Interaction of heavy ions: the form of the potential and estimate of its parameters

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The existing theoretical approaches to the interaction of nuclei, and also experimental information about heavy-ion collisions at low energies, are analyzed. A new, physically grounded parametrization of the optical potential is proposed and considered as one of the possible ways of treating heavy-ion optical potentials from a systematic and unified point of view. The universal parameters that occur in the potential are estimated. The most characteristic feature of the proposed potential is the short-range repulsion due to the influence of the Pauli principle in the case of overlapping wave functions of the fermions composing the systems.

INTRODUCTION

The problem of determining the potential of the interaction between heavy ions is one of the important and as yet not completely solved problems of low-energy nuclear physics. On the one hand, knowledge of the potential is needed to understand the mechanisms of nuclear transformations—transfer reactions, fusion, deep inelastic collisions—and the analysis of these processes in the framework of the existing theoretical models. On the other hand, the potential contains information about the properties of nuclear matter, the nucleon distribution in nuclei, and the nucleon–nucleon interaction, i.e., by studying nuclear collisions, we add to our knowledge about the fundamental characteristics of nuclear matter.

A systematic microscopic approach to the calculation of the interaction potential of two nuclei consisting of A_P and A_T nucleons, respectively, requires exact solution of a many-particle ($A = A_P + A_T$ nucleons) problem with a realistic nucleon–nucleon interaction and averaging of this interaction over the many-particle wave function describing the $A_P + A_T$ collision in the entrance channel. At the present time, it is difficult to do this, since we do not know how to solve with the necessary accuracy the quantum-mechanical problem of the motion of a large number of nucleons with an arbitrary two-body interaction. Howevèr, the problem of the *NN* interaction itself still awaits a solution. Further, it is not clear in advance to what extent many-particle correlations and forces in the nucleus can be ignored.

One of the most common ways of obtaining information about the interaction of nuclei is to analyze the experimental data on the elastic scattering of heavy ions phenomenologically in the framework of the optical model. Besides the well-known difficulties associated with the ambiguity in the determination of the parameters of the potential on the basis of an incomplete set of experimental data, the result of the analysis is to a large degree predetermined by the particular choice of the radial dependence of the potential. The overwhelming majority of studies are based on the simple Woods–Saxon form of the potential, which was originally used to describe elastic neutron–nucleus scattering.

The recently discovered change in the nature of elastic scattering on the transition from the "light" ion ^{7}Li to the "heavy" ^{9}Be (in the framework of the Woods–Saxon parametrization) leads to an abrupt jump in the depth of the real part of the optical potential, and this jump has not found an adequate explanation on the basis of present-day physical ideas. In our view, this difficulty of the optical model is one of the manifestations of the inadequacy of the Woods–Saxon potential for describing heavy-ion collisions. We also take it to be the obstacle in the way of a systematic representation of heavy-ion optical potentials. Adjusting the parameters of the Woods–Saxon potential using specific experimental data, we achieve a correct description of the behavior of the potential in a narrow range of distances r, i.e., in the region most sensitive to the given experiment. For a "physically justified" parametrization, the behavior of the potential outside the region of sensitivity must be stable with respect to small changes in the masses of the colliding nuclei. In contrast, an "incorrect" parametrization may result in a phenomenological potential which becomes physically meaningless outside the region of sensitivity; in particular, it may be manifested in an abrupt change in the depth of the Woods–Saxon potential on the transition from the ion ^{7}Li to the ion ^{9}Be, i.e., on the addition of just two nucleons.

Alongside the "model-free" methods of analysis of heavy-ion elastic scattering currently under development, the search for an alternative to the Woods–Saxon potential must be based on the results of theoretical calculations of the ion–ion potential that, despite their approximate nature, can correctly reflect the qualitative dependence of the interaction on the nuclear masses, the collision energy, and the distance between the nuclei.

Because the many-particle problem of heavy ion collisions is so complicated, it is, as a rule, analyzed in the framework of two extreme approximations—adiabatic collision and sudden collision. The adiabatic models, for example, the liquid-drop model, lead to a correct connection between the properties of normal nuclear matter and the behavior of the potential at the edge of the nucleus. However, they are not applicable to the description of the interior region of the potential corresponding to overlapping of the wave functions of the colliding ions. Models based on the sudden-collision approximation with allowance for the Pauli principle predict the existence in the interaction potential of a strong repulsion at short distances, this preventing the nuclei from penetrating each other. This phenomenon is a general property of composite systems of fermions and is not related to the repulsive core in the nucleon–nucleon potential. The existence of a short-range repulsion follows from the existing

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microscopic calculations of ion–ion interaction based on approximate solution of the many-particle Schrödinger equation with a two-body nucleon–nucleon potential.

At the end of this paper we propose a general expression for the real part of the optical potential, separating what are in our view the main dependences on the mass numbers of the colliding nuclei and on the energy of their relative motion. After separation of the corresponding factors, the properties of the potential become universal and are intimately related to the characteristics of infinite nuclear matter. It is then possible to make quite specific estimates of the parameters that occur in the potential.

1. PHENOMENOLOGICAL OPTICAL POTENTIALS

Nucleons, Deuterons, and the Nuclei ^{3}He and ^{3}H

In 1954, Feshbach, Porter, and Weisskopf[1] showed that the basic features of the cross sections (averaged over the resonances) for elastic scattering of neutrons with energy up to 3 MeV by a number of nuclei can be described by means of a complex potential in the form of a rectangular well:

$$V=\begin{cases} -V_0(1+i\xi), & r\leqslant R, \\ 0, & r>R, \end{cases} \tag{1}$$

where $V_0 = 42$ MeV, $\xi = 0.03$, and $R = 1.45\,\mathrm{F}\cdot\mathrm{A}^{1/3}$. Since then, experimental material on the scattering of nucleons by nuclei of almost all elements has been accumulated. It has been found that the data on the total neutron cross sections and on the differential cross sections of elastic scattering of nucleons and their polarization can be explained in the framework of the optical model of elastic scattering[1-8] with an absorbing potential:

$$i\hbar\frac{\partial\Psi}{\partial t}=\left[\frac{\hat{p}^2}{2m}+V(r)+iW(r)\right]\Psi. \tag{2}$$

In the process of collision with a nucleus, a nucleon exchanges energy with many nuclear degrees of freedom, and this leads to the excitation of numerous complicated configurations. Under such conditions, the coupling of the incident nucleon to the target acquires an irreversible nature and can be taken into account as an effective damping of the single-particle motion described by the imaginary part $W(r)$ of the optical potential. The absorption determined by the imaginary component can be associated with the mean free path λ of the nucleon in the nucleus, which is given by $\lambda^{-1} = -2W(r)/\hbar v$ and describes the characteristic distance over which the wave function Ψ of the motion of a distinguished nucleon in the average field of the target is "transformed" into complicated many-particle states.

Although in the general case the optical potential must be nonlocal, to describe the experimental angular distributions of the elastic scattering of nucleons it is sufficient to use the simple form of the potential

$$V(r)=-\frac{V_0(1+i\xi)}{1+\exp[(r-R_V)/a_V]}, \tag{3}$$

which was proposed by Woods and Saxon[8] on the basis of the argument that the average field of the nucleus must approximately follow the distribution of the density of nucleons. The diffuseness a_V of the potential is close to the diffuseness of the charge distribution ($a_V \simeq 0.6$ F), while the radius R_V exceeds the characteristic nuclear radius by approximately the range of the nucleon–nucleon forces.

Because of the simplicity of the calculations and the relative perspicuity of the model, the phenomenological analysis of experimental elastic-scattering data became one of the most common methods for obtaining information about nucleon–nucleus interactions, an indispensable part of any experimental study of nucleon–nucleus interaction. The most complete compilation of optical potentials obtained by various authors up to 1975 inclusive is given in the review of Ref. 9, in which one can also find examples of the systemization of the parameters of the potentials for different targets and collision energies.

The success in the description of nucleon scattering stimulated the application of the optical model to the analysis of scattering by nuclei of more complicated particles—deuterons and the ^{3}He and ^{3}H nuclei. In this case too, the simple parametrization (3) makes it possible to obtain good agreement with experiment and a systematic set of parameters.[9] During the 30-year existence of the optical model, the Woods–Saxon parametrization of the potential has operated so successfully that, despite its being proposed originally for nucleon scattering with no direct relation to interaction between heavy ions, it is still widely used to analyze heavy-ion collisions.

Alpha Particles

Doubly ionized ^{4}He atoms were already used to study nuclear interactions more than 70 years ago. Nuclear physics itself owes its existence to Rutherford's famous experiment.[10] The remarkable properties of the α particle—the high binding energy, the compact magic structure, and the absence of spin—make it the only light particle that one can attempt to compare with heavy ions. Many important results and ideas gleaned from α-particle experiments were transferred to heavy-ion collisions. There has recently been published[11] a detailed analysis of α-particle–nucleus elastic scattering and therefore in the present work we shall mention only the most important results that remain significant in heavy-ion physics.

Systematic analysis of the scattering of 40-MeV α particles by Ar, Cu, and Pb nuclei showed[12,13] that the behavior of the scattering cross section in the forward hemisphere is determined by a "tail" of the optical potential whose real part can be represented at large distances ($r \gtrsim 7$ F) by [see (3)]

$$V(r)=-1100\exp\left[-\left(\frac{r-1.17A^{1/3}}{0.574}\right)\right], \tag{4}$$

and that there is a continuous nonuniqueness in the choice of the parameters of the potential, this being expressed in the fact that an equally good description of the elastic scattering of the α particles in the forward hemisphere can be obtained by means of potentials for which the values of $V_0 \times \exp(R_V/a_V)$ are approximately the same. It was also conjectured in Refs. 12 and 13 that the small-angle elastic scattering is insensitive to the behavior of the potential within the nucleus.

Study of the scattering of α particles with energy E$\simeq$25 MeV by a number of nuclei from O to U in the range of angles $\theta < 80°$ established[14] that there is not only a continuous but also a discrete nonuniqueness in the choice of the potential, i.e., it is possible to construct a discrete set of potentials with strongly different depths of the real part, giving an equally good description of the experimental data.

It was pointed out in Ref. 15 that the interior part of the potential has a stronger influence on the phase shifts of partial waves with small values of the orbital angular momentum l (because of the decrease in the centrifugal barrier, these waves penetrate deeper). The relative smallness of the contribution of these waves to the forward-scattering cross section has the consequence that any potential which correctly reproduces the phase shifts of a few surface waves, with $l \simeq kR$, dominant in the small-angle scattering is suitable for the description of the experimental data.

