Geometrical analysis of nonlinear equations in the theory of relativistic strings

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The geometrical approach to explicitly and completely integrable nonlinear partial differential equations with two independent variables that arise in the theory of relativistic strings is reviewed. The geometry of two-dimensional minimal surfaces in n-dimensional pseudo-Euclidean space is considered. The description of these surfaces by differential forms makes it possible to obtain two series of systems of nonlinear equations whose general solutions can be constructed explicitly. A geometrical method of obtaining new nonlinear equations that admit a Lax representation is presented. These equations describe a relativistic string in de Sitter space–time, a sphere in a three-dimensional unimodular affine space, and a special parametrization on an ordinary sphere in three-dimensional Euclidean space.

INTRODUCTION

The experimental data on high-energy interactions of elementary particles obtained at modern accelerators indicate more and more convincingly that the theory which describes hadron dynamics is quantum chromodynamics (QCD). However, the successes of the theoreticians in solving fundamental problems in QCD such as, for example, the clarification of the mechanism of quark confinement within hadrons and nonperturbative calculations are as yet quite modest in comparison with the experimental results in this field. In such a situation, there arose naturally a requirement for models that represent a simplification or approximation to QCD. A whole series of such models was in fact created. These are all possible potential models, which postulate a particular confining potential for quarks in hadrons, bag models, the parton model, models that take into account phenomenologically the nontrivial properties of the vacuum in QCD, and so forth.

The relativistic string model gives a clear picture of the quark-confinement mechanism in hadrons. It is as follows. If the distance between quarks is appreciably less than the hadron diameter, the quarks are essentially free. At a greater distance,the main role in the interaction of the quarks with one another comes to be played by configurations of the gluon field in which this field is concentrated in the form of a filament or tube along the line joining the quarks. Ignoring the transverse dimensions of the filament, we obtain a one-dimensionally extended object connecting the quarks; this has become known as a relativistic string. The action of the string must obviously be an appropriate approximation to the action of the non-Abelian gluon field. However, the requirement of relativistic invariance and the analogy with the action for a point particle already make it possible to write down the string action almost uniquely. The action of a relativistic string is taken to be proportional to the area of the world-surface spanned by the string as it moves in Minkowski space.

The fundamental difficulties that arise in the construction of a quantum theory of a relativistic string (the unphysical dimension of space–time, tachyon states) stimulated searches for nonstandard approaches to this model. One such approach—the geometrical one— is based on description of the world-surface of a relativistic string by differential forms satisfying conditions of integrability that are the nonlinear partial differential equations of Gauss, Peterson–Codazzi, and Ricci. In the geometrical approach, these equations are regarded as equations of motion specifying the string dynamics. A remarkable property of these equations in relativistic string theory is the possibility of constructing their general solution.

The simplest equations here are the nonlinear Liouville equation and the simplified system of two nonlinear Lund–Regge equations. The Liouville equation arises not only in relativistic string theory but also, for example, in the investigation of instanton solutions in non-Abelian gauge theories, while the Lund–Regge system is of interest from the point of view of the nonlinear two-dimensional σ model with the symmetry group $SO(4)$. In the general case, the geometrical approach in relativistic string theory leads to new systems of nonlinear equations for which an algorithm for constructing general solutions can be given.

Before turning to the quantization of a relativistic string in the framework of the geometrical approach, it is necessary to investigate at the classical level the nonlinear equations which describe the string dynamics in this approach. The present review is devoted mainly to this question.

We begin by listing briefly some methods of constructing explicit solutions of nonlinear partial differential equations developed in recent years.

It should be pointed out immediately that the possibility of constructing a general solution for a nonlinear partial differential equation is an extreme rarity. The modern theory of such equations can at best establish whether a solution exists and is unique. But for numerous nonlinear equations, particularly ones important from the physical point of view, for example, the Navier–Stokes equation, it is even possible to do this exhaustively. The search for general solutions of nonlinear partial differential equations became "a backwater far from the mainstream of mathematics, where amateurs can dabble to their heart's content undisturbed by competition from the professionals."[1] In this situation, the progress in the investigation of nonlinear partial differential equations achieved during the last 15 years was entirely unexpected. We are referring to the inverse scattering method, which was created during this time and was successfully applied to numerous nonlinear equations.[2–6]

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To solve mathematical problems—to construct general solutions to nonlinear partial differential equations—use was made of techniques well developed in theoretical physics—spectral analysis of the Schrödinger and Dirac equations and the procedure for recovering a potential using experimental results on particle scattering by the potential. The success of the inverse scattering method demonstrated once more how fruitful is intimate collaboration between pure mathematics and theoretical physics, a collaboration that has unfortunately become extremely rare in recent years.[1]

As frequently happens in such a situation, the success of the inverse scattering method attracted the attention of many mathematicians and theoretical physicists to the problem. There is already a very extensive literature on this subject.[2–6]

Besides the inverse scattering method, other approaches to the construction of general solutions of nonlinear partial differential equations was proposed. A purely group-theoretical method of constructing such solutions was developed in a series of papers by Leznov and Savel'ev.[7–10]

The present authors have proposed a method of constructing general solutions of nonlinear equations[11–14] using the methods of classical differential geometry.[15–18] In addition, a method of constructing a whole series of nonlinear equations admitting a Lax representation was developed.[19–24] The essential point here was the use of the mathematical formalism of the theory of surfaces.[16,17]

In this review, we shall attempt to present briefly the ideas and main results of this purely geometrical approach to explicitly and completely integrable nonlinear equations.

It should be mentioned that in all the approaches listed above one is concerned solely with nonlinear partial differential equations of a special type. The class of these equations will always remain very small compared with the complete manifold of nonlinear equations. Nevertheless, methods of constructing general solutions of nonlinear equations of even a special form undoubtedly warrant the attention of both mathematicians and physicists.

The nonlinear equations with which we shall be dealing arise in the theory of minimal surfaces. We shall first give a geometrical definition of a minimal surface and then indicate the branches of theoretical physics in which such surfaces arise.

At each point of a two-dimensional surface, its curvature is characterized by two invariants: the *total*, or *Gaussian curvature*, and the *mean curvature*.[15,16,25] The first variant determines the *intrinsic curvature* of the surface, i.e., a curvature that does not depend on the bending of the surface. By bending is meant a deformation of the surface that preserves the metric relations on it (distances between two points, angles between intersecting curves). Putting it differently, one can say that the *first quadratic form of the surface* is conserved under bending. The mean curvature of the surface characterizes its extrinsic curvature, and this invariant depends essentially on the manner in which the surface is embedded in the enveloping space.

Surfaces with constant Gaussian curvature and constant mean curvature were the subject of investigation for many generations of geometers. Examples of surfaces of the first type are the sphere, hyperboloids, and pseudospheres. It is interesting to note that surfaces of *constant total curvature* are intimately related to the nonlinear sine–Gordon equation[16,26–28]:

$$\varphi_{xy} = \sin \varphi, \qquad (1)$$

which is a classical equation solvable by the inverse scattering method.[1)]

A *minimal surface*[25,31,77] gives an example of a surface with *constant mean curvature*, moreover zero curvature. The definition of a minimal surface admits a variational formulation. The problem of finding a surface with least area and given boundary (plateau problem[32]) leads to the finding of such a surface. In ordinary three-dimensional Euclidean space, soap films spanning rigid wire frames give a perspicuous representation of minimal surfaces.

The history of minimal surfaces goes back to studies of Lagrange (1760), but even today this problem attracts the interest of mathematicians.[33–35] It is probably still too early to speak of the end of the "story youth" of this problem and the beginning of its 'tired old age."[36]

Minimal surfaces in pseudo-Euclidean space started to be considered in theoretical physics following the formulation of the theory of one-dimensionally extended relativistic objects—relativistic strings.[37,38] A relativistic string is determined by its action, which is proportional to the area of the world-surface swept out by the string in the process of its motion in Minkowski space. The model of a relativistic string arises in different branches of theoretical physics: in the investigation of nonlinear two-dimensional models of Born–Infeld type,[39] in the dual-resonance approach to hadron physics,[37,38] in the study of quark interaction in hadrons,[40,41] and in cosmology in the elucidation of the mechanism of galaxy formation.[42,43]

The plan of our exposition is as follows. In the first section, we give basic facts from classical differential geometry relating to the theory of surfaces. The main attention is devoted to the description of a moving basis on a surface, by means of which a surface can be specified by its fundamental differential forms. The Gauss, Peterson–Codazzi, and Ricci equations, which these forms must satisfy,[17,18] are derived.

In the second section, we define a minimal surface (Refs. 15, 17, 31, and 33) and construct an expansion for its coordinates in a special basis. As a result, the Cartesian coordinates of a minimal surface embedded in an n-dimensional pseudo-Euclidean space are expressed in terms of $2(n-2)$ arbitrary functions of one variable.[11,44]

In the third section, we write down explicitly the nonlinear embedding equations for minimal surfaces in 3-, 4-, 5-, and 6-dimensional pseudo-Euclidean space. We also construct general solutions of these equations.[12–14,45]

In the fourth section, minimal surfaces are treated in a

[1)]One of the first studies of this equation was the original communication "On the cutting of clothes" of the Russian mathematician P. L. Chebyshev, written in Paris in 1878.[29] In this connection, it would be appropriate to call Eq. (1) the Chebyshev equation, particularly since the now standard name for it—the sine–Gordon equation—is little more than jargon in rather poor taste. Unfortunately, such a proposal[30] did not receive support even in the Soviet literature.

special parametrization (the so-called $t = \tau$ gauge in the theory of relativistic strings). We here obtain a new series of nonlinear equations for which a general solution can be constructed.[12–14]

In the fifth section, we show how the classical theory of surfaces can be used to obtain nonlinear equations that admit a Lax representation.[21,24]

In the sixth section, we give a geometrical interpretation of the nonlinear equation $\varphi_{,11} - \varphi_{,22} = e^{\varphi} - e^{-2\varphi}$ and construct for it a linear spectral problem in matrices from the Lie algebra of the group $SL(3, R)$.[20]

In the seventh section, we obtain a new system of two nonlinear equations describing a minimal surface in a space-time of constant curvature.[19] For this system, a Lax representation is also explicitly constructed.

In the eighth section, we show how a new nonlinear completely integrable equation can be obtained on the basis of the geometry of the ordinary sphere.[22]

In the conclusions, we briefly summarize the results of the geometrical approach to nonlinear explicitly and completely integrable equations.

In the Appendix, we show how it is possible to construct in a simple manner the general solution of the nonlinear Liouville equation by considering the geometry of an ordinary sphere in three-dimensional Euclidean space.

1. BASIC FACTS FROM THE THEORY OF SURFACES

Suppose we have some n-dimensional flat space with arbitrary signature of the metric, the coordinates of which we shall denote by x^μ, $\mu = 0, 1, \ldots, n-1$. If in this space there is given an m-dimensional surface ($m \leqslant n$), this means that we are given a set of n functions of m variables $u^1, \ldots, u^m$

$$x^\mu (u^1, u^2, \ldots, u^m), \quad \mu = 0, 1, \ldots, n-1, \tag{2}$$

where

$$\mathrm{Rank} \left| \frac{\partial x^\mu}{\partial u^i} \right| \equiv \mathrm{Rank} \, | x^\mu_{,i} | = m. \tag{3}$$

In this case, one speaks of a parametric specification of the m-dimensional surface, and the parameters $u^1, \ldots, u^m$ play the part of curvilinear coordinate on it. The functions (2) give exhaustive information about the surface.[17,18]

However, in a number of cases one does not require such a detailed description of the surface, for example, when one is considering a complete class of surfaces characterized by some general criterion, or when one is studying the local properties of a surface at given point. For these purposes and in many other cases it is convenient to specify the surface by its basic *differential forms*.

In the theory of surfaces, one uses *quadratic* and *linear* differential forms. The former were introduced into geometry by Gauss[46]; the latter, by Darboux[47] and Cartan.[48] These forms arise naturally when one considers *a moving basis* on the surface.

At each point of the surface with coordinates $\{u^1, \ldots, u^m\}$ we construct an *orthonormal basis* formed by a set of n vectors:

$$e^\mu_a (u^1, \ldots, u^m), \qquad a = 1, 2, \ldots, n, \tag{4}$$

$$\sum_{\mu=0}^{n-1} e^\mu_a e^\mu_b c_\mu = \varepsilon_a \delta_{ab}, \; \varepsilon_a = \pm 1, \; c_\mu = \pm 1 \tag{5}$$

[there is no summation over a in (5)]. The first m vectors e^μ_i, $i = 1, \ldots, m+1, \ldots, n$, are *tangent* to the surface (2), and the remaining $n - m$ vectors e^μ_α, $\alpha = m+1, \ldots, n$, *are normals* to it. The factors c_μ and ε_α in (4) take into account the signature of the metric of the enveloping space. We put the origin of the basis (4) at the point $x^\mu(u^1, \ldots, u^m)$. We recall that a set of m vectors tangent to the surface (2) is given by the partial derivatives $x^\mu_{,i}$, $i = 1, 2, \ldots, m$. In the general case, these vectors are not orthonormal, but by virtue of (2) they are certainly linearly independent at each point of the surface. Therefore, the basis (4) can always be constructed.

