mu}(x) d^3x = \text{const}$, the bag energy

$$H_q^{\mathrm{MIT}} \equiv P^0 = \int_V \left(\frac{1}{2}\overline{\Psi}_\alpha \gamma^0 \; \mathrm{i}\; \overleftrightarrow{\partial_0} \Psi_\alpha + B \right) d^3x = E_q + BV \tag{44}$$

including the volume contribution BV. Thus, the term of the action $-B\int d^4x \vartheta_B(x)$ plays in the relativistic appraoch the same role as the potential energy $\Sigma_{i<j} \lambda_i \lambda_j V^c(r_{ij})$ in the nonrelativistic Hamiltonian (2). However, there is an important difference, since quark confinement in the MIT model is ensured not only by the infinitely increasing energy of the bag as $R \to \infty$ but also by the linear boundary conditions (42), which are equivalent to the requirement of vanishing of the color flux through the surface of the bag (for each quark species α):

$$\overline{\Psi}_\alpha \mathrm{i} n_\mu \gamma^\mu \Psi_\alpha = \overline{\Psi}_\alpha \Psi_\alpha = -\overline{\Psi}_\alpha \Psi_\alpha = 0, \quad r = R$$

(no summation over α). (45)

In addition, it follows from (41) [if Eq. (40) are taken into account] that

$$-\frac{1}{2} n^\mu \partial_\mu (\overline{\Psi}\Psi) - B, \quad r = R, \tag{46}$$

which is a constraint on the fields $\overline{\Psi}$ and Ψ (the condition of equilibrium of the pressures of the quark gas and the gluon condensate at the interface).

Color magnetic interaction

Not a few studies have been devoted to calculation of the gluon propagator in the closed bag. However, we are interested here only in the static limit, which in any approach must lead to the magnetostatics of the quark currents,[36] i.e., to an interaction proportional to the operator $\Sigma_{i<k}\lambda_i\lambda_k j_i j_k$, where j_i is the total angular momentum of quark i. It can be shown (see Ref. 14) that the well-known magnetostatic formula of the MIT model,[35,36] which was derived for the color magnetic interaction of the quark orbitals $1s_{1/2}$, $2p_{1/2}$, also holds for the $2p_{3/2}$ orbitals.

$$H_{CM}^{\mathrm{MIT}} = -\frac{\alpha_s}{R}\sum_{i<k}^{N}\frac{\lambda_i\lambda_k}{4}(2j_i 2j_k)\,M(\omega_i,\ \omega_k), \tag{47}$$

$$M(\omega_i,\ \omega_k) = 3\,\frac{\mu_i(R)\,\mu_k(R)}{R^3} \times\left[1+\frac{2R^3}{\mu_i(R)\,\mu_k(R)}\int_0^R\frac{\mu_i(r)\,\mu_k(r)}{r^4}\,dr\right], \tag{48}$$

where $\mu_i(r)$ is expressed in terms of the density of the quark color magnetic moment $\mu_i'(r)$ in the bag:

$$\mu_i^\alpha = \lambda_i^\alpha 2j_i\int_0^R\mu_i'(r)\,dr,\ \ \mu_i(r) = \int_0^r\mu_i'(\rho)\,d\rho, \tag{49}$$

$$\mu'(r) = \begin{cases}\frac{2}{3}N_n^2 j_0\left(\omega_n\frac{r}{R}\right)j_1\left(\omega_n\frac{r}{R}\right)r^3 & \text{for } 1s_{1/2},\ 2p_{1/2};\\ \frac{4}{15}N_n^2 j_1\left(\omega_n\frac{r}{R}\right)j_2\left(\omega_n\frac{r}{R}\right)r^3 & \text{for } 2p_{3/2}\end{cases} \tag{50}$$

Here, ω_n are the eigenfrequencies of the solutions satisfying the boundary condition (42). We take into account the $2p_{3/2}$ modes, since without them the algebraic treatment would be incomplete (for example, it is impossible to go over from jj to LS coupling). The $2p_{3/2}$ states are also needed when one considers translational modes.[63]

We mention some differences between the quark Hamiltonian in the bag model and the nonrelativistic Hamiltonian (2); 1) in the qq interaction term (47) allowance is made for "orbital color magnetism," which is not the case in the potential models: 2) the boundary condition (42) automatically leads to a spin–orbit splitting

$$R^{-1}[\omega(p_{1/2}) - \omega(p_{3/2})] \approx 100\ \text{MeV}. \tag{51}$$

Combining all the contributions, we obtain the following generalization of the energy operator (44) of the bag:

$$H_q^{\mathrm{MIT}} = \frac{4}{3}\pi R^3 B + \frac{1}{R}\Big(\sum_{u=s_{1/2},\,p_{1/2},\,p_{3/2}} N_u\,\omega_u - Z_0\Big) + H_{CM}^{\mathrm{MIT}} \tag{52}$$

(the term $-Z_0/R$ is the so-called Casimir energy, the result of subtracting the zero-point vibrations of the fields confined in the finite spherical bag). For a spherical bag, the pressure equilibrium condition (46) is equivalent to the condition of a minimum of the energy of the bag when its radius R is varied,

$$dH_q^{\mathrm{MIT}}/dR = 0,\ \ R = \overline{R}, \tag{53}$$

and the hadron mass is determined by Eq. (52) for $R = \overline{R}$.

The spectrum of the six-quark bag with allowance for orbital excitation was studied in Refs. 14, 44, and 65. It was shown in Ref. 44 that the color magnetic interaction leads to aproximate degeneracy of the lowest levels of two configurations: $s_{1/2}^4\,p_{1/2}^2$ and $s_{1/2}^6$. This result was subsequently confirmed by a calculation[14] using a complete basis of the configurations $s_{1/2}^4\,p_{1/2}^2$ and $s_{1/2}^4\,p_{3/2}^2$. Such calculations are rather difficult because the color magnetic interaction (47) breaks the CS symmetry, and therefore it is not possible to use the technique of the Casimir invariants of the groups (3). Table II gives the values of the orbital integral $M(\omega_u, \omega_v)$ of the color magnetic interaction in different binary orbital states uv. We shall consider here the general scheme of application of the technique of coefficients of fractional parentage and the Clebsch–Gordan coefficients of the unitary groups of higher rank that was used in Refs. 13, 14, and 44.

Classification of the states

In the jj coupling scheme, we use the quantum numbers of the groups $U(m)$ from the reduction chain

$$U(48)_{JCI} \supset U(8)_J \times U(6)_{CI} \supset U(8)_J \times U(3)_C \times U(2)_I. \tag{54}$$

We here regard the eight different projections of the angular momenta of the orbitals $s_{1/2}$, $p_{1/2}$, and $p_{3/2}$ as the projections of one spinor q_J of the group $U(8)_J$. For this group, we use the system of quantum numbers corresponding to the reduction to the direct sum $s_{1/2}\oplus p_{1/2}\oplus p_{3/2}$ of subspaces:

$$U(8)_J \supset U(2)_{s1} \times U(2)_{p1} \times U(4)_{p3}. \tag{55}$$

We shall label the state vector (to be specific, we consider the $6q$ state with the deuteron quantum numbers J^P, $I = 1^+, 0$) by the permutational Young tableaux $[f]$ in all the subspaces (54) and by the values of the angular momenta J_{s1}, J_{p1}, and J_{p3} of the shell $s_{1/2}^4$, $p_{1/2}^2$, and $p_{3/2}^2$. We shall denote the angular momentum of the mixed shell $p_{1/2}\,p_{3/2}$ by J_p^+ or J_p^-, depending on the permutational symmetry of the orbitals. Thus, in the jj coupling scheme we denote the basis vectors of the configurations $s_{1/2}^4 p_{n/2}^2$, $n = 1, 3$ as follows:

$$\Psi_i^{jj} = |s_{1/2}^4 p_{n/2}^2\,[f_J]\,(J_{s1},\ J_{pn})\,J,\ [2^3]_C I\,|f_{CI} = \tilde{f}_J] : [1^6]_{JCI}\rangle. \tag{56}$$

TABLE II. The matrix element (48).

uv	$s_{1/2}^2$	$p_{1/2}^2$	$p_{3/2}^2$	$s_{1/2}p_{1/2}$	$s_{1/2}p_{3/2}$
$\frac{\alpha_s}{4}M(\omega_u,\ \omega_v)$	0.09736	0.06454	0.01837	0.06131	0.04201

In our case $I=0$, and therefore $[f_I]=[3^2]$, and there are altogether four different CI states:

$$[2^3]_C\circ[3^2]_I=[1^6]_{CI}+[2^21^2]_{CI}+[3^2]_{CI}+[41^2]_{CI}. \quad (57)$$

In accordance with the Pauli principle $[f_J]=[\tilde{f}_{CI}]$, and therefore the orbital symmetries $[f_J]=[6],[42],[2^3],[31^3]$ are allowed. Another restriction on the quantum numbers of the vector (56) follows from the relations

$$J_s+J_p=J,\ [f_s]\times[f_p]\to[f_J]. \quad (58)$$

In Table III we have listed all the states that satisfy Eqs. (57) and (58) (total angular momentum $J=1$). In the same table we give the quantum numbers of the basis vectors in the LS coupling scheme,

$$\Psi_i^{LS}=|s^4p^2[f_{XS}]([f_X]LS,\omega_{XS})J,\ [2^3]_C I[f_{CI}=\tilde{f}_{XS}]:[1^6]_{XSCI}\rangle, \quad (59)$$

where for the classification of the state we use the reduction

$$U(48)_{XSCI}\supset U(8)_{XS}\times U(6)_{CI}\supset U(8)_{XS}\times U(3)_C\times U(2)_I, \quad (60)$$

$$U(8)_{XS}\supset U(2)_S\times U(4)_X, \quad (61)$$

in which $U(4)_X$ is the orbital group acting on the direct sum of spaces $s\oplus P$; ω_{XS} is the repetition index for the multiple representations in the inner product $[f_X]\circ[f_S]$.