In the light of this, it is obvious that the choice of the potential is nonunique. Since only the dominant surface waves are important in low-energy small-angle scattering, any potential that in the region $r \gtrsim 7$ F has a behavior near that of (4) is satisfactory from the point of view of the experimental data. Having in (3) the three independent parameters V_0, R_V, and a_V, we can always fix one of them, for example, R_V, and choose the values of the remaining two in such a way that for $r \gtrsim 7$ F the real part of the potential decreases approximately in accordance with the law (4). Varying R_V within reasonable limits, we obtain in this manner a continuous family of potentials that each describe the experiment well.

However, if the number of important waves is not large, we can, by changing the depth of the real part of the potential strongly, choose it in such a way that the logarithmic derivatives of the wave functions corresponding to the dominant orbital angular momenta take the necessary values at the edge of the nucleus. This can be done in several ways by changing the number of nodes of the wave functions within the nucleus. Potentials that generate different numbers of nodes but nearly equal logarithmic derivatives at the edge of the nucleus give phase shifts for the dominant waves that are correct from the point of view of experiment, and this leads to the discrete nonuniqueness. The associated possible strong change in the phase shifts with the small orbital angular momenta is not manifested in the behavior of the cross section.

Thus, to obtain more reliable information about the potential, it is necessary to use experimental data on scattering with a large number of important waves. A greater importance of small impact parameters is equivalent to an advance of the lower limit of the region of sensitivity of the potential (to the experiment) within the nucleus. If for this limit r_s we choose some quantitative definition, for example, we assume that for $r < r_s$ a change in the potential in the neighborhood of the point r does not lead to a significant increase in the χ^2 test characterizing the agreement between calculation and experiment, then r_s will depend on the collision energy E and on the maximal detection angle θ in the given experiment. Then, if for a definite pair of colliding nuclei two experiments are made for which $E_1 \geqslant E_2$ and $\theta_1 \geqslant \theta_2$, it is to be expected that r_s will be a monotonic function, i.e., $r_s(\theta_1, E_1) \leqslant r_s(\theta_2, E_2)$.

Thus, by raising the energy and making experiments at larger angles, we shift the limit r_s, of sensitivity in the direction of shorter distances, increasing thereby the amount of information about the potential. Since, as a rule, the potential is sought in a given class of functions, experiment restricts the region of acceptable values of the free parameters. In the review of Ref. 11, a number of α-particle elastic-scattering experiments were analyzed, and it was shown that successive increase in the collision energy and the maximal detection angle leads, on the one hand, to a decrease in the number of phase-equivalent potentials associated with the discrete nonuniqueness and, on the other, to a restriction of the families of potentials associated with the continuous nonuniqueness. Naturally, there is a continuous family near each potential of the discrete set.

This result offers hope for the existence of experimental conditions under which the optical potential may be determined uniquely in a given class of functions. This conjecture took on a definite form in the work of Goldberg, Smith, and Burdzik.[16,17] Using Ford and Wheeler's [8] quasiclassical model of elastic scattering, in which the deflection angle of the classical trajectory with orbital angular momentum l is expressed in terms of the phase shift δ_l calculated in the quasiclassical approximation, $\theta(l) = 2d\delta_l/dl$, they found that for sufficiently high ($E \gtrsim 100$ MeV) collision energy a maximal deflection angle θ_R arises, its value decreasing with increasing energy.

In this case, the experimental elastic-scattering cross section, divided by the Rutherford cross section, has a monotonic, almost exponential decrease for angles $\theta > \theta_R$. At the same time, the usual diffraction oscillations are observed in the region of small angles $\theta < \theta_R$. By analogy with optics, this phenomenon is called rainbow scattering.

It is important that the position of the angle θ_R of rainbow scattering is determined by the real part of the optical potential, and this simplifies the problem of determining the parameters of the potential. In particular, use of experimental data on the rainbow scattering eliminates the discrete nonuniqueness in the choice of the potential, since a strong increase in the depth of the real part in a given class of potentials is accompanied by an increase in the angle θ_R to $\theta_R > 180°$, and this leads to disappearance of the rainbow. In this connection, a "prescription" was formulated in Refs. 16 and 17. In accordance with this prescription, to eliminate the discrete nonuniqueness it is necessary to determine the parameters of the potential by analyzing experimental data at sufficiently high energies, when rainbow scattering occurs, and to use in this case experimental cross sections measured at angles $\theta > \theta_R$. This prescription was successfully used to determine the optical potential of elastic scattering of α particles with energy from 90 to 172 MeV by nuclei from ^{12}C to ^{208}Pb inclusive.[19,20]

Heavy Ions

As the experimental investigations developed, information was accumulated about the parameters of the heavy-ion optical potentials. The most detailed compilation of data ob-

tained in this manner is given in Ref. 9. Similar tables were recently published in Ref. 21, which collects together data on the elastic scattering of the ^{6}Li and ^{7}Li ions obtained in recent years. Experimental studies of heavy-ion collisions are currently being developed strongly in three main directions, namely, the set of accelerated ions is being extended, their energy is being raised, and the measurements are being extended to the largest scattering angles. Despite all the efforts that have been made, there is as yet no systematic set of heavy-ion optical potentials for different masses and collision energies.

It was shown in Refs. 22–24 that the angular distributions of the elastic scattering of heavy ions and surface reactions are to a large degree determined by the rainbow-scattering angle. Therefore, the shape of the scattering cross sections in the forward hemisphere can be reproduced[23,26] if one knows the classical trajectories that pass near the rainbow-scattering trajectory. Since at the distances characteristic of these trajectories the imaginary part of the optical potential is, as a rule, small, it can be assumed that the shape of the angular distributions depends mainly on the real part.

On the basis of these considerations, Christensen and Winther[27] took the following parametrization for the real part of the optical potential at the edge of the nucleus:

$$V_N(r) = V_0 \exp(-r/\alpha). \quad (5)$$

Assuming that the nuclear attraction is small compared with the nuclear interaction at sufficiently large distances, they used classical perturbation theory to obtain an analytic expression for the rainbow-scattering angle θ_R as a function of the parameters α and V_0. They found that all potentials intersecting at a point just closer to the origin than the point of closest encounter for the rainbow-scattering trajectory have the same angles θ_R. Analyzing about 60 empirical Woods–Saxon potentials obtained by different authors in an analysis of the scattering of heavy ions with $A_p = 10$–84, Christensen and Winther proposed the following expression for the real part of the optical potential:

$$V_N(r) = 50 \frac{R_p R_T}{R_p + R_T} \exp\left(-\frac{r - R_p - R_T}{\alpha}\right) \quad (6)$$

(the suffix p refers to the projectile, the suffix T to the target nucleus), this giving the correct rainbow-scattering angles for all the considered cases with good accuracy. The geometrical parameters in (6) are determined as follows: $R_{p,T} = 1.233A^{1/3}_{p,T} - 0.978A^{-1/3}_{p,T}$ F and $\alpha = 0.63$ F. Thus, in our view the expression (6) is an important step toward the construction of a systematic set of heavy-ion potentials.

However, in the search for such a systematization of the potentials it is important to remember that the concept of the sensitivity radius, i.e., the point at which all the potentials suitable for the given set of experimental data intersect, has meaning only for a given functional form of its real part, as was very clearly demonstrated in Ref. 28 for the example of ^{16}O + ^{208}Pb elastic scattering at 192 MeV. Therefore, in speaking of the systematics of the heavy-ion optical potentials we actually always mean a certain class of functions by means of which the main qualitative characteristics of the interaction between nuclei are represented. The suitable class of functions must be chosen on the basis of the results of calculations of the ion–ion interaction from first principles. Measurements of the elastic scattering of 156–MeV ^{6}Li ions by the ^{12}C, ^{28}Si, ^{40}Ca, ^{90}Zr, and ^{208}Pb nuclei[29-32] led to the observation of a characteristic picture of rainbow scattering similar to the analogous phenomenon for α particles. It was found that by using potentials of Woods–Saxon type it is possible to obtain good agreement between calculation and experiment only if the depth of the real part is ~ 150 MeV.[29-32] The ratio of the depths of the imaginary and real parts of the potential at the edge of the nucleus was found to be small: $W/V \simeq 0.2$. The parameters of the potential for Li depend strongly on the collision energy. The elastic scattering of ^{7}Li ions has similar properties.[33]

Systematic measurements of the elastic scattering of the heavier ^{9}Be, ^{12}C, and ^{16}O ions by the ^{28}Si nucleus at collision energies $E \gtrsim 200$ MeV showed[34-36] that in these cases rainbow scattering is not yet observed. The corresponding experimental data can be described by means of shallow optical potentials with a real part of depth $\lesssim 40$ MeV and ratio $W/V \simeq 1$. At the same time, the parameters of the potentials hardly depend on the collision energy. We note that this last result holds only for forward scattering.

Thus, the available experimental data indicate that on the transition from the "light" ^{7}Li ion to the "heavy" ^{9}Be ion there is an abrupt change in the nature of the elastic scattering.[32,33] If the standard Woods–Saxon potentials are used, there is a resulting unphysical "jump" in the depth of the real part (this, moreover, being in the direction of a weaker interaction) when just two nucleons are added to the projectile. In other words, the assumption of a Woods–Saxon form of the real part of the optical potential leads in the given case to a strange result, and this confirms the need for a more careful estimate of the form of the real part of the heavy-ion optical potentials.

2. ADIABATIC AND SUDDEN COLLISIONS

As the colliding ions approach each other, two main processes develop simultaneously and with comparable rates. There is the mutual penetration of the ions into each other and the breakup of their internal structure. The first of these leads to the production of elastically scattered ions, and the second to their excitation or breakup. In the framework of the optical model, all inelastic channels accompanied by changes in the structure of the colliding ions are effectively taken into account by the introduction of the imaginary part into the ion–ion interaction potential.

In view of the tremendous complexity of the many-particle heavy-ion collision problem, two extreme approximations are, as a rule, employed—those of an adiabatic and a sudden collision. The first approximation presupposes a development of the events in which the approach toward each other of the centers of mass of the colliding ions is accompanied by a smooth adiabatic change in the internal structure of the ions and "equalization" of the nucleon densities is energetically advantageous (at the given instantaneous distance). If an adiabatic collision is to occur, it is necessary (but not sufficient) for the relaxation rate of the internal degrees

of freedom of the nucleus to be large compared with the translational velocity of the projectile. A sudden collision occurs so rapidly that the internal structure of the ions cannot change significantly during the interaction time. In this case, the nucleon densities of the nuclei in the region of their overlap are simply added, and this, as we shall see below, unavoidably leads to the occurrence of a strong short-range repulsive core in the potential.

In order to estimate whether a collision is adiabatic or sudden, we can consider the ratio k/k_F, where $k = [(2m/\hbar^2)(E/A)]^{1/2}$ is the mean translational momentum of a nucleon in the projectile, and k_F is the limiting Fermi momentum for the internal motion of the nucleons in the nuclei. If $k >> k_F$, a sudden collision can be expected; in the opposite case $k << k_F$, an adiabatic collision.