It can be shown that if such a basis is known at every point of the surface, then the surface can be recovered from it. If we are interested in the local properties of the surface, it is sufficient to find the differential equations that determine the change in the basis (4) as its origin $x^\mu(u^1, \ldots, u^m)$ moves over the surface. These equations describe the variation of the radius vector of the surface,

$$dx^\mu = \omega^i e^\mu_i, \qquad i = 1, \ldots, m, \tag{6}$$

and the change in the basis unit vectors e^μ_a,

$$de^\mu_a = \Omega_{a\cdot}^{\cdot b} e^\mu_b \tag{7}$$

as the basis $\{x^\mu; e^\mu_a, a = 1, \ldots, n\}$ moves over the surface. Here, ω^i and $\Omega_{a\cdot}^{\cdot b}$ are *linear differential forms*:

$$\omega^i = \omega^i_j (u^1, \ldots, u^m) \, du^j, \tag{8}$$

$$\Omega_{a\cdot}^{\cdot b} = \Omega_{a\cdot|k}^{\cdot b} (u^1, \ldots, u^m) \, du^k; \tag{9}$$

$$i, j, k = 1, \ldots, m; \quad a, b = 1 \ldots, n.$$

Differentiation of the relations (5) gives

$$\Omega_{a\cdot}^{\cdot b} \varepsilon_b + \Omega_{b\cdot}^{\cdot a} \varepsilon_a = 0, \quad a, b = 1, \ldots, n$$

(no summation over a and b).

Equations (6) and (7) form a system of linear first-order partial differential equations of the form

$$d\theta_r (u) = \sum_{i=1}^{m} \psi^i_{rs} (u) \, \theta_s (u) \, du^i \tag{10}$$

or, in equivalent notation,

$$\frac{\partial \theta_r}{\partial u^i} = \psi^i_{rs} \theta_s (u), \qquad i = 1, \ldots, m, \tag{11}$$

where $\theta_r(u)$ denotes the set of variables $\{x^\mu, e^\mu_a\}$. On the functions $\theta_r(u)$ the conditions (5) are imposed. Therefore, Eqs. (10), (11), and (5) are a system of *mixed type*.[16] The conditions of integrability of Eqs. (11) reflect the fact that

$$\theta_{r,ij} = \theta_{r,ji}, \quad i, j = 1, \ldots, m. \tag{12}$$

These conditions lead to the following requirements on the coefficient matrices ψ^i in Eqs. (11):

$$\psi^i_{,j} - \psi^j_{,i} + [\psi^i, \psi^j] = 0, \qquad i, j = 1, \ldots, m. \tag{13}$$

To rewrite the integrability conditions (12) in terms of the differential forms ω^i and $\Omega_{a\cdot}^{\cdot b}$, it is convenient to use the formalism of exterior differentiation.[36,48–53] We require only the rule for exterior differentiation of linear forms. Let a be a linear differential form in the basis $\{du^1, \ldots, du^m\}$:

$$a = a_i (u) \, du^i, \quad i = 1, \ldots, m.$$

Then the exterior differential da of this form is defined by

$$da = da_i \wedge du^i = \frac{\partial a_i(u)}{\partial u^j} du^j \wedge du^i$$
$$= \sum_{i<j} (a_{i,j} - a_{j,i}) du^j \wedge du^i, \tag{14}$$

where $\wedge$ is the symbol of the exterior product:

$$du^i \wedge du^j = -du^j \wedge du^i. \tag{15}$$

As a result of the exterior differentiation of the linear form a, we obtain an exterior form of second degree, or a 2-form. Further exterior differentiation leads to forms of higher degree. A function $f(u^1, \ldots, u^m)$ is a form of zeroth degree. Its exterior differential is identical to the ordinary differential:

$$df = f_{,i} du^i.$$

Using (14), we can readily show that

$$d^2 f \equiv d(df) = 0, \tag{16}$$

since $f_{ji} = f_{,ij}$. From (14) and (15) there follow the rules of differentiation of the exterior product of two forms. If a and b are exterior forms of degree p and q, respectively, then

$$d(a \wedge b) = da \wedge b + (-1)^p a \wedge db. \tag{17}$$

We require the special case of Eq. (17) when $p = 1, q = 0$, *i.e.*, b is simply a function $f(u^1, \ldots, u^m)$:

$$d(a \cdot f) = da \cdot f - a \wedge df. \tag{18}$$

The conditions of integrability of Eqs. (10), which are given by Eq. (12), can be expressed in the language of exterior forms in accordance with (16) as

$$d^2\theta_r = d(d\theta_r) = 0. \tag{19}$$

Returning to Eqs. (6) and (7), we can rewrite their conditions of integrability (19) as

$$d^2 x^\mu = d(dx^\mu) = 0, \quad d^2 e_a^\mu = d(de_a^\mu) = 0. \tag{20}$$

From (20), using (6), (7), and (18), we obtain

$$\omega^j \wedge \Omega_{j\cdot}^{\cdot\alpha} = 0; \tag{21}$$
$$d\omega^i = \omega^j \wedge \Omega_{j\cdot}^{\cdot i}; \tag{22}$$
$$d\Omega_{a\cdot}^{\cdot b} = \Omega_{a\cdot}^{\cdot c} \wedge \Omega_{c\cdot}^{\cdot b}; \tag{23}$$

$i, j, k, \ldots = 1, \ldots, m$; $a, b, c, \ldots = 1, \ldots, n$; $\alpha, \beta, \gamma, \ldots = m + 1, \ldots n$. In component form, these equations become

$$\omega_i^j \Omega_{j\cdot|k}^{\cdot\alpha} - \omega_k^j \Omega_{j\cdot|i}^{\cdot\alpha} = 0; \tag{24}$$
$$\omega_{k,l}^i - \omega_{l,k}^i = \omega_l^j \Omega_{j\cdot|k}^{\cdot i} - \omega_k^j \Omega_{j\cdot|l}^{\cdot i}; \tag{25}$$
$$\Omega_{a\cdot|i,k}^{\cdot b} - \Omega_{a\cdot|k,i}^{\cdot b} = \Omega_{a\cdot|k}^{\cdot c} \Omega_{c\cdot|i}^{\cdot b} - \Omega_{a\cdot|i}^{\cdot c} \Omega_{c\cdot|k}^{\cdot b}. \tag{26}$$

In fact, Eqs. (24)–(26) represent a different form of expression of Eqs. (13) taking into account the explicit form of the matrices ψ.

In courses of differential geometry[16–18,53] it is shown that the system of equations of mixed type (6), (7), and (5) has, when the conditions of integrability (21)–(23) are satisfied, a solution $\{x^\mu(u^1, \ldots, u^m), e_\alpha{}^\mu(u^1, \ldots, u^m)\}$ that depends on $n(n + 1)/2$ constants of integration, where n is the dimension of the enveloping space. Different choices of the constants of integration correspond to displacements and rotations of the surface in the space as a whole.

Thus, we have the following *basic theorem* in the theory of surfaces (Bonnet's theorem). Differential forms ω^i and $\Omega_{a\cdot}^{\cdot b}$ satisfying the conditions of integrability (24)–(26) determine a surface $x^\mu(u^1, \ldots, u^m)$, apart from transformations in the group of motions of the enveloping space.

Besides linear forms, *quadratic differential forms* have been used in the theory of surfaces since Gauss's studies. The first quadratic form g_{ij} determines the *intrinsic geometry* of the surface:

$$dx^\mu dx^\mu c_\mu = x^\mu_{,i} x^\mu_{,j} c_\mu du^i du^j = \sum_{i,j=1}^m g_{ij}(u) du^i du^j,$$
$$g_{ij} = x^\mu_{,i} x^\mu_{,j} c_\mu. \tag{27}$$

The second quadratic forms $b_{\alpha|ij}$ determine the *extrinsic geometry* of the surface and are given by

$$\nabla_j x^\mu_{,i} = \sum_{\alpha=m+1}^n \varepsilon_\alpha b_{\alpha|ij} e_\alpha^\mu. \tag{28}$$

where ∇_j denotes *covariant differentiation* with respect to the metric g_{ij}:

$$\nabla_j x^\mu_{,i} = x^\mu_{,ij} - \Gamma_{ij}^k x^\mu_{,k}, \tag{29}$$

Here, Γ_{ij}^k are the Christoffel symbols for g_{ij}.[17]

The first (g_{ij}) and second ($b_{\alpha|ij}$) quadratic forms determine the motion over the surface of the basis formed by the set of m tangent vectors $x^\mu_{,i}$, $i = 1, \ldots, m$, and $n - m$ normals e_α^μ, $\alpha = m + 1, \ldots, n$, from Eq. (4). The motion of this basis is described by Eq. (28) and analogous equations for the normals e_α^μ, $\alpha = m + 1, \ldots, n$, which in terms of g_{ij} and $b_{\alpha|ij}$ have the form

$$\frac{\partial e_\alpha^\mu}{\partial u^i} = -b_{\alpha|ij} g^{jk} x^\mu_{,k} + \sum_\beta \varepsilon_\beta \nu_{\beta\alpha|i} e_\beta^\mu. \tag{30}$$

Here, besides the quadratic forms g_{ij} and $b_{\alpha|ij}$ we have introduced the so-called torsion vectors $\nu_{\beta\alpha|i} = -\nu_{\alpha\beta|i}$, $\alpha, \beta = m + 1, \ldots, n$; $i = 1, \ldots, m$.

The conditions of integrability of the linear equations (28) and (30) can be readily written down by using the well-known expression for the commutator of two covariant derivatives[17]:

$$(\nabla_i \nabla_j - \nabla_j \nabla_i)\lambda_k = R^l_{\cdot kij} \lambda_l, \tag{31}$$

where $R^l_{\cdot kij}$ is the Riemann–Christoffel curvature tensor for the metric g_{ij}, and λ_k is an arbitrary covariant vector. These conditions are given by the equations of Gauss

$$R_{ijkl} = \sum_{\alpha=m+1}^n \varepsilon_\alpha (b_{\alpha|ik} b_{\alpha|jl} - b_{\alpha|il} b_{\alpha|jk}), \tag{32}$$

of Peterson and Codazzi,

$$\nabla_k b_{\alpha|ij} - \nabla_j b_{\alpha|ik} = \sum_\beta \varepsilon_\beta (\nu_{\beta\alpha|k} b_{\beta|ij} - \nu_{\beta\alpha|j} b_{\beta|ik}) \tag{33}$$

and of Ricci

$$\nu_{\beta\alpha|j,k} - \nu_{\beta\alpha|k,j} + \sum_\gamma \varepsilon_\gamma (\nu_{\gamma\beta|j} \nu_{\gamma\alpha|k} - \nu_{\gamma\beta|k} \nu_{\gamma\alpha|j})$$
$$+ g^{lm} (b_{\beta|lj} b_{\alpha|mk} - b_{\beta|lk} b_{\alpha|mj}) = 0. \tag{34}$$

Thus, if the set of functions g_{ij}, $b_{\alpha|ij}$, and $\nu_{\alpha\beta|i}$ satisfies Eqs. (32)–(34), then it determines the surface $x^\mu(u^1, \ldots, u^m)$, apart from its motion in the space as a whole.

Equations (32)–(34) are completely equivalent to the conditions of integrability (24)–(26) and are actually the expression of these conditions in terms of other variables.

Between the linear and quadratic forms of a surface the following relation holds:

$$g_{ij} = \sum_{k=1} \varepsilon_k \omega_i^k \omega_j^k; \quad (35)$$

$$b_{\alpha|ij} = -\sum_{k=1}^{m} \varepsilon_k \omega_i^k \Omega_{\alpha \cdot |j}^{\cdot k}; \quad (36)$$

$$\nu_{\gamma\alpha|i} = \varepsilon_\gamma \Omega_{\alpha \cdot |i}^{\cdot \gamma}, \quad (37)$$

where $i, j, k, l \ldots = 1, \ldots, m; \alpha, \beta, \gamma \ldots = m+1, \ldots, n$.

We shall be dealing with two-dimensional minimal surfaces embedded in n-dimensional pseudo-Euclidean space with signature $(+ - - - \ldots)$ of the metric. Such is the problem to which one is led by the investigation of a *relativistic strong model*[38] whose world-surface is a two-dimensional minimal surface in Minkowski space. Thus, in the considered case

$$\varepsilon_1 = -\varepsilon_s = 1, \; s = 2, \ldots, n, c_0 = -c_r = 1,$$

$$r = 1, \ldots, n-1.$$

Compared with submanifolds of higher dimensions, two-dimensional surfaces are distinguished from the point of view of the simplicity of the embedding equations (24)–(26) or (32)–(34). On a two-dimensional surface one can always choose a curvilinear coordinate system $\{u^1, u^2\}$ in which the metric tensor g_{ij} has a conformally flat form:

$$g_{11} = \pm g_{22}, \; g_{12} = g_{21} = 0.$$

Moreover, the curvature tensor R_{ijkl} for a two-dimensional surface has just one essential component: R_{1212}.

