On the physical vacuum in QCD

V. N. Pervushin

Joint Institute for Nuclear Research, Dubna
Fiz. Elem. Chastits At. Yadra **15**, 1073–1110 (September–October 1984)

The aim of the review is to give a popular exposition of modern theoretical ideas about the vacuum in QCD.

INTRODUCTION

The concept of the vacuum is one of the central concepts in modern quantum field theory, the term being used to mean the state of the system with the lowest energy. As a rule, the vacuum energy is calculated by means of perturbation theory, which uses the notion of the virtual creation and annihilation of particles. Here, one can also speak of the concept of the "virtual vacuum" of perturbation theory if by "vacuum" one understands a physical substance that cannot be directly measured by physical devices but provides an exhaustive explanation for a number of phenomena when they are treated as the result of an interaction with it. For example, by interaction with a virtual vacuum one can explain experimental facts in quantum electrodynamics (QED) such as the Lamb shift of the energy levels in atomic spectra, the anomalous magnetic moments of particles, and so forth.

The idea of the virtual vacuum of perturbation theory was tranferred to quantum chromodynamics (QCD), i.e., the theory of the interaction of quarks and gluons.[1,2] The concept of the virtual vacuum, augmented by the property of invariance of physical quantities with respect to renormalizations of the quantum fields and coupling constants,[3] explains in QCD the phenomenon of asymptotic freedom, i.e., the decrease in the effective quark–gluon coupling constant with increasing momentum transfer. The formula for asymptotic freedom is

$$\alpha(q^2)=\frac{1}{b\ln(q^2/\Lambda^2)};\quad b=\frac{1}{4\pi}\left(11-\frac{2}{3}n_f\right). \tag{1}$$

Here, Λ is a dimensional parameter that arises when boundary conditions are imposed on the solution of the renormalization-group equations, and n_f is the number of quark flavors.[3] (The fact that the dimensional parameter Λ appears is called dimensional transmutation.)

Since the coupling constant (1) becomes infinite at $q=\Lambda$, the parameter Λ simultaneously determines the limits of applicability of the perturbation theory. The unbounded growth of the coupling constant at large distances $R\sim\Lambda^{-1}$ is called the infrared catastrophe, and the confinement of quarks in hadrons is attributed to it. Thus, both asymptotic freedom and confinement found, at least qualitatively, an explanation in the framework of the notion of the virtual vacuum of perturbation theory. In conjunction with renormalization-group methods, this explanation provided a justification of the quark–parton phenomenology of strong interactions at short distances.[4]

The successful development of QCD during recent years (see Ref. 1) has helped to strengthen its logical and physical foundations. A deep belief in the renormalization-group methods of perturbation theory is the basis of modern predictions of an asymptotic desert in unified theories and even modern variants of cosmogony and cosmology.[5] However, this system of widely accepted physical ideas about QCD is encountering more and more difficulties of theoretical and experimental nature, the most important of which is the absence of a constructive proof of confinement with a quantitative description of hadron physics at large distances.

We may here mention the fact that the experimental value of Λ depends strongly on the experiment and is slipping further and further into the low-energy region ($\Lambda\leqslant 100$ MeV), while the data of the CERN–JINR experiment NA-4 even permits a value of Λ equal to zero.[6] It can be seen from the expression (1) that in terms of distances a tending of Λ to zero is tantamount to an unlimited extension of the region of applicability of perturbation theory, which begins to include the confinement region. In such a case, the confinement problem cannot be related to the question of the applicability of perturbation theory, since in the confinement region the renormalization-group coupling constant (1) is still sufficiently small and the infrared catastrophe does not exist. The virtual vacuum of perturbation theory cannot then be regarded as the physical cause of confinement, and we are forced to admit our ignorance of the cause.

Recently, hopes of finding the unknown mechanism of confinement have come to be associated more and more with the construction of the "true nonperturbative vacuum of QCD."

At the present time, the dominant idea in the attempt to solve this problem is the analogy between quantum field theory and exactly solvable models of statistical physics like the Ising model, which describe phase transitions in a ferromagnet as the temperature varies. It is assumed that these phase transitions reproduce in the main features the "phase transition" in QCD from short to large distances. The main argument for such analogies is the identity of quantum field-theory models having an imaginary time (i.e., in Euclidean space–time) and models of statistical physics. This approach has already begun to provide its first results, for example, a successful computer calculation of hadron masses in lattice approximations of QCD.[7]

In the framework of this direction, it is implicitly assumed that all the main problems of QCD are concentrated in the region of the phenomena that can be described by the transition to Euclidean space. However, if we make this transition for quantum mechanics, we obtain the theory of Brownian motion. There is in fact an entire class of problems, for example, the calculation of critical exponents or the behavior of correlation functions (Green's functions), for

 0090-4759/84/050481-16$04.40

which quantum mechanics is identical to the theory of Brownian motion. However, the latter is not capable of describing interference phenomena such as the recently confirmed Aharonov–Bohm effect, the Bell and Josephson effects, and other purely quantum effects and phenomena.

Further, it is not obvious that the final solution to the problem of the vacuum and large distances lies in the region described by Euclidean field theory. In particular, in quantum electrodynamics, which served as the original analogy in the formulation of QCD, the elimination of infrared divergences (i.e., the same large-distance problem) cannot in principle be definitively solved in Euclidean space.

In the present review, an attempt is made to give a popular exposition of the theoretical ideas and principles associated with a "purely quantum" solution of the vacuum problem in QCD. The level of the exposition in the review is aimed at readers who are qualified experimentalists and are beginning their careers as theoreticians. Section 1 gives a brief history of the development of gauge theories and an introduction to their basic principles. Section 2 presents the main features of the "quantum" solution to the problem of large distances in QED. Readers familiar with this material can immediately turn to Secs. 3–6, which present the main body of the review.

1. BRIEF HISTORY OF GAUGE THEORIES

It is helpful to recall the main stages in the creation and development of theoretical electrodynamics. Modern field theory began with the ideas of Michael Faraday.

One is struck by the unprejudiced view that Faraday took of the results of his own experiments; it enabled him to develop his ideas about the physical vacuum (i.e., about a substance that he did not observe directly but whose reality he did not doubt). His ideas also had a fruitful influence on the shaping of the directions of his further experimental investigations, in particular, his study of the influence of a magnetic field on light.[1)] This example can serve as a good illustration of an answer to the following question: Why must an experimentalist know about the properties of the vacuum?

"Faraday, in his mind's eye, saw lines of force traversing all space where the mathematicians saw centers of force attracting at a distance: Faraday saw a medium where they saw nothing but distance; Faraday sought the seat of the phenomena in real actions going on in the medium, they were satisfied that they had found it in a power of action at a distance impressed on the electric fluids."[8]

Faraday's deep conviction of the reality of the electromagnetic field and the unity of the forces of nature became the guiding principles of the entire further development of physics.

The physical stage of the formation by Faraday of the basic ideas, concepts, and representations of electromagnetic phenomena was succeeded by the mathematical stage of the creation of the theory of electromagnetic phenomena, which is entirely due to Maxwell.

We can illustrate by examples the influence of the Faraday–Maxwell electrodynamics on the further development of physics.

1. Lorentz, Poincaré, and Einstein found the group of relativistic transformations of Maxwell's equations, and they generalized classical mechanics. The special theory of relativity was created.

2. The idea was born that what is important for the theoretician is not the equations themselves but the symmetry principles of these equations. This idea was successfully realized by Einstein in the construction of the general theory of relativity, which introduces concepts for the gravitational field.

3. Weyl[9] pointed out that the symmetry principles in electrodynamics (special relativity) and in general relativity play different roles with respect to the interaction. In contrast to electrodynamics, the symmetry principle in general relativity makes it possible to elucidate the form of the interaction itself. Weyl attempted to find a similar "dynamical" symmetry principle for electrodynamics in the period from 1919 to 1930. It is remarkable that he succeeded in finding the group of such "dynamical" transformations, from which the form of the electromagnetic interaction follows, only after the construction of quantum mechanics.

Weyl became acquainted with Fock's paper of Ref. 10, which established the form of the operator of the quantum-mechancial momentum in the presence of the electromagnetic field,

$$-i\hbar\frac{\partial}{\partial x_\mu}+\frac{e}{c}A_\mu,$$

and guessed that the required dynamical transformations are the phase transformations of the wave functions of particles in quantum mechanics:

$$\psi'=\exp\left(i\frac{e}{\hbar c}\lambda\right)\psi;\quad \lambda=\lambda(x,\ t);\qquad(2)$$

$$A'_\mu=\exp\left(-i\frac{e}{\hbar c}\lambda\right)\left(A_\mu+\frac{\hbar c}{ie}\partial_\mu\right)\exp\left(i\frac{e}{\hbar c}\lambda\right).\qquad(3)$$

4. Dirac found his famous equation for a spinor particle and proposed quantum electrodynamics with the Lagrangian

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}^2+\bar\psi\left(i\hbar\partial_\mu+\frac{e}{c}A_\mu\right)\gamma^\mu\psi+M\bar\psi\psi;\qquad(4)$$
$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu,$$

which is invariant with respect to the transformations (2) and (3). These transformations became known as gauge transformations.

5. Yang and Mills[11] proposed a so-called non-Abelian gauge theory, in which the Weyl phase λ in the transformations (2), (3) is replaced by a matrix of an internal symmetry group of isotopic type:

$$-i\lambda\frac{e}{\hbar c}\Rightarrow\hat\lambda(x,\ t)=\frac{e}{\hbar c}\sum_{a=1}^{3}\frac{\tau^a\lambda^a(x,t)}{2i},\qquad(5)$$

where τ^a is a Pauli matrix. As a result, the gauge field A_μ itself was replaced by the matrix

[1)]In 1938 there was found in the archives of the Royal Society a letter written by Faraday in 1832 that he had requested should be opened one hundred years later and in which he predicted the electromagnetic origin of light (we recall that Maxwell was born in 1831).

$$\hat{A}_\mu = \frac{e}{\hbar c}\sum_{a=1}^{3}\frac{\tau^a}{2i}A^a_\mu \quad (6)$$

with the transformation law

$$\hat{A}'_\mu = e^{\hat{\lambda}}(\hat{A}_\mu + \partial_\mu)e^{-\hat{\lambda}}, \quad (7)$$

$$\psi' = e^{-\hat{\lambda}}\psi. \quad (8)$$

A Lagrangian of the type (4) can be invariant with respect to such transformations only if a self-interaction of the gauge fields is included:

$$F^a_{\mu\nu} - \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \frac{e}{\hbar c}\epsilon^{abc}A^b_\mu A^c_\nu. \quad (9)$$

Non-Abelian gauge fields are charged and can, as it were, radiate themselves.