Some of the states listed in Table III correspond to excitations of the center of mass of the complete system. The separation of the spurious states in a relativistic system is a special problem,[63,64] which we shall not attack here, since the MIT model does not pretend to quantitative predictions for orbital excitations.

2. Fractional parentage and Clebsch–Gordan coefficients

The fractional-parentage decomposition $q^N\to q^{N'}\times q^{N''}$ $(N'+N''=N)$ of an arbitrary basis vector [(56) or (59)] can be expressed symbolically in the form

$$|q^N[1^N]\alpha\rangle=\sum_{\alpha',\alpha''}\Gamma(\alpha',\alpha'':\alpha)\times\{|q^{N'}[1^{N'}]\alpha'\rangle|q^{N''}[1^{N''}]\alpha''\rangle\}_{JCI}, \quad (62)$$

where $\alpha=\alpha_{JCI}$ (or α_{XSCI}) is a complete set of internal quantum numbers of the group $U(48)_{JCI}$ [or $U(48)_{XSCI}$] for the reduction (54) [or (60)]; the quantum numbers identified by single and double primes refer to the subsystems $q^{N'}$ and $q^{N''}$, respectively. The curly brackets $\{\ldots\}_{JCI}$ mean, as usual, the operation of addition of all three-dimensional moments [C is a "color moment," an invariant of the group $O(3)$ that appears in the reduction $U(3)_C\supset O(3)_C$ which is used here].

The relation (62) can be regarded as a special case of the general formula for reducing a direct product of irreducible representations of the unitary group $U(m)$, $m=48$, by means of the Clebsch–Gordan coefficients of this group (see

TABLE III. Complete basis of the configurations s^6 and s^4p^2 ($J, I=1, 0$).

Quantum numbers	Number of states	$[f]_J$ or $[f]_{XS}$				Configuration
		[6]	[42]	[23]	[31³]	
$J_sJ_p^*$	$s^6_{1/2}$		10			1
	$s^4_{1/2}p^2_{1/2}$	21	21, 11, 10, 01	01	10	7
	$s^4_{1/2}p^2_{3/2}$	23, 21	23, 21, 12, 11, 10, 01	01	12, 10	11
	$s^4_{1/2}p_{1/2}p_{3/2}$	$22^+, 21^+$	$22^+, 21^+, 12^\pm, 11^\pm, 01^+$	01^+	$11^-, 12^-$	12
LS_ω^{**}	$s^6[6]_X$		01			1
	$s^4p^2[6]_X$	23	21, 01			3
	$s^4p^2[51]_X$	22, 12	22, 21, 12, 11, 10, 01			8
	$s^4p^2[42]_X$	21, 01	$23, 22, 21_1, 21_2, 01_1, 01_2$	21, 01	21, 01	12
	$s^4p^2[41^2]_X$		12, 11, 10	10	12, 11, 10	7

*J_s, J_p in the jj coupling scheme.
**LS, ω_{XS} in the LS coupling scheme.

Sec. 4):

$$|q^N[f]\,\alpha,\ \gamma\rangle = \sum_{\alpha',\,\alpha''}\left[\begin{matrix} f' \times f'' \\ \alpha' \quad \alpha'' \end{matrix}\middle|\begin{matrix} f,\ \gamma \\ \alpha \end{matrix}\right]|q^{N'}[f']\,\alpha'\rangle|q^{N''}[f'']\,\alpha''\rangle. \tag{63}$$

The decomposition coefficients (62) and (63) are related by the obvious equation

$$\left[\begin{matrix} 1^{N'} \times 1^{N''} \\ \alpha' \quad\quad \alpha'' \end{matrix}\middle|\begin{matrix} 1^N \\ \alpha \end{matrix}\right] = \Gamma(\alpha',\ \alpha'':\alpha)\,\langle J'J'_zJ''J''_z|JJ_z\rangle \times \langle C'C'_zC''C''_z|CC_z\rangle\,\langle I'I'_zI''I''_z|II_z\rangle. \tag{64}$$

Thus, the required fractional-parentage coefficient $\Gamma(\alpha', \alpha'':\alpha)$ is the factor of the Clebsch–Gordan coefficient of the group $U(48)_{JCI}$, which is invariant with respect to the subgroup $O(3)_J \times O(3)_C \times SU(2)_I$. In Sec. 4, we shall continue the fractional-parentage factorization, following the reduction (54) [or (60)], and give a method for calculating this factor. Here, taking the coefficients $\Gamma(\alpha', \alpha'':\alpha)$ to be known, we shall solve the variational problem for the MIT Hamiltonian (52) in the *jj* coupling scheme (56).

Diagonalization of the MIT Hamiltonian by means of the fractional-parentage technique

The spectrum of the configurations $s^4_{1/2}\,p^2_{1/2}$ and $s^4_{1/2}\,p^2_{3/2}$ was determined by diagonalizing the energy operator (52) in the basis (56):

$$\mathrm{Det}\,\langle s^4_{1/2}p^2_{k/2}\,[1^6]\,\alpha_{JCI}|H^{\mathrm{MIT}}_q(R) - E|s^4_{1/2}p^2_{k/2}\,[1^6]\,\bar{\alpha}_{JCI}\rangle = 0 \tag{65}$$

with subsequent minimization of the eigenvalues $E_n(R)$ with respect to the bag radius:

$$dE_n/dR = 0,\ R = \bar{R}. \tag{66}$$

This led to a mass spectrum

$$M_n = E_n(\bar{R}) \tag{67}$$

for each configuration.

All the terms of the energy operator (52) except the last, $H_{\mathrm{CM}}(R)$, are diagonal in the *jj*-coupling basis (56). The color magnetic operator (47) is not diagonal, since the orbital integral $M(\omega_u, \omega_v)$ has different values for different quark pairs (see Table II). The two-particle coefficients of fractional parentage for $q^6 \to q^4 \times q^2$ make it possible to reduce the summation over the quark pairs ik in (47) to a sum over all states α''_{JCI} of one pair (for example, the pair $ik = 56$) in the fractional-parentage decomposition (62). Substituting the decomposition (62) for the case $N' = 4$, $N'' = 2$ in the bra and ket of the matrix element (65), we obtain the following decomposition of the matrix element of the color magnetic operator (47):

$$\langle s^4_{1/2}p^2_{n/2}\,[1^6]\,\alpha_{JCI}|H^{\mathrm{MIT}}_{\mathrm{CM}}|s^4_{1/2}p^2_{n/2}\,[1^6]\,\bar{\alpha}_{JCI}\rangle = \tfrac{1}{2}N(N-1)\frac{1}{R}\sum_{\alpha'_{JCI}}\sum_{\alpha''_{JCI}}\sum_{\bar{\alpha}''_{JCI}}\Gamma(\alpha'_{JCI},\ \alpha''_{JCI}:\alpha_{JCI})\times\Gamma(\alpha'_{JCI},\ \bar{\alpha}''_{JCI}:\bar{\alpha}_{JCI})\,\langle[f'_C]|-\sum_{\alpha=1}^{8}\lambda^\alpha_5\lambda^\alpha_6|[f''_C]\rangle\,\langle J''|2j_5\times 2j_6|J'''\rangle \times\langle s^{N''_{s1}}_{1/2}p^{N''_{pn}}_{n/2}\,[f''_J]\left|\frac{\alpha_s}{4}M(\omega_5,\ \omega_6)\right|s^{N''_{s1}}_{1/2}p^{N''_{pn}}_{n/2}\,[f'_J]\rangle. \tag{68}$$

To calculate (68), it is necessary to know not only the coefficients of fractional parentage but also the simplest matrix elements of the two-particle operators $-\Sigma^8_{\alpha=1}\lambda^\alpha_i\lambda^\alpha_k$ and $2j_i 2j_k$ in symmetric and antisymmetric two-body states, for example.

$$\langle[f''_C]\,|-\sum_{\alpha=1}^{8}\lambda^\alpha_i\lambda^\alpha_k\,|\,[f'_C]\rangle = \begin{cases} -\frac{4}{3}, & [f''_C] = [2], \\ \frac{8}{3}, & [f''_C] = [1^2], \end{cases} \tag{69}$$

$$\langle J''\,|\,2j_i\cdot 2j_k\,|\,J'\rangle = -15\left(j_i = j_k = \frac{3}{2},\ J'' = 0\right)\ \ldots, \tag{70}$$

and also the orbital integrals $M(\omega_u, \omega_v)$ for all possible combinations $s^{N''_{s1}}_{1/2}\,p^{N''_{pn}}_{n/2}, N''_{s1} + N''_{pn} = 2$, $n = 1,3$ (see Table II). After diagonalization of the resulting matrix, we arrive at the spectrum given in Table IV.