2. MINIMAL SURFACES

Minimal surfaces[17,33] are characterized by the vanishing of the mean curvature along the directions of all the normals $e_\alpha^\mu(u^1, u^2)$, $\alpha = 3, \ldots, n$, to the surface at the given point $\{u^1, u^2\}$. The mean curvature h_α along the direction e_α^μ is the trace of the corresponding tensor of the second quadratic form $b_{\alpha|ij}$:

$$h_\alpha = b_{\alpha|ij} g^{ij}. \quad (39)$$

Thus, for a minimal surface

$$h_\alpha = 0. \quad (40)$$

The problem of finding a minimal surface can be formulated as a variational problem for the functional

$$S = \int d^m u \, | g |^{1/2}, \quad (41)$$

where $g = \det\|g_{ij}\|$, and $g_{ij}(u)$ is the induced metric (27) on the surface $x^\mu(u^1, \ldots, u^m)$. Indeed, one can show[17] that the conditions (40) follow from the Euler equations for (41):

$$\frac{\delta \sqrt{|g|}}{\delta x_\mu} = \sqrt{|g|} \nabla^i \nabla_i x^\mu = \sqrt{|g|} g^{ij} \nabla_i x^\mu_{,j}$$

$$= \sqrt{|g|} \sum_{\alpha=m+1}^{n} \varepsilon_\alpha g^{ij} b_{\alpha|ij} e_\alpha^\mu = 0. \quad (42)$$

We have here used Eq. (28), which determines the coefficients of the second quadratic forms $b_{\alpha|ij}$. Since the normals e_α^μ $\alpha = m+1, \ldots, n$ are linearly independent,

$$h_\alpha = g^{ij} b_{\alpha|ij} = 0, \quad \alpha = m+1, \ldots, n. \quad (43)$$

It follows from Eq. (42) that the coordinates of the minimal surface $x^\mu(u^1, \ldots, u^m)$ are harmonic functions of the parameters $u^1, \ldots, u^m$:

$$\nabla^i \nabla_i x^\mu (u^1, \ldots, u^m) = 0, \quad \mu = 0, \ldots, n-1, \quad (44)$$

where

$$\nabla^i \nabla_i = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial u^i} \left(\sqrt{|g|} g^{ij} \frac{\partial}{\partial u^j} \right) \quad (45)$$

is the Laplace–Beltrami operator for the metric g_{ij}.

It can also be seen from (42) that among the n equations (44) at least *m are consequences of the remainder*. The same conclusion can be obtained by applying the second part of Noether's theorem[54,55] to the functional (41), which is invariant with respect to the transformations

$$\bar{u}^i = f^i (u^1, \ldots, u^m) \quad (46)$$

with m arbitrary functions f^i. Indeed, for (42) and (44) there hold the m Noether identities

$$c_\mu x^\mu_{,j} \nabla^i \nabla_i x^\mu = 0, \quad j = 1, \ldots, m. \quad (47)$$

In the general case, Eqs. (44) for the coordinates $x^\mu(u^1, \ldots, u^m)$ of the minimal surface are nonlinear second-order partial differential equations, since the metric tensor g_{ij} is expressed in terms of x^μ in accordance with (27). But if the minimal surface is *two-dimensional*, then on it, as on every two-dimensional surface, one can choose the conformally flat system (38) of coordinates u^1 and u^2, in which Eqs. (44) are linearized:

$$x^\mu_{,11} \pm x^\mu_{,22} = 0, \quad \mu = 0, \ldots, n-1. \quad (48)$$

To be specific, we shall consider a hyperbolic metric g_{ij}, and therefore in (38) and (48) we take the lower sign. Thus, in our case the coordinates of the two-dimensional minimal surface will satisfy the equations

$$x^\mu_{,11} - x^\mu_{,22} = 0, \quad \mu = 0, \ldots, n-1, \quad (49)$$

$$x^2_{,1} = g_{11} = -g_{22} = -x^2_{,2} = e^{-q}, \quad g_{12} = c_\mu x^\mu_{,1} x^\mu_{,2} = 0. \quad (50)$$

In accordance with the previous section, the minimal surface can be described either by its radius vector $x^\mu(u^1, u^2)$, which satisfies Eqs. (49) and (50), or by the set of variables g_{ij}, $b_{\alpha|ij}$, $\nu_{\alpha\beta|i}$, which satisfy the minimality conditions (43) and the Gauss, Peterson–Codazzi, and Ricci equations (32)–(34). Knowing the soltuion of (49) and (50), we can recover g_{ij}, $b_{\alpha|ij}$, $\nu_{\alpha\beta|i}$ and obtain a solution of the corresponding nonlinear partial differential equations (32)–(34). Further, the resulting solution of the Gauss, Peterson–Codazzi, and Ricci equations will be *general* if the representation for $x^\mu(u^1, u^2)$ satisfying (49) and (50) contains a sufficient number of arbitrary functions.

For this purpose, it is convenient to introduce in the enveloping space a special basis[44,53] formed by two isotropic vectors η_1^μ and η_2^μ, $\eta_i^2 = 0$, i = 1, 2, $(\eta_1, \eta_2) = 1$ and $n-2$ mutually orthogonal spacelike unit vectors η_α^μ, $\eta_\alpha^2 = -1$, with $(\eta_1, \eta_\alpha) = (\eta_2, \eta_\alpha) = 0$, $(\eta_\alpha, \eta_\beta) = 0$, $\alpha \neq \beta$, $\alpha, \beta = 3, 4,$

$\ldots, n$. In this basis, the nonlinear conditions (50) are algebraic conditions on the coefficient functions and can be readily solved.

Before we turn to the construction of a decomposition for $x^\mu(u^1, u^2)$ in the basis $\{\eta^\mu_\alpha\}$, we use the invariance of Eqs. (49) and (50) under the conformal transformations

$$\bar{u}^\pm = f_\pm(u^\pm), \quad u^\pm = u^1 \pm u^2 \tag{51}$$

of the parameters u^1 and u^2 with arbitrary functions $f_\pm$ and impose on the string coordinates $x^\mu(u^1, u^2)$ in addition to (50) the conditions[12–14,19]

$$(x^\mu_{,11} \pm x^\mu_{,12})^2 = -q^2, \tag{52}$$

where q^2 is an arbitrary positive constant. Other conditions can be chosen instead of (52). One possibility will be considered in Sec. 4.

We now construct the decomposition for $x^\mu(u^1, u^2)$ in the basis $\{\eta^\mu_\alpha\}$. Let

$$x^\mu(u^1, u^2) = \psi^\mu_+(u^+) + \psi^\mu_-(u^-) \tag{53}$$

be the general solution of Eqs. (49). Then by virtue of (50), $\psi'_+(u^+)$ and $\psi'_-(u^-)$ must be isotropic vectors:

$$(\psi'_\pm)^2 = 0. \tag{54}$$

We shall denote by a prime differentiation with respect to the argument of a function. To satisfy (52), we must require

$$(\psi''_\pm)^2 = -q^2/4. \tag{55}$$

We seek expansions for $\psi'_\pm(u^\pm)$ in the form

$$\psi'_\pm(u^\pm) = A_\pm(u^\pm)\left[\eta_1 + B_\pm(u^\pm)\eta_2 \pm \sum_{r=1}^{n-2} f_{\pm r}(u^\pm)\eta_{r+2}\right]. \tag{56}$$

Substituting (56) in (54) and (55), we obtain

$$A_\pm(u^\pm) = \frac{q}{2\sqrt{\sum\limits_{r=1}^{n-2}[f'_{\pm r}(u^\pm)]^2}}, \quad B_\pm(u^\pm) = \frac{1}{2}\sum_{s=1}^{n-2} f^2_\pm(u^\pm). \tag{57}$$

Thus, we have for the vectors $\psi'_\pm(u^\pm)$ the representation[12–14]

$$\psi'_\pm(u^\pm) = \frac{q}{2\sqrt{\sum\limits_{r=1}^{n-2}[f'_{\pm r}(u^\pm)]^2}}\left[\eta_1 + \frac{1}{2}\sum_{s=1}^{n-2} f^2_{\pm s}(u^\pm)\eta_2 \pm \sum_{p=1}^{n-2} f_{\pm p}(u^\pm)\eta_{p+2}\right]. \tag{58}$$

By means of these formulas we can obtain an expression for the metric of a two-dimensional minimal surface in n-dimensional pseudo-Euclidean space:

$$e^\varphi = (g_{11})^{-1} = -(g_{22})^{-1} = (2\psi'_+(u^+)\psi'_-(u^-))^{-1} = \frac{4}{q^2}\frac{\sqrt{\sum\limits_{r=1}^{n-2}[f'_{+r}(u^+)]^2\sum\limits_{s=1}^{n-2}[f'_{-s}(u^-)]^2}}{\sum\limits_{p=1}^{n-2}[f_{+p}(u^+)+f_{-p}(u^-)]^2} \equiv \Lambda_{n-2}(u^1, u^2). \tag{59}$$

Thus, Eqs. (58) and (59) make it possible to express the differential forms of the minimal surface in terms of $2(n-2)$ arbitrary functions of one variable.

3. NONLINEAR EQUATIONS IN THE THEORY OF MINIMAL SURFACES AND THEIR GENERAL SOLUTIONS

We now turn to the explicit finding of nonlinear equations whose general solutions will be constructed by means of the decompositions (58) and (59). We begin by noting that these equations are in no way trivial, i.e., they cannot be reduced by a substitution to linear partial differential equations. The simplest of them will be the nonlinear Liouville equation.[56,57]

We consider Eqs. (32)–(34) for the two-dimensional minimal surface in the conformally flat coordinate system (50). In these coordinates,the conditions of minimality (43) become

$$b_{\alpha|11} = b_{\alpha|22}, \quad \alpha = 3, \ldots, n. \tag{60}$$

By simple manipulations, we can obtain from the Peterson–Codazzi equations (33) the relations[19,45]

$$\frac{\partial}{\partial u^\mp}\sum_{\alpha=3}^{n}(b_{\alpha|11} \pm b_{\alpha|12})^2 = 0. \tag{61}$$

On the other hand, by virtue of (28)

$$\sum_{\alpha=3}^{n}(b_{\alpha|11} \pm b_{\alpha|12})^2 = -[\nabla_1(x^\mu_{,1} \pm x^\mu_{,2})]^2. \tag{62}$$

This equation can be taken further by using the explicit form of the Christoffel symbols for the metric (50):

$$\Gamma^1_{11} = \Gamma^1_{22} = \Gamma^2_{12} = -\frac{\varphi_{,1}}{2}, \quad \Gamma^2_{11} = \Gamma^2_{22} = \Gamma^1_{12} = -\frac{\varphi_{,2}}{2}. \tag{63}$$

Indeed, substitution of (63) in (62) gives with allowance for (52)

$$\sum_{\alpha=3}^{n}(b_{\alpha|11} \pm b_{\alpha|12})^2 = -[\nabla_1(x^\mu_{,1} \pm x^\mu_{,2})]^2 = -(x^\mu_{,11} \pm x^\mu_{,12})^2 = q^2. \tag{64}$$

Thus, the condition (52) on the coordinates $x^\mu(u^1, u^2)$ of the minimal surface agrees with Eq. (33).

We consider first the simplest case—a three-dimensional enveloping space. By virtue of (64), Eq. (33) is satisfied identically, and the Gauss equation (32) reduces to the Liouville equation

$$\varphi_{,11} - \varphi_{,22} = 2q^2 e^\varphi. \tag{65}$$

Here, we have used the following expression for the curvature tensor in the metric (50):

$$R_{1212} = -\frac{1}{2}e^{-\varphi}(\varphi_{,11} - \varphi_{,22}). \tag{66}$$

The representation (59) for the metric tensor of the minimal surface with $n = 3$ gives the well-known[53,56,57] general solution of the Liouville equation(65):

$$e^\varphi = \Lambda_1 = \frac{4}{q^2}\frac{f'_+(u^+)f'_-(u^-)}{[f_+(u^+)+f_-(u^-)]^2}. \tag{67}$$

Another way of obtaining the general solution of Eq. (65), also geometrical, is given in the Appendix.

When the dimension of the enveloping space in which the minimal surface is embedded is increased, the number of equations in the system (32)–(34) becomes much larger, and, moreover, the number of functions $g_{ij}, b_{\alpha|ij}, \nu_{\alpha\beta|i}$ exceeds the number of equations. However, this system of equations simplifies appreciably if one chooses a moving basis on the minimal surface appropriately. Indeed, nothing is changed in the derivation of the Gauss, Peterson–Codazzi, and Ricci equations (32)–(34) if we go over from the basis

$$x^{\mu}_{,1},\ x^{\mu}_{,2},\ e^{\mu}_3, \ldots, e^{\mu}_n \tag{68}$$

to a new basis obtained from (68) by a rotation in the group $SO(1,1)\times SO(n-2)$ that does not mix the tangent space of the surface $\{x^{\mu}_{,1}, x^{\mu}_{,2}\}$ with its normal space $\{e^{\mu}_3, \ldots, e^{\mu}_n\}$.

