Unstable quantum systems and Feynman integrals

P. Exner

Joint Institute for Nuclear Research, Dubna

Fiz. Elem. Chastits At. Yadra **15**, 121–155 (January–February 1984)

General properties of unstable systems (particles, nuclei, etc.) are considered. A basic criterion for the existence of a solution to the inverse decay problem is formulated. The influence of repeated measurements on the exponential nature of the decay laws and the measured lifetime is considered. The order of magnitude of these effects is estimated in the framework of a model of the decay of charged kaons in a bubble chamber. These considerations provide a physical basis for the semigroup approximation in the description of the dynamics of unstable systems. It is shown that every such semigroup can be characterized by a so-called pseudo-Hamiltonian. The connection between this pseudo-Hamiltonian and the total Hamiltonian of the system is found. Rigorous approaches to the definition of Feynman integrals are reviewed. It is shown how in the framework of some of these approaches it is possible to express the evolution operator corresponding to a pseudo-Hamiltonian of Schrödinger type with a local complex potential. A multidimensional damped harmonic oscillator is considered as an example.

INTRODUCTION

The importance of attempts at complete theoretical description of unstable systems follows from the simple fact that in the majority of cases the objects considered in microphysics are unstable. For example, if we restrict ourselves to elementary particles, then among the 20 that are conditionally stable five are genuinely stable (in the framework of the experimentally achieved accuracy: γ, ν_e, ν_μ, e, p); in addition, recent experimental and theoretical results [indications of a nonvanishing $\bar{\nu}_e$ rest mass, attempts to combine the strong and electroweak interactions in the framework of the groups SU(5), SO(10), etc.] suggest that the neutrino and proton may also decay.

First, some words about the history of the problem. After the discovery by Becquerel of natural radioactivity, Elster and Geitel formulated in 1899 an empirical law according to which the amount of a radioactive preparation decreases exponentially with the time:

$$N(t) = N(0) \exp(-\Gamma t), \tag{1}$$

each form of decay being characterized by a corresponding decay time Γ^{-1}. The elementary explanation proposed by Schweidler in 1905 is based on the assumption that the decay probability of each of the nuclei is independent of the time of its formation. At that period, this was a rather daring conjecture, since it presupposed that the decay of an individual nucleus cannot be described deterministically. However, by means of classical physics it is impossible to explain the *ad hoc* assumption made by Schweidler, to say nothing of the determination of the constant Γ. Such a possibility first appeared after the creation of quantum mechanics. The first attempt of this kind was made in 1928 by Gamow in his semiclassical theory of α radioactivity.[1]

The first completely quantum theory of decay was created in 1930 by Weisskopf and Wigner,[2] who considered the damping of the radiation of an excited atom. Their theory is based on the assumption, deduced from experiment, that the decay law, i.e., the probability of finding an atom at the time t in the initial (excited) state, has the form (1). Then the distribution of the radiation intensity is given by a relation of Breit–Wigner type, in which the width of the spectral line is equal to the sum of the probabilities of all energetically allowed transitions from the excited state. The Weisskopf–Wigner method proved to be convenient for describing the decay of a number of other systems. In some cases (for example, for decays with violation of CP parity[3]) it is necessary to replace the original assumption by a more general one (the so-called *Weisskopf–Wigner condition*), in accordance with which the time development in the state space of an unstable system is determined by a family of operators $\{V(t): t \geqslant 0\}$ that form a semigroup:

$$V(t)\,V(t') = V(t+t'), \quad t, t' \geqslant 0. \tag{2}$$

Thus, the generalized Weisskopf–Wigner method, in which the condition (2) is assumed to hold and the generator of the semigroup $V(\cdot)$ can be calculated by means of perturbation theory, has a very wide range of practical application. However, there are also associated with it some serious difficulties which follow from the condition (2) itself, which is incompatible with semiboundedness of the spectrum of the operator of the total energy.[4–12] Therefore, the semigroup description of the time development of unstable systems must be regarded as only approximate if we do not wish to depart from the fundamental postulates of quantum theory.

Among the other problems encountered in attempts to construct a general theory of unstable quantum systems on the basis of "first principles," we mention some of the most important.

a) Relativistic Invariance. The main problem is the choice of representation of the Poincaré group $\mathscr{P}$ in the state space $\mathscr{H}_u$ of the unstable system. The simplest possibility is to take a nonunitary representation with exponential decrease of the norm of vectors under time shifts.[13] Similar results are also obtained in other approaches,[14] in particular when one considers the so-called Poincaré semigroup.[15] However, for a complete description it is necessary to take

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the state space $\mathscr{H}$ (including the decay products—see Sec. 1) and specify on it a unitary reducible representation of $\mathscr{P}$.[16] An attempt to relate the representations in $\mathscr{H}_u$ and $\mathscr{H}$ was made in Ref. 8, but it suffers from a serious shortcoming.[17]

b) Formulation of the Theory at the Quantum-Field Level. The formalism of quantum field theory is actually used in the calculation of the transition probabilities needed for the approximate Weisskopf–Wigner description. However, a physicist with a theoretical bent will immediately want to know what is a field whose fundamental quanta are unstable particles. There have been various attempts to construct such theories, but they were either purely phenomenological,[18] or reduced to the case of a free field.[5] A less ambitious but in practice more successful program is the consideration of solvable models (see, for example, Refs. 6, 7, and 11 for Lee's model, Ref. 19 for van Kampen's model, etc.). We must also mention here numerous studies of the process of attainment of thermodynamic equilibrium; for example, we mention Refs. 20 and 21.

c) Connection with Scattering Theory. In the first place, this offers a possibility of describing metastable particles and resonances on a common basis. A good illustration is the theory of nonrelativistic potential scattering (see, for example, Ref. 22). From it, we know that there exists an important difference between two cases: If the energy resolution is sufficiently good that a resonance can be determined from the scattering cross section and the phase shift, then the corresponding measurement time is such that there is no sense in speaking of time evolution of the given state as an independent object, and vice versa. Similar conclusions can also be drawn for relativistic unstable particles.[16] On the other hand, a unified phenomenological description of resonances and metastable particles is helpful, for example, in analytic S-matrix theory[23] and in quark models.[24] This emphasizes the importance of studying the connection between scattering and decay processes.

In considering problems of this type, it is convenient to identify the properties that are common to all unstable quantum systems. This approach (the so-called *kinematic concept*—it was first clearly formulated by Williams[8]) gives, first, a possibility of determining the framework for studying concrete dynamical properties of decaying micro-objects and, second, makes possible wide use of the methods of functional analysis (see, for example, Refs. 4, 8–10, 12, 16, 17, 25, and 26). A brief review of some results of this approach is given in the following two sections.

Before we begin the exposition, let us say a few words about the content of the present review. Sections 3 and 4 consider the case when the unstable system is subjected to multiple measurements. Such a situation is frequently encountered in experiments and can serve as one of the ways of explaining the difficulties associated with the exponential nature of decay laws. It can be shown that a nonexponential behavior can be manifested in experiments of this kind in the dependence of the measured mean lifetime on the density of measurements. To estimate the order of magnitude of this effect, we consider in Sec. 4 a model that describes decay of charged kaons in a bubble chamber. This shows that even under optimal experimental conditions the effect is very small and difficult to record. In Sec. 5, we return to the "kinematic concept" and discuss the so-called bounded-energy approximation. The arguments of Secs. 1–5 provide the physical basis for the semigroup approximation in the description of the time development of unstable systems. Every such semigroup can be characterized by its generator or, in other words, a Schrödinger equation with a phenomenological non-self-adjoint Hamiltonian (pseudo-Hamiltonian). In Sec. 6, we discuss the properties of pseudo-Hamiltonians, in particular, their connection with the corresponding total-energy operators.

The second part of the review is devoted to Feynman integrals and their application to the description of the time development of unstable systems. First (in Sec. 7), we briefly review the most important rigorous approaches to the definition of Feynman integrals, restricting ourselves to integrals in Lagrangian form in a flat configuration space. We then consider how Feynman integrals can be used to express the solution of a Schrödinger equation with a pseudo-Hamiltonian containing a local complex potential. As an example, we consider in Sec. 9 a multidimensional damped harmonic oscillator described by a complex quadratic potential, find the explicit form of the propagator, calculating the corresponding Feynman integral, and discuss the properties of this solution in the one-dimensional case. In the final section, we discuss some results related to "Feynman paths," which suggest that the possibilities of path integration for describing the dynamics of quantum systems, including unstable systems, are still far from exhausted.

1. BASIC CONCEPTS OF THE QUANTUM KINEMATICS OF UNSTABLE SYSTEMS

We begin by formulating the postulates. We denote by $\mathscr{H}_u$ and $\mathscr{H}$, respectively, the Hilbert state spaces of the considered unstable system and the isolated system consisting of the unstable system and its decay products. Further, let $\{U(t): t\in \mathbf{R}\}$ be a unitary (strongly continuous) representation of the one-parameter group of time shifts (the evolution operator),

$$U(t) = \exp(-iHt),\ t\in \mathbf{R} \tag{3}$$

(for simplicity, we shall generally use a system of units with $c=\hbar=1$), where H is the total Hamiltonian of the system. We assume that:

a) $\mathscr{H}_u$ is a proper subspace in $\mathscr{H}$;

b) there does not exist $t>0$ for which the subspace $\mathscr{H}_u = E_u\mathscr{H}$ is invariant with respect to $U(t)$.

The *reduced evolution operator* is defined by the relation $V(t) = E_u U(t) E_u$, from which it folows that $\{V(t)\}$ is a strongly continuous one-parameter contractive family, and

$$V^+(t) = V(-t),\ t\in \mathbf{R}. \tag{4}$$

At the same time, $\{V(t)\}$ is not a group, since (2) cannot hold for all $t, t'\in \mathbf{R}$ on account of postulate (b). The *decay law* of the system, whose state at the initial time $t=0$ is described by the density matrix ρ, $\mathrm{Ran}\,\rho\subset\mathscr{H}_u$, is given by

$$P_\rho:\ P_\rho(t) = \mathrm{Tr}(V^+(t)V(t)\rho). \tag{5}$$

This is a continuous function satisfying the condition $0\leqslant P_\rho(t)\leqslant P_\rho(0)=1$.