However, real heavy-ion collisions evidently have properties intermediate between these two cases. Further, the proximity to the one or other extreme approximations depends both on the collision energy and on the properties of the nuclei themselves, in particular on their binding energies and structures.

The nature of heavy-ion collisions can be determined precisely, on the one hand, by means of a microscopic calculation of the collision dynamics on the basis of an effective nucleon–nucleon potential. Note that we here mean a truly microscopic calculation based on approximate solution of the equations of motion of all the nucleons. Another natural approach is to assume the existence of a repulsive core and to look for manifestations of it in experimental data on heavy-ion collisions.

3. LIQUID-DROP MODEL

The methods based on a representation of the nucleus as a liquid drop of incompressible nuclear matter must be regarded as the simplest example of the adiabatic approximation in the problem of calculating the interaction potential between heavy ions. The assumption of incompressibility means that in a collision of heavy ions one can admit all volume preserving deformations of their shape, the density distribution being uniform. It is then obvious that the volume part of the interaction energy does not change during the collision, and the only possible changes in the potential energy are due to the change in the surface area of the system of the two coalescing nuclear drops, and also the change in the Coulomb interaction.

The difference between the surface energy of a spherical drop of $A_1 + A_2$ nucleons and two drops of A_1 and A_2 nucleons separated by a large distance is

$$V_0 = b_{\text{surf}}[(A_1+A_2)^{2/3} - A_1^{2/3} - A_2^{2/3}], \tag{7}$$

where $b_{\text{surf}} \simeq 17$ MeV is a known constant.[37] The existence of surface energy is a general property of finite systems associated with the distinguished nature of particles at the surface. Usually, the quantity (7) is associated with the value of the ion–ion potential at the point $r = 0$.

If it is assumed that the ion–ion potential has the Woods–Saxon form (3), then the parameter V_0 is determined by the relation (7). The determination of the radius R_V and the diffuseness a_V requires additional assumptions.

The Wilczynskis proposed[38] that the maximal force of attraction between nuclei acts at the distance of the sum of their half-density radii, $R_V = R_0 = R_1 + R_2$, and that its magnitude can be estimated by means of the expression[39]

$$(dV/dr)_{r=R_0} = 4\pi\gamma R_1 R_2/(R_1 + R_2), \tag{8}$$

where $\gamma = 0.95$ MeV/F^2 is the surface-tension coefficient of nuclear matter. They calculated the radii $R_{1,2}$ in accordance with Myers's formula[40]

$$R_i = 1.128A_i^{1/3}(1 - 0.786A_i^{-2/3}). \tag{9}$$

Using (4) and (8), we can readily determine the diffuseness parameter:

$$a = \frac{V_0 R_0}{16\pi\gamma R_1 R_2} = 0.356\,\frac{R_0}{R_1 R_2}[(A_1+A_2)^{2/3} - A_1^{2/3} - A_2^{2/3}]. \tag{10}$$

Thus, the expressions (7)–(10) give an estimate for the interaction potential between heavy ions that is obtained from general considerations without any adjustable parameters.

The review of Ref. 41 gives an example, due to Bondorf, of an estimate obtained in a similar spirit: $R_i = 1.3\,\text{F}\cdot A_i^{1/3}$, $a = 0.9$ F, obtained by simple interpolation between the energy of two infinitely separated spherical drops and the energy of the compound nucleus, the Woods–Saxon potential (3) being used.

The ion–ion potentials constructed on the basis of the liquid-drop model give a good description of the form of the interaction at the edge of the nucleus. In particular, they give a very good description of the interaction barriers between heavy ions obtained from analysis of the experimental excitation functions for fusion reactions.[38]

Recently, Krappe, Nix, and Sierk[42,43] proposed a generalized liquid-drop model, in which the surface energy of a nucleus consisting of A nucleons has the form

$$E_s = -\frac{C_s}{8\pi^2 r_0^2 a^3}\int\int d\mathbf{r}\,d\mathbf{r}'\left(\frac{\sigma}{a} - 2\right)\sigma^{-1}\exp(-\sigma/a), \tag{11}$$

where $\sigma = |\mathbf{r} - \mathbf{r}'|$. In this expression, the double integration is extended over the volume of the nucleus, which is assumed to be unchanged by deformations and equal to $(4/3)\pi r_0^3 A$. The dependence of the energy on the nucleon composition of the nucleus is contained in the factor $C_s = a_s(1 - k_s I^2)$, where $I = (N - Z)/A$ is the neutron excess. Such a determination of the surface energy is somewhat different from the usual one and is constructed with allowance for the requirement of minimality of the surface energy of the two half-spaces of nuclear matter at the instant when they touch. Taking the free parameters to have the values $r_0 = 1.18$ F, $a = 0.65$ F, $a_s = 21.7$ MeV, and $k_s = 3.0$, Krappe, Nix, and Sierk[42,43] were able to obtain within the framework of the generalized liquid-drop model a good description of a large collection of experimental data relating to the nuclear masses and deformations, the fusion and fission barriers, and the differential cross sections of heavy-ion small-angle elastic scattering.

To conclude this section, we note that models based on the representation of the nucleus as an incompressible liquid drop evidently lead to the correct connection between the

properties of normal nuclear matter and the behavior of the interaction potential at the edge of the nucleus. In this sense, they can serve as guides in judging whether ion– ion potentials obtained phenomenologically or in other models are reasonable. At short distances, the potentials constructed by means of the liquid-drop model give weak attraction and do not have a repulsive core.

However, it was recently shown[44] that if one takes into account (in the framework of a liquid-drop model with a diffuse edge) the possibility of addition of the nucleon densities of the two nuclei at the instant when they touch, then a nonlinear Schrödinger equation is obtained:

$$\frac{\hbar^2}{2\mu}\Delta\Psi-(V-E)\,\Psi+K/9\,(\rho/\rho_0-1)\,\Psi=0, \qquad (12)$$

where $\rho_0=\Sigma_i\,\Psi_i^{(0)}\Psi_i^{(0)*}$ is the normal density corresponding to the solutions of Eq. (12) for $\rho=\rho_0$, $\rho=\Sigma_i\,\Psi_i\Psi_i^*$ is the density calculated by means of the solutions of (12) for $\rho\neq\rho_0$, and K is the compressibility of nuclear matter. The force that arises through the compression of the nuclear matter in the framework of such a generalization of the liquid-drop model is an analog of a repulsive core.

4. FOLDED POTENTIALS

The nonuniqueness in the determination of the parameters of potentials using experimental data makes it necessary to have physical estimates of when the results obtained are reasonable. As a very simple estimate of the dependence of the potential on the projectile mass, some authors[45,46] have used a procedure of unary folding:

$$V(r)=\int d\mathbf{r}_p\,\rho_p(\mathbf{r}_p)\,V_{NT}(|\mathbf{r}-\mathbf{r}_p|), \qquad (13)$$

in accordance with which the potential $V_{NT}(r)$ of the interaction of a nucleon with the target nucleus, integrated over the volume of the projectile together with the nucleon-density distribution $\rho_p(r)$ in it, is identified with the interaction potential of the two heavy ions.

If we know sufficiently well (from experiments on the scattering of electrons or Hartree–Fock calculations) the nucleon-density distributions $\rho_p(r)$ for many nuclei, then the optical potential $V_{NT}(r)$ itself is the product of phenomenological analysis of experimental data on the scattering of nucleons by the corresponding target, as a rule, moreover, at very low energies, where the nonuniqueness of the optical model is most pronounced. In addition, the rough nature of the approximation (13) has the consequence that the unary folded potentials do not agree with the behavior of the phenomenological optical potentials near the strong-absorption radius.[47]

From the physical point of view, it is more consistent to estimate the behavior of the potential at the edge of the nucleus by means of the double folding procedure:

$$V(r)=\int\int d\mathbf{r}_p\,d\mathbf{r}_T\rho_p(r_p)v_{NN}(|\mathbf{r}+\mathbf{r}_p-\mathbf{r}_T|)\,\rho_T(r_T), \qquad (14)$$

where $\rho_p(r)$ and $\rho_T(r)$ are the nucleon-density distributions in the colliding nuclei, and v_{NN} is the effective nucleon–nucleon interaction.

If the attempt is made to use for $v_{NN}(r)$ the realistic nucleon–nucleon interactions successfully employed in calculations of the structure of light nuclei and nucleon scattering, the procedure (14) leads to ion–ion potentials that near the strong-absorption radius overestimate the depth of the real part of the corresponding phenomenological potentials by 20–30%. To a large extent, this discrepancy is due to the large range of the one-pion exchange potential in realistic *NN* interactions.

Satchler and Love[49] proposed to use for $v_{NN}(r)$ the effective interaction constructed by Bertsch *et al.*[50] using the elements of the G reaction matrix [see (40)] generated by Reid's realistic *NN* interaction.[51]

The double-folded potential (14), obtained in this manner and denoted in the literature by M3Y, has proved effective in the description of the experimental angular distributions of heavy-ion elastic scattering,[52-58] including rainbow scattering. However, the calculations of Refs. 59–61 showed that in the case of scattering of the weakly bound nuclei ^{6}Li, ^{7}Li, and ^{9}Be satisfactory description of the experiments is possible only if the potential (14) is multiplied systematically (irrespective of the target and the collision energy) by a factor $N\simeq 0.5$–0.7.

To conclude this section, we note that the folded potentials considered here are constructed in the sudden approximation and do not take into account at least two important factors—the density dependence of the *NN* interaction and the Pauli principle, which are important at distances corresponding to overlapping of the nuclear volumes. Therefore, they are meaningful only at the edge of the nucleus. Within the nucleus the potentials are characterized by strong attraction, the real part having a depth of the order of hundreds of mega-electron-volts.

5. BRUECKNER APPROACH

In the theory of Brueckner *et al.*,[62,63] the total energy of a system of interacting fermions is represented in the form of a functional of the local single-particle energy density:

$$\langle\Psi|\hat{H}|\Psi\rangle=\int d\mathbf{r}\,\varepsilon(\rho). \qquad (15)$$

This formalism proved to be effective in the description of the binding energies of nuclei. In particular, it is attractive because it uses for different nuclei a universal expression for the energy density $\varepsilon(\rho)$ as a function of the nucleon density ρ. Moreover, the form of the function $\varepsilon(\rho)$ for finite nuclei is intimately related to the basic properties of infinite nuclear matter.

The success in the description of bound states stimulated the application of this approach to the calculation of the interaction between two nuclei. In the sudden approximation, the interaction energy of the nuclei has the form[64]

$$V(R)=\int d\mathbf{r}\,\{\varepsilon(\rho_1+\rho_2)-\varepsilon(\rho_1)-\varepsilon(\rho_2)\}, \qquad (16)$$

i.e., as the distance R between the centers of mass of the nuclei changes and their nucleon densities ρ_1 and ρ_2 are therefore added, there is a change in the energy which it is attractive to identify with the interaction potential of the heavy ions.