6. Weinberg, Salam, and Glashow proposed a unified theory of weak and electromagnetic interactions on the basis of the principle of non-Abelian gauge symmetry.[12]

7. In the fifties, a further symmetry principle of electrodynamics was discovered, namely, the symmetry, mentioned in the Introduction, of the theory with respect to renormalization of the physical quantities.[3]

8. The phenomenon of asymptotic freedom of non-Abelian guage theories was discovered[2] by means of renormalization-group methods, and Fritzsch, Gell-Mann, and Leutwyler[13] proposed the formulation of the quark–parton ideas that had already been developed in the form of modern quantum chromodynamics.[2)]

Thus, the strategy of modern physics became the search for simple basic symmetry principles, and the most fruitful tactic in the construction of virtually all viable theories became the analogy with Faraday–Maxwell electrodynamics; and the more we learnt about the symmetry of electrodynamics, the more fruitful were these analogies.

2. ON THE PHYSICAL VACUUM IN QED

As was said above, modern physical ideas are based on the concept of the virtual vacuum of peturbation theory. In quantum electrodynamics, it is also necessary to take into account nonperturbative effects associated with interaction with both the virtual and real infrared photons. The point is that the corrections from the virtual photon vacuum become infinitely large in the infrared region. These divergences are due to the fact that perturbation theory becomes invalid when applied to processes in which long-wavelength, or soft, photons participate.[15] (It is well known that the number of photons emitted by a charged particle per unit interval of energies tends to infinity as the photon energy tends to zero, whereas in pertrubation theory it is assumed that the emission of one photon is more probable than the emission of two.)

The effects of interaction with soft virtual photons can be calculated exactly outside the framework of perturbation theory, since in this case it is possible to ignore the recoil of the charge following emission and absorption of long-wavelength photons. As a result, the probabilities W_i of all the studied proceses involving charged particles can be represented in the form of the product of the probability of the given process calculated in accordance with perturbation theory with allowance for the hard photons, $W_{(\mathrm{p.t.})i}$, and the exact probability that in the given process no infrared photon is emitted at all, P_0:

$$W_i = W_{(\mathrm{p.t.})i}P_0. \quad (10)$$

The probability P_0 depends on the momentum transfer ΔP of the charged particle, the mass M of this particle, and the infrared-cutoff parameter λ. For example, for elastic scattering of a photon by an electron this factor has the form[15]

$$P_0 = \exp\left\{-\frac{2\alpha}{3\pi}\left(\frac{\Delta P}{M}\right)^2\ln\frac{M}{\lambda}\right\} \equiv \exp\left(-\bar{\nu}\left(\frac{M}{\lambda}\right)\right). \quad (11)$$

When the infrared cutoff is lifted, all these probabilities are freed of infinities, but they become equal to zero:

$$\lim_{\lambda\to 0} P_0 = 0. \quad (12)$$

Thus, the "exact theory" is not capable of solving the problem of the infrared divergences. The problem can be solved by bearing in mind that the separation of a particle from the field radiated by it has an approximate nature in perturbation theory and that any detecting device has a finite energy resolution $\Delta\varepsilon$. If this device detects the investigated process, it detects all the inelastic processes with *soft real photons* that accompany this process. For correct comparison with experiment, it is necessary to take into account both the elastic and the inelastic cross section. The physical device measures the cross section of the process, averaged over all inelastic channels.

$$\overline{W}_i = W_{(\mathrm{p.t.})i}P_0\sum_{n=0}^{\infty}P_{(n)}, \quad (13)$$

where $P_{(n)}$ is the probability of emission of n photons with total energy from λ to the resolution $\Delta\varepsilon$ of the instrument; the emission takes place in a statistically independent manner and satisfies the Poisson distribution:

$$P_{(n)} = \frac{\left[\bar{\nu}\left(\frac{\Delta\varepsilon}{\lambda}\right)\right]^n}{n!}; \qquad \sum_n P_{(n)} = \exp\left\{\bar{\nu}\left(\frac{\Delta\varepsilon}{\lambda}\right)\right\}. \quad (14)$$

It can be seen from (11)–(14) that the dependence on the infrared-cutoff parameter disappears in the averaged cross section (13):

$$\overline{W}_i = W_{(\mathrm{p.t.})i}\exp\left\{-\frac{2\alpha}{3\pi}\left(\frac{\Delta p}{M}\right)^2\ln\frac{M}{\Delta\varepsilon}\right\}. \quad (15)$$

Thus, the physics at large distances in QED is hidden by the resolution of the physical device and depends on the phenomenological parameter $\Delta\varepsilon$ and on the conditions of the experiment. Such a dependence also exists in the experimental determination of the basic parameter of the theory—the fine-structure constant $\alpha = 1/137$, which is measured in the elastic scattering of photons by electrons in the Thomson limit of an infinitesimally small photon momentum.

In accordance with (15), for any photon energy k_0 and momentum transfer $\Delta p \sim k_0$ one can make a measurement for the differential cross section with arbitrarily small resolution $\Delta\varepsilon$ such that the results of the measurement of the fine-structure constant will differ appreciably from the Thomson limit. The Thomson limit is attained only in the case when the relative energy resolution $\Delta\varepsilon/k$ of the device is

[2)]It should be stressed that the quantum number inherent in quarks—color—was discovered by Bogolyubov, Struminskiĭ, and Tavkhelidze.[14]

fixed in the limit $k \to 0$. Thus, before one gives a definite physical meaning to quantitative predictions of an experiment in QED it is necessary to check carefully all the experimental conditions. It is here appropriate to recall the following words of Born about quantum theory[16]: "The crucial point is that quantum mechanics describes not the objective external world itself but only a quite definite experiment set up to observe a part of the external world. Without this idea, it is quite impossible to even formulate any dynamical problem in quantum theory. But if this idea is adopted, the fundamental indeterminism of the physical predictions becomes natural, since no experimental device can ever be absolutely accurate." From the above, the following conclusions can be drawn:

1. In QED it is not possible, strictly speaking, to distinguish particle and field, and what is usually called the perturbative vacuum ($A = 0$) does not exist. Besides the virtual vacuum of perturbation theory, there exist real infrared photons that cannot be obseved by a given physical device, but the interaction with them must be taken into account nonperturbatively in order to eliminate the infrared singularities and determine the dependence of the cross sections of processes on the experimental conditions. Such infrared photons can also be called the vacuum in the sense of the definition given in the Introduction; we shall call them the *physical vacuum*.

2. Quantum electrodynamics has as its low-energy limit the classical theory as the theory of real physical phenomena with a coupling constant determined by means of the Thomson limit of the cross section of elastic Compton scattering of photons by electrons. The coupling constant in QED is obtained by extrapolating quantitative measurements under quite definite conditions on the experimental arrangement that eliminate to the maximal extent the nonperturbative influence of the infrared photons. Strictly speaking, in QED the coupling constant depends on the parameter of the device and on the conditions of the experiment in which it is measured. Although this dependence is vanishingly small, its existence is important as a precedent indicating the phenomenological nature of the physics of large distances in QED.

3. In accordance with the invariance of QED with respect to renormalization-group transformations, the physical results in QED do not depend on the choice of the normalization point of the coupling constant: $\alpha(q^2 = 0)$ or $\alpha(q^2 = \Lambda^2 \neq 0)$.

3. THE PROBLEM OF LARGE DISTANCES IN QCD

In its foundations, QCD is based to a large degree on analogy with QED, and uses perturbation theory in exactly the same way. To avoid infrared divergences in perturbation theory, one generally chooses a normalization point for the effective coupling constant (1) at which there are no infrared singularities. (We recall that in QED this step is definitely inadequate for the construction of matrix elements without divergences on the mass shell, and it is also necessary to take into account the interaction with the physical vacuum.) The main argument for perturbation theory in QCD is the property of asymptotic freedom (1), which does not occur in QED. However, as a rule all asymptotically free theories are unstable[3]: The ordinary vacuum $A = 0$ of perturbation theory is not the lowest stable state (i.e., it contradicts the first definition of the vacuum given at the start of the Introduction). For QCD, this was shown for the first time by Savvidi and Matinyan.[17] They obtained the effective Hamiltonian (H) of QCD in the presence of constant magnetic fields (B) in the single-loop approximation:

$$H(gB) = bg^2B^2 \left| \ln \frac{gB}{\Lambda_s^2} - \frac{1}{2} \right|; \quad g = \frac{e}{\hbar c}$$

and they showed that the renormalization-group vacuum ($B = 0$) has a higher energy ($H = 0$) than the energy of a constant magnetic field:

$$H(gB = \Lambda_s^2) = -\frac{1}{2} b\Lambda_s^2. \tag{16}$$

In other words, renormalization-group QCD in the single-loop approximation is a color ferromagnet with ground state characterized by spontaneous magnetization $B \neq 0$. Nielsen and Olesen[18] showed that the Savvidi vacuum is also unstable, its energy having an imaginary part. Allowance for the collective oscillation of the field leads to a new minimum of the energy and a new vacuum with a much more complicated structure. Variants of this vacuum are called the θ_3 vacuum and the spaghetti vacuum.[18] It is difficult to investigate the stability of the Nielsen vacuum because of its complexity; evidently, it too is not the final step in the hierarchy of infrared unstable vacua generated by the single-loop approximation of the renormalization-group scheme of QCD.

We see that attempts to elucidate the instability of perturbation theory within the framework of perturbation theory indicate the existence of nonperturbative infrared vacuum fields—the analog of the physical vacuum in QED.

Among the competing and mutually complementary phenomenological models and theoretical schemes in the region of large distances in QCD, the concept of a "nonperturbative vacuum" is being used more and more frequently. Precisely what is meant by it is still very unclear, just like the foundations of the phenomenological models themselves.

In the work of Ref. 17 mentioned above, the vacuum is a magnetic field. In the work of the group at the Institute of Theoretical and Experimental Physics, vacuum expectation values are understood, ultimately, as the parameters in a power-law expansion of the exact quark and gluon propagators.[19] Lee and his collaborators[20] regard the vacuum as a dielectric medium with different properties inside hadrons and outside them, while the hadrons are themselves regarded as nonlinear soliton waves in this medium. (Lee's model is a generalization of the most popular bag models: JINR,[21] MIT,[22] SLAC.[23]) Adler[24] (see also Refs. 25–27) proposed for the gluon vacuum of QCD equations of potential type,

$$E_i^a \sim B_i^a \sim \nabla_i^{ab}\Phi^b, \tag{17}$$

where

$$E_i^a = \partial_0 A_i^a - \partial_i A_0^a + g\epsilon^{abc} A_0^b A_i^c \equiv \partial_0 A_i^a - \nabla_i^{ab} A_0; \tag{18}$$

$$B_i^a = \epsilon_{ijk} \left(\partial_j A_k^a + \frac{1}{2} g\epsilon^{abc} A_j^b A_k^c \right) \tag{19}$$

are the electric and magnetic field, respectively, and

$$\nabla_i^{ab}\Phi^b = \partial_i\Phi^a + g\epsilon^{abc} A_i^b\Phi^c; \quad g = e/\hbar c. \tag{20}$$

Attempts to justify these equations in the spirit of the first papers on superfluidity theory[28–30] were made in Refs. 26 and 27. To obtain Eqs. (17), it is important to recognize, following QED, the fact of the existence of unobservable infrared gluon fields, the interaction with which must be taken into account in order to eliminate the infrared divergences. However, in contrast to QED, the infrared limit

$$(\partial_\mu A_\nu - \partial_\nu A_\mu) \to 0 \tag{21}$$

of QCD implies a strong interaction of the gluons for practically any arbitrarily small coupling g, since the terms describing the interaction in (18) and (19) become equal to, or greater than, the "kinetic" term (21) in the limit (21). Because of the strong coupling, the quantum behavior of the infrared gluons must be correlated in the entire region of space that they occupy. The ground state of such a system must be described by a single wave function like the wave function in the theory of quantum liquids.[28–30]

The equations of the quantum hydrodynamics of gluons can be derived, following Landau,[29] from the very formulation of the problem of "finding the spectrum of stationary states" (which presupposes the existence of a coordinate system in which the gluoun fields are stationary, $E\sim\nabla\Phi$) and from relativistic covariance (i.e., the assumption that the electric and magnetic fields are on an equal footing: $B\sim\nabla\Phi$). At the same time, the covariant momenta of the potential Φ vanish identically $[\nabla^2\Phi = 0]$, and for certain coefficients in (17) the formal expression for the energy–momentum tensor of such fields is equal to zero, in agreement with the requirement that the vacuum be unobservable.