As we have already noted, for a large number of unstable systems the validity of a decay law of the type (1), $P(t) = \exp(-\Gamma t)$, $t \geqslant 0$, has been confirmed experimentally. In such a case, the *initial decay rate* $\dot{P}(+0)$ is clearly nonvanishing. On the other hand, formal differentiation of the relation (5) gives $\dot{P}_\rho(0) = 0$. The difficulty resides in the formality of this argument; indeed, Horwitz *et al.*[9] showed that for ρ corresponding to a pure state ψ the relation $\dot{P}_\psi(0) = 0$ holds if $\dim \mathscr{H}_u < \infty$, $\{V(t)\}$ is a contractive semigroup, and the expectation value of the energy in the state ψ is finite. However, it can be shown that among these conditions only the last is important. We introduce the concept of a *state with finite energy*: This is any state ρ for which the integral

$$\int_{\mathbf{R}} \lambda d \operatorname{Tr}(\rho E_\lambda) \tag{6}$$

exists (in the proper sense); here, $E_\lambda \equiv E_H[(-\infty, \lambda)]$ denotes a resolution of the identity of the Hamiltonian H. We denote the set of such states by $M(H)$. The following theorem holds[17,25]:

THEOREM 1. If $\rho \in M(H)$, then $\dot{P}_\rho(0) = 0$. To an exponential decay law there correspond energy distributions of Breit–Wigner type, for which the integral (6) exists only in the principal-value sense; therefore, the paradox noted above does not occur. It must, however, be emphasized that the above theorem *does not* make it possible to confirm or refute the condition (2). The point is that experimentally one can measure (with a definite accuracy) only the behavior of the decay law over time intervals, and not the initial decay rate directly. On the other hand, one cannot establish experimentally whether a given state belongs to the set $M(H)$[17,25] (this makes more precise the assertion about physically realizable states contained in the classical monographs of Refs. 27 and 28; see Sec. 5 and the more detailed discussion in Ref. 29, Sec. 7.2.3).

2. INVERSE DECAY PROBLEM AND PROPERTIES OF THE SPECTRUM OF THE HAMILTONIAN

The following question arises naturally: using only data on one unstable system, can one recover a complete description of the decay? A rigorous formulation of this problem appeared for the first time in Refs. 8 and 9 as the so-called *inverse decay problem*; one is given a one-parameter family $\{V(t)\}$ of contractive operators on the space $\mathscr{H}_u$ satisfying the condition (4) and asks under what conditions there exist a Hilbert space $\mathscr{H} \supset \mathscr{H}_u$ and a one-parameter strongly continuous group $\{U(t)\}$ on $\mathscr{H}$ such that for all $t \in \mathbf{R}$ the relation $V(t) = E_u U(t) \upharpoonright \mathscr{H}_u$ holds. The basic criterion for the existence and uniqueness of a solution follows from the Szökefalvi-Nagy theory of unitary continuations (dilatations) (see, for example, Ref. 30, Chap. 1, and also Refs. 8, 9, and 17). Important here is the condition of *positive definiteness*: An operator function $V: \mathbf{R} \to \mathscr{B}(\mathscr{H}_u)$ is regarded as positive definite if for all natural n and $t_1, \ldots, t_n \in \mathbf{R}, \varphi_1, \ldots, \varphi_n \in \mathscr{H}_u$.

$$\sum_{i,j=1}^{n} (\varphi_i, V(t_i - t_j)\varphi_j) \geqslant 0. \tag{7}$$

THEOREM 2. For a function $V: \mathbf{R} \to \mathscr{B}(\mathscr{H}_u)$ there exists a triplet $\{\mathscr{H}, U(\cdot), E_u\}$ with the required properties if and only if v is positive definite, weakly continuous, and $V(0) = I_u$. In addition, if the *minimality condition*

$$\mathscr{H} = \overline{\left[\bigcup_{t \in \mathbf{R}} U(t)\, \mathscr{H}_u\right]_{\lim}} \tag{8}$$

is satisfied, then this triplet is unique up to isometric isomorphism. It is called the *minimal unitary extension* of the operator function V.

The theorem is taken from the monograph of Ref. 30, where it is also shown that every continuous contractive semigroup on $\mathscr{H}_u$ satisfies the conditions, i.e., admits a minimal unitary extension. Postulate (b) from the previous section entails the condition

$$V^+(t)\, V(t) \neq I_u, \quad t > 0. \tag{9}$$

Among the operator functions allowed by this condition and Theorem 2, we can choose reduced evolution operators on the basis of further (physically motivated) requirements. One of them is semiboundedness of the spectrum $\sigma(H)$ of the Hamiltonian; however, there are cases known when this condition *is not satisfied*[4,7–10]:

a) $\{V(t): t \geqslant 0\}$ is a continuous contractive semigroup;

b) $V(\cdot)$ is continuous and Eq. (2) holds for all $t \geqslant T_r \geqslant 0$, $t' \geqslant 0$;

c) there exists a nonzero vector $\varphi \in \mathscr{H}_u$ and $\gamma > 0$ such that for all $t \geqslant 0$ the relation $\|V(t)\varphi\| \leqslant \exp(-\gamma t)$ holds;

d) $\dim \mathscr{H}_u = 1$ and the integral $\int_{\mathbf{R}} \ln|(\varphi, V(t)\varphi)| dt / 1 + t^2$ diverges.

Any one of the conditions (a)–(d) means that $\sigma(H)$ is unbounded; if, in addition, one of the conditions (a)–(c) is satisfied, then $\sigma(H) = \mathbf{R}$. From this there follow *necessary conditions* for $\sigma(H)$ to be bounded below; they are clearly not satisfied for reduced evolution operators of Weisskopf–Wigner type.

One can also give a *sufficient condition* (which does not require the explicit form of the minimal unitary extension, the finding of which is, in general, a very complicated problem). For this, we need a generalization of the Bochner–Khinchin theorem,[12] according to which every function V satisfying the conditions of Theorem 2 can be expressed as follows:

$$V(t) = \int_{\mathbf{R}} e^{-i\lambda t}\, dF_\lambda, \tag{10}$$

where $\lambda \to \int_\Delta dF_\lambda$ is some positive operator Borel measure on $\mathbf{R}$, whose support we denote by $\sigma[V]$. It is important that the operators can be found by means of the following generalization of the Fourier–Stieltjes transformation:

$$\Phi(\xi + i\eta) = \int_0^\infty V(t)\, e^{it(\xi + i\eta)}\, dt, \quad \eta > 0; \tag{11a}$$

$$\frac{1}{2}[\Omega_0(\lambda - 0) + \Omega_0(\lambda + 0)] = \frac{1}{\pi} \lim_{\eta \to 0+} \int_0^\lambda \operatorname{Re} \Phi(\xi + i\eta)\, d\xi; \tag{11b}$$

$$E_\lambda = -w - \lim_{\varkappa \to -\infty} \Omega_0(\varkappa) + \Omega_0(\lambda + 0). \tag{11c}$$

The required condition for semiboundedness of $\sigma(H)$ is given

by the following theorem[12]:

THEOREM 3. Let V be a reduced evolution operator and H a Hamiltonian corresponding to a *minimal* unitary extension. Then $\sigma(H) = \sigma[V]$.

In contrast to V, knowledge of the decay laws, i.e., knowledge of the operator function $t \to V^+(t)V(t)$, is not sufficient to recover a complete description of the decay.[12] We give some other properties of the spectrum[9,10]:

THEOREM 4. a) If $w - \lim_{t\to\infty} V(t) = 0$, then $\sigma(H)$ is continuous.

b) If, in addition, $(\psi, V(\cdot)\psi) \in L(\mathbf{R})$ for any $\psi \in \mathscr{H}_u$, then $\sigma(H)$ is absolutely continuous.

Using the transformation (11) and the boundedness of $|(\psi, V(\cdot)\psi)|$, we can readily show that under the conditions of Theorem 4b there exists for every $\psi \in \mathscr{H}_u$ a function $g_\psi \in L(\mathbf{R}) \cap L^2(\mathbf{R})$, such that

$$(\psi,\ V(t)\psi) = \int_{\mathbf{R}} e^{-i\lambda t} g_\psi(\lambda)\, d\lambda,$$

from which $g_\psi(\cdot)$ can be obtained by inverse Fourier transformation.

3. REPEATED MEASUREMENTS

The above difficulties with the exponential decay law led to the idea that the exponential behavior is actually due to the fact that in the majority of real experiments an unstable system is subjected, not to one, but to many repeated measurements (as, for example, in photoemulsions, bubble chambers, etc.—see Refs. 16 and 31–34). This idea is easy to accept: The classical derivation by Schweidler of an exponential decay law was based on the assumption that the system does not remember its history. A nonexponential behavior, in contrast, indicates the existence of memory; naturally, repeated measurements could weaken or destroy this memory.

To describe the influence of repeated measurements, we must in the first place know what occurs on reduction of a state. We assume the following:

A) If in a given measurement the unstable system is found to be *undecayed*, then the result of the reduction is its *original state* (see Refs. 16, 31, and 32 for the physical arguments in favor of this assumption).

In view of the situation in real experiments, it is sensible to take into account the possibility that the measurements can be made at random instants of time. We therefore describe the structure of the measuring system [for simplicity, we shall speak of a chamber) by a function $W(\cdot,\cdot)$, whose values $W(t,t')$ give the probability that an unstable system prepared at the time t *has not been measured* during the time interval (t,t')]. We denote by $E(\tau,t)$ the probability that an unstable system prepared at the time τ survives to the time t, i.e., that it will be found undecayed in all measurements made up to this time. In particular, we call the function $E(\cdot) = E(0,\cdot)$ *the measured decay law*. More precisely, the measured decay law is determined by

$$N(t) = \int_{\tau_0}^{0} G_0(\tau)\, E(\tau,\ t)\, d\tau,$$

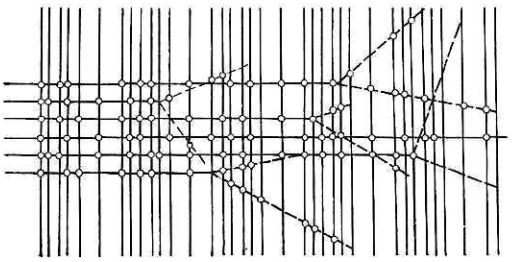

FIG. 1. Arrangement of experiment with random repeated measurements. The open circles indicates the measurements, and the broken lines the decay products.

since the preparation of the initial state must also be regarded as a random process described by the density $G_0(\cdot)$ of the first measurement. However, we can allow such a simplification, because the properties in which we are interested are transferred completely from $E(\cdot)$ to $N(\cdot)$. We call the function $P(\cdot)$, whose values give the probability that an unstable system prepared at $t = 0$ and *not then subjected to measurement* will be found undecayed in a measurement at time t the *primary decay law*. In accordance with the results of Sec. 1, we only assume that it is continuous and that $0 \leqslant P(t) \leqslant P(0) = 1$.

