Interrelation of representation spaces and Wigner–Racah calculus for compact Lie groups

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New analytic methods in the Wigner–Racah calculus for compact groups of second and higher rank are discussed. Relations of equivalence, proportionality, and analytic continuation based on the use of the complementary groups are considered. Optimal expressions are given in the form of multiple finite series for the isofactors and other transformation coefficients. The operations of Wigner–Racah algebras with multiple irreducible representations are simplified on the basis of biorthogonal systems of noncanonical bases and the Clebsch–Gordan coefficients of the Lie groups.

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INTRODUCTION

The theory of group representations has long been a powerful tool of modern theoretical physics. Since the classical work of Weyl, Wigner, and Racah there has been a steady extension of the classes of physical problems for whose solution compact groups and Lie algebras of higher than the first rank are used. We can mention here the well-known applications of irreducible representations (IRs) of Lie groups of different ranks for the classification of many-particle states in the shell models of the atom and the nucleus, and also SU_3 symmetry of the elementary particles.

The assumption that the Hamiltonian has an approximate symmetry makes it possible to use a basis with definite transformation properties in the many-particle Hilbert state space. Irreducible representations of unitary, orthogonal, and symplectic groups of different ranks and their subgroups characterize the states of nuclei in microscopic models (such as the translationally invariant shell model, the model of K harmonics, and the method of generalized hyperspherical functions) as well as in various models describing collective excitations (see, for example, Ref. 1–11). The unitary and other classical Lie groups are also widely used to describe the coherence properties of many-level quantum systems (see Refs. 12 and 13), and also new families of elementary particles (see Refs. 14 and 15).

For effective investigation of quantum systems with complex symmetry one needs a special mathematical formalism in order to express the matrix elements of the tensor operators of the physical quantities in the corresponding bases of the irreducible representations (i.e., in the multidimensional state spaces of the physical systems). Such a formalism (the theory of angular momentum) has achieved a certain degree of completion for the rotation group SO_3 (SU_2) of three-dimensional space (see Refs. 16–18).

The calculation of the matrix elements of tensor operators acting on irreducible representation spaces of groups is based on the Wigner–Eckart theorem, which makes it possible to expand them with respect to the Clebsch–Gordan (CG) coefficients of the corresponding groups (see, for example, Refs. 19 and 20). The expansion coefficients are called submatrix or reduced matrix elements. (In the case of simply reducible groups, one of which is SU_2, and also in a number of other cases, the matrix elements can be expressed as products of the above quantities.)

In the investigation of the matrix elements of operators that depend on the variables of several subsystems, and are also expressed in the form of products of other operators, there arise naturally the matrices of transformation of the coupling schemes (recoupling matrices) of the IR bases of the corresponding groups, these being generalizations of the $3nj$ coefficients of the theory of angular momentum and expressible as sums of products of CG coefficients.

Finally, many mathematical calculations can be more readily performed using the so-called canonical IR bases with subsequent transformation of the results to the bases corresponding to physical schemes of reduction of the group on subgroups. The transformations between different IR bases of complicated chains of subgroups are also realized by coefficients of fractional parentage, and, in essence, by CG coefficients and recoupling matrices. It is therefore expedient to call the entities of all three types—the CG coefficients, the elements of the recoupling matrices, and the elements of the actual matrices of the transformations between the IR bases—transformation coefficients or Wigner–Racah functions.

The formal rules of vector addition (coupling)—based on the unitarity properties of the transformation coefficients—of the IR bases by CG coefficients, of coupling of IR bases, and of transformation between IR bases corresponding to different reduction schemes can be naturally generalized for the representation spaces of any compact group and can be combined into the Wigner–Racah (WR) algebra of this group. For the WR algebras of groups that are not simply reducible, it is generally necessary to introduce additional matrix indices to distinguish multiple irreducible representations in the transformation coefficients.

The mathematical realization of the WR algebras of specific Lie groups—the explicit construction of the IR bases corresponding to physical reduction schemes, and the finding of the explicit form of the CG coefficient and other transformation coefficients—presents much more complicated and specific problems, which can

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be solved in the framework of the WR calculus of the corresponding groups.

The investigations of the mathematically simpler canonical IR bases of unitary and orthogonal groups are based on the results of Gel'fand, Tsetlin, and Graev and were developed further by Biedenharn, Moshinsky, Vilenkin, Klimyk, and others (see, for example, Refs. 19 and 20). Various methods of constructing noncanonical bases important for physical applications were proposed by Bargmann, Moshinsky, Elliott, Sharp and their collaborators, initially for SU_3 (see, for example, Refs. 1, 2, and 21), and later for other Lie groups. Elliott's scheme for constructing noncanonical bases was significantly improved by Smirnov, Asherova, and Tolstoĭ, who constructed and used projection operators of irreducible representations of Lie algebras (see, for example, Ref. 10).

Various methods have been used to construct the CG coefficients of canonical IR bases of compact Lie groups more complicated than SU_2: the method of generating invariants[22-25] (significantly advanced by Shelepin and Karasev), the recursive methods of expression intimately related to it,[24-29] the method of integrating over the group of products of the IR matrices (effectively realized for the first time by Klimyk and Gavrilik, see Ref. 20), and the method of Asherova, Smirnov, and Tolstoi based on the use of projection operators of Lie algebras.[30-32]

The theory of other transformation coefficients of the representation spaces of the higher Lue groups has so far been developed to a far smaller extent.

The fundamentally greater complexity (compared with the theory of angular momentum) of the WR calculus of simple Lie groups of second and higher ranks and, in particular, the inequivalence of the different constructions of noncanonical IR bases and CG coefficients of these groups are due to the appearance of multiple irreducible representations of the group or subgroups, respectively, in the decomposition of the direct product of two irreducible representations or on restriction on noncanonical chains of subgroups. The basis functions of the noncanonical bases constructed directly by analytic methods are, as a rule, like the various transformation coefficients, mutually nonorthogonal for different values of the index ρ distinguishing the states of the multiple irreducible representations. In fact, initially one can find only linear or bilinear combinations of the transformation coefficients (or noncanonical basis functions) with different values of ρ with subsequent separation of orthonormalized coefficients (or basis functions) by numerical orthogonalization, which does not guarantee a common numbering of the states of multiple irreducible representations, or by solving an eigenvalue problem for several operators, which is possible only if one gives up analyticity.

The reason why the many known expressions for the CG coefficients and isofactors of the higher groups are rather ineffective is to be sought in the excess of these expressions. As the expansion coefficients of the linear combinations we mentioned, one usually obtains transformation coefficients with a rather complicated structure, and the entire manifold of states of multiple irreducible representations is realized by variations of the parameters of these coefficients in a definite region of power r_0 (r_0 is the multiplicity of the corresponding irreducible representation). As will be shown below, the most convenient algorithms for constructing the WR functions and the basis functions of the irreducible representations are based on a hierarchiral approach. For different levels of generality of the basis functions or transformation coefficients, for example, depending on the IR classes and the appearance of multiple irreducible representations in the corresponding expansions, one or other method of a combination of them gives the best result.