If the dimension of the enveloping space is $n\geqslant 4$, then in the space normal to the two-dimensional minimal surface there are two mutually orthogonal spacelike vectors which can be associated naturally with the minimal surface.[14] They are $\nabla_1 x^{\mu}_{,1}$ and $\nabla_1 x^{\mu}_{,2}$. In accordance with (28) and (60), the vector $\nabla_2 x^{\mu}_{,2}$ is identical to $\nabla_1 x^{\mu}_{,1}$. Indeed, it follows from Eq. (64) that

$$(\nabla_1 x_{,1}\cdot\nabla_1 x_{,2}) = 0. \tag{69}$$

Therefore, it is natural to direct the two normals to the minimal surface along the vectors $\nabla_1 x^{\mu}_{,1}$ and $\nabla_1 x^{\mu}_{,2}$; for example, e^{μ}_3 along $\nabla_1 x^{\mu}_{,1}$ and e^{μ}_4 along $\nabla_1 x^{\mu}_{,2}$. Then we obtain directly from (28)

$$\begin{aligned} &b_{3|12} = b_{4|11} = b_{4|22} = b_{\alpha|ij} = 0,\\ &\alpha = 5, \ldots, n,\ i,\ j = 1,2. \end{aligned} \tag{70}$$

To satisfy the condition (64), we can set

$$b_{3|11} = q\cos\frac{\theta}{2}, \qquad b_{4|12} = q\sin\frac{\theta}{2}. \tag{71}$$

The Gauss equation (32) then becomes

$$\varphi_{,11}-\varphi_{,22} = 2q^2 e^{\varphi}\cos\theta, \quad e^{-\varphi} = x^2_{,1} = -x^2_{,2}, \tag{72}$$

and this equation remains valid for any dimension of the enveloping space in which the minimal surface is embedded.

By means of the Peterson–Codazzi equations (33) for $\alpha = 3, 4$ we can express in terms of the function $\theta(u^1, u^2)$ the torsion vector $\nu_{34|i}$, $i = 1, 2$:

$$\nu_{34|1} = \frac{\theta_{,2}}{2}, \quad \nu_{34|2} = \frac{\theta_{,1}}{2}. \tag{73}$$

In four-dimensional space–time, the system (32)–(34) reduces to the two nonlinear equations

$$\varphi_{,11}-\varphi_{,22} = 2q^2 e^{\varphi}\cos\theta, \quad \theta_{,11}-\theta_{,22} = 2q^2 e^{\varphi}\sin\theta, \tag{74}$$

where the function $\varphi(u^1, u^2)$ determines the conformally flat metric $g_{11} = -g_{22} = e^{-\varphi}$, $g_{12} = 0$, on the minimal surface, and $\theta(u^1, u^2)$ determines the second quadratic forms $b_{3|ij}$ and $b_{4|ij}$ $(i, j = 1, 2)$ and the torsion vector $\nu_{34|i}$ in accordance with Eqs. (70), (71), and (73).

The special choice of the normals e^{μ}_3 and e^{μ}_4 to the minimal surface has made it possible to obtain directly from the general system (32)–(34) a system of two equations for the two functions φ and θ, this contrasting with Refs. 19 and 24, in which the normals e^{μ}_3 and e^{μ}_4 were in no way fixed. In the quoted papers, it was necessary to introduce auxiliary functions $\alpha_{\pm}$, which occurred in the final equations only in the form of the difference $\theta = \alpha_+ - \alpha_-$.

We obtain the general solution of the system (74) by means of the expressions (58) and (59), which give[2)]

$$e^{\varphi} = \Lambda_2, \tag{75}$$

$$\theta = \operatorname{arctg}\frac{b_{4|12}}{b_{3|11}} = \operatorname{arctg}\sqrt{\frac{(\nabla_1 x_{,2})^2}{(\nabla_1 x_{,1})^2}}. \tag{76}$$

For the explicit expression of θ in terms of the arbitrary functions $f_{\pm}(u^{\pm})$ it is convenient, instead of using Eqs. (53), (58), and (76), to determine θ from the first equation of (74), having substituted there (75). The result is

$$\theta(u^1, u^2) = \arccos\Delta_2, \tag{77}$$

where

$$\Delta_r = \frac{2}{\sqrt{\sum_{k=1}^{r} f'^2_{+k}\sum_{j=1}^{r} f'^2_{-j}}} \times\left[\frac{\sum_{i=1}^{r}(f_{+i}+f_{-i})f'_{+i}\sum_{j=1}^{r}(f_{+j}+f_{-j})f'_{-j}}{\sum_{i=1}^{r}(f_{+i}+f_{-i})^2} - \sum_{i=1}^{r} f'_{+i}f'_{-i}\right], \tag{78}$$

$$f_{\pm i} \equiv f_{\pm i}(u^{\pm}).$$

In Ref. 76, the general solution of the system (74) was found by reducing these equations in a single Liouville equation for the complex function $\varphi + i\theta$.

A new system of three nonlinear equations, not previously given in the literature, is obtained when one applies the present method to two-dimensional minimal surfaces in five-dimensional pseudo-Euclidean space. Choosing the normals e_3 and e_4 in the manner described above, we obtain the expressions (70), (71), and (73) and Eq. (72). Besides the variables that occur in the four-dimensional case, there now appear two torsion vectors $\nu_{35|i}$ and $\nu_{45|i}$, $i = 1, 2$. The Peterson–Codazzi equations (33) with $\alpha = 5$ give

$$\nu_{35|2}\cos\frac{\theta}{2} = \nu_{45|1}\sin\frac{\theta}{2}, \quad \nu_{45|2}\sin\frac{\theta}{2} = \nu_{35|1}\cos\frac{\theta}{2}. \tag{79}$$

To satisfy these equations, we must set

$$\left.\begin{aligned} &\nu_{35|2} = h(u^1, u^2)\sin\frac{\theta}{2}, \quad \nu_{45|1} = h(u^1, u^2)\cos\frac{\theta}{2},\\ &\nu_{45|2} = p(u^1, u^2)\cos\frac{\theta}{2}, \quad \nu_{35|1} = p(u^1, u^2)\sin\frac{\theta}{2}. \end{aligned}\right\} \tag{80}$$

The Ricci equations (34) can now be expressed in the form

$$\theta_{,11}-\theta_{,22}-(h^2-p^2)\sin\theta = 2q^2 e^{\varphi}\sin\varphi, \tag{81}$$

$$\sin\frac{\theta}{2}(h_{,1}-p_{,2}) + \cos\frac{\theta}{2}(h\theta_{,1}-p\theta_{,2}) = 0, \tag{82}$$

$$\cos\frac{\theta}{2}(p_{,1}-h_{,2}) + \sin\frac{\theta}{2}(h\theta_{,2}-p\theta_{,1}) = 0. \tag{83}$$

The substitution

$$h = \varkappa_{,2}\left(\sin\frac{\theta}{2}\right)^{-2}, \quad p = \varkappa_{,1}\left(\sin\frac{\theta}{2}\right)^{-2} \tag{84}$$

transforms Eq. (82) into an identity, and (83) gives

$$\left(\operatorname{ctg}^2\frac{\theta}{2}\varkappa_{,1}\right)_{,1} = \left(\operatorname{ctg}^2\frac{\theta}{2}\varkappa_{,2}\right)_{,2}. \tag{85}$$

Finally, the system (32)–(34) in the case of a five-dimen-

[2)]*Translator's Note.* The Russian notation for the trigonometric, inverse trigonometric, hyperbolic trigonometric functions, etc., is retained here and throughout the article in the displayed equations.

sional enveloping space reduces to the following three nonlinear equations:

$$\left.\begin{aligned} &\varphi_{,11}-\varphi_{,22}=2q^2 e^{\varphi}\cos\theta,\\ &\theta_{,11}-\theta_{,22}+2\frac{\cos\frac{\theta}{2}}{\sin^3\frac{\theta}{2}}(\varkappa^2_{,1}-\varkappa^2_{,2})=2q^2 e^{\varphi}\sin\theta,\\ &\left(\operatorname{ctg}^2\frac{\theta}{2}\varkappa_{,1}\right)_{,1}=\left(\operatorname{ctg}^2\frac{\theta}{2}\varkappa_{,2}\right)_{,2}. \end{aligned}\right\} \quad (86)$$

It is interesting to note that the last equation in (86) is identical to the second equation in the nonlinear Lund–Regge system.[58]

We write down the general solution of the system (86):

$$\left.\begin{aligned} &e^{\varphi}=\Lambda_3,\quad \theta=\arccos\Delta_3,\\ &\frac{\varkappa_1^2}{\sin^2\frac{\theta}{2}}=q^2\cos^2\frac{\theta}{2}e^{\varphi}-\frac{\theta^2_{,2}}{4}-\left[\left(\frac{\nabla_1 x^{\mu}_{,1}}{q\cos\frac{\theta}{2}}\right)_{,1}\right]^2,\\ &\frac{\varkappa^2_{,2}}{\sin^2\frac{\theta}{2}}=-q^2\cos^2\frac{\theta}{2}e^{\varphi}-\frac{\theta^2_{,1}}{4}\left[\left(\frac{\nabla_1 x^{\mu}_{,1}}{q\cos\frac{\theta}{2}}\right)_{,2}\right]^2. \end{aligned}\right\} \quad (87)$$

Here, Λ_3 and Δ_3 are determined by (59) and (78), respectively, and the covariant derivative $\nabla_1 x^{\mu}_{,1}$ must be constructed by means of the decompositions (53) and (58).

If the dimension of the enveloping space is increased by one, i.e., we go over to $n=6$, we again encounter a gauge arbitrariness associated with the possibility of rotating the normals e_5 and e_6 through an arbitrary angle α. Then $\nu_i=\nu_{56|i}=-\Omega^{\cdot 5}_{6\cdot|i}$, $i=1, 2$, transform as follows:

$$\bar{\nu}_i=\nu_i-\alpha_{,i},\quad i=1,\ 2. \quad (88)$$

The angle $\alpha(u^1, u^2)$ can always be chosen to make the transformed quantities $\bar{\nu}_i$, $i=1, 2$, satisfy, for example, the "Lorentz condition"

$$\bar{\nu}_{1,1}-\bar{\nu}_{2,2}=0. \quad (89)$$

It is obvious that for this it is sufficient to take as the angle α the solution of the equation

$$\alpha_{,11}-\alpha_{,22}=\nu_{1,1}-\nu_{2,2}.$$

Equation (89) is satisfied if

$$\bar{\nu}_1=\psi_{,2},\ \bar{\nu}_2=\psi_{,1}. \quad (90)$$

Introducing in addition to $h(u^1, u^2)$ and $p(u^1, u^2)$ in the substitution (80) the new functions $r(u^1, u^2)$ and $s(u^1, u^2)$,

$$\left.\begin{aligned} &\nu_{36|1}=s(u^1, u^2)\sin\frac{\theta}{2},\quad \nu_{36|2}=r(u^1, u^2)\sin\frac{\theta}{2},\\ &\nu_{46|1}=r(u^1, u^2)\cos\frac{\theta}{2},\quad \nu_{46|2}=s(u^1, u^2)\cos\frac{\theta}{2}, \end{aligned}\right\} \quad (91)$$

we can reduce the system of Gauss, Peterson–Codazzi, and Ricci equations (32)–(34) to the nonlinear equations

$$\left.\begin{aligned} &\varphi_{,11}-\varphi_{,22}=2q^2 e^{\varphi}\cos\theta;\\ &\theta_{,11}-\theta_{,22}-(h^2+r^2-p^2-s^2)\sin\theta=2q^2 e^{\varphi}\sin\theta;\\ &\psi_{,11}-\psi_{,22}=(pr-hs)\cos\theta;\\ &r_{,2}-s_{,1}+\psi_{,1}h-\psi_{,2}p+\operatorname{tg}\frac{\theta}{2}(s\theta_{,1}-r\theta_{,2})=0;\\ &h_{,2}-p_{,1}+\psi_{,2}s-\psi_{,1}r+\operatorname{tg}\frac{\theta}{2}(p\theta_{,1}-h\theta_{,2})=0;\\ &s_{,2}-r_{,1}+\psi_{,1}p-\psi_{,2}h+\operatorname{ctg}\frac{\theta}{2}(s\theta_{,2}-r\theta_{,1})=0;\\ &p_{,2}-h_{,1}+\psi_{,2}r-\psi_{,1}s+\operatorname{ctg}\frac{\theta}{2}(p\theta_{,2}-h\theta_{,1})=0. \end{aligned}\right\} \quad (92)$$

The system (92) consists of seven equations, three of which are of the second order and four of the first. The unknowns are the seven functions φ, θ, ψ, p, h, r, s. There probably exists a substitution analogous to (84) that reduces the last four equations in (92) to a single second-order equation. The general solution of the system of nonlinear equations (92) can be constructed by means of (49) and (78) and the decompositions (53) and (58). The functions φ, θ, ψ, p, h, r, s will be expressed in terms of eight functions $f_{\pm j}(u^{\pm})$, $j=1, \ldots, 4$, of the single variable $u^{\pm}$.

Another method of reducing the system of equations (32)–(34) in the case of a two-dimensional minimal surface embedded in six-dimensional pseudo-Euclidean space was considered in Ref. 45.

It is obvious that the method presented here can also be applied to enveloping spaces of higher dimension $n>6$. The general solution of Eqs. (32)–(34) can again be expressed by means of (59) and (78) and the decompositions (53) and (58) in terms of $2(n-2)$ arbitrary functions $f_{\pm j}(u^{\pm})$, $j=1, \ldots, n-2$, of one variable $u^{\pm}$.

A system of $n-2$ nonlinear equations describing a relativistic string in n-dimensional space–time was obtained in Ref. 78 by complete elimination of the functional arbitrariness in the embedding equations (32)–(34). A different approach to this problem was considered in Ref. 79.