In calculating $E(t)$, it is necessary to take into account the contributions from the processes when the unstable system is not measured during the time $(0,t)$, when it is measured once, twice, etc. (and always found to be undecayed—see Fig. 1). Expressing these contributions by means of P and W and summing the corresponding series, we obtain[34,35]

$$E(t) = W(0,\ t) + \int_0^t \dot{W}(\tau,\ t)\, f(\tau)\, d\tau, \tag{12a}$$

where f is the solution of the Volterra equation of the second kind

$$f(t) = -W(0,\ t)\, P(t) - \int_0^t \dot{W}(\xi,\ t)\, P(t-\xi)\, f(\xi)\, d\xi; \tag{12b}$$

and $\dot{W}(t,t') \equiv \partial W(t,t')/\partial t$. We assume the following:

B) $W(t,t')$ is continuous with respect to each of the arguments and does not decrease with respect to the first, $0 \leqslant W(t,t') \leqslant W(t',t') = 1$, and the derivative $\dot{W}(t,t')$ is bounded and measurable with respect to t.

Then from the basic theorem for Volterra equations it follows that for Eqs. (12) *there exists exactly one solution* $E(\cdot)$, which is continuous, nonincreasing, and has $E(0) = 1$.

However, to find this solution and study its properties is in general not a simple problem. We therefore mention some important special cases, from which one can get an idea of the behavior of the general solution (see the discussion in Ref. 36):

I. Chamber of Periodic Structure and Exponential Primary Decay Law. We assume that $P(t) = \exp(-\Gamma t)$, $t \geqslant 0$, and, in addition,

$$W(t,\ t') = \exp\left\{-\int_t^{t'} \varphi(\tau)\, d\tau\right\}, \tag{13}$$

where φ is a bounded measurable t_c-periodic function. It follows from the relation (13) that

$$W(t,\ t')\ W(t',\ t'') = W(t,\ t''),\ t \leqslant t' \leqslant t'' \qquad (14)$$

(independently of the periodicity assumption). This condition is not always satisfied in real experiments. It can be used, for example, in the case of bubble chambers (except for points near the ends of the tracks), but it is not suitable for measurements in photoemulsions.

II. Idealized Spark Chamber. We assume that the points at which a measurement can be made are separated by the distance t_c from each other, $t_n = nt_c$, and that W is a step function: $W(t_n,t) = \exp(-a(m-n))$ for $t \in (t_m, t_{m+1})$. The above existence result cannot be applied to this case, since assumption B is not satisfied. However, Eqs. (12) now reduce to an expression for $E(t)$ in the form of a finite sum. It is readily deduced[37] that the function $E(\cdot)$ in this case is also a step function and does not increase.

III. Homogeneous Chamber[32,36]: $W(t,t') = \exp[-\lambda \times (t'-t)]$.

The behavior of the measured decay law in these cases can be described as follows.[36,37] If the density of measurements, which can be characterized by

$$\nu_\varphi = t_c^{-1} \int_0^{t_c} \varphi(t)\, dt,$$

is sufficiently high, then there exists a positive γ such that

$$E(t) = \Phi(t) \exp(-\gamma t), \quad t \geqslant 0,$$

and the function $\Phi(\cdot)$ is automatically periodic with period t_c. At the same time, the density of measurements can be compared with the reciprocal (primary) mean lifetime of the given unstable system,

$$T = \int_0^\infty P(t)\, dt.$$

In the first case we require $\nu_\varphi > \Gamma = T^{-1}$, and in the third case $\nu_\varphi = \lambda > T^{-1}$ (in the second case, the scale for comparison is also given by T^{-1}, but the condition for ν_φ is more complicated; see Ref. 37). In addition, in the third case the period t_c can be assumed to be any positive number:

$$\Phi_{\mathrm{III}}(t) = A + B(t),\ A > 0,$$

and there exists $\alpha_0 > 0$ such that for all $\alpha \leqslant \alpha_0$ the relation $\lim_{t\to\infty} B(t) e^{\alpha t} = 0$ holds. With regard to the measured reciprocal mean lifetime γ, in the first case $\gamma = \Gamma$. On the other hand, γ in general *depends not only on $P(\cdot)$ but also on the structure of the chamber.*[16,38] We obtain the equations[32,36,37]

$$(e^a - 1) \sum_{k=1}^{\infty} P(kt_c) \exp(k(\gamma t_c - a)) = 1, \qquad (15)$$

$$\lambda \int_0^\infty P(t) \exp[(\gamma - \lambda)t]\, dt = 1 \qquad (16)$$

for cases II and III, respectively. Under the conditions noted above, each of these equations has one and only one solution $\gamma \in (0, \nu_\varphi)$. It is of interest to consider limiting cases for Eq. (15): $a = \nu_\varphi t_c \gg 1$ corresponds to Ref. 16, while on the other hand Eq. (15) for fixed ν_φ and $t_c \to 0$ goes over into (16). In the case of a homogeneous chamber, the following important proposition[36] follows from the properties of Eq. (16): For nonexponential $P(\cdot)$ the solution $\gamma(\lambda)$ not only can, but *must*, depend on the frequency λ of the measurements; more precisely, the relation $\gamma(\lambda) = \Gamma$ on some open interval of λ values is true if and only if $P(t) = \exp(-\Gamma t)$, $t \geqslant 0$.

One further limiting case must be mentioned. Suppose the initial decay rate is zero. At a high density of measurements it is then natural to expect[16] that γ^{-1} will be much greater than T, and in the limit of continuous measurement decay will not be observed at all. It can be shown that this phenomenon is again due to the semiboundedness of the Hamiltonian; it constitutes the essence of the so-called quantum *Zeno paradox* (see Ref. 39, and also Ref. 40). However, under the conditions of real experiments we are still far from this paradoxical situation[35,38,41]; in the following section, we shall illustrate this by a simple model. Other questions related to continuous observation will be discussed in the final section.

4. A MODEL: DECAY OF CHARGED KAONS IN A BUBBLE CHAMBER

We consider in more detail the behavior of unstable particles in a bubble chamber. In Eqs. (12) we substitute $W(t,t') = \exp[-\lambda(t)(t'-t)]$, where λ varies slowly with the time, i.e., $|d\lambda^{-1}(t)/dt| \ll 1$. If the functions P and f vary slowly (in the same time scale), then the values of the integrals in (12) are not very sensitive to replacement of the function W by another function possessing the following properties:

a) $W(t,t')$ is essentially nonzero only for $t' - t \lesssim \lambda^{-1}(t)$;

b) $\int_t^\infty W(t,t')dt' = \lambda^{-1}(t)$, i.e., the mean distance between bubbles (measurements) is conserved.

The best modification is

$$W(t,\ t') = \begin{cases} 1, & t \leqslant t' < t + \lambda^{-1}(t); \\ 0, & t + \lambda^{-1}(t) \leqslant t', \end{cases} \qquad (17a)$$

which makes it possible to find an explicit expression for $E(\cdot)$. We assume further that the primary decay law is expressed in the form

$$P(t) = N^2 (e^{-\Gamma t} + M(t)). \qquad (18)$$

We assume that an almost monoenergetic beam passes through the chamber. Then it is natural to express all quantities as functions of the position x of the particle. In addition, the primary decay law must be expressed in the proper time: If the particle moves between two measurements at the points x_1 and x_2 with velocity βc, then the corresponding time difference is

$$t_2' - t_1' = (c\beta\gamma)^{-1}(x_2 - x_1),\ \gamma = (1 - \beta^2)^{-1/2}.$$

Our model assumption (17a) states that the measurements are made at the points $x_1, x_2, x_3, \ldots$, and

$$x_{i+1} - x_i = L(x_i), \quad \left|\frac{d}{dx} L^{-1}(x)\right| \ll 1. \qquad (17b)$$

From the assumptions (17) and (18) we can readily deduce[35] the form of the measured decay law:

$$E(x) = E_0(x)\, e^{K(x)},$$

where

$$E_0(x) = \exp\left\{-\Gamma \int_{x_0}^{x} \frac{d\xi}{c\beta(\xi)\gamma(\xi)}\right\}$$

is the exponential part, modified by the relations between the proper time and the laboratory time, and the correction due to the nonexponential behavior of P is

$$K(x) \approx \int_{x_0}^{x} [2 \ln N + M(\Delta t'(\xi))\, e^{\Gamma \Delta t'(\xi)}] \frac{d\xi}{L(\xi)}, \qquad (19)$$

where $\Delta t'(\xi) = L(\xi)[c\beta(\xi)\gamma(\xi)]^{-1}$. As the primary decay law, we consider the simplest modification of an exponential law:

$$P(t) = \left| \frac{N\delta}{\pi} \int_{-\varepsilon}^{\varepsilon} e^{-iET} \frac{dE}{E^2 + \delta^2} \right|^2, \qquad (20)$$

where $\delta = \frac{1}{2}\Gamma$ and N is a normalization factor. For us, only the case $\varepsilon \gg \delta$, $\delta t \gg 1$ is important; for it, we readily find

$$N \approx 1 + \frac{2\delta}{\pi\varepsilon}; \qquad (21a)$$

$$|M(t)| \leqslant \frac{4\delta}{\pi\varepsilon}. \qquad (21b)$$

The function $M(\cdot)$ can be expressed explicitly.[35] If we restrict ourselves to the terms of lowest order in δ/ε, δt, we obtain

$$M(t) \approx \frac{4\delta}{\pi\varepsilon} [\cos(\varepsilon t) + \varepsilon t\ \mathrm{si}(\varepsilon t)]\, e^{-\delta t}. \qquad (21c)$$

This function oscillates with a decreasing amplitude; its graph is shown in Fig. 2 together with the corresponding $P(\cdot)$. As particles, we choose charged kaons:

$$m = 493\ \mathrm{MeV}/c^2,\ \Gamma = 5.32 \cdot 10^{-14}\ \mathrm{MeV}.$$

The chamber is characterized by the mean distance between the bubbles, which is defined by

$$L = C\beta^2. \qquad (22a)$$

Further, it is necessary to know the dependence of β on x. It is given by the energy losses, for which we take instead of the Bethe–Bloch formula the approximate expression

$$-dE/dx = D\beta^{-2}. \qquad (22b)$$

We choose the parameters to make them correspond to a hydrogen chamber under normal conditions[42]:

$$C = 0.07\ \mathrm{cm},\quad D = 0.155\ \mathrm{MeV/cm}.$$

Then for $\beta \lesssim 0.9$, the error in the approximate expression does not exceed 20%, which is sufficient for a rough estimate. By means of (21) and (22) we can calculate[35] the correction (19):

$$K(x) \approx \frac{2}{\pi C} \frac{\delta}{\varepsilon} \left\{x - \frac{mc^2}{D}\left[\sqrt{F^2 - 4} - \sqrt{\left(F - \frac{D}{mc^2}x\right)^2 - 4}\right]\right\},$$

where $F = \gamma(0) + \gamma(0)^{-1}$ and $\gamma(0)$ is the Lorentz factor at $x = 0$; a graph of the function $K(\cdot)$ for four values of the initial momentum of the kaon is shown in Fig. 3. Figure 4 shows the arrangement of an experiment in which one can (in principle at least) measure the correction $K(x)$. A change in the primary decay law can be achieved artifically by means of an energy filter, the role of which could be played, for example, by an apparatus like a mass spectrometer. To dispense with $E_0(\cdot)$, it is necessary to compare the fractions of particles that pass through the plane x for different widths ε_1 and ε_2 of the energy gap. In particular, for $\varepsilon_1 \ll \varepsilon_2$ we obtain

$$\frac{N(\varepsilon_1, x) - N(\varepsilon_2, x)}{N(\varepsilon_2, x)} \approx K(\varepsilon_1, x).$$

Bearing in mind the attainable energy resolutions ε, we conclude[35] that the considered effect has the order $\lesssim 10^{-7}$. This number can be increased by considering instead of kaons suitable unstable nuclei, for which the ionization, i.e., the density of measurements, is Z^2 times greater. However, even in this case it is doubtful whether an experimental confirmation can be achieved.