On the other hand, a global approach is very important in the investigation of the WR calculus of higher Lie groups and their individual links. Thus, to investigate the recursive expressions for the CG coefficients of the unitary groups it is necessary to use definite recoupling matrices, by means of which one can not only simplify the orthonormalization of these CG coefficients but also understand the interrelation of the recursive expressions (and, accordingly, the classification of the multiple irreducible representations in them) and the expressions (and clasification) obtained by integration over the group or by other means. Without using special CG coefficients of a noncanonical basis one cannot explicate the fundamental duality relation between the different realization of the noncanonical basis corresponding to the reduction $SU_3 \supset SO_3$ presented in the review of Ref. 21. (This relation was not noted in Ref. 21). Finally, investigation of the WR calculus of a Lie group of definite rank frequently yields new information about the WR calculus of a group of lower or higher rank (even in the theory of angular momentum too), this information sometimes being unobtainable within the framework of the WR calculus of the latter group.

In fact, there exist between the WR calculi of definite Lie groups of different ranks and names much deeper and more varied interrelations than the natural inclusion in them of the WR calculi of the subgroups[1)] or the expression of certain transformation coefficients (for example, the recoupling matrices) in accordance with the rules of the WR algebras in the form of combination of other WR functions.

By the word "interrelation" in the title of this paper we understand several aspects of the mutual relationships of the representation spaces. First, it is clear that the WR algebra realizes some interrelationship between the IR spaces of a group, transforming one space into another of the same dimension. Second, as we shall show, there exist relations of equivalence, pro-

[1)]We recall that the CG coefficients of a given group G can always be expanded with respect to the CG coefficients of a subgroup G'; we call the expansion coefficients the isofactors of the chain $G \supset G'$. When the CG coefficients of the subgroup are not orthonormalized with respect to the indices of multiple irreducible representations, the indices of the multiple irreducible representations of the subgroup appearing in them and in the isofactors of the group are dual.

portionality, and analytic continuation between the different transformation coefficients of groups of different ranks and names. Most of these relations are a consequence of the interrelationships of the representation spaces of the so-called complementary groups (these spaces can be constructed simultaneously by a general procedure). Third, the manifold of realizations of the spaces of multiple irreducible representations that are more or less effective in one or other respect calls for elucidation of their interrelation. The solution of this problem is greatly simplified by using the concept of biorthogonal or dual (with respect to the indices of multiple irreducible representations) systems of IR bases, and also CG coefficients. The use of analytic biorthogonal systems leads to further generalization of the methods of WR algebras of groups that are not simply reducible with a certain easing of the problems of the WR calculus associated with orthogonalization.

The present paper is an attempt to generalize these aspects of the interrelationships of the representation spaces for the development of analytic methods in the WR calculus of compact Lie groups, above all to find optimal expressions for the CG coefficients and other transformation coefficients of these groups, and also for the investigation of the properties of noncanonical bases of irreducible representations.

1. COMPLEMENTARY GROUPS AND RELATIONSHIPS BETWEEN TRANSFORMATION COEFFICIENTS OF DIFFERENT GROUPS

It is shown in the books of Weyl[33] and Littlewood[34] that the theory of finite-dimensional irreducible representations of the general linear group $GL(n, C)$ and its unitarity subgroup U_n is intimately related to the theory of the representations of the group of permutations (the symmetric group) S_N. Although elementary operations with tensors were long known (see, for example, Refs. 35–37) various authors only gradually[38-42] became convinced of the equivalence of apparently independent quantities such as the recoupling matrices of the irreducible representations of the group U_k and the matrices of transition between different bases of irreducible representations of both symmetric (see Ref. 3 36) and unitary groups restricted in accordance with schemes of the type $S_N \supset S_{N'} \dotplus S_{N''}$ or $U_n \supset U_{n'} \dotplus U_{n''}$ $(N' + N'' = N,\ n' + n'' = n)$,[2)] respectively. In the monograph of Ref. 5, the equivalence of the isofactors of the chains $U_{n'n''} \supset U_{n'} \times U_{n''}$ and $S_N \supset S_{N'} \dotplus S_{N''}$ is proved. On the basis of the connection between the IR bases of the unitarity and symmetric groups it is established that not only the above-mentioned transformation coefficients but also the isofactors of the semicanonical chains do not depend on n, provided, of course, their signatures (the set of Young diagrams of the irreducible representations) have meaning for the given group U_n and its subgroups. Some interesting results based on the use of the interrelation between the bases of the linear and symmetric groups were obtained by Jucys[41,43,44] and Sullivan[45-47] (see also Refs. 48–51).

Somewhat unexpected was the appearance of $6j$ coefficients and CG coefficients of the group SU_2 with multiple 1/4 parameters of the type of angular momentum and its projections respectively, as matrices of "tree transplantation" (transition between different semicanonical bases) of class-1 irreducible representations of the orthogonal groups[52] (see also Ref. 53) and matrices of transition between the bases of class-1 irreducible representations of the unitary groups reduced on different chains of unitary and orthogonal subgroups.[54] These relations between the isofactors, and also between other transformation coefficients of the orthogonal and symplectic groups,[55-57,49] could be understood and generalized (see Refs. 58 and 59) on the basis of the concept of complementary groups proposed by Moshinsky and Quesne.[60,61]

Complementary Groups

Two groups $\overline{G}$ and $\overline{\overline{G}}$ are said to be complementary with respect to a nontrivial representation of the group G if in its decomposition every irreducible representation $\overline{\lambda} \times \overline{\overline{\lambda}}$ of the subgroup $\overline{G} \times \overline{\overline{G}}$ appears not more than once for a one-to-one correspondence between the irreducible representations $\overline{\lambda}$ and $\overline{\overline{\lambda}}$ of both subgroups $\overline{G}$ and $\overline{\overline{G}}$ (cf. Refs. 60 and 58).

This definition is satisfied by the groups U_m and U_n with respect to symmetric or antisymmetric irreducible representations of the group U_{mn}. [In the first case, the Young diagrams of the irreducible representations of U_m and U_n are identical, and a common basis is realized in the form of a polynomial in boson operators (cf. Refs. 62–64].[3)] In the second case, the Young diagrams are conjugate (they can be obtained from one another by interchanging the rows and columns), and a basis can be realized by using fermion operators. Examples of groups that are complementary with respect to the IRs $[\frac{1}{2} \ldots \frac{1}{2}, \frac{1}{2}]$ and $[\frac{1}{2} \ldots \frac{1}{2}, -\frac{1}{2}]$ of the group SO_{8l+4} are the quasispin group SU_2 and the group Sp_{4l+2}, which is used in the theory of many-electron atomic shells (cf. Refs. 37, 60, 67, and 68).

The groups U_n and S_N are complementary with respect to the space of tensors of rank N of n-dimensional space. Irreducible tensors simultaneously realize IR bases of both complementary groups.

Moshinsky and Quesne[60,61] showed that the orthogonal and the symplectic group $[O(n)$ and $Sp(2k)]$, one of which is noncompact, are complementary with respect to the infinite-dimensional irreducible representations $\langle \frac{1}{2} \ldots \frac{1}{2} \rangle$ and $\langle \frac{1}{2} \ldots \frac{1}{2}, 3/2 \rangle$ of the noncompact group $Sp(2kn)$. The O_n (or Sp_{2m}) invariant bilinear forms of boson creation or annihilation operators are generators of the complementary groups $Sp(2k, R)$ [or

[2)]We use the symbol $\dotplus$ if we have in mind a direct sum of Lie algebras or group matrices. We shall call bases corresponding to maximal embeddings of direct sums of group matrices semicanonical.

[3)]We should mention the result of Holman[63] (cf. 64–66), who succeeded in constructing explicitly a polynomial canonical basis (and thus any finite-dimensional irreducible representation) of the groups $GL(n)$ and U_n in the form of a polynomial of boson operators (of elements of group matrices) with special CG coefficients as expansion coefficients.