To conclude this section, we mention that the system of nonlinear equations (86) was obtained, and their general solution was constructed in the framework of a purely group-theoretical approach, in Ref. 9. Another form of expression of these equations was proposed in Ref. 10.

The investigation of the equations that describe minimal surfaces in pseudo-Euclidean space with dimension $n\geqslant 4$ is not only of purely mathematical interest, as a method of obtaining explicitly integrable nonlinear equations, but is also of interest from the point of view of the relativistic string model. Bearing in mind that there is as yet no satisfactory quantum theory of this object, it would be interesting to consider, by analogy with nonlinear sigma models, the $1/n$ expansion in the relativistic string model as well.

4. THE $t=\tau$ PARAMETRIZATION IN THE THEORY OF MINIMAL SURFACES

In the relativistic string model, it is interesting to consider the description of a two-dimensional minimal surface in Minkowski space when one of the curvilinear coordinates $u^1=\tau$ on the surface is identical to the time coordinate of the enveloping space, $x^0=t$. This is the so-called $t=\tau$ gauge.[24,38] The geometrical description of the minimal surface by means of differential forms in the $t=\tau$ parametrization leads to new nonlinear equations whose general solution can be found by a method similar to the one described in Sec. 3.

In the $t=\tau$ parametrization, a two-dimensional minimal surface embedded in n-dimensional pseudo-Euclidean Minkowski space is described by an $(n-1)$-dimensional Euclidean vector $\mathbf{x}=\{x^1, x^2, \ldots, x^{n-1}\}$, which depends on two parameters $u^1=\tau=x^0=t$ and $u^2=\sigma$. Equations (49) and (50), which determine the minimal surface, are now written as[12,14]

$$\mathbf{x}_{,11}-\mathbf{x}_{,22}=0, \tag{93}$$

$$\mathbf{x}_{,1}^2+\mathbf{x}_{,2}^2=1, \quad \mathbf{x}_{,1}\mathbf{x}_{,2}=0. \tag{94}$$

The conditions (94) dictate the following form of the metric tensor on the surface $\mathbf{x}(u^1, u^2)$:

$$g_{11}=\mathbf{x}_{,1}^2=\sin^2\theta, \quad g_{22}=\mathbf{x}_{,2}^2=\cos^2\theta, \quad g_{12}=0. \tag{95}$$

In what follows, we shall need the Christoffel symbols corresponding to this metric:

$$\left.\begin{array}{ll}\Gamma^1_{11}=\Gamma^1_{22}=\theta_{,1}\operatorname{ctg}\theta, & \Gamma^1_{12}=\Gamma^1_{21}=\theta_{,2}\operatorname{ctg}\theta,\\ \Gamma^2_{11}=\Gamma^2_{22}=-\theta_{,2}\operatorname{tg}\theta, & \Gamma^2_{12}=\Gamma^2_{21}=-\theta_{,1}\operatorname{tg}\theta.\end{array}\right\} \tag{96}$$

The two-dimensional surface $\mathbf{x}(u^1, u^2)$ in the $(n-1)$-dimensional Euclidean space determined by Eqs. (93) and (94) is not minimal from the point of view of this space, i.e., it does not satisfy the conditions

$$g^{ij}b_{\alpha|ij}=0, \quad \alpha=3, \ldots, n-1. \tag{97}$$

Nevertheless, Eqs. (93) and (94), when rewritten in terms of the quadratic forms g_{ij} and $b_{\alpha|ij}$, lead to the same equations as (60):

$$b_{\alpha|11}=b_{\alpha|22}, \quad \alpha=3, \ldots, n-1. \tag{98}$$

The single essential component R_{1212} of the curvature tensor has in the metric (95) the form[24]

$$R_{1212}=\frac{1}{2}\sin 2\theta\,(\theta_{,11}-\theta_{,22}). \tag{99}$$

The Gauss equation (32) can be written as

$$\frac{1}{2}\sin 2\theta\,(\theta_{,11}-\theta_{,22})=\sum_{\alpha=3}^{n-1}(b^2_{\alpha|11}-b^2_{\alpha|12}). \tag{100}$$

The construction of the general solutions of the system of equations (33), (34), and (100) will be based, as in Sec. 3, on expression of the solutions of Eqs. (93) that satisfy the nonlinear conditions (94) in a special basis.

We begin by considering the simplest case of three-dimensional space–time in which the minimal surface is embedded. In the $t=\tau$ parametrization, the coordinates $x^1(u^1, u^2)$ and $x^2(u^1, u^2)$ specify a plane that is the projection of the minimal surface from the space $\{x^0, x^1, x^2\}$ onto the coordinate plane Ox^1x^2. In this case, $b_{\alpha|ij}=0$ and $\nu_{\alpha\beta|i}=0$. The only nontrivial equation in the system (33), (34), and (100) is the Gauss equation (100), which reduces to the d'Alembert equation

$$\theta_{,11}-\theta_{,22}=0. \tag{101}$$

For this simple sample, we shall demonstrate the complete scheme of obtaining general solutions of Eqs. (32)–(34). We take the solution of Eqs. (93) and (94) for the two-dimensional radius vector $\mathbf{x}(u^1, u^2)$ in the form

$$\left.\begin{array}{l}\mathbf{x}(u^1, u^2)=\boldsymbol{\psi}_+(u^+)-\boldsymbol{\psi}_-(u^-), \quad (\boldsymbol{\psi}'_\pm)^2=1,\\ u^\pm=u^1\pm u^2,\end{array}\right\} \tag{102}$$

$$\boldsymbol{\psi}'_\pm(u^\pm)=\frac{1}{\sqrt{2}}\{\cos\varphi_\pm(u^\pm), \ \pm\sin\varphi_\pm(u^\pm)\}. \tag{103}$$

In accordance with (95), $\theta(u^1, u^2)$ is determined by

$$\theta(u^1, u^2)=\operatorname{arctg}\left(\frac{\mathbf{x}^2_{,1}}{\mathbf{x}^2_{,2}}\right)^{\frac{1}{2}}. \tag{104}$$

Substituting here (102) and (103), we obtain the general solution of Eq. (101):

$$\theta(u^1, u^2)=\operatorname{arctg}\left[\frac{1-\cos(\varphi_++\varphi_-)}{1+\cos(\varphi_++\varphi_-)}\right]^{\frac{1}{2}}=\frac{1}{2}\,[\varphi_+(u^+)+\varphi_-(u^-)]. \tag{105}$$

We now consider a four-dimensional pseudo-Euclidean space in which the minimal surface is embedded. In this case, Eqs. (33), (34), and (100) will not be so trivial as Eq. (101) considered above.

In the $t=\tau$ parametrization, the minimal surface is described by a three-component Euclidean vector $\mathbf{x}(u^1, u^2)$, which gives the projection of this surface from the four-dimensional Minkowski space into the ordinary three-dimensional Euclidean space. It is easy to show that with allowance for (96) and (98) the system of equations (33), (34), and (100) reduces to the two nonlinear equations

$$\left.\begin{array}{l}\theta_{,11}-\theta_{,22}+\dfrac{\cos\theta}{\sin^3\theta}(\varkappa^2_{,1}-\varkappa^2_{,2})=0,\\ (\operatorname{ctg}^2\theta\,\varkappa_{,1})_{,1}=(\operatorname{ctg}^2\theta\,\varkappa_{,2})_{,2},\end{array}\right\} \tag{106}$$

where the function $\varkappa(u^1, u^2)$ determines the coefficients of the second quadratic form:

$$b_{11}=b_{22}=\operatorname{ctg}\theta\,\varkappa_{,2}, \quad b_{12}=\operatorname{ctg}\theta\,\varkappa_{,1}. \tag{107}$$

The system (106) differs from the well-known Lund–Regge equations[58] by the absence in the first equation of the term $\sin\theta\cos\theta$. Equations (106) can be obtained from the system (86) by setting there $\varphi\to-\infty$, $\theta\to 2\theta$. However, it is not clear how one is to find a general solution for (106) by proceeding from the solution for the system (86). Therefore, we shall construct the general solution for the system of nonlinear equations (106) from the beginning.

We again represent the vector $\mathbf{x}(u^1, u^2)$ by means of (102), and for $\boldsymbol{\psi}'_\pm(u^\pm)$ we take spherical coordinates in the three-dimensional Euclidean space:

$$\boldsymbol{\psi}'_\pm(u^\pm)=\frac{1}{\sqrt{2}}\{\sin\omega_\pm\cos\varphi_\pm, \ \pm\sin\omega_\pm\sin\varphi_\pm, \ \pm\cos\omega_\pm\},$$
$$\omega_\pm\equiv\omega_\pm(u^\pm), \quad \varphi_\pm\equiv\varphi_\pm(u^\pm). \tag{108}$$

For the function $\theta(u^1, u^2)$ we obtain in accordance with (104)

$$\theta(u^1, u^2)=\operatorname{arctg}\left\{\frac{1-[\sin\omega_+\sin\omega_-\cos(\varphi_++\varphi_-)-\cos\omega_+\cos\omega_-]}{1+[\sin\omega_+\sin\omega_-\cos(\varphi_++\varphi_-)-\cos\omega_+\cos\omega_-]}\right\}^{\frac{1}{2}}. \tag{109}$$

Using the cosine theorem for a spherical triangle,[59] we can readily show that 2θ in (109) is the angle in the spherical triangle opposite the side $\varphi_++\varphi_-$, for which the adjacent angles are ω_+ and ω_-.

The system (108) contains only the partial derivatives of the function $\varkappa(u^1, u^2)$. In accordance with the derivation formulas (28), they are determined by

$$\varkappa_{,1}=\operatorname{tg}\theta\sqrt{(\nabla_1\mathbf{x}_{,2})^2}, \quad \varkappa_{,2}=\operatorname{tg}\theta\sqrt{(\nabla_1\mathbf{x}_{,1})^2}. \tag{110}$$

It is obvious that the decompositions (102) and (108) make it possible to express $\varkappa_j$, $i=1, 2$, in terms of four arbitrary functions $\varphi_\pm(u^\pm)$ and $\omega_\pm(u^\pm)$ of one variable, and therefore we shall not give here these lengthy expressions.

Generalizing the representations (103) and (108) to the n-dimensional case, we can construct general solutions of the system of nonlinear equations (33), (34), (100) in the $t = \tau$ parametrization for arbitrary dimension of the enveloping space in which the minimal surface is embedded.

5. THEORY OF SURFACES AND LAX REPRESENTATION FOR NONLINEAR EVOLUTION EQUATIONS

If a nonlinear partial differential equation is to be investigated by the inverse scattering method, it is necessary to find for it a Lax operator representation, i.e., to represent this equation as the result of the compatibility of some linear spectral problem.[2–6] General algorithms for constructing a Lax representation for a given equation do not exist. Generally, one argues in the opposite direction. One takes a class of matrix integro-differential operators and considers the condition of their compatibility. This yields a set of nonlinear equations for which the inverse scattering method can certainly be used.

As was noted in Refs. 19–24, the theory of surfaces provides one such series of nonlinear equations. Indeed, the Gauss, Peterson–Codazzi, and Ricci equations (32)–(34), which are nonlinear second-order partial differential equations for the differential forms of the surface, are in accordance with their derivation the condition of compatibility of the linear equations that describe the motion of the basis over the surface. The question of the introduction of a spectral parameter into the corresponding linear equations here remains open. However, in a number of cases such a parameter can be introduced by using some symmetry of the nonlinear equation.

Well known is the connection of the sine–Gordon equation and the corresponding linear spectral problem to surfaces of constant curvature.[24] It can be shown that a similar connection can be established for other nonlinear equations solvable by the inverse scattering method. Below, we consider some examples of such equations.

6. GEOMETRICAL INTERPRETATION OF THE EQUATION $\varphi_{,11} - \varphi_{,22} = e^{\varphi} - e^{-2\varphi}$

A complete series of studies[60–62] has been devoted to this equation, which has become known as the Dodd–Bullough equation. An exact S matrix has been found[63] for the corresponding two-dimensional field theory.

The Dodd–Bullough equation is also related to surfaces of constant curvature, not in ordinary Euclidean space, but in three-dimensional unimodular affine space.[20]

Without pretending to completeness of exposition, we give here the basic ideas of affine differential geometry.[53,64–67] This theory operates with an affine n-dimensional space whose elements are points (or vectors) with coordinates $(x^1, x^2, \ldots, x^n)$. In this space there is defined the group of affine transformations

$$x'^i = c^i_j x^j + b^i \tag{111}$$

with the condition $\det\|c^i_j\| \neq 0$.

According to Klein,[66] affine geometry studies the invariants of the transformations (111). The theory of surfaces in affine space, i.e., differential affine geometry, can be constructed by analogy with the differential geometry of ordinary Euclidean space.[67] The theory is based on forms that must be *invariants* under the transformations (111) of the space.

For our purposes, it will be sufficient to restrict ourselves to the case of a three-dimensional unimodular affine space. The transformations (111) for the unimodular affine group $SL(n, R)$ satisfy the condition

$$\det \| c^i_j \| = 1. \tag{112}$$

We shall denote the vectors of this space, like the vectors of ordinary three-dimensional Euclidean space, by $\mathbf{r}$. We shall specify the two-dimensional surface in the three-dimensional affine space in the parametric form $\mathbf{r} = \mathbf{r}(u^1, u^2)$.