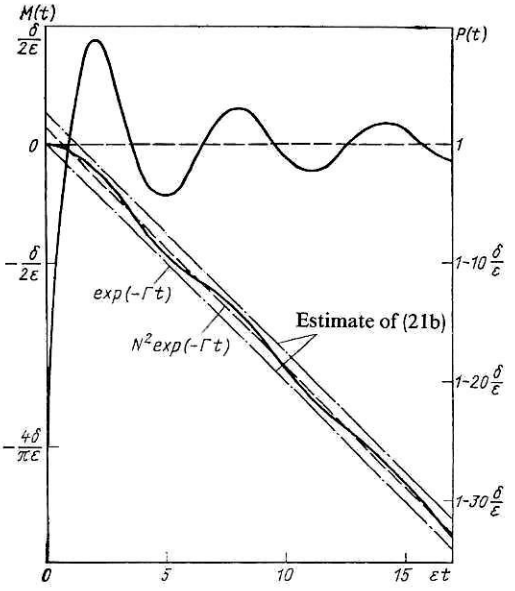

FIG. 2. The decay law (20) and the function (21b) for $\delta \ll \epsilon$, $\delta t \ll 1$.

5. BOUNDED-ENERGY APPROXIMATION

The results of the previous section show that the influence of the "energy cutoff," i.e., the approximation of the Hamiltonian by a bounded operator, on the behavior of the decay law may be slight. This is probably true in all cases that can be realized in practice even if the deviations are enhanced in the process of repeated measurements. On the other hand, one must consider how such an approximation

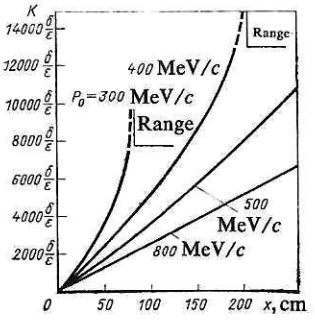

FIG. 3. The correction (19) to the measured decay law.

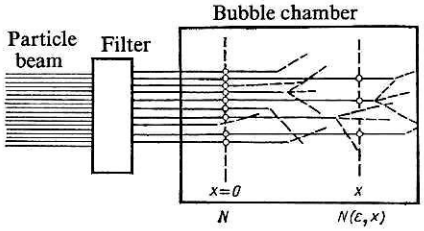

FIG. 4. Arrangement of measurement of deviations from an exponential decay law.

influences other problems of decay theory, in particular, the inverse decay problem.

We introduce the concept of a *state with bounded energy*: This is any state ρ for which there exists $b>0$ such that

$$\mathrm{Tr}\,(\rho E_\lambda) = \begin{cases} 0, \ldots \lambda \leqslant -b, \\ 1, \ldots \lambda > b \end{cases}$$

[see (6)]; we denote the set of such states by $B(H)$. The simple properties of such states reduce to the following[26]:

THEOREM 5. a) For any state ρ there exists a one-parameter family $\{\rho_b\} \subset B(H)$ such that $\lim_{b\to\infty} \mathrm{Tr}|\rho - \rho_b| = 0$.

b) For a state of an unstable system $\rho \in B(H)$, the decay law $P_\rho(\cdot)$ is a restriction of an entire function to $[0,\infty)$.

The first proposition shows how it is possible for $\rho \in B(H)$ to strengthen Theorem 1; at the same time, the question of the boundedness of the energy in a given state again cannot be resolved experimentally. The family $\{\rho_b\}$ can be constructed explicitly; for example,

$$\rho_b = N_b E^{(b)} \rho E^{(b)}, \tag{23}$$

where $E^{(b)} = E_H(\Delta_b), \Delta_b = (-b, b)$ and N_b is a normalization factor. The relation (23) is in fact tantamount to an "energy cutoff." We obtain the state ρ_b after performing (with a positive result) on the system in the sate ρ a "yes–no" experiment[43]: "Does the value of the total energy belong to the interval Δ_b?"

We now consider how the approximation influences the description of decay. In accordance with postulate (b) in Sec. 1 we have $[E_u, E^{(b)}] \neq 0$ in the general case, i.e., the inclusion $\mathrm{Ran}\,\rho_b \subset \mathscr{H}_u$ may not hold. In connection with the interpretation mentioned above, it is natural to take $\mathscr{H}_u^b = \overline{E^{(b)}\mathscr{H}_u}$ as an *approximate state space* (for the given unstable systems and the parameter b). Similarly, we define an *approximation reduced evolution operator*,

$$V_b : V_b(t) = E_u^{(b)} U(t) E_u^{(b)}, \quad t \in \mathbf{R}.$$

The main question is the sense in which $E_u^{(b)}$ approximates E_u as $b\to\infty$. Above all, the following possibilities are of interest:

i) the strong operator limit (S): $s - \lim_{b\to\infty} E_u^{(b)} = E_u$;

ii) the limit in the operator norm (N): $u - \lim_{b\to\infty} E_u^{(b)} = E_u$.

Of course, (S) follows from (N). If $\dim \mathscr{H}_u < \infty$, it can be shown[26] that (N) is satisfied. On the other hand, (N) is in general false for $\dim \mathscr{H}_u = \infty$; the question of the validity of (S) remains open in this case. We have the following theorems[26]:

THEOREM 6. It follows from (S) that $s - \lim_{b\to\infty} V_b(t) = V(t)$ and $\lim_{b\to\infty} P_{\rho_b}(t) = P_\rho(t)$ for all $t \in \mathbf{R}$, $\mathrm{Ran}\,\rho \subset \mathscr{H}_u$. If, in addition, (N) holds, then $u - \lim_{b\to\infty} V_b(t) = V(t)$ and the limit for the decay laws is uniform in $\mathbf{R}$.

THEOREM 7. If $\{\mathscr{H}, U(\cdot), E_u\}$ is a minimal unitary extension of the operator function $V(\cdot)$, then the minimal unitary extension of the approximate reduced evolution operator $V_b(\cdot)$ is $\{\mathscr{H}^b = E^{(b)}\mathscr{H}, U_b(\cdot) = E^{(b)}U(\cdot), E_u^{(b)}\}$.

These results admit a natural physical interpretation, a detailed discussion of which is given in Ref. 26. The main value of the considered approximation is that it enables us to justify the use of the Weisskopf–Wigner method, despite its noted shortcomings, on deeper theoretical foundations.

6. QUANTUM-MECHANICAL PSEUDO-HAMILTONIANS

In quantum mechanics, non-self-adjoint operators are frequently used for the phenomenological description of nonisolated systems. An example is the optical model for particle scattering by nuclei.[44–47] The time development of the system is then determined by a Schrödinger-type equation,

$$\mathrm{i}\frac{d\psi}{dt} = H_p\psi, \tag{24}$$

where H_p is the phenomenological non-self-adjoint Hamiltonian on the Hilbert space $\mathscr{H}_p$. As a rule, we are interested only in dissipative systems, i.e., systems for which

$$\mathrm{Im}\,(\psi, H_p\psi) \leqslant 0, \quad \psi \in D(H_p). \tag{25}$$

If H_p does not depend on the time, then it follows formally from (24) and (25) that the time evolution in $\mathscr{H}_p$ is described by the contractive semigroup

$$V(t) = \exp(-\mathrm{i}H_p t), \quad t \geqslant 0. \tag{26}$$

Thus, there is a natural connection between non-self-adjoint Hamiltonians and the Weisskopf–Wigner condition. If, in the spirit of the foregoing sections, we wish to include a phenomenological description in the standard quantum-theoretical formalism, it is necessary to find a minimal unitary extension of the semigroup $\{V(t)\}$. The difficulties associated with the unboundedness of the Hamiltonian that is then obtained can be overcome as before—the phenomenological description does not pretend to absolute accuracy, and therefore a weak violation of the condition (2) is perfectly acceptable.

In practice, one most often encounters phenomenological Hamiltonians of the type

$$H_p = -\frac{1}{2}\Delta + U, \tag{27}$$

where Δ is the Laplace operator on $L^2(\mathbf{R}^d)$ and U is a "potential," in particular, an operator of multiplication by a complex function $U(\cdot)$, $\mathrm{Im}\,U(x) \leqslant 0$. To such cases, one cannot

directly apply the arguments of the previous section [one should first verify the fulfillment of condition (S)]. However, the approximate nature of the condition (2) can also be interpreted in another way, namely, instead of assuming that the total Hamiltonian is given and changing the state (by adding an energy filter), we can assume that the set of states is given (i.e., the space $\mathscr{H}_p$) and regard the "approximation" as a manifestation of the fact that it is not possible to determine the total Hamiltonian with complete accuracy. If we assume that the semigroup $\{V(t)\}$ is continuous and that the accuracy of the semigroup approximation depends on the state of the system, we can formulate the requirement given above more concretely: For all $\varepsilon>0$, $T>0$ and an arbitrary finite subset $M\subset\mathscr{H}_p$ there exists a triplet $\{\mathscr{H}', H', P'\}$ such that:

a) $\mathscr{H}_p = P'H'$;

b) H' is self-adjoint and semibounded;

c) $\|(P'e^{-iH't} - V(t))\psi\| < \varepsilon$ for all $\psi\in M$, $t\in[0,T]$.