The Brueckner approach to the calculation of the potential was developed by various authors, who used for this purpose different ways to construct the expression $\varepsilon(\rho)$ and choose the nucleon densities $\rho_{1,2}$.

In Refs. 65 and 66, the energy density was taken in the form

$$\varepsilon(\rho) = \tau_{TF} + \rho V(\rho,\alpha) + \frac{\hbar^2}{2m}\eta(\nabla\rho)^2 + \frac{1}{2}eV_c\rho_c - 0{,}7386e^2\rho_c^{4/3}, \quad (17)$$

where τ_{TF} is the kinetic-energy density in the Thomas–Fermi approximation:

$$\tau_{TF} = \frac{3}{5}\frac{\hbar^2}{2m}\left(\frac{3\pi^2}{2}\right)^{2/3}\rho^{5/3}\frac{1}{2}[(1-\alpha)^{5/3} + (1+\alpha)^{5/3}]. \quad (18)$$

Here and below, $\alpha = (\rho_n - \rho_p)/(\rho_n + \rho_p)$, i.e., it is the neutron excess in the system. The term $V(\rho,\alpha)$ is the potential energy of nuclear matter per nucleon:

$$V(\rho,\alpha) = b_1(1 + a_1\alpha^2)\rho + b_2(1 + a_2\alpha^2)\rho^{4/3} + b_3(1 + a_3\alpha^2)\rho^{5/3}, \quad (19)$$

and the gradient term in (17) is responsible for the finiteness of the range of the nuclear forces. The two last terms correspond to the direct and exchange Coulomb interaction. The parameters $b_1 = -818.25$ MeV · F^3, $b_2 = 1371.06$ MeV · F^4, $b_3 = -556.55$ MeV · F^5, $a_1 = -0.316$, $a_2 = 0.2$, $a_3 = -1.646$, and $\eta = 15.2$ were taken from Ref. 67, where they were obtained from the condition of best description of the binding energies of the ^{16}O, ^{40}Ca, ^{208}Pb, and ^{238}U nuclei in the framework of Brueckner's theory. The nucleon densities $\rho_{1,2}$ were taken from the Hartree–Fock calculations made in Ref. 68. They agree with the data on electron scattering by the corresponding nuclei.

By means of the above potential, Ngô *et al.* calculated the interaction barriers for many pairs of colliding nuclei and obtained good agreement with the experimental data. They also proposed a convenient parametrization for the ion–ion potentials obtained in the framework of such an approach:

$$V_N(r) = \frac{A_p^{1/3}A_T^{1/3}}{A_p^{1/3} + A_T^{1/3}}U_N(s), \quad (20)$$

where $V_N(r)$ corresponds to the potential (16) without allowance for the Coulomb interaction, $A_{p,T}$ are the mass numbers of the nuclei, and $U_N(s)$ is a universal function, dependent only on the distance s between the spherical surfaces of the nuclei:

$$U_N(s) = \begin{cases} -V_0\exp(-0.27s^2), & s \geqslant 0, \\ -V_0 + 6.3s^2, & s \leqslant 0, \end{cases} \quad (21)$$

where $s = r - r_0(A_p^{1/3} + A_T^{1/3})$. It was assumed that for all the studied nuclei (medium and heavy) one can take $V_0 \simeq 30$ MeV and $r_0 \simeq 1$ F to good accuracy.

The method of calculating the interaction potential considered above is fairly complicated for practical use, since it employs Hartree–Fock nucleon densities, the determination of which is in itself a laborious task. It was shown recently[69] that similar results can be obtained if the simple analytic expressions proposed by Myers[40] are taken for the nucleon densities, namely,

$$\rho_{n,p}(r) = \rho_{n,p}(0)\{1 + \exp[(r - C_{n,p})/0.55]\}^{-1}, \quad (22)$$

where $C_{n,p} = R_{n,p}(1 - 1/R_{n,p}^2)$ and $R_{n,p} = r_{n,p}^{(0)}A^{1/3}$. The parameters $r_{n,p}^{(0)}$ have the values

$$r_p^{(0)} = 1.128 \text{ F}, \quad r_n^{(0)} = 1.1375 + 1.875\cdot10^{-4}A \text{ F} \quad (23)$$

for the proton and neutron distributions, respectively. The normalization constants are

$$\rho_n(0) = \frac{3}{4\pi}\frac{N}{A}r_n^{(0)-3}, \quad \rho_p(0) = \frac{3}{4\pi}\frac{Z}{A}r_p^{(0)-3}. \quad (24)$$

In Ref. 69, the same expressions (17)–(19) were taken for the energy density $\varepsilon(\rho)$, but the values of the numerical parameters were determined differently. For $\alpha = 0$, the energy density

$$\tilde{\varepsilon}(\rho) = \tau_{TF} + V(\rho, \alpha = 0) \quad (25)$$

in fact reproduces the equation of state of nuclear matter. Therefore, requiring fulfillment of the restrictions

$$\tilde{\varepsilon}(\rho_0) = -15.6 \text{ MeV}, \quad \rho_0 = 0.17 \text{ F}^{-3}, \quad K = 250 \text{ MeV} \quad (26)$$

[where the well-known values of the equilibrium energy density $\varepsilon(\rho_0)$ and the nuclear-matter density ρ_0, and also a special assumption about the compressibility K have been used], Ngô *et al.*[69] obtained the values of three parameters: $b_1 = -588.75$ MeV · F^3, $b_2 = 563.56$ MeV · F^4, and $b_3 = 160.92$ MeV · F^5. The other group of parameters, $a_1 = -0.424$, $a_2 = -0.0973$, and $a_3 = -2.25$ was obtained from the condition of agreement with the calculation of the binding energy of a neutron gas,[70] i.e., for $\alpha = 1$. The last parameter $\eta = 7.23$, which is important for finite nuclei, was obtained by fitting the binding energy of the ^{40}Ca nucleus in the framework of Brueckner's theory to the densities $\rho_{n,p}(r)$ determined above.

6. SKYRME INTERACTION

A conceptually similar approach to the calculation of heavy-ion interaction was developed by Stancu and Brink,[71] who determined the energy density $\varepsilon(\rho)$ by means of the effective density-dependent nucleon–nucleon Skyrme interaction S III (Ref. 72):

$$\varepsilon(\rho) = \frac{\hbar^2}{2m}\tau + \frac{1}{2}t_0\left[\left(1 + \frac{1}{2}x_0\right)\rho^2 - \left(x_0 + \frac{1}{2}\right)(\rho_n^2 + \rho_p^2)\right] + \frac{1}{4}(t_1 + t_2)\rho\tau + \frac{1}{8}(t_2 - t_1)(\rho_n\tau_n + \rho_p\tau_p) + \frac{1}{16}(t_2 - 3t_1)\rho\nabla^2\rho + \frac{1}{32}(3t_1 + t_2)(\rho_n\nabla^2\rho_n + \rho_p\nabla^2\rho_p) + \frac{1}{4}t_3\rho_n\rho_p\rho, \quad (27)$$

where the numerical values of the parameters $t_0 = -1128.75$ MeV · F^3, $t_1 = 395.0$ MeV · F^5, $t_2 = -95.0$ MeV · F^5, $t_3 = 14\,000.0$ MeV · F^6, and $x_0 = 0.45$ were chosen in such a way that the binding energies of magic nuclei obtained by means of $\varepsilon(\rho)$ were close to the experimental values. In such an approach, good agreement is simultaneously obtained for the charge distributions in the nuclei and the spectra of the single-particle energies.

The kinetic-energy density $(\hbar^2/2m)\tau$ in (27) was taken in the Thomas–Fermi approximation $\tau = (3/5)(3\pi^2)^{2/3}\rho^{5/3}$; the quantities $\tau_{n,p}$ were expressed similarly in terms of ρ_n =

$(N/A)\rho$ and $\rho_p = (Z/A)\rho$, respectively. Further, it was assumed that the nucleon density $\rho(r)$ for each nucleus has a simple Fermi distribution with respect to r, and the parameters of the distributions were fitted in such a way as to reproduce the densities obtained as a result of calculations by the Skyrme–Hartree–Fock method.[72]

Using this approach, Stancu and Brink[71] calculated the interaction potentials for all possible pairs that can be formed from the magic nuclei ^{60}O, ^{40}Ca, ^{48}Ca, ^{56}Ni, ^{90}Zr, and ^{208}Pb. As in the case of the model of Ngô et al. based on the use of the Brueckner functional of the energy density, the Stancu–Brink potentials at distances less than the sums of the rms radii of the nuclei have a strong short-range repulsion. At large distances, the ordinary nuclear attraction is obtained; this can be approximately described by the Woods–Saxon potential (3) with parameters

$$\left.\begin{aligned} V_0 &= D\,(11.92-0.41D),\\ R_V &= 1.153\,(A_p^{1/3}+A_T^{1/3}),\\ a_V &= 0.015\,(A_p^{1/3}+A_T^{1/3})+0.5, \end{aligned}\right\} \qquad (28)$$

where $D = A_p^{2/3} + A_T^{2/3} - (A_p + A_T)^{2/3}$.

To conclude this section, we note that, irrespective of the particular method used to construct the Brueckner energy-density functional or the Skyrme energy functional all calculations of ion–ion potentials based on the sudden approximation lead to qualitatively similar results (for additional examples, see Refs. 73–75).

The most characteristic feature of the potentials obtained in this manner is the strong short-range repulsion at distances at which there is appreciable overlapping of the nuclear densities. On the one hand, the existence of this repulsive core is due to the allowance for the antisymmetrization dictated by the Pauli principle directly in the stage of calculations of the necessary Brueckner or Skyrme functionals, when, by fitting the free parameters using the properties of finite nuclei, we attempt to satisfy the condition of saturation of the nuclear forces. On the other hand, the hypothesis that the nuclear densities are summed additively, i.e., the collision is sudden, is important for the occurrence of the repulsive core.

Another important result of the studies considered in this and the previous section is, in our view, the possibility that is established in them of qualitative estimation of the behavior of the ion-ion potential at large distances as a function of the masses of the colliding nuclei; also important is the similar behavior in the exterior region exhibited by the potentials obtained by different authors.