$SO^*(2k)$]. The irreducible representations of class k (dependent on k parameters) of the group O_n correspond to irreducible representations in the discrete positive series of the group $Sp(2k, R)$, and an irreducible representation of class k of the group Sp_{2m} corresponds to an irreducible representation of the complementary group $SO^*(2k)$ (cf. Refs. 61 and 59). The group U_{2m}, which includes the two subgroups O_{2m} and Sp_{2m}, has the group U_k as a complementary group in the case of an irreducible representation of class k. It is shown in Ref. 69 that the common subgroup U_m—the interesection of the groups SO_{2m} and Sp_{2m}—has the pseudounitary group $U(k, k)$, which includes the subgroups $Sp(2k, R)$ and $SO^*(2k)$, as a complementary group with respect to infinite-dimensional irreducible representations of class 1 of the group $U(mk, mk)$. The basis states of the mixed tensor irreducible representations of class k, $\bar{k}$ of this group U_m are realized as zero-trace tensors of covariant and contravariant boson creation operators, and their invariant forms are generators of the noncompact group $U(k, k)$.

In what follows, we shall use two forms of parametrization of basis functions and transformation coefficients (WR functions) of irreducible representations of the groups U_n, Sp_{2m}, and O_n; these are the regular and the homogeneous forms.[59] In the regular form, the class of each irreducible representation of the group (and a subgroup of the group) is equal to its rank. In the case of the homogeneous parametrization of the complete matrix of transformation coefficients, the IR classes are added in vector addition; on reduction on the chains $U_n \supset U_{n'} \dot{+} U_{n''}$, $O_n \supset O_{n'} \dot{+} O_{n''}$, $Sp_{2m} \supset Sp_{2m'} \dot{+} Sp_{2m''}$, $U_n \supset O_n$, $U_{2m} \supset Sp_{2m}$ the IR class of each individual subgroup is equal to the IR class of the group; on reduction on the chains $O_{2m} \supset U_m$, $Sp_{2m} \supset U_m$, irreducible representations of class k of the groups O_{2m} and Sp_{2m} are decomposed with respect to the irreducible representations of class k, $\bar{k}$ of the group U_m (cf. Ref. 71). A nondegenerate homogeneous parametrization appears naturally when the classes of all irreducible representations do not exceed the ranks of the groups, i.e., as long as nonstandard Young diagrams cannot appear when the characters are expanded in accordance with Littlewood's rules[34] (see Refs. 37 and 71). In the degenerate case, the class of some irreducible representations may formally exceed the rank of the group. These irreducible representations are expressd in the form of nonstandard symbols (see Refs. 37 and 71) with a sufficient number of zeros adjoined.

Realized in the form of boson polynomials and parametrized in the homogeneous form, the basis functions of the irreducible representations of class k (respectively, $k\bar{k}$ or k' and k'') of complicated chains of subgroups, including the standard reductions

$$\left.\begin{array}{l} U_n \supset O_n,\ U_{2m} \supset Sp_{2m},\ O_{2m} \supset U_m,\ Sp_{2m} \supset U_m,\ U_n \times U_n \supset U_n, \\ O_n \times O_n \supset O_n, Sp_{2m} \times Sp_{2m} \supset Sp_{2m},\ U_n \supset U_{n'} \dot{+} U_{n''}, \\ O_n \supset O_{n'} \dot{+} O_{n''},\ Sp_{2m} \supset Sp_{2m'} \dot{+} Sp_{2m''}, \end{array}\right\} \tag{1a}$$

are simultaneously the basis functions of the irreducible representations of chains, including the reductions

$$\left.\begin{array}{l} U_k \subset Sp(2k, R),\ U_k \subset SO^*(2k),\ Sp(2k, R) \subset U(k, k), \\ SO^*(2k) \subset U(k, k),\ U_{k'} \dot{+} U_{k''} \subset U_k, \\ Sp(2k', R) \dot{+} Sp(2k'', R) \subset Sp(2k, R), \\ SO^*(2k') \dot{+} SO^*(2k'') \subset SO^*(2k), \\ U_k \subset U_k \times U_k,\ Sp(2k, R) \subset Sp(2k, R) \times Sp(2k, R), \\ SO^*(2k) \subset SO^*(2k) \times SO^*(2k), \end{array}\right\} \tag{1b}$$

parametrized in regular form. It follows from this that the transformation coefficients that realize the transformations of the IR bases of the numbered irreducible representations of the chains (1a) (including the isofactors and the recoupling matrices) simultaneously realize the transformations of the IR bases of the numbered irreducible representations of the chains (1b) of the complementary groups.

Instead of the correspondence between an irreducible representation of a compact group and an irreducible representation of the discrete positive series of the complementary noncompact group it is expedient to use the correspondence between the irreducible representations with highest weight of these groups:

$$O_n\colon [\lambda_1\lambda_2 \ldots \lambda_k\dot{0}] \text{ and } Sp(2k, R)\colon \tag{2a}$$

$$\left\langle -\lambda_k - \frac{n}{2}, \ldots, -\lambda_2 - \frac{n}{2}, -\lambda_1 - \frac{n}{2} \right\rangle; \tag{2b}$$

$$\begin{array}{c} Sp_{2m}\colon \langle \lambda_1 \ldots \lambda_k \dot{0} \rangle \text{ and } SO^*(2k)\colon [-\lambda_k - m, \ldots, -\lambda_1 - m]; \\ U_m\colon \{\lambda_1 \ldots \lambda_k \dot{0}\} \text{ and } U_k\colon \left\{ -\lambda_k - \frac{m}{2}, \ldots, -\lambda_1 - \frac{m}{2} \right\} \end{array} \tag{2c}$$

(in the case of covariant tensor irreducible representations), and

$$\begin{array}{c} U_n\colon \{\lambda_1 \ldots \lambda_k \dot{0}, -\mu_k, \ldots, -\mu_1\} \text{ and } \\ U(k, k)\colon \left\{ -\lambda_k - \frac{n}{2}, \ldots, -\lambda_1 - \frac{n}{2}, \mu_1 + \frac{n}{2}, \ldots, \mu_k + \frac{n}{2} \right\} \end{array} \tag{2d}$$

(in the case of mixed tensor irreducible representations of the group U_n).

The following theorem can now be formulated (cf. Ref. 59).

THEOREM 1. The transformation coefficients, parametrized in homogeneous form, of the representation spaces numbered by the irreducible representations of chains of subgroups of the type (1a) can, if the general conditions of analytic continuation are observed, be expressed as the analytic continuation of the regularly parametrized conjugate WR functions numbered by the irreducible representations of chains of the type (1b) of compact forms of complementary groups with equivalent indices or multiple irreducible representations and corresponding [in accordance with (2)] parameters of the irreducible representations. Analytic continuations in the opposite direction is possible only when the homogeneous WR functions are not degenerate.

Because WR functions of the same kind of Lie groups of different orders, parametrized in homogeneous form, can be obtained by analytic continuation of the same regular WR function of a complementary group, Theorem 2 holds.

THEOREM 2. Provided the same general conditions are satisfied, a WR function in homogemeous form is equal to the analytic continuation of another WR function of the same type for which the parameters of the irreducible representation of class k and the orders of the corresponding groups and subgroups are replaced in such a way that the sum of the parameters of the irreducible representation and half the order of the group remains the same, and the indices of multiple irreducible representations are equivalent.