The finiteness of the set M and of the time interval $[0,T]$ does not present difficulties, since in every real experiment the accuracy of the semigroup approximation is verified under precisely such conditions. It can be shown that $\mathscr{H}'$ and P' can be chosen independently of ε, T, M (Ref. 48):

THEOREM 8. Let $\{V(t)\}$ be a continuous contractive semigroup and $\{\mathscr{H}, U(t) = e^{-iHt}, P\}$ be its minimal unitary extension. For all $\varepsilon>0$, $T>0$ and finite $M\subset\mathscr{H}_p$ there exists a sequence $\{f_n\}$ of real Borel functions such that:

a) $\{\mathscr{H}, f_n(H), P\}$ satisfies conditions (a)–(c) for all sufficiently large n;

b) $f_n(H)$ converges to H in the strong resolvent sense.

Thus, we can formulate a definition: A linear operator H_p on $\mathscr{H}_p$ is a *pseudo-Hamiltonian* if iH_p generates a continuous contractive semigroup $\{V(t)\}$ on $\mathscr{H}_p$ (in particular, this means that H_p is densely defined and closed). We shall call the self-adjoint operator H on $\mathscr{H}\supset\mathscr{H}_p$ that generates the minimal unitary extension of the semigroup $\{V(t)\}$ the *quasi-Hamiltonian* corresponding to H_p.

The basic criterion for a densely defined closed operator H_p to be a pseudo-Hamiltonian follows directly from the Hille–Yosida theorem (see Refs. 48 and 49, Theorem X.48). In particular, for this, fulfillment of the conditions (25) and $\mathrm{Im}(\psi, H_p^+\psi)\leqslant 0$ for all $\psi\in D(H_p^+)$ are sufficient. We shall consider in what follows generalized Schrödinger operators of the type (27) with a local complex potential U. We denote by H_0 the operator on $\mathscr{H}_p$ whose action on vectors of the domain of definition $D(\Delta)\cap D(U)$ is given by (27). If we also denote by $\overline{H}$ the closure of the operator H_0, then we have the following theorem[48]:

THEOREM 9. Let H be a generalized Schrödinger operator on $L^2(\mathbf{R}^d)$ with a local complex potential U. The condition

$$\mathrm{Im}\ U(x)\leqslant 0 \tag{28}$$

almost everywhere in $\mathbf{R}^d$ is necessary if H is to belong to the class of pseudo-Hamiltonians. If, in addition, H^+ is the generalized Schrödinger operator corresponding to the complex-conjugate potential $\overline{U}$, i.e., if $H^+ = \overline{-1/2\Delta + U^+}$, then the condition (28) is at the same time sufficient.

Other sufficient conditions more suitable for practical applications can be deduced by means of perturbation theory, in the first place by the following lemma of Kato–Rellich type (see Refs. 48 and 49, Sec. X.8):

LEMMA. Let H be a densely defined operator that admits closure and such that $\overline{H}$ is a pseudo-Hamiltonian, and let B be a closed accretive operator, $D(B)\supset D(H)$. If there exist non-negative constants $a<1$ and b such that

$$\|B\varphi\|\leqslant a\|H\varphi\| + b\|\varphi\| \quad \text{for all} \quad \varphi\in D(H),$$

then $D(\overline{H})\subset D(B)$ and the operator $\overline{H} - iB$ is closed and belongs to the class of pseudo-Hamiltonians.

From this, for example, we obtain a sufficient condition for generalized Schrödinger operators on $L^2(\mathbf{R}^d)$, $d\leqslant 3$ (Ref. 48):

THEOREM 10. Let H be a generalized Schrödinger operator on $L^2(\mathbf{R}^d)$, $d\leqslant 3$, with a local complex potential U satisfying the condition (28). If $U\in L^2(\mathbf{R}^d) + L^\infty(\mathbf{R}^d)$, then $D(H) = D(\Delta)$ and $H = -\frac{1}{2}\Delta + U$ is a pseudo-Hamiltonian (without closure!).

Other properties of pseudo-Hamiltonians, in particular the connection between the eigenvalues of a pseudo-Hamiltonian and the spectral characteristics of the quasi-Hamiltonian corresponding to it, are considered in Ref. 48. We note further that not only local absorptive potentials are of interest from the point of view of physical applications. As an example, we may mention Feshbach's nonlocal optical potential.[45,47,50] The scattering theory for pseudo-Hamiltonians is still far from completion. As yet, there are only a number of interesting special results.[47,50–52]

7. FEYNMAN INTEGRALS—GENERAL OUTLINE

The mathematical methods generated by physics are frequently doubly fruitful—first as a convenient but more or less formal computational technique, but then as a theory based on a rigorous treatment of the original idea in a suitable mathematical language. For example, the δ function is extremely helpful for calculations, but the full strength of this idea (which undoubtedly exceeds Dirac's expectation) appeared only after the creation of the theory of generalized functions. It is therefore easy to understand the attraction of attempts to construct a mathematically correct theory of the Feynman integral. We here discuss several approaches to this problem; a more complete treatment can be found in a large number of reviews and monographs; see, for example, Refs. 53–56, which discuss the mathematical aspect of the problem, and Refs. 57–59 and 107, which give some physical applications. We mention also the monographs of Refs. 60 and 61, which are devoted to the application of the mathematically well-defined methods of functional integration (in the first place, the Wiener integral) in quantum physics.

The point of interest in physics applications is the description of dynamics by means of path integrals. If we restrict ourselves to the simplest example of a single spinless particle with free Hamiltonian $H_0 = -\hbar^2/2m\Delta$ interacting with an external field described by the potential V, then Feynman's famous result (see Refs. 62 and 63, Chap. 3) states that the wave function at the time t is equal to

$$\left(\exp\left[-\frac{\mathrm{i}}{\hbar}(H_0+V)t\right]\psi\right)(x)=\int_{\Gamma_x}\exp\left\{\frac{\mathrm{i}}{\hbar}S(\gamma)\right\}\psi(\gamma(0))\,D\gamma, \tag{29}$$

where ψ is the wave function at the initial time $t=0$,

$$S(\gamma)=\int_0^t\left[\frac{1}{2}m\,|\,\dot{\gamma}(\tau)\,|^2-V(\gamma(\tau))\right]d\tau \tag{30}$$

is the classical action along γ, and Γ_x denotes the space of all paths γ whose ends lie at the point x (in what follows, we shall again assume $\hbar=c=1$ and where possible also $m=1$). Thus, the main task is to interpret the formal expression on the right-hand side of Eq. (29) or more general expressions of the type

$$\int_{\Gamma_x}\exp\left\{\frac{\mathrm{i}}{2s}\int_0^t|\,\dot{\gamma}(\tau)\,|^2\,d\tau\right\}f(\gamma)\,D\gamma, \tag{31}$$

where f is a complex functional on the space of paths Γ_x and s is a real parameter, or, finally, even more general operator Feynman integrals.[64–66]

The first and apparently most popular way to achieve the result was proposed by Feynman in his original paper.[62] It consists of replacing the set of all paths by a set of broken paths [the particle velocity is assumed constant in the intervals of time $(j\delta, j\delta+\delta)$, $\delta=t/n$, $j=0,1,\ldots,n-1$]; in this case, (31) can be defined naturaly as an integral over the corresponding finite-dimensional space of paths. The construction is completed by going to the limit $n\to\infty$. Clearly, it is here not necessary to make a restriction to equidistant partitions; any sequence of partitions $\sigma=\{\tau_j:0=\tau_0<\tau_1<\cdots<\tau_n=t\}$ of the interval $[0,t]$ is suitable for this purpose, provided the lengths of the subintervals in it tend to zero. Above all, the following property is important: For cylindrical functionals (roughly speaking, these are the functionals that depend only on a "finite number of variables") the following relation holds:

$$\int_{\Gamma_x}\exp\left\{\frac{\mathrm{i}}{2s}\int_0^t|\,\dot{\gamma}(\tau)\,|^2\,d\tau\right\}f(\gamma(\tau_0),\ \ldots,\ \gamma(\tau_{n-1}))\,D\gamma$$
$$=\prod_{j=0}^{n-1}(2\pi\mathrm{i}s\delta_j)^{-d/2}\int_{\mathbf{R}^d}\exp\left\{\frac{\mathrm{i}}{2s}\sum_{k=0}^{n-1}$$
$$\times\,|\,\gamma_{k+1}-\gamma_k\,|^2\,\delta_k^{-1}\right\}f(\gamma_0,\ \ldots,\ \gamma_{n-1})\,d^d\gamma_0,\ \ldots,\ d^d\gamma_{n-1}, \tag{32}$$

where $\delta_j=\tau_{j+1}-\tau_j$ and d denotes the dimension of the configuration space. This relation is so natural that it seems reasonable to require its fulfillment for any definition of the functional integral (31).

We note that one of the possible constructions of the Wiener measure begins precisely with the relation (32) for $s=-i$ (see Ref. 49, Sec. X.11, and Ref. 61). In this case, however, one can carry out the task in the framework of measure theory, despite the fact that in an infinite space there does not exist a translationally invariant measure of Lebesgue type (see Refs. 55 and 67, Chap. 1)—roughly speaking, the singularities of the exponential term and $D\gamma$ cancel and the formal expression

$$\exp\left\{-\frac{1}{2}\int_0^t|\,\dot{\gamma}(\tau)\,|^2\,d\tau\right\}D\gamma$$

can be replaced by $dw(\gamma)$, where w is the Wiener measure. On the other hand, analogous arguments cannot be applied to the Feynman integral, since the exponential term in this case is an oscillating function. This difficulty could be overcome by defining the Feynman integral as a limit (when the parameter s reaches the real axis from below) if suitable measures existed for s in the lower complex half-plane (this suggestion is due to Gel'fand and Yaglom[68]). Unfortunately, these measured exist if and only if $s=-ic$, $c>0$.

Wiener integrals could themselves also be used to define and calculate Feynman integrals. Such a method is based on analytic continuation: Assume, for example, that we know the Wiener integral $I_S(f)=\int_\Gamma f(\gamma)dw_{is}(\gamma)$, where Γ is the corresponding space of paths and w_{is} is the Wiener measure with dispersion is (see Ref. 67, §13) for $1-\delta\leqslant is\leqslant 1$ for some $\delta>0$ and that $I_s(f)$ are the boundary values of a function $s\to I_s(f)$ analytic (for example) in the region shown in Fig. 5; then the Feynman integral of f can be defined as the analytic continuation of $I_s(f)$ to the point $s=1$. Note that since s is proportional to m^{-1}, this definition reduces to analytic continuation with respect to the mass; it is due to Cameron,[71] and various modifications of it can be found, for example, in Refs. 64–66.