7. PROXIMITY FORCE THEOREM

The so-called macroscopic approach to the problem of the interaction between heavy ions aims at establishing the general features of the way in which the parameters of the potential vary with the nuclear masses and the collision energy. The estimates obtained in this way can be used as a basis for more detailed investigations taking into account the individual features of the specific nuclei. In the category of "macroscopic" we may include the approaches considered above based on the liquid-drop model and on the use of an energy-density functional. The development of the macroscopic approach presupposes the validity of two main assumptions. First, in the considered system there must be many nucleons, $A \gg 1$, this making it possible to employ statistical methods and to ignore the discreteness of the nuclear structure. Second, the nuclei must have a thin surface layer, $b \ll R$, so that the nucleon density over the complete volume of the nucleus is almost constant. These two assumptions together ultimately offer hope of establishing a connection between the interaction of specific nuclei and the properties of nuclear matter.

The macroscopic philosophy was formulated in its most definitive form by Randrup *et al.*[76,77] on the basis of a "proximity force theorem," applied to the interaction of two undeformable bodies whose surfaces have small curvature and diffuseness. In accordance with the theorem, the interaction force between such bodies as a function of the shortest distance s between their surfaces can be represented as a product of two factors. The first factor is the interaction potential per unit of surface between plane and parallel layers of the same material as the considered bodies. The second factor has a purely geometrical origin and is associated with the curvature of the surfaces of the interacting bodies.

Indeed, under the two assumptions made above, the interaction potential of two spherical bodies can be expressed as

$$V(s) \simeq \int d\sigma e(D), \qquad (29)$$

where $e(D)$ is the surface density of the interaction energy of two plane layers separated by distance D, and the surface integral is along the gap separating the considered bodies.

It is obvious that $e(D)$ is a function that depends only on the properties of the material of which the bodies are made. In the case of large $D > b$, i.e.), at distances that exceed the diffuseness of the edge, $e(D) = 0$, while for $D = 0$ we have $e(0) = -2\gamma$, where γ is the surface-tension coefficient of the given material. Further, $D = 0$ corresponds to touching of the bodies, for which the sum of their boundary densities is equal to the matter density in the central regions of each of them.

Using (29) and the Taylor expansion of D near $s = r - R_1 - R_2$ we can reduce the surface integral to the one dimensional integral

$$V(s) = 2\pi \overline{R}_{12} \int_{D=s}^{\infty} dD e(D), \qquad (30)$$

where $R_{1,2}$ are the radii of the spherical bodies, r is the distance between their centers of mass, and $\overline{R}_{12} = R_1 R_2/(R_1 + R_2)$ is the reduced radius of curvature of the surfaces.

Equation (30) is essentially a compact expression of the theorem mentioned above. Differentiation of this expression with respect to s gives the interaction force:

$$F(s) = -dV/ds = -2\pi \overline{R}_{12}\, e(s). \qquad (31)$$

Since $e(0) = -2\gamma$, the maximal force of attraction between the bodies in contact is

$$F(0) = -4\pi \overline{R}_{12}\gamma. \qquad (32)$$

The last relation has a general nature and applies to bodies consisting of different substances. Examples of its experimental verification for particles of rubber, mica, and gelatine are given in Refs. 78 and 79.

In the case of the nuclear interaction, $b \simeq 1$ F and $\gamma \simeq 1$ MeV/F^2. It is therefore convenient to go over to measurement of lengths in units of b, and energies in units of 2γ, i.e., the function $e(D)$ can be represented in the dimensionless form

$$e(\xi b) = 2\gamma\varphi(\xi), \tag{33}$$

where $\xi = s/b$ is the distance between the surfaces in units of b. Then the interaction potential between the spherical nuclei takes the form

$$V(r) = 4\pi\gamma b\bar{R}_{12}\Phi(\xi), \tag{34}$$

where $\Phi(\xi) = \int_\xi^\infty d\xi'\varphi(\xi')$.

Despite the fact that Eq. (34) is general in nature, the specific form of the universal function $\varphi(\xi)$ depends on the model chosen to describe the properties of nuclear matter. However, it is possible to understand the qualitative behavior of $\varphi(\xi)$ on the basis of simple considerations. At large distances $\xi \gg 1$, it must fall rapidly to zero because of the finite range of the nuclear forces. As the two layers of nuclear matter approach each other, an attraction develops between them, and the function $\varphi(\xi)$ increases in absolute magnitude, remaining negative. When the two layers touch, $\xi = 0$, it reaches its minimum $\varphi(0) = -1$. Further penetration of the layers into each other ($\xi < 0$) leads to strong repulsion and an increase of $\varphi(\xi)$ in the direction of positive values.

In Ref. 76, the Thomas–Fermi approximation was used to calculate $\varphi(\xi)$. It was assumed that the densities of the layers of nuclear matter are summed additively (sudden collision). The interaction potential was calculated by means of the nucleon–nucleon potential

$$V_{NN}(r) = -C\frac{\exp(-r/b)}{r/b}(1 - p^2/p_0^2), \tag{35}$$

which depends on the relative momentum p of the nucleons; this imitates the influence of the short-range nucleon–nucleon core. The parameters, $b = 0.62567$ F, $p_0 = 392.48$ MeV/c, and $C = 300$ MeV, were taken from Ref. 82, where they were obtained from the condition of best description of the properties of nuclear matter and a number of nuclei with the potential (35) in the framework of the Thomas–Fermi method.

The function $\Phi(\xi)$ calculated in this manner is tabulated in Ref. 76. For practical purposes, one can use the analytic expression for it given in Ref. 77:

$$\Phi(\xi) = \begin{cases} -\frac{1}{2}(\xi - 2.54)^2 - 0.0852(\xi - 2.54)^3, & \xi \leqslant 1.2511, \\ -3.437\exp(-\xi/0.75), & \xi \geqslant 1.2511. \end{cases} \tag{36}$$

Using the values $\gamma = 0.9517$ MeV/F^2 and $R_i = \tilde{R}_i(1 - \tilde{R}_i^{-2})$, where $\tilde{R}_i = (1.13 + 0.0002A_i)A_i^{1/3}$ F, Randrup *et al.*,[76,77] using Eqs. (34) and (36), constructed an interaction potential for heavy ions in qualitative agreement with the results of the Brueckner calculations considered above (see Eq. (20). We note also that the "contact" parametrization (34) works excellently for the phenomenological potentials (6) and the potentials obtained using the Skyrme interaction (27). In our view, all this confirms the reasonableness of the assumptions made in the derivation of the expression (34) for the case of nuclear matter, i.e., that the proximity force theorem holds for the ion–ion potential. A generalization of the theorem for the case when the curvature of the surfaces is not assumed to be small is considered in Ref. 83.

8. ENERGY DEPENDENCE OF THE POTENTIAL

In all the above, in discussing the connection between the ion–ion potential and the general properties of infinite nuclear matter or the effective nucleon–nucleon interaction, we took no account of the velocity of the relative motion of the colliding nuclei. However, from the most general physical considerations it is to be expected that the parameters of the optical potential and even its qualitative behavior may change significantly with increasing projectile energy.

In particular, at low energies, when the mean momentum of the translational motion of a nucleon in the projectile nucleus, $k = [(2m/\hbar^2)(E/A)]^{1/2}$; is small compared with the limiting Fermi momentum k_F for the internal motion of the nucleons in nuclei, there is a strong overlapping of the Fermi spheres of the colliding nuclei in the momentum space. In this case, the fraction of the nucleons that must be subject to the influence of the Pauli principle at the time of the collision is large. In the opposite case $k > 2k_F$, the Fermi spheres do not overlap, and the Pauli principle does not hinder the nucleons of the two nuclei from being localized in a volume of the order of that of the compound nuclear system. In other words, from simple estimates we may conclude that the contribution of the Pauli principle to the compressibility of nuclear matter decreases with increasing collision energy.

Beck, Müller, and Köhler,[84] using the Brueckner–Goldstone reaction-matrix formalism,[85,86] put such estimates in a more precise quantitative form. In the calculation of the interaction potential between heavy ions in Ref. 84 it was assumed that during the collision the nucleon–density distributions in the two nuclei do not change, i. e., that the sudden approximation holds. But it was assumed in accordance with the Pauli principle that during the collision nucleons entering the region of overlap of the Fermi spheres of the two nuclei must be rearranged in the momentum space in such a way that none of the states is occupied twice. It was assumed that the rearrangement occurs in the simplest possible manner, i.e., by an increase in the radii of the Fermi spheres of each of the nuclei.

A further simplifying assumption was the adoption of the approximation of a local density, i.e., the interaction potential between two nuclei separated by distance D was determined by the expression

$$V(D) = \int \varepsilon(k_{F_1}, k_{F_2}, k)\,\rho(\mathbf{r}, D)\,d\mathbf{r}, \tag{37}$$

where the limiting Fermi momenta k_{F_1} and k_{F_2} correspond to the local densities of the corresponding nuclei, and ρ, which depends on the distance D, is the total density of nucleons; $\varepsilon(k_{F_1}, k_{F_2}, k)$ denotes the interaction energy per nu-

cleon of two layers of nuclear matter whose relative velocity is characterized by the mean momentum k. It was determined as the difference between the energies of the compound system and the two systems separated by a large distance:

$$\varepsilon = \Delta\tau + \Delta\pi = \tau_{1+2} + \pi_{1+2} - (\tau_1 + \tau_2) - (\pi_1 + \pi_2). \tag{38}$$

The kinetic-energy difference $\Delta\tau$ is due to the rearrangement of the nucleons in the momentum space in accordance with the Pauli principle. The potential energies π were calculated in the framework of the reaction-matrix (G) formalism:

$$\pi = \frac{2}{(2\pi)^3} \int\limits_{\mathscr{F}} d\mathbf{k}_1\, d\mathbf{k}_2\, \langle \mathbf{k}_1\mathbf{k}_2 | G | \mathbf{k}_1\mathbf{k}_2 \rangle \Big/ \int\limits_{\mathscr{F}} d\mathbf{k}_1\, d\mathbf{k}_2, \tag{39}$$

where the integration in the momentum space is over the region $\mathscr{F}$ of all occupied states. The reaction matrix G is related as follows to the bare nucleon–nucleon interaction v:

$$G = v + v\frac{Q}{e + i\eta} G, \tag{40}$$

where Q is the Pauli operator, and e is the self-consistent energy denominator. The bare interaction v was taken to be the soft core nucleon–nucleon potential of Reid.[51]

This scheme was used to calculate the interaction potential of two ^{16}O nuclei and also of the nuclei ^{40}Ar and ^{121}Sb.[84] The calculations gave the following qualitative picture of the change in the real part of the optical potential with the energy. At low energies ($k \to 0$), the Pauli principle gives rise to a repulsive core at distances of the order of the sum of the radii of the colliding nuclei. At large distances (at the edge of the nuclei), a surface potential well is formed with a depth of the order of several tens of mega-electron-volts. With increasing energy (for example, at $k \simeq 1\ \mathrm{F}^{-1}$) the repulsive core at short distances persists. However, the strength of the repulsion, which is determined by the slope of the curve of the ion–ion potential, decreases, this corresponding to the weaker influence of the Pauli principle. Simultaneously, the attractive well at the edge of the nucleus becomes deeper. With further increase in the collision energy ($k \simeq 2\ \mathrm{F}^{-1}$) the potential becomes everywhere attractive down to the shortest distances, and its depth exceeds 100 MeV. The change in the ion–ion potential in this energy region ($k \lesssim 2\ \mathrm{F}^{-1}$) is largely determined by the energy dependence of the kinetic term $\Delta\tau$ in (38), this energy dependence arising from the rearrangement of the nucleons in the momentum space. At very high energies ($k \simeq 3\ \mathrm{F}^{-1}$), it follows from the calculations of Ref. 84 that the Pauli principle has a stronger influence on the potential term $\Delta\pi$ of the expression (38), and in principle this may lead to a repulsive potential. However, such high energies lie outside the ambit of the present paper.