General Conditions of Analytic Continuation of Transformation Coefficients.

These conditions are also taken into account in the continuation of the IR signatures (the elements of the groups of permutations of the IR parameters being used, see Refs. 72–74). We note first of all that the WR functions of compact groups depend on discrete values of the parameters, in the case of finite multiplicities of the irreducible representations in the corresponding expansions can be expressed in the form of elementary functions of polynomials in the IR signatures, and are uniquely determined by the boundary values in a subregion of power equivalent to the region of indices of the multiple irreducible representations (see Ref. 29 and the examples in Secs. 3 and 4 of the present paper).

The WR functions, represented in the form of factorial sums (generalized multidimensional hypergeometric series), can be analytically continued if the number of terms in the individual sums is bounded by linear combinations of differences between the lengths of neighboring rows of the Young diagrams and by the number of contractions of the corresponding tensors on reduction or decomposition (i.e., by reductions in the total number of cells in the Young diagrams). Linear combinations of these quantities also determine the asymptotic multiplicity of irreducible representations, this being equal to the actual multiplicity for sufficiently large values of the last parameter of each of the irreducible representations of the chains of groups listed in (1a) and (1b). The asymptotic multiplicities of the irreducible representations for conjugate reductions depend on the same way on the parameters related by the substitution (2) and determine the number of functions that in the different regions of the parameters give the values of the conjugate homogeneous and regular WR functions with the same index of the multiple irreducible representations. If the actual multiplicity of the irreducible representations is less than the asymptotic multiplicity, it is necessary to take into account the linear dependences that then arise between the previous boundary values of the WR functions (see Secs. 3 and 4). These dependences are taken into account automatically when one considers bilinear forms (pairwise products summed over the indices of multiple irreducible representations) of WR functions of the same kind.

A certain care is required for analytic continuation into the region of degenerate homogeneous WR functions. First, the actual and asymptotic multiplicities of the irreducible representations can in this case differ appreciably. Second, there may be complications due to the appearance of several nonstandard symbols denoting the same irreducible representations, or due to discontinuities of the analytic expressions for the dimensions of irreducible representations of class m of the groups O_{2m} and the appearance of IR signatures characteristic of SO_{2m}. Therefore, in these cases factors of the type $\sqrt{2}$ may appear, and the procedure of analytic continuation also makes it possible to find WR functions which depend on the irreducible representations of the groups SO_n and in the main homogeneity regions are equal to the WR functions of the groups O_n. In general, it is expedient to examine the degenerate cases separately for each concrete pair of conjugate WR functions. Nevertheless, the relations of analytic continuation are also helpful in the degenerate cases, for example, in the investigation of the WR functions of the canonical basis and closely related variants of the semicanonical bases of the irreducible representations of the groups SO_n, O_n, and Sp_{2m}. Thus, on the transition to the canonical basis there appears the IR parameter δ of the subgroup O_1, which takes the values 0 or 1, depending on the parity of the difference of the sums of parameters of the irreducible representations of the groups O_n and O_{n-1}. It is also expedient to retain this parameter in the case of the reduction $SO_n \supset SO_{n-1}$, writing it formally in the form $SO_n \supset SO_{n-1} + SO_1$. Then instead of (2a) we establish the correspondence

$$SO_{1(i)}: [\delta_i] \text{and}\, Sp\,(2k, R): \left\langle -\tfrac{1}{2},\ \ldots,\ -\tfrac{1}{2},\ -\tfrac{1}{2}-\delta_i \right\rangle. \quad (3)$$

The advantage of the relations of analytic continuation based on Theorem 2 is that the IR signatures of the subgroups whose order does not change may be degenerate. It is therefore rather convenient to use these relations in the case of the canonical basis as well. In addition, if the orders of the corresponding groups and subgroups differ by even numbers, Theorem 2 establishes an identity between homogeneously parametrized WR functions of groups of different ranks, i.e., makes it possible to compare the numerical values of these functions.

We note that the relations are very helpful when one is using WR functions of groups of high rank, for example, when one is describing the states of many-nucleon systems.

We discuss the behavior of the factorial sums. Some arguments of the factorials (apart from those that limit the intervals of summation) may be transformed into half-integral arguments. In such a case, it is convenient to use double factorials. Thus, the expressions for the resulting CG coefficients and the $6j$ coefficients of the group SU_2 with multiple 1/4 parameters (which correspond to Young diagrams of half-integer numbers) can be readily expressed in double factorials: All the arguments of the factorials (like the simple factors of the type of the dimension of the irreducible representations of SU_2) are multiplied by 2, and the symbol ! is replaced by !!, but the exponents of the powers of (-1) are not changed. In the case of a $6j$ coefficient, it is necessary to add a factor 2.

Some Examples of the Relations of Analytic Continuation of the Transformation Coefficients of Various Groups

Information about the WR functions of compact Lie groups of high rank can be obtained in the case of restricted classes of irreduible representations when expressions are known for the conjugate WR functions of a group of low rank; in particular, this can be done by using the theory of representations of the groups SU_2, SO_4, Sp_4 (SO_5), and SU_4.

Because the groups SU_2 and Sp_2 are isomorphic, the elements of the matrix of the transition between the bases of the symmetric irreducible representation of the group U_n, numbered by the irreducible representations of the chains $U_n \supset SO_n \supset SO_{n'} + SO_{n''}$ and $U_n \supset U_{n'} + U_{n''} \supset SO_{n'} + SO_{n''}$, can be expressed as the analytic continuation of the CG coefficient of the group SU_2 (isofactor of the reduction $Sp_2 \supset U_1$):

$$\left\langle \begin{matrix} \{p+q\}_n \\ \{p\}_{n'} \{q\}_{n''} \\ [l]_{n'} [k]_{n''} \end{matrix} \middle| \begin{matrix} \{p+q\}_n \\ [L]_n \\ [l]_{n'} [k]_{n''} \end{matrix} \right\rangle$$

$$= \begin{bmatrix} -\frac{1}{2}l-\frac{1}{4}n' & -\frac{1}{2}k-\frac{1}{4}n'' & -\frac{1}{2}L-\frac{1}{4}n \\ -\frac{1}{2}p-\frac{1}{4}n' & -\frac{1}{2}q-\frac{1}{4}n'' & -\frac{1}{2}(p+q)-\frac{1}{4}n \end{bmatrix} = (-1)^{\frac{1}{2}(L-l-q)}$$

$$\times \begin{bmatrix} \frac{1}{4}(p+q+l+k+n)-1 & \frac{1}{4}(p+q-l-k) & \frac{1}{2}L+\frac{1}{4}n-1 \\ \frac{1}{4}(p-q+l-k+n'-n'') & \frac{1}{4}(q-p+l-k) & \frac{1}{2}(l-k)+\frac{1}{4}(n'-n'') \end{bmatrix} \tag{3a}$$

(cf. Ref. 54). A similar expression can be written down for the matrix of the transition between bases of an irreducible representation of class 2 of the group U_{2m} reduced on the chains $U_{2m} \supset Sp_{2m} \supset Sp_{2m'} + Sp_{2m''}$ and $U_{2m} \supset U_{2m'} + U_{2m''} \supset Sp_{2m'} + Sp_{2m''}$ (see Ref. 69). For this purpose, one continues analytically the isofactors of $SO_4 \supset U_2$, these being equal to the ordinary CG coefficients of SU_2.