Thus, the methods of Sec. 2 are generalized. Casimir invariants were used in earlier studies of the spectrum of the $6q$ bag[35,60,61] only for the configuration $s^6_{1/2}$, for which the orbital part $M(\omega(s_{1/2}), \omega(s_{1/2}))$ can be taken in front of the sum in (47). There is now a possibility of making calculations for orbitally excited configurations as well (tables of coefficients of fractional parentage were published in Refs. 13 and 14).

The spectra given in Table IV show that in complete agreement with the results of the nonrelativistic estimates of Sec. 1 the lowest states of the configurations $s^4_{1/2}\,p^2_{1/2}$ and $s^4_{1/2}\,p^2_{3/2}$ are approximately degenerate with the state 2161 MeV of the configuration $s^6_{1/2}$. There is a difference from the nonrelativistic calculation only in that the spin–orbit splitting separates the poles of the state $s^4_{1/2}\,p^2_{3/2}$ and $s^4_{1/2}\,p^2_{1/2}$ from each other by about 0.3 GeV [in accordance with the splitting (51)].

The *P*-matrix approach

In the case of the $6q$ bag, the boundary condition (42) of the MIT model does not correspond to the physical situation, since quarks can leave the bag in colorless fragments, forming $3q$ clusters. Therefore, the discrete spectrum of the $6q$ bag, which is the result of solution of a problem with artificial boundary conditions, cannot correspond to any observable particles (for example, dibaryon resonances). Such eigenstates are now usually called "primitive states."

Jaffe and Low[24] pointed out that primitive states must in fact be present in hadron scattering states in the sense that they can serve as a good approximation to the wave function in the interior region at an energy E of the system close to the eigenenergy E_n of a primitive state.

This was illustrated in Ref. 24 by a simplified example

TABLE IV. Spectrum of primitive states (MeV) $J, I = 1, 0$.

n	0	1	2	3	4	5	6	7	8	9	10	11
$s^6_{1/2}$	2161											
$s^4_{1/2}p^2_{1/2}$		2373	2557	2578	2640	2649	2742	2869				
$s^4_{1/2}p^2_{3/2}$		2045	2193	2208	2275	2403	2409	2471	2530	2535	2631	2744

of particle scattering by a rectangular well having a depth insufficient for the formation of a bound state. If for the wave function $\Psi_E(r)$ of this problem one uses an artificial boundary condition of "confinement," setting $\Psi_E(b) = 0$ at the edge of the well ($r = b$), then one obtains a discrete spectrum of primitive states in the well: $E_n = k_n^2/2\mu$, $\sqrt{k_n^2 + 2\mu V_0} = n\pi/b$. It can be shown that scattering by the well can be described in terms of this spectrum. The logarithmic derivative of the scattering wave function $\Psi_E^{(+)}(r)$ at the point $r = b$ can be calculated in this case analytically:

$$P(k, b) \equiv \frac{d[r\Psi_E^{(+)}(r)}{\Psi_E^{(+)}(r)\,dr}\Big|_{r=b} = qb\cot qb = 1 - 2q^2b^2\sum_{n=1}^{\infty}\frac{1}{q^2b^2 - n^2\pi^2}, \quad (71)$$

where $q = \sqrt{k^2 + 2\mu V_0}$. The last equation in (71) is the expansion of $x\cot x$ into simple fractions. It can be seen from this expansion that the P matrix defined by the first equation in (71) has poles at the points E_n of the primitive spectrum. Moreover, as $k \to k_n$ the wave function $\Psi_E^{(+)}(r)$ in the region $r \leqslant b$ is identical to the primitive state $\Psi_{E_n}(r)$.

In the application to the problem of scattering of $3q$–$3q$, $3q$–$3q\bar{q}$, $q\bar{q}$–$q\bar{q}$ quark clusters, this analogy with scattering by a rectangular well led to the formulation of the P-matrix approach, in which the poles of the P matrix are identified with the primitive spectra of the corresponding MIT bags (q^6, $q^4\bar{q}$, $q^2\bar{q}^2$). Of course, for cluster systems of the type $3q + 3q$ there is no rigorous proof, as in the case (71), that the P matrix has poles at the points of the primitive spectrum and that the scattering wave function in the region $r \leqslant b$ is identical to a primitive state of the MIT bag. This is only an assumption, though it can be realized in the model by formulating appropriately an approximation to the many-particle scattering problem.

In the composite quark bag model proposed by Simonov,[25] the connection between the hadron component $\Psi_h(r)$ (nucleon in the given case) and the quark component $\Psi_q(r_1, \ldots, r_6)$ of the wave function on the NN system is established by means of a phenomenological surface "channel-coupling" potential

$$V_{hq}\,|\,\Psi_q^{\nu}\rangle = C_{\nu}\frac{\delta(r-b)}{r}, \quad (72)$$

where b plays the part of an (artificial) range of the NN channel ($b \gtrsim R$). In this model, Ψ_h decreases rapidly for $r < b$, since in the NN interaction channel (72) there appears an effective potential barrier (analog of a core). In its turn, Ψ_q^{ν} satisfies the artificial boundary conditions of the MIT model. Because of this, the spectrum E_{ν} of the $6q$ system is discrete, and the Green's function in the quark channel has the form

$$G_q(E) = \sum_{\nu}\frac{|\,\Psi_q^{\nu}\rangle\langle\Psi_q^{\nu}\,|}{E - E_{\nu}}. \quad (73)$$

In Simonov's model,[25] the pole expansion

$$P_h(k, b) = \sum_{\nu}\frac{C_{\nu}^2}{E - E_{\nu}} + P_0(k, b) \quad (74)$$

automatically follows from (72) and (73) after the standard procedure of projection[50,69] of the quark channel from the equations for Ψ_h.

Comparison of the expansion (74) with the experimental phase shifts of NN scattering [in the case of S waves, $P_h(k, b) = kb\cot(kb + \delta(k))$] it was shown[25–28] that the $^3S_1$ and $^1S_0$ phase shifts in the wide range of energies $0 \leqslant E_{\text{lab}} \lesssim 500$ MeV are completely determined by just one pole $E_1 \approx 2.1$–2.2 GeV of the P matrix (for example, for $b = 1.44$ F, we have $R = 1.32$ F). Qualitatively, this agrees with the spectra of Table IV, since the lowest $6q$ states of the configurations s^6 and s^4p^2 lie above the NN threshold for $E_{\text{lab}} \gtrsim 450$ MeV. However, it cannot be also asserted that the lowest primitive state belongs to the configuration $s^6_{1/2}$. It is very probable that a situation is realized analogous to the one obtained in nonrelativistic calculations in Harvey's approach or using the resonating-group method, i.e., the wave function of the $6q$ system in the region $r < b$ is a superposition of $s^6_{1/2}$ and $s^4_{1/2}\,p^2_{3/2}$. The admixture of the $s^4_{1/2}\,p^2_{3/2}$ state is analogous to the D-wave admixture in the deuteron. The $s^4_{1/2}\,p^2_{3/2}$ configuration is spherically asymmetric and has nonzero quadrupole moment. This is an interesting question, since it makes it possible to reexamine the connection between the quadrupole moment of the deuteron and its anomalous magnetic moment; however, on the basis of the MIT model it is unfortunately meaningless to make quantitative predictions for orbitally excited configurations. Our treatment is decidedly qualitative in nature.

3. GENERALIZATION TO THE CASE OF $9q$, $12q$, AND $3Aq$ SYSTEMS

It is interesting to consider at the quark microscopic level the evolution of the color magnetic attraction on the transition to the nearest heavier nuclei ^{3}H–^{3}He, ^{4}He, etc. It is difficult to anticipate the result, since the Young tableaux $[f_X]$ and $[f_{CS}]$ acquire several rows, and all the relations

look much more complicated. Therefore, at present it is not expedient to study systems such as $9q$, $12q$, $15q$, etc., on the basis of the Clebsch–Gordan technique; rather, it is better to use the approximation of Casimir operators. We begin with the multiplication of Young tableaux.

Auxiliary theorem

To obtain the inner products $[f_C]\circ[f_S]$ and $[f_{CS}]\circ[f_I]$ we exploit the simplicity of the situation associated with the fact that one of the multiplied Young tableaux has two rows. It is then convenient to use Littlewood's theorem,[70] according to which the following relation holds for mixed products:

$$([\lambda]\times[\mu])\circ[\nu]=\sum_{\rho,\sigma}\Gamma_{\rho\sigma\nu}([\lambda]\circ[\rho])([\mu]\circ[\sigma]). \quad (75)$$

The coefficients $\Gamma_{\rho\sigma\nu}$ "select" from all possible schemes $[\rho]$ and $[\sigma]$ only those that in the outer product $[\rho]\times[\sigma]=\Sigma_{\nu'}\Gamma_{\rho\sigma\nu'}[\nu']$ give among other $[\nu']$ the Young tableau $[\nu]$ ($\Gamma_{\rho\sigma\nu'}$ is the multiplicity of $[\nu']$).