An alternative approach is based on analytic continuation with respect to the time; among its applications, we can mention in the first place the constructive methods of Euclidean quantum field theory, which have been so actively developed during the last decade.[66,72] The description of the dynamics in this approach starts from an expression for $\exp(-(H_0+V)t)$ analogous to (29). This is the so-called Feynman–Kac formula, which was first obtained by Kac in 1951 (see Ref. 49, Sec. X.11).

The following group of definitions develops at a rigorous level Feynman's original idea in the sense that "integrals" of the type (31) are obtained in them "sequentially," i.e., as the limit of some sequence of "finite-dimensional" integrals. In this way, Nelson[64] was one of the first who succeeded in deriving a rigorous variant of Eq. (29). His method was based on the Lie–Trotter formula; this necessitated a small departure from Feynman's heuristic proposal, namely, the functional $\exp\{-i\int_0^t V(\gamma(\tau))d\tau\}$ had to be replaced

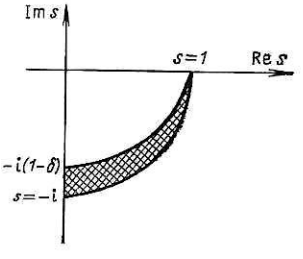

FIG. 5. Definition of the Feynman integral by analytic continuation with respect to the mass.

in the approximating integrals by expressions containing Darboux–Riemann sums $\exp\{-i\sum_{j=0}^{n-1} V(\gamma(\tau_j))\delta_j\}$, where $\tau_j = jt/n$, $\delta_j = t/n$, $j = 0,\ldots,n-1$. If one takes, for example, dimension $d \leqslant 3$ of the configuration space, Nelson's result applies to potentials $V \in L^2(\mathbf{R}^d) + L^\infty(\mathbf{R}^d)$. It can be further generalized to the case of potentials that depend on the time,[73] for the harmonic oscillator,[74] etc. Generally speaking, sequential methods of this type enjoy the greatest popularity in the physics literature.

There are also definitions that do not require "finite-dimensional" approximations. Instead of this, one adds to the formal "Feynman measure" a further exponential factor, which transforms it into a good Gaussian measure on the given space of paths; this factor is then eliminated in a limiting process. Such an approach is due to Itô[75]; in the framework of this approach one can prove the basic dynamical relation (29) for $d = 1$ for various potentials: $V \in \mathcal{F}(\mathbf{R})$, i.e., for potentials that are the Fourier transform of some bounded measure on the configuration space (note that potentials of this class are bounded and continuous), and also for a linear potential and the harmonic-oscillator potential. We mention also the method recently proposed by Tarski (see Ref. 54, pp. 254–279), which to some extent combines the "finite-dimensional" (cylindrical) approximations with Itô's method.

The next group of definitions essentially uses Fourier transformations. It includes the DeWitt–Morette method,[76,77] in which the nonexisting Feynman measure is replaced by a "projective generalized function" (projective distribution); the latter is defined by its Fourier transform. By this method one can calculate various path integrals of interest in physics.[55] For us, the most important is the definition of Alberverio and Höegh-Kron (see Refs. 53 and 54, pp. 8–13), in which the space of paths has the structure of a Hilbert space (we denote it by $\mathcal{H}$) and the "integrable" functionals are those that are the Fourier transforms of finite measures on $\mathcal{H}$ (more detailed information will be given in the next section). The basic dynamical relation (29) has been proved in the framework of this approach for potentials $V \in \mathcal{F}(\mathbf{R}^d)$.

However, the original definition of Alberverio and Köegh-Kron does not make it possible to "integrate" some functionals important from the physics point of view, for example, the exponential functional corresponding to the harmonic-oscillator potential. Therefore, various generalizations have been proposed. The first of them is again due to Alberverio and Höegh-Kron (see Refs. 53 and 54, pp. 19–23); it is based on replacement of the standard scalar product in the space of paths [see (34)] by a quadratic form that is not necessarily positive definite. In other words, the exponential term in the "Feynman measure" may in this approach also contain a factor from the potential energy. A serious shortcoming of the generalized definition is that it in general is not an extension of the original definition. A further generalization in this direction can be found in Ref. 86.

One can attempt to extend the class of "integrable" functionals by directly adding to it the Fourier transforms of generalized functions. Partial results have been obtained in this direction[78]; the main obstacle here is the fact that little is known about generalized functions of an "infinite number of arguments." Approximation by broken paths present another possibility of extension. The pioneering work in this approach is due to Truman (see Refs. 70, 79, 80, and 54, pp. 73–102, and also Ref. 69); a further generalization and detailed discussion can be found in Ref. 81. An attractive feature of this approach is that it is closer to Feynman's heuristic arguments than Nelson's method, since in the approximations one encounters integrals of the exponential of the exact classical action and not the Darboux–Riemann sums that approximate it.

We shall discuss in more detail the approaches to the definition of Feynman integrals of the type proposed by Nelson, Alberverio, Höegh-Kron, and Truman. To conclude this section, we mention some other definitions and problems. It is sometimes convenient to consider, not the Feynman integral separately, but so-called Feynman maps[70,82] (this concept is intimately related to Cameron's complex Wiener integral[66,69,71]), i.e., a family of maps characterized by a complex parameter s [see (31)]. This approach makes it possible to combine to a certain extent the methods based on analytic continuation and the sequential methods and to treat (for a definite class of functionals) Feynman and Wiener integrals on a common basis. We note further that in the above arguments we restricted ourselves to functional integrals in the configuration space, i.e., "Lagrangian" integrals. With regard to functional integrals in phase space, or "Hamiltonian" integrals, which were also introduced by Feynman,[83] we can refer, for example, to Refs. 56, 84, and 85 and the literature given in them. The Feynman integral in phase space can also be defined by means of Fourier transformation—the exponential of the potential term generates (for a certain class of potentials) a generalized measure with respect to which the exponential of the kinetic part of the action is integrated.[87] We note also that the Hamiltonian form of the Feynman integrals is convenient for considering questions related to quantization,[56,84] though this circumstance does not play an important part for Hamiltonians of Schrödinger type, since for them all quantizations are the same. Finally, we shall not consider here the quasiclassical approximation or the use of Feynman integrals to find the classical limit of the dynamics of quantum systems; this is discussed, for example, in Refs. 88–90.

8. SCHRÖDINGER PSEUDO-HAMILTONIANS AND FEYNMAN INTEGRALS

We begin with a more detailed description of some of the above definitions of the Feynman integral. The essence of the Alberverio–Höegh-Kron (AH) approach is a relation like Parseval's equation:

$$(2\pi i)^{-n/2}\int_{\mathbf{R}^n} \exp\left\{\frac{i}{2}|x|^2\right\} f(x)\, d^n x = \int_{\mathbf{R}^n} \exp\left\{-\frac{i}{2}|y|^2\right\} \tilde{f}(y)\, d^n y, \tag{33}$$

where $f(x) = \int_{\mathbf{R}^n} e^{-ix\cdot y}\tilde{f}(y)d^n y$, which is satisfied, for example, for any rapidly decreasing function, $f \in \mathcal{S}(\mathbf{R}^n)$. The left-hand side of the equation is an expression of the type of a Feynman integral [see (31)], and therefore it is natural to use

the right-hand side as a definition in the case when $\mathbf{R}^n$ is replaced by the (infinite-dimensional) space of paths and the left-hand side becomes meaningless. Thus, the point of departure of the AH definition is the choice of the space $\mathscr{H}$ of paths, which is here ascribed the structure of a real separable Hilbert space. In accordance with Eqs. (31) and (32), it is expedient for a quantum-mechanical system with d degrees of freedom, its configuration space being $\mathbf{R}^d$, to take as $\mathscr{H}$ the set $AC_0[J^t;\mathbf{R}^d]$, i.e., the set of all paths γ in $\mathbf{R}^d$ corresponding to the time interval $J^t=[0,t]$ that are absolutely continuous, have square-integrable derivatives, and for which $\gamma(t)=0$. The norm in the space $\mathscr{H}$ is given by

$$\|\gamma^2\| = (\gamma,\gamma) = \int_0^t |\dot{\gamma}(\tau)|^2\, d\tau. \tag{34}$$

Note that paths in the space $\mathscr{H}$ can be characterized equivalently by trigonometric series.[79,91] We denote the set of "integrable functionals" by $\mathscr{F}(\mathscr{H})$; it consists of all $f:\mathscr{H}\to\mathbf{C}$ of the form

$$f(\gamma) = \int_{\mathscr{H}} e^{i(\gamma,\gamma')}\, d\mu_f(\gamma'), \tag{35}$$

where μ_f is some finite complex Borel measure on the space of paths $\mathscr{H}$. For such functionals, we define by analogy with Eq. (33) the *Feynman integral* $I_{\mathrm{AH}}:\mathscr{F}(\mathscr{H})\to\mathbf{C}$ by

$$I_{\mathrm{AH}}(f) \equiv \overset{\mathrm{AH}}{\int_{\mathscr{H}}} \exp\left\{\frac{i}{2}\|\gamma\|^2\right\} f(\gamma)\, D\gamma := \int_{\mathscr{H}} \exp\left\{-\frac{i}{2}\|\gamma\|^2\right\} d\mu_f(\gamma). \tag{36}$$

The properties of the mapping $I_{\mathrm{AH}}(\cdot)$ are considered in detail in, for example, Refs. 53, 79, and 82. We mention some of the most important properties:

i) *normalization*: for the unit functional the relation $I_{\mathrm{AH}}(1)=1$ holds;

ii) $I_{\mathrm{AH}}(f)$ for a cylindrical functional f can be expressed by a formula of the type (32);

iii) when $\mathscr{H}$ can be decomposed into an orthogonal sum $\mathscr{H}_1\oplus\mathscr{H}_2$, there is an *analog of Fubini's theorem*;

iv) there are *transformation formulas* for linear inhomogeneous substitution of the "integration variable" γ analogous to the Cameron–Martin formulas for the Wiener integral.