The solutions of Eq. (17) are nonanalytic in the coupling constant $g = e/\hbar c$ and vanish in the classical limit $\hbar\to 0$. For example, the spherically symmetric solutions have the form

$$\Phi = \beta A_0; \quad A_\mu^a = (A_0,\ A_i) = \frac{\hbar c}{e}\,\Sigma_{\mu\nu}^a\partial^\nu \ln\rho; \quad \Box\rho = 0; \tag{22}$$

$$\rho = e^{-k_0 t}\,\frac{\operatorname{sh} k_0 r}{r}; \quad \left(\rho = e^{ik_0 t}\,\frac{\sin k_0 r}{r}\right). \tag{23}$$

Here, k_0 and β are arbitrary parameters, $r = |x|$, and

$$\Sigma_{ij}^a = \epsilon^{aij}; \quad \Sigma_{0j}^a = -\Sigma_{j0}^a = i\delta_{ja}. \tag{24}$$

In Refs. 24–27, these solutions are used to justify the bag model.[21–23]

A common feature of the models of the QCD vacuum mentioned above is the use of ideas, concepts, and terminology from theories of *quantum liquids*.[28–30] (For Maxwell, one of the most fruitful analogies was the analogy between the electromagnetic field and the theory of a *classical liquid* proposed by Rayleigh.)

The main difficulty in understanding superfluidity phenomena is the extension of the region of applicability of quantum notions to macroscopic objects. The creators of quantum mechanics clearly recognized that its unusual laws are laws of the microscopic world, whereas the laws of classical physics describe the macroscopic world. London, Landau, Bogolyubov, and others[28–30] extended these quantum ideas to macroscopic objects, namely, superconductors and superfluids. This step was fruitful for explaining many mysterious phenomena that became known as macroscopic quantum phenomena. One of the tendencies of the development of modern theoretical physics is the gradual recognition of the quantum nature of our macroscopic world and transition to the language formulated in the theories of superfluidity and superconductivity.

4. THIS IS A QUANTUM WORLD!

Quantum ideas about the physical world are the basis of modern physical concepts, and without a clear understanding of quantum theory it is impossible to make a step even at the Faraday level of the investigation of the vacuum (see Sec. 1).

We consider in this section some examples that lay bare the essence of quantum theory in order to formulate it in the form of a clear and simple proposition.

Problem. A "train" of mass M travels from the point A to the point B during the time T with constant velocity. Give the classical and quantum descriptions of its motion. For the classical description, it is sufficient to find the trajectory of the motion:

$$X(t) = \frac{X_B - X_A}{T}\,t + X_A. \tag{25}$$

The quantum description consists of finding the wave function from the Schrödinger equation

$$\hat{H}\Psi = i\hbar\partial_t\Psi; \quad H = \frac{\hat{p}^2}{2M}; \quad i[\hat{p},\ \hat{X}] = \hbar$$

and the energy spectrum of the stationary states:

$$H\Psi_\varepsilon(X) = \varepsilon\Psi_\varepsilon(X); \quad \Psi(X,\ t) = \Psi_\varepsilon(X)\,e^{-i\varepsilon t}.$$

The answer can be written down at once:

$$\Psi_\varepsilon(X) = \frac{1}{\sqrt{2\pi}}\,e^{ipX/\hbar}; \quad \varepsilon = \frac{p^2}{2M}, \tag{26}$$

where p is an eigenvalue of the momentum operator.

The analog in quantum theory of the trajectory (25) is the probability amplitude of transition of the "train" from the point A to the point B during the time T; it can be written in the form of the spectral representation

$$G(X_A,\ 0 \mid X_B,\ T) = \sum_\varepsilon e^{-i\varepsilon T}\Psi_\varepsilon(X_A)\,\Psi_\varepsilon^*(X_B) = \sqrt{\frac{M\hbar}{2\pi i}}\exp\left\{\frac{iS_{cl}(X_A,\ X_B)}{\hbar}\right\}; \tag{27}$$

$$S_{cl} = \frac{(X_B - X_A)^2 M}{2T}, \tag{28}$$

where S_{cl} is the classical action function for the trajectory (25). However, the formulation of the problem given above admits a quite different quantum description. The point is that in the conditions of the problem we said nothing at all about the properties of the space $\{X\}$ as a whole, i.e., about the properties of the complete line in which the interval between A and B lies. We implicitly assumed that this line is infinite and not closed. Now suppose we have a closed line. Then the points X and $X + L$, where L is the length of the complete closed line, coincide, i.e., are physically equivalent (Fig. 1). This means that the wave function at the point X is equal, apart from a phase $e^{i\theta}$, $|\theta| \leqslant \pi$, to the wave function at the point $X + L$:

$$\Psi(X + L) = e^{i\theta}\cdot\Psi(X). \tag{29}$$

is obtained in the same way. In this example we see that the use of the variational method in field theory (the derivation of the Euler equation, the conservation laws, etc.) presupposes not only a vanishing of the fields A themselves and their variations at spatial infinity but also smoothness and continuity of the physical transformations (43) and their vanishing at infinity. With regard to the basic entity of the Weyl transformations, their vanishing at infinity,

$$\lim_{x\to\pm\infty} \exp\left(\mathrm{i}\Lambda^{(n)}(x)/\hbar\right) = 1;$$
$$\Lambda^{(n)}(+\infty) - \Lambda^{(n)}(-\infty) = 2\pi n\hbar, \tag{48}$$

signifies that all the "paths" $P_1(x) = \exp(-\mathrm{i}\Lambda(x)/\hbar)$ are *closed*. And, as we said above, all closed continuous paths on a circle are divided into classes with index $n = \pm(0,1,2,\ldots)$ and it is in no way possible to pass by continuous transformations from one class to another. Since these transformations are applied to the fields A_1., all fields are also divided into classes connected by finite gauge transformations with index n:

$$A_1^{(n)} = P_1^{(n)}(x)^{-1}\left(A_1^0 + \mathrm{i}\hbar\,\frac{\partial_1}{e}\right)P_1^{(n)}(x). \tag{49}$$

The configurations $A^{(0)}(x), A^{(1)}(x),\ldots,A^{(n)}(x),\ldots$ are physically equivalent "points" in the functional configuration space, and therefore the wave functions at these "points" are identical apart from the phase:

$$\Psi(A^{(1)}) = e^{\mathrm{i}\theta}\,\Psi(A^{(0)}). \tag{50}$$

It is fairly easy to find a solution of the Schrödinger equation (44) with the conditions (47) and (50). An eigenfunction of the Hamiltonian operator (44) is simultaneously an eigenfunction of the "momentum" operator,

$$\hat{E}(x)\,\Psi \equiv \frac{\hbar}{\mathrm{i}}\,\frac{\delta}{\delta A(x)}\,\Psi = E(x)\,\Psi,$$

and the latter is, as is well known, a plane wave:

$$\Psi(A) \sim \exp\left(\mathrm{i}\int dx\,A(x)\,E(x)/\hbar\right).$$

The transversality condition (47) leads to the constant value $E(x) = \text{const}$, i.e., to the same field momentum for the fields at all points of space. The transversality condition plays the role of a strong interaction, which correlates the excitations of the fields in the whole of space and, in other words, leads to a collective motion of the field in complete analogy with the case (32) of superfluidity:

$$\Psi(A) \sim \exp\left\{\mathrm{i}\left[\int dx\,A(x)\right]E/\hbar\right\},$$

the role of the collective variable of the "centroid" in the field space being played by the functional

$$N[A] = \frac{e}{2\pi\hbar}\int dx\,A(x), \tag{51}$$

which changes by an integer under the transformations (43) and (49) with nontrivial "gauge paths" $P_1^{(n)}$:

$$N[A^{(n)}] = \frac{e}{2\pi\hbar}\int dx\left(A^{(0)}(x) + \frac{\partial_1\Lambda^{(n)}}{e}\right)$$
$$= N[A^{(0)}] + \frac{\Lambda^{(n)}(+\infty) - \Lambda^{(n)}(-\infty)}{2\pi\hbar} = N[A^{(0)}] + n. \tag{52}$$

Thus, the solution of the Schrödinger equation depends on a single variable,

$$\Psi(A) \equiv \Psi(N) = \frac{1}{\sqrt{2\pi}}\exp\left\{\mathrm{i}\left(E\,\frac{2\pi}{e}\right)N\right\}, \tag{53}$$

and the spectrum of the constant electric field (53) is determined from the condition (50), just as the train spectrum (30) is determined in the space with annular topology:

$$E = \frac{e}{2\pi}(2\pi k + \theta); \quad k = \pm(0, 1, 2\ldots). \tag{54}$$

As a result, we obtain a quantum system whose observable (54) is relativistically invariant and has finite energy density:

$$w = \frac{1}{2}E^2 V \quad \left(V = \int dx\right)$$

This system satisfies all the principles of quantum field theory, but yet it has a remarkable nonclassical feature. Its ground state (i.e., the state with minimal energy—the vacuum) corresponds to a nonvanishing constant electric field:

$$E_{\min} = \frac{e\theta}{2\pi}; \quad |\theta| \leqslant \pi,$$

i.e., to an undamped motion in the space of fields if there are nonvanishing jumps $\theta \neq 0$ of the phase in (50). We have obtained here a field analog of the Josephson effect—the existence of a current without external sources of this current (32) and the existence of a constant field without external sources of this field, i.e., plates of a flat capacitor, which are not present in the original Lagrangian. These constant electric fields are the real infrared vacuum in the given model. The occurrence of the vacuum is a purely quantum effect, related to the Josephson effect, and the region of applicability of the quantum theory is here equal to the size of the space $V = \int dx$. This fact can be directly seen by calculating the amplitude of transition from the point N_0 to the point N_T by means of the spectral representation (27). The upshot is that we obtain the already known expression (31) in the form of a sum over paths in the cyclic space $\{N\}$:

$$G(N_0 0\,|\,N_T, T) = \sqrt{\frac{M}{2\pi\mathrm{i}}}\sum_{n=-\infty}^{\infty} e^{\mathrm{i}\theta n}\exp\{\mathrm{i}S(N_0, N_T + n)/\hbar\}, \tag{55}$$

where

$$S(N_0, N_T) = \frac{1}{2}\left(\frac{N_0 - N_T}{T}\right)^2 MT \equiv \int_0^T dt\,\frac{\dot{N}^2}{2}M, \tag{56}$$

and M is the effective mass of the system,

$$M = \left(\frac{2\pi\hbar}{e}\right)^2\frac{1}{V},$$

which determines the limits of applicability of the quantum theory of order V. Thus, the quantum-mechanical ideas that led to the formulation of the principle of gauge invariance and to modern "classical" relativistic field theory must be extended to the entire macroscopic region V in which the gauge fields are defined and, in particular, one must introduce the concept of the common wave function of this quantum world. Weyl's proposal of gauge symmetry as the symmetry of quantum theory becomes for two dimensions a generalization of quantum notions to the entire two-dimensional world. Instead of the hiding of the physics of large distances in ordinary electrodynamics, we obtain in two dimensions a description of this physics by means of the topological properties of the gauge transformations. The criterion for the existence of nontrivial topological properties of the theory is the vanishing of the Lorentz-invariant functional known as the Pontryagin index:

$$\nu[A] = \frac{e}{4\pi\hbar} \int dx\, dt\, \epsilon^{\mu\nu} F_{\mu\nu} \equiv \frac{e}{2\pi\hbar} \int dx\, dt\, F_{01}. \quad (57)$$

In the case $A_0 = 0, F_{01} = \partial_0 A_1$ this functional for (55) is equal to

$$\nu[A] \equiv \int_0^T dt \frac{\partial}{\partial t} N = N_T - N_0 \neq 0. \quad (58)$$

We now turn to a more general method of quantizing the gauge theory by means of the constraint equation (38), in which there is an indeterminacy in the infrared region ($\partial_1^2 = 0$). There exists a unique way of satisfying the criterion (58) and describing quantum collective dynamics—to take into account the *singular solution* of the homogeneous equation $\partial_1^2 A_0 = 0$ in the general solution of the constraint equation (38):

$$A_0(x, t) = C(t)\, x + \frac{1}{\partial^2} (\partial_1 \partial_0 A_1 - e j_0). \quad (59)$$

For example, in the theory without currents, $j_0 = 0$, substituting (59) in the original Lagrangian (34) and the definition (57), (58) of the Pontryagin index, we obtain immediately the well-known Lagrangian of the collective motion (56):

$$\left. \begin{aligned} L &= \frac{1}{2} \int dx\, (\partial_0 A_1 - \partial_1 A_0) = \frac{1}{2} V C^2(t); \\ \dot{N} &= \frac{e}{2\pi\hbar} C(t)\, V \mapsto L = \frac{1}{2} \dot{N}^2 \left(\frac{2\pi\hbar}{e} \right)^2 \frac{1}{V}. \end{aligned} \right\} \quad (60)$$

The infrared vacuum here plays the part of singular external fields $A_0 \sim Cx$, which are present in the theory without a "source" (plates of a flat capacitor) by virtue of the Josephson effect. The classical theory (60) is fictitious, needed only as an intermediate step in the construction of the Hamiltonian

$$H = \frac{1}{2} \left(\frac{\hat{p} e}{2\pi} \right)^2 V; \quad \hbar p = \frac{\delta L}{\delta \dot{N}}; \quad \mathrm{i}\, [\hat{P}, \hat{N}] = 1$$

of the quantum theory, which has a real meaning. We have become accustomed to the idea that the laws of quantum theory hold for the microscopic world, and the classical laws for the macroscopic world. The quantum macroscopic phenomena (superfluidity, superconductivity, etc.) are in this view regarded as exceptions to these rules that can be obtained under laboratory conditions. But we have here encountered an example of relativistic field theory for the description of which the field of quantum laws must be extended to the entire world. We recall that in such a quantum world the motion of a physical object is not only determined by local equations but is also essentially governed by the topology of the space. Speaking figuratively, we can say that nonlocal quantum objects, in contrast to classical objects, know in advance how and where they move; their motion is harmonized with the properties of the world as a whole.

The recognition of the quantum nature of our macroscopic world may radically change our view of the world and our understanding of the harmonic connection between the microcosmos (man) and the macrocosmos. Some idea of this can be gleaned from the articles and papers of Blokhintsev.[32,33] We have neither the possibility nor sufficient preparation to go into these questions here.

5. GIVE US A VACUUM WITHOUT PARTICLES, PLEASE!

Modern theoretical ideas are generally divided roughly into three parts:

1. Relativistic quantum mechanics (generalization of the Schrödinger equation).
2. Relativistic classical field theory (generalization of classical electrodynamics).
3. Relativistic quantum field theory (a generalization of the first two).

It is a historical phenomenon that quantum field theory, which arose as a relativistic generalization of the Schrödinger equation, subsequently began to develop fruitfully as a generalization of classical field theory. The inner logic in the structure of any physical theory repeats the logic of the historical development of all physics: first classical, then quantum physics. The passing of some theory (for example, the physics of strong interactions) in the process of theoretical construction through all stages of the history of physics by no means implies that in all stages this theory has a consistent interpretation and is destined to describe real physical phenomena. Many unresolved paradoxes and blind alleys of theoretical physics frequently arise precisely because of attempts to interpret a physical theory at one of the intermediate stages as the real theory. There exist, for example, attempts to construct a systematic relativistic one-or two-particle quantum theory or a relativistic classical theory (without quantization).

In Sec. 4, we encountered a field-theory model that has a nontrivial interpretation only in terms of the physics of macroscopic quantum phenomena—superfluidity and superconductivity. A similar situation arises in the study of Dirac fields, which do not have consistent single-particle or classical interpretations. It is well known that the Dirac equation leads to solutions with both positive and negative energies. A quantum-mechanical system with negative energies is unstable with respect to weak perturbations of external fields. Thus, a single-particle theory has meaning only in the absence of interaction, i.e., for an isolated free particle. We cannot use such a theory to describe real dynamical problems, in particular, the hydrogen atom. Therefore, the Lagrangian for a free Dirac particle in, for example, two dimensions [cf. (33) and (35)]

$$L_0 = \int dx\, \bar{\psi} \hbar \mathrm{i}\, \hat{\partial} \psi = \int dx\, \mathrm{i}\hbar\, [\psi_1^+ (\partial_0 - \partial_1) \psi_1 + \psi_2^+ (\partial_0 + \partial_1) \psi_1] \quad (62)$$

is interpreted as the Lagrangian of a classical field with the Hamiltonian

$$\begin{aligned} H &= \int dx (\pi_1 \psi_1 + \pi_2 \psi_2) - L_0 = \\ &= \mathrm{i}\hbar \int dx\, [\psi_1^+ \partial_1 \psi_1 - \psi_2^+ \partial_1 \psi_2]; \\ \pi_i &= \frac{\delta L}{\delta \partial_0 \psi_i(x)} \end{aligned} \quad (63)$$

with a view to subsequent quantization. We rewrite this expression in the more convenient "momentum" representation

$$\psi_i(x, t) = \frac{1}{\sqrt{2\pi}} \int dk\, e^{ikx} a_{ik}(t); \quad (64)$$

$$\psi_i^+(x, t) = \frac{1}{\sqrt{2\pi}} \int dk\, e^{ikx} a_{ik}^+(t);$$

$$H = \int_{-\infty}^{\infty} dk\, \hbar k\, (a_{1k}^+ a_{1k} - a_{2k}^+ a_{2k}). \quad (65)$$

It is readily seen that this Hamiltonian is not positive definite. We shall quantize this theory by imposing anticommutation relations for fermions in accordance with

$$\{\pi_i(x),\ \psi_j(y)\} = i\delta(x-y)\,\delta_{ij} \;|\!\Rightarrow\; \{a_{ik}^+,\ a_{jq}\} = \delta(k-q)\,\delta_{ij}. \quad (66)$$

We define the vacuum as the state with zero energy:

$$\langle 0 \mid H \mid 0 \rangle = 0. \quad (68)$$

Application of the particle creation and annihilation operators to the vacuum gives zero:

$$a_{k_i} \mid 0\rangle = 0; \quad \langle 0 \mid a_{k_i}^+ = 0. \quad (68)$$

It can be seen from the expression (65) that the operators $a_1^+(-k)$, $a_2^+(k)$ $(k>0)$ create particles with negative energy $\varepsilon = -k$. By itself, negative energy in field theory is not a serious defect. The difficulty can be eliminated by redefining the energy . What is bad is that the physical system (65) has no ground state. This system is unstable with respect to weak perturbations. Dirac proposed the only acceptable solution to this problem, one that is impossible outside the framework of quantum field theory. He formulated a "theory of holes," in which the negative-energy problem is solved by filling with particles all negative-energy states in accordance with the Pauli exclusion principle.

The new vacuum is a state in which all negative-energy levels are filled. This ensures the stability of the ground state of physical systems, in particular the hydrogen atom, since it follows from the Pauli principle that no article with positive energy can fall into the sea—a hole—is treated as an antiparticle with positive energy. Mathematically, this means that the old annihilation operators of articles with negative energy become antiparticle creation operators (C^+), while the meaning of the annihilation operators of particles with positive energy (b) is unchanged. Such a transition to the new vacuum can be written in the form

$$\left.\begin{array}{ll} a_{1p} = b_p\theta(p) + c_p^+\theta(-p); & a_{2p} = b_p\theta(-p) + c_p^+\theta(p); \\ a_{1p}^+ = b_p^+\theta(p) + c_p\theta(-p); & a_{2p}^+ = b_p^+\theta(-p) + c_p\theta(p), \end{array}\right\} \quad (69)$$

where $\theta(p)$ is the step function: $\theta(p \geqslant 0) = 1$, $\theta(p<0) = 0$.

The new vacuum

$$b \mid 0\rangle = c \mid 0\rangle = 0 \quad (70)$$

is simultaneously the state with lowest energy, and the problem of negative energies is resolved. The Dirac sea possesses infinite energy ε_0 and charge Q_0. However, according to Dirac's hypothesis they are unobservable, and only the differences

$$(\varepsilon - \varepsilon_0), \quad (Q - Q_0)$$

are observable.