Using the fact that the two-row Young tableau $[N-m, m]$ in which we are interested is contained in the outer product

$$[N-m]\times[m]=[N]+[N-1,1]+[N-2,2]\quad\ldots\quad\ldots+[N-m,m], \quad (76)$$

which can be readily calculated, substitution of $[N-m]\times[m]$ in Eq. (75) on the left leads to an important simplification of this equation:

$$([N-m]\times[m])\circ[\nu]_N=\sum_{\rho,\sigma}\Gamma_{\rho\sigma\nu}[\rho]_{N-m}\times[\sigma]_m. \quad (77)$$

We have used the fact that $[N-m]\circ[\rho]=[\rho]$ and that $[\rho]$ must be a Young tableau with $(N-m)$ cells [this is indicated by the subscript in (77)], and $[m]\circ[\sigma]=[\sigma]$, where $[\sigma]=[\sigma]_m$ It follows from (81) that $[N-m, m]=[N-m]\times[m]-[N-m+1]\times[m-1]$, and hence, using (77), we obtain the final result

$$[N-m,\ m]\circ[\nu]_N=\sum_{\rho,\sigma}\Gamma_{\rho\sigma\nu}[\rho]_{N-m}\times[\sigma]_m-\sum_{\alpha,\beta}\Gamma_{\alpha\beta\nu}[\alpha]_{N-m+1}\times[\beta]_{m-1}. \quad (78)$$

The relation (78) makes it possible to reduce the inner multiplication by the two-row Young tableau to outer products of auxiliary Young tableaux, for the calculation of which simple rules exist.[40,70]

The most symmetric states in the *CS* space

We now turn to the ^{3}He–^{3}H nucleus and consider the corresponding microscopic states of the $9q$ system, for which we have the Young tableaux $[3^3]_C$, $[54]_S$, and $[54]_I$ corresponding to the values $S=\frac{1}{2}$, $I=\frac{1}{2}$ and the color neutrality condition. Using (78), we calculate the product

$$[54]_S\circ[3^3]_C=[621]+[54]+[531]+[521^2]+[432]+[431^2]+[42^21]+[421^3]+[3^221]+[32^21^2]+[32\,1^4]+[2^41], \quad (79)$$

in which $[621]_{CS}$ is the most symmetric CS state. We now determine the orbital state $s^{9-m}p^m[f_X]$ with the minimal number of p excitations, which will satisfy the Pauli principle, $[f_X]=[\tilde{f}_{CSI}]$, if $[f_{CS}]=[621]$. All possible Young tableaux $[f_{CSI}]$ are contained in the product, which we again calculate by means of (78):

$$[621]_{CS}\circ[54]_I=[72]+[71^2]+2[63]+3[621]\quad\ldots\quad\ldots+[3^3]+[3^221]+[3^21^3]. \quad (80)$$

The smallest number of p excitations corresponds to the last term in (80), for which $[f_X]=[52^2]$ by the Pauli principle. Thus, in the $9q$ system for $S=1/2$, $I=1/2$ the state

$$\Psi_1(9q)=|s^5p^4[52^2]_X,\ [3^3]_CS=\tfrac{1}{2}[621]_{CS}I=\tfrac{1}{2}[3^21^3]_{CSI}:[1^9]_{XSCI}\rangle \quad (81)$$

is energetically distinguished by the color magnetic attraction. In contrast, the "ground" state, corresponding to the configuration $s^9[9]_X$, belongs to the energetically disadvantageous Young tableau $[2^41]_{CS}$, the last in the Clebsch–Gordan series (79):

$$\Psi_0(9q)=|s^9[9]_X,\ [3^3]_CS=\tfrac{1}{2}[2^41]_{CS}I=\tfrac{1}{2}[1^9]_{CSI}:[1^9]_{XSCI}\rangle. \quad (82)$$

Going over to the $12q$ system with the quantum numbers of the ^{4}He nucleus ($S=0, I=0$), we make the same operations with the Young tableaux $[4^3]_C$, $[6^2]_S$, $[6^2]_I$:

$$[4^3]_C\circ[6^2]_S=[82^2]+[741]+\ldots+[3^22^21^2]+[2^6]$$
$$[82^2]_{CS}\circ[6^2]_I=[84]+[831]+\ldots+[4^22^2]+[4^21^4]. \quad (83)$$

We find that the maximal color magnetic attraction acts in the configuration $s^6p^6[62^3]_X$ for $[f_{CS}]=[82^2]$, i.e., the state

$$\Psi_{\text{II}}(12q)=|s^6p^6[62^3]_X,\ [4^3]_CS=0[82^2]_{CS}I=0[4^21^4]_{CSI}:[1^{12}]_{XSCI}\rangle \quad (84)$$

is energetically advantageous, and in the ground state

$$\Psi_0(12q)=|s^{12}[12]_X,\ [4^3]_CS=0[2^6]_{CS}I=0[1^{12}]_{CSI}:[1^{12}]_{XSCI}\rangle \quad (85)$$

color magnetic repulsive forces are dominant.

Rough estimate of the color magnetic effect

In Tables V and VI we give the values of the color magnetic contribution

$$\Delta([f_{CS}])=\langle[f_{CS}]|-\sum_{i<j}^{N}\lambda_i\lambda_j\sigma_i\sigma_j|[f_{CS}]\rangle, \quad (86)$$

calculated by means of the Casimir invariants in the systems $N=9$ and 12. Using expressions of the type (10) and (11), we obtain the following energies of the states (81), (82) and (84),

TABLE V. $9q$, $S=\frac{1}{2}$, $I=\frac{1}{2}$.

$[f]_{CS}$	$[2^41]$	$[321^4]$	$[32^21^2]$	$[3^221]$	$[421^3]$	$[42^21]$	$[431^2]$	$[432]$	$[521^3]$	$[531]$	$[54]$	$[621]$
$[f]_X$	[9]	[72]	[81]	[72]	[63]	[72]	[63]	[63]	[621]	[531]	$[4^21]$	$[52^2]$
s^np^m	s^9	s^7p^2	s^8p^1	s^7p^2	s^6p^3	s^7p^2	s^6p^3	s^6p^3	s^6p^3	s^5p^4	s^4p^5	s^5p^4
$\Delta([f]_{CS})$	40	40	24	4	8	−4	−12	−20	−24	−40	−56	−56
L	0	0,2	1	0,2	1,3	0,2	1,3	1,3	1,2	1, 2, 3	1, 2, 3, 4	0,2

(85) above the $3N$ and $4N$ thresholds, respectively [in an approximation independent of the form of the potentials in (2)]:

$$\begin{aligned} E_0(s^9) &= \langle \Psi_0(9q) \mid H_q \mid \Psi_0(9q) \rangle - 3\langle N \mid H_q \mid N \rangle \\ &= 64\langle V \rangle, \\ E_0(s^{12}) &= 128\langle V \rangle, \\ E_1(s^2p^6) &= 6\hbar\omega - 64\langle V \rangle. \end{aligned} \tag{87}$$

These relations must be compared with the results for the $6q$ system:

$$\left.\begin{aligned} E_0(s^6) &= \langle \Psi_0(6q) \mid H_q \mid \Psi(6q) \rangle - 2\langle N \mid H_q \mid N \rangle \\ &= \frac{56}{3}\langle V \rangle, \\ E_1(s^4p^2) &= \langle \Psi_1(6q) \mid H_q \mid \Psi_1(6q) \rangle - 2\langle N \mid H_q \mid N \rangle \\ &= 2\hbar\omega - \frac{40}{3}\langle V \rangle. \end{aligned}\right\} \tag{88}$$

Here, $\hbar\omega$ is the excitation energy of the p state in the $3Aq$ system. It can be estimated from the $N^*(1535)$ and $N(940)$ mass difference, $\hbar\omega \approx 600$ MeV. With increasing A, this quantity decreases, since the size of the $3Aq$ systems increases with increasing A. Taking for the $6q$, $9q$, and $12q$ systems the same value $\hbar\omega = 400$ MeV, at which the levels (88) are approximately degenerate, we obtain (for $\langle V \rangle \approx 19$ MeV)

$$\left.\begin{aligned} E_0(s^6) &\approx 350\ \text{MeV}, & E_1(s^4p^2) &\approx 550\ \text{MeV}, \\ E_0(s^9) &\approx 1200\ \text{MeV}, & E_1(s^5p^4) &\approx 1000\ \text{MeV}; \\ E_0(s^{12}) &\approx 2400\ \text{MeV}, & E_1(s^6p^6) &\approx 1200\ \text{MeV}. \end{aligned}\right\} \tag{89}$$

The resulting gain in the total energy due to the color magnetic attraction (calculated per nucleon pair) that the configurations s^4p^2, s^5p^4, s^6p^6, etc., give compared with the configurations s^6, s^9, s^{12}, etc., is shown in Fig. 6. The sharp minimum for the nucleus $A=4$ shows that in this sense the α particle is a "quark magic nucleus," and this is so, not because the lowest quark shell s^{12} is filled, but because of the pronounced lowering of the configuration s^6p^6 compared with s^{12}. The quark configuration $s^6p^6[62^3]_X$ in the ^{4}He nucleus can be represented in such a way that the configuration $s^4p^2[42]_X$ is contained with respect to each pair of nucleons. In the nucleon sector ($4N$) this appears in the form that with respect to each pair of nucleons the wave function has a node, i.e., is orthogonal to the $0S$ state. This is very similar to the description of the ^{12}C nucleus in the 3α model with forbidden states.[71]

Thus, the quark approach to the ^{4}He nucleus may be equivalent to using NN potentials with forbidden states of the type of Ref. 22. As can be seen from (89), the strength of the color magnetic attraction is not yet sufficient to draw all the four nucleons into the quark region $R_q \approx r_h$; the wave function is suppressed in this region (see Fig. 4, which shows a similar situation), and the logarithmic derivative of the function $\Psi(4N)$ at the boundary of the region is large for each two-body NN coordinate. It appears to us that this preliminary picture indicates ways of explaining the ^{4}He "puzzles."

