Quantum statistics of systems interacting with electromagnetic fields

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The equilibrium and dynamic properties of macroscopic systems interacting with a field of electromagnetic radiation are investigated. The regular methods of asymptotically exact solution associated with the use of approximating Hamiltonians and the elimination of boson variables from the kinetic equations are reviewed. Some generalizations of the Dicke model are considered.

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INTRODUCTION

During the last decade, there has been a strong growth of interest in the theoretical investigation of the properties of macroscopic two- and many-level systems interacting with electromagnetic fields. There are many reasons for this, and the main ones are as follows. First, the use of powerful sources of coherent radiation have now made it possible to develop intensively experiments in the field of quantum radio physics and nonlinear optics. In particular, much attention is paid to the study of cooperative spontaneous emission in two-level systems and the nonlinear effects associated with it. It must be emphasized that two-level systems may differ strongly from the physical point of view, being, for example, molecular, atomic, or nuclear. However, the characteristic behavior of such macroscopic systems does not depend on their particular physical nature, which is primarily manifested in the conditions under which the field induces in each emitter only one two-level transition and correlation arises between the emitters. This circumstance opens up the possibility of a theoretical description of the processes that take place in such systems on the basis of a general approach, i.e., makes it possible to use models with similar mathematical structure for systems of different physical natures.

Second, the simplest model problem describing a two-level system in resonant interaction with a radiation field—the so-called Dicke model[1]—could be solved exactly in the thermodynamic limit (i.e., in the limit of an infinite system). It is well known that the number of exactly solvable model problems of statistical mechanics is very small, while it is precisely such problems that, as a rule, have a decisive influence on the development of theory. Study of even simplified models of macroscopic two-level systems makes it possible to construct an adequate description of many physical processes and to predict a number of basically new phenomena.

The investigation of cooperative phenomena in macroscopic two-level systems began in 1954, when Dicke's pioneering paper was published.[1] The model problem he proposed is formulated as follows. Suppose a system of N two-level emitters, which has volume V, is situated in a resonator. The Hamiltonian of the system is

$$H=\sum_{f=1}^{N}\frac{1}{2m}\left(\mathbf{P}_f+\frac{e}{c}\mathbf{A}\right)^2+\sum_q \hbar\omega_q a_q^+ a_q, \qquad (1)$$

where a_q^+ and a_q are the operators of creation and annihilation of photons with frequency ω_q, $\mathbf{P}_f$ is the momentum of the electron bound to the f-th two-level atom, e is the electron charge, c is the velocity of light, and $\mathbf{A}$ is the purely transverse vector potential of the electromagnetic field:

$$\mathbf{A}=\frac{1}{\sqrt{V}}\sum_q \tau_q c\left(\frac{2\pi\hbar}{\omega_q}\right)^{1/2}(a_q \exp[i\mathbf{q}\mathbf{f}]+a_q^+\exp[-i\mathbf{q}\mathbf{f}]).$$

Here, the sum over q is taken with allowance for the ultraviolet cutoff, and τ_q is a polarization unit vector. Further, in the dipole approximation one usually ignores the term $e^2/2mc\,\mathbf{A}^2$ because of its smallness[2,3] and restricts the treatment to a two-dimensional state space for each atom, which corresponds to allowing for only one two-level transition. This makes it possible to use quasispin operators to represent the two-level emitters.[1,2] As a result, after some simplifications an effective Hamiltonian of the following form is obtained (the Dicke model):

$$H=\sum_q \hbar\omega_q a_q^+ a_q+\sum_{f=1}^{N}\left\{\frac{1}{2}\hbar\omega_0\sigma_f^z+\frac{\lambda}{\sqrt{N}}(a_0\sigma_f^++a_0^+\sigma_f^-)\right\}, \qquad (2)$$

which corresponds to the so-called rotating-wave approximation.[2] Here, σ_f^z, $\sigma_f^{\pm}=1/2(\sigma_f^x\pm i\sigma_f^y)$ are Pauli operators describing the two-level emitters:

$$\sigma_f^{\pm}|\pm\rangle_f=0,\ \sigma_f^{\pm}|\mp\rangle_f=|\pm\rangle_f,\ \sigma_f^z|\pm\rangle_f=\pm|\pm\rangle_f,$$

and λ is the quasispin–photon coupling constant, defined by

$$\lambda=\omega_0 d\sqrt{\frac{2\pi\hbar}{\omega_0\mathscr{V}}}, \qquad (3)$$

where d is the electric dipole moment of the transition, $d=\langle+|\mathbf{x}_f|-\rangle$ and $\mathscr{V}\equiv V/N$ is the specific volume.

An important property of this model problem, first pointed out by Dicke,[1] is that it describes a transition to an excited "super-radiant" state, which is characterized by spontaneous coherent emission with intensity proportional to N^2. This intensity is due to the correlation between the individual emitters. The spontaneous super-radiance phenomenon predicted by Dicke[1] in 1954 was discovered experimentally only in 1972 in gaseous HF.[4] In addition, it proved possible to ob-

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serve a number of similar phenomena—photon echo,[2,5] self-induced transparency,[2] etc., in which, however, an intensity dependence of the type N^2 is due to external reasons (coherent pumping).

The rigorous mathematical investigation of the equilibrium properties of the model problem with the Hamiltonian (1) began in 1973.[6,7,131] In particular, it was shown that when allowance is made for only one resonance mode of the photon field and the strong coupling condition

$$\lambda > \hbar\omega_0 \qquad (4)$$

is satisfied, the system makes an equilibrium phase transition to a state characterized by spontaneous polarization in the subsystem of the emitters (M system) amd macroscopic population of the resonance mode of the field (F system). In numerous subsequent studies the equilibrium properties of the Dicke model (1) and its generalizations were investigated (Refs. 8–47). For this, the method proposed and developed in Refs. 11 and 20, which makes it possible to obtain exact expressions for the thermodynamic potentials and equilibrium mean values for the M and F systems, proved to be very fruitful. Conceptually, this method takes it origins from Bogolyubov's studies on the theory of a weakly nonideal Bose gas[48,49] and the approach associated with the introduction of a so-called approximating Hamiltonian.[50]

In Refs. 3, 19, 26, and 34, the influence of the A^2 term on the phase transition to the super-radiant state was discussed. In Refs. 18 and 51 the analogies that exist between the phase transition in the Dicke model and the phenomenon of ferroelectric ordering with condensation of a soft phonon mode were discussed. A generalization of the Dicke model to the case of a many-level system was proposed by Gilmore.[52] The problem of describing many-photon processes was investigated in Refs. 53–56. In Ref. 57, the breaking of $SU(3)$ symmetry on the super-radiant phase transition was considered.

As we have already pointed out, the phenomenon of super-radiance attracts great interest in connection with the possibility of its practical application—to invert the front of an electromagnetic wave,[58] to generate coherent radiation in single-path resonator-free lasers,[59] etc. In this connection, we must mention especially the papers of Khokhlov[60,61] who first proposed the idea of making lasers that operate in the x-ray and gamma ranges by using the super-radiance effect in Mössbauer crystals.

Besides the exact solution to the equilibrium problem describing the super-radiant phase transition, there is also undoubted interest in investigating the nonequilibrium properties of macroscopic two-level systems. The adequate description of the dynamics of such systems is of special importance, in particular, in connection with the determination of the working regime, the choice of the active medium, and estimation of the power of super-radiant lasers, in the first place for the x-ray and gamma ranges.[62,63] The derivation of an exact kinetic equation for a super-radiant system is a very complicated mathematical problem. In the description of the dynamics of such systems, numerous simplifying physical assumptions are therefore usually made. Such is the approach based on a Markov master equation and developed by Bonifacio *et al.*[64-67] The dynamics of the photon system with allowance for incoherent pumping was studied on the basis of Zwanzig's approach[68] in Refs. 69 and 70. Willis[71] used the Bogolyubov–Krylov method of solution of nonlinear equations.

Recently, in connection with the investigation of the polaron problem, an approach has been developed for the construction of an exact kinetic equation for systems interacting with a boson thermal bath.[72,73] Use of the deep analogies in the mathematical formulation of the model problems in polaron theory and in the theory of super-radiance made it possible to extend the method to macroscopic two-level M–F systems like the Dicke model.[75-77] It proved possible to obtain a number of important results for the determination of the constant and collective frequency shifts and the characteristic relaxation (conversion) times in such systems.

It should be noted that the theoretical investigation of collective processes in macroscopic super-radiant M–F systems has developed recently into an independent scientific direction occupying an intermediate position between statistical mechanics and quantum radio physics. A very great number of original papers and a number of reviews and specialized collections have been devoted to this subject. However, the world literature does not contain a sufficiently complete survey of regular methods that enable one to obtain exact results for model problems in the theory of super-radiant M–F systems. The present paper is to be regarded as an attempt to fill this gap.

The paper is arranged as follows. In Sec. 1, we review the mathematically rigorous methods of solution of the model Dicke problem in the equilibrium case and consider some properties of this model and its generalizations. Section 2 is devoted to the application of the Dicke model and its generalizations to the description of coherent physical systems. In particular, we here investigate the realization of a strong-coupling condition of the type (4) in physical systems. Finally, in Sec. 3 we describe a method for constructing an exact kinetic equation for super-radiant systems and review the main results relating to the dynamical properties of such systems.

1. METHOD OF EXACT INVESTIGATION OF THE EQUILIBRIUM PROPERTIES OF SUPER-RADIANT SYSTEMS

1.1. Equilibrium properties of the Dicke model

We here discuss the equilibrium (thermodynamic) properties of the Dicke model with the Hamiltonian (2). In 1973, Hepp and Lieb[7,131] showed that the Dicke model admits an exact solution in the thermodynamic limit ($N \to \infty$), and they described the "super-radiant" phase transition. If the strong-coupling condition (4) is satisfied, there is simultaneous ordering below the critical temperature in the subsystem of two-level atoms and the photons. The state of the photon subsystem can be interpreted as a classical coherent state with superimposed temperature-dependent "noise", the photon mode being macroscopically populated: $\langle a^+a\rangle \propto N$ (see Refs. 6, 7, 11, and 16). Simultaneously, the quasispin subsystem of the atoms is also macroscopically ordered. Above the critical temperature, the ordering disappears (phase transition of the second kind).

Hepp and Lieb's work stimulated interest in the study of various modifications and generalizations of the Dicke model and similar models in other branches of statistical physics and solid-state theory and laid the foundation of a significant development of rigorous methods of investigation of the equilibrium properties of such systems.

In the description of the equilibrium properties of the system, a central part is played by the free-energy functional $f[H]$,[1)] and also various physically important mean values characterizing the ordering in the system.

Since the two-level atoms in the Dicke model interact only with one resonance mode of the photon field, we shall in what follows consider only the main part of the Hamiltonian (2), which can be written in the form

$$H=\omega a^+a+\sqrt{N}\,\lambda\,(aS^+ + a^+S^-)+\varepsilon NS^z, \tag{5}$$

where $\omega=\hbar\omega_0$, $\varepsilon=\hbar\omega_0$, and

$$S^{\pm}=S^x\pm iS^y,\ S^{x,y,z}=\frac{1}{2N}\sum_{j=1}^{N}\sigma_j^{x,y,z}. \tag{5a}$$

Hepp and Lieb's result[7,13] for the free energy of the Dicke model (5) is as follows.[2)] We construct for the Hamiltinian (5) an "approximating" Hamiltonian, replacing the photon operators by C numbers (variational parameters) in accordance with the rule

$$\frac{a}{\sqrt{N}}\to-\frac{\lambda}{\omega}C,\ \frac{a^+}{\sqrt{N}}\to-\frac{\lambda}{\omega}C^*.$$

As a result, we obtain the Hamiltonian

$$H_A(C)=\varepsilon NS^z-N\frac{\lambda^2}{\omega}(CS^+ + C^*S^-)+N\frac{\lambda^2}{\omega}|C|^2. \tag{6}$$

It can be shown that the free energies $f[H]$ for the original system (5) and $f[H_A(C)]$ for the approximating system are equal in the limit $N\to\infty$ if the parameters C and C^* are chosen using the condition of an absolute minimum of the expression $f_{N=\infty}[H_A(C)]$.[3)] We denote the point of the minimum by $\bar{C}$,

$$f_\infty[H_A(\bar{C})]=abs\min_C\{f_{N=\infty}[H_A(\bar{C})]\}. \tag{7}$$

Thus,

$$\lim_{N\to\infty}f_N[H]=f_\infty[H_A(\bar{C})]. \tag{8}$$

Since the approximating Hamiltonian (6) is a linear form in the Pauli matrices, the free energy $f[H_A(C)]$ can be readily calculated[4),5)]:

$$f[H_A(C)]=-\theta\ln\left(2\,\mathrm{ch}\frac{\sqrt{\varepsilon^2+4\frac{\lambda^4}{\omega^2}|C|^2}}{2\theta}\right)+\frac{\lambda^2}{\omega}|C|^2. \tag{9}$$

The necessary condition for a minimum $(\partial f/\partial C)$ gives an equation for $\bar{C}$, $\bar{C}^*$:

$$C=\langle S^-\rangle_{H_A(C)},\ C^*=\langle S^+\rangle_{H_A(C)}, \tag{10}$$

or, explicitly,

$$|\bar{C}|=\frac{\lambda^2}{\omega}\frac{|\bar{C}|}{\sqrt{\varepsilon^2+4\frac{\lambda^4}{\omega^2}|\bar{C}|^2}}\,\mathrm{th}\frac{\sqrt{\varepsilon^2+4\frac{\lambda^4}{\omega^2}|\bar{C}|^2}}{2\theta}. \tag{11}$$

The phase φ of the parameter $\bar{C}=|\bar{C}|\,e^{i\varphi}$ here remains arbitrary. For $\theta<\theta_c$, where

$$\theta_C=\frac{\varepsilon}{2}\left(\mathrm{arcth}\frac{\omega\varepsilon}{\lambda}\right)^{-1}, \tag{12}$$

Eq. (11) has a nontrivial solution $|\bar{C}|\neq0$, this realizing an absolute minimum of the free energy (9). For $\theta\geqslant\theta_C$, the unique solution of Eq. (11) is trivial: $|\bar{C}|\equiv0$, $\theta\geqslant\theta_C$. Substututing $C=\bar{C}$ in (9), we obtain an explicit expression for the free energy of the Dicke model as $N\to\infty$. At the level of the free energies, the Hamiltonians $H(5)$ and $H_A(\bar{C})$ are equivalent as $N\to\infty$.

It is clear from (10) that $\bar{C}^{\#}\equiv\langle S^{\pm}\rangle_{H_A(\bar{C})}$. One can show here too that averaging over $H_A(\bar{C})$ can be replaced by averaging over H as $N\to\infty$. Thus,

$$\bar{C}^{\#}\equiv\langle S^{\pm}\rangle_H|_{N\to\infty}. \tag{13}$$

The following relation also holds:

$$\left\langle\frac{a^+a}{N}\right\rangle_H=\frac{\lambda^2}{\omega^2}\langle S^+S^-\rangle_H\underset{N\to\infty}{=}\frac{\lambda^2}{\omega^2}|\bar{C}|^2. \tag{14}$$

Equations (13) and (14) follow from the general results of Sec. 1.3 [we note that Eq. (13) is not to be understood in the sense of ordinary mean values, but in the sense of "quasiaverages," i.e., with allowance for spontaneous symmetry breaking; for more details, see Sec. 1.3].

Thus, for $\theta<\theta_C$ ordering occurs in the quasispin subsystem of the atoms, the ordering being characterized by the order parameter $|\bar{C}|\equiv|\langle S^{\pm}\rangle_H|$. Figure 1 shows the dependence of $|\bar{C}|$ on the temperature. As follows from (14), such ordering is accompanied by the appearance of a macroscopic number of coherent photons: $\langle a^+a\rangle_H \propto N|\bar{C}|^2$. At low temperatures, the system is in the super-radiant state, which vanishes when $\theta\geqslant\theta_C$.

As we have shown, the model Dicke Hamiltonian describes a phase transition to the super-radiant state

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1)For a system with Hamiltonian H and temperature $\theta=kT$, where k is Boltzmann's constant and T is the absolute temperature, we have by definition $f[H]=-(\theta/N)\ln\mathrm{Tr}\exp(-H/\theta)$, where N is the number of particles $(N\to\infty)$; $\mathrm{Tr}\exp(-H/\theta)$ is the partition function.

2)We present these results here in a form somewhat different from the original and corresponding to the discussion in Sec. 1.2 of this paper.

3)For the description of the thermodynamics and the phase transition it is in fact only the asymptotic properties in the limit of an infinitely large system, $N\to\infty$, that are important.

4)*Translator's Note.* The Russian notation for the trigonometric, inverse trigonometric, hyperbolic trigonometric functions, etc., is retained here and throughout the article in the displayed equations.

5)It is necessary to take into account the formulas $\exp(a\sigma)=ch|a|+\frac{a\sigma}{|a|}sh\,|a|$, $|a|=\sqrt{(a\cdot a)}$, $\mathrm{Tr}\exp(a\sigma)=ch\,|a|$.

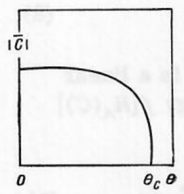

FIG. 1. Temperature dependence of the order parameter in the Dicke model.

only if the strong-coupling condition (4) is satisfied. This presupposes the presence of a high density $\rho \equiv v^{-1}$ or a low resonance frequency ω_0. These conditions may be a serious obstacle to the experimental realization of such a phase transition in physical systems. There is therefore much interest in considering physical situations that lead to a change in the strong-coupling conditions. Thus, it was pointed out in Ref. 78 that if one gives up the rotating-wave approximation, i.e., if one takes into account "counter-rotating"[19] interaction terms of the form $a^+\sigma^+$ and $a\sigma^-$, the simple substitution

$$\lambda \to \tilde{\lambda} = 2\lambda$$

is made in the strong-coupling condition (4). In a more general case, one can consider a Hamiltonian with counter-rotating terms of the form[13]

$$H = \hbar\omega_0 a^+ a + \frac{\lambda}{\sqrt{N}} \sum_{j=1}^{N} [a\sigma_j^+ + a^+\sigma_j^- + \mu(a\sigma_j^- + a^+\sigma_j^+)] + \frac{1}{2}\hbar\omega_0 \sum_{j=1}^{N} \sigma_j^z.$$

Then the condition for the existence of a phase transition is

$$\lambda(1+\mu) > \hbar\omega_0.$$

We recall that the Dicke model is a simplified version of the original Hamiltonian (1). For less radical simplifications, one should take into account in the Hamiltonian the additional A^2 term present in (1). The influence of this term on the phase transition has been discussed in a number of papers (Refs. 3, 9, 26, 28, and 34). In connection with the problem of allowing for the A^2 term, Gilmore and Bowden[19] proposed a canonical transformation to eliminate the term as a result of renormalization of the parameters of the Hamiltonian. Indeed, when the A^2 term is retained, the Dicke Hamiltonian (1) must be augmented by the expression[26]

$$k(a^+ + a)^2. \tag{15}$$

In this case, a transformation of the boson variables of the form[19]

$$\begin{pmatrix} \alpha \\ \alpha^+ \end{pmatrix} = \begin{pmatrix} \text{ch}\,\gamma & e^{i\varphi}\,\text{sh}\,\gamma \\ e^{-i\varphi}\,\text{sh}\,\gamma & \text{ch}\,\gamma \end{pmatrix} \begin{pmatrix} a \\ a^+ \end{pmatrix}$$

with the parameter γ chosen using the condition

$$\text{th}\,\gamma = -(\sqrt{F}-1)/(\sqrt{F}+1), \quad F \equiv 1 + 4k/\hbar\omega$$

makes it possible to eliminate the term (15) from the Hamiltonian. The resulting renormalization of the parameters of the Dicke Hamiltonian (for $\mu = 0$) has the form

$$\hbar\omega \to \hbar\omega' = \hbar\omega\sqrt{F};$$

$$\lambda \to \lambda' = \frac{\lambda}{2}(F^{1/4} + F^{-1/4}).$$

Thus, even when the A^2 term is taken into account we arrive at a standard structure of the type (5).