It was already discovered in the thirties[34] that the introduction of a Dirac sea has more serious consequences, namely, a change in the commutation relations between currents:

$$\left.\begin{array}{l} j_{5,1}(x) = j_0(x) \equiv \bar\psi\gamma_0\psi = \bar\psi\gamma_5\gamma_1\psi = (\psi_1^+\psi_1 + \psi_2^+\psi_2); \\ -j_{5,0}(x) = j_1(x) \equiv \bar\psi\gamma_1\psi = -\bar\psi\gamma_5\gamma_0\psi = (\psi_2^+\psi_2 - \psi_1^+\psi_1). \end{array}\right\} \quad (71)$$

In particular, let us consider the commutation relations between the currents $\rho_1(p)$ and $\rho_1(-p)$, where $\rho_1(p)$ is defined by

$$\psi_1^+(x)\,\psi_1(x) = \frac{1}{2\pi} \int dp\, e^{ipx} \rho_1(p) \quad (72)$$

and has the form

$$\left.\begin{array}{l} p>0; \quad \rho_1(p) = \displaystyle\int_{-\infty}^{\infty} dk\, a_{1,k+p}^+ a_{1,k}; \\ \rho_1(-p) = \displaystyle\int_{-\infty}^{\infty} dk\, a_{1,k}^+ a_{1,k+p}. \end{array}\right\} \quad (73)$$

Using the commutation relations (66), we can readily represent the commutator

$$[\rho_1(p),\ \rho_1(-p')] = \int_{-\infty}^{\infty} dq\, dq\ [a_{p+q}^+ a_q,\ a_{q'}^+ a_{q'+p'}]$$

as a difference of two expressions,

$$[\rho_1(p), \rho_1(-p')] = \int_{-\infty}^{\infty} dq\, (a_{p+q}^+ a_{p'+q}) - \int_{-\infty}^{\infty} dq\, (a_{p-p'+q}^+ a_q), \quad (74)$$

one of which can be reduced to the other by means of the linear change of variables $q \to p' + q$. As a result, we obtain zero:

$$[\rho_1(p),\ \rho_1(-p')] = 0. \quad (75)$$

However, this is not so for the Dirac vacuum (69), (70). We substitute on the right-hand side of (74) the definition (69) and arrange the creation operators to the left of the annihilation operators [i.e., we reduce (74) to a normal product with respect to the vacuum (70), denoting it by ::], obtaining as a result

$$[\rho_{1D}(p),\ \rho_{1D}(-p')] = :[\rho_{1D}(p),\ \rho_{1D}(-p')]:$$

$$+\,\delta(p-p')\left[\int_{-\infty}^{\infty} dq\,\theta(p+q) - \int_{-\infty}^{\infty} dq\,\theta(q)\right] = -p\delta(p-p'). \quad (76)$$

It can be seen from the integral (76) that a linear change of variables leading to zero as in (75) is not possible for all functions. It must be borne in mind that the operator relations, taken by themselves, are formal and acquire meaning only when applied to the function spaces on which they are defined. The redefinition of the operators (69) that accompanies the introduction of the new vacuum changes our original implicit assumptions about the properties of the class of functions on which the creation and annihilation operators a^+, a act.

In the coordinate representation (71), the relation (76) takes the form of the commutators for scalar fields:

$$i\,[j_{5,0}(x, t),\quad j_{5,1}(y, t)] = \frac{1}{\pi}\,\partial_x\delta(x-y). \quad (77)$$

If we make in (77) the redefinition

$$J_{5\mu} = \frac{1}{\pi\hbar}\,\partial_\mu\Phi, \quad (78)$$

we obtain

$$i\,[\partial_0\Phi\,(x,\ t),\ \partial_1\Phi\,(y,\ t)] = \hbar\partial_x\delta\,(x-y).$$

During the fifties, commutation relations for currents of the type (77) came to be called Schwinger anomalous commutators. We see here one further example of how changes in the global characteristics of the theory (the vacuum) change its local properties.

For an interacting system of Dirac particles, the different vacua lead to theories corresponding to the same Hamiltonian but with different symmetry. Consider, for example, the Hamiltonian for the theory (33), (34). We showed in Sec. 4 that the elimination of the nondynamical fields leads to Coulomb interaction of currents and to external fields (infrared vacuum). The Hamiltonian of the complete theory has the form

$$H = H_0 + H_I;$$

$$\left.\begin{aligned} H_0 &= \int dx\,\hbar\bar{\psi}i\gamma_1\,\partial_1\psi + \frac{1}{2}\left(\frac{ep}{2\pi}\right)^2 V;\quad p = 2\pi k + \theta;\\ H_I &= \int dx\,\frac{e^2 p}{2\pi}\,(\partial_1^{-1}j_0) + \frac{e^2}{2}\int dx\,(\partial_1^{-1}j_0)^2, \end{aligned}\right\} \qquad (79)$$

where the last term in (79) describes the Coulomb interaction.

Using the commutation relations (77), we find the Heisenberg equation of motion for the current $j_{5,0}$ of (71):

$$\hbar\partial_t j_{5,0} = i\,[H,\ j_{50}] \equiv i\,[H_0,\ j_{5,0}] + i\,[H_I,\ j_5].$$

If there were no interaction or if the commutation relations (75) for the unstable vacuum held, we would obtain the ordinary continuity equation

$$\partial_t j_{5,0} - \partial_x j_{5,1} \equiv \partial_\mu j_{5\,\mu} = 0, \qquad (80)$$

which is known as the conservation law of the axial current. In the case of (77), it is readily seen that $[H_I, j_{5,0}] \neq 0$, and the conservation law (80) is not satisfied:

$$\partial_\mu j_{5,\,\mu} = \frac{e^2}{\pi\hbar}\left(\partial_1^{-1}j_1 + \frac{p}{2\pi}\right) \equiv \frac{e^2}{\pi\hbar}\left(\frac{\Phi}{\sqrt{\pi\hbar}} + \frac{p}{2\pi}\right). \quad (81)$$

Taking into account (78), we can rewrite this relation in the form of the relativistic equation for a scalar particle

$$\hbar^2\partial_\mu^2\Phi = M^2\left(\Phi + \frac{p\sqrt{\pi\hbar}}{2\pi}\right) \qquad (82)$$

with mass

$$M^2 = \hbar\,\frac{e^2}{\pi}. \qquad (83)$$

The pseudoscalar uncharged field Φ has been formed as a bound state of charged quarks, which are unobservable in the sense that they are not local perturbations in the given model. The result (83) is identical to the exact solution of this model given by Schwinger.[35] (The model of electrodynamics in two-dimensional space–time is itself called the Schwinger model.)

The effective Hamiltonian for the "observable" objects

$$H_{\rm eff} = \frac{1}{2}\hbar^2\,[(\partial_0\Phi)^2 + (\partial_1\Phi)^2] + \frac{1}{2}M^2\left[\Phi + \sqrt{\pi\hbar}\,\frac{p}{2\pi}\right]^2 \quad (84)$$

describes a free pseudoscalar field over a condensate $\sqrt{\pi\hbar}\ P/\pi$,, which reflects the existence of infrared vacuum fields. The quantum collective motions of the vacua of the bosons and fermions are intimately related.[36] The expression (83) is invariant with respect to simultaneous transformations of the infrared vacuum and the Dirac vacuum,

$$\exp\{i\,(Q_5 + 2N)\,\alpha\}\,H_{\rm eff}\exp\{-i\,(Q_5 + 2N)\,\alpha\} = H_{\rm eff}, \qquad (85)$$

where N is the collective variable of the gauge field, and Q is the so-called chiral charge difference of the numbers of left-handed, ψ_2, and right-handed, ψ_1, fermions:

$$Q_5 = \int dx\,j_{5,\,0} = \frac{1}{\sqrt{\pi\hbar}}\int dx\,(\partial_0\Phi) = \int dx\,(\psi_2^+\psi_2 - \psi_1^+\psi_1).$$

The nonconservation of the axial current (81) leads to nonconservation of the difference between the numbers of left- and right-handed fermions:

$$\partial_t Q_5 \neq 0.$$

Such a phenomenon also exists in four-dimensional models, and the Rubakov effect[37] recently discovered in unified field theories is based on it. It is well known that in unified theories the left- and right-handed fermions belong to different multiplets and have different quantum numbers. The presence of external fields of monopole type "polarizes" the Dirac vacuum and induces transitions between left- and right-handed fermions with different quantum numbers. In particular, these transitions can lead to proton decay

In the Schwinger model, the axial symmetry is broken without the occurrence of an additional zero-mass particle, called a Goldstone particle. At the present time, the large mass of the ninth pseudoscalar meson [the so-called $U(1)$ problem][38] is attributed to such a phenomenon.

We have presented an interpretation of the Schwinger model based mainly on papers of the twenties and thirties.[36] This interpretation is attractive in that it clearly demonstrates the purely quantum nature of the physical results, this being indicated by their definite dependence of the Planck constant $\hbar$.

The papers of Refs. 38–41 are based on a somewhat different interpretation of the same results and especially the θ vacuum; this interpretation involves not only the topological properties of the gauge field but also a partial restoration of the axial symmetry by means of a change in the definition of physical states. Such a change is not dictated by an inner necessity and, as was pointed out by Coleman,[40] does not provide a physical justification for introduction of the parameter θ in the massive Schwinger model.

The dramatic history of axial anomalies in the four-dimensional case is well set forth in Ref. 42. Only recently, Gribov[43] showed that the origin of the axial anomalies in the four-dimensional case, as in the two-dimensional, is the Dirac vacuum.

We called this section "Give us a vacuum without particles, please." If by particles one understands observable excitations of fields, then the question "without which particles should we give you the vacuum?" becomes quite reasonable and logical for quantum field theory.

6. INSTANTONS, ETC.

The recognition of the part played by topology in the theory of gauge fields is the most characteristic feature of modern theoretical physics. One of the first results in this direction was the discovery by Polyakov *et al.*[44,45] of the nontrivial topology of the gluon fields (9) described by a finite action

$$S=\int d^4x\left[-\frac{1}{4}F^{a2}_{\mu\nu}\right];\quad F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+g\epsilon^{abc}A^b_\mu A^c_\nu \tag{86}$$

in Euclidean space ($d^4x \Rightarrow -i\,d\tau\,d^3x$). Such Euclidean fields were called "instantons." They indicated degeneracy of the classical gluon vacuum and came to be regarded as quasiclassical "paths" (in the space of fields) describing tunneling quantum transitions between different vacua.[45–48] The discovery of instantons gave rise to a flood of papers on the application of the quasiclassical instanton approximation in QCD and its justification. We should mention here: 1) the picture of phase transitions of the instanton vacuum,[49] which to a certain degree served as a confirmation of the bag model[21–23]; 2) the application of the topological degeneracy to the solution of the problem of the large mass of the ninth pseudoscalar meson[0,51]; 3) the formulation of, and various solutions to, the problem of preserving the *CP* invariance of the strong interactions, including predictions of a new neutral particle—the axion.[52,53] All these results have been fairly widely discussed in reviews (see, for example, Refs. 19, 51, and 53). We shall here discuss the main difficulties in understanding the topology of non-Abelian fields, its relation to quantization, and alternatives to the instanton realization of topological degeneracy of the vacuum in a gauge theory.