Similar in spirit to Ref. 84 are the calculations of Refs. 87 and 88, in which the same Reid potential was used to obtain a self-consistent solution of the Bethe–Goldstone equation (40) in the momentum space; these calculations led to the same qualitative picture of the dependence of the ion–ion potential on the collision energy.

9. MICROSCOPIC APPROACHES

Introductory Comments

We consider now in more detail what information about the interaction of heavy ions can be obtained from the existing microscopic approaches in the theory of nuclear reactions. We recall that we apply the epithet "microscopic" only to the approaches that are based on approximate solution of a many-particle Schrödinger equation with a two-body nucleon–nucleon potential. Another characteristic feature of these approaches is the aim of taking into account systematically the Pauli principle in the wave functions and equations.

In the general case, the solution of the many-channel problem of heavy-ion collisions leads to an effective interaction which is nonlocal and depends on the energy and the orbital angular momentum of the relative motion. With increasing mass of the colliding nuclei, these dependences become weaker. Therefore, in the case of heavy ions it is, as a rule, possible to simplify the problem by using a local potential with a weak energy dependence.

In recent years, the development of the microscopic approaches has made it possible to go over from the study of collisions of the lightest nuclei to calculations for more complicated systems such as $\alpha + {}^{40}\mathrm{Ca}$, $^{12}\mathrm{C} + {}^{12}\mathrm{C}$, $^{12}\mathrm{C} + {}^{16}\mathrm{O}$, $^{16}\mathrm{O} + {}^{16}\mathrm{O}$, etc. Despite the approximate nature of the calculations and the uncertainty in our knowledge of the nucleon–nucleon potential, this has made it possible to investigate the behavior of the ion–ion potential directly on the basis of the many-particle Schrödinger equation.

Interpolation Approach

It is very convenient to investigate the interaction of heavy ions in the framework of the interpolation approach[89-94] to nuclear theory, since in this method one uses from the very beginning a translationally invariant collective variable ρ, defined by

$$\rho^2 = \frac{1}{A} \sum_{i>j}^{A} (\mathbf{r}_i - \mathbf{r}_j)^2 \equiv A\overline{r^2}, \tag{41}$$

which is unaffected by all permutations of the nucleons. In this definition, r_i is the coordinate of nucleon i, and r^2 is the rms radius of the system of A nucleons.

In the simplest case, when no allowance is made for compound-nucleus states and just a single binary reaction channel is considered, the completely antisymmetric wave function corresponding to the collision of two ions of A_1 and A_2 nucleons, respectively, can be written in the form

$$\Psi(\rho) = \Phi(\rho)\, w(\rho), \tag{42}$$

where the symbol $\boldsymbol{\rho} = \{\rho, \Omega\boldsymbol{\rho}\}$ denotes the radius vector in the $(3A-3)$–dimensional space of the relative coordinates of the $A_1 + A_2$ nucleons. The length of the vector $\boldsymbol{\rho}$ is determined by the expression (41), and the $3A-4$ angles denoted by $\Omega\boldsymbol{\rho}$ specify its direction. The actual choice of these angles is here unimportant.

The function $w(\boldsymbol{\rho})$ in (42) is given by

$$w(\boldsymbol{\rho}) = \frac{1}{\sqrt{N}} \hat{A}\, \{Z^L Y_{LM}(\mathbf{Z})\, \varphi_1(\boldsymbol{\rho}_1)\, \varphi_2(\boldsymbol{\rho}_2)\}, \tag{43}$$

where $Z = (A_1, A_2/A)^{1/2}(\mathbf{r}_2 - \mathbf{r}_1)$ is the "reduced" distance between the centers of mass of the ions, whose coordinates are denoted by $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively; L is the orbital angular momentum of the relative motion of the ions. The wave function $\varphi_1(\boldsymbol{\rho}_1)$ describes the motion of the A_1 nucleons in one of the ions, and the vector $\boldsymbol{\rho}_1 = \{\rho_1, \Omega\boldsymbol{\rho}_1\}$ is defined in exactly the same way as the vector $\boldsymbol{\rho}$ but for the $(3A_1-3)$-dimensional space of the relative coordinates of the nucleons. The function $\varphi_2(\boldsymbol{\rho}_2)$ and the vector $\boldsymbol{\rho}_2$ refer to the A_2 nucleons of the second ion. The operator A antisymmetrizes the expression in the curly brackets with respect to all $N = A!/A_1!A_2!$ permutations that carry nucleons from one ion to the other.

If the function $w(\boldsymbol{\rho})$ is assumed known, then $\Phi(\rho)$, which describes the relative motion of the ions, can be found from the many-particle Schrödinger equation

$$\left\{-\frac{\hbar^2}{2m}\sum_{i=1}^{A}\frac{\partial^2}{\partial r_i^2}+\sum_{i>j}^{A}V(i,j)-E\right\}\Phi(\rho)\,w(\boldsymbol{\rho})=0 \quad (44)$$

with two-body nucleon–nucleon potential $V(i,j)$. Multiplying this equation from the left by $w^+(\boldsymbol{\rho})$ and integrating over the angles $\Omega\boldsymbol{\rho}$, we obtain an equation for the unknown function $\Phi(\rho)$:

$$-\frac{\hbar^2}{2m}\left[\Phi''+\frac{\mu'}{\mu}\Phi'\right]+[V(\rho)-\varepsilon]\,\Phi=0, \quad (45)$$

where the prime denotes differentiation with respect to ρ; $V(\rho)$ is the averaged interaction of the nucleons in the different ions, the direct analog of the double-folded potentials considered above; ε is the energy of the relative motion. The symbol μ denotes the ρ-dependent function

$$\mu = \rho^{3A-4}\int d\Omega\boldsymbol{\rho}\;|\,w(\boldsymbol{\rho})\,|^2. \quad (46)$$

Introducing the new function $\chi(\rho)=\mu^{-1/2}\Phi(\rho)$, we obtain instead of (45) the simple equation

$$-\frac{\hbar^2}{2m}\chi''+[V(\rho)+U(\rho)-\varepsilon]\,\chi=0, \quad (47)$$

which has the form of the radial Schrödinger equation for the motion of a particle in the potential $V(\rho)+U(\rho)$.

In (47), the potential

$$U(\rho)=\frac{\hbar^2}{2m}\frac{1}{2\sqrt{\mu}}\left(\frac{\mu'}{\sqrt{\mu}}\right)' \quad (48)$$

describes the repulsion of the ions due to the Pauli principle. Indeed, for all the complexity of the integration in (46) it can be shown,[92] using the K-harmonic technique, that for $\rho \leqslant \rho_0$, where ρ_0 is the value of the variable ρ at which the ions "touch," the quantity (48) can be estimated in the general case. On the transition to the variable $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$, i.e., the distance between the centers of mass of the ions, the potential $U(\rho)$ takes the form of a repulsive core with a radius approximately equal to the sum $R = R_1 + R_2$ of the radii of the colliding ions. The height of the core is close to the value of the kinetic energy of the A nucleons localized within a spherical volume with radius $\rho_0/\sqrt{A}$.

Resonating-Group Method

The most fully developed microscopic approach to the solution of many-nucleon problems is the resonating-group method. The most detailed description of this method and examples of application to definite systems are given in the monograph of Ref. 95.

In the framework of this approach, the wave function of a many-nucleon system in the continuous spectrum is described in the completely antisymmetrized form

$$\Psi=\hat{A}\left\{\sum_i \varphi(A_i)\,\varphi(B_i)\,F(\mathbf{R}_i)+\sum_m a_m\varphi_m(A)\right\}, \quad (49)$$

where $\varphi(A_i)$ and $\varphi(B_i)$ are the wave functions of the clusters A_i and B_i in channel i, $F(\mathbf{R}_i)$ is an unknown function describing the relative motion of the clusters, and $\mathbf{R}_i$ is the distance between their centers of mass. Thus, the first sum in (49) is the part of the total wave function containing information about the motion in the open binary reaction channels. The second sum is an expansion with respect to states $\varphi_m(A)$ of compound-nucleus type with unknown coefficients a_m. The functions $\varphi_m(A)$ decrease rapidly outside the region of overlapping of the clusters. This part of the total wave function takes into account effectively the possibility of polarization of the clusters in the collision.

In contrast to the interpolation approach considered above, the function $F(\mathbf{R}_i)$ is not invariant with respect to permutations and cannot be taken in front of the antisymmetrization operator $\hat{A}$. Therefore, after substitution of the wave function (49) in the Schrödinger equation and the standard operation of projection, a complicated system of integro-differential equations is obtained for the unknown $F(\mathbf{R}_i)$ and a_m.

By successive elimination (using the Green's-function formalism) from this system of equations of all the inelastic channels and the compound states, it is possible to obtain a single integro-differential equation for the wave function $F(\mathbf{R})$, which describes the relative motion of the clusters in the elastic channel. The effective interaction which then arises is nonlocal, depends on the energy, and has an imaginary absorbing part. Essentially, this complex interaction must be associated with the optical potential.

However, this method of calculating the optical potential is complicated and in practice cannot be implemented. Instead of this, by solving the equations by the resonating-group method one can calculate the elastic-scattering phase shifts and then choose a local optical potential that leads to similar values of the phase shifts subject to the condition of a good description of the experimental data on the angular distributions of elastic scattering and polarization.

Such a program was realized[96–98] for scattering in the systems $^3\mathrm{H}+\alpha$, $\alpha+\alpha$, and $\alpha+{}^{16}\mathrm{O}$. The calculations showed that for satisfactory reproduction of the results of the resonating-group method the real part of the "equivalent" optical potential must be split with respect to the parity (for light ions) in its long-range part. In addition, at short distances it must have a repulsive core to prevent the appearance of local bound states and ensure correct behavior of the function of the relative motion, this behavior being distinguished by rapid damping at short distances in the case of overlapping of the volumes of the clusters.