In the theory of surfaces of Euclidean space, the main role is played, as we pointed out in Sec. 2 by two quadratic forms:

$$\Phi_1 = \sum_{i,j=1}^{2} g_{ij}\, du^i\, du^j, \quad g_{ij} = \mathbf{r}_{,i}\mathbf{r}_{,j}, \quad \mathbf{r}_{,i} = \partial\mathbf{r}/\partial u^i \tag{113}$$

and

$$\Phi_2 = \sum_{i,j=1}^{2} b_{ij}\, du^i\, du^j, \quad b_{ij} = \mathbf{r}_{,ij}\mathbf{n}, \tag{114}$$

where $\mathbf{n}$ is the normal to the surface: $\mathbf{n} = [\mathbf{r}_{,1} \times \mathbf{r}_{,2}]/\sqrt{\det\|g_{ij}\|}$.

In unimodular affine geometry, one can introduce the concept of the invariant volume of the parallelepiped constructed on three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$:

$$V_{\text{inv}} = (\mathbf{abc}) = \begin{vmatrix} a^1 & a^2 & a^3 \\ b^1 & b^2 & b^3 \\ c^1 & c^2 & c^3 \end{vmatrix}. \tag{115}$$

Using this, one constructs invariant forms that determine the surface $\mathbf{r}(u^1, u^2)$ in the given geometry. The first quadratic form

$$\Psi_1 = \sum_{i,j=1}^{2} \tilde{g}_{ij}\, du^i\, du^j \tag{116}$$

is given by the tensor

$$\tilde{g}_{ij} = \frac{a_{ij}}{(|a|)^{\frac{1}{4}}}, \quad a_{ij} = (\mathbf{r}_{,ij}\mathbf{r}_{,1}\mathbf{r}_{,2}), \quad a = \det\|a_{ij}\|. \tag{117}$$

In contrast to the Euclidean case, the second form in affine unimodular geometry is *cubic*:

$$\Psi_2 = \sum_{i,j,k=1} T_{ijk}\, du^i\, du^j\, du^k, \tag{118}$$

where the tensor T_{ijk} is determined by means of an auxiliary vector

$$\mathbf{m} = \frac{[\mathbf{r}_{,1} \times \mathbf{r}_{,2}]}{(|a|)^{\frac{1}{4}}} \tag{119}$$

exterior to the surface as follows:

$$T_{ijk} = \nabla_i \nabla_j \mathbf{r}_{,k}\mathbf{m}. \tag{120}$$

Here, ∇_j denotes the covariant differentiation with respect to the tensor $\tilde{g}_{ij}$.

The tensors $\tilde{g}_{ij}$ and T_{ijk} are symmetric with respect to their indices and are connected by the apolarity relation

$$\tilde{g}^{ij}T_{ijk} = 0. \tag{121}$$

Besides the exterior vector **m** an important part in the theory is played by the *affine normal*

$$\mathbf{N} = \frac{1}{2}\tilde{g}^{ij}\nabla_i\nabla_j\mathbf{r} = \frac{1}{2}\Box\mathbf{r}, \tag{122}$$

where $\Box$ is the covariant Laplace–Beltrami operator for the metric tensor $\tilde{g}_{ij}$.

On the surface $\mathbf{r}(u^1, u^2)$ of the affine space, the moving frame is formed by the two tangent vectors $\mathbf{r}_{,1}(u^1, u^2)$ and $\mathbf{r}_{,2}(u^1, u^2)$ and the affine normal $\mathbf{N}(u^1, u^2)$. The displacement of this frame over the surface is described by the derivation equations of Gauss,

$$\nabla_i\mathbf{r}_{,j} = T^k_{ij}\mathbf{r}_{,k} + \tilde{g}_{ij}\mathbf{N}, \tag{123}$$

and Weingarten

$$\mathbf{N}_{,i} = A^j_i\mathbf{r}_{,j}. \tag{124}$$

As usual, indices are raised and lowered by means of the tensor $\tilde{g}_{ij}$, and the quantities A_{ij} are detrmined by

$$A_{ij} = \nabla_k T^k_{ij} - H\tilde{g}_{ij}, \tag{125}$$

where $2H = -A^i_i$.

The invariant H is called the mean curvature of the surface, and in the geometry of the unimodular affine group it plays the same part as the mean curvature

$$h = -\frac{1}{2}b^i_i \tag{126}$$

in the Euclidean theory of surfaces. A second invariant associated with the tensors A_{ij} and $\tilde{g}_{ij}$ is the total curvature

$$K = \det\|A^j_i\| = \frac{\det\|A_{ij}\|}{\det\|\tilde{g}_{ik}\|}. \tag{127}$$

Its analog in the Euclidean theory is the total or Gaussian curvature of the surface:

$$k = \det\|b^j_i\| = \frac{\det\|b_{ij}\|}{\det\|g_{ik}\|}. \tag{128}$$

According to *Rado's theorem*,[53] the forms (116) and (118), which are connected by the relation (121), determine the surface up to unimodular affine transformations if the conditions of compatibility of Eqs. (123) and (124) are satisfied.

We now turn to the determination of the affine sphere in terms of the invariants K and H. We recall first that the ordinary Euclidean sphere $\mathbf{r}^2 = R^2$ is specified in the language of k and h by the conditions

$$h = \text{const} = \frac{1}{R}, \quad k = \text{const} = \frac{1}{R^2} = h^2. \tag{129}$$

Indeed, from (113), (114), (125), and (128) and using the fact that $\mathbf{n} = \mathbf{r}/R$ for the sphere, we have

$$b_{ij} = -\frac{1}{R}g_{ij}, \quad h = -\frac{1}{2}b^i_i = \frac{1}{R}, \quad k = \frac{1}{R^2}. \tag{130}$$

In the geometry of the unimodular affine group, spheres are determined as surfaces in which the affine normals intersect at one point:

$$\mathbf{r} + R\mathbf{N} = \mathbf{c}, \tag{131}$$

where $\mathbf{c}$ is a constant vector in the affine space. In complete analogy with (129), one can show that from (131) there follow[64,65]

$$H = \text{const} = \frac{1}{R}, \quad K = H^2. \tag{132}$$

For clarity, we give two examples of surfaces that are affine spheres with centers at the origin:

$$z(x^2 + y^2) = 1, \quad xyz = 1.$$

On the affine sphere, we choose the coordinate system in the curvature lines[64]:

$$\tilde{g}_{11} = -\tilde{g}_{22} = e^{\varphi}, \quad \tilde{g}_{12} = \tilde{g}_{21} = 0. \tag{133}$$

It follows from the condition (121) that all the components of the tensor T_{ijk} can be expressed in terms of two quantities, which we denoted by A and B:

$$T_{111} = T_{221} = A, \quad T_{222} = T_{112} = B. \tag{134}$$

For our purposes, it is sufficient to assume that A and B are constants:

$$A = \frac{1}{2}, \quad B = 0, \quad H = -\frac{1}{2}.$$

The moving basis on the affine sphere formed by the two tangent vectors $\mathbf{r}_{,1}$ and $\mathbf{r}_{,2}$ and the normal $\mathbf{N}$ (122) is not normalized. It can be normalized by replacing $\mathbf{N}$ by $\mathbf{e}_3 = e^{-\varphi}\mathbf{N}$. Then by means of Eqs. (117), (122), and (133) it is readily verified that the basis $\{\mathbf{r}_{,1}, \mathbf{r}_{,2}, \mathbf{e}_3\}$ is normalized by the condition

$$(\mathbf{r}_{,1}\mathbf{r}_{,2}\mathbf{e}_3) = 1. \tag{135}$$

We now write down explicitly equations that determine the motion of the basis $\{\mathbf{r}_{,1}, \mathbf{r}_{,2}, \mathbf{e}_3\}$ over the affine sphere, using (123) and (124) and making the following change of the parameters u^1, u^2:

$$u^1 \pm u^2 \to \lambda^{\pm 1}(u^1 \pm u^2),$$

where λ is a constant. In what follows, it will play the part of a spectral parameter.[68] These equations have the form

$$\frac{\partial}{\partial u^j}\begin{pmatrix}\mathbf{r}_{,1}\\ \mathbf{r}_{,2}\\ \mathbf{e}_3\end{pmatrix} = \Omega^j\begin{pmatrix}\mathbf{r}_{,1}\\ \mathbf{r}_{,2}\\ \mathbf{e}_3\end{pmatrix}, \quad j = 1, 2, \tag{136}$$

where the matrices Ω^j, $j = 1, 2$, belong to the Lie algebra of the group $SL(3, R)$ and are determined by

$$\left.\begin{aligned}
\Omega^1 &= \left\|\begin{matrix} \frac{\varphi_{,1}}{2} + \lambda_+ e^{-\varphi} & \frac{\varphi_{,2}}{2} - \lambda_- e^{-\varphi} & 2\lambda_+ e^{2\varphi} \\ \frac{\varphi_{,2}}{2} + \lambda_- e^{-\varphi} & \frac{\varphi_{,1}}{2} - \lambda_+ e^{-\varphi} & -2\lambda_- e^{2\varphi} \\ \lambda_+ e^{-\varphi} & \lambda_- e^{-\varphi} & -\varphi_{,1} \end{matrix}\right\|, \\
\Omega^2 &= \left\|\begin{matrix} \frac{\varphi_{,2}}{2} + \lambda_- e^{-\varphi} & \frac{\varphi_{,1}}{2} - \lambda_+ e^{-\varphi} & 2\lambda_- e^{2\varphi} \\ \frac{\varphi_{,1}}{2} + \lambda_+ e^{-\varphi} & \frac{\varphi_{,2}}{2} - \lambda_- e^{-\varphi} & -2\lambda_+ e^{2\varphi} \\ \lambda_- e^{-\varphi} & \lambda_+ e^{-\varphi} & -\varphi_{,2} \end{matrix}\right\|,
\end{aligned}\right\} \tag{137}$$

$$\lambda_\pm = \frac{1}{4}\left(\lambda \pm \frac{1}{\lambda}\right).$$

It is readily verified that the condition of compatibility for the linear equations (136),

$$\Omega^1_{,2} - \Omega^2_{,1} + [\Omega^1, \Omega^2] = 0, \tag{138}$$

gives

$$\varphi_{,11} - \varphi_{,22} = e^{\varphi} - e^{-2\varphi}. \tag{139}$$

This equation is the concrete expression in the coordinate system (133) of a general equation in the geometry of affine spheres,[65]

$$\Box \ln J = 6(H + J),$$

where $J = 1/2T^{ijk}T_{ijk}$ is the Picard invariant.

In Ref. 60, a Lax representation was constructed for Eq. (139) in 3×3 matrices by a method different from the one considered here.

It is not possible to lower the dimension of the matrix equations (136) in the same way as was done for the sine–Gordon equation[24,68,69] by going over to the spinor representation of the rotation group $SO(3)$, since the group of unimodular affine transformations $SL(3, R)$ in three-dimensional affine space does not have matrix representations of dimension lower than three.

7. LAX REPRESENTATION FOR NONLINEAR EQUATIONS DESCRIBING A TWO-DIMENSIONAL MINIMAL SURFACE IN DE SITTER SPACE

We consider here the system of the two nonlinear equations[19]

$$\left.\begin{aligned}\varphi_{,11}-\varphi_{,22}&=e^{\varphi}\cos\theta+\varepsilon e^{-\varphi},\\ \theta_{,11}-\theta_{,22}&=e^{\varphi}\sin\theta,\quad \varepsilon=\pm 1.\end{aligned}\right\}\tag{140}$$

These equations describe a two-dimensional minimal surface in space–time of constant curvature. Such surfaces arise, for example, in the theory of a relativistic string in a de Sitter universe. Setting $\theta = 0$ in (140), we obtain the equation

$$\varphi_{,11}-\varphi_{,22}=2\begin{cases}\sinh\varphi, & \varepsilon=-1,\\ \cosh\varphi, & \varepsilon=+1,\end{cases}\tag{141}$$

which describes the minimal surface in three-dimensional de Sitter space–time. Equation (141) is the generalization to the case of an indefinite metric of the well-known result in differential geometry on the embedding of a minimal surface in a three-dimensional space of constant curvature.[27]

We obtain a Lax representation for the system (140) by using the geometrical nature of these equations. A Lax representation for these equations was found by a different method in Ref. 70. We shall show below why it is not possible to construct a general solution for the system (140) by the method described in Secs. 2–4.