As we have already seen for the example of the Schwinger model, the topology of gauge fields relates to quantum theory rather than to classical theory, and it would be more consistent to introduce topology in the construction of the quantum equations[26,48]; for example, in the formulation of the operator conditions of gauge invariance (7) on the wave "function" $\Psi(A)$ satisfying the Schrödinger equation in the $A_0=0$ gauge [see Eqs. (18), (19), (40), and (44)],

$$\hat{H}\Psi_\varepsilon(A)=\varepsilon\Psi_\varepsilon(A);\quad \hat{H}=\int d^3x\,\frac{1}{2}\left\{\left(\frac{\delta}{i\hbar\delta A^a_i}\right)^2+[B^a_i]^2\right\}. \tag{87}$$

We recall that in a gauge theory invariance of the Hamiltonian H with respect to the time-independent gauge transformations (7)

$$\hat{A}'_i(x,t)=e^{\hat{\lambda}(x)}(\hat{A}_i(x,t)+\partial_i)\,e^{-\hat{\lambda}(x)} \tag{88}$$

imposes a condition of invariance on the wave function: $\Psi(A')=\Psi(A)$.[4)] In solving (87), it is important for us to formulate this condition in operator form. For this, we consider infinitesimally small transformations (88):

$$A'^a_i=A^a_i+\nabla^{ac}_i(A)\lambda^c\equiv A^a_i+\delta A^a_i; \tag{88}$$

$$\nabla^{ac}_i(A)\equiv\delta^{ac}\partial_i+g\epsilon^{abc}A^b_i) \tag{89}$$

and expand $\Psi(A+\delta A)$ in a series in δA, equating to zero the first term of the expansion in δA (see (45), (46)):

$$\delta\Psi(A)\equiv\int d^3x\,\frac{\delta\Psi}{\delta A^a_i(x)}\nabla^{ac}_i(A)\lambda^c+O(\lambda^2)=0. \tag{91}$$

If the function $\lambda^c(x)$ are smooth and vanish at infinity sufficiently rapidly, the condition (91) can be written in the local form

$$\nabla^{ac}_i(A)\frac{\delta\Psi}{\delta A^c_i}=0. \tag{92}$$

[4)]Generally speaking, one requires that Ψ be a representation of the invariance group of H, though not necessarily a "scalar" representation, i.e., an invariant.

The analogous condition in QED is known as the tranversality condition. We must here emphasize a very important point. It is the condition $\Psi(A')=\Psi(A)$ and the integral equation (91) that are primary. Equivalent to them are Eq. (92) and, simultaneously, the condition of restriction of the possible class of transformations to smooth functions that vanish at infinity:

$$\lim_{|x|\to\infty}\lambda^a(x)=0. \tag{93}$$

As already stressed in Sec. 4, in quantum theory *global properties* of gauge transformations of the type (93) become important. The generalization of the condition (93) to a complete gauge transformation,

$$\lim_{|x|\to\infty}e^{-\hat{\lambda}(x)}=1, \tag{94}$$

leads to a theory with nontrivial topological properties essentially equivalent to the topological properties of the Schwinger model considered in Sec. 4.

The matrix $\exp(\hat{\lambda}(x))$ is defined in the domain of values of the group SU(2), which is a closed sphere of three dimensions. When (x) ranges over the complete three-dimensional space $R(3)$, the matrix

$$P_3(x)=e^{-\hat{\lambda}(x)} \tag{95}$$

ranges over the three-dimensional sphere, i.e., describes a three-dimensional path on the three-dimensional sphere, just as the function (43) is a one-dimensional path on the circle. The closed-three dimensional paths (94) have exactly the same topological properties as the one-dimensional paths $P_1(x)$. The paths $P_3(x)$ are characterized by an integral number of circuits around the sphere:

$$n=-\frac{1}{12\pi^2}\int d^3x\,\epsilon_{ijk}\,\mathrm{tr}\,(P_3^{-1}\partial_iP_3)(P_3^{-1}\partial_jP_3)(P_3^{-1}\partial_kP_3). \tag{96}$$

[It should be noted that in ordinary QED there is only the topologically trivial mapping ($n=0$) of three-dimensional space onto the one-dimensional circle of values of the gauge phase.]

All gauge fields can also be divided into classes of fields with index n:

$$A^{(n)}_i=P^{(n)}_3(x)^{-1}(A^{(0)}_i+\partial_i)P^{(n)}_3(x). \tag{97}$$

Fields from different classes cannot be connected by a continuous sequence of infinitesimally small transformations (89). Therefore, the Schrödinger equation (87) with a Hamiltonian invariant with respect to (97) must be augmented by the condition of physical identity of the "points" $A^{(0)}_i$, $A^{(1)}_i$, etc.:

$$\Psi(A^{(1)})=e^{i\theta}(A^{(0)}) \tag{98}$$

[cf. (47) and (50)]. Following the analogy with the Schwinger model in the $A_0=0$ gauge, one can give the operator form of this condition:

$$T\Psi(A)=e^{i\theta}\Psi(A);\quad T=\exp\left(\frac{d}{dN}\right), \tag{99}$$

where $N[A]$ is a functional of the field A:

$$N[A]=\frac{g^2}{8\pi^2}\int d^3x\,\epsilon_{ijk}\left[\frac{1}{2}\partial_iA^a_kA^a_j+\frac{g}{6}\epsilon^{abc}A^a_iA^b_kA^c_j\right], \tag{100}$$

which changes by an integral number under the transformations (97),[54]

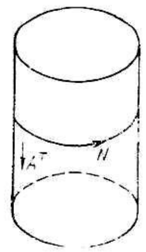

FIG. 3. Configuration space $\{A\}$ of non-Abelian gauge theory.

$$N[A^{(n)}] = N[A^{(0)}] + n, \tag{101}$$

i.e., it carries all the information about the topology of the gauge field. The variation of $N[A]$ with respect to the field A is the magnetic field:

$$\delta N[A]/\delta A_a^i(x) = \frac{g^2}{8\pi^2} B_i^a(x). \tag{102}$$

Faddeev[54] proposed that the quantity (100) should be regarded as a collective dynamical variable, and he repesented the configuration space of the field variables pictorially in the form of an infinite-dimensional cylinder. The cyclic coordinate on the cylinder corresponds to N, and the longitudinal coordinates to the transverse non-Abelian fields (Fig. 3).

It is to be expected that the phase indeterminacy of the wave function (98) will lead, as in the Josephson effect, to "undamped currents" in this "cylinder," i.e., to nonvanishing vacuum gluon fields ($B \neq 0, E \neq 0$), which could explain confinement, bags, mass spectra, and so forth.

The realization of this possibility by means of instantons was the subject of Refs. 44–49. The main assumption of these papers is that the transition amplitude between the "points" $A(x)_1$ and $A(x)_2$,

$$G\left(A_{(1)};\ -\frac{T}{2}\right|\left.A_{(2)};\ \frac{T}{2}\right) = \langle A_1 \mid e^{-iTH} \mid A_2\rangle$$
$$\equiv \sum e^{-i\varepsilon T}\Psi_\varepsilon(A_{(1)})\Psi_\varepsilon^*(A_{(2)}), \tag{103}$$

can be represented as a sum over classical paths

$$G \sim \sum_{\text{(paths)}} e^{i\theta n} \exp\left\{i\frac{S_{\text{cl}}(A_{(1)}^{(0)};\ A_{(2)}^{(n)})}{\hbar}\right\}, \tag{104}$$

where S_{cl} is the classical action function for the path from $A_{(1)}^{(0)}$ to $A_{(2)}^{(n)} = P_3^{(n)}(A_{(2)}^{(0)} + \partial)P_3^{(n)-1}$ [see (28), (31)]. A second assumption is that the main contribution to the sum (104) in the limit $T = -i\tau \to -i\infty$, $A_{(1)}^{(0)} = A_{(2)}^{(0)} = 0$, is made by the solutions of the classical equations $\sigma S_{\text{cl}}/\delta A = 0$ with finite action, i.e., instantons. Comparison of (104) with the spectral representation (103) in which only the first term with minimal energy survives in the limit $T \to -i\infty$, led to a calculation of the vacuum energy: $\varepsilon_\theta \sim \cos\theta$.[45–49]

As yet, the only rigorous argument for the validity of the instanton approximation is quantum mechanics, in which classical solutions like instantons play the part of paths of a tunneling transition between different minima of a potential.

It should be noted that the representation of the amplitude (103) in the form of the sum over paths (104), which was done for non-Abelian theory by Faddeev, Popov, *et al.* (see Ref. 55), has been rigorously proved only in the framework of perturbation theory.[56] (For example, Gribov[57] established that the Fadeev–Popov representation changes if one takes into account the indeterminacy of the non-Abelian fields that remains after the fixing of the gauge and arises outside the framework of perturbation theory.)

However, already at the level of the spectral representation (103) the quasiclassical philosophy of the instanton approximation encounters the difficulty noted in Ref. 26. The point is that the system of Schwinger equations and subsidiary conditions (87), (92), and (99) [in contrast to the analogous equations (44), (47), and (50) of the Schwinger model] is redefined and cannot have physical solutions, i.e., strictly speaking the transition probability (103) vanishes exactly. The reason for the redefinition of the system of equations (87), (92), and (99) is the dependence of the functional N on the transverse variables (102). It is readily verified that the operator (99), which changes N by an integer, $\hat{T}N = N + 1$, does not commute with the Hamiltonian, $[\hat{T},\hat{H}] \neq 0$, whereas the classical transformations $N' = N + 1$ left the Hamiltonian invariant. A similar example in which the symmetry of the quantum theory is not identical to the classical symmetry was analyzed in Sec. 5, in which we discussed the axial anomaly. We are here dealing with a topological anomaly. They both arise outside the framework of perturbation theory.

The operators $\hat{T}$ $(\hat{T}\Psi = e^{i\theta}\Psi)$ and $\hat{H}$ $(\hat{H}\Psi = \varepsilon\Psi)$, $[\hat{T},\hat{H}] \neq 0$, cannot have common eigenfunctions Ψ except for the case $\varepsilon = 0$. In this case, the wave function is well known to us as a plane wave:

$$\Psi \sim e^{ipN}; \quad p = 2\pi k + \theta; \quad \frac{\delta N}{\delta A} = \frac{g^2}{8\pi^2} B.$$

However, the momentum p is imaginary:

$$(\hbar p)^2 = -(g^2/8\pi^2)^2 \tag{105}$$

and the corresponding wave function cannot have a probabilistic interpretation—it is not normalizable. Such solutions are not taken into account in the sum (103). As a result, as we have already noted, the transition probability (103) is strictly zero because of the topological anomaly.

The Schwinger model indicates one further way of constructing a topological gauge theory, namely, by introducing an independent topological variable in exactly the same way as was done in Sec. 4 [see Eqs. (38), (59), and (60)]. The main conceptual difficulty of such a realization of topology is that we must expect a consistent relativistic interpretation of the theory only at the quantum level, although our original point of departure is classical quantities and concepts that may contradict the principles of relativistic invariance. One such concept is the dynamical variable of a simultaneous excitation of the system as a whole, which arises in the solution of the constraint equations:

$$\delta S/\delta A_0 = 0; \quad (\nabla_i)^2 A_0 = \nabla_i \partial_0 A_i + j_0; \tag{106}$$

$$A_0 = C(t)\Phi + \left(\frac{1}{\Delta_i^2}\right)(\nabla_i \partial_0 A_i + j_0); \quad \nabla^2\Phi = 0. \tag{107}$$

Here, j_0 are color currents, and $C(t)$ is in its mathematical meaning the variable mentioned above that does not fit in the framework of relativistic classical principles.