We make some comments about the methods of studying the equilibrium properties of Dicke-type models. In their original paper, Hepp and Lieb[7] used a rather complicated "direct" method which was actually based on the construction of the complete spectrum of the Hamiltonian using special properties of the model. Therefore, new approaches making it possible to obtain the same physical results were investigated in many subsequent studies. Almost immediately, Hepp and Lieb's thermodynamic results were reproduced by a simpler method based on the use of Glauber coherent states by Wang and Hioe[6] (see also Refs. 130 and 131). Vertogen and De Vries[8] based their work on an analysis of the equations of motion in the approach of C^* algebras. In Ref. 13, Brankov, Zagrebnov, and Tonchev used the technique of approximating Hamiltonians of N. N. Bogolyubov, Jr.[50]; Moshchinskii and Fedyanin[23] and Popov and Fedotov[47,132] used the method of steepest descent in a functional integral; other methods were also developed.

New methods were proposed by N. N. Bogolyubov, Jr. and Plechko,[11,20] who studied from a unified point of view an entire class of systems including the Dicke model and its generalizations, and also other related models as special cases. It was found that many characteristic properties of the Dicke model hold generally for all systems in which one or several modes of a boson field interact with some subsystem ("matter"). These general methods are discussed below in Secs. 1.2 and 1.3. In Sec. 1.4, we briefly discuss some models related to the standard Dicke model.

1.2. A class of exactly solvable model Hamiltonians with interaction between matter and a boson field. Calculation of the free energy

We consider the problem of calculating the free energy for a generalized class of model systems describing a subsystem (matter) that is not particularized, with a finite number of modes of a boson field.[11] This class of systems includes the Dicke model and related models.

We consider the class of model Hamiltonians of the form

$$H = \sum_{\alpha=1}^{S} \omega_\alpha a_\alpha^+ a_\alpha + \sqrt{N} \sum_{\alpha=1}^{S} (\lambda_\alpha a_\alpha^+ L_\alpha + \lambda_\alpha^* a_\alpha L_\alpha^+) + T - N \sum_{\alpha=1}^{S} \varkappa_\alpha L_\alpha^+ L_\alpha, \tag{16}$$

where:

α_α^+ and a_α are the operators of creation and annihilation of mode α of the boson field; they satisfy the commutation relations

$$a_\alpha a_\beta^+ - a_\beta^+ a_\alpha = \begin{cases} 1, & \alpha = \beta; \\ 0, & \alpha \neq \beta; \end{cases} \tag{17}$$

$T \equiv T^+$, L_α, L_α^+ are operator constructions representing the N-particle "M subsystem" (matter), which is not particularized; on these constructions there are merely imposed the following general subsidiary conditions:

$$\left.\begin{array}{l}\| L_\alpha \| \leqslant K_1, \| L_\alpha T - T L_\alpha \| \leqslant K_2, \\ \| L_\alpha L_\beta^\# - L_\beta^\# L_\alpha \| \leqslant K_3/N,\end{array}\right\} \tag{18}$$

where $\|\ldots\|$ denotes the norm of an operator, and K_1, K_2, K_3 are certain N-independent constants. These conditions are not in fact restrictive and hold in concrete cases[6)];

N is the number of particles in the M subsystem. We shall make our calculations for finite fixed N, going to the thermodynamic limit, $N \to \infty$, at the very end;

$\omega_\alpha > 0$ are the frequencies of the boson field; $\varkappa_\alpha \geqslant 0$ are the "coupling constants" in the additional L^2 interaction in the matter subsystem (there is no analog of this term in the Dicke model (1); such a term appears, for example, in some superconducting models[83]).

The Hamiltonian H (16) is defined on the space

$$\mathscr{H}_{M \otimes B} = \mathscr{H}_M \otimes \mathscr{H}_B, \tag{19}$$

where $\mathscr{H}_M$ is the state space of the M subsystem (matter), and $\mathscr{H}_B$ is the Fock space of the boson field. It is clear that the operators of the M subsystem always commute with $\{a_\alpha, a_\alpha^+\}$.

The Hamiltonian (16) describes S modes of the boson field interacting with the M subsystem. Different specific models are obtained for appropriate choices of the operators $\{T, L_\alpha, L_\alpha^+\}$ of the M subsystem. In the Dicke model, we have one field mode, $S=1$, and $T = \varepsilon N S^z$; $L = S^-$, $L^+ = S^+$; $\varkappa = 0$.

We shall show that the free energy for the Hamiltonian (16) can be calculated on the basis of the following approximating Hamiltonian:

$$H_A(C) = T - N \sum_{\alpha=1}^{S} g_\alpha (C_\alpha L_\alpha^+ + C_\alpha^* L_\alpha) + N \sum_{\alpha=1}^{S} g_\alpha |C_\alpha|^2, \tag{20}$$

$$g_\alpha \equiv \varkappa_\alpha + |\lambda_\alpha|^2/\omega_\alpha,$$

where C_α and C_α^* are variational parameters chosen to ensure an absolute minimum of the free energy $f[H_A(C)]$. We denote the values of the parameter that realize the minimum by $\{\overline{C}_\alpha\}$:

$$f_N[H_A(\overline{C})] = \text{abs}\min_{C} f_N[H_A(C)] \tag{21}$$

(we note that the operations of going to the limit $N \to \infty$ and minimization with respect to $\{C_\alpha\}$ commute[11]). Our aim is to prove the relation[7)]

$$|f_{N}[H_A(\overline{C})] - f_N[H]| \xrightarrow[N\to\infty]{} 0. \tag{22}$$

For the proof, we require two fundamental theorems.

1. For all Hermitian operators (Hamiltonians) $\mathfrak{U}_1$ and $\mathfrak{U}_2$ the following inequalities hold between the equilibrium mean values and the free energies (Bogolyubov, 1956):

$$\frac{1}{N}\langle \mathfrak{A}_1 - \mathfrak{A}_2 \rangle_{\mathfrak{A}_1} \leqslant f[\mathfrak{A}_1] - f[\mathfrak{A}_2] \leqslant \frac{1}{N}\langle \mathfrak{A} - \mathfrak{A}_2 \rangle_{\mathfrak{A}_2}. \tag{23}$$

2. The free-energy theorem (N. N. Bogolyubov, Jr.[50]). We introduce an auxiliary Hamiltonian of the form

$$\widetilde{H} = T - N \sum_{\alpha=1}^{S} g_\alpha L_\alpha^+ L_\alpha, \; g_\alpha \geqslant 0, \tag{24}$$

where T, L_α, L_α^+ satisfy the relations (18); then the free energies $f_N[N]$ and $f_N[H_A(C)]$ (21) are equal to the limit $N \to \infty$:

$$0 \leqslant f_N[H_A(\overline{C})] - f_N[\widetilde{H}] \leqslant \varepsilon_N \xrightarrow[N\to\infty]{} 0, \tag{25}$$

where $\varepsilon_N \propto N^{-2/5}$ (if desired, the estimate ε_N can be improved by imposing stronger conditions than (18); see Ref. 25).

The relations (22) and (25) indicate that the free energy for the class of systems H (16) is equal in the limit $N \to \infty$ not only to (21) but also to $f[\widetilde{H}]$; however, in specific systems $f[\widetilde{H}]$ cannot be directly calculated because of the L^2 term in (24), and our final aim is to reduce the problem to the calculation of $f[H_A(\overline{C})]$ (21). The Hamiltonian $\widetilde{H}$ plays an important auxiliary role.

Equation (22) is proved by the chain of inequalities[11]

$$-\zeta_N \leqslant f[H]_{M\otimes B} - f[\widetilde{H}]_M \leqslant f[H_A(\overline{C})]_M - f[\widetilde{H}]_M \leqslant \varepsilon_N, \tag{26}$$

where $\xi_N \to 0$, $\varepsilon_N \to 0$, $N \to \infty$. Here, the indices M and M×B (and B below) identify the state space on which the corresponding Hamiltonian acts, and the trace is taken in the calculation of $f[\ldots]$.

The extreme right-hand inequality follows from the theorem (25). To prove the central inequality, we must show that

$$f[H]_{M\otimes B} \leqslant f[H_A(\overline{C})]_M. \tag{27}$$

We introduce the auxiliary Hamiltonian

$$H_B(\overline{C}) = N \sum_{\alpha=1}^{S} \omega_\alpha \left(\frac{a_\alpha^+}{\sqrt{N}} + \frac{\lambda_\alpha^*}{\omega_\alpha}\overline{C}_\alpha^*\right)\left(\frac{a_\alpha}{\sqrt{N}} + \frac{\lambda_\alpha}{\omega_\alpha}\overline{C}_\alpha\right). \tag{28}$$

In the inequalities (23) we choose $\mathfrak{U}_1 = H$ and $\mathfrak{U}_2 = H_A(\overline{C}) + H_B(\overline{C})$; calculating $\mathfrak{U}_1 - \mathfrak{U}_2$, we obtain from (23)

$$\begin{aligned} &f[H]_{M\otimes B} - f[H_A(\overline{C}) + H_B(\overline{C})]_{M\otimes B} \\ &\leqslant -\sum_{\alpha=1}^{S} \varkappa_\alpha \langle (L_\alpha^+ - \overline{C}_\alpha^*)(L_\alpha - \overline{C}_\alpha)\rangle_{H_A(\overline{C})+H_B(\overline{C})} \\ &+ \sum_{\alpha=1}^{S} \left\langle \lambda_\alpha \left(\frac{a_\alpha^+}{\sqrt{N}} + \frac{\lambda_\alpha^*}{\omega_\alpha}\overline{C}_\alpha^*\right)(L_\alpha - \overline{C}_\alpha) + \text{h.c.}\right\rangle_{H_A(\overline{C})+H_B(\overline{C})}. \end{aligned} \tag{29}$$

The first term on the right-hand side is here nonpositive, and the second term is equal to zero, since the mean value with respect to $H_A(\overline{C}) + H_B(\overline{C})$ factorizes into the product $\langle\ldots\rangle_{H_A(\overline{C})}\langle\ldots\rangle_{H_B(\overline{C})}$ of the mean values of the corresponding operators, and, obviously, $\langle a_\alpha^+/\sqrt{N} + \lambda_\alpha^*/\omega_\alpha \overline{C}_\alpha^*\rangle_{H_B(C)} \equiv 0$. Thus,

$$f[H]_{M\otimes B} \leqslant f[H_A(\overline{C}) + H_B(\overline{C})]_{M\otimes B}. \tag{30}$$

Further,

$$f[H_A(\overline{C}) + H_B(\overline{C})]_{M\otimes B} = f[H_A(\overline{C})]_M + f[H_B(\overline{C})]_B,$$

where

$$f[H_B(\overline{C})]_B = \frac{\theta}{N}\sum_{\alpha=1}^{S} \ln\left(1 - e^{-\frac{\omega_\alpha}{\theta}}\right) \leqslant 0. \tag{31}$$

From this the inequality (26) follows.

It remains to prove the left-hand inequality in (26):

$$-\zeta_N \leqslant f[H]_{M\otimes B} - f[\widetilde{H}]_M. \tag{32}$$

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6) For example, in the Dicke model we have $\|S^\pm\| \leqslant 1$, and by virtue of the additivity $[S^-, S^+] \propto 1\, N$.

7) The relation (9) is a special case of (22). In Secs. 1.2 and 1.3 we do not dwell on the motives for the steps that are taken, since they are evident from the analogy with the Dicke model discussed in Sec. 1.1.

Note that $H(16)$ can be represented in the form

$$H=\sum_{\alpha=1}^{S} x_\alpha\omega_\alpha a_\alpha^+ a_\alpha+\widetilde{H}_x$$
$$+N\sum_{\alpha=1}^{S}(1-x_\alpha)\,\omega_\alpha\left(\frac{a_\alpha^+}{\sqrt{N}}+\frac{\lambda_\alpha^* L_\alpha^+}{\omega_\alpha(1-x_\alpha)}\right)\left(\frac{a_\alpha}{\sqrt{N}}+\frac{\lambda_\alpha L_\alpha}{\omega_\alpha(1-x_\alpha)}\right), \quad (33)$$

where

$$\widetilde{H}_x=T-N\sum_{\alpha=1}^{S}\left(\varkappa_\alpha+\frac{|\lambda_\alpha|^2}{(1-x_\alpha)\,\omega_\alpha}\right)L_\alpha^+ L_\alpha. \quad (34)$$

Here, we have introduced the parameter x_α, $0<x_\alpha\leqslant 1$ and we have separated the auxiliary boson term [the first in (33)] in order to have the possibility of applying the inequality (23).

In (23), we choose

$$\mathfrak{A}_1=H,\quad \mathfrak{A}_2=\widetilde{H}_x+\sum_{\alpha=1}^{S} x_\alpha\omega_\alpha a_\alpha^+ a_\alpha. \quad (35)$$

Bearing in mind that [see (33)] $\mathfrak{U}_1-\mathfrak{U}_2\geqslant 0$, we have

$$0\leqslant f[H]_{M\otimes B}-f[\widetilde{H}_x+\sum_{\alpha=1}^{S} x_\alpha\omega_\alpha a_\alpha^+ a_\alpha]_{M\otimes B}. \quad (36)$$

At the same time, it is obvious that

$$f[\widetilde{H}_x+\sum_{\alpha=1}^{S} x_\alpha\omega_\alpha a_\alpha^+ a_\alpha]_{M\otimes B}=f[\widetilde{H}_x]_M+f[\sum_{\alpha=1}^{S} x_\alpha\omega_\alpha a_\alpha^+ a_\alpha]_B$$
$$=f[\widetilde{H}_x]_M+\frac{\theta}{N}\sum_{\alpha=1}^{S}\ln\left[1-\exp\left(-x_\alpha\frac{\omega_\alpha}{\theta}\right)\right]\geqslant \quad (37)$$
$$\geqslant f[\widetilde{H}_x]_M-\frac{\theta}{N}\sum_{\alpha=1}^{S}\left(\frac{x_\alpha\omega_\alpha}{\theta}+\ln\frac{\theta}{x_\alpha\omega_\alpha}\right).$$

We now estimate the difference between the free energies $f[\widetilde{H}]_M$ and $f[\widetilde{H}_x]_M$. Setting $\mathfrak{U}_1=\widetilde{H}$, $\mathfrak{U}_2=\widetilde{H}_x$ in (23) and using the boundedness of the operators L_α and L_α^+ in the norm (18), we obtain

$$0\leqslant f[\widetilde{H}]_M-f[\widetilde{H}_x]_M\leqslant K_1^2\sum_{\alpha=1}^{S}\frac{x_\alpha}{1-x_\alpha}\frac{|\lambda_\alpha|^2}{\omega_\alpha}. \quad (38)$$

On the basis of (36)–(38) we have

$$-\sum_{\alpha=1}^{S}\left(\frac{x_\alpha}{1-x_\alpha}\frac{|\lambda_\alpha|^2}{\omega_\alpha}K_1^2+\frac{x_\alpha\omega_\alpha}{N}+\theta\ln\frac{\theta}{\omega_\alpha}+\frac{\theta}{N}\ln\frac{1}{x_\alpha}\right) \quad (39)$$
$$\leqslant f[H]_{M\otimes B}-f[\widetilde{H}]_M.$$

The right-hand side of (39) does not contain the parameters x_α. Therefore, we can choose x_α on the left-hand side at our discretion. We choose $x_\alpha=1/N$ for all α. As a result, we obtain the lower bound (30) for

$$\zeta_N=s\theta\frac{\ln N}{N}+\frac{K_1^2}{N-1}\sum_{\alpha=1}^{S}\frac{|\lambda_\alpha|^2}{\omega_\alpha} \quad (40)$$
$$+\frac{1}{N}\sum_{\alpha=1}^{S}\left(\theta\ln\frac{\theta}{\omega_\alpha}+\frac{\omega_\alpha}{N}\right)\propto\frac{\ln N}{N}\xrightarrow[N\to\infty]{}0.$$

The inequalities (26) are thus proven.