First, there is the fact that the ^{4}He nucleus fails to be bound by 4–5 MeV in calculations with "realistic" NN forces.[72] The example of the 3α system shows[71] that the replacement of the $\alpha\alpha$ potentials with a repulsive core by a

TABLE VI. $12q$, $S=0$, $I=0$.

$[f]_{CS}$	$[2^6]$	$[42^4]$	[5421]	$[5^21^2]$	$[62^3]$	$[631^3]$	[6321]	[642]	$[6^2]$	$[7312^2]$	[741]	$[82^2]$
$[f]_X$	[12]	[10,2]	[741]	[642]	$[82^2]$	[642]	[732]	[642]	$[4^3]$	[6321]	[5421]	$[62^3]$
s^np^m	s^{12}	$s^{10}p^2$	s^7p^5	s^6p^6	s^8p^4	s^6p^6	s^7p^5	s^6p^6	s^4p^8	s^6p^6	s^5p^7	s^6p^6
$\Delta([f]_{CS})$	96	40	−24	−40	−24	−24	−36	−56	−96	−64	−84	−96
L	0	0,2	1, 2, 3, 4	$0,2^2, 3, 4$	0,2	$0,2^2, 3, 4$	1, 2, 3	$0,2^2, 3, 4$	0, 2, 4	1,2	1,2	0

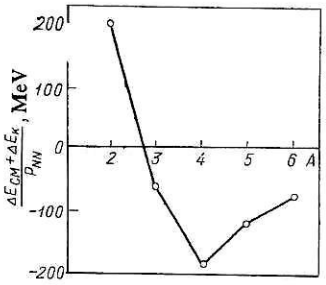

FIG. 6. Energy gain in the many-quark systems $3Aq$ due to color magnetic attraction in orbitally excited configurations $S^{3A\text{-}npn}$. The energy is calculated per nucleon pair, $\Delta E = (\Delta E_{CM} + \Delta E_k)/P_{NN}$, where ΔE_{CM} and ΔE_k are the change in the color magnetic energy and the kinetic energy on the transition from s^{3A} to the energetically most advantageous configuraiton $s^{3A\text{-}npn}$ (when $A > 4$, with allowance for the occupation of the quark shell s^{12} and excitation of the $2s$–$2d$ shell).[37]

deep attractive potential with forbidden states increases the binding energy of the system appreciably. A first attempt in this direction in the $4N$ system gave for the ^{4}He nucleus similar results ($E_{bd} \simeq 42$ MeV).[73] In Ref. 73, only an S-wave central potential[74] with one forbidden state was used. However, the inclusion of tensor forces usually decreases E_{bd} by about 8–9 MeV.[72] Therefore, it is necessary to go further in improving the accuracy of the results of Refs. 73 and 74.

Second, it is well known that it is necesary to introduce a "quark enhancement" into the electromagnetic form factors in the pre-asymptotic region.[75] It is possible that the quark exchange effects in the elastic form factors are small,[76] and then the need for "quark enhancement" means simply that the gradients of the wave function $\Psi(4N)$ at short distances between two (or three and four) nucleons are in reality much greater than in the case of "realistic" NN potentials.[43] This would correspond to a growth in the contribution of the s^4p^2 configurations on the transition from $2N$ to $4N$.

Third, there is the very interesting phenomenon of the dip in the density at the center of the ^{4}He nucleus with a radius of about 1 F.[77] It is natural to assume that this dip is due to the suppression of $\Psi(4N)$ in the quark region R_q (see Fig. 4) and to the influence of this irregularity in a certain neighborhood of R_q (there is a "healing" region, in which the logarithmic derivative is still large). Of course, such arguments are not rigorous, but they do demonstrate the value of the quark concepts in nuclei.

4. ALGEBRAIC METHODS OF THE QUARK MODEL

The Casimir-operator technique (see Sec. 1) makes it possible to solve[35] only a restricted number of problems. To calculate in the general case the matrix elements of single-quark, two-quark, etc., operators, to determine the weights of different baryon pairs (NN, $\Delta\Delta$, N^*N^*, etc.), hidden color,[33] etc., it is necessary to use a fractional-parentage decomposition $q^N \to q^{N'} \times q^{N''}$ ($N' + N'' = N, N'' = 1, 2, 3, \ldots$) of the form (62). In accordance with (64), the required coefficient of fractional parentage is the invariant (scalar) factor of the Clebsch–Gordan coefficient of the group $U(48)$ for the reduction

$$U(48)_{JCI} \supset O(3)_J \times O(3)_C \times SU(2)_I, \tag{90}$$

or

$$U(48)_{XSCI} \supset O(3)_X \times SU(2)_S \times O(3)_C \times SU(2)_I. \tag{91}$$

In view of the high ranks of the unitary groups (54) and (60), we also need methods of calculating coefficients of fractional parentage (or Clebsch–Gordan coefficients) that are: 1) independent of the rank of the group $U(m)$; 2) valid in the case of noncanonical reductions of the form that we have considered: (54), (55); (60), (61); (90), (91). In recent years, this problem has been solved using the formalism of the permutation group S_N (see Refs. 6, 13, 14, 66–68, and 78).

The group S_N began to be used for calculation of coefficients of fractional parentage as early as the beginning of the fifties in work on the nuclear shell model.[79] Such methods are now well known and are well represented in the monographs of Refs. 40 and 80. Nevertheless, deeper understanding of the interconnection between the groups $U(m)$ and S_N continued to be achieved (Refs. 13, 66–68, 78, and 81), and quark models gave a new impetus to this progress. We shall here consider only the fundamental questions needed for an understanding of the present state of the problem.

Historical remarks

The most important fundamental result is that of Weyl,[54] who established a correspondence between the irreducible representations of the groups $U(m)$ and S_N in the tensor space

$$q^N = \underbrace{q \times q \times \ldots \times q}_{N}$$

[here, $q = (q_1 q_2 \ldots q_m)$ is a spinor of the group $U(m)$]. The tensors q^N form a basis of an irreducible representation of the group $[U(m)]^N = U(m) \times U(m) \times \ldots \times U(m)$ that is reducible with respect to the subgroup $U(m) \times S_N$. The irreducible representations of the group $U(m)$ (we denote their Young tableau by $[f]$) and of the group S_N (with corresponding notation $\{f\}$) are combined into one irreducible representation of the group $U(m) \times S_N$ (notation $([f], \{g\})$) in such a way that: 1) $f = g$; 2) there are no multiplicities in the Clebsch–Gordan series

$$([1], \{1\})^N = \sum_f ([f], \{f\}). \tag{92}$$

In (92) on the left we have the direct product of spinor representations of the group $U(m) \times S_N$, i.e., we actually have the complete space q^N. The summation on the right in (92) is over all possible N-cell Young tableaux f containing not more than m rows ($f_1 + f_2 + \ldots + f_m = N$).

Pairs of groups possessing the listed properties were studied by Moshinsky and Quesne in Ref. 82, in which such pairs of groups were said to be mutually complementary and a general assertion was formulated, namely, in the theory of representations of mutually complementary groups all invariant quantities that have the same meaning are equal, and therefore they can be calculated in the formalism of the one group as well as in the formalism of its complementary group. A trivial example is an invariant such as the Young tableau $[f]$ (or $\{f\}$), which according to Weyl[54] characterizes irreducible representations of the group $U(m)$ and the

group S_N simultaneously. A less trivial example is provided by the invariant symbols of these groups (Racah, Fano coefficients, etc.). Each of the $6f$, $9f$, etc., symbols[83,84] depends only on the Young tableaux and on the repetition indices and, as was shown in Refs. 85 and 86, has the same value in the formalism of the group $U(m)$ and in the formalism of S_N. A full investigation of these questions was completed quite recently by Alishauskas.[81]

In recent years the consequences of this complementarity of the groups has also been followed at the level of invariant quantities such as the Clebsch–Gordan coefficients of the groups $U(m)$ and S_N. The Clebsch–Gordan coefficients depend both on the invariants (Young tableaux) and on "projections"—the internal quantum numbers of these groups; however, the "projections" are also invariants, though with respect to definite subgroups of the original group, and the complementarity principle can also be extended to them. In Ref. 87 a number of theorems were proved that established relations between contractions of the Clebsch–Gordan coefficients of the groups $U(mn) \supset U(m) \times U(n)$ and $S_N \supset S_{N'} \times S_{N''}$. Finally, in Refs. 13, 14, and 66–68 it was shown that the scalar factors of the Clebsch–Gordan coefficients of the groups $U(m)$ and S_N are numerically equal (for an appropriate choice of the systems of internal quantum numbers.[1)]

This last result makes it possible to reduce the calculation of the Clebsch–Gordan coefficients of a unitary group $U(m)$ of arbitrary rank m to the solution of the known equations for the Clebsch–Gordan coefficients of the group S_N (the equations for the K matrix from the book of Ref. 40). We shall here describe such a method for calculating the Clebsch–Gordan coefficients of the groups $U(m)$ for reductions of the type (54), (55) and (60), (61), following Refs. 13 and 14.