The matrices of transformation between different variants of a semicanonical basis of symmetric irreducible representations of the orthogonal groups are considered in Refs. 49, 52, 53, 57, and 58 and expressed in the form of the analytic continuation of $3nj$ coefficients with multiple 1/4 parameters. Note that the matrices of transformation of the types of the semicanonical bases of the irreducible representations of class 2 of the symplectic groups can be expressed by analytic continuation of the recoupling matrices of the irreducible representation of SO_4, i.e., in terms of products of pairs of ordinary $3nj$ coefficients of the theory of angular momentum.

It follows from Theorem 1 that the isofactors of the semicanonical and canonical bases, which reduce the direct product of two symmetric (one-parameter) irreducible representations of the orthogonal groups, can be expressed by analytic continuation of the isofactors of the canonical basis of an irreducible representation of the group Sp_4 (SO_5) (see Refs. 55 and 56):

$$\begin{bmatrix} SO_n & l_1 \quad l_2 & [L_1 L_2] \\ SO_{n'} \dot{+} SO_{n''} & l_1' l_1'' \quad l_2' l_2'' u; & [L_1' L_2'] \; [L_1'' L_2''] \end{bmatrix}$$

$$= \begin{bmatrix} \left\langle -\frac{2L_2'+n'}{4}, -\frac{2L_1'+n'}{4} \right\rangle \left\langle -\frac{2L_2''+n''}{4}, -\frac{2L_1''+n''}{4} \right\rangle & \left\langle -\frac{2L_2+n}{4}, -\frac{2L_1+n}{4} \right\rangle^u \\ -\frac{2l_1'+n'}{4}, -\frac{2l_2'+n'}{4} \quad -\frac{2l_1''+n''}{4}, -\frac{2l_2''+n''}{4} & -\frac{2l_1+n}{4}, -\frac{2l_2+n}{4} \end{bmatrix}. \tag{4}$$

In the isofactor of $Sp_4 \supset SU_2 \times SU_2$ (i.e., $Sp_4 \supset Sp_2 + Sp_2$), whose analytic continuation appeared on the right-hand side of (4), the irreducible representations of the group are denoted by $(K\Lambda)$, the maximal values of the angular momenta I and J characterizing the irreducible representations of the subgroups SU_2 and SU_2 (see Ref. 74). The index u distinguishes the multiple irreducible representations of not only the subgroups $SO_{n'} + SO_{n''}$ in the decomposition of the irreducible representations of the group SO_n but also the group Sp_4 in the decomposition of the direct product. Some cases of the application of the relation (4) are discussed in Sec. 2.

Theorem 2 makes it possible to write down a relation of analytic continuation of the isofactors of a different type:

$$\begin{bmatrix} SO_n & l_1 l_2 [L_1 L_2] \\ SO_{n-1} & l_1' l_2' [L_1' L_2'] \end{bmatrix}$$

$$= \begin{bmatrix} SO_5 & l_1+\frac{n-5}{2} \; l_2+\frac{n-5}{2} \left[L_1+\frac{n-5}{2}, \; L_2+\frac{n-5}{2}\right] \\ SO_4 & l_1'+\frac{n-5}{2} \; l_2'+\frac{n-5}{2} \left[L_1'+\frac{n-5}{2}, \; L_2'+\frac{n-5}{2}\right] \end{bmatrix}. \tag{5}$$

The relation analogous to (4) between the isofactors of the semicanonical bases of the symplectic groups and the isofactors of $SO_4 \supset SO_2 + SO_2$ makes it possible to express the special isofactors of the groups Sp_{2m} which realize vector addition of the bases of symmetric irreducible representations, in the form of a product of two CG coefficients of the group SU_2 (see Ref. 59), and to understand their unusual factorization (see Eq. (36) in Ref. 74).

From Theorem 2 there follow relations between the isofactors of the chain $SU_n \supset SO_n$ (cf. Ref. 75):

$$\begin{bmatrix} SU_n & p_1 \; p_2 \; \{h_1 h_2\} \\ SO_n & l_1 \; l_2 \; \omega [L_1 L_2] \end{bmatrix}$$

$$= \begin{bmatrix} SU_4 & p_1+\frac{n}{2}-2 \;\; p_2+\frac{n}{2}-2 \left\{h_1+\frac{n}{2}-2, \; h_2+\frac{n}{2}-2\right\} \\ SO_4 & l_1+\frac{n}{2}-2 \;\; l_2+\frac{n}{2}-2 \; \omega \left[L_1+\frac{n}{2}-2, \; L_2\frac{n}{2}-2\right] \end{bmatrix}; \tag{6a}$$

$$\begin{bmatrix} SU_n & p \; \{\dot{0}, -q\} \; \{\lambda, \dot{0}, -\mu\} \\ SO_n & l_1 \quad l_2 \qquad \omega [L_1 L_2] \end{bmatrix}$$

$$= \begin{bmatrix} SU_4 & p+\frac{n}{2}-2 \left\{\dot{0}, \; -q-\frac{n}{2}+2\right\} \left\{\lambda+\frac{n}{2}-2, \; 0, \; 0, \; -\mu-\frac{n}{2}+2\right\} \\ SO_4 & l_1+\frac{n}{2}-2 \quad l_2+\frac{n}{2}-2 \quad \omega \left[L_1+\frac{n}{2}-2, \; L_2+\frac{n}{2}-2\right] \end{bmatrix}, \tag{6v}$$

the value of which is due to the fact that the groups SO_4 and $SU_2 \times SU_2$ are isomorphic and the existence of expressions for the corresponding isofactors of the supermultiplet basis.[75-77]

Finally, we give an example of a relation that makes it possible to find a WR function in regular form—the most general matrix of transition between the canonical basis of an irreducible representation of Sp_4 and the five-dimensional quasispin basis, the bases being numbered by the irreducible representations of the subgroups $Sp_4 \supset SU_2 \times SU_2 \supset SO_2 \times SO_2$ and $Sp_4 \supset U_2 \supset U_1$:

$$\left\langle \begin{matrix} \langle K\Lambda \rangle \\ IM \; JN \end{matrix} \middle| \begin{matrix} \langle K\Lambda \rangle \\ \omega; \; VTM_T \end{matrix} \right\rangle$$

$$= \begin{bmatrix} SU_4 & -2M-2 \{\dot{0}, \; 2N+2\} \{-V-T-2, \; 0, \; 0, -V+T+2\} \\ SO_4 & -2I-2 \quad -2J-2 \qquad \omega [-2\Lambda-2, \; -2K-2] \end{bmatrix}. \tag{7}$$

Here, $M_T = M+N$, $V = M-N$. The expression (7) is used in Ref. 78, and its inverse in Ref. 75. To prove Eqs. (7) and (6b), it is necessary to consider, besides the pairs of chains of complementary groups of the type (1a) and (1b), pairs of the type

$$\left.\begin{array}{l} U_n \times U_n \supset U_n \text{ and } U_{k'} + U_{k''} \subset U(k', k''); \\ U_n \supset O_n \text{ and } Sp(2k, R) \subset U(k, k). \end{array}\right\} \tag{8}$$

It should be noted that the simplest phase relations represented by Eqs. (3)–(6) hold only for comparatively large values of n ($n \geqslant 4$ or 5). The phase relation for $n = 2, 3, 4$ are considered in the quoted papers (in particular, in Ref. 75).