We have therefore shown that the calculation of the free energy $f[H]_{M\times B}$ for the class of model systems (16) as $N\to\infty$ reduces to the calculation of the free energy $f[H_A(\overline{C})]_M$. The Hamiltonian $H_A(C)$ does not contain the boson operators at all and is linear in the operators $\{L_\alpha\}$. Because of the comparative simplicity of the structure of $H_A(C)$ in concrete systems the free energy $f[H_A(C)]$ can be readily calculated. The necessary condition for a minimum $(\partial f/\partial C_\alpha=0)$ gives a system of equations for determining the parameters $\{\overline{C}_\alpha\}$:

$$\left.\begin{array}{l} C_\alpha=\langle L_\alpha\rangle_{H_A(C)},\\ C_\alpha^*=\langle L_\alpha^+\rangle_{H_A(C)}.\end{array}\right\} \quad (41)$$

We conclude by noting some relations for the mean values. We introduce the notation

$$\left.\begin{array}{l} B_\alpha=\dfrac{a_\alpha}{\sqrt{N}}+\dfrac{\lambda_\alpha}{\omega_\alpha}L_\alpha,\\ B_\alpha^+=\dfrac{a_\alpha^+}{\sqrt{N}}+\dfrac{\lambda_\alpha^*}{\omega_\alpha}L_\alpha^+.\end{array}\right\} \quad (42)$$

Note that such constructions occur in the representation (33), from which it can be seen that when the left-hand side of (36) is written out in more detail the mean value $\langle B_\alpha^+ B_\alpha\rangle$ can be estimated. Finally, we can obtain the estimate[11]

$$\sum_{\alpha=1}^{S}\omega_\alpha\langle B_\alpha^+ B_\alpha\rangle_H\leqslant(\varepsilon_N+\zeta_N)\left(1+0\left(\frac{1}{N}\right)\right)\xrightarrow[N\to\infty]{}0, \quad (43)$$

whence, using the well-known inequality[80]

$$|\langle\mathfrak{A}\mathfrak{B}\rangle|^2\leqslant\langle\mathfrak{A}\mathfrak{A}^+\rangle\langle\mathfrak{B}^+\mathfrak{B}\rangle, \quad (44)$$

we readily obtain the following asymptotically exact equations[11]:

$$\left\langle\frac{a_\alpha^+ a_\alpha}{N}\right\rangle_H=\frac{|\lambda_\alpha|^2}{\omega_\alpha^2}\langle L_\alpha^+ L_\alpha\rangle_H\Big|_{N\to\infty}. \quad (45)$$

We see that the number of quanta of the boson field is related to the mean values of the L operators, which usually characterize the ordering in the "matter" [see Ref. 11 and the discussion in Sec. 1.4].

1.3. On the calculation of quasiaverages

The relation (22) proves the equivalence of H and $H_A(\overline{C})$ in the limit $N\to\infty$ at the level of the free energies. One can also consider whether H and $H_A(\overline{C})$ are equivalent for the calculation of mean values. In physical models, in particular, one is generally very interested in the mean values of the operators $\{L_\alpha, L_\alpha^+\}$. Since $\langle L_\alpha\rangle_{H_A(\overline{C})}\equiv\overline{C}_\alpha$ [see (41)], it can be assumed that $\langle L_\alpha\rangle_H=\overline{C}_\alpha$. This equation does indeed hold, but for "quasiaverages" and not for ordinary mean values.

The concept of quasiaverages was introduced by Bogolyubov,[79,80] who showed that in systems with degeneracy some physically important mean values are identically equal to zero for finite N by virtue of the symmetry. In such cases, physical meaning attaches not to the ordinary Gibbs mean values $\langle\ldots\rangle_H$ but to "quasiaverages" $\prec\ldots\succ_H$, which are defined as follows[79,80]. In the Hamiltonian one introduces small symmetry-breaking terms determined by the parameters $\{\tau_\alpha\}$; one then calculates the ordinary mean values $\langle\ldots\rangle_{H_\tau}$, goes to the limit $N\to\infty$, and only then sets $\{\tau_\alpha\}\to 0$:

$$\prec\ldots\succ_H=\lim_{\tau\to 0}\lim_{N\to\infty}\langle\ldots\rangle_{H_\tau}. \quad (46)$$

Thus, quasiaverages are mean values calculated with allowance for spontaneous symmetry breaking.

In specific cases, the symmetry-breaking terms ("the sources") usually have a direct physical meaning.[8)] However, when one is considering systems that are not particularized in detail the question of the choice of the "sources" is not so simple.[50] Methods for introducing quasiaverages for the class of Hamiltonians (16) were proposed in Ref. 20. Quasiaverages are defined in accordance with the rule (46) on the basis of the Hamiltonian

$$H_\tau = H + 2N \sum_{\alpha=1}^{S} \tau_\alpha \omega_\alpha \left(\frac{a_\alpha^+}{\sqrt{N}} + \frac{\lambda_\alpha^*}{\omega_\alpha} \overline{C}_\alpha^*\right)\left(\frac{a_\alpha}{\sqrt{N}} + \frac{\lambda_\alpha}{\omega_\alpha}\overline{C}_\alpha\right), \quad (47)$$

where $\{\tau_\alpha\}$ are real small parameters, $\tau_\alpha > 0$.

Using methods similar to those employed in Sec. 1.2, we can obtain the estimate

$$\sum_{\alpha=1}^{S} \omega_\alpha \Big[\Big\langle \Big(\frac{a_\alpha^+}{\sqrt{N}} + \frac{\lambda_\alpha^*}{\omega_\alpha} L_\alpha^+\Big)\Big(\frac{a_\alpha}{\sqrt{N}} + \frac{\lambda_\alpha}{\omega_\alpha} L_\alpha\Big)\Big\rangle_{H_\tau} + \tau_\alpha \Big\langle \Big(\frac{a_\alpha^+}{\sqrt{N}} + \frac{\lambda_\alpha^*}{\omega_\alpha} \overline{C}_\alpha^*\Big)\Big(\frac{a_\alpha}{\sqrt{N}} + \frac{\lambda_\alpha}{\omega_\alpha}\overline{C}_\alpha\Big)\Big\rangle_{H_\tau}\Big] \leqslant \xi_{N,\tau} \xrightarrow[\tau_\alpha>0,\ N\to\infty]{} 0. \quad (48)$$

Proceeding from (48) and the inequality (44), we can readily obtain relations for the quasiaverages of the operators $L_\alpha^\#$, $a_\alpha^\#/\sqrt{N}$ and the parameters $\overline{C}_\alpha^\#$ of the following type[20]:

$$\left.\begin{aligned} &\prec L_\alpha \succ_H = \overline{C}_\alpha, \\ &\prec \frac{a_\alpha}{\sqrt{N}} \succ_H = -\frac{\lambda_\alpha}{\omega_\alpha}\overline{C}_\alpha, \\ &\prec \frac{a_\alpha^+ a_\alpha}{N} \succ_H = \frac{|\lambda_\alpha|^2}{\omega_\alpha^2}|\overline{C}_\alpha|^2 \end{aligned}\right\} \quad (49)$$

etc. In the general case, these relations can be expressed in the form of the following "rules of substitution" for calculating unary and binary mean values:

$$\frac{a_\alpha^\#}{\sqrt{N}} \rightleftarrows -\frac{\lambda_\alpha^\#}{\omega_\alpha} L_\alpha^\# \rightleftarrows -\frac{\lambda_\alpha^\#}{\omega_\alpha}\overline{C}_\alpha^\#. \quad (50)$$

In Ref. 39, these rules were extended to arbitrary mean values of the form

$$\Big\langle \dots \frac{a_\alpha^+}{\sqrt{N}} \dots \frac{a_\beta}{\sqrt{N}} \dots L_\alpha \dots L_\beta^+ \dots \Big\rangle. \quad (51)$$

In the calculation of all such mean values, the operators $a_\alpha^\#/\sqrt{N}$, $L_\alpha^\#$ and the parameters $\overline{C}_\alpha^\#$ are modified in accordance with the rule (50) (at the level of the quasiaverages).

Note that in the case of the ordinary Dicke model (5) we have a case of degeneracy. It can be shown that for finite N at all temperatures $\langle S^\pm\rangle_H \equiv 0$ because of the hidden symmetry of the Hamiltonian (5). (This symmetry is manifested indirectly in that the free energy $f[H_A(C)]$ (9) does not depend on the phase of the parameter $\overline{C}$, and the phase $\overline{\varphi}$ for $\overline{C} = |\overline{C}|e^{i\overline{\varphi}}$ can be chosen arbitrarily.) However, at the level of quasiaverages $|\langle S^\pm\rangle| \equiv |\overline{C}| \neq 0$ for $\theta \leqslant \theta_C$, as follows from the general results (49). Note that on the transition to the more general model (17) (for $\mu \neq 0$) the symmetry is lost, the parameters $\overline{C}, \overline{C}^*$ are determined uniquely (here $\varphi = 0$), and the quasiaverages are equal to the ordinary mean values.

Note that the symmetry breaking in the Hamiltonian (47) (if it occurs) is actually realized only by the linear terms $\tau_\alpha \overline{C}_\alpha^* a_\alpha$, $\tau_\alpha c_\alpha a_\alpha^+$.

In specific systems, the mean values and quasiaverages of the operators L_α usually have the meaning of "order parameters" in the matter subsystem. At the same time, the relations (49) and (50) indicate that the occurrence of ordering in the matter ($\langle L_\alpha\rangle \equiv \overline{C}_\alpha \neq 0$) is necessarily accompanied in all cases by macroscopic population of the boson modes, for which effectively $a_\alpha \propto \sqrt{N}$, $a_\alpha^+ a_\alpha \propto N$. This is a universal property and does not depend on the specific features of the matter subsystem.[20]

Note also that, as can be seen from the form of the Hamiltonian $\tilde{H}$ (24), the interaction of the matter with the boson field leads to an additional effective "attraction" [the negative L^2 term in (24)], which facilitates the ordering.

1.4. Other models. Some remarks

The class of systems considered in Secs. 1.2 and 1.3 includes various modifications of the Dicke model and some other models.

It includes, for example, the following generalized variant of the Dicke model (5):

$$H = \sum_{m=1}^{S} \omega_m a_m^+ a_m + \frac{1}{\sqrt{N}} \sum_{m=1}^{S}\sum_{j=1}^{N} [a_m(\lambda_{mj}^* \sigma_j^+ + \mu_{mj}\sigma_j^-) + a_m^+(\lambda_{mj}\sigma_j^- + \mu_{mj}^*\sigma_j^+)] + \sum_{j=1}^{N} (\varepsilon_j^x \sigma_j^x + \varepsilon_j^y \sigma_j^y + \varepsilon_j^z \sigma_j^z), \quad (52)$$

a special case of which is (17) and also the models in Sec. 3 [Eqs. (72) and (82)]. Other examples are the Kobayashi model in the theory of ferroelectrics with hydrogen bonds,[81,82] the Fröhlich model in the theory of superconductivity,[83] models of a ferromagnetic crystal with phonon instability,[84-86] the model of the anomalous ferroelectric transition in proustite Ag_3AsS_3,[87] etc. We emphasize that the term "superradiant phase transition" is here to be understood with respect to not only dipole–photon systems but also systems interacting with phonon fields.[24] For example, in Ref. 24 a crystal with N paramagnetic impurity centers with spin S in a constant field H_0 was considered. The crystal represents a resonator of longitudinal acoustic waves with frequency ω resonator for $\Delta M = 2$ transitions in the Zeeman spectrum of the system (M is the magnetic quantum number). The Hamiltonian of such a system has the form[24]

$$H = \hbar\omega\left(a_k^+ a_k + \frac{1}{2}\right) + \sum_{j=1}^{N} \hbar\omega_0 S_j^z + \frac{1}{\sqrt{N}} \sum_{j=1}^{N} [G(S_j^+)^2 a_k e^{i\mathbf{k}\mathbf{r}_j} - G^*(S_j^-)^2 a_k^+ e^{-i\mathbf{k}\mathbf{r}_j}], \quad (53)$$

where a_k^+ and a_k are the operators of phonons with wave vector **k**, ω_0 is the frequency of the Zeeman splitting, S_j are operators of the spin S, and the terms $G(S_j^+)^2$ and $G^*(S_j^-)^2$ represent the spin part of the tensor spin–phonon interaction (for more detail, see Ref. 24). Such a Hamiltonian belongs to the general class considered in the present section, and it describes a phase transition. Experimentally, such a transition can be ob-

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8) For example, in isotropic ferromagnets the spontaneous magnetization should be calculated with the inclusion of a weak magnetic field, which fixes the direction of the magnetization and is removed only after taking the limit $N \to \infty$.

served in the pronounced enhancement of the Mandelstam–Brillouin scattering near θ_C.[24]

As we already noted in Sec. 1.3, a common feature of such systems is the simultaneous ordering in the matter and field subsystems. In the models of Refs. 82 and 83, in which the quasispin or fermion subsystem interacts with the phonons of a crystal lattice, there is a simultaneous ordering in the matter and a structural rearrangement in the lattice when the phase transition occurs.

Various physical models are also discussed in the following section.

2. GENERALIZATIONS OF THE DICKE MODEL AND DESCRIPTION OF PHYSICAL SYSTEMS

2.1. The problem of the realization of the strong-coupling conditions in a many-component system

In the previous section, we already pointed out that the phase transition to the super-radiant state is possible only if a strong-coupling condition of the type (4), which has a physical nature, is satisfied. This can be a serious obstacle to the realization of such a transition in definite physical systems. In this connection, it was argued in Ref. 32 that in a two-component system fulfillment of the strong-coupling condition for only one of the components is sufficient for a phase transition in both components. For a system with an arbitrary number of components, the problem of the existence of the super-radiant phase transition was investigated in Ref. 37.

Following Ref. 37, we consider the system with the Hamiltonian

$$H^{(m,\,n)}=\sum_{i=1}^{m}\hbar\omega_i a_i^{\dagger}a_i+\sum_{j=1}^{n}\varepsilon_j\sigma_j^z+\frac{1}{\sqrt{N}}\sum_{i=1}^{m}\sum_{j=1}^{n}\lambda_{ij}(\sigma_j^- a_i^{\dagger}+\sigma_j^{\dagger}a_i). \qquad (54)$$

Here, $\sigma_j^{\cdots}\equiv\Sigma_f\sigma_{fj}^{\cdots}$, m is the number of modes, n is the number of components, i.e., the number of "species" of two-level emitters (m and n are fixed numbers), and the parameters λ_{ij} are determined in the usual manner for the Dicke model. It is assumed that each component consists of the same macroscopic number N of emitters. In accordance with the method set forth in the previous section, we make a renormalization of the field operators:

$$a_i^{\dagger}\to\tilde{a}_i^{\dagger}=a_i^{\dagger}+\frac{1}{\hbar\omega_i\sqrt{N}}\sum_{j=1}^{n}\lambda_{ij}\sigma_j^{\dagger};\quad a_i\to\tilde{a}_i=a_i+\frac{1}{\hbar\omega_i\sqrt{N}}\sum_{j=1}^{n}\lambda_{ij}\sigma_j^{-}.$$

As a result, the effective Hamiltonian (as $N\to\infty$) of the M system can be represented in the form (a special case of the Hamiltonian $\bar{H}$ from Sec. 1.2)

$$H_M^{(m,\,n)}=\sum_j\varepsilon_j\sigma_j^z-\frac{1}{N}\sum_{i,\,j}\frac{\lambda_{ij}^2}{\hbar\omega_i}\sigma_j^{\dagger}\sigma_j^{-}-\frac{1}{N}\sum_{j,\,k}\frac{\lambda_{ij}\lambda_{ik}}{\hbar\omega_i}\sigma_j^{\dagger}\sigma_k^{-},\quad j\neq k. \qquad (55)$$

The last term in (55) is distinguished, since it describes the interaction of quasispins of different species. An interaction of this kind is encountered in various problems in the theories of magnetism and ferroelectricity.[88,89] In the investigation of systems with such interaction, one cannot use the standard variational principle associated with minimization of the free energy; instead, one must use the more complicated minimax principle introduced in Ref. 90, since otherwise unphysical results can be obtained.[89] The minimax principle was extended to quasispin systems of fairly general form in Ref. 91. The basic idea of the approach is as follows. Suppose for simplicity that the Hamiltonian contains the interaction of particles of only two species, which are described by the quasispin operators $\Lambda^{(1)}$ and $\Lambda^{(2)}$, respectively:

$$H=-\frac{\lambda}{N}\Lambda^{(1)}\Lambda^{(2)},\ \Lambda^{(j)}=\sum_{f=1}^{N}\sigma_{fj}^{\pm},\ [\Lambda^{(1)},\ \Lambda^{(2)}]=0.$$

This Hamiltonian can be represented by means of an identity transformation in the form

$$H=-\frac{\lambda}{2N}P^2+\frac{\lambda}{2N}\{(\Lambda^{(1)})^2+(\Lambda^{(2)})^2\}, \qquad (56)$$

where $P\equiv\Lambda^{(1)}+\Lambda^{(2)}$. The approximating, thermodynamically equivalent Hamiltonian for (56) is chosen in the form

$$H_0=-\frac{\lambda}{2}(2P-\xi)\xi+\frac{\lambda}{2}\sum_{j=1}^{2}(2\Lambda^{(j)}-\eta_j)\eta_j,$$

the variational parameters ξ and η_j being determined from the condition[88,89]

$$\min_{(\xi)}\max_{(\eta_j)}F_0,\quad F_0\equiv\lim_{N\to\infty}\left\{-\frac{\Theta}{N}\ln\mathrm{Tr}\,e^{-H_0/\Theta}\right\}.$$

Note that the order in which the extrema are taken is here important.[50,90]

In the case of the Hamiltonian (55), the last term can be represented by analogy with (56) in the form

$$-\frac{1}{N}\sum_{i=1}^{m}\sum_{j,\,k=1}^{n}\frac{\lambda_{ij}\lambda_{ik}}{2\hbar\omega_i}(1-\delta_{jk})\,[(\sigma_j^-+\sigma_k^{\dagger})^2-(\sigma_j^-)^2-(\sigma_k^{\dagger})^2].$$

Using now the minimax principle, we can readily show that the Hamiltonian of the M system in (54) is thermodynamically equivalent to the approximating Hamiltonian

$$H_M^{(m,\,n)}(\xi)=\sum_i\varepsilon_j\sigma_j^z-\sum_i\sum_{j,\,k}\frac{\lambda_{ij}\lambda_{ik}}{\hbar\omega_i}(\sigma_j^-\xi_k+\sigma_j^{\dagger}\xi_j^*-\xi_j^*\xi_k), \qquad (57)$$

where $\xi_k=\langle\sigma_k^+/N\rangle$. The parameter ξ_k is related in a simple way to the spontaneous polarization P_k in component k: $P_k\propto|\xi_k|$. Therefore, the quantities ξ_k can be regarded as order parameters in the M system. For the B system, the order parameter characterizing the population of mode i is $\sum_j\lambda_{ij}|\xi_j|$. The Hamiltonian (57) is linear in the operators $\sigma^{\pm}$ and can be readily diagonalized. After simple calculations, we obtain for the free energy of the M system

$$F_M=\sum_i\sum_{j,\,k}\frac{\lambda_{ij}\lambda_{ik}}{\hbar\omega_i}\xi_j^*\xi_k-\Theta\sum_j\ln 2\,\mathrm{ch}\frac{E_j}{\Theta},$$

where

$$E_j=\sqrt{\varepsilon_j^2+|A_j|^2},$$

$$A_j=\sum_i\left\{\frac{\lambda_{ij}^2}{\hbar\omega_i}\xi_j+\sum_k(1-\delta_{jk})\frac{\lambda_{ij}\lambda_{ik}}{\hbar\omega_i}\xi_k\right\}.$$

The order parameters ξ_j are solutions of the system[37]

$$A_j=A_j\sum_{i,\,k}\frac{\lambda_{ij}\lambda_{ik}}{\hbar\omega_i}\frac{\mathrm{th}\,(E_j/\Theta)}{2E_j};\quad j=1,\ 2,\ \ldots,\ n. \qquad (58)$$

It is clear that for any temperature $\Theta\geqslant 0$ we have the solution

$$\xi_1 = \xi_2 = \ldots = \xi_n = 0,$$

which corresponds to the absence of ordering. We shall not seek nontrivial solutions. Suppose that at a definite temperature some $\xi_j = \xi_{j_0} \neq 0$, whereas all the remaining $\xi_j = 0$. Then instead of (58) we have

$$\sum_i \frac{\lambda_{ij}\lambda_{ij_0}}{\hbar\omega_i} \xi_{j_0} = \Big(\sum_i \frac{\lambda_{ij}\lambda_{ij_0}}{\hbar\omega_i} \xi_{j_0} \Big) \sum_{i,\,k} \frac{\lambda_{ij}\lambda_{ik}}{\hbar\omega_i} \frac{\operatorname{th}(\widetilde{F}_j/\Theta)}{2\widetilde{F}_j},$$

$$\widetilde{F}_j \equiv \sqrt{\varepsilon_j^2 + \Big| \sum_i \frac{\lambda_{ij}\lambda_{ij_0}}{\hbar\omega_i} \xi_{j_0} \Big|^2}.$$

Since $\xi_{j_0} \neq 0$ by hypothesis,

$$2\widetilde{E}_j = \sum_{i,\,k} \frac{\lambda_{ij}\lambda_{ik}}{\hbar\omega_i} \operatorname{th} \frac{\widetilde{F}_j}{\Theta}, \quad j = 1,\ 2,\ \ldots,\ n.$$

This system admits a nontrivial solution only if the coupling conditions

$$\forall_j,\ k = 1,\ 2,\ \ldots,\ n, \quad \varepsilon_j/\operatorname{Ar\,th} B_j = \varepsilon_k/\operatorname{Ar\,th} B_k, \tag{59}$$

where $B_j \equiv \sum_{i,k} \lambda_{ij}\lambda_{ik}/\hbar\omega_i$, are satisfied. At the same time, the critical temperature for component j_0 is

$$\Theta_C = \varepsilon_{j_0}/\operatorname{Ar\,th} B_{j_0}. \tag{60}$$

Since the conditions (59) do not depend on the choice of j_0 ($1 \leqslant j_0 \leqslant n$), it follows by virtue of (60) that the phase transition occurs at the same critical temperature in all the components. This is quite clear from the physical point of view; for the occurrence of spontaneous polarization in any of the components leads to the appearance of an "average field," in which all the remaining components are polarized.