Factorization of Clebsch–Gordan coefficients

In accordance with Racah's (factorization) lemma,[88] a Clebsch–Gordan coefficient can be represented as a product of invariant (scalar) factors corresponding to the links of the reduction chain of subgroups and Clebsch–Gordan coefficients of subgroups of lower rank. In the case of the reductions (54), (55) and (60), (61) it is necessary to consider reduction links of two types:

$$U\ (mn)_{ab} \supset U\ (m)_a \times U\ (n)_b; \tag{93}$$

$$U\ (m+n)_{a+b} \supset U\ (m)_a \times U\ (n)_b. \tag{94}$$

The spaces ab and $a+b$ in (93) and (94) form the Kronecker product ($q_{ab} = q_a \circ q_b$) or the direct sum ($q_{a+b} = q_a \oplus q_b$) of the subspaces a and b ($q_{ab}, q_{a+b}, q_a, q_b$ are spinors of the considered groups).

In the case of the reduction (93), the factorization formula has the form [the notation is the same as in (63)]

$$\begin{bmatrix} f'_{ab} \times f''_{ab} & f_{ab} & \gamma_{ab} \\ \alpha'_{ab} & \alpha''_{ab} & \alpha_{ab} \end{bmatrix} = \sum_{\gamma_a, \gamma_b} F^U_{a \cdot b} \begin{bmatrix} f'_a \times f''_a & f_a \gamma_a \\ \alpha'_a & \alpha''_a \end{bmatrix} \begin{bmatrix} f'_b \times f''_b & f_b & \gamma_b \\ \alpha'_b & \alpha''_b & \alpha_b \end{bmatrix}, \tag{95}$$

where $F^U_{a \cdot b}$ is a scalar factor that depends only on the invariants of the groups $U(mn)$, $U(m)$, and $U(n)$:

$$F^U_{a \cdot b} = \left(\begin{pmatrix} f'_a f'_b f'_{ab} & (\omega'_{ab}) \\ f''_a f''_b f''_{ab} & (\omega''_{ab}) \\ f_a f_b f_{ab} & (\omega_{ab}) \\ (\gamma_a) & (\gamma_b)\ (\gamma_{ab}) \end{pmatrix} \right). \tag{96}$$

Here, γ and ω are repetition indices identifying multiple representations in outer and inner products of Young tableaux, respectively:

$$[f'] \times [f''] = \sum_f \nu_f^{\text{out}} [f], \quad \gamma = 1, 2, \ldots, \nu_f^{\text{out}}, \tag{97}$$

$$[f_a] \circ [f_b] = \sum_{f_{ab}} \nu_f^{\text{in}} [f_{ab}], \quad \omega_{ab} = 1, 2, \ldots, \nu_f^{\text{in}}. \tag{98}$$

Because of the multiplicities in (97), the factorization of the Clebsch–Gordan coefficients in (95) is incomplete and summation over the repetition indices γ_a and γ_b is also understood.

In the case of the reduction (94), we have an analogous factorization formula, and the corresponding scalar factor F^U_{a+b} is (apart from the normalization) identical to the $9f$ symbol introduced in Ref. 83,

$$F^U_{a+b} = (-1)^{N''_a N'_b} \binom{N_a}{N''_a}^{1/2} \binom{N_b}{N''_b}^{1/2} \binom{N_{a+b}}{N''_{a+b}}^{-1/2} \times \left(\frac{n_{f'_a} n_{f''_a}}{n_{f_a}} \right)^{1/2} \left(\frac{n_{f'_b} n_{f''_b}}{n_{f_b}} \right)^{1/2} \left(\frac{n_{f'_{a+b}} n_{f''_{a+b}}}{n_{f_{a+b}}} \right)^{-1/2} \times \begin{pmatrix} f'_a f'_b f'_{a+b} \\ f''_a f''_b f''_{a+b} \\ f_a f_b f_{a+b} \end{pmatrix}^{\gamma' \gamma'' \gamma}_{\gamma_a \gamma_b \gamma_{a+b}}. \tag{99}$$

In Eq. (99), which is taken from Ref. 12, n_f is the dimension of the irreducible representation $\{f\}$ of the group S_N, $N = N_a + N_b = N'_{a+b} + N''_{a+b}$, $N_a = N'_a + N''_a$, $N_b = N'_b + N''_b$, $N'_{a+b} = N'_a + N'_b$, $N''_{a+b} = N''_a + N''_b$.

Because the transformation (95) is unitary, the scalar factors are normalized by

$$\sum_{\gamma_a, \gamma_b} \sum_{f'_a, f''_a} \sum_{f'_b, f''_b} |F^U_{a \cdot b}|^2 = 1, \quad \sum_{\gamma_a, \gamma_b} \sum_{f'_a, f''_a} \sum_{f'_b, f''_b} |F^U_{a+b}|^2 = 1. \tag{100}$$

Examples of other scalar factors. Shell coefficients of fractional parentage

Another example of a scalar factor is the isoscalar factor introduced by De Swart.[89] It corresponds to the reduction $SU(3)_F \supset SU(2)_I$. The scalar factors are orbital coefficients of fractional parentage of the shell model.[12] The single-shell and two-shell coefficients of fractional parentage correspond to the reductions

$$U\ (2j+1) \supset O\ (3), \tag{101}$$

$$U\ (2j_1 + 1 + 2j_2 + 1) \supset U\ (2j_1 + 1) \times U\ (2j_2 + 1) \supset O\ (3)_1 \times O\ (3)_2 \supset O\ (3)_{1+2}, \tag{102}$$

[1)]It is interesting to note that this result was obtained almost simultaneously and independently in four groups involved in quark calculations,[13,66–68] but it may also possibly be implicit in the earlier studies of Refs. 40, 80, and 84. In any case, the logic of the quark calculations necessarily leads to formulation of the theorem, since without this result the technical problems are difficult to overcome.

where j, j_1, j_2 are the single-particle angular momenta of the shells. Let F_j^U be the single-shell coefficient of fractional parentage:

$$F_{ij}^U = \langle f'J'\omega' \times f''J''\omega'' \mid fJ\omega \rangle, \tag{103}$$

where f is the orbital Young tableau, J is the total angular momentum of the shell, and ω is some aditional quantum number.[12] Then the two-shell coefficient of fractional parentage can be represented in the factorized form [in accordance with the reduction (102)]

$$F_{j_1+j_2}^U = F_{1+2}^U F_{j_1}^U F_{j_2}^U F_{1+2}^O, \tag{104}$$

where F_{1+2}^U is the factor associated with the first link of the reduction in (102) [it can be calculated using (99)], and $F_{j_1}^U$ and $F_{j_2}^U$ are the factors corresponding to the second link of the reduction in (102) [they are determined in (103)]. The last link of the reduction in (102) is analogous to (94), and the corresponding factor F_{1+2}^O is related to the $9j$ symbol by an expression of the type (99),

$$F_{1+2}^O = \sqrt{(2J'+1)(2J''+1)(2J+1)} \begin{pmatrix} J_1' & J_2' & J' \\ J_1'' & J_2'' & J'' \\ J_1 & J_2 & J \end{pmatrix}, \tag{105}$$

and is normalized by the relation $\Sigma_{J',J''} |F_{1+2}^O|^2 = 1$.

Equations (99) and (103)–(105) solve the problems associated with the reduction (94) used in (55) and the reduction (102), which is used in (90) and (91). This solution uses only the standard shell technique.[12,80] The only difficulties are associated with the calculation of the $9f$ symbols. In the case when the Young tableaux in a $9f$ symbol contain not more than two rows ($[f] = [f_1 f_2]$) the $9f$ symbol is identical to a $9j$ symbol (apart from the normalization).

$$\sqrt{n_{f_a'} n_{f_a''} n_{f_b'} n_{f_b''}} \begin{pmatrix} f_a' & f_b' & f_{a+b}' \\ f_a'' & f_b'' & f_{a+b}'' \\ f_a & f_b & f_{a+b} \end{pmatrix}$$

$$= \sqrt{(2J_a'+1)(2J_a''+1)(2J_b'+1)(2J_b''+1)} \begin{pmatrix} J_a' & J_b' & J_{a+b}' \\ J_a'' & J_b'' & J_{a+b}'' \\ J_a & J_b & J_{a+b} \end{pmatrix}, \tag{106}$$

where $J = \frac{1}{2}(f_1 - f_2)$. In the case of more complicated Young tableaux, the recursion relation of Ref. 84 can be used to calculate the $9f$ symbol.