The last remark can be illustrated by the wave function of the relative motion of two α particles in the state with L

$= 0$ at low energies.[95] The proton-density distributions in the ^{8}Be nucleus obtained with antisymmetrized and nonantisymmetrized wave functions were investigated. The results showed that if the calculations are made with a nonantisymmetrized function, then the relative-motion function in it is better taken in the form $R^4 \exp(-bR^2)$, i.e., sufficiently strongly preventing penetration of the two clusters into each other.

Thus, we see that in the attempt to reproduce the results of calculations in the resonating group method by means of an "equivalent" local potential (for pointlike structureless particles, as is actually assumed in the optical model) it is necessary to introduce a repulsive core to prevent mutual penetration of the colliding nuclei. The core arises through the antisymmetrization of the many-particle wave function in accordance with the Pauli principle.

Orthogonality-Condition Method

The use of the resonating group method or the closely related generator-coordinate method to calculate heavy-ion collisions entails great computational difficulties in connection with the antisymmetrization of the wave function of a many-nucleon system. This stimulated the development of approaches that are much simpler in practical application but still lead to many-particle wave functions having a behavior close to that given by the results of the resonating-group method.

It was noted that for two interacting compound systems of fermions there are so-called forbidden states that the system cannot take on account of the Pauli principle. One of the ways of constructing such states was proposed as early as 1962 by Feshbach.[99] A very complete review of related questions was recently published by Kukulin, Neudachin, and Smirnov.[100]

For actual calculations, wide use is made of the orthogonality-condition method proposed by Saito.[101] The essence of this approach is to replace the complicated exchange right-hand side of the equation of the resonating-group method for the relative-motion function of the clusters by the condition that this function be orthogonal to all the so-called redundant solutions. When the antisymmetrization operator is applied to the wave function of the resonating-group method (with orthogonality conditions), these solutions disappear. Thus, the orthogonality-condition method reduces the problem to the solution of the direct (without allowance for exchange effects) part of the Schrödinger equation for the wave function of the relative motion of the clusters in the subspace orthogonal to all the forbidden states.

In Ref. 102, this method was used to study the systems $\alpha + \alpha$, $\alpha + {}^{16}$O, and ${}^{16}\text{O} + {}^{16}\text{O}$. In particular, the corrections to the direct interaction of the clusters due to the requirement of orthogonality to the forbidden states were investigated. It was found that in the case of $\alpha + \alpha$ scattering the resonating-group method of calculation leads to functions of the relative motion in the s and d states that in the region of overlapping of the clusters have almost energy-independent oscillations, which are usually regarded as a manifestation of a repulsive core with a radius of the order of the distance from the coordinate origin to the outermost node of the function of the relative motion. Such behavior of the wave functions for $\alpha + \alpha$ scattering can be obtained on account of their orthogonality to the forbidden states, and the correction to the direct interaction of the two α particles in the orthogonality-condition method is small. On the transition to the heavier ${}^{16}\text{O} + {}^{16}\text{O}$ nuclei the oscillations in the wave function of the relative motion in the region of overlapping of the clusters are strongly suppressed even for nonresonant continuum states at low ($\lesssim 20$ MeV) energies. This can be interpreted as a manifestation of a repulsion effect in the subspace of the allowed states. Moreover, the strength of this effect decreases with increasing collision energy. Thus, it is found that the effective interaction of the two ^{16}O nuclei determined by the orthogonality-condition method in the subspace of the allowed states has at distances $r \lesssim 4$ F a strong (about 250 MeV) repulsion, this completely covering the region of the coordinate space in which the wave functions of the forbidden states are localized. Therefore, for heavy ions it is possible to ignore the orthogonality conditions and effectively replace them by a repulsive core in the local ion–ion potential.

Another interesting possibility of obtaining an effective interaction potential of the nuclei in the subspace of allowed states was demonstrated in Refs. 103 and 104. For the pairs of light nuclei $d + {}^3\text{H}$, $d + \alpha$, $\alpha + {}^3\text{H}$, $\alpha + \alpha$ the number of forbidden states is small. It is therefore possible to construct for them effective potentials with large depth in such a way that the forbidden states correspond to truly bound states in these potentials. Then, by virtue of the orthogonality of the continuum and discrete-spectrum wave functions of a Hermitian Hamiltonian, the forbidden states are automatically orthogonal to the wave function describing the scattering. Calculations in the framework of this approach led to good agreement between the theoretical and experimental scattering phase shifts and cross sections. We emphasize once more that the effective potentials obtained by this method are determined in the subspace of allowed states, since otherwise they have truly bound states of pairs of clusters with quantum numbers forbidden by the Pauli principle.

As was noted in Ref. 100, this approach cannot be used in the case of the heavy systems ${}^{12}\text{C} + {}^{12}\text{C}$, ${}^{12}\text{C} + {}^{16}\text{O}$, and ${}^{16}\text{O} + {}^{16}\text{O}$, to a large degree because of the strong suppression of the wave functions of the relative motion of the clusters in the region of their overlapping. In these problems, it is physically justified to use a repulsive core to imitate the effect of the Pauli principle. In the intermediate case of $\alpha + \alpha$ scattering, one can work either with a repulsive core or with deep potentials ensuring orthogonality of the relative-motion wave function to the forbidden states. The choice of the approach is dictated by how one intends subsequently to use the effective potential.[102,103]

To conclude this section, we note that orthogonalization of the wave function of the relative motion of the clusters to a large number of forbidden states, low-lying, as a rule, with respect to the energy and localized in the region of overlapping of the clusters, is tantamount to rapid damping of the relative-motion wave function at short distances

between the colliding ions, i.e., the interaction potential of these clusters has a repulsion. Therefore, in the case of heavy ions it is more convenient to work with nonantisymmetrized functions, introducing into the potential a short-range repulsive core, which can be regarded from the point of view of the orthogonality-condition method as a quasiclassical imitation of the effect of the Pauli principle.

Other Microscopic Approaches

As we are unable to dwell in detail on all the existing microscopic approaches to calculation of the interaction between heavy ions, we shall now mention a number of studies that in our view have the greatest interest.

Greiner and Pruess[105] studied the $^{12}C + {}^{12}C$ and $^{16}O + {}^{16}O$ interactions in the framework of the two-center shell model and demonstrated that in the case of a rapid collision between these nuclei a strong repulsion develops. A similar result was obtained by Vershinin and Cherdantsev[106] by taking into account deviations of the shape of the nuclei from sphericity and also the possible deformation of the compound nucleus.

In Ref. 107, Yukawa calculated the real part of the interaction potential of two ^{16}O ions in the framework of a method similar to the generator-coordinate method with a two-body *NN* interaction of Volkov type. At short distances, the resulting potential has a repulsive core with radius of order 4 F, and at the edge of the nucleus there is a comparatively shallow ($\sim$24 MeV) attractive potential well. Yukawa[107] regards the appearance of the core as a general phenomenon and notes that the surface nature of direct nuclear reactions can be one of the manifestations of the short-range repulsion between nuclei.

A calculation of the interaction of two deformed ^{12}C nuclei by the Hartree–Fock method with a Skyrme *NN* interaction[108] showed that, irrespective of the orientation of the symmetry axes of the ^{12}C nuclei, the potential has a short-range repulsive core and a shallow attractive well at the edge of the nucleus.

Recently, Filippov[109] proposed in the framework of the resonating-group method a new and very promising approach using an expansion of the wave functions of the relative motion with respect to an oscillator basis and also asymptotic expressions for the expansion coefficients. In our view, this method makes it possible to advance significantly in the direction of heavier ions in the calculation of the effective interaction potential, including the region of overlapping of the volumes of the colliding nuclei.

To conclude this section, we note that the majority of the microscopic approaches to the estimation of the ion–ion interaction potential in which the Pauli principle and the many-nucleon nature of the problem are taken into account systematically lead to the following qualitative picture of the interaction. First, these potentials have a surface nature, i.e., a shallow attractive well at the edge of the nucleus; second, in the region of overlapping of the wave functions of the nuclei there is a repulsive core with radius of the order of the sum of the radii of the nuclei. Thus, although the existing microscopic approaches cannot yet pretend to a high accuracy of the calculations for heavy-ion collisions, they nevertheless quite unambiguously indicate the validity of the qualitative estimates of the properties of the real part of the heavy-ion optical potential made on the basis of the macroscopic approaches considered in the previous sections. Therefore, we believe that in the construction of phenomenological optical potentials these features of the heavy-ion interactions must be taken into account from the very beginning.

We emphasize that to obtain the form of the potential considered here it is important to take into account the many-particle nature of the problem and the antisymmetry of the total wave function at the nucleon level. Thus, the semimicroscopic approach proposed by Zelenskaya and Gurbanovich[110,111] to calculate the interaction potential of 1*p*-shell nuclei takes into account the clustering effect of the colliding ions in the language of a problem of three or four structureless bodies. The deep "bare" interaction potentials of the clusters and the incomplete allowance for the many-particle nature of the problem lead in the framework of this approach to deep heavy-ion potentials without a repulsive core.

10. CONCLUSIONS

The analysis of the various theoretical approaches to the problem of the interaction of nuclei, and also the available information on heavy-ion collisions obtained by phenomenological analysis of the experimental data, leads us to conclude that the following structure of the real part of the heavy-ion optical potential is the one that is most physically justified:

$$V(k, r) = V_{\text{cor}}(k, r) + V_N(k, r) + V_{\text{coul}}(r), \qquad (50)$$

where $V_{\text{cor}}(k, r)$ is the short-range repulsive core, $V_N(k, r)$ is the potential of the nuclear attraction, and $V_{\text{coul}}(r)$ is the Coulomb interaction, i.e., we assume that for heavy ions one can with reasonable accuracy take a local optical potential with parameters that depend on the velocity of the relative motion of the colliding nuclei. In this connection, the expression (50) includes a dependence of the potential on the mean momentum $k = [(2m/\hbar^2)(E/Ap)]^{1/2}$ of the translational motion of the nucleons of the projectile.

We emphasize that the interaction potential contains a repulsive core for general physical reasons. The short-range repulsion is a manifestation of the Pauli principle, which prevents the overlapping of the wave functions of two composite systems of fermions. This was pointed out by Zel'dovich already in 1959.[112] In the physics of the interaction of atoms this phenomenon has long been known.[113]

As we have shown above, allowance for the Pauli principle in theoretical models of the ion–ion interaction based on the sudden approximation necessarily leads to the appearance of a repulgive core in the interaction potential of nuclei too. Moreover, if the Brueckner approach or a density-dependent Skyrme interaction is used, partial allowance for the Pauli principle occurs already in the stage of the construction of the energy-density functional or the effective nucleon–nucleon forces, when, in fitting the free parameters by means of the properties of finite nuclei, we attempt to satisfy the condition of saturation of the nuclear forces. The

existence of a core also follows from the existing microscopic calculations of effective local ion–ion potentials by the resonating-group method, by means of hyperspherical functions, generator coordinates, the two-center shell model, and so forth.