Groups of Permutations of the Parameters of Irreducible Representations and Simplification of the Limiting Cases of Transformation Coefficients

It is well known that the Schur functions, and thus the characters of the general linear andunitray groups, are invariant up to the sign under permutations of the parameters of the irreducible representations,

$$m_{kn} \to m_{l_k n} - l_k + k, \tag{9}$$

where $k \to l_k$ is a permutation of the numbers of the rows in the Young diagram $\{m_{1n}, \ldots, m_{kn}, \ldots, m_{nn}\}$. As is shown in Ref. 79, these substitutions leave invariant the eigenvalues of the Casimir operators and realize equivalence transformations for the IR bases. These transformations form a group isomorphic to the Weyl group of the Lie algebra and are phase relations that can be readily established by using the matrix elements of the infinitesimal operators (see Refs. 72 and 73).

The groups of permutations of the parameters of the irreducible representations of the groups O_n include in addition to permutations of the type (9) the reflections

$$m_{in} \to -m_{in} - n + 2i \tag{10}$$

[for the irreducible representations of the groups SO_n for n even, the number of reflections (10) is even], and the groups of substitutions of the parameters of the irreducible representations of the groups Sp_{2m} include in addition to the permutations (9) the reflections

$$m_{im} \to -m_{im} - 2m + 2i - 2. \tag{11}$$

Permutations of the parameters applied to some of the signatures of the isofactors or other transformation coefficients make it possible to find relations of analytic continuation that are an effective means of extension of the limits of applicability and are also simplifications of the expressions for special cases of the WR functions. In particular, the parameter permutations $j \to -j - 1$ have long been used in the theory of angular momentum (see Ref. 17). Elementary examples of the use of permutations of the parameters of irreducible representations of Lie groups of low rank are given in Refs. 80 and 74. In Secs. 3 and 4 we shall discuss very important cases of relations of this type used in Refs. 75, 81, and 82.

We consider one further class of equivalence relations between the WR functions of different Lie groups (cf. Refs. 49 and 57). In what follows in this subsection we shall have in mind only the following WR functions: the recoupling matrices of the irreducible representations of the classical Lie groups (including the exceptional groups); the isofactors of the semicanonical bases of the irreducible representations of the groups U_n, O_n, and Sp_{2m}; and the matrices of transition between different semicanonical bases of the irreducible representations of the groups U_n, O_n, and Sp_{2m}. Then the following theorem holds for them.

THEOREM 3. If all the signatures of one of the above WR functions (possibly after application to some of them of certain elements of permutation groups) are equal to the corresponding signatures of a WR function of the same kind of the group U_n, then these functions are themselves equal. If after the deletion in the signatures of the WR function of the group U_n of the lowest rows of the Young diagram not permitted for the WR function of the same kind of the group U_k ($k < n$) there remains a set of signatures that has meaning for the WR function of the group U_k, then the original WR function is equal to the product of the WR function of the group U_k obtained after the deletion and the WR function of the group U_{n-k} with signatures formed from the deleted parts of the Young diagrams.

The criterion of coincidence of the signatures of a WR function of any compact group and a unitary group is equivalent to the requirement that contractions should not appear on vector addition and reduction on semicanonical subgroups.

Examples of relations that are consequences of Theorem 3 can be found in Refs. 74, 80, and 83.

Proportionality of the Isofactors and Elements of the Recoupling Matrices of Irreducible Representations of the Unitary Groups

A consequence of the complementarity of the unitary and symmetric groups is the relation

$$\begin{aligned} &\begin{bmatrix} \lambda_{(1)} & \lambda_{(2)} & \lambda^u \\ \alpha_1;\ \mu_{(1)}\nu_{(1)} & \alpha_2;\ \mu_{(2)}\nu_{(2)} & \alpha;\ \mu^v\nu^w \end{bmatrix} \\ &= \left[\frac{\mathfrak{M}(\lambda_{(1)})\,\mathfrak{M}(\lambda_{(2)})\,\mathfrak{M}(\mu)\,\mathfrak{M}(\nu)}{\mathfrak{M}(\mu_{(1)})\,\mathfrak{M}(\mu_{(2)})\,\mathfrak{M}(\nu_{(1)})\,\mathfrak{M}(\nu_{(2)})\,\mathfrak{M}(\lambda)}\right]^{1/2} \\ &\times \langle \mu_{(1)}\nu_{(1)}(\lambda_{(1)}^{\alpha_1}),\ \mu_{(2)}\nu_{(2)}(\lambda_{(2)}^{\alpha_2});\ \lambda^u \mid \mu_{(1)}\mu_{(2)}(\mu^v),\ \nu_{(1)}\nu_{(2)}(\nu^w);\ \lambda^\alpha \rangle, \end{aligned} \tag{12}$$

which was proved in most general form for the first time in Ref. 48 (see also Ref. 49). On the left-hand side, we have the isofactor of the semicanonical basis of an irreducible representation of U_n, and on the right an element of the recoupling matrix of bases of four irreducible representations of U_n (generalization of a $9j$ coefficient). The irreducible representations of the group and subgroups are denoted by the Young diagrams $\lambda_{(i)}, \mu_{(i)}, \nu_{(i)}$. Multiple irreducible representations in the decompositions of the direct products are distinguished by superscripts α_i, u, v, w, and multiple irreducible representations of the subgroup $U_{n'} + U_{n''}$ are distinguished by indices α_i in front of these irreducible representations. The proportionality factor is

$$\mathfrak{M}(\lambda) = \frac{N!}{d_\lambda} = \frac{\prod_{i=1}^{n} (h_i + n - i)!}{\prod_{1 \leqslant i < j \leqslant n} (h_i - h_j - i + j)}, \tag{13}$$

where d_λ is the dimension of the irreducible representation $\lambda \equiv \{h_1, h_2, \ldots, h_n\}$ of the group S_N.

In Refs. 48 and 49, the relations (12) were proved by the generalized method of Jucys[43] of construction of IR bases of the unitary groups by means of the projection operators

$$O^{\lambda}_{\rho\eta} = \mathfrak{M}^{-1}(\lambda) \sum_{s} D^{\lambda}_{\rho\eta}(s)\, s \tag{14}$$

of the symmetric groups. Here, $D^{\lambda}_{\rho\eta}(s)$ is the matrix element of the irreducible representation λ of the group S_N corresponding to the group element s. As is shown in Ref. 43, the functions $O^{\lambda}_{\rho\eta}F(\mathrm{m})$ form an orthogonal basis of the irreducible representation λ of the group U_n numbered by the weight $\mathrm{m} = (m_1, m_2, \ldots, m_n)$ and certain classes of indices η for constant ρ. Here, $F(\mathrm{m})$ is a component of a tensor of rank N with ordered indices, and m_k is the number of indices equal to k. The functions (see Ref. 49)

$$\left[\prod_{k=1}^{n} m_k!\right]^{-\frac{1}{2}} \mathfrak{M}^{\frac{1}{2}}(\lambda)\, O^{\lambda}_{\rho\eta}F(\mathrm{m}) \tag{15}$$

form an orthonormalized semicanonical basis of the irreducible representation λ of the group U_n if the index η represents basis characteristics of the complementary chain of symmetric groups ending with the invariance subgroup of the tensor $F(\mathrm{m})$, $S_{m_1} + S_{m_2} + \ldots + S_{mn}$, the irreducible representations of all S_{m_k} being identity representations.

When (15) is used, one first finds certain auxiliary isofactors and then proves Eq. (12). A relation equivalent to (12) was also proved to Sullivan,[45] who actually used a generalization of Frobenius's theorem that was first proved by Burneĭka.[84]

A special case of (12) is the relation between the doubly stretched $9j$ coefficients and CG coefficients of the group SU_2 (Eq. (31.15a) in Ref. 17). Other special cases are obtained in Refs. 41 and 80.