In the Schwinger model, this difficulty was avoided by the relativistic normalization of C by means of the Pontryagin index. The corresponding expression for the Pontryagin index in QCD has the form

$$\nu[A]=\int dt\,\dot{\nu};\quad \dot{\nu}=\frac{g^2}{8\pi^2}\int d^3x\,(\partial_0A_i-\nabla_iA_0)\,B_i. \qquad (108)$$

To simplify the treatment of such a theory as (106)–(108), we make an approximation in accordance with which all the transverse variables vanish:

$$\partial_0A_i^a=0. \qquad (109)$$

In QED this approximation is called electrostatics; in its, one actually ignores the effects of the virtual vacuum, and only the Coulomb part of the interaction remains in the Lagrangian. In QED this is sufficient to describe with good accuracy the spectrum of atoms and to understand many phenomena in atomic and molecular physics. We shall refer to the corresponding approximation in QCD without the virtual vacuum as chromostatics.

The main difference between chromostatics and electrostatics is that in QCD there exist stationary solutions of the equations of the gluon fields, $(\delta S/\delta A_i)=0$, without external sources of these fields. For example, in Adler's papers[24] devoted to Euclidean chromostatics the solutions of the following equations are regarded as such vacuum gluon fields:

$$B_i^a=(\nabla_i\Phi)^a.$$

The direct generalization

$$B=i\nabla_i\Phi \qquad (110)$$

of these equations to Minkowski space leads to complex non-Abelian fields. (It should be mentioned that hitherto all attempts to generalize a nontrivial topology of gauge fields to Minkowski space have used, as a rule, complex fields.[26,49]) The main argument for choosing the vacuum equations (110) for the theory (101)–(108) is the relativistically invariant form of the interaction of the vacuum (110) with color currents (see the Appendix).

In obtaining the Hamiltonian in the theory (101)–(108) it must be borne in mind that the Pontryagin index on the solutions of (110) becomes imaginary, and for self-consistency of the quantum theory of the collective excitation the canonically conjugate momentum

$$\hbar p=\delta S/\delta\dot{\nu}$$

must also be imaginary. For the momentum defined by Eq. (105) the Hamiltonian of the vacuum vanishes, whereas the Hamiltonian of the interaction with the $A_\mu=(\Phi,A_i)$ vacuum has the form

$$H=\int d^3x\,j_\mu A^\mu+\frac{1}{2}\int d^3x\,j_0\frac{1}{\nabla^2}j_0+O(g^2). \qquad (111)$$

Here, we have separated from the term of order $O(g^2)$ the analog of the Coulomb interaction in QED. The first term in (111) does not depend on the coupling constant.

In choosing the solutions in Eq. (110), it must be borne in mind that the field Φ plays a "quantum" role of one of the coefficients of the operator expansion of the electric field (107),

$$\hat{E}_i\sim\hat{p}\nabla_i\Phi+\dots,$$

and must be normalizable,

$$\int d^3x\,(\nabla_i\Phi)^2<\infty,$$

in accordance with the principle of the probabilistic interpretation of quantum theory. As Adler shows,[24] there exists a unique solution to equations of the type $B\sim\nabla\Phi$ that has a finite norm:

$$A_\mu^a=\frac{\hbar c}{e}\Sigma_{\mu\nu}^a\partial^\nu\ln\rho(x);\quad \rho(x)=e^{-k_0t}\frac{\sinh k_0|x+X|}{k_0|x+X|}, \qquad (112)$$

where X_i and k_0 are arbitrary parameters of position and scale [see Eqs. (22)–(24)]. Solutions of such type were considered for the first time in Ref. 58. (For brevity, following Polyakov, we shall call them "hedgehogs.") The normalized spectrum of the operator $(i\nabla_i)^2$ in (111) is strictly positive, which indicates stability of the vacuum (112).

We shall show that the considered approximation of chromostatics (in the absence of the virtual vacuum) is sufficient to establish one of the criteria for confinement in Minkowski space—the absence of poles of the exact Green's function[5)]:

$$G(p)(2\pi)^4\delta^4(p-q)=\frac{1}{\int d^3X}\int d^3X\int d^4x\,d^4y\,e^{ipx-iqy}G(x,y). \qquad (113)$$

Here, $G(x,y)$ satisfies the condition

$$\gamma_\mu(\partial_\mu+gT^aA_\mu^a)\,G(x,y)=-\delta^4(x-y), \qquad (114)$$

where T^a are the matrices of the representation of the color group for the quark. In (113), we have averaged only over the position parameters of the hedgehog, since in a more complete theory these parameters play the part of dynamical variables. The scale k_0 cannot play such a part (since the corresponding variable cannot be normalized[24]) and must be regarded as a phenomenological parameter of the type of the "instrument size" in QED (see Sec. 2).

In accordance with Ref. 60, in which complete sets of normalizable solutions of the Dirac equations in Minkowski space

$$\gamma_\mu(\partial_\mu+gT^aA_\mu^a)\psi=0;\quad \bar{\psi}\gamma_\mu(-\overleftarrow{\partial}_\mu+gT^aA_\mu^a)=0$$

for left-handed (L) and right-handed (R) quarks were found, the exact Green's functions have the form

$$G_L(x,y)=\left(\frac{\rho(x)}{\rho(y)}\right)^T G_0(x-y);$$

$$G_R(x,y)=\left(\frac{\rho(y)}{\rho(x)}\right)^T G_0(x-y),$$

where G_0 is the propagator of a free quark, and T is the number that determines the dimension of the representation $(T=\frac{1}{2},1,\dots)$.

The expression (113) can be readily calculated by using the remarkable formula

$$\lim_{V\to\infty}\frac{1}{\int_V d^3X_V}\int d^3X\left(\frac{\rho(x)}{\rho(y)}\right)^T=\rho(T(x-y))|_{X=0},$$

which can be proved by an expansion term by term with respect to $(\mathbf{x}-\mathbf{y})$ and the neglect of terms that vanish in the infinite volume V. As a result, for the exact propagator we obtain the estimate

---

5)A phenomenological model of hadrons based on this fact is considered in Ref. 59.

$$G(p) = \int d^4y\, e^{ipy} \rho(Ty)\, G_0(y)$$
$$\sim \frac{1}{2|p|2Tk_0}\left(\ln \frac{p_0-|p|+2iTk_0}{p_0+|p|+2iTk_0} - \ln \frac{p_0-|p|+i\varepsilon}{p_0+|p|+i\varepsilon}\right). \qquad (115)$$

We have omitted here the unimportant algebraic factors of the type $\hat{p}$, which reflect the multicomponent nature of the quarks. We have here a typical confinement propagator, which goes over in the limit $Tk_0 \to 0$ into the retarded propagator of a free quark:

$$\lim_{Tk_0\to 0} G(p) \sim \frac{1}{2|p|}\left(\frac{1}{p_0-|p|+i\varepsilon} - \frac{1}{p_0+|p|+i\varepsilon}\right) = \frac{1}{p_0^2-|p|^2+i\varepsilon p_0}.$$

This limit of Tk_0 means either no color, $T = 0$, or the region of short distances, $p^2 \to \infty$.

It is important to note that the integral (115) has a mathematical meaning (i.e, does not diverge) only in Minkowski space and for retarded Green's functions. (In Elucidean space, there exist quite different criteria for choosing "physical" solutions of the Dirac equations and, accordingly, the Green's functions.[24] This fact explains the opposite conclusion drawn by Adler in Ref. 24 about the absence of confinement in a Euclidean vacuum of hedgehog type.)

Such a physical picture of chromostatics can be regarded for the time being as an example of an alternative version of confinement not associated with the behavior of the effective coupling constant (1) and phase transitions. It must be borne in mind that a consistent theory is still in the stage of formulation. At the same time, a number of proposals have been made to extend the range of values of physical quantities that appear as unusual from the classical point of view as did Dirac's in his time to the effect that the range of values of the energy spectrum should be extended to negative numbers.

However, already at the present state of chromostatics one can recognize some positive features.

1. There exist chiral transformations that "undress" the exact quark propagators within Feynman diagrams; these propagators become free in the massless limit, and in this sense the perturbation theory differs little from the perturbative theory in QED or in ordinary QCD.

2. These chiral transformations leave the $U(1)$ anomaly and the possibility of solving the problem of the ninth pseudoscalar meson by analogy with the Schwinger model.

3. An effective dependence of the quark mass on distances arises.

CONCLUSIONS

We have attempted here to describe some tendencies in our understanding of the QCD vacuum. The essence of these tendencies is to be found in the extension of quantum concepts and the gradual recognition that the so-called classical relativistic field theory is a product of quantum-mechanical ideas of the thirties and cannot have a consistent classical interpretation any more than the Dirac equation can. We have given a quantum version of an "explanation" of confinement in the spirit of the Josephson effect. We have not considered numerical computer calculations, which are inseparably tied to the renormalization-group formulation of QCD, the attempt to represent Yang–Mills theory as a completely integrable system, and classical way of solving the quark-confinement problem in the spirit of Wilson's criterion, etc.

What is the most fruitful way of considering the vacuum problem: as a difficult mathematical problem for a computer or as a secret for the human mind? Only time will tell.

The present review was written while I was giving lectures for young scientists of the Laboratory of Nuclear Problems of the JINR and for members and students of the Department of Theoretical Physics at the University of Tashkent. I should like to express my gratitude to L. I. Lapidus, B. M. Pontecorvo, and M. M. Musakhanov for the opportunity of giving these lectures.

I thank S. Adler, A. V. Efremov, A. M. Polyakov, and V. A. Rubakov for discussing a number of questions reflected in the review.