De Sitter space–time can be regarded as a hyperboloid in a pseudo-Euclidean five-dimensional space with coordinates z^{μ}, $\mu = 0, \ldots, 4$:

$$(z^0)^2-(z^1)^2+(z^2)^2-(z^3)^2+\varepsilon(z^4)^2=\varepsilon R^2,\tag{142}$$

where $\varepsilon = +1$ for the de Sitter universe of the first kind and $\varepsilon = -1$ for the de Sitter universe of the second kind.[71]

Any submanifold of de Sitter space–time can also be decribed by differential forms g_{ij}, $b_{\alpha|ij}$, and $\nu_{\alpha\beta}|_i$, which must satisfy Gauss, Peterson–Codazzi, and Ricci equations. The Peterson–Codazzi (33) and Ricci (34) equations keep the same form, while the Gauss equation (32) is modified as follows[17]:

$$R_{ijkl}=\sum_{\sigma=3}^{4}\varepsilon_{\sigma}(b_{\sigma|ik}b_{\sigma|jl}-b_{\sigma|il}b_{\sigma|jk})+\frac{\varepsilon}{R^2}(g_{ik}g_{jl}-g_{il}g_{kj}).\tag{143}$$

Equations (33), (34) and (143) are the conditions of compatibility of the linear equations

$$\nabla_j z^{\mu}_{,i}=\sum_{\sigma=3}^{4}\varepsilon_{\sigma}b_{\sigma|ij}e^{\mu}_{\sigma}-\frac{\varepsilon}{R^2}g_{ij}z^{\mu},\tag{144}$$

$$e^{\mu}_{\sigma,i}=-b_{\sigma|ij}g^{jk}z^{\mu}_{,k}-\sum_{\tau}\varepsilon_{\tau}\nu_{\tau\sigma|i}e^{\mu}_{\sigma},\tag{145}$$

$$i, j, k\ldots=1, 2, \quad \tau, \sigma\ldots=3, 4,$$

which describe the motion over the surface $z^{\mu} = z^{\mu}(u^1, u^2)$ of the basis

$$z^{\mu}_{,1},\ z^{\mu}_{,2},\ e^{\mu}_3,\ e^{\mu}_4,\ z^{\mu}.\tag{146}$$

By virtue of (142), the radius vector $z^{\mu}(u^1, u^2)$ of the surface itself plays the part of the third normal in this basis.

As before, the condition of minimality of the surface[17] is given by Eqs. (43):

$$g^{ij}b_{\alpha|ij}=0,\quad \alpha=3, 4.\tag{147}$$

Contraction of Eq. (144) with the metric tensor g^{ij} leads with allowance for (147) to a well-known result in the theory of minimal surfaces embedded in spaces of constant curvature,[72] namely, the action of the Laplace–Beltrami operator $\nabla_i\nabla^i$ on the vector z^{μ} is determined by the equation

$$\nabla_i\nabla^i z^{\mu}=-2\frac{\varepsilon}{R^2}z^{\mu}.\tag{148}$$

In contrast to (44), this equation cannot be linearized in any system of curvilinear coordinates u^1, u^2 on the minimal surface. Therefore, it is not possible to construct its general solution or, therefore, to use the method of obtaining general solutions of the nonlinear equations (33), (34), and (143) described in Secs. 2–4. However, on the basis of the geometrical nature of these equations it is possible to construct a Lax representation for them.

We first show how the system (140) follows from (33), (34), and (143).

On the minimal surface, we again take a conformally flat coordinate system,

$$g_{11}=-g_{22}=e^{-\varphi},\quad g_{12}=g_{21}=0,\tag{149}$$

in which the minimality conditions (147) become

$$b_{\alpha}|_{11}=b_{\alpha}|_{22},\quad \alpha=3, 4.\tag{150}$$

If we eliminate from the Peterson–Codazzi equations (33) the torsion vectors $\gamma_{34|i}$, $i = 1, 2$, and take into account (150), we arrive at (61):

$$\frac{\partial}{\partial u^{\pm}}\sum_{\alpha=3}^{4}(b_{\alpha|11}\pm b_{\alpha|12})^2=0.\tag{151}$$

We satisfy these equations if we set

$$\sum_{\alpha=3}^{4}(b_{\alpha|11}\pm b_{\alpha|22})^2=q^2,\tag{152}$$

where q is a constant. Using (144) and (149), we can rewrite Eqs. (152) in the form

$$\left[\left(\nabla_1 z^{\mu}_{,1}+\frac{\varepsilon}{R^2}e^{-\varphi}z^{\mu}\right)\pm\nabla_2 z^{\mu}_{,1}\right]^2=q^2.\tag{153}$$

Thus, in the enveloping five-dimensional space we have two mutually orthogonal vectors related naturally to the coordinates of the minimal surface. They are the vectors

$$\nabla_1 z^{\mu}_{,1} + \frac{\varepsilon}{R^2} e^{-\varphi} z^{\mu} \text{ and } \nabla_2 z^{\mu}_{,1}. \tag{154}$$

Directing the normal e_3^{μ} along the first of them and e_4^{μ} along the second, we obtain

$$b_{3\,|\,12} = b_{4\,|\,11} = b_{4\,|\,22} = 0. \tag{155}$$

To satisfy the condition (153), we can set

$$b_{3\,|\,11} = q\cos\frac{\theta}{2}, \quad b_{4\,|\,12} = q\sin\frac{\theta}{2}. \tag{156}$$

It is now easy to show that the Gauss (143) and Ricci (34) equations reduce to the system (140), while the Peterson–Codazzi equations (33) with $\alpha = 3, 4$ determine the torsion vector $\nu_{34|i}$, $i = 1, 2$, in terms of the function θ:

$$\nu_{34\,|\,1} = \frac{\theta_{,2}}{2}, \quad \nu_{34\,|\,2} = \frac{\theta_{,1}}{2}. \tag{157}$$

We now turn to the derivation of the Lax representation for the system (140). For this, we construct from the basis (146) a new orthonormal basis $\bar{e}_a$, $a = 1, \ldots, 5$:

$$\begin{gathered}\bar{e}_1 = z_{,1} e^{\varphi/2}, \quad \bar{e}_2 = i z_{,2} e^{\varphi/2}, \quad \bar{e}_3 = i e_3, \\ \bar{e}_4 = i e_4, \quad \bar{e}_5 = \sqrt{\varepsilon}\,\frac{z}{R}, \quad \bar{e}_a \bar{e}_b = \delta_{ab}.\end{gathered} \tag{158}$$

Using Eqs. (144), (145), and (158), we obtain the following equations, which describe the motion of the basis $\{\bar{e}_a\}$ over the minimal surface:

$$\frac{\partial \bar{e}_a}{\partial u^j} = \sum_{b=1}^{5} \omega^j_{ab} \bar{e}_b, \quad a, b = 1, \ldots, 5, \quad j = 1, 2, \tag{159}$$

where ω^j are 5×5 matrices belonging to the Lie algebra of the group $SO(5)$:

$$\omega^1 = \begin{vmatrix} 0 & \frac{i}{2}\varphi_{,2} & iqe^{\varphi/2}\cos\frac{\theta}{2} & 0 & -\frac{\sqrt{\varepsilon}}{R}e^{\varphi/2} \\ -\frac{i}{2}\varphi_{,2} & 0 & 0 & -qe^{\varphi/2}\sin\frac{\theta}{2} & 0 \\ -iqe^{\varphi/2}\cos\frac{\theta}{2} & 0 & 0 & \frac{\theta_{,2}}{2} & 0 \\ 0 & qe^{\varphi/2}\sin\frac{\theta}{2} & -\frac{\theta_{,2}}{2} & 0 & 0 \\ -\frac{\sqrt{\varepsilon}}{R}e^{\varphi/2} & 0 & 0 & 0 & 0 \end{vmatrix}, \tag{160}$$

$$\omega^2 = \begin{vmatrix} 0 & \frac{i}{2}\varphi_{,1} & 0 & iqe^{\varphi/2}\sin\frac{\theta}{2} & 0 \\ -\frac{i}{2}\varphi_{,1} & 0 & -qe^{\varphi/2}\cos\frac{\theta}{2} & 0 & i\frac{\sqrt{\varepsilon}}{R}e^{\varphi/2} \\ 0 & qe^{\varphi/2}\cos\frac{\theta}{2} & 0 & \frac{\theta_{,1}}{2} & 0 \\ -iqe^{\varphi/2}\sin\frac{\theta}{2} & 0 & -\frac{\theta_{,1}}{2} & 0 & 0 \\ 0 & -i\frac{\sqrt{\varepsilon}}{R}e^{\varphi/2} & 0 & 0 & 0 \end{vmatrix}. \tag{161}$$

We now use the fact that the group SO(5) admits a representation by 4×4 matrices. In this representation, the generators of $SO(5)$ are given by[73]

$$I_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu], \quad \mu, \nu = 1, \ldots, 5, \tag{162}$$

where γ_μ are the well-known Dirac matrices,

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}, \quad \mu, \nu = 1, \ldots, 5. \tag{163}$$

For what follows,it is convenient to take a representation of the γ matrices in which the matrix $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ is diagonal:

$$\gamma_k = \begin{pmatrix} 0 & i\sigma_k \\ -i\sigma_k & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}, \tag{164}$$

where σ_k are the Pauli matrices, $k = 1, 2, 3$.

We can associate Eqs. (159) in a one-to-one manner with the following two systems of linear equations, which each now contain only four equations:

$$\frac{\partial \psi_a}{\partial u^j} = \frac{1}{2}\sum_{\mu,\nu>1}^{5} \omega^j_{\mu\nu}(\gamma_\mu\gamma_\nu)_{ab}\psi_b = \frac{1}{4}\Omega^j_{ab}\psi_b, \qquad j = 1, 2. \tag{165}$$

Here, $\psi_a(u^1, u^2)$ is a four-component function column, and Ω^j are 4×4 matrices of the form

$$\Omega^j = 2\sum_{\mu>\nu=1}^{5} \omega^j_{\mu\nu}\gamma_\mu\gamma_\nu, \qquad j = 1, 2. \tag{166}$$

To express these matrices explicitly, we introduce the notation

$$\Omega^j = \begin{pmatrix} a^j_{11} & a^j_{12} \\ a^j_{21} & a^j_{22} \end{pmatrix}, \tag{167}$$

where a^j_{kl}, in their turn, are 2×2 matrices. Using (159)–(166), we obtain for a^j_{kl} in Eqs. (167) the following decompositions with respect to the matrices σ_i:

$$\left.\begin{aligned} &a^1_{11} = Q^*\sigma_1 - \varkappa^*_{,2}\sigma_3, && a^2_{11} = -iQ^*\sigma_1 - \varkappa^*_{,1}\sigma_3, \\ &a^1_{12} = a^1_{21} = -i\sigma_1\frac{\sqrt{\varepsilon}}{R}e^{\varphi/2}, && a^2_{12} = a^2_{21} = -\sigma_2\frac{\sqrt{\varepsilon}}{R}e^{-\varphi/2}, \\ &a^1_{22} = Q\sigma_2 - \varkappa_{,2}\sigma_3, && a^2_{22} = -iQ\sigma_1 - \varkappa_{,1}\sigma_3, \end{aligned}\right\} \tag{168}$$

where $Q = qe^{\varkappa}$, $2\varkappa = \varphi + i\theta$, $\varepsilon = \pm 1$.

The compatibility conditions for the linear equations (165),

$$\Omega^1_{,2}-\Omega^2_{,1}+\frac{1}{4}[\Omega^1,\ \Omega^2]=0, \tag{169}$$

reduce to the system of nonlinear equations (140). To introduce a spectral parameter in (165), it is necessary to go over to the new variables $\bar{u}^1$, $\bar{u}^2$:

$$\bar{u}^1+\bar{u}^2=\lambda(\bar{u}^1+u^2),\quad \bar{u}^1-\bar{u}^2=\lambda^{-1}(u^1-u^2). \tag{170}$$

Then Eqs. (140) keep their previous form, while $\lambda+1/\lambda$ combinations occur in the matrix elements Ω^j_{ab}. We shall not perform these elementary transformations here.

8. NEW NONLINEAR EQUATION ASSOCIATED WITH THE SPHERE IN THREE-DIMENSIONAL EUCLIDEAN SPACE

Reading the material of the previous sections, the reader might get the impression that to every surface of a particular type there corresponds a definite nonlinear equation or system of nonlinear equations. However, this is not so. Different nonlinear equations may correspond to one and the same surface. On a surface, one can choose different curvilinear coordinate systems and, as a result, obtain different nonlinear equations. Let us consider, for example, the ordinary sphere in three-dimensional space:

$$\mathbf{r}^2=1,\quad \mathbf{r}=\mathbf{r}(u^1,\ u^2). \tag{171}$$

If on it we choose the so-called Chebyshev coordinate system,[67]

$$ds^2=(dv^1)^2+2\cos\omega\, dv^1\, dv^2+(dv^2)^2, \tag{172}$$

then the Gauss equation (32) gives the sine–Gordon equation

$$\omega_{,12}=\sin\omega. \tag{173}$$

But if on the sphere we take the conformally flat coordinate system

$$ds^2=e^{\varphi}[(du^1)^2+(du^2)^2], \tag{174}$$

then from (32) we obtain the Liouville equation (see the Appendix)

$$\varphi_{,11}+\varphi_{,22}=-2e^{\varphi}. \tag{175}$$

We consider on the sphere (171) a curvilinear coordinate system in which the coordinate lines are geodesic ellipses and hyperboloids.[26,74] The square of the line element in this case has the form

$$ds^2=\frac{(du^1)^2}{\sin^2(\theta/2)}+\frac{(du^2)^2}{\cos^2(\theta/2)}. \tag{176}$$

In the coordinate system (176), the moving frame $\{\mathbf{r}_{,1},\mathbf{r}_{,2},\mathbf{e}_3\}$ is orthogonal but not orthonormal:

$$\mathbf{r}^2_{,1}=\left(\sin\frac{\theta}{2}\right)^{-2},\quad \mathbf{r}^2_{,2}=\left(\cos\frac{\theta}{2}\right)^{-2},\quad \mathbf{r}_{,1}\mathbf{r}_{,2}=0.$$

Nevertheless, such a basis is convenient for introducing a spectral parameter in the Lax representation obtained below.