The models of ion–ion interaction considered in the present paper lead to qualitatively similar but nevertheless quantitatively differing estimates of the height, radius, and diffuseness of the core, this being due not only to the differences between the theoretical approaches themselves but also to the uncertainty in our knowledge about the interaction of nucleons in a nuclear medium or the properties of nuclear matter. The actual form of the repulsive core and its intensity depend strongly on the extent to which the real collisions of heavy ions are adiabatic or sudden. A further uncertainty with regard to the core parameters is associated with the influence of the individual characteristics of the considered nuclei, including the binding energies, the shape, and the nucleon distribution.

Under these conditions of strong uncertainty of the specific theoretical predictions, the following scheme for determining the properties of the short-range repulsive core is, in our view, expedient. On the basis of simple physical assumptions, we choose a convenient parametrization of the core, and then, by analyzing the experimental data on heavy-ion collisions, we determine the unknown parameters of the model.

To this end, we propose the following parametrization of the repulsive core:

$$V_{\text{cor}}(k, r) = V_{\text{cor}}\alpha(k) f_{\text{cor}}(r), \tag{51}$$

which is constructed on the basis of simple physical considerations. We take two heavy ions with mass numbers A_{p} and A_{T}, respectively, with $A_{\text{p}} \leqslant A_{\text{T}}$. We suppose that in the collision process their structure is unchanged, and to simplify the calculations we assume that the nuclei have a spherical shape with radii $R_{\text{p,T}} = r_0 A_{\text{p,T}}^{1/3}$ and a uniform distribution of nucleons, i.e., the nucleon density in each of the nuclei is described by

$$\rho(r) = \begin{cases} 0, & r > R, \\ \rho_0, & r \leqslant R, \end{cases} \tag{52}$$

where ρ_0, the equilibrium nucleon density, is the same for both nuclei, and $R = r_0 A^{1/3}$ is the radius of the nucleus with mass number A.

It is natural to expect the Pauli principle to be more strongly manifested in a collision process with greater overlapping of the wave functions of the nuclei. Therefore, we assume that the repulsive core $V_{\text{cor}}(k, r)$ as a function of the distance between the centers of mass of the colliding nuclei is proportional to the volume of their region of overlapping, i.e.,

$$f_{\text{cor}}(r) = \begin{cases} 1, & x \leqslant x_0, \\ \frac{3}{4} a(a+1) \frac{(1-x)^2}{x}\left[1 - \frac{1}{3} d(1-x)(3+x)\right], & x_0 \leqslant x \leqslant 1, \\ 0, & x \geqslant 1, \end{cases} \tag{53}$$

in which $x = r/R_V$ is the ratio of the distance between the centers of mass of the colliding nuclei to the radius of the core (which is of the order of the sum of the radii of the colliding nuclei): $R_V = R_{\text{p}} + R_{\text{T}} = r_0(A_{\text{p}}^{1/3} + A_{\text{T}}^{1/3})$; $a = R_{\text{T}}/R_{\text{p}} = A_{\text{T}}^{1/3}/A_{\text{p}}^{1/3} \geqslant 1$ is the ratio of the radius of the target nucleus to the radius of the projectile nucleus; $d = (a + 2 + 1/a)/4$; $x_0 = (a-1)/(a+1)$.

It is obvious that $x \geqslant 1$ corresponds to distances r at which the two volumes of the nuclei do not overlap. In this case, $f_{\text{cor}}(r) = 0$. For $x \leqslant x_0$, there is complete overlapping (the projectile "is within" the target). Then the effect of the Pauli principle is maximal, and $f_{\text{cor}}(r) = 1$.

In the framework of the assumptions made above, a region of overlapping with the doubled nucleon density $2\rho_0$ is formed once the distance between the ions in the collision process becomes less than $r = R_V$. The adding of the densities increases the energy of the nucleons in this region, and therefore increases the energy of the complete system. In the case of complete overlapping (for $r \leqslant R_{\text{T}} - R_{\text{p}}$) the increase is obviously

$$\Delta V = 2A_p [\varepsilon(2\rho_0) - \varepsilon(\rho_0)], \tag{54}$$

where $\varepsilon(\rho)$ is the total energy of the nuclear matter per nucleon at the given nucleon density ρ.

We now relate the height of the repulsive core V_{cor} to ΔV. The equation of state of nuclear matter, i.e., the dependence $\varepsilon(\rho)$, is determined by the behavior of the kinetic energy $t(\rho)$ of the Fermi gas and its potential energy $v(\rho)$ as the nucleon density ρ changes:

$$\varepsilon(\rho) = t(\rho) + v(\rho). \tag{55}$$

Therefore, in the framework of our assumptions we must identify ΔV with the value of the heavy-ion interaction potential at the coordinate origin:

$$\Delta V = V_{\text{cor}}(k, 0) + V_N(k, 0). \tag{56}$$

In (55) and (56), Coulomb forces are not taken into account.

Because the depth of the attractive potential is small compared with the height of the repulsive core for sufficiently heavy projectile ions ($A_{\text{p}} \geqslant 6$), we can in the low-energy limit ($k \to 0$) set

$$2A_p [\varepsilon(2\rho_0) - \varepsilon(\rho_0)] = V_{\text{cor}}. \tag{57}$$

The present state of the theory of nuclear matter does not permit the obtaining of a unique dependence $\varepsilon(\rho)$ even for low densities $\rho \lesssim 2\rho_0$. Therefore, expressing (57) in terms of the compressibility $K = 9\rho_0^2 (d^2\varepsilon/d\rho^2)_0$, of nuclear matter, for the values of which there are fairly weak limits, $K = 100$–300 MeV,[114] we obtain the following rough estimate for the height of the repulsive core:

$$V_{\text{cor}} \simeq A_p \frac{1}{9} K. \tag{58}$$

Separating the regular factor A_{p},

$$V_{\text{cor}} = A_p v_{\text{cor}}, \tag{59}$$

for v_{cor}, we obtain a fairly large region of acceptable values: $v_{\text{cor}} = 10$–30 MeV.

As we have already said, raising the collision energy weakens the effect of the Pauli principle. The smaller the

translational momentum k, the larger the fraction of the nucleons that are subject to the effect of the Pauli principle when the nuclear volumes overlap. Without attempting to simulate the rearrangement of the "redundant" nucleons in the momentum space, we propose to take into account the energy dependence of the height of the core by means of a factor $\alpha(k)$ proportional to the volume of overlapping (in the momentum space) of the Fermi spheres of colliding nuclei whose centers are separated by the "distance" k, namely,

$$\alpha(k)=\begin{cases} 1-\frac{3}{2}(k/2k_F)+\frac{1}{2}(k/2k_F)^3, & k\leqslant 2k_F, \\ 0, & k>2k_F, \end{cases} \tag{60}$$

where k_F is the limiting Fermi momentum corresponding to the normal density ρ_0, i.e., $k_F \simeq 1.36\ \mathrm{F}^{-1}$. For $k \ll k_F$, it follows from (60) that the effect of the Pauli principle is maximal. As the ratio $k/2k_F$ increases, the effect decreases. For $k > 2k_F$, the core disappears.

Thus, the final expression for the core is

$$V_{cor}(k, r) = A_p v_{cor} \alpha(k) f_{cor}(r). \tag{61}$$

All the quantities that occur here have been defined above. The expression itself contains two free parameters: $r_0 \simeq 1$ F and $v_{cor} \simeq 10$–30 MeV, which must be determined using experimental data.

With allowance for the results of the studies discussed above, we use a parametrization of the nuclear-attraction potential $V_N(k, r)$ (50) based on the proximity force theorem:

$$V_N(k, r) = 2\pi \frac{R_p R_T}{R_p + R_T} v_N(k, s), \tag{62}$$

where $s = r - R_p - R_T$, and the radii $R_{p,T}$ were defined above.

On the basis of the results of Ngô *et al.*, we propose for $v_N(k, s)$, in contrast to the traditional Woods–Saxon form, the expression

$$v_N(k, s) = \begin{cases} -v_N(k), & s\leqslant 0, \\ -v_N(k)\exp[-(r-R_V)^2/a_V^2], & s\geqslant 0, \end{cases} \tag{63}$$

where the radius R_V is defined above, and a_V is the diffuseness of the potential, a free parameter of the model to be determined from experimental data. In accordance with (21), we can expect $a_V \simeq 1.9$ F.

To take into account the dependence of the potential V_N on the collision energy, we choose the simplest expression

$$v_N(k) = v_N[1 + \beta(k/2k_F)^2], \tag{64}$$

where $v_N \simeq 4.8$ MeV $\cdot$ F^{-1} in accordance with (21), and β is a constant. Both parameters v_N and β are assumed to be free. We emphasize that we have chosen the form of the change in the attractive potential (64) with the energy on the basis of intuitive considerations. However, in real situations, when heavy-ion elastic scattering is analyzed in comparatively small ranges (about 200 MeV) of the collision energy, the momentum does not change strongly. Therefore, we hope that a simple linear approximation will make it possible to encompass the main features of the energy dependence of the attraction potential.

We represent the Coulomb interaction in the standard manner in the form of the potential of a uniformly charged sphere of radius R_V.

Thus, the real part of the heavy-ion optical potential determined by Eqs. (50), (53), and (60)–(64) contains five free parameters: v_{cor}, r_0, v_N, a_V, and β. We expect that none of these parameters, which basically depend on the properties of nuclear matter, will change strongly on the transition from one pair of colliding heavy ions to another. The expected values of these parameters are given above.

To conclude this review, we note that the introduction of a repulsive core into the heavy-ion interaction potential made it possible in a number of studies (see, for example, Refs. 115–123) to achieve a good description of the anomalous large-angle scattering of ions and individual characteristics of nuclear molecules. In a systematic study of the properties of the ion–ion interaction it is necessary to apply the parametrization proposed here for the real part of the optical potential in consistent analysis of the experimental data on heavy-ion elastic scattering in a wide range of collision energies and scattering angles.

The first attempts to apply this philosophy to the analysis of the elastic scattering of ^{6}Li and ^{16}O ions by ^{28}Si nuclei were successful.[122-125]

At the same time, it should be noted that to obtain a sufficiently detailed picture of the dependence of the properties of the ion–ion potential on the nuclear masses and on the collision energy new experimental information is needed, in particular, on the elastic scattering of ^{6}Li, ^{9}Be, ^{10}B, ^{14}N ions by the ^{12}C, ^{28}Si, ^{40}Ca nuclei; this would make it possible to follow in detail the change in the anomalous large-angle scattering of these ions as a function of the collision energy and the masses of the projectile and target.

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