It should, however, be emphasized that the designation "integral" for the mapping $I_{\mathrm{AH}}(\cdot)$ is only nominal. For example, it does *not* have one of the most important properties of a Lebesgue integral—the theorem on majorizable convergence does not hold, and a counterexample is demonstrated in Ref. 82. This has certain consequences, for example, for the rigorous justification of heuristic arguments about the classical limit of quantum mechanics.[79]

We now turn to the extension of the AH definition by means of broken paths. Let σ be some partition $\{\tau_k:0=\tau_0<\tau_1<\ldots<\tau_n=t\}$ of the interval J^t. We denote $\delta_k=\tau_{k+1}-\tau_k$ and $\delta(\sigma)=\max_{0\le k\le n-1}\delta_k$. To each σ there corresponds a projection operator $P(\sigma)$ on the space $\mathscr{H}$:

$$(P(\sigma)\gamma)(\tau) = \gamma(\tau_k) + [\gamma(\tau_{k+1})-\gamma(\tau_k)]\,\delta_k^{-1}(\tau-\tau_k) \tag{37}$$

(if $\tau\in[\tau_k,\tau_{k+1}]$), which associates with any γ a broken path $P(\sigma)\gamma$, which passes through the points $\gamma(\tau_k)$, $k=0,1,\ldots,n$. Then we can define the mapping $I_{PP}(\cdot)$:

$$I_{PP}(f) := \lim_{\delta(\sigma)\to 0} I_{\mathrm{AH}}(f\circ P(\sigma)), \tag{38}$$

if $f\circ P(\sigma)\in\mathscr{F}(\mathscr{H})$ for every partition σ and the given limit exists. The definition (38) is an extension of the AH definition—by the reproduction-kernel technique one can show[81] that $I_{PP}(f)$ exists and is equal to $I_{\mathrm{AH}}(f)$ for any $f\in\mathscr{F}(\mathscr{H})$.

Another formulation is also possible. Using the fact that the functional $f\circ P(\sigma)$ is cylindrical and the properties of $I_{\mathrm{AH}}(\cdot)$, we can rewrite the relation (38) in the form

$$I_{PP}(f) = \lim_{\delta(\sigma)\to 0} [(2\pi i)^n \delta_0\delta_1\ldots\delta_{n-1}]^{-d/2} \times \int_{\mathbf{R}^d}\ldots\int_{\mathbf{R}^d} \exp\left\{\frac{i}{2}\sum_{k=0}^{n-1}|\gamma_{k+1}-\gamma_k|^2\,\delta_k^{-1}\right\} \times f_\sigma(\gamma_0,\ \ldots,\ \gamma_{n-1})\, d^d\gamma_0\ldots d^d\gamma_{n-1}, \tag{39}$$

where $\gamma_k=[P(\sigma)\gamma](\tau_k)$ and $f_\sigma(\gamma_0,\ldots,\gamma_{n-1})=f[P(\sigma)\gamma]$. The multiple integral may also exist (in the principal-value sense) for some functionals f for which $f\circ P(\sigma)$ does not belong to $\mathscr{F}(\mathscr{H})$. Therefore, the formulation of the definition[69,70,80] based directly on Eq. (39) is more general than the one given above, but in it the connection with the AH definition is lost. In addition, the calculation of multidimensional improper integrals is sensitive to the limit procedure,[55,82] i.e., one can here readily make an error.

Finally, we mention the definition of the Feynman integral on the basis of a Lie–Trotter product-type formula: For functionals $f=f_{V,\varphi,x,t}$ of the form

$$f(\gamma) = \exp\left\{-i\int_0^t V(\gamma(\tau)+x)\, d\tau\right\}\varphi(\gamma(0)+x) \tag{40}$$

we define

$$I_{LT}(f) := \lim_{\delta(\sigma)\to 0} [(2\pi i)^n \delta_0\delta_1\ldots\delta_{n-1}]^{-d/2} \times \int_{\mathbf{R}^d}\ldots\int_{\mathbf{R}^d} \exp\{iS_\sigma(\gamma_0,\ \ldots,\ \gamma_n)\}\,\varphi(\gamma_0+x)\, d^d\gamma_0\ldots d^d\gamma_{n-1}, \tag{41}$$

where $\gamma_n\equiv 0$; the improper integrals are defined as $\int_{\mathbf{R}^d} d^d\gamma \equiv \lim_{c\to\infty}\int_{|\gamma|<c} d^d\gamma$, all limits are understood in the L^2 sense, and

$$S_\sigma(\gamma_0,\ \ldots,\ \gamma_n) = \sum_{k=0}^{n-1}\left[\frac{1}{2}\left(\frac{\gamma_{k+1}-\gamma_k}{\delta_k}\right)^2 - V(\gamma_k+x)\right]\delta_k. \tag{42}$$

If the potential V depends explicitly on the time, then $V(\gamma_k+x)$ in the last formula must be replaced by $V(\gamma_k+x,\tau_k)$ (see Refs. 73, 92). Comparison of $I_{PP}(f)$ for functionals of the type (40) with $I_{\mathrm{LT}}(f)$ shows the difference that we mentioned above: In the first case, the exponential contains the *exact action*, but in the other the *Riemann approximation* to it. In the considered cases, the two methods give the same results (see Theorem 11), it is an extremely

difficult problem to estimate their relative rate of convergence.[93]

After this preliminary review, we now consider how one can express the dynamics of a system describes by a Schrödinger pseudo-Hamiltonian by means of Feynman integrals. We begin with the case when all the Feynman integrals defined above exist and have the same values.[92,94]

THEOREM 11. Suppose $V \in \mathcal{F}(\mathbf{R}^d)$, i.e., that V is the Fourier transform of some finite complex Borel measure on $\mathbf{R}^d$. If, further, V satisfies the condition (28) of being dissipative, then $H_p = -1/2\,\Delta + V$ is a pseudo-Hamiltonian and the function ψ: $\psi(x,t) = I_{\mathrm{AH}}(f) = I_{PP}(f) = I_{\mathrm{LT}}(f)$, is given by Eq. (40), satisfies Eq. (24) with the initial data $\psi(\cdot,0) = \varphi$.

If a restriction is made to only one or two of the above definitions, the proposition of the theorem admits a number of generalizations:

i) Potentials in $\mathcal{F}(\mathbf{R}^d)$ are bounded and continuous, but the last assumption can be dropped: The theorem will hold for *any bounded* measurable potential V if I_{LT} is taken as the Feynman integral.[92] We note also that similar results were obtained recently[95–97] by means of the Cameron–Storvick analytic Feynman integral.[65,66,71]

ii) The *requirement of boundedness of the potential can also be given up*. In Ref. 92, for example, the theorem is proved for $V \in L^2(\mathbf{R}^d) + L^\infty(\mathbf{R}^d)$ when $d \leqslant 3$ (see Theorem 10) for the Feynman integral I_{LT} (the restriction on the dimension can here be lifted by replacing L^2 by a suitable L^p, since this is done for real-valued potentials[73]).

iii) For I_{LT} again one can admit *potentials that depend on the time*, provided the dependence is sufficiently smooth and the condition of being dissipative remains in force during the considered time interval.[92] A similar result for the Feynman integral I_{AH} and potentials with values in $\mathcal{F}(\mathbf{R}^d)$ follows from Ref. 97.

(iv) The *damped harmonic oscillator* for I_{LT} and I_{pp}, which will be considered in the following section.

9. EXAMPLE: DAMPED HARMONIC OSCILLATOR

To illustrate the above results, we consider as an example the multidimensional damped harmonic oscillator described by the pseudo-Hamiltonian

$$H = -\frac{1}{2}\Delta + x\cdot(A - \mathrm{i}W)x, \tag{43}$$

where Δ, as always, is the Laplace operator on $\mathbf{R}^d$ and A and W are $d \times d$ matrices with positive eigenvalues. In contrast to the majority of approaches to the description of a damped oscillator (see the review of Ref. 98), we do not attempt to obtain the pseudo-Hamiltonian (43) by quantizing a classical damped oscillator. Such an approach is meaningful only when one is considering a concrete quantum system for which the damping mechanism is to a reasonable degree similar to the classical case. In the spirit of the general discussion of pseudo-Hamiltonians (see Sec. 6), we shall only assume that the presence of an imaginary part in the potential is a phenomenlogical description of how our oscillator interacts with the medium surrounding it. Note also that compared with the known results we assume neither a time dependence of the frequency[99] nor the presence of external forces.[100] On the other hand, we do not restrict the dimension—and the generalization to the case $d > 1$ for a complex harmonic potential is nontrivial, since the matrices A and W do not in general admit simultaneous diagonalization.

It is first necessary to verify that (43) is indeed a pseudo-Hamiltonian, since $V(x) = x\cdot(A - iW)x$ does not belong to the class of potentials covered by Theorem 10. This can be proved by iterative application of the lemma from Sec. 6 (and is done in Ref. 101). It is then necessary to calculate the corresponding Feynman integral. This can be done in different ways. The simplest is to use the Lie–Trotter formula, i.e., to calculate the right-hand side of (41) for a sequence of equidistant partitions, as is done in Ref. 101. At the same time, the special choice of a sequence convenient for calculations does not have fundamental significance: The existence of $I_{\mathrm{LT}}(f)$ in this case follows from the theorem proved in Ref. 92. In addition, the use of the Lie–Trotter formula guarantees that the obtained expression is a contractive semigroup, i.e., that it satisfies an equation of the Schrödinger type (24) with pseudo-Hamiltonian (43). The calculation $I_{pp}(f)$ is more complicated. The right-hand side of the expression (39), where f is defined by (40) and (43), can again be calculated for equidistant partitions; this is done in Ref. 94, and the result agrees with $I_{\mathrm{LT}}(f)$. However, the expressions corresponding to a general sequence of partitions are too complicated; to establish the existence of $I_{pp}(f)$, it would be necessary to verify that $f \in \mathcal{F}(\mathcal{H})$, as is done in Ref. 97 for $A = 0$.