In the considered many-component system, the standard strong-coupling condition (4) is replaced by

$$\sum_{k=1}^{n} \sum_{i=1}^{m} \frac{\lambda_{ij}\lambda_{ik}}{\hbar\omega_i} > 2\varepsilon_j, \tag{61}$$

and the conditions (59) must be satisfied simultaneously. It must be particularly emphasized that the critical temperature (60) is higher than the corresponding temperature for each component if the latter is regarded as "pure," i.e., is treated independently of the others (Fig. 2).

Thus, an increase in the number of components (in the number of species of emitters) in the system can indeed play the part of an additional ordering factor and lead to a significant change in the conditions for the existence of a phase transition.

In crystalline super-radiant systems, the thermal vibrations of the emitters near the equilibrium positions have a significant influence on the conditions and nature of the phase transition. Allowance for such vibrations requires the investigation of Hamiltonians more general than the standard Dicke model (2). We now turn to the study of such generalizations.

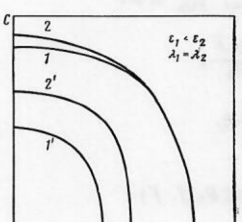

FIG. 2. Result of numerical solution of Eq. (30) for a two-component system. Curves 1′ and 2′ correspond to the "pure" components.

In connection with generalizations of the Dicke model which are of interest from the point of view of physical applications, we also mention the so-called super-radiant models with trilinear interaction considered in a number of papers by Kir'yanov and Yarunin.[133,134] The investigated Hamiltonian has the form

$$H = \hbar\omega a^+ a + \sum_{f=1}^{N} \Big\{ \varphi_f^\dagger \Big[\frac{\hbar\Omega}{2} \sigma^z + \frac{\lambda}{\sqrt{N}} (\sigma^+ a + a^+ \sigma^-) \Big] \varphi_f \Big\},$$

where $\varphi_f = (\alpha_f, \beta_f)$, $\alpha_f \alpha_{f'}^+ - \varkappa \alpha_{f'}^+ \alpha_f = \beta_f \beta_{f'}^+ - \varkappa \beta_{f'}^+ \beta_f = \delta_{ff'}$, i.e., in the system the photon field interacts with lattice oscillators having frequency Ω. It is found that in the case of a Bose lattice ($\varkappa = 1$) the threshold value of the coupling parameter λ decreases with increasing

$$\langle n \rangle = \frac{1}{N} \Big\langle \sum_f (\alpha_f^\dagger \alpha_f + \beta_f^\dagger \beta_f) \Big\rangle,$$

which characterizes the average population number of the lattice (per site). This result suggests that with increasing number of levels there is a weakening of the strong-coupling condition analogous to the case considered here of an increased number of components.

2.2. Models of a macroscopic two-level system in a crystal

The behavior of super-radiant systems in the presence of phonon degrees of freedom has often been considered (see, for example, Refs. 22, 27, and 38 and the references given in them). Here, we shall consider only three main approaches to the investigation of this problem.

It is well known that Mandelstam–Brillouin scattering always occurs in crystals—a plane electromagnetic wave is diffracted by elastic waves. A modification of the Dicke model to take into account such scattering was proposed in Ref. 92. The corresponding Hamiltonian has the form

$$H = \sum_q \hbar\omega_q a_q^+ a_q + \frac{1}{2}\hbar\omega_0 \sum_f \sigma_f^z + \sum_{q,\,f} \frac{\lambda_q}{\sqrt{N}} (a_q \sigma_f^+ + a_q^+ \sigma_f^-)$$
$$\times \Big\{ 1 + \frac{1}{\sqrt{N}} \sum_k (K_k b_k + K_k^* b_k^+) \Big\} + \sum_k \hbar\Omega_k b_k^+ b_k. \tag{62}$$

Here, b_k^+ and b_k are the operators of creation and annihilation of a phonon with quasimomentum k and frequency Ω_k. The coefficients K_k and K_k^* can be determined from the expression for the Brillouin energy[93]:

$$-\sum_f \mu_f E(x_f)(u_f \nabla)\,\rho(x_f),$$

where μ_f is the dipole-moment operators, x_f are the coordinates of emitter f, $E(x_f)$ is the transverse electromagnetic field, u_f is the displacement operator of emitter f, and $\rho(x_f)$ is the mean density. For such a model when only a finite number of modes for the boson fields are taken into account it could be shown[92] that the presence of the phonon contribution leads to a change in the standard form of the strong-coupling condition; if the phonon contribution to the energy of the system is sufficiently large, the phase transition to the super-radiant state changes from one of the second to one of the first kind.

Another effect associated with the influence of the phonons is due to the possible dependence of the coupling constant λ in the Dicke model on the coordinates of the emitters: $\lambda \to \lambda(x_f)$. In this connection, we mention Refs. 22 and 94, which considered a generalization of the Dicke model for systems with fluctuating dipole-photon coupling constants $\lambda(x_f)$ under fairly weak conditions on the distribution function, a compressible Dicke model, taking into account classical lattice vibrations, being investigated as an example. Quantum fluctuations of the "emitters" were investigated in the approximation of a single phonon mode in Ref. 27 on the basis of the same approach as in Refs. 22 and 94. We emphasize that the presence of a dependence $\lambda(x_f)$ can play an important part only when the wavelength of the electromagnetic field is comparable with the distance a between neighboring lattice sites (Ref. 22).[9)] For such a generalization of the Dicke model, it was shown that the presence of phonon degrees of freedom can change the strong-coupling condition and the kind of the phase transition in this case too.[22,27,94] In addition, it was shown that strong interaction of the photons with the elastic vibrations in the crystal can lead to macroscopic deformation of the lattice, i.e., to a structural phase transition.

We note that the condition $\Lambda \propto a$, where Λ is the wavelength of the electromagnetic radiation, corresponds to the x-ray frequency range, whereas the currently existing sources of coherent electromagnetic radiation have much larger wavelengths. Indeed, the mean distance between atoms in solids is $a=(3-5)10^{-8}$ m, whereas for the optical range, for example, $\Lambda \sim 10^{-5}$ m. When

$$\Lambda \ll a, \tag{63}$$

the coupling constant λ varies very weakly when there are thermal vibrations of the emitters in the crystal. Therefore, if the conditions (63) are satisfied it is necessary to take into account other physical mechanism coupling the dipole, photon, and phonon subsystems in the crystal. One such mechanism is the direct electrostatic dipole-dipole interaction.[33,35,38]

Indeed, a coherent optical wave creates in a crystal a system of parallel electric dipoles.[96] The interaction of such dipoles can be written down by analogy with a ferromagnet[97] by means of the operator

$$\sum_{ff'}\{\Phi_{ff'}\sigma_f^+\sigma_{f'}^- + \Phi_{ff'}^*\sigma_f^-\sigma_{f'}^+ + \Psi_{ff'}\sigma_f^z\sigma_{f'}^z\},$$

where Φ and Ψ are functions of the distance between the corresponding dipoles:

$$\Phi_{ff'} = \Phi(|x_f - x_{f'}|), \quad \Psi_{ff'} = \Psi(|x_f - x_{f'}|).$$

These functions can be expanded with respect to the relative displacements of the atoms,[33] which makes it possible to introduce an operator of dipole-phonon interaction in the Hamiltonian.

Following Refs. 33, 35, and 38, we consider the generalization of the Dicke model characterized by the Hamiltonian

$$H = H_{\text{phot}} + H_{d\text{-phot}} + H_{\text{phon}} + H_d + H_{d\text{-phon}}. \tag{64}$$

Here, the operator $H_{d\text{phot}}$ describes the resonance mode of the electromagnetic field,

$$H_{\text{phot}} = \hbar\omega_0 a_0^+ a_0,$$

the operator H_d describes the dipole subsystem,

$$H_d = \frac{1}{2}\hbar\omega_0\sum_f \sigma_f^z + \sum_{ff'}\Phi(|x_f - x_{f'}|)\sigma_f^-\sigma_{f'}^+,$$

the operator $H_{\text{d-phot}}$ describes the ordinary interaction in the Dicke model,

$$H_{\text{d-phot}} = \frac{1}{\sqrt{N}}\sum_f \lambda(\sigma_f^- a_0^+ + a_0\sigma_f^+)$$

and the operator H_{phon} describes the free phonon field in the approximation of a finite number of modes:

$$H_{\text{phon}} = \sum_k \hbar\Omega_k b_k^+ b_k.$$

To construct the operator $H_{\text{d-phon}}$, we expand, as assumed above, the function $\Phi(\cdot)$ with respect to the relative displacements $u_{ff'} = u_f - u_{f'}$, $u_f = x_f - f$. For simplicity, we restrict ourselves to the harmonic approximation

$$H_{d\text{-phon}} = \frac{1}{\sqrt{N}}\sum_{ff'}\sigma_f^+\sigma_{f'}^- \sum_{k'} B_k(f, f')(b_k^+ + b_k),$$

where

$$B_k(f, f') = B_{-k}^*(f, f') = \frac{-i}{N^2\sqrt{4M\hbar\Omega_k}}\sum_\nu \tau_k\nu\tilde{\Phi}(\nu)(e^{ikf} - e^{ikf'});$$

M is the mass of an "atom," τ_k is the unit polarization vector of the phonons, and $\tilde{\Phi}(\nu)$ is the Fourier transform of the function $\Phi(\cdot)$.

To obtain an exact solution, we now assume that

$$\Phi(|f - f'|) = N^{-1}\varphi(|f - f'|),$$

where $\varphi(\cdot)$ is a bounded function satisfying certain special conditions.[50,98,99] In this case, using the methods set forth in Sec. 2 of the present paper, we can construct for (64) an effective, thermodynamically equivalent Hamiltonian of the form

$$\tilde{H} = \tilde{H}_d + \tilde{H}_{\text{phon}} + \tilde{H}_{\text{phot}}, \tag{65}$$

where

$$\tilde{H}_d = -\sum_{ff'} G(f, f')\sigma_f^+\sigma_{f'}^- + \frac{1}{2}\hbar\omega_0\sum_f \sigma_f^z + \sum_k \frac{N}{\hbar\Omega_k}\eta_k\eta_k^*,$$

$$\tilde{H}_{\text{phon}} = \sum_k \hbar\Omega_k \tilde{b}_k^+\tilde{b}_k, \quad \tilde{b}_k^+ = b_k^+ + \frac{\eta_k^*\sqrt{N}}{\hbar\Omega_k},$$

$$\tilde{H}_{\text{phot}} = \hbar\omega_0\tilde{a}_0^+\tilde{a}_0, \quad \tilde{a}_0 = a_0 + \frac{\lambda\sqrt{N}}{\hbar\omega_0}\xi.$$

Here

$$G(f, f') = \frac{\lambda^2}{N\hbar\omega_0} - \varphi(f, f') + \sum_k \frac{2}{\hbar\Omega_k}\operatorname{Re}\{\eta_k^* B_k(f, f')\}.$$

The complex parameters ξ and η_k will be determined below.

In accordance with the method of approximating Ham-

[9)]Usually, a standing electromagnetic wave in a resonator is considered. The physical system is quasi-one-dimensional, since phonons with $k\|q$ play the main part.[95]

iltonians,[50] the Hamiltonian of the dipole subsystem in (65) can be replaced by a thermodynamically equivalent Hamiltonian of the form

$$H_d^{(0)} = -\sum_{ff'} G(f, f')(\sigma_f^+\xi + \sigma_f^-\xi^* - \xi\xi^*) + \frac{1}{2}\hbar\omega_0 \sum_f \sigma_f^z + \sum_k \frac{N}{\hbar\Omega_k}\eta_k\eta_k^*,$$

in which the parameters ξ and η_k are determined from the self-consistency equations[38]

$$\left.\begin{aligned} x &= \frac{(\Delta + Bx)\,x}{2E}\,\mathrm{th}\,\frac{E}{\Theta},\\ x &= |\xi|^2,\\ \eta_k &= \left\langle \frac{1}{N}\sum_{ff'} B_k(f, f')\,\sigma_f^+\sigma_{f'}^- \right\rangle = \widetilde{B}_k x, \end{aligned}\right\} \quad (66)$$

where

$$\Delta \equiv \frac{\lambda^2}{\hbar\omega_0} - \frac{1}{N}\sum_p \varphi(p), \quad B \equiv 2\sum_k \frac{1}{N}\sum_p B_k(p)/\hbar\Omega_k;$$

$$E \equiv \sqrt{\left(\frac{1}{2}\hbar\omega_0\right)^2 + (\Delta + Bx)^2 x^2}.$$

The Gibbs free-energy density corresponding to $H_d^{(0)}$ is[35]

$$\overline{F}(\Delta, B) = (\Delta + Bx)\,x + \frac{1}{2}Bx^2 - \Theta \ln 2\,\mathrm{ch}\,\frac{E}{\Theta}. \quad (67)$$

It is readily seen that for fixed Θ and $\hbar\omega_0$ the total differential of the function (67) can be represented in the form

$$d\overline{F} = -x\,d\Delta - \frac{1}{2}x^2\,dB.$$

Therefore, x, which is conjugate to the "external parameter" Δ, can be regarded as an order parameter of the dipole subsystem, whereas $y \equiv x^2/2$ is the parameter conjugate to the "external" phonon parameter B.

We consider first the case $B = 0$, i.e., we ignore the phonon contribution. Then Eq. (66), which always has the trivial solution, will also admit the existence of a nonvanishing solution, provided the strong-coupling condition

$$\Delta > \hbar\omega_0 \quad (68)$$

is satisfied and the temperature satisfies

$$\Theta < \Theta_C = \hbar\omega_0 / \ln\left(\frac{\Delta + \hbar\omega_0}{\Delta - \hbar\omega_0}\right).$$

At the point Θ_C, there is a phase transition of the second kind to the super-radiant state characterized by the order parameter $x > 0$ (Fig. 3, curve 1). Note that allowance for the electrostatic dipole interaction Φ has led to a change in the strong-coupling condition (68) compared with the standard case (4) and to a lowering of the critical temperature.

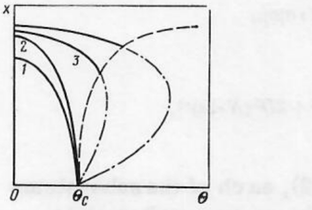

FIG. 3. Temperature dependence of the order parameter for fixed $\Delta > \hbar\omega_0$ and different B. The broken curve indicates the points of the transition of the first kind, and the chain curves represent unstable solutions.

Now suppose the parameter Δ is fixed and $B \neq 0$. If $B \leqslant B_C$, where

$$B_C = \frac{\Delta^2}{(\hbar\omega_0)^2\,\Theta_C}\left[\Delta(3\Delta + \Theta_C) + (\hbar\omega_0)^2\right], \quad (69)$$

then the phase transition in the system is of the second kind, but the curve $x(\Theta, B_1)$ lies above the curve $x(\Theta, B_2)$ if $B_1 < B_2$ (Fig. 3, curve 2). But if $B > B_C$, then the nature of the phase transition in the system is changed. The temperature Θ_{tr} of the transition of the first kind is determined from the condition

$$\overline{F}(\Theta_{tr}) = -\Theta_{tr} \ln 2\,\mathrm{ch}\,\frac{\hbar\omega_0}{2\Theta_{tr}}$$

(Fig. 3, curve 3), and $\Theta_{tr} > \Theta_C$.

For sufficiently large values of B, a phase transition of the first kind is possible in the system even when the strong-coupling condition (68) is not satisfied, when a transition of the second kind is impossible in principle. The corresponding situation is shown in Fig. 4.

Figure 5 shows the dependence of the order parameter x on the temperature for fixed B and varying Δ. It can be seen that with increasing Δ the temperature of the transition of the first kind increases, reaching at $\Delta = \Delta_C$, which is determined by the condition $B = B_C(\Delta_C)$, the temperature Θ_C, after which the transition becomes one of the second kind when Δ is increased.