Scalar factors as weight factors

We now consider the reduction (93), which occurs in all links of the chains (54) and (60). By means of the factorization lemma, the quark coefficient of fractional parentage can be written down [with allowance for (54) and (94)]:

1) in the jj coupling scheme, for the reduction (54)

$$\Gamma^{jj} = F_{J\cdot CI}^U F_{J_s+J_p}^U F_{C\cdot I}^U F_C^U; \tag{107}$$

2) in the LS coupling scheme, for the reduction (60)

$$\Gamma^{LS} = F_{XS\cdot CI}^U F_{X\cdot S}^U F_{s+p}^U F_{C\cdot I}^U F_C^U. \tag{108}$$

The first factor in these relations is the well-known[12] weight factor $F_{J\cdot CI}^U = F_{XS\cdot CI}^U = \sqrt{n_{f'} n_{f''}/n_f}$, formed from the quantities $\sqrt{1/n_f}$, which are Clebsch-Gordan coefficients of the group S_N (see Ref. 40) for the formation of a completely antisymmetric state in the product $\{f\} \circ \{\tilde{f}\} \to \{1^N\}$ of conjugate representations. We shall show below that any scalar factors $F_{a\cdot b}^U$ also have the meaning of weight factors formed from Clebsch–Gordan coefficients of the groups S_N, $S_{N'}$, and $S_{N''}$, but for the more complicated reduction $\{f_a\} \circ \{f_b\} \to \{f_{ab}\}$, where $\{f_{ab}\} \neq \{1^N\}$. Therefore, the product of three Clebsch–Gordan coefficients will be completed by the operation of contraction over "projections." The ordinary weight factor $\sqrt{n_{f'} n_{f''}/n_f}$ can also be regarded as the result of contraction over the projections of the representation $\{f'\}$ and $\{f''\}$, this being equivalent in the given case to multiplication by the dimensions of these representations: $\sqrt{n_{f'} n_{f''}/n_f} = n_{f'} n_{f''} \sqrt{1/n_{f'}} \sqrt{1/n_{f''}} \sqrt{1/n_f}$.

Scalar factors as contractions of Clebsch–Gordan coefficients

It follows from the definition (95) and the orthogonality properties of the Clebsch–Gordan coefficients that the factor $F_{a\cdot b}^U$ is equal to the contraction of the Clebsch–Gordan coefficients of the unitary groups $U(mn)_{ab}$, $U(m)_a$, $U(n)_b$ with respect to the internal quantum numbers of the groups U_a and U_b:

$$F_{a\cdot b}^U = \sum_{\alpha_a', \alpha_a''} \sum_{\alpha_b' \alpha_b''} \left[\begin{matrix} f_{ab}' \times f_{ab}'' \\ \alpha_{ab}' \quad \alpha_{ab}'' \end{matrix} \middle| \begin{matrix} f_{ab} & \gamma_{ab} \\ \alpha_{ab} & \end{matrix} \right]$$

$$\times \left[\begin{matrix} f_a' \times f_a'' \\ \alpha_a' \quad \alpha_a'' \end{matrix} \middle| \begin{matrix} f_a & \gamma_a \\ \alpha_a & \end{matrix} \right] \left[\begin{matrix} f_b' \times f_b'' \\ \alpha_b' \quad \alpha_b'' \end{matrix} \middle| \begin{matrix} f_b & \gamma_b \\ \alpha_b & \end{matrix} \right]. \tag{109}$$

In Eqs. (95) and (109), it is assumed that the system of internal quantum numbers of the group $U(mn)_{ab}$ includes the quantum numbers of the subgroups $U(m)_a$ and $U(n)_b$:

$$\alpha_{ab} = ([f_a]\, \alpha_a \circ [f_b]\, \alpha_b,\ \omega_{ab}). \tag{110}$$

Note that substitution of (110) in (109) leads automatically after contraction to an invariant that depends only on the Young tableaux and on the repetition indices, as is fixed in the form (96).

Use of transformation matrices

For Clebsch-Gordan coefficients of the groups S_N, $S_{N'}$, and $S_{N''}$ ($N = N' + N''$) it is possible to define a contraction with respect to projections analogous to (109), but for this it is necessary to use in the group S_N the system of internal quantum numbers corresponding to the nonstandard reduction

$$S_N \supset S_N \times S_{N''}. \tag{111}$$

Suppose the Clebsch–Gordan coefficients of the group S_N are given, as usual, in the Young–Yamanouchi basis[40] corresponding to the standard reduction

$$S_N \supset S_{N-1} \supset \ldots \supset S_2 \supset S_1, \tag{112}$$

in which we use as projections the Yamanouchi symbols[40]

$$y = n_N n_{N-1} \ldots n_{N''} n_{N''-1} \ldots n_1. \tag{113}$$

The bases of the standard (112) and nonstandard (111) reductions are related by a unitary transformation[85]:

$$|f,\ y\rangle \equiv |f,\ Y''y'\rangle = \sum_{\gamma} \sum_{f'',\ y''} \langle f\gamma,\ f'y' \times f''y'' \,|\, f,\ y\rangle\, |f\gamma,\ f'y' \times f''y''\rangle. \tag{114}$$

In (114), the notation of the basis vectors is analogous to the notation $|j,\ m\rangle$, in which the projection is separated by a comma from the invariant of the given irreducible representation. The transformation (114) is diagonal with respect to the Yamanouchi symbol y', which is part of the original symbol y in (113) (we denote the remaining part of the symbol by Y''):

$$y = Y''y',\ Y'' = n_N n_{N-1} \ldots n_{N'+1},\ y' = n_{N'} n_{N'-1} \ldots n_1. \tag{115}$$

The transformation matrices $\langle f\gamma, f'y' \times f''y'' \,|\, f, y\rangle$ were introduced already in the first studies using the technique of coefficients of fractional parentage in the nuclear shell model,[79] and they were later investigated in detail by Kaplan,[85] who published tables of their values.[85] The transformation (114) makes it possible to go over in the Clebsch–Gordan coefficients of the group S_N to the basis of the nonstandard reduction (111):

$$C(S_N \supset S_{N'} \times S_{N''}) \equiv \left\{ \begin{matrix} f_a & \circ & f_b \\ (f'_a y'_a \times f''_a y'',\ \gamma_a) & & (f'_b y'_b \times f''_b y''_b,\ \gamma_b) \end{matrix} \middle| \begin{matrix} f_{ab} \omega_{ab} \\ (f'_{ab} y'_{ab} \times f''_{ab} y''_{ab},\ \gamma_{ab}) \end{matrix} \right\}$$

$$= \sum_{Y''_a,\ Y''_b Y''_{ab}} \left\{ \begin{matrix} f_a \circ f_b \\ y_a \quad y_b \end{matrix} \middle| \begin{matrix} f_{ab} \ \omega_{ab} \\ y_{ab} \end{matrix} \right\} \langle f_{ab}\gamma_{ab},\ f'_{ab}y'_{ab} \times f''_{ab}y''_{ab} \,|\, f_{ab},\ y_{ab}\rangle \times \langle f_a\gamma_a,\ f'_a y'_a \times f''_a y''_a \,|\, f_a,\ y_a\rangle \langle f_b\gamma_b,\ f'_b y'_b \times f''_b y''_b \,|\, f_b,\ y_b\rangle. \tag{116}$$

Scalar factor of the Clebsch–Gordan coefficients of the group $S_N \supset S_N \times S_N$

The scalar factor $F^S_{a\cdot b}$ corresponding to the reduction (111) can be expressed in the form of a contraction analogous to (109). We use for this the Clebsch–Gordan coefficients of the group S_N in the form (116):

$$F^S_{a\cdot b} = \sum_{y'_a,\ y'_b\ y''_a,\ y''_b} C(S_N \supset S_{N'} \times S_{N''}) \left\{ \begin{matrix} f'_a \circ f'_b \\ y'_a \quad y'_b \end{matrix} \middle| \begin{matrix} f'_{ab} \ \omega'_{ab} \\ y'_{ab} \end{matrix} \right\} \times \left\{ \begin{matrix} f''_a \circ f''_b \\ y''_a \quad y''_b \end{matrix} \middle| \begin{matrix} f''_{ab} \ \omega''_{ab} \\ y''_{ab} \end{matrix} \right\}. \tag{117}$$

We have obtained the result that $F^S_{a\cdot b}$ depends on the same invariants as $F^U_{a\cdot b}$ in (96), and in accordance with the complementarity principle $F^S_{a\cdot b}$ and $F^U_{a\cdot b}$ must be the same quantities. The proof of the equation

$$F^S_{a\bullet b} = F^U_{a\cdot b} \tag{118}$$

is now a purely technical problem, and it is solved, for example, in Ref. 14 (see also Refs. 6, 13, 67, 68, and 78).