We now turn to a description of the results:

THEOREM 12. The operator (43) with natural domain of definition is a pseudo-Hamiltonian. The explicit form of the corresponding semigroup $V(t) = \exp(-iHt)$ can be obtained from the formula $[V(t)\varphi](x) = I_{\mathrm{LT}}(f_{V,\varphi,x,t})$ in the form

$$(V(t)\varphi)(x) = \int_{\mathbf{R}^d} G_t(x,\ y)\,\varphi(y)\,d^d y; \tag{44a}$$

$$G_t(x,\ y) = (2\pi\mathrm{i})^{-d/2}[\det(\Omega^{-1}\sin\Omega t)]^{-1/2} \times \exp\left\{\frac{\mathrm{i}}{2}[x\cdot(\Omega\cot\Omega t)x + y\cdot(\Omega\cot\Omega t)\,y] - \mathrm{i}y\cdot(\Omega\operatorname{cosec}\Omega t)\,x\right\}; \tag{44b}$$

$$\Omega = -[2(A - \mathrm{i}W)]^{1/2}. \tag{44c}$$

We consider the basic properties of the solution (44). We begin with the "*undamped*" *limit*: we assume $d = 1$, $\Omega = \omega - i\nu$ and that the function φ has compact support. If $\omega t \neq k\pi$, then

$$\lim_{\nu\to 0+}(V(t)\varphi)(x) = \int_{\mathbf{R}^d} K_t(x,\ y)\,\varphi(y)\,d^d y;$$

$$K_t(x,\ y) = (2\pi\mathrm{i})^{-d/2}\left(\frac{\omega}{|\sin\omega t|}\right)^{1/2} \times \exp\left\{\frac{\mathrm{i}}{2\sin\omega t}[(x^2+y^2)\cos\omega t - 2xy]\right\} \times \exp\left\{-\frac{\pi\mathrm{i}}{2}\operatorname{Ent}\frac{\omega t}{\pi}\right\},$$

where the last factor is none other than the Maslov correction. On the other hand, for $\omega t = k\pi$ and $\varphi \in \mathcal{S}(\mathbf{R}^d)$ we have

$$\lim_{\nu\to 0+}(V(t)\varphi)(x) = \exp\left\{-\frac{\mathrm{i}}{2}k\pi\right\}\varphi((-1)^k x).$$

Thus, we have obtained a new (and very natural) derivation of the "extended" Feynman formula.[102]

We consider further the *classical limit*. For simplicity, we restrict ourselves to wave packets in the form of a "displaced ground state" $\varphi = \varphi_{\alpha,\varkappa}$:

$$\varphi(x) = (\pi\lambda^2)^{-1/4} \exp\left\{-(2\lambda^2)^{-1}(x-\alpha)^2 + \frac{i}{\hbar}\varkappa x\right\}, \quad (45)$$

where $\Lambda^2 = \hbar(m\Omega)^{-1}$, $\lambda^2 = \hbar(m\omega)^{-1}$ (we again assume $d = 1$ and $\Omega = \omega - i\nu$). The propagator corresponding to nonunit values of the mass m and Planck's constant $\hbar$ can be obtained from (44b) by the substitution $t \to \hbar t m^{-1}$, $\Omega \to m\Omega\hbar^{-1}$. It follows from (44) and (45) that[101]

$$|(V(t)\varphi)(x)|^2 = (\pi\lambda^2)^{-1/2} \exp\{-\nu t + \lambda^2 (x - x_0(t))^2 + y(t)\},$$

where

$$x_0(t) = (\alpha \cos \omega t + \beta \sin \omega t)\, e^{-\nu t}, \beta = (m\omega)^{-1}(\varkappa - m\alpha\nu)$$

and the function y tends rapidly to a finite limit with increasing t. Thus, we obtain a Gaussian wave packet with the following properties:

a) the height of the peak decreases approximately as $e^{-\nu t}$;

b) its width λ is conserved; in the classical limit, when $\alpha^2 + \beta^2 \gg \lambda^2$, it can be ignored;

c) the peak moves along the path $x = x_0(t)$.

As we noted at the start of the section, the operator (43) is not obtained as a result of quantization, and therefore the classical limit may differ from the behavior of a classical damped oscillator. In fact, there is such a difference: The path corresponds to a classical oscillator with initial position α but with momentum $\varkappa - 2m\alpha\nu$ instead of $\varkappa$, which is the expectation value of the quantum-mechanical momentum operator in the state φ.

We note, however, that this difference can be ignored in the case of weak damping, when $\nu \ll \omega$.

For $d = 1$, it is also easy to find the *point spectrum* of the pseudo-Hamiltonian (43): The eigenvalues are $\lambda_n = \Omega(n + 1/2)$, $n = 0,1,2,...$ and they correspond to the eigenvectors

$$\psi_n : \psi_n(x) = N_{nn}^{-1/2} H_n(\Omega^{1/2}x) \exp\left(-\frac{1}{2}\Omega x^2\right),$$

where H_n are the Hermite polynomials. It can be seen that the eigenvalues lie on a half-line passing from the origin through the fourth quadrant of the complex plane. Thus, the larger n, the shorter the lifetime of the eigenstate ψ_n. It is important that because the pseudo-Hamiltonian H is nonnormal its eigenvectors are in general nonorthogonal; their scalar products $(\psi_n, \psi_m) = N_{nn}^{-1/2} N_{mm}^{-1/2} N_{nm}$ are given in Ref. 101.

10. INSTEAD OF CONCLUSIONS: SOME COMMENTS ON "FEYNMAN PATHS"

We return once more to the problem of continuous observation mentioned at the end of Sec. 3. We consider a quantum system with Hilbert state space $\mathscr{H}$ whose time development is determined by the Hamiltonian H. We assume given a projection-valued function $E(\cdot)$ on the interval $[0,b]$. Then for any partition σ of the interval $[0,t] \subset [0,b]$ we can introduce

$$U_E(t, 0; \sigma) = E(t)\, e^{-iH\delta_{n-1}} E(\tau_{n-1})\, e^{-iH\delta_{n-2}} \ldots e^{-iH\delta_0} E(0)$$

(see the notation introduced in Secs. 7 and 8). It is easy to give this expression a physical meaning if $E(\tau)$ is understood as a "yes–no" experiment made on the system at the time τ. Therefore, it is reasonable to regard the operator function

$$U_E : U_E(t, 0) = \lim_{\delta(\sigma)\to 0} U_E(t, 0; \sigma) \quad (46)$$

(when it exists) as the *evolution operator* of the system *in the presence of continuous measurement* described by the function $E(\cdot)$.

The results known that bear on this problem concern above all the case when the function E is constant. The best known among them is the "Zeno paradox" that we mentioned above.[39] This result can also be deduced from weaker assumptions[103] by using the Chernoff product formula[104]:

THEOREM 13. Suppose $E(\tau) = P$ for all τ, where P is an orthogonal projection operator on $\mathscr{H}$. Suppose H is a semibounded self-adjoint operator on $\mathscr{H}$ whose domain of definition $D(H)$ is dense in $P\mathscr{H}$, and suppose there exists an antiunitary operator θ such that $\theta P \theta^{-1} = P, \theta e^{-iHt} \theta^{-1} = e^{iHt}$ for all $t \in \mathbf{R}$. If $U_E(t,0)$ exists for $0 \leqslant t \leqslant b$ for at least one $b > 0$, then there exists a self-adjoint operator H_P that is an extension of PHP and is such that $U_E(t,0) = \exp(-iH_P t)P$ for all $t \geqslant 0$.

Thus, if the conditions of the theorem are satisfied, continuous observation forces the system to remain within the subspace of states $P\mathscr{H}$.

Recently, Aharonov and Vardi[105] considered the case when the instrument function $E(\cdot)$ is not constant and its values are one-dimensional projection operators. They concluded that continuous measurement again forces the state of the system to move along the "path in $\mathscr{H}$" defined by the function $E(\cdot)$, this giving it the meaning of an "individual Feynman path." In particular, if H is a Schrödinger Hamiltonian in $L^2(\mathbf{R})$ and $E(\cdot)$ corresponds to a Gaussian wave packet moving along the path $x_0(\cdot)$ in $\mathbf{R}$, then Aharonov and Vardi conclude that $U_E(t,0)$ multiplies this wave packet by the well-known exponential factor $\exp[iS(x_0)]$, where $S(x_0)$ is the action along x_0. It can be shown that the results of Ref. 105 can be derived rigorously and, most importantly, they do not depend on a number of the special assumptions made in Ref. 105 about the measuring instrument and the Hamiltonian. For example, the following theorem holds[103]:

THEOREM 14. Suppose $\psi \in \mathscr{S}(\mathbf{R}^d)$ and $x_0 \in C^1 \times [[0,b]; \mathbf{R}^d]$. Suppose that for every $t \in [0,b]$ the one-dimensional projection operator $E(t)$ corresponds to the vector $\psi_t : \psi_t(x) = \psi(x - x_0(t))$, and that H is a self-adjoint operator (a Hamiltonian) or pseudo-Hamiltonian on $\mathscr{H} = L^2(\mathbf{R}^d)$ such that $D(H) \supset \mathscr{S}(\mathbf{R}^d)$, and the function $t \to H\psi_t$ is continuous on $[0,b]$. Then for $\varphi_t \equiv U_E(t,0)\varphi$ the relation

$$\varphi_t = \exp\left\{i \int_0^t \left[\sum_{k=1}^d \dot{Q}_k(\tau)\dot{\mathcal{P}}_k(\tau) - \mathcal{E}(\tau)\right] d\tau\right\} (\psi_0, \varphi)\psi_t$$

holds for all $t \in [0,b)$ and $\varphi \in \mathscr{H}$, where

$$\mathcal{E}(t) = (\psi_t, H\psi_t), \quad \mathcal{P}_k(t) = (\psi_t, P_k\psi_t), \quad Q_k(t) = (\psi_t, Q_k\psi_t).$$

It is desirable to find analogous proposition for more general functions $E(\cdot)$, including ones that are infinitely dimensional. The following case is particularly important: Suppose $H = -\frac{1}{2}\Delta$ is the free Hamiltonian on $L^2(\mathbf{R}^d)$ and that $E(t)$ projects onto $L^2(M_t)$, $M_t \subset \mathbf{R}^d$. Using the definitions (41) and (46), we can readily find

$$(U_E(t,\ 0)\,\psi)(x) = \chi_{M_t}(x) \int_{\Gamma(M)}^{LT} \psi(\gamma(0) + x) \exp\left\{\frac{\mathrm{i}}{2}\|\gamma\|^2\right\} D\gamma,$$

where χ_{M_t} is the characteristic function of the set M_t and $\Gamma(M)$ consists of all paths in the space $AC_0[J';\mathbf{R}^d]$ (see Sec. 8) for which $\gamma(\tau) + x \in M_\tau$ for every $\tau \in [0,t]$. Thus, $U_E(t,0)$ can in a certain sense *replace the nonexistent Feynman measure* (see Ref. 106 and the analogous problem for the Wiener measure[61]).

Another important property of the above result is that it does not require the Hamiltonian to have a definite form (for example, of a Schrödinger operator), and it does not even need to be self-adjoint. This offers hope that the Feynman-integral method will find applications also for more general Hamiltonians (pseudo-Hamiltonians) than the dissipative Schrödinger operators with local complex potentials considered in the foregoing sections.

We note finally that a more complete discussion of the material of the present review, and also many other problems, can be found in the monograph of Ref. 108.

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