We now consider the Hamiltonian $H_d^{(0)}$. After transition to the diagonal representation, it takes the form

$$H_d^{(0)} = \sum_f E S_f^z + N\frac{B}{2}x^2 + N(\Delta + Bx)\,x,$$

where $\sum_f S_F^z$ is the operator of the difference between the level populations:

$$S_f^z = (u^2 - v^2)\,\sigma_f^z - 2uv\,(\sigma_f^+ + \sigma_f^-),$$

$$u = \sqrt{\frac{E + \hbar\omega_0/2}{2E}}, \quad v = \sqrt{\frac{E - \hbar\omega_0/2}{2E}}.$$

Then

$$S = \left\langle \frac{1}{N}\sum_f S_f^z \right\rangle = 1 - 2\,(e^{2E/\Theta} + 1)^{-1}.$$

The dependence of S on Θ is shown in Fig. 6.

Further, as is readily seen, there is in the system a coherent quasi-Larmor precession of the quasispins (mean dipole moments)

$$\frac{d\mathbf{S}_f}{dt} = [\mathbf{S}_f \times \mathscr{E}], \quad \mathscr{E} = \{0, 0, E\}$$

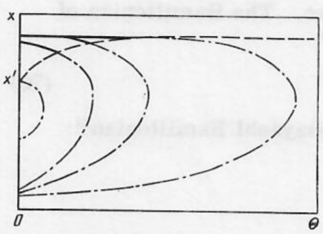

FIG. 4. Temperature dependence of the order parameter for fixed $\Delta < \hbar\omega_0$ and different B,

$$x' = \frac{1}{6B_0}\left(B_0 - 2\Delta + \sqrt{4\Delta^2 + 8\Delta B_0 + B_0^2 - 24B_0\hbar\omega_0}\right).$$

The broken curve indicates the points of the transition of the first kind; the chain curves represent unstable solutions.

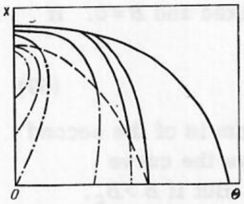

FIG. 5. Temperature dependence of the order parameter for fixed B and different $\hbar\omega_0$. The broken curve corresponds to the points of the transition of the first kind, and the chain curves correspond to unstable solutions.

with frequency E whose temperature dependence is shown in Fig. 7.

Thus, allowance for the direct electrostatic interaction of the dipoles and the effects associated with the presence of the phonon subsystem leads to a significant change in the strong-coupling conditions and in a number of cases to a change in the type of the transition to the super-radiant state. In addition, the transition to the super-radiant state in the considered model can be accompanied by macroscopic deformation of the lattice, i.e., by a structural transition. Note that such a deformation can in a number of cases be regarded as analogous to super-radiance (see Sec. 1.3 and Ref. 24).

2.3. The problem of super-radiant generation in ferroelectrics

We consider one of the possible physical realizations of the model problem investigated in the previous subsection. We note first that the interaction describing ferroelectric ordering in crystals is usually chosen in a form that agrees with H_d in (64) (see, for example, Ref. 100). Since the parallel ordering of the dipoles in ferroelectrics arises spontaneously when the temperature is lowered below the Curie point Θ_S, this analogy suggests that in the system consisting of the ferroelectric resonator and the electromagnetic field super-radiant condensation of the resonant mode of the electromagnetic field can take place below Θ_S.[44]

Following Ref. 46, we consider the simplest quasi-spin model of a ferroelectric of order–disorder type interacting with a resonant mode of the electromagnetic field, and we investigate the possibility of super-radiant generation in such a system. Suppose the system consists of a resonator (a crystal of KH_2PO_4 type) and a standing electromagnetic wave. The Hamiltonian of such a system has the form[46]

$$H = H_K + H_{\rm phot} + H_{d\text{-phot}}. \qquad (70)$$

Here, H_K is the so-called Kobayashi Hamiltonian[81]:

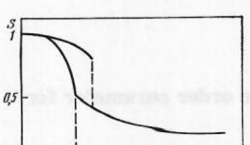

FIG. 6. Temperature dependence of the difference between the populations. The continuous curve corresponds to a transition of the second kind.

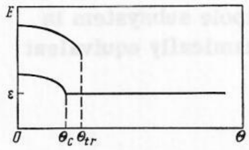

FIG. 7. Temperature dependence of the frequency of collective precession. The continuous curve corresponds to a transition of the second kind.

$$H_{\rm K} = \hbar\Omega b^+ b - \varepsilon \sum_f \sigma_f^x - \frac{1}{2}\sum_{ff'} J(f, f')\,\sigma_f^z\sigma_{f'}^z + \frac{1}{\sqrt{N}} K \sum_f \sigma_f^z (b^+ + b),$$

where $\hbar\Omega$ is the energy of an optical phonon, $\varepsilon > 0$ is the tunneling integral of a proton in the two-well potential of the hydrogen bond, J is the parameter of the ferroelectric interaction, and K is the quasispin–phonon coupling constant. The operator $H_{\rm phot}$ describes the energy of the resonance mode of the electromagnetic field:

$$H_{\rm phot} = \hbar\omega a^+ a.$$

The operator of the energy of the interaction of the dipoles with the resonant photon mode has the form

$$H_{d\text{-phot}} = \frac{1}{\sqrt{N}} D \sum_f \sigma_f^z (a^+ + a),$$

where D is the parameter of the dipole–photon coupling:

$$D = d\sqrt{2\pi\hbar\omega\rho}.$$

Here, ρ is the density of the medium and d is the electric dipole moment of the transition between the symmetric and antisymmetric states of the proton. Further

$$\hbar\omega = E_{\rm ant} - E_{\rm sym},$$

where $E_{...}$ is the corresponding energy. Taking into account the definition of the tunneling integral,[100] we obtain from this for the frequency of the resonant mode

$$\omega = 2\varepsilon/\hbar. \qquad (71)$$

As was shown in Ref. 82, the dipole–phonon interaction in the Kobayashi model can be effectively taken into account by a renormalization of the dipole–dipole interaction parameter:

$$J(f, f') \to J(f, f') + 2K^2/N\hbar\Omega.$$

Moreover, since (70) is a special case of the Hamiltonian (64) of the previous section, it can be replaced in the thermodynamic limit by an effective Hamiltonian of the form

$$\left.\begin{aligned} &\widetilde{H} = \widetilde{H}_{\rm phon} + \widetilde{H}_{\rm phot} + \widetilde{H}_d;\\ &\widetilde{H}_{\rm phon} = \hbar\Omega \tilde{b}^+\tilde{b};\ \tilde{b} = b + K\sqrt{N}\,\xi\,(\hbar\Omega)^{-1};\\ &\widetilde{H}_{\rm phot} = 2\varepsilon\tilde{a}^+\tilde{a},\ \tilde{a} = a + D\sqrt{N}\,\xi\,(2\varepsilon)^{-1},\\ &\widetilde{H}_d = -\varepsilon\sum_f \sigma_f^x - \frac{1}{2}\sum_{ff'} \mathcal{J}(f, f')\,\sigma_f^z\sigma_{f'}^z, \end{aligned}\right\} \qquad (72)$$

where

$$\mathcal{J}(f, f') = J(f, f') + 2K^2(N\hbar\Omega)^{-1} + 2D^2(N\cdot 2\varepsilon)^{-1},$$
$$\xi = \left\langle N^{-1}\sum_f \sigma_f^z\right\rangle.$$

In the effective Hamiltonian (72), each of the subsystems is defined in such a way that the corresponding state spaces do not intersect; therefore each of the subsystems makes an independent additive contribution to the total free energy of the system. In accordance with the result of the previous section, the condensation of the resonant photon mode is determined by

$$P = N^{-1}\langle a^{+}a\rangle = D^2(2\varepsilon)^{-2}\xi^2. \quad (73)$$

A super-radiant state occurs in the system when $P>0$. We shall find the conditions under which $P>0$. In this connection we note that it is customary in the theory of ferroelectricity to use an average-field approximation, which leads to satisfactory agreement with the experiments.[101,102] In accordance with the method of approximating Hamiltonians,[50] this approximation leads to an exact result for a quasispin system with Hamiltonian of the type $\tilde{H}_d$ when the kernel $\mathcal{J}(f,f')$ describes an interaction of "infinite range." In the simplest case, it is sufficient to assume that

$$\mathcal{J}(f, f') = \mathcal{J}N^{-1},\ \mathcal{J} = \mathrm{const},$$

i.e.,

$$J(f, f') = \mathrm{const}\cdot N^{-1}.$$

Then for the order parameter we readily obtain the equation

$$\xi = \mathcal{J}\xi E^{-1}\,\mathrm{th}\,(E\Theta^{-1}), \quad (74)$$

where $E=\sqrt{\varepsilon^2+\mathcal{J}^2\xi^2}$. It is readily seen that $\xi>0$, provided

$$\varepsilon<\mathcal{J},\ \Theta<\Theta_S=\varepsilon/\mathrm{Arth}\,(\varepsilon\mathcal{J}^{-1}). \quad (74a)$$

We emphasize that the relation (74a) here plays the part of the strong-coupling condition (4). If (74a) is satisfied, ferroelectric ordering characterized by the order parameter (74) occurs in the system. By virtue of (73), this corresponds to macroscopic population of the resonant photon mode. Thus, the occurrence of spontaneous polarization in a ferroelectric can be regarded as the obtaining of population inversion by means of purely thermal pumping. Coherent electromagnetic superradiance in such a system can be realized, for example, by rapid repolarization of the ferroelectric in an external classical field. Such an effect has evidently been observed in $BaTiO_3$ single crystals below Θ_S.[103] A certain analogy can also be made with the lasing process in paramagnets realized for the first time in $La_2Mg_3(3O_3)_{12}\cdot 24H_2O$(LaMN) crystals with CO^+ and Ce^{3+} admixtures.[104]

Thus, when $\Theta<\Theta_S$ and an external classical field is abruptly reversed, population inversion occurs in a ferroelectric of KDP type, i.e., such a system can be regarded as a laser with thermal pumping. The radiation power in such a system can be estimated by means of the relation

$$U=|\mathscr{E}_S-\mathscr{E}_P|\tau, \quad (75)$$

where $\mathscr{E}_S=1/2\mathcal{J}\xi^2-E\tanh(E\Theta^{-1})$, $\mathscr{E}_P=-\varepsilon\tanh(\varepsilon\Theta^{-1})$, and τ is the characteristic relaxation time. The problem of determining τ will be investigated in detail in the following section.

We note further that in accordance with (71) the radiation frequency is determined by the tunneling integral. This changes appreciably from one substance to another. In addition, because of the isotope effect,[102] the radiation frequency must change on deuteration. The dependence of the tunneling integral on the mass of the tunneling particle is determined by the relation[105]

$$\varepsilon=\frac{\Omega_0}{2}\left\{1-\left[1-\frac{D}{\Omega_0}\Phi\left(\frac{D}{2\rho\Omega_0}\right)\right]+\frac{2\rho}{\sqrt{\pi}}\exp\left[-\left(\frac{D}{2\rho\Omega_0}\right)^2\right]\right\}e^{-\rho^2},$$

where

$$\rho^2=m\Omega_0\frac{(2\delta)^2}{\hbar};\quad \Phi(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}dy;$$

in which m is the mass of the tunneling particle, Ω_0 is the oscillator frequency in one of the minima of the potential of the hydrogen bond, 2δ is the distance between the minima ($2\delta - 3.5\times10^{-8}$ m for KDP), and D is the asymmetry parameter ($D\sim 250-350$ cm^{-1}).

In the considered case, there is also coherent quasi-Larmor precession of the quasispins described by the equation[46]

$$\frac{d\xi}{dt}=[\xi(t)\times\mathscr{E}(t)],\ \mathscr{E}=\{2\varepsilon,\ 0,\ 2\mathcal{J}\xi_z(t)\}.$$

The trajectory of the vector ξ is determined by the equations

$$\xi_x^2+\xi_y^2+\xi_z^2=C_1=\mathrm{const};\ \varepsilon\xi_x+\frac{1}{2}\mathcal{J}\xi_z^2=C_2=\mathrm{const};$$

$$t=\frac{-\mathcal{J}}{4\sqrt{2\varepsilon}\sqrt[4]{\varepsilon^2+\mathcal{J}(C_1\mathcal{J}-2C_2)}}\{F(\varphi(t),\ k)-F(\varphi(0),\ k)\},$$

where $F(\varphi,k)$ is an elliptic integral of the first kind, and

$$\varphi(t)=\arcsin\left\{\xi_z(t)\frac{2\mathcal{J}\sqrt{\varepsilon(C_2\mathcal{J}-\varepsilon^2+\gamma)}\sqrt[4]{\gamma}}{\sqrt{(C_1\varepsilon^2-C_2^2)[2\mathcal{J}^2(\gamma+\varepsilon^2-C_2\mathcal{J})+\xi_z^2(t)]}}\right\},$$

$$k\equiv\frac{\sqrt{C_1\varepsilon^2-C_2^2}}{2\mathcal{J}\sqrt{C_2\mathcal{J}-\varepsilon^2+\gamma}\sqrt[4]{\gamma}};\quad \gamma\equiv\varepsilon\sqrt{\varepsilon^2+\mathcal{J}(C_1\mathcal{J}-2C_2)}\ \text{(Fig. 8)}.$$

Further, using the relations (50) for the phonon operators, we can readily obtain the following expression for the deformation of the lattice.

$$\langle u_f\rangle=2\,k\xi/\hbar\Omega,$$

where $\langle u_f\rangle$ is the mean displacement of emitter f.

Our treatment can be readily generalized to ferroelectrics of types different from KDP.

Since ferroelectrics are characterized by a strong coupling between the dipole and phonon subsystems, striction and piezoelectric effects occur in them. The latter can also be used to create population inversion for $\Theta>\Theta_S$.[46] On the other hand, the effect of a short-wave classical field can lead under certain conditions to partial or complete compensation of the striction effects.[46] We now turn to a consideration of this possibility.

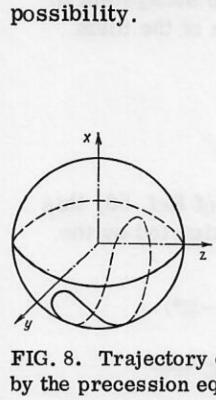

FIG. 8. Trajectory of the end of the vector $\xi(t)$ determined by the precession equation.

2.4. Dicke model on a lattice in an external classical field

The influence of an external classical field on the phase transition in the Dicke model was considered by Gilmore and Bowden.[19] In this case, the standard Dicke Hamiltonian must be augmented by a term of the form

$$\nu(x_f)(\sigma_f^+ + \sigma_f^-), \tag{76}$$

where $\nu(\cdot)$ are parameters that introduce the classical field at the point x_f. In Ref. 19, the simplest case $\nu(x_f) = \nu = \text{const}$ was considered and it was shown that for the order parameter in the M system the trivial solution is then not realized. More interesting situations are, however, realized when the emitters are placed at the sites of a compressible lattice and the wavelength of the classical field is comparable with the lattice parameter.[46] In this case, the total Hamiltonian of the system has the form

$$H = \hbar\omega a^+a + \sum_f \left\{ \frac{1}{2}\hbar\omega\sigma_f^z + \lambda N^{-1/2}(\sigma_f^+ a + \sigma_f^- a^+) + \nu(x_f)(\sigma_f^+ + \sigma_f^-) \right\} + \sum_q \hbar\Omega_q b_q^+ b_q. \tag{77}$$

We restrict our treatment to the case when the system is a resonator and the field ν is a sinusoidal standing wave with phase φ and wave vector k:

$$\nu(x_f) = \nu_0 \sin(kx_f - \varphi), \quad \nu_0 = \text{const}.$$

Let u_f be the displacement of emitter f: $u_f = x_f - f$. We expand (x_f) in the displacements u_f:

$$\nu(x_f) = \nu(f) + \sum_{n=1}^{\infty} \frac{1}{n!}(u_f \nabla_{u_f})^n \nu(f).$$

We restrict ourselves to the approximation linear in u_f and, going over in the usual manner to the second-quantization representation, we obtain instead of the last term in (77)

$$N^{-1/2}\sum_f (\sigma_f^- + \sigma_f^+) \sum_q [B_q(f)(b_q + b_{-q}^+) + B_q^*(f)(b_{-q} - b_q^+)];$$

$$B_q(f) = \tau_q k (2m\Omega_q)^{-1/2} e^{iqf} \nu_0 \cos(kf - \varphi).$$

Here, m is the mass of the "emitter," and τ_q is the unit polarization vector of the phonon with quasimomentum q. Assuming the presence of strong coupling between the dipole and phonon subsystems and instability of one or several phonon modes at the super-radiant phase transition, we restrict ourselves in the following to a finite number of phonon modes. Then, using the above methods, we obtain for the dipole subsystem in (77) an equivalent effective Hamiltonian of the form

$$H_d = \sum_f \left\{ \frac{1}{2}\hbar\omega\sigma_f^z + \nu(f)\sigma_f^x - N^{-1}\sum_{f'} \left[\frac{\lambda^2}{\hbar\omega}\sigma_f^+\sigma_{f'}^- + 2\sum_q \frac{B_q(f)B_q^*(f')}{\hbar\Omega_q}\sigma_f^x\sigma_{f'}^x \right] \right\}.$$

In accordance with the general method of Ref. 50, this Hamiltonian, in its turn, can be approximated by the operator

$$H_d^{(0)} = \sum_f \left\{ \frac{1}{2}\hbar\omega\sigma_f^z + \nu(f)\sigma_f^x - \frac{\lambda^2}{\hbar\omega}(\sigma_f^-\xi^* + \sigma_f^+\xi - \xi\xi^*) - \sum_q \frac{2}{\hbar\Omega_q}[B_q(f)\eta_q^* + B_q^*(f)]\sigma_f^x \right\} + \sum_q 2N\frac{1}{\hbar\Omega_q}\eta_q\eta_q^*.$$

The variational parameters ξ and η_q which occur here are determined from self-consistency equations of the form

$$\left.\begin{aligned} \xi &= -N^{-1}\sum_f z_f \frac{\text{th}(E_f/\Theta)}{E_f}, \\ \eta_q &= -N^{-1}\sum_f B_q(f)\,\text{Re}\, z_f \frac{\text{th}(E_f/\Theta)}{E_f}, \end{aligned}\right\} \tag{78}$$

where

$$E_f = \frac{1}{2}\sqrt{(\hbar\omega)^2 + 4z_f z_f^*},$$

$$z_f = \nu(f) - \frac{\lambda^2}{\hbar\omega}\xi - \sum_q \frac{2}{\hbar\Omega_q}[B_q(f)\eta_q^* + B_q^*(f)\eta_q].$$

Equations (78) make it possible to determine not only the modulus of the dipole order parameter ξ, as in the standard Dicke model, but also the argument $\arg\xi = \Psi$. The solutions of the system (78) depend strongly on the wavelength and the phase of the classical field. We shall consider only some characteristic situations. For simplicity and clarity, we restrict ourselves to the case when only one mode $\Omega_q = \Omega$ of the phonon field is effective, and we shall assume that the vector k is collinear with the Ox axis (quasi-one-dimensional crystal).