APPENDIX TO SECTION 6. CHROMOSTATICS

We consider the theory of non-Abelian fields described by the action and Pontryagin index

$$S = \int d^4x \left[\frac{1}{2}(\partial_0 A_i - \nabla_i A_0)^2 - \frac{1}{2} B_i^2 + j_\mu^a A_\mu^a\right]; \qquad (A.1)$$

$$\nu = \int d^4x \frac{g^2}{8\pi^2}(\partial_0 A_i - \nabla_i A_0)^a B_i^a. \qquad (A.2)$$

We make the approximation $\partial_0 A_i = 0$ and substitute in (A.1) and (A.2) the solution of the equations of motion

$$\nabla_i^2(A) A_0 = j_0; \quad A_0 = C(t)\,\Phi + \frac{1}{\nabla_i^2} j_0; \quad (\nabla_i^2 \Phi = 0). \qquad (A.3)$$

As a result, we obtain $S = \int dtL$, $\nu = \int dt\dot{\nu}$, where

$$L(A) = \int d^3x\left[-\frac{1}{2}B^2 - j_i^a A_i^a + \frac{1}{2} j_0 \frac{1}{\nabla^2} j_0\right] + \frac{1}{2}C^2(t)\left[\int d^3x (\nabla\Phi)^2\right] + C(t)\int d^3x\left[j_0\Phi + \nabla_i\Phi\left(\nabla_i \frac{1}{\nabla^2} j_0\right)\right]; \qquad (A.4)$$

$$\dot{\nu} = \frac{g^2}{8\pi^2}\int d^3x\left[C(t)\,\nabla_i\Phi + \nabla_i \frac{1}{\nabla^2} j_0\right] B_i. \qquad (A.5)$$

By virtue of (A.3), the Lagrangian (A.4) does not depend locally on the field Φ. This fact determines the rule for transferring derivatives in expressions of the type

$$-\int d^3x (\nabla_i \Phi)\left(\nabla_i \frac{1}{\nabla^2} j_0\right) = \int d^3x\, j_0 \Phi. \qquad (A.6)$$

In (A.4), we express the variable $C(t)$ in terms of the covariant variable (A.5) and calculate the Hamiltonian:

$$H = \dot{\nu} \hbar p - L; \quad \hbar p = \delta L/\delta\dot{\nu}.$$

As a result, we obtain an expression analogous to the Hamiltonian (79) in the Schwinger model (79):

$$H_{tot} = H_{vac} + H_j;$$

$$H_{vac} = \int d^3x \frac{1}{2} B^2(x) + \frac{1}{2}\left[\left(\int d^3x B\nabla\Phi\right)^2 \Big/ \int d^3x (\nabla\Phi)^2\right]\left(\hbar p \frac{g^2}{8\pi^2}\right)^2; \qquad (A.7)$$

$$H_j = \int d^3x\left[-\frac{1}{2} j_0 \frac{1}{\nabla^2} j_0 + j_i A_i - \left(\hbar p \frac{g^2}{8\pi^2}\right) B_i \left(\nabla_i \frac{1}{\nabla^2} j_0\right)\right].$$

Using (A.6), we can readily see that in the class of vacuum fields $B = i\nabla\Phi$, and for the coupling constants (105), the Hamiltonian (A.7) takes the form (111).

[1]A. V. Efremov and A. V. Radyushkin, Riv. Nuovo Cimento 3, No. 2 (1980).
[2]D. Gross and F. Wilczek, Phys. Rev. D 8, 3633 (1973).
[3]N. N. Bogolyubov and D. V. Shirkov, Vvedenie v teoriyu kvantovannykh poleĭ, Nauka, Moscow (1976); English translation: Introduction to the Theory of Quantized Fields, 3rd Ed., Wiley, New York (1980); A. A. Vladimirov and D. V. Shirkov, Usp. Fiz. Nauk **129** 407 (1979) [Sov. Phys. Usp. **22**, 860 (1979)].
[4]R. P. Feynman, Photon–Hadron Interactions, Addison–Wesley, Reading Mass. (1972) (Russian translation published by Mir, Moscow (1975)).
[5]M. Gell-Mann, P. Ramond, and R. Slansky, Rev. Mod. Phys. **50**, 721 (1978); V. Rubakov and M. Shaposhnikov, in: Tr. XV Mezhdunarodnoĭ shkoly po fizike vysokikh énergiĭ, Dubna (1982) (Proc. of the 15th Intern. School on High Energy Physics, Dubna 1982), Dubna-D2, 4-83-179 (1983), p. 5.
[6]D. Bollini, J. Savin, *et al.*, Phys. Lett. **104B**, 403 (1981).
[7]I. Montvay, DERSY-83-001 Preprint, Hamburg (1983).
[8]J. C. Maxwell, Izbrannye sochineniya po teorii élektromagnitnogo polya (Selected Papers on the Theory of the Electromagnetic Field; Russian translation), Fizmatigiz, Moscow (1954).
[9]H. Weyl, Z. Phys. **56**, 330 (1929).
[10]V. Fock, Z. Phys. **39**, 226 (1927).
[11]C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954).
[12]S. Weinberg, Rev. Mod. Phys **52**, 515 (1980); A. Salam, Rev. Mod. Phys. **52**, 525 (1980); S. H. Glashow, Rev. Mod. Phys. **52**, 539 (1980).
[13]H. Fritzsch, M. Gell-Mann, and H. Leutwyler, Phys. Lett. **47B**, 375 (1973).
[14]N. N. Bogolyubov, B. V. Struminskiĭ, and A. N. Tavkhelidze, Preprint D-1968 [in Russian], JINR, Dubna (1965).
[15]J. D. Bjorken and S. D. Drell, Relativistic Quantum Fields, McGraw-Hill, New York (1964) (Russian translation published by Nauka, Moscow (1976)).
[16]M. Born, My Life and my Views, Scribner, New York (1968) (Russian translation published by Progress, Moscow (1973)).
[17]G. K. Savvidy, Phys. Lett. **71B**, 133 (1977); S.G. Matinyan and G. K. Savvidy, Nucl. Phys. **B134**, 539 (1978).
[18]N. K. Nielsen and P. Olesen, Nucl. Phys. **B144**, 376 (1978); H. B. Nielsen, in: Particle Physics (eds. I. Andric *et al.*), North-Holland, Amsterdam (1980), p. 67.
[19]A. I. Vaĭnshteĭn *et al.*, Usp. Fiz. Nauk **136**, 553 (1982) [Sov. Phys. Usp. **25**, 195 (1982)].
[20]T. D. Lee, Particle Physics and Introduction to Field Theory, Harwood, New York (1981).
[21]P. N. Bogolubov, Ann. Inst. Henri Poincaré **8**, 163 (1967).
[22]A. Chodos *et al.*, Phys. Rev. D **9**, 239 (1974).
[23]W. A.Bardeen *et al.*, Phys. Rev. D **4**, 1094 (1975).
[24]S. Adler, Phys. Rev. D **17**, 3212 (1976); **18**, 411 (1976).
[25]A. M. Protogenov, Yad. Fiz. **29**, 534 (1979) [Sov. J. Nucl. Phys. **29**, 269 (1979)].
[26]V. N. Pervushin, Teor. Mat. Fiz. **45**, 394 (1980).
[27]V. N. Pervushin, Yad. Fiz. **36**, 262 (1982) [Sov. J. Nucl. Phys. **36**, 152 (1982)].
[28]F. London, Phys. Ref. **54**, 947 (1938).
[29]L. D. Landau, J. Phys. USSR **5**, 71 (1941).
[30]N. N. Bogolubov, J. Phys. USSR **9**, 23 (1947).
[31]P. A. M. Dirac, Proc. R. Soc. London, Ser. A **117**, 610 (1928).
[32]D. I. Blokhintsev, Usp. Fiz. Nauk **42**, 78 (1950); **42**, 104 (1951).
[33]D. I. Blokhintsev, Tekh. Molodezhi No. 3, **42** (1982).
[34]P. Jordan, Z. Phys. **93**, 464 (1935); **98**, 759 (1936); **99**, 109 (1936); **102**, 243 (1936); **105**, 114, 229 (1937); A. A.Socolow, Phys. Z. Sowjetunion **2**, 148 (1937).
[35]J. Schwinger, Phys. Rev. **128**, 2425 (1962).
[36]N. P. Ilieva and V. N. Pervushin, Preprint E2-3-283 [in English], JINR, Dubna (1983); Yad. Fiz. **39**, 1011 (1984) [Sov.J. Nucl. Phys. **39**, 638 (1984)].
[37]V. A.Rubakov, Nucl. Phys. **B203**, 311 (1982).
[38]J. Kogut and L. Susskind, Phys. Rev. D **11**, 3594 (1975).
[39]L. Lowenstein and A. Swieca, Ann. Phys. (N.Y.) **68**, 172 (1971).
[40]S. Coleman, Ann. Phys. (N.Y.) **101**, 239 (1976).
[41]N. V. Krasnikov *et al.*, Teor. Mat. Fiz. **45**, 313 (1980).
[42]S. B. Treiman, R. Jackiw, and D. J. Gross, Lectures on Current Algebra and its Applications, Princeton University Press, Princeton (1972) (Russian translation published by Atomizdat, Moscow (1978), p. 112).
[43]V. N. Gribov, Preprint KFKI-1981-66.
[44]A. M. Polyakov, Phys. Lett. **59B**, 83 (1975); A. A.Belavin *et al.*, Phys. Lett. **59B**, 85 (1975).
[45]A. M. Polyakov, Nucl. Phys. **B120**, 429 (1976).
[46]G. 't Hooft, Phys. Rev. Lett. **37**, 8 (1976); Phys. Rev. D **14**, 3432 (1976).
[47]C. G. Callan, Jr. *et al.*, Phys. Lett. **63B**, 334 (1976).
[48]R. Jackiw and C. Rebbi, Phys. Rev. Lett. **37**, 172 (1976).
[49]C. G. Callan, Jr. *et al.*, Phys. Rev. D **19**, 1826 (1979); M. Ilgenfritz, D. I. Kazakov, and M. Mueller-Preussker, Yad. Fiz. **31**, 1107 (1980) [Sov. J. Nucl. Phys. **31**, 571 (1980)].
[50]E. Witten, Nucl. Phys. **B156**, 269 (1979); C. Veneziano, Nucl. Phys. **B159**, 213 (1979).
[51]M. K. Volkov, Fiz. Elem. Chastits At. Yadra **13**, 1070 (1982) [Sov. J. Part. Nucl **13**, 446 (1982)].
[52]S. Weinberg, Phys. Rev. Lett. **40**, 223 (1976).
[53]N. V. Krasnikov, V. A. Matveev, and A. N. Tavkhelidze, Fiz. Elem. Chastits At. Yadra **12**, 100 (1981) [Sov. J. Part. Nucl. **12**, 38 (1981)].
[54]L. D. Faddeev, in: Materialy IV Mezhdunar. soveshch. po nelokal'nym teoriyam polya. 1976 (Proc. of the Fourth Intern. Symposium on Nonlocal Field Theories, 1976), D1-9788, JINR, Dubna (1977), p. 267.
[55]A. A. Slavnov and L. D. Faddeev, Vvedenie v kvantovuyu teoriyu kalibrovochnykh poleĭ, Nauka, Moscow (1976); English translation: Gauge Fields, Introduction to Quantum Theory, Benjamin/Cummings, Reading, Mass (1980).
[56]A. A. Slavnov, Teor. Mat. Fiz. **22**, 177 (1975).
[57]V. N. Gribov, Nucl. Phys. **B106**, 189 (1976).
[58]A. M. Polyakov, Pis'ma Zh. Eksp. Teor. Fiz. **20**, 430 (1974) [JETP Lett. **20**, 194 (1974)].
[59]G. V. Efimov and M. A. Ivanov, Fiz. Elem. Chastits At. Yadra **12**, 1220 (1981) [Sov. J. Part. Nucl. **12**, 489 (1981)].
[60]V. N. Pervushin and J. Horjesi, Preprint E2-82-316 [in English], JINR, Dubna (1982).

Translated by Julian B. Barbour