The possibility of choosing the coordinate system (176) on the sphere is proved in classical courses of differential geometry.[47,74] However, the function $\theta(u^1,u^2)$ cannot be arbitrary; it must satisfy a certain nonlinear second-order partial differential equation.[22] This equation is the Gauss equation for the two-dimensional Riemannian manifold of constant curvature with the metric (176).

The motion of the frame $\{\mathbf{r}_{,1},\mathbf{r}_{,2},\mathbf{e}_3\}$ over the sphere is determined by Eqs. (28) and (30):

$$\nabla_j\mathbf{r}_{,i}=b_{ij}\mathbf{e}_3, \tag{177}$$

$$\partial_i\mathbf{e}_3=-b^j_i\mathbf{r}_{,j},\quad i,\ j,\ k=1,\ 2. \tag{178}$$

We here take into account the fact that in the case of a sphere the second quadratic form is proportional to the first[15,24]:

$$\begin{aligned} b_{11}&=-\frac{1}{R}g_{11}=-\frac{1}{R}\left(\sin\frac{\theta}{2}\right)^{-2},\\ b_{22}&=-\frac{1}{R}g_{22}=-\frac{1}{R}\left(\cos\frac{\theta}{2}\right)^{-2},\\ b_{12}&=-\frac{1}{R}g_{12}=0,\end{aligned} \tag{179}$$

and the Christoffel symbols for the metric (176) are given by

$$\left.\begin{aligned}&\Gamma^1_{11}=-\varphi_{,1}\operatorname{ctg}\varphi,\ \Gamma^1_{12}=\Gamma^1_{21}=-\varphi_{,2}\operatorname{ctg}\varphi,\ \Gamma^1_{22}=-\varphi_{,1}\operatorname{tg}^3\varphi,\\ &\Gamma^2_{11}=\varphi_{,2}\operatorname{ctg}^3\varphi,\quad \Gamma^2_{12}=\Gamma^2_{21}=\varphi_{,1}\operatorname{tg}\varphi,\quad \Gamma^2_{22}=\varphi_{,2}\operatorname{tg}\varphi,\\ &\varphi=2\theta.\end{aligned}\right\} \tag{180}$$

Equations (177) and (178) now take the form

$$\frac{\partial}{\partial u^j}\begin{pmatrix}\mathbf{r}_{,1}\\ \mathbf{r}_{,2}\\ \mathbf{e}_3\end{pmatrix}=\omega^j\begin{pmatrix}\mathbf{r}_{,1}\\ \mathbf{r}_{,2}\\ \mathbf{e}_3\end{pmatrix},\quad j=1,\ 2, \tag{181}$$

where the 3×3 matrices ω^j are determined by

$$\left.\begin{aligned}\omega^1&=\begin{vmatrix}-\varphi_{,1}\operatorname{ctg}\varphi & \varphi_{,2}\operatorname{ctg}^3\varphi & -R^{-1}(\sin\varphi)^{-1}\\ -\varphi_{,2}\operatorname{ctg}\varphi & \varphi_{,1}\operatorname{tg}\varphi & 0\\ R^{-1} & 0 & 0\end{vmatrix},\\ \omega^2&=\begin{vmatrix}-\varphi_{,2}\operatorname{ctg}\varphi & \varphi_{,1}\operatorname{tg}\varphi & 0\\ -\varphi_{,1}\operatorname{tg}^3\varphi & \varphi_{,2}\operatorname{tg}\varphi & -R^{-1}(\cos\varphi)^{-2}\\ 0 & R^{-1} & 0\end{vmatrix}.\end{aligned}\right\} \tag{182}$$

The condition of compatibility of the linear equations (181),

$$\omega^1_{,2}-\omega^2_{,1}=[\omega^2,\ \omega^1], \tag{183}$$

reduces to the following nonlinear equation of hyperbolic type for the function $\varphi(u^1,u^2)$:

$$(\varphi_{,1}\operatorname{tg}^2\varphi)_{,1}+(\varphi_{,2}\operatorname{ctg}^2\varphi)_{,2}=\frac{1}{R^2}(\sin 2\varphi)^{-1}. \tag{184}$$

Equation (184) is the Gauss equation (32),

$$R_{1212}=b_{12}b_{12}-b_{11}b_{22},$$

where R_{1212} is the essential component of the Riemann–Christoffel curvature tensor for the metric (176). The Peterson–Codazzi equations for the sphere are satisfied identically, so that instead of the three linear compatibility conditions for the linear system (181) there is the single condition (184).

If we wish to use the matrices ω^i (182) to construct a linear auxiliary spectral problem, which is needed for the solution of Eq. (184) by the inverse scattering method, we must introduce in ω^i a dependence on the spectral parameter λ. For this, we take into account the fact that the nonlinear equation (184) is invariant with respect to the transformations

$$u^i\to\lambda u^i,\quad i=1,\ 2,\quad R\to\lambda R,\quad \varphi\to\varphi, \tag{185}$$

where λ is a constant. The linear equations (181) are not invariant under such substitutions, and as a result the matrices ω^i come to depend on the spectral parameter λ.

Thus, with the nonlinear equation (184) we can associate the linear spectral problem

$$\frac{\partial\psi}{\partial u^i}=\Omega^i(\lambda)\psi, \quad i=1, 2, \tag{186}$$

where $\psi(u^1, u^2)$ is a function column with three components, and the matrices $\Omega^i(\lambda)$ have the form

$$\left.\begin{aligned}
\Omega^1(\lambda)&=\begin{vmatrix} -\varphi_{,1}\,\mathrm{ctg}\,\varphi & \varphi_{,2}\,\mathrm{ctg}^3\varphi & -\frac{\lambda}{R}(\sin\varphi)^{-2} \\ -\varphi_{,2}\,\mathrm{ctg}\,\varphi & \varphi_{,1}\,\mathrm{tg}\,\varphi & 0 \\ (\lambda R)^{-1} & 0 & 0 \end{vmatrix},\\
\Omega^2(\lambda)&=\begin{vmatrix} -\varphi_{,2}\,\mathrm{ctg}\,\varphi & \varphi_{,1}\,\mathrm{tg}\,\varphi & 0 \\ -\varphi_{,1}\,\mathrm{tg}^3\varphi & \varphi_{,2}\,\mathrm{tg}\,\varphi & -\frac{\lambda}{R}(\cos\varphi)^{-2} \\ 0 & (\lambda R)^{-1} & 0 \end{vmatrix}.
\end{aligned}\right\} \tag{187}$$

By means of the gauge transformation

$$\overline{\Omega}^i=g^{-1}\Omega^i g-g^{-1}\partial_i g, \quad i=1, 2,$$

with the matrix $g=\mathrm{diag}((\sin\varphi)^{-1}, (\cos\varphi)^{-1}, 1)$ one can go over to a linear spectral problem analogous to (186) with matrices Ω^i from the Lie algebra of the group $SL(3, R)$: $\overline{\Omega}^1=\varphi_{,2}\,\mathrm{ctg}^2\varphi X_1+R^{-1}(\sin\varphi)^{-1}X_2$, $\quad\overline{\Omega}^2=\varphi_{,1}\,\mathrm{tg}^2\varphi X_1+R^{-1}(\cos\varphi)^{-1}X_3$, where

$$X_1=\begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \quad X_2=\begin{vmatrix} 0 & 0 & -\lambda \\ 0 & 0 & 0 \\ \lambda^{-1} & 0 & 0 \end{vmatrix}, \quad X_3=\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & -\lambda \\ 0 & \lambda^{-1} & 0 \end{vmatrix},$$

$$[X_1, X_2]=-X_3, \quad [X_2, X_3]=-X_1, \quad [X_3, X_1]=-X_2.$$

The metric (176) on a surface of constant curvature was also considered in Ref. 80 with a view to obtaining a new nonlinear integrable equation. However, because of errors in the calculations the equation obtained in Ref. 80 does not agree with (184).

9. CONCLUSIONS

The foregoing examples show that the classical theory of minimal surfaces gives a fairly rich series of systems of nonlinear equations for which a general solution can be constructed explicitly. It is important that the method of constructing the solution is purely algebraic, and this explains its exceptional simplicity.

Besides helping us to find explicitly integrable nonlinear equations, the classical theory of surfaces is also helpful in finding new nonlinear equations that can be investigated by the inverse scattering method, as was shown in Secs. 6–8.

APPENDIX

We shall show here how it is possible to obtain the general solution of the nonlinear Liouville equation on the basis of the intrinsic geometry of the ordinary sphere in three-dimensional Euclidean space.[24] We consider the elliptic case, i.e., Eq. (175). As was already noted in Sec. 8, if the conformally flat system of curvilinear coordinates (174) is chosen on the sphere (171), the function $\varphi(u^1,u^2)$ will satisfy the Liouville equation (175). Stereographic coordinates on the sphere are the simplest example of a conformally flat system. These coordinates are defined as follows. The points of the sphere are projected by straight lines emanating from the "north" pole onto the plane passing through the equator of the sphere. A rectangular system of coordinates u^1 and u^2 with origin at the center of the sphere is introduced on the plane. The stereographic coordinates of a point on the sphere are the coordinates u^1, u^2 of its projection onto the equatorial plane. If θ and φ are spherical coordinates, then the stereographic coordinates u^1 and u^2 are related to them by

$$u^1=\mathrm{ctg}\frac{\theta}{2}\sin\varphi, \quad u^2=\mathrm{ctg}\frac{\theta}{2}\cos\varphi. \tag{A.1}$$

The line element on the sphere in these coordinates has the form

$$ds^2=d\theta^2+\sin^2\theta\, d\varphi^2=\frac{4}{[1+(u^1)^2+(u^2)^2]^2}[(du^1)^2+(du^2)^2]$$
$$=\frac{4\,dz\,d\bar z}{(1+|z|^2)^2}, \tag{A.2}$$

where $z=u^1+iu^2$, and the bar denotes the complex conjugate. Thus the coordinates u^1, u^2 on the sphere are conformally flat. It follows from comparison of (174) and (A2) that a particular solution of the Liouville equation (175) is determined by the expression

$$e^{\varphi}=\frac{4}{[1+(u^1)^2+(u^2)^2]^2}=\frac{4}{(1+|z|^2)^2}. \tag{A.3}$$

We now take into account the fact that the conformally flat form of the metric (A.2) is preserved under the following transformations of the coordinates u^1, u^2:

$$u^1+iu^2\to\Phi(u^1+iu^2) \quad \text{or} \quad z\to\Phi(z), \tag{A.4}$$

where $\Phi(z)$ is an arbitrary analytic function of the complex variable z satisfying the Cauchy–Riemann conditions

$$\frac{\partial\Phi}{\partial\bar z}=\frac{\partial\overline{\Phi}}{\partial z}=0. \tag{A.5}$$

The transformation (A.4) reduces the metric (A.2) to the form

$$ds^2=\frac{4|\Phi_z'|^2}{(1+|\Phi|^2)^2}dz\,d\bar z. \tag{A.6}$$

From this there follows directly the general solution of the Liouville equation (175):

$$e^{\varphi}=\frac{4|\Phi_z'|^2}{(1+\Phi^2)^2}. \tag{A.7}$$

In addition, we obtain a remarkable property of the solutions of Eq. (175): If $f(z)$, $z=u^1+iu^2$, determines a vector solution of this equation in accordance with

$$e^{\varphi}=f(z),$$

then the relation

$$e^{\varphi}=f(\Phi(z))\,|\Phi_z'|^2,$$

where $\Phi(z)$ is an arbitrary analytic function of the variable z, will also determine a solution of Eq. (175) (see Ref. 75).

The Liouville equation (175) and the sine–Gordon equation (173) describe the metric of the same surface [the sphere (171)] but in the different coordinate systems (172) and (174). It is therefore natural to attempt to go over from the conformally flat coordinates (174) to the Chebyshev coordinate grid (172) in order to obtain the general solution of the sine–Gordon equation (173) by means of the general solution (A7) of the Liouville equation. However, the problem of finding such a transformation of the parameters

$$u^1 = u^1(v^1, v^2), \quad u^2 = u^2(v^1, v^2) \tag{A.8}$$

is less simple than the solution of the sine–Gordon equation itself. Indeed, for the transition from (174) to (172) the functions (A.8) must satisfy the well-known *Servant–Bianchi* equations[67]:

$$\frac{\partial^2 u^1}{\partial v^1 \partial v^2} + \Gamma^1_{ij} \frac{\partial u^i}{\partial v^1} \frac{\partial u^j}{\partial v^2} = 0, \quad \frac{\partial^2 u^2}{\partial v^1 \partial v^2} + \Gamma^2_{ij} \frac{\partial u^i}{\partial v^1} \frac{\partial u^j}{\partial v^2} = 0, \tag{A.9}$$

where Γ^i_{jk} are the Christoffel symbols for the metric (174),

$$\Gamma^1_{11} = -\Gamma^1_{22} = \Gamma^2_{12} = \frac{\varphi_{,1}}{2}, \quad \Gamma^2_{22} = \Gamma^1_{12} = -\Gamma^2_{11} = \frac{\varphi_{,2}}{2}, \tag{A.10}$$

and it is necessary to substitute the general solution (A.7) of the Liouville equation in (A.9) and (A.10).

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