Case 1. The wavelength Δ of the classical field is equal to the lattice constant a and $\varphi = -(\pi/2)$ (see Fig. 8a). Then $\nu(f) = \nu_0$, $B(f) = 0$. The second equation in (78) has only the trivial solution $\eta = 0$. In other words, for such a choice of the field the system is "insensitive" to small thermal vibrations of the emitters in the crystal. From the first equation in (78) we have

$$\left.\begin{aligned} \text{Re}\,\xi &\equiv \xi_x = -\frac{\nu_0 - \lambda^2\xi_x/\hbar\omega}{E}\,\text{th}\frac{E}{\Theta}, \\ \text{Im}\,\xi &\equiv \xi_y = \frac{\lambda^2\xi_y}{\hbar\omega E}\,\text{th}\frac{E}{\Theta}, \end{aligned}\right\} \tag{79}$$

$$E \equiv \frac{1}{2}\sqrt{(\hbar\omega)^2 + 4(\nu_0 - \lambda^2\xi_x/\hbar\omega)^2}.$$

Hence, $\text{Im}\,\xi = 0$. The dependence of ξ_x on Θ for different relationships between ν_0 and $\lambda^2/\hbar\omega$ is shown in Fig. 9b. For each fixed ν_0 and $\lambda^2/\hbar\omega$ there exist two solution branches, one of which corresponds to the case $\xi_x > 0$ (broken curves), and the other to the case $\xi_x < 0$ (continuous curves). As $\nu_0 \to 0$, the branches approach each other, going over for $\nu_0 = 0$ into the ordinary solution for the Dicke model (chain curve). This situation gives a perspicuous illustration of Bogolyubov's concept of quasiaverages,[80] according to which an arbitrarily small perturbation can lead to a qualitative change in the behavior of the order parameter. Investigating the corresponding free energies, we can show that the solution with $\xi_x < 0$ is absolutely

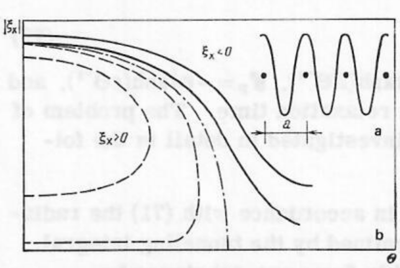

FIG. 9. Linear chain of emitters in a classical field with wavelength a and phase $\varphi = -\pi/2$ (a) and temperature dependence of the real order parameter ξ_x (b).

stable. We note that a similar result was obtained by Gilmore and Bowden[19] for a constant external classical field.

In the considered case, a phase transition in the system is absent (for $\nu>0$); the state of the system is characterized by a monotonically decreasing order parameter $|\xi_x|$.

Case 2. $\Lambda=a, \varphi=n\pi, n=0,1,\ldots$ (Fig. 10a). In this case

$$\nu(f)=0,\ B(f)=\nu_0 A(\hbar\Omega)^{-1/2},\ A\equiv 2\pi a^{-1}(2m)^{-1/2}.$$

Equations (78) become

$$\left.\begin{aligned}\xi&=\left[\frac{\lambda^2}{\hbar\omega}\xi+2\nu_0\frac{A}{(\hbar\Omega)^{3/2}}(\eta+\eta^*)\right]\frac{\operatorname{th}(E/\Theta)}{E};\\ \eta&=4\nu_0A(\hbar\Omega)^{-3/2}\left[\frac{\lambda^2}{\hbar\omega}\xi_x+4\nu_0\frac{A}{(\hbar\Omega)^{3/2}}\operatorname{Re}\eta\right]\frac{\operatorname{th}(E/\Theta)}{E},\end{aligned}\right\}\tag{80}$$

where

$$E\equiv\frac{1}{2}\sqrt{(\hbar\omega)^2+4\left|\frac{\lambda^2}{\hbar\omega}\xi+4\nu_0\frac{A}{(\hbar\Omega)^{3/2}}\operatorname{Re}\eta\right|^2}.$$

It can be seen from the second equation that $\eta=\operatorname{Re}\eta$, $\operatorname{Im}\eta=0$. Further, comparing the first and second equation in (80), we find

$$\eta=4\nu_0\frac{A}{(\hbar\Omega)^{3/2}}\xi_x.$$

We assume that $\xi_y\neq 0$. In this case we have $\xi_x=0$ for $\nu_0\neq 0$. For ξ_y we have the equation

$$\frac{1}{2}\xi_y\sqrt{(\hbar\omega)^2+4\lambda^4\xi_y^2/(\hbar\omega)^2}=\frac{\lambda^2}{\hbar\omega}\operatorname{th}\left(\frac{1}{2\Theta}\sqrt{(\hbar\omega)^2+4\lambda^4\xi_y^2/(\hbar\omega)^2}\right).$$

Thus, for $\Theta<\Theta_C=\hbar\omega/2\tanh^{-1}[(\hbar\omega)^2/2\lambda^2]$ there arises in the system an ordering characterized by the purely imaginary order parameter $\xi=i\xi_y$, the phonon order parameter vanishing: $\eta=0$.

Now suppose $\xi_x\neq 0$. In this case, $\xi_y=0$ and

$$\frac{1}{2}\xi_x\sqrt{(\hbar\omega)^2+4[\lambda^2/\hbar\omega+16\nu_0^2A^2(\hbar\Omega)^{-3}]^2\xi_x^2}=\left[\frac{\lambda^2}{\hbar\omega}+16\nu_0^2\frac{A^2}{(\hbar\Omega)^3}\right]\times\operatorname{th}\left\{\frac{1}{2\Theta}\sqrt{(\hbar\omega)^2+4[\lambda^2/\hbar\omega+16\nu_0^2A^2(\hbar\Omega)^{-3}]^2\xi_x^2}\right\}.$$

For

$$\Theta<\Theta_C'=h\omega/2\operatorname{Arth}\left\{\hbar\omega/2\left[\frac{\lambda^2}{\hbar\omega}+16\nu_0^2\frac{A^2}{(\hbar\Omega)^3}\right]\right\}$$

there arises in the system an ordering characterized by the purely real order parameter $\xi=\xi_x$. It is clear that $\Theta_C<\Theta_C'$, and that it is the solution with $\xi_x\neq 0$ that is absolutely stable (Fig. 10b). The strong-coupling condition takes the form

$$\hbar\omega<\frac{\lambda^2}{\hbar\omega}+16\nu_0^2\frac{A^2}{(\hbar\Omega)^3},$$

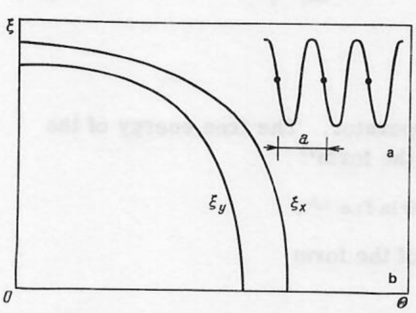

FIG. 10. Linear chain of emitters in a classical field with wavelength a and phase $\varphi=0$ (a) and temperature dependence of the real and imaginary parts of the order parameter ξ (b).

i.e., the presence of the external classical field in the special case $\Lambda=a$ and $\varphi=0$ leads to a significant change in the condition for a phase transition to the superradiant state.

Case 3. $\Lambda=2a$, $\varphi=+\pi/2$ (Fig. 11). In this case,

$$\sin(kf-\varphi)=\begin{cases}+1, & f=2n+1,\ n=1,2\ldots,\\ -1, & f=2n,\end{cases}\qquad \cos(kf-\varphi)=0$$

whence $B(f)=0$, i.e., $\eta=0$. The first equation in (78) takes the form

$$\xi=\left(\frac{\lambda^2}{\hbar\omega}\xi-\nu_0\right)\frac{\operatorname{th}(E_+/\Theta)}{2E_+}+\left(\frac{\lambda^2}{\hbar\omega}\xi+\nu_0\right)\frac{\operatorname{th}(E_-/\Theta)}{2E_-},$$

where

$$E_\pm=\sqrt{(\hbar\omega)^2+4|\nu_0\mp\lambda^2\xi^2(\hbar\omega)^{-1}|}.$$

The solutions of this equation agree qualitatively with those of case 1.

Thus, the presence of the short-wave external classical field in the system leads to a change in the strong-coupling condition and to a qualitative change in the behavior of the order parameter.

2.5. Calculation of macroscopic characteristics of nonlinear optical conversion

We must mention one further important example of a physical system for which the methods considered in Sec. 1 make it possible to obtain an exact (as $N\to\infty$) solution. We are referring to the description of many-photon processes in many-level systems in connection with the problem of nonlinear optical conversion of the frequency of coherent electromagnetic radiation from the infrared range to the region of visible light. This problem is very important in connection with the development of devices for visualization and detection of infrared radiation, which is of undoubted interest for applied problems of infrared spectroscopy, optical communication, infrared astronomy, etc.[106] We illustrate the process for the example of frequency conversion in sodium vapor, which was recently realized experimentally.[107,108]

The scheme of the working levels of the sodium atom used for the conversion is shown in Fig. 12. The $3S-4S$ transition is a dipole-forbidden transition, and is therefore excited by two-photon pumping by laboratory lasers (photons ω_1 and ω_2). The frequency of the $4S-4P$ transition corresponds to a transparency window of the atmosphere in the infrared range ($\Lambda_3\sim 1.06$ μm). The $4S-4P$ transition is excited by an external signal. The $4P-3S$ transition emission occurs at the total frequency, which corresponds to the visible part of the spectrum.

This process corresponds to conversion of a signal

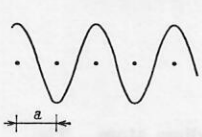

FIG. 11. Linear chain of emitters in a classical field with wavelength $2a$ and phase $\varphi=-\pi/2$.

with fixed frequency. However, such a many-level system can also be used for infrared spectroscopy. In this case, the investigated signal plays the part of one of the photons in exciting the $3S$–$4S$ transition (for example, ω_2), while the $4S$–$4P$ transition is excited by pumping from a laboratory laser. The pumping at the frequency ω_1 must be done by a frequency tunable laser.[108] In the scheme of this spectroscope, the frequency of the investigated signal is determined from the resonance conditions.

We should say that quantum nonlinear frequency conversion has already been realized experimentally in numerous gaseous and crystalline media, in particular, in proustite Ag_3AsS_3,[109] silver thiogalate, mercury thiogalate, sodium vapor,[122] etc. Important features of such conversion are the fact that it is instantaneous and that it has a low noise level.[107]

One of the main tasks associated with the description of the process is the determination of the conversion efficiency, i.e., the dependence of the intensity of the radiation field on the intensities of the signal and pumping fields. This necessarily involves the problem of describing a many-photon process in a many-level system. Such processes are usually described either by means of semiclassical perturbation theory[107,108] or by approximate methods of nonlinear mechanics.[109]

On the other hand, to describe m-photon resonance in a two-level system a microscopic approach was developed in Refs. 55, 56, 110, and 111 on the basis of the use of interaction terms of the form

$$g\sum_f \left(\sigma_f^- \prod_{i=1}^{m} a_i^\dagger + \sigma_f^+ \prod_{i=1}^{m} a_i\right),$$

where g is a parameter proportional to the dipole matrix element of the m-photon transition in the two-level atom (see also Ref. 112).

However, to describe the two-photon excitation in the system shown in Fig. 12, one can use a different device, namely, in accordance with Salour's idea[54] one can represent the two-photon pumping of the $3S$–$4S$ transition as a combination of two single-photon excitations, i.e., one can introduce an intermediate virtual level. We shall then be dealing with an effective four-level system with four-photon interaction.[43] A model for describng a many-level system was developed by Gilmore[213,214] and is characterized by a Hamiltonian of the form

$$H=\sum_{1\leqslant i<j\leqslant n}\hbar\omega_{ji}a_{ji}^{\dagger}a_{ji}+\sum_{f=1}^{N}\sum\varepsilon_i M_i(f)$$
$$+\frac{1}{\sqrt{N}}\sum_{f=1}^{N}\sum_{1\leqslant i<j\leqslant n}\lambda_{ji}\left[a_{ji}^{\dagger}E_{ij}(f)+a_{ji}E_{ji}^{\dagger}(f)\right]. \qquad (81)$$

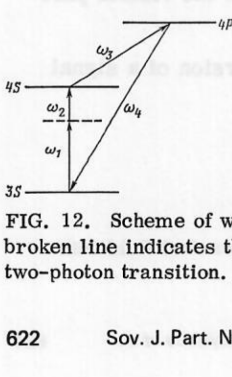

FIG. 12. Scheme of working levels of the sodium atom. The broken line indicates the virtual level used to describe the two-photon transition.

Here, n is the number of levels, and the number of modes is $\binom{n}{2}$, since to each pair of levels there corresponds one mode; the photon operators satisfy commutation relations of the form

$$a_{ji}=a_{ij}^{\dagger},\ [a_{j'i'},\ a_{ji}^{\dagger}]=\delta_{j'j}\delta_{i'i}.$$

The operator $E_{ji}(f)$ describes the transition from state i to state j in emitter f. The operators M and E are generators of the group $SU(n)$. We emphasize that the idea of describing the state of systems of n-level emitters by means of representations of the group $SU(n)$ was proposed by Shelepin.[115]

In Ref. 19, generalizations of the model (81) associated with allowance for external classical fields and "classical currents" were considered. The latter are introduced into the Hamiltonian by means of additional terms of the form[19]

$$h\sqrt{N}\,a^{+}+h^{*}\sqrt{N}\,a.$$

We now consider our effective four-level system with four-photon interaction (Fig. 12). In this case, the Hamiltonian (81) can be written in the form[43]

$$H=\sum_{i=1}^{4}\hbar\omega_i a_i^{\dagger}a_i+\sum_f\Big[\varepsilon_i M_i(f)+\frac{\lambda_i}{\sqrt{N}}\left(a_i^{\dagger}E_i(f)+a_iE_i^{\dagger}(f)\right)-\nu_i\left(E_i(f)+E_i^{\dagger}(f)\right)\Big]\Big\}. \qquad (82)$$

Here, ε_i is the energy of the corresponding level, and the parameters λ_i are determined by the relation

$$\lambda_i=\Delta\varepsilon_i\,d_i\sqrt{2\pi\hbar/\omega_i\mathscr{V}},$$

where $\Delta\varepsilon_i$ is the difference between the energies of the corresponding levels, and d_i is the matrix element of the operator of the dipole moment of the transition. The parameters ν_i describe the classical fields,

$$\nu_i=d_i\mathscr{E}_i,$$

where $\mathscr{E}_i$ is the intensity of the corresponding field. The operators $M_i(f)$ and $E_i(f)$ are, respectively, diagonal and step (nondiagonal) generators of the group $SU(4)$ corresponding to emitter f (Na atom).

The solution to the model problem with the Hamiltonian (82) can also be found by means of the general method set forth in Sec. 1 of the present paper. The approximating, thermodynamically equivalent Hamiltonian for (82) can be represented in the form

$$\widetilde{H}=\sum_i\Big\{\hbar\omega_i\tilde{a}_i^{\dagger}\tilde{a}_i+\sum_f\Big[\varepsilon_iM_i(f)-\Big(\frac{\lambda_i^2}{\hbar\omega_i}\xi_i+\nu_i\Big)\left(E_i(f)+E_i^{\dagger}(f)\right)\Big]+N\frac{\lambda_i^2}{\hbar\omega_i}\xi_i^2\Big\}, \qquad (83)$$

where

$$\tilde{a}_i=a_i+\frac{\lambda_i}{\hbar\omega_i}\xi_i\sqrt{N}\,1$$

and 1 is the identity operator. The free energy of the M system in (83) has the form[43]

$$F(\Theta,\ \xi)=\sum_i\frac{\lambda_i^2}{\hbar\omega_i}\xi_i^2-\Theta\ln\operatorname{Tr}e^{-Q/\Theta},$$

where Q is a matrix of the form

$$\begin{pmatrix}\varepsilon_1&\beta_1&0&\beta_4\\ \beta_1&\varepsilon_2&\beta_2&0\\ 0&\beta_2&\varepsilon_3&\beta_3\\ \beta_4&0&\beta_3&\varepsilon_4\end{pmatrix}$$

and $\beta_i \equiv -(\nu_i + \lambda_i^2/\hbar\omega_i \xi_i)$. The parameters ξ_i are determined from the condition of minimality of the function $F(\Theta, \xi)$, which leads to the system of equations

$$\xi_\alpha = \left\{-\sum_i \frac{\partial\varphi_i}{\partial\xi_\alpha} e^{-\varphi_i/\Theta}\right\}\left\{2\frac{\lambda_\alpha^2}{\hbar\omega_\alpha}\sum_i e^{-\varphi_i/\Theta}\right\}, \quad \alpha = 1, 2, 3, 4,$$

where φ_i are the roots of the characteristic equation $\det(Q - \varphi 1) = 0$, which is an algebraic equation of the fourth degree. Analytic solution of the system of equations for ξ_α for arbitrary Θ is not possible. Therefore, the following calculations must be made on a computer, which was done in Ref. 43 for numerical values of the parameters corresponding to the experimental situation for conversion in Na vapor.[107,108] Analysis of the numerical solution leads to the conclusion that there exists a dependence of the form

$$\mathscr{E}_4 = \gamma \mathscr{E}_1 \mathscr{E}_2 \mathscr{E}_3,$$

where $\gamma \propto \omega_4^2$. Such a dependence agrees well with the results of the experiments of Refs. 107 and 108. The nature of the dependence remains the same even when there is an appreciable change in the intensities of the pumping and signal fields. For the efficiency of the conversion of a signal with frequency ω_3 we have

$$\varkappa \equiv \left|\frac{\mathscr{E}_4}{\mathscr{E}_3}\right|^2 = \gamma^2 |\mathscr{E}_1 \mathscr{E}_2|,$$

which also agrees well with the results of the experiments under near-resonance conditions.

Thus, our study in the present section of "realistic" examples shows that the methods of exact solution set forth in Sec. 1 make it possible to obtain an adequate description for numerous concrete physical phenomena.

We now turn to the consideration of dynamical processes in super-radiant systems.