The relation (12) is used in Ref. 85 in a fractional-parentage expansion of the wave functions of some shells in an atom or nucleus. Some other applications of (12), as well as other cases of proportionality of isofactors and other transformation coefficients of representation spaces of different groups, will be discussed in the following sections.

2. MATRIX ELEMENTS OF POWERS OF GROUP GENERATORS AND FUNDAMENTAL TRANSFORMATION COEFFICIENTS

The complexity of the general expressions for the CG coefficients, isofactors, and other transformations coefficients of irreducible representations of higher groups produced a need for various recursion relations and expressions. The recursion relations establish linear dependences between WR functions whose parameters change in small steps. In the case of a recursion expression, a transformation coefficient is represented as a function of simpler cases (boundary values) of the same coefficient and some auxiliary WR functions. In particular, for isofactors there exist two classes of recursion relations and two classes of recursion expressions. In the first case, only the basis characteristics of the irreducible representations of the group change, but not the signatures of the actual irreducible representations. To construct recursion relations of such type, one uses infinitesimal operators (see Ref. 20, 86, and 87), and to construct recursion expressions of the same class one uses operators that lower the weight in the form of polynomials of ordered infinitesimal operators.[30,88-90] In the construction of expressions of this type, and also when one uses the projection operators of irreducible representations of Lie algebras in the form of Löwdin and Shapiro[91,92] (in the case of the subgroup SU_2), and Asherova, Smirnov, and Tolstoĭ,[90,93,94] it is expedient to use submatrix (reduced matrix) elements of powers and ordered products of infinitesimal operators. The corresponding matrix elements for the irreducible representations of the unitary groups are equivalent to the unitarized general β functions of Gel'fand and Graev[95]; the use of factorization makes it possible to optimize their expressions.

The recursion relations (see, for example, Ref. 29) and expressions (see Refs, 24–28) of the second class make it possible to change both the basis characteristics and the actual signatures of the irreducible representations. The recursion expressions (like the relations) of this type are constructed by the methods of the Wigner–Racah algebra in the form of combinations (i.e., products summed over the basis characteristics of definite irreducible representations) of simpler isofactors and elements of the recoupling matrices of the irreducible representations of the subgroups. Necessary constructive elements of the optimal recursion expressions for the isofactors are fundamental (which reduce simply, i.e., without multiple irreducible representations in the decomposition of a direct product; cf. Ref. 24) and closely related WR functions, which can be expressed in the form of multiple finite series.

The fundamental isofactors and other WR functions presented in this section are characterized by great universality and symmetry properties that go beyond the ordinary permutations of irreducible representations and relations of contragredience.

Matrix Elements of Powers of Infinitesimal Operators of the Unitary Groups

The generators of the group U_n satisfy the commutation relations

$$[E_{ik}, E_{lm}] = \delta_{kl}E_{im} - \delta_{im}E_{lk}.$$

Matrix elements of powers of these operators in a nonunitary basis were found for the first time in Ref. 95, and submatrix elements of a unitary basis were found in Refs. 89 and 96. The reduced matrix elements of the operators

$$\left[\frac{p!}{\prod_{i=1}^{n-1} \alpha_i!}\right]^{1/2} \prod_{i=1}^{n-1} E^{\alpha_i}_{in} \left(\sum_{i=1}^{n-1} \alpha_i = p\right) \tag{16}$$

are equal to each other (see Ref. 89):

$$\left\langle \begin{matrix}\lambda\\ \mu'\end{matrix} \middle\| E^{p}_{n-1,\,n} \middle\| \begin{matrix}\lambda\\ \mu\end{matrix} \right\rangle = \frac{[p!\,d_{n-1}(\mu)]^{1/2}\, S_{n,\,n-1}(\lambda;\,\mu)}{S_{n-1,\,n-1}(\mu';\,\mu)\, S_{n,\,n-1}(\lambda;\,\mu')}. \tag{17}$$

Here, we have used the notation

$$_{l}d_n(\lambda) = \prod_{\substack{1 \leqslant i < j \leqslant n\\ i \neq l,\, j \neq l}} (\lambda_i - \lambda_j - i + j) = {}_{l}S^{2}_{n,\,n}(\lambda;\,\lambda); \tag{18}$$

$$
{}_{l}S_{n,\,m(k)}(\lambda;\,\mu) = {}_{-l}S^{-1}_{m,\,n(-k-1)}(\mu;\,\lambda)
= \left[\frac{\prod\limits_{j=1(j\neq -l)}^{m}\ \prod\limits_{i=1(i\neq l)}^{\min(j+k,\,n)} (\lambda_i-\mu_j-i+j+k)!}{\prod\limits_{i=1(i\neq l)}^{n}\ \prod\limits_{j=1(j\neq -l)}^{\min(i-k-1,\,m)} (\mu_j-\lambda_i+i-j-k-1)!}\right]^{1/2} \tag{19}
$$

(n and m are natural numbers, $-m \leqslant l \leqslant n$). The vanishing parameters l and k can be omitted, and then the notation (19) is identical to the expression (19) of Ref. 96). In addition, ${}_{n}S_{n,m(k)}(\lambda;\mu)=S_{n-1,m(k)}(\lambda;\mu)$; ${}_{-m}S_{n,m(k)}(\lambda;\mu)=S_{n,m-1(k)}(\lambda;\mu)$. To calculate the matrix element of the operator $E^{p}_{n-1,n}$, we also need the special isofactor of $U_{n-1}\supset U_{n-2}$

$$
\begin{bmatrix} \mu & p & \mu' \\ \nu & 0 & \nu \end{bmatrix} = \frac{[p!\,d_{n-1}(\mu')]^{1/2}\, S_{n-1,\,n-2}(\mu';\,\nu)}{S_{n-1,\,n-1}(\mu';\,\mu)\, S_{n-1,\,n-2}(\mu;\,\nu)}. \tag{20}
$$

The matrix elements of the remaining operators (16) can be expressed by means of the special isofactors given below.

Isofactors for Vector Addition of the Bases of a General and a Symmetric Irreducible Representation of the Group U_n

On the reduction of the direct product of two irreducible representations of a unitary group, one of which is symmetric or antisymmetric, the resulting irreducible representations do not occur more than once. Isofactor of the second type are considered in Ref. 97. The use of the commutation properties of powers of group generators made it possible in Ref. 89 to obtain two classes of expressions for isofactors of the first type. To combine these expressions into a single algorithm, we introduce the notation

$$
\begin{aligned}
&W_{n(l)}\left(\begin{matrix}\lambda_{(1)r_1k_1}; & \lambda_{(2)r_2k_2} \\ \lambda_{(3)r_3k_3}; & \lambda_{(4)r_4k_4}\end{matrix}\,\middle|\, f\right)
= \prod_{a=1}^{2}\prod_{b=3}^{4} S_{n-r_a,\,n-r_b(k_a+k_b)}(\lambda_{(a)};\,\lambda_{(b)}) \\
&\times \sum_{\sigma} (-1)^{\sum\limits_{i=1(i\neq l)}^{n}\sigma_i}\, {}_{l}d_n(\sigma)\, f(\lambda_{(1)};\,\lambda_{(2)};\,\lambda_{(3)};\,\lambda_{(4)};\,\sigma) \\
&\times \prod_{a=1}^{2} {}_{-l}S^{-2}_{n-r_a,\,n(k_a)}(\lambda_{(a)};\,\sigma) \prod_{b=3}^{4} {}_{l}S^{-2}_{n,\,n-r_b(k_b)}(\sigma;\,\lambda_{(b)}).
\end{aligned} \tag{21}
$$

Here, $\lambda_{(a)}$, $\lambda_{(b)}$, σ are highest weights (of the Young diagram). It is expedient to take the σ-independent factor of the function f in front of the summation sign.