Calculation of scalar factors

The identity of the quantities $F^S_{a\cdot b}$ and $F^U_{a\cdot b}$ makes it possible to use the formalism of the group S_N to calculate $F^U_{a\cdot b}$. We start from the equations for the Clebsch–Gordan coefficients of Ref. 90, which require knowledge of the representation matrices (D functions). The form of these equations does not depend on the specific group. Transforming the symbols under the Young–Yamanouchi basis for the group S_N, we write the original in the form

$$\left\{ \begin{matrix} f_a \circ f_b \\ y_a \quad y_b \end{matrix} \middle| \begin{matrix} f_{ab} \ \omega_{ab} \\ y_{ab} \end{matrix} \right\} = \sum_{\bar{y}_a,\ \bar{y}_b,\ \bar{y}_{ab}} D^{f_a}_{y_a \bar{y}_a}(P)\, D^{f_b}_{y_b \bar{y}_b}(P)\, D^{f_{ab}}_{y_{ab} \bar{y}_{ab}}(P) \left\{ \begin{matrix} f_a \circ f_b \\ \bar{y}_a \quad \bar{y}_b \end{matrix} \middle| \begin{matrix} f_{ab} \ \omega_{ab} \\ \bar{y}_{ab} \end{matrix} \right\}. \tag{119}$$

To calculate $F^S_{a\cdot b}$ (117) it is sufficient to use in (119) the binary permutation $P = P_{N',N'-1}$, for which the representation matrix $D^f_{y\bar{y}}(P) = \langle f, y|P\,|f, y\rangle$ can be readily calculated in the Young–Yamanouchi basis for any irreducible representation $\{f\}$ by means of the standard rules described, for example, in Ref. 40. This circumstance is an undoubted advantage of the approach based on the group S_N. If Eq. (119) were required for an arbitrary Clebsch–Gordan coefficient of the unitary group $U(m)$, the problem of calculating the D functions on the group $U(m)$ would present serious difficulties.

Going over in (119) to the basis of the nonstandard reduction (111) by means of the transformations (114) and (115) and defining

$$\tilde{D}^{f_a}_{aa} \equiv \langle f_a\gamma_a,\ f'_a y'_a \times f''_a y''_a \,|\, P_{N',N'+1} \,|\, f_a\gamma_a,\ \bar{f}'_a \bar{y}'_a \times \bar{f}''_a \bar{y}''_a\rangle \tag{120}$$

and similarly $\tilde{D}^{f_b}_{bb}$, $\tilde{D}^{f_{ab}}_{ab\ \overline{ab}}$, we obtain a system of equations for $F^S_{a\cdot b}$ (we omit the repetition indices γ and ω)[14]:

$$\left(\begin{pmatrix} f'_a & f'_b & f'_{ab} \\ f''_a & f''_b & f''_{ab} \\ f_a & f_b & f_{ab} \end{pmatrix} \right) = \sum_{\bar{f}'_a,\ \bar{f}'_b} \sum_{\bar{f}''_a,\ \bar{f}''_b} C_{\bar{f}'_a \bar{f}''_a \bar{f}'_b \bar{f}''_b} \left(\left(\begin{pmatrix} \bar{f}'_a & \bar{f}'_b & f'_{ab} \\ \bar{f}''_a & \bar{f}''_b & f''_{ab} \\ f_a & f_b & f_{ab} \end{pmatrix} \right) \right), \tag{121}$$

where the coefficients $C_{\bar{f}'_a \bar{f}''_a \bar{f}'_b \bar{f}''_b}$ are the following contraction of the $\tilde{D}$ functions (120) and Clebsch–Gordan coefficients of the groups $S_{N'}$ and $S_{N''}$:

$$C_{\bar{f}'_a \bar{f}''_a \bar{f}'_b \bar{f}''_b} = \sum_{\bar{y}'_a,\ \bar{y}'_b} \sum_{\bar{y}''_a,\ \bar{y}''_b} \sum_{y'_a,\ y'_b,\ y'_{ab}} \sum_{y''_a,\ y''_b,\ y''_{ab}} \tilde{D}^{f_a}_{aa} \tilde{D}^{f_b}_{bb} \tilde{D}^{f_{ab}}_{ab\overline{ab}} \times \left\{ \begin{matrix} f'_a \circ f'_b \\ y'_a \quad y'_b \end{matrix} \middle| \begin{matrix} f'_{ab} \\ y'_{ab} \end{matrix} \right\} \left\{ \begin{matrix} f''_a \circ f''_b \\ y''_a \quad y''_b \end{matrix} \middle| \begin{matrix} f''_{ab} \\ y''_{ab} \end{matrix} \right\} \left\{ \begin{matrix} \bar{f}'_a \circ \bar{f}'_b \\ \bar{y}'_a \quad \bar{y}'_b \end{matrix} \middle| \begin{matrix} f'_{ab} \\ \bar{y}'_{ab} \end{matrix} \right\} \left\{ \begin{matrix} \bar{f}''_a \circ \bar{f}''_b \\ \bar{y}''_a \quad \bar{y}''_b \end{matrix} \middle| \begin{matrix} f''_{ab} \\ \bar{y}''_{ab} \end{matrix} \right\}. \tag{122}$$

The system of equations (121) has the advantage over the equations for the K matrix in Ref. 40 in that it permits calculation of the required scalar factors $F^S_{a\cdot b}$ for the reduction $S_N \supset S_{N'} \times S_{N-2}$, avoiding the intermediate stages $S_N \supset S_{N-1}$, $S_{N-1} \supset S_{N-2}$, etc.; tables of the scalar factors of the Clebsch–Gordan coefficients of the group S^6 calculated in this manner are given in Refs. 13 and 14.

CONCLUSIONS

In this review, we have considered questions related to the interpretation of the interaction in the region of the core as an essentially many-particle phenomenon that cannot be reduced to an NN repulsion. We have analyzed the existing

quark calculations, using the formalism of the groups $SU(6)_{CS}$ and S_N, and we have shown that in these calculations full allowance was not yet made for the channels of color magnetic attraction associated with the orbital symmetry $[42]_X$ (for example, in the resonating-group calculations one should take into account the coupling to the channel N^*N^* or take into account the polarization terms with maximal *CS* symmetry). When all the states are fully taken into account, the orbital symmetry $[42]_X$ will evidently play the decisive role in *NN* scattering. Thus, the interpretation of the E dependence of the *NN* scattering phase shifts proposed in Ref. 22 is confirmed provisionally. In particular, we may have here the reason for the strong spin–orbit coupling in nuclei. In addition, replacement of "realistic" potentials of the *NN* interaction[43] by color magnetic attraction with forbidden states can produce an appreciable effect in the nuclei ^{3}H–^{3}He and ^{4}He. A preliminary indication of this kind was obtained in Ref. 73. At the same time, it would be interesting to test potentials with forbidden states in the description of the cross sections $d\sigma/d\Omega$ and the polarizations $P(\vartheta)$ in *NN* scattering in the wide energy range 0.5–2 GeV.

We mention some general questions. Once the algebraic structure of the problem has been clarified and a general picture has been obtained of the configurations of the system that are the most important, the main obstacle to unambiguous predictions is the absence of definite information about the exact form of the quark Hamiltonian in the case of exotic systems ($6q$, $9q$, $12q$, etc.). In interaction terms of different tensor dimension it is currently necessary to use phenomenological parameters matched to the spectrum of the ordinary hadrons, and there are as yet no indpendent "multiquark" experimental results with which a comparison could be made.

It is very probable that the quark degrees of freedom make an important contribution to all processes in nuclei when the momentum transfers are large: $Q^2 \gtrsim 1$ GeV$^2/c^2$ (characteristic dimensions $r < r_c$). However, as yet such experimental data have not acquired an unambiguous interpretation. For example, the inclusive data on deep inelastic scattering of electrons[91] ($Q^2 = 1$–3 GeV$^2/c^2$) and muons[92] ($Q^2 \approx 100$ GeV$^2/c^2$) obtained recently for light nuclei (d, ^{3}He, ^{4}He, ^{12}C) in a region kinematically forbidden for scattering by individual nucleons can be interpreted in terms of few-nucleon correlations[11] of Jastrow type, but there also exists an alternative point of view,[7,93–95] according to which scattering into the kinematically forbidden region occurs on multiquark components of the nuclear wave function ($6q$, $9q$, $12q$). The deep inelastic data for the scattering of strongly interacting particles by nuclei have even more interpretations (see Refs. 7, 11, and 94–99) in terms of the cumulative effect[100] and nuclear scaling.[101] A similar situation is now being found for the data on scattering of leptons by nuclei (EMC effect).

We end by noting some advantages of the quark approach that are usually missed if the data at large Q^2 are interpreted using only the orbitally symmetric configurations $s^6[6]_X$, $s^9[9]_X$, $s^{12}[12]_X$. First, the *NN* correlations that follow from the multiquark picture when allowance is made for the symmetry $[42]_X$ (see Fig. 6, curve 3) are no less rigid than in the case of a hard core (large positive logarithmic derivative for $r \gtrsim r_c$), but it is not necessary to introduce at the same time an unphysical infinite repulsive potential. Second, the probability of the $6q$ component in the nuclei need not be small, in contrast to the case of *NN* repulsion.

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