3. DYNAMICAL PROPERTIES OF MACROSCOPIC TWO-LEVEL SYSTEMS INTERACTING WITH ELECTROMAGNETIC FIELDS

3.1. Exact kinetic equation for the generalized Dicke model

As we mentioned in the Introduction, an exact description of the dynamics of a two-level M–F system was obtained only comparatively recently[75-77] through the extension to such systems of the methods developed by Bogolyubov[72,73] in connection with the investigation of the polaron problem. Following Ref. 75 we consider a system characterized by a Hamiltonian of the form

$$\left.\begin{aligned}
&H_t = H_M + H_F + H_{MF};\\
&H_M = \sum_{f=1}^{N} \frac{1}{2}\hbar\Omega_f \sigma_f^z + H_0;\\
&H_F = \sum_k \hbar\omega_k a_k^+ a_k;\\
&H_{MF} = \frac{1}{\sqrt{N}}\sum_{k,f} \lambda_k e^{\varepsilon t}\{a_k e^{ikx_f}(\sigma_f^+ + \mu\sigma_f^-) + a_k^+ e^{-ikx_f}(\sigma_f^- + \mu\sigma_f^+)\}.
\end{aligned}\right\} \tag{84}$$

Here, H_0 is the kinetic-energy operator of the emitters, t is the time, $\varepsilon \in \mathbf{R}$, and the "counter-rotating" terms with μ are introduced, as in Sec. 1, for generality.

We denote by $\mathscr{D}_t$ the statistical operator of the M–F system (84), which satisfies the Liouville equation

$$i\hbar\frac{\partial}{\partial t}\mathscr{D}_t = [H_t, \mathscr{D}_t] \tag{85}$$

with the initial conditions of the form

$$\left.\begin{aligned}
&\mathscr{D}_{t_0} = \rho(M)\mathscr{D}(F);\\
&\mathscr{D}(F) = e^{-H_F/\Theta}\Big/\operatorname*{Tr}_{(F)} e^{-H_F/\Theta};\\
&\operatorname*{Tr}_{(M)}\rho(M) = 1,
\end{aligned}\right\} \tag{86}$$

corresponding to the equilibrium state of the field, its interaction with the M system being assumed to be switched on at the initial time $t = t_0$. As is readily seen from (85) and (86), $\mathrm{Tr}\,\mathscr{D}_t = 1$.

Following Ref. 75, we introduce the Heisenberg representation for the dynamical variable $\mathfrak{U}(t, \mathrm{M}, \mathrm{F})$, which is given in the Schrödinger representation by

$$\mathfrak{A}(t, M_t, F_t) = U^{-1}(t, t_0)\,\mathfrak{A}(t, M, F)\,U(t, t_0),$$

where the unitary operator $U(t, t_0)$ is determined by the equation

$$i\hbar\frac{\partial}{\partial t}U(t, t_0) = H_t U(t, t_0);\quad U(t_0, t_0) = 1.$$

Let $\mathscr{O}(\mathrm{M})$ be an operator that acts on the eigenfunctions of the Hamiltonian (84) only as functions of the variables relating to the M system. The equation of motion for the operator $\mathscr{O}(\mathrm{M})$ in the Heisenberg representation can be transfc ..ned by the method of elimination of the boson variables[73] to the form

$$\begin{aligned}
&\operatorname*{Tr}_{(\bar{M})}\left\{\mathcal{O}(M)\frac{\partial\rho_t(M)}{\partial t} + (i\hbar)^{-1}\left[H_0 + \frac{1}{2}\sum_f \hbar\Omega_f\sigma_f^z,\ \mathcal{O}(M)\right]\rho_t(M)\right\}\\
&= N^{-1}\sum_k \lambda_k^2 \int_{t_0}^{t} d\tau \operatorname*{Tr}_{(M,F)} e^{-i\omega_k(t-\tau)} e^{\varepsilon(t+\tau)}\left\{N_k \sum_f e^{-ikx_f}[\sigma_f^-(\tau) + \mu\sigma_f^+(\tau)]\right.\\
&\times\left[\mathcal{O}(M_t),\ \sum_f e^{ikx_f}(\sigma_f^+(t) + \mu\sigma_f^-(t))\right] + (1 + N_k)\sum_f e^{ikx_f}[\sigma_f^+(t)\\
&\left. + \mu\sigma_f^-(t),\ \mathcal{O}(M_t)]\sum_f e^{-ikx_f}[\sigma_f^-(\tau) + \mu\sigma_f^+(\tau)]\right\}\mathscr{D}_{t_0} + N^{-1}\sum_k \lambda_k^2\\
&\times\int_{t_0}^{t} d\tau \operatorname*{Tr}_{(M,F)} e^{i\omega_k(t-\tau)} e^{\varepsilon(t+\tau)}(1 + N_k)\sum_f e^{ikx_f}(\sigma_f^+(\tau) + \mu\sigma_f^-(\tau))\\
&\times\mathcal{O}(M_t),\ \sum_f e^{-ikx_f}(\sigma_f^-(t) + \mu\sigma_f^+(t))\\
&+ N_k\left[\sum_f e^{-ikx_f}(\sigma_f^-(t) + \mu\sigma_f^+(t)),\ \mathcal{O}(M_t)\right]\sum_f e^{ikx_f}(\sigma_f^+(\tau) + \sigma_f^-(\tau)\Big\}\mathscr{D}_{t_0}.
\end{aligned}$$

Здесь $\rho_t(M) = \operatorname*{Tr}_{(M)}\mathscr{D}_t$ и

$$N_k = \frac{e^{-\hbar\omega_k/2\Theta}}{2\,\mathrm{sh}(\hbar\omega_k/2\Theta)}. \tag{87}$$

Here, $\rho_t(\mathrm{M}) = \mathrm{Tr}_{(\mathrm{M})}\mathscr{D}_t$ and

$$\nu_\alpha = 2\pi n_\alpha/L_\alpha, \quad \alpha = x, y, z,$$

In a number of cases, it is more convenient to use the formalism of collective operators of the emitters[75,116] instead of the individual quasispin variables σ_f. Suppose the vector ν corresponds to modes in the working volume of the resonator:

$$R_\nu^\pm = \sum_f \sigma_f^\pm e^{\pm i\nu x_f};\quad R_\nu^z = \sum_f \sigma_f^z e^{i\nu x_f};\quad \Omega_\nu = \sum_f \Omega_f e^{i\nu x_f}.$$

where n_α are integers, and $\Pi_\alpha L_\alpha = V$. Then the collective operators of the emitters can be defined by

$$N^{-1}\sum_f e^{i(\nu-\nu')x_f} = \delta_{\nu\nu'};\quad N^{-1}\sum_f e^{i\nu(x_f - x_{f'})} = \delta_{ff'},$$

Taking into account the obvious relations we can reduce Eq. (87) after transition to the collective variables to the form

$$\operatorname*{Tr}_{(M)}\left\{\mathcal{O}\, M)\frac{\partial\rho_t(M)}{\partial t}+(i\hbar)^{-1}\left[\sum_\nu \frac{1}{2}\hbar\Omega_\nu R^z_\nu+H_0,\ \mathcal{O}(M)\right]\rho_t(M)\right\}$$
$$=N^{-1}\sum_{k,\nu}\lambda_k^2\int_{t_0}^{t} d\tau \operatorname*{Tr}_{(M,F)} e^{-i\omega_k(t-\tau)}e^{\varepsilon(t+\tau)}$$
$$\times\{N_k Q^+_\nu(\tau)[\mathcal{O}(M_t),\ Q_\nu(t)]+(1+N_k)[Q_\nu(t)\mathcal{O}(M_t)]Q^+_\nu(\tau)\}\mathcal{D}_{t_0} \tag{88}$$
$$+N^{-1}\sum_{k,\nu}\lambda_k^2\int_{t_0}^{t} d\tau \operatorname*{Tr}_{(M,F)} e^{i\omega_k(t-\tau)}e^{\varepsilon(t+\tau)}$$
$$\times\{(1+N_k)Q_\nu(\tau)[\mathcal{O}(M_t),\ Q^+_\nu(t)]+N_k[Q^+_\nu(t),\ \mathcal{O}(M_t)]Q_\nu(\tau)\}\mathcal{D}_{t_0},$$

where we have used the notation

$$Q_\nu(t)\equiv R^+_\nu(t)\,\varphi(k-\nu)+\mu R^-_\nu(t)\,\varphi(k+\nu);\ \ \varphi(z)\equiv N^{-1}\sum_f e^{izx_j}.$$

Equation (87) and the equivalent equation (88) in the limit $t_0\to-\infty$ are a generalized kinetic equation for the dynamical system with the Hamiltonian (84).

We now suppose that

$$\mathcal{D}_F=|0><0|;\quad H_0=0, \tag{89}$$

i.e., at the initial time there is no radiation in the system and the emitters are rigidily fixed at the sites of the crystal lattice. We shall also assume that the distribution of the eigenfrequencies Ω_f of the emitters is symmetric about some frequency Ω and does not depend on the emitter coordinates x_f. We denote by $J(\Omega_f)$ the distribution function of the frequencies and average $\exp(i\Omega_f t)$ with allowance for the inhomogeneous Lorentz broadening[2]:

$$\langle e^{i\Omega_f t}\rangle\equiv e^{i\Omega t}\int_{-\infty}^{+\infty} e^{iWt}J(W)\,dW=e^{i\Omega t}e^{-\frac{|t|}{2T}},$$

where T is the so-called oscillator lifetime. We note that usually only the simplest case $\Omega_f=\Omega=\text{const}$ is considered. We assume further that $\varepsilon=0$ and ignore the rapidly oscillating terms of the type R^+R^+ and R^-R^-. We shall assume that $T\gg t\gg t_{\max}\sim L_\alpha/c$, where c is the velocity of light,[2] and we bear in mind that the function $\varphi(k-\nu)$ has a sharp peak at $k=\nu$. Then Eq. (88) can be transformed to[75]

$$\operatorname*{Tr}_{(M)}\left\{\mathcal{O}(M)\frac{\partial\rho_t(M)}{\partial t}+(i)^{-1}\frac{1}{2}\Omega\sum_\nu[R^z_\nu,\ \mathcal{O}(M)]\rho_t(M)\right\}$$
$$=\frac{1}{2}\sum_\nu\{\Gamma^-_\nu[R^+_\nu,\ \mathcal{O}(M)]R^-_\nu+\Gamma^+_\nu[R^-_\nu,\ \mathcal{O}(M)]R^+_\nu+\text{h.c.}\}\mathcal{D}_{t_0}, \tag{90}$$

where

$$\Gamma^\pm_\nu\equiv\gamma^\pm_\nu-i\Omega^\pm_\nu;$$
$$\gamma^\pm_\nu\equiv 2(2\pi)^{-3}\int dk\frac{2T\lambda_k^2\varphi^2(k-\nu)}{1+4T^2(\omega_k\pm\Omega)^2};$$
$$\Omega^\pm_\nu\equiv 8(2\pi)^3T^2\int dk\,\lambda_k^2\varphi^2(k-\nu)\frac{\omega_k\pm\Omega}{1+4T^2(\omega_k\pm\Omega^2)}.$$

We define the operator

$$S_\nu\equiv\sum_{ff'}\sigma^+_f\sigma^-_{f'}e^{i\nu(x_f-x_{f''})}.$$

Then from (90) we have

$$\operatorname*{Tr}_{(M)}\left\{\mathcal{O}(M)\frac{\partial\rho_t(M)}{\partial t}+i(\Omega+\Omega')\left[\mathcal{O}(M),\ \sum_\nu R^z_\nu\right]\rho_t(M)\right.$$
$$-\frac{i}{2}\sum_\nu\tilde{\Omega}_\nu[\mathcal{O}(M),\ S_\nu]\rho_t(M)$$
$$=\frac{1}{2}\sum_\nu\gamma^-_\nu\{[R^+_\nu,\ \mathcal{O}(M)]R^-_\nu+R^+_\nu[\mathcal{O}(M),\ R^-_\nu]\}\mathcal{D}_{t_0} \tag{91}$$
$$+\frac{1}{2}\sum_\nu\gamma^+_\nu\{[R^-_\nu,\ \mathcal{O}(M)]R^+_\nu+R^-_\nu[\mathcal{O}(M),\ R^+_\nu]\}\mathcal{D}_{t_0}.$$

Here,

$$\Omega'\equiv\frac{1}{2}\sum_\nu(\Omega^+_\nu+\Omega^-_\nu)$$

determines a constant frequency shift—the so-called Bethe part of the Lamb shift[116] whereas

$$\tilde{\Omega}_\nu\equiv\Omega^+_\nu+\Omega^-_\nu$$

determines the collective shift of the frequencies. Note that in the limit $T\to+\infty$ Eq. (91) is identical to the Markov master equation of Ref. 116 obtained in the weak-coupling approximation. The simplest variant of Eq. (91) was obtained for the single-mode case in this approximation in Ref. 65. Such an equation describes collective spontaneous emission.[116] We emphasize that Eq. (91), which was established in Ref. 75, is obtained as a special case of the generalized kinetic equation (88) in collective variables with the initial conditions (89) for inhomogeneous Lorentz broadening and a system of emitters rigidly fixed at the sites of the crystal lattice. On the basis of the generalized equation (88) one can also obtain other results of the theory of super-radiant systems, for example, an equation for the Bloch angle ψ, which is introduced by the relation[2]

$$R_\nu=\frac{1}{2}N\cos\psi.$$

Determining the dependence of the radiation intensity on the time by

$$I(t)=-\frac{\partial}{\partial t}\langle R^z\rangle,$$

we can also readily obtain from (91) Eberly's equation for $I(t)$.[117] We emphasize that the approach to the description of the dynamics of a two-level macroscopic M–F system developed in Refs. 75–77 and based on the use of the exact kinetic equation (87) or (88) is the most general.

3.2. Relaxation processes in macroscopic two-level systems

In the investigation of the dynamics of super-radiant systems, it is very important to have an adequate determination of the characteristic times of conversion (relaxation to the de-excited state), since this is one of the determining factors in the choice of the working medium and the regime and also in estimating the power of super-radiant lasers[63,63] (see also Ref. 46). To this end, the generalized kinetic equation (88) was used in Ref. 76 to obtain an equation for the single-particle distribution function[118,119]

$$W_t(p,\ m)=\operatorname*{Tr}_{(2,\ldots,N)}\rho_t(p_1,\ m_1;\ \ldots,\ P_N,\ m_N),$$

where p_t is the momentum of particle f, and m_f is the index of the "quasispin state." Following Ref. 76, we define the function $\mathcal{O}(M)$ as $\mathcal{O}(p_f,\sigma^2_f)$. Such operators satisfy commutation relations of the form

$$e^{ikx_f}\mathcal{O}(p_f)=\mathcal{O}(p_f-\hbar k)\,e^{ikx_f};$$
$$\mathcal{O}(p_f)\,e^{ikx_f}=e^{ikx_f}\mathcal{O}(p_f+\hbar k);$$
$$\sigma^+_f\mathcal{O}(\sigma^z_f)=\mathcal{O}(\sigma^z_f-1)\,\sigma^+_f;$$
$$\sigma^-_f\mathcal{O}(\sigma^z_f)=\mathcal{O}(\sigma^z_f+1)\,\sigma^-_f.$$

On the other hand, for $f\neq f'$

$$[\sigma_f^{\pm} e^{\pm ikx_f};\ \mathcal{O}(p_f,\ \sigma_f^z)] = 0.$$

With allowance for these commutation relations, the commutators on the right-hand side of the kinetic equation (87) can be represented in the form

$$[\mathcal{O}(p_f,\ \sigma_f^z),\ R_k^{\pm}] = \{\mathcal{O}(p_f,\ \sigma_f^z) - \mathcal{O} p_f \mp \hbar k,\ \sigma_f^z \mp 1\}\, \sigma_f^{\pm} e^{\pm ikx_f}. \quad (92)$$

We now assume that the interaction in H_{MF} is small, and we express $R_k^{\pm}(\tau)$ approximately in terms of $R_k^{\pm}(t)$. In the "zeroth approximation"[73,76] we have

$$x_f(\tau) = x_f(t) - \frac{p_f(t)}{m}(t-\tau);$$
$$\sigma_f^{\pm}(\tau) = \sigma_f^{\pm}(t)\, e^{\mp i\Omega(t-\tau)}.$$

For simplicity, we here consider the single-mode case: $\Omega_f \equiv \Omega$.

By means of the operator relation[119,120]

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,\ B]}$$

we can now readily obtain

$$e^{ikx_f(\tau)} = e^{-\frac{i}{\hbar}\frac{p_f^2(t) - [p_f(t) - \hbar k]^2 (t-\tau)}{2m}}\, e^{ikx_f(t)}$$
$$= e^{ikx_f(t)}\, e^{-\frac{i}{\hbar}\frac{[p_f(t)-\hbar k]^2 - p_f^2(t)}{2m}(t-\tau)}.$$

For $R_k^{\pm}(\tau)$, we have

$$R_k^{\pm}(\tau) = \sum_f \sigma_f^{\pm}(\tau)\, e^{\pm ikx_f(\tau)}$$
$$= \sum_f \sigma_f^{\pm}(t)\, e^{\pm ikx_f(t)}\, e^{-\frac{i}{\hbar}\left\{\frac{[p_f \pm \hbar k]^2 - p_f^2}{2m} \pm \hbar\Omega\right\}(t-\tau)} \quad (93)$$
$$= \sum_f e^{-\frac{i}{\hbar}\left\{\frac{[p_f(t) \mp \hbar k]^2 - p_f^2(t)}{2m} \mp \hbar\Omega\right\}(t-\tau)}\, \sigma_f^{\pm}(t)\, e^{\pm ikx_f(t)}.$$

Using (92) and (93) to calculate the commutators on the right-hand side of Eq. (87) and bearing in mind that by virtue of our assumption that the interaction in H_{MF} is small the terms with factors of the type

$$e^{\pm ik(x_{f'} - x_f)}\, \sigma_{f'}^{\pm} \sigma_f^{\mp}$$

will give only a small contribution after averaging and summation, we can represent the kinetic equation in the form

$$\begin{aligned}
\mathop{\mathrm{Tr}}_{(M)} \mathcal{O}(p_f, \sigma_f^z)\, \frac{\partial \rho_t(M)}{\partial t} &= N^{-1} \sum_k |\lambda_k|^2 \int_{t_0}^{t} d\tau \mathop{\mathrm{Tr}}_{(M)} e^{i\omega_k \tau}(1+N_k) \\
&\quad \times \exp\left\{\frac{i}{\hbar}\left[\frac{(p_f - \hbar k)^2 - p_f^2}{2m} - \hbar\Omega\right](t-\tau)\right\} \\
&\quad \times \{\mathcal{O}(p_f - \hbar k,\ \sigma_f^z - 1) - \mathcal{O}(p_f,\ \sigma_f^z)\}\, \sigma_f^{+}\sigma_f^{-}\rho_t \\
&\quad + N^{-1}\sum_k |\lambda_k|^2 \int_{t_0}^{t} d\tau \mathop{\mathrm{Tr}}_{(M)} e^{-i\omega_k\tau} N_k \\
&\quad \times \exp\left\{\frac{i}{\hbar}\left[\frac{(p_f + \hbar k)^2 - p_f^2}{2m} + \hbar\Omega\right](t-\tau)\right\} \\
&\quad \times \{\mathcal{O}(p_f + \hbar k,\ \sigma_f^z + 1) - \mathcal{O}(p_f,\ \sigma_f^z)\}\, \sigma_f^{-}\sigma_f^{+}\rho_t,
\end{aligned} \quad (94)$$

where $\rho_t \equiv \mathrm{Tr}_{(F)} \mathscr{D}_{t_0}$. Since

$$\sigma_f^{+}\sigma_f^{-} = \frac{1}{2}(1 + \sigma_f^z),\ \sigma_f^{-}\sigma_f^{+} = \frac{1}{2}(1 - \sigma_f^z),$$

we have

$$\mathop{\mathrm{Tr}}_{(M)} \{\mathcal{O}(p_f,\ \sigma_f^z)\, \sigma_f^{\pm}\sigma_f^{\mp}\rho_t\}$$
$$= \sum_m \int d\rho\, \mathcal{O}(p,\ m)\, \delta\left(m \mp \frac{1}{2}\right) W_t(p,\ m).$$

Therefore, using the relations (93) and (94), we obtain an equation for the single-particle distribution function $W_t(p, m)$:

$$\sum_m \int dp\, \mathcal{O}(p,\ m)\, \frac{\partial}{\partial t} W_t(p,\ m)$$
$$= 2\pi\hbar N^{-1} \sum_m \sum_k |\lambda_k|^2 \int dp \left\{(1+N_k)\, \delta\right.$$
$$\times \left(\hbar\omega_k - \hbar\Omega + \frac{p^2 - (p+\hbar k)^2}{2m}\right) \delta\left(m + \frac{1}{2}\right) W_t(p + \hbar k,\ m+1)$$
$$- (1+N_k)\, \delta\left(\hbar\omega_k - \hbar\Omega - \frac{p^2 - (p+\hbar k)^2}{2m}\right) \delta\left(m - \frac{1}{2}\right) W_t(p,\ m)$$
$$+ N_k \delta\left(\hbar\omega_k - \hbar\Omega - \frac{p^2 - (p+\hbar k)^2}{2m}\right) \delta\left(m - \frac{1}{2}\right) W_t(p + \hbar k,\ m-1)$$
$$\left. - N_k \delta\left(\hbar\omega_k - \hbar\Omega + \frac{p^2 - (p+\hbar k)^2}{2m}\right) \delta\left(m + \frac{1}{2}\right) W_t(p,\ m)\right\} \mathcal{O}(p,\ m).$$

Since $\mathcal{O}(p, m)$ is an arbitrary function, we obtain from here the following equation for the single-particle distribution function:

$$\frac{\partial}{\partial t} W_t(p,\ m) = 2\pi\hbar N^{-1} \sum_k |\lambda_k|^2 \left\{(1+N_k)\, W_t\left(p + \hbar k,\ \frac{1}{2}\right)\right.$$
$$\left. - N_k W_t\left(p,\ -\frac{1}{2}\right)\right\} \delta\left(m + \frac{1}{2}\right) \delta\left(\hbar\omega_k - \hbar\Omega + \frac{p^2 - (p+\hbar k)^2}{2m}\right)$$
$$+ 2\pi\hbar N^{-1} \sum_k |\lambda_k|^2 \Big\} N_k W_t\left(p + \hbar k,\ -\frac{1}{2}\right) - (1+N_k)$$
$$\times W\left(p,\ \frac{1}{2}\right)\Big\}\, \delta\left(m - \frac{1}{2}\right) \delta\left(\hbar\omega_k - \hbar\Omega + \frac{(p+\hbar k)^2 - p^2}{2m}\right).$$

Taking into account the explicit form of the function λ_k and going over from summation over k to integration, we obtain the final form of the equation for the single-particle distribution function[76]:

$$\left.\begin{aligned}
&\frac{\partial}{\partial t} W_t\left(p,\ -\frac{1}{2}\right) \\
&= \frac{d^2}{\pi} \int dk\, \frac{1}{\omega_k}\left\{(1+N_k)\, W_t\left(p + \hbar k,\ \frac{1}{2}\right)\right. \\
&\left. - N_k W_t\left(p,\ -\frac{1}{2}\right)\right\} \delta\left(\hbar\omega_k - \hbar\Omega + \frac{p^2 - (p+\hbar k)^2}{2m}\right); \\
&\frac{\partial}{\partial t} W_t\left(p,\ \frac{1}{2}\right) = \frac{d^2}{\pi} \int dk\, \frac{1}{\omega_k}\left\{N_k W_t\left(p + \hbar k,\ -\frac{1}{2}\right)\right. \\
&\left. - (1+N_k)\, W_t\left(p,\ \frac{1}{2}\right)\right\} \delta\left(\hbar\omega_k - \hbar\Omega + \frac{(p+\hbar k)^2 - p^2}{2m}\right).
\end{aligned}\right\} \quad (95)$$

Comparing Eq. (95) with the so-called Pauli kinetic-balance equation (see, for example, Ref. 121), we can readily show that the probability that an emitter with momentum p absorbs a photon and goes over to the state $|+\rangle$ is

$$\Gamma_+ = \frac{d^2}{\pi} \int dk\, \frac{1}{\omega_k} N_k \delta\left(\hbar\omega_k - \hbar\Omega + \frac{p - (p+\hbar k)^2}{2m}\right).$$

The probability of transition to the state $|-\rangle$ with emission of a photon has the form

$$\Gamma_- = \frac{d^2}{\pi} \int dk\, \frac{1}{\omega_k}(1+N_k)\, \delta\left(\hbar\omega_k - \hbar\Omega + \frac{(p+\hbar k)^2 - p^2}{2m}\right).$$

The corresponding characteristic relaxation times are defined as

$$\tau_{\pm} = \Gamma_{\pm}^{-1}.$$

Further, using the well-known property $\delta(ax) = 1/|a|\,\delta(x)$ of the δ function and the relation $\omega_k = kc$, we obtain

$$\delta\left(\hbar\omega_k - \hbar\Omega \mp \frac{(p+\hbar k)^2 - p^2}{2m}\right)$$
$$= \frac{m}{\hbar pk}\, \delta\left(\frac{mc}{p} - \frac{m\Omega}{pk} \mp \left(\cos\varphi + \frac{\hbar k}{2p}\right)\right),$$

where φ is the angle between the vectors k and p. Making the natural assumption $\hbar\Omega \ll mc^2$, we obtain for $\tau_{\pm}$

$$\tau_{\pm} = \beta\, \frac{\hbar^2 pc^2}{2md}\left\{\ln \frac{e^{\mp\beta\hbar k_- c} - 1}{e^{\mp\beta\hbar k_+ c} - 1}\right\},$$

where

$$k_{\pm} = \Omega/(c \pm p/m).$$

For nonrelativistic particles ($p \ll mc$) we obtain from this in the case of high temperatures ($\beta \ll 1/\hbar\Omega$)

$$\tau_+ = \tau_- = \tau = \frac{\hbar^2 c^3}{4d^2}\beta.$$

This expression corresponds to the relaxation time in a system of uncorrelated emitters.

At low temperatures ($\beta \gg 1/\hbar\Omega$), when the super-radiant state is realized in the system, we obtain

$$|\tau_+ \to \infty, \quad \tau_- = \hbar c^3/4d^2\Omega. \tag{96}$$

This expression for the conversion time τ_- agrees well with the standard estimate for the time corresponding to the emission maximum in a super-radiant laser.[62,63]

3.3. Estimate of the radiation power in a ferroelectric super-radiant laser

We now show how relations of the type (96) can be used to estimate the lasing power in a super-radiant laser. As an example, we consider the laser effect that arises on the repolarization of a ferroelectric of KDP type as considered in Sec. 2.2. As we pointed out, the radiation power in such a system can be estimated by means of the relation (75). We now estimate the conversion time τ on the right-hand side of (75). Note that for this we would proceed as in Secs. 3.1 and 3.2 for the system with the Hamiltonian (70). There is, however, a different way of determining the probability Γ_- and, therefore, the conversion time τ; it is associated with the use of the formalism of two-time thermal Green's functions, which was introduced into statistical mechanics by Bogolyubov and Tyablikov.[123]

For generality, we consider a system characterized by a Hamiltonian of the form

$$H = \sum_k \hbar\omega_k a_k^+ a_k + \hbar\Omega \sum_f \sigma_f^z + \frac{\lambda}{2\sqrt{N}} \sum_{k,f} (a_k^+\sigma_f^- + a_k\sigma_f^+) - \sum_{f,f'} J(f, f')\,\sigma_f^+\sigma_{f'}^-. \tag{97}$$

Here, the last term describes the direct dipole–dipole interaction. Following Ref. 124, we consider a retarded Green's function of the form

$$G_{gf}(t-t') = \langle\langle \sigma_g^-(t) \mid \sigma_f^+(t') \rangle\rangle$$

and write down for it the equation of motion

$$\begin{aligned} i\hbar \frac{d}{dt} G_{gf}(t-t') &= \delta(t-t')(1-2\langle n\rangle) + \hbar\Omega G_{gf}(t-t') + \frac{\lambda}{2\sqrt{N}} \sum_k G_k^{(1)}(t-t') \\ &- \frac{\lambda}{\sqrt{N}} \sum_k G_k^{(2)}(t-t') + \sum_p J(g,p)\, G_{pf}(t-t') \\ &- 2\sum_p J(p,f)\,\langle\langle \sigma_g^+\sigma_g^-\sigma_p^- | \sigma_f^+\rangle\rangle. \end{aligned} \tag{98}$$

Here

$$\langle n\rangle \equiv \langle \sigma_g^+(0)\,\sigma_g^-(0)\rangle;$$
$$G_k^{(1)}(t-t') \equiv \langle\langle a_k(t) | \sigma_f^+(t')\rangle\rangle;$$
$$G_k^{(2)}(t-t') \equiv \langle\langle a_k(t)\,\sigma_g^+(t)\,\sigma_g^-(t) | \sigma_f^+(t')\rangle\rangle.$$

The equations of motion for the functions $G_k^{(1)}$ and $G_k^{(2)}$ are

$$i\hbar \frac{d}{dt} G_k^{(1)}(t-t') = 2\delta(t-t')\langle a_k(0)\,\sigma_f^+(0)\rangle + \hbar\omega_k G_k^{(1)}(t-t') + \frac{\lambda}{2\sqrt{N}} \sum_p G_{pf}(t, t'); \tag{99}$$

$$i\hbar \frac{d}{dt} G_k^{(2)}(t-t') = \delta(t-t')\langle a_k(0)\,\sigma_f^+(0)\,n_g(0)\rangle + \frac{\lambda}{2\sqrt{N}} \sum_{p\neq g} \langle\langle n_g\sigma_p^- | \sigma_f^+\rangle\rangle - \frac{\lambda}{2\sqrt{N}} \sum_q \langle\langle a_q^+ a_k \sigma_g^- \sigma_f^+\rangle\rangle. \tag{100}$$

Assuming, as in the previous subsection, that the interaction is weak, and making the decouplings

$$\langle\langle n_g\sigma_p^- | \sigma_f^+\rangle\rangle = \langle n\rangle\, G_{pf}(t-t');$$
$$\langle\langle a_q^+ a_k \sigma_q^- | \sigma_f^+\rangle\rangle = \delta_{qk}\langle \nu_k\rangle\, G_{gf}(t-t');$$
$$\langle \nu_k\rangle \equiv \langle a_k^+ a_k\rangle,$$

we obtain after transition to the E representation

$$\begin{aligned} EG_{gf}(E) &= \frac{1}{2\pi}(1-2\langle n\rangle)\,\delta_{gf} + \hbar\Omega G_{gf}(E) + \frac{\lambda}{2\sqrt{N}} \sum_k \{G_k^{(1)}(E) - 2G_k^{(2)}(E)\} \\ &+ \sum_p J(g,p)\, G_{pf}(E) - 2\sum_p J(g,p)\langle n\rangle\, G_{pf}(E); \end{aligned}$$

$$EG_k^{(1)}(E) = \hbar\omega_k G_k^{(1)}(E) + \frac{\lambda}{2\sqrt{N}} \sum_p G_{pf}(E) + \frac{1}{\pi}\langle a_k\sigma_f^+\rangle;$$

$$EG_k^{(2)}(E) = \hbar\omega_k G_k^{(2)}(E) + \frac{\lambda}{2\sqrt{N}} \sum_{p\neq g} \langle n\rangle\, G_{pf} - \frac{\lambda}{2\sqrt{N}}\langle \nu_k\rangle\, G_{gf}(E) + \frac{1}{2\pi}\langle a\sigma_f^+ n_g\rangle.$$

From this we find

$$\begin{aligned} (E-\hbar\Omega)\,G_{gf}(E) &= \frac{1}{2\pi}(1-2\langle n\rangle)\,\delta_{gf} + \frac{\lambda^2}{4N}\sum_k \frac{1}{E-\hbar\omega_k}\sum_p G_{pf}(E) - \frac{\lambda^2}{2N}\sum_p \frac{1}{E-\hbar\omega_k}\sum_p \langle n\rangle\, G_{pf}(E) \\ &+ \frac{\lambda^2}{2N}\langle n\rangle \sum_k \frac{\langle n\rangle + \langle \nu_k\rangle}{E-\hbar\omega_k} G_{gf}(E) \\ &+ \sum_p J(g,p)\, G_{gf}(E) - 2\sum_p J(g,p)\langle n\rangle\, G_{gf}(E). \end{aligned} \tag{101}$$

For a system on a lattice of the type of a ferroelectric with hydrogen bonds, it is necessary to take into account the presence of translational invariance, whence

$$G_{gf}(E) = N^{-1}\sum_q e^{i(g-f,\,q)}\,G_q(E);\quad \delta_{gf} = N^{-1}\sum_q e^{i(g-f,\,q)}.$$

From (101), we now obtain an equation for the function G_q in the E representation:

$$\begin{aligned} (E-\hbar\Omega)\,G_q(E) &= \frac{1}{2\pi}(1-2\langle n\rangle) + \frac{\lambda^2}{4N}\sum_k \frac{1-2\langle n\rangle}{E-\hbar\omega_k} G_q(E) \\ &+ \frac{\lambda^2}{2N}\sum_k \frac{\langle n\rangle + \langle \nu_k\rangle}{E-\hbar\omega_k} G_q(E) + J(q)(1-2\langle n\rangle)\,G_q(E), \end{aligned} \tag{102}$$

where

$$J(q) \equiv \sum_p J(g-p)\, e^{i(g-p,\,q)}.$$

From Eq. (102), we have

$$G_q(E) = \frac{1}{2\pi}\,\frac{1-2\langle n\rangle}{E-\hbar\Omega - J(q)(1-2\langle n\rangle) - M(E)},$$

where the mass operator is

$$M(E) = \frac{\lambda^2}{4N}\sum_k \frac{1+2\langle \nu_k\rangle}{E-\hbar\omega_k}.$$

For small damping, we obtain from here for the probability of transition to the state $|-\rangle$ with emission of a photon

$$\Gamma = \pi\frac{\lambda^2}{4N}\sum_k (1+2\langle \nu_k\rangle)\,\delta(\hbar\Omega - \hbar\omega_k - J(q)(1-2\langle n\rangle)).$$

Bearing in mind that

$$\langle \nu_k\rangle = (e^{\beta\hbar\omega_k} - 1)^{-1}$$

and going over from summation over k to integration, we obtain finally the general expression for the probability of transition to the de-excited state:

$$\Gamma = \frac{2d^2}{\hbar^2 c^3}\{\hbar\Omega + J(q)(1-2\langle n\rangle)\}\{1+2[e^{\beta(\hbar\Omega + J(q)(1-2\langle n\rangle))}-1]^{-1}\}.$$

In the special case of low temperatures, when the system goes over into the ferroelectric phase, we have

$$\Gamma \cong \frac{2d\Omega}{\hbar c^3}\left\{1+\frac{J(q)}{\hbar\Omega}(1-2\langle n\rangle)\right\}.$$

Accordingly, for the conversion time as $\beta \to \infty$ we find

$$\tau_{\beta\to\infty} = \frac{\hbar c^3}{2d\Omega}\left\{1+\frac{J(q)}{\hbar\Omega}(1-2\langle n\rangle)\right\}^{-1}. \quad (103)$$

Substituting now (103) in the expression (75), we obtain for the radiation power the estimate

$$U = \left|\frac{1}{2}\mathcal{J}\xi^2 - \mathrm{E th}\,(E\beta) - \varepsilon\,\mathrm{th}\,(\varepsilon\beta)\right| \frac{\hbar c^3}{2d\Omega}\left\{1+\frac{J(q)}{\hbar\Omega}(1-2\langle n\rangle)\right\}^{-1}. \quad (104)$$

Substituting in (104) the parameters of a specific ferroelectric crystal of KDP type, we can estimate the radiation power in a fairly wide range of temperatures.

CONCLUSIONS

We briefly summarize our study in the present paper of the methods of investigating the equilibrium and dynamical properties of macroscopic systems interacting with boson fields.

We note first that the methods set forth in Secs. 1 and 3 apply for the investigation of a very large class of model problems. Besides the problems of quantum radio physics considered in the review, we can mention in this connection the problem of the structural phase transition in media with magnetic ordering,[84-86] the problem of phase transitions of the metal–dielectric type in crystals with phonon instability (see, for example, Refs. 125 and 126), the problem of a paramagnetic impurity in crystals,[127] and many others. On the other hand, these methods make it possible to obtain an adequate description of many concrete physical phenomena in two-level and many-level systems, including conversion of the frequency of infrared radiation upward in nonlinear optical media.

It is characteristic that whereas the first experimental studies were directed toward the direct observation of Dicke's "classical" super-radiance, which results from the correlation of the emitters in the process of photon exchange,[34,128,129] there have more recently been active experimental and theoretical investigations of systems in which the ordering of electric dipoles resulting from thermal phase transitions or other reasons can be used to generate super-radiance. As is shown by our study here of the process of super-radiant generation in ferroelectrics of KDP type, the developed exact methods make it possible to construct estimates for important physical characteristics of lasers with thermal pumping such as the radiation power, the frequency shift, etc.

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