Then the isofactors of the chain $U_n\supset U_{n-1}$ can be expressed as follows:

$$
\begin{aligned}
\begin{bmatrix}\alpha & p & \lambda \\ \beta & q & \mu\end{bmatrix} &= W_{n(n)}\left(\begin{matrix}\alpha_{0,0} & \mu_{1,0} \\ \beta_{1,0} & \lambda_{0,-1}\end{matrix}\,\middle|\, f_0\right) \\
&= W_{n(l)}\left(\begin{matrix}\lambda_{0,0} & \beta_{1,-1} \\ \mu_{1,0} & \alpha_{0,0}\end{matrix}\,\middle|\, f_{(l)}\right) \\
&(l=1,\,2,\,\ldots,\,n);
\end{aligned} \tag{22a}
$$

$$
\begin{aligned}
f_0 &= (-1)^{\sum\limits_{i=1}^{n-1}\beta_i}\,[(p-q)!\,d_n(\lambda)\,d_{n-1}(\beta)]^{1/2} \\
f_{(l)} &= (-1)^{\sum\limits_{i=1}^{l-1}(\alpha_i-\beta_i+\mu_i)+\sum\limits_{i=l+1}^{n}\lambda_i}\left[\frac{d_n(\lambda)\,d_{n-1}(\beta)}{(p-q)!}\right]^{1/2}.
\end{aligned} \tag{22b}
$$

In the expression (22a) (cf. Refs. 89 and 96), the lengths of the intervals over which the summation parameters σ_i range are determined by the differences

$$
\min(\alpha_i,\,\mu_i) - \max(\lambda_{i+1},\,\beta_i), \tag{23a}
$$

and in the l-th expression of the type (22b) (cf. Ref. 89) they are determined by the differences ($i\neq l$)

$$
\min(\lambda_i,\,\beta_{i-1}) - \max(\alpha_i,\,\mu_i). \tag{23b}
$$

Up to elementary factors, the expressions of both types have symmetry of Regge type with respect to permutations of the parameters paired in the brackets. Depending on the lengths of the intervals, it is expedient to use one of the $n+1$ expressions for the considered isofactors. Thus, only (22b) for $l=n$ gives an expression without a sum if the Young diagram μ takes the maximal value.

Choosing (22b) for $l=1$ and using the relation (12), we obtain an expression containing $n-1$ sums for special elements of the recoupling matrix of four irreducible representations of the group U_n, three of which are symmetric (cf. Ref. 65):

$$
\begin{aligned}
&\langle \alpha q(\beta),\ r\ \ p-q\,(p-q+r);\ \lambda|\alpha r(\mu),\ q\ \ p-q\,(p);\ \lambda\rangle \\
&\qquad = W_{n(n)}\left(\begin{matrix}\alpha_{0,0} & \lambda_{0,1} \\ \beta_{0,-1} & \mu_{0,-1}\end{matrix}\,\middle|\, f'\right),
\end{aligned} \tag{24}
$$

where

$$
f' = (-1)^{\sum\limits_{i=2}^{n}\lambda_i}\left[\frac{r!\,q!\,d_n(\mu)\,d_n(\beta)}{p!\,(p-q+r)!}\right]^{1/2}.
$$

The *Weyl coefficients* of the canonical basis of an irreducible representation of the group U_{n+1} [these coefficients realize a permutation of the n-th and $(n+1)$-th components of the vector space of the group U_{n+1}] can be expressed in thesimplest form as a special case of this quantity for the choice $p=q$ (cf. Ref. 98).

We note that Eqs. (13.1b), (13.1c), (29.1b), and (32.18) of Ref. 17 are special cases of Eqs. (22a), (22b), and (25).

Some Other Recoupling Matrices of Irreducible Representations of the Groups U_n

To construct a semicanonical basis corresponding to the reduction $U_n\supset U_{n-2}+U_2$ (cf. the case $U_4\supset U_2+U_2$ in Ref. 99), it is expedient to use the projection operators[91,92] of the Lie subalgebra of the group U_2 that transforms the $(n-1)$-th and n-th coordinates. The corresponding matrix elements of the projection operators that realize the transformation between the canonical and semicanonical bases are simultaneously a bilinear form of the recoupling matrices of three irreducible representations of U_n, the second and the third irreducible representations being symmetric (cf. Ref. 100):

$$
\begin{aligned}
&\sum_{\rho}\langle \alpha p_1(\beta)\,p_2,\ \lambda|\alpha,\ p_1p_2(\varepsilon);\ {}^{\rho}\lambda\rangle\langle \alpha p_1'(\mu)\,p_2',\ \lambda|\alpha,\ p_1'p_2'(\varepsilon);\ {}_{\rho}\lambda\rangle \\
&\qquad = W_{n(0)}\left(\begin{matrix}\alpha_{0,-1} & \lambda_{0,0} \\ \beta_{0,0} & \mu_{0,0}\end{matrix}\,\middle|\, f_{s1}\right),
\end{aligned} \tag{25}
$$

where

$$
\begin{aligned}
f_{s1}(\sigma) &= (-1)^{\sum\limits_{i=1}^{n}\alpha_i-\varepsilon_1}(\varepsilon_1-\varepsilon_2+1)\left[\frac{(p_1-\varepsilon_2)!\,(p_1'-\varepsilon_2)!\,d_n(\beta)\,d_n(\mu)}{(\varepsilon_1-p_1)!\,(\varepsilon_1-p_1')!}\right]^{1/2} \\
&\times \frac{(\varepsilon_1-p_1+z)!\,(\varepsilon_1-p_1'+z)!}{z!\,(\varepsilon_1-\varepsilon_2+z+1)!};\qquad z=\sum_{i=1}^{n}(\sigma_i-\alpha_i)-\varepsilon_1.
\end{aligned}
$$

This expression and also the expression for bilinear forms of recoupling matrices obtained in Ref. 82 by permutation of the parameters of the irreducible representations,

$$\sum_{\rho} \langle \alpha\,[\dot{0},\ -q]\,(\beta)\,p,\ \lambda|\alpha,\ [\dot{0},\ q]\,p\,([p_0,\ \dot{0},\ -q_0]);\ {}^{\rho}\lambda\rangle$$
$$\times \langle \alpha\,[\dot{0},\ -q']\,(\mu)\,p',\ \lambda|\alpha,\ [\dot{0},\ -q']\,p'\,[p_0,\ \dot{0},\ -q_0];\ {}_{\rho}\lambda\rangle$$
$$= W_{n(0)}\left(\begin{matrix}\alpha_{0,0} & \lambda_{0,0}\\ \beta_{0,0} & \mu_{0,0}\end{matrix}\middle| f_{m1}\right), \tag{26}$$

where

$$f_{m1}(\sigma) = (-1)^{q_0+\sum\limits_{i=1}^{n}\alpha_i}\left[\frac{d_n(\beta)\,d_n(\mu)}{(q-q_0)!\,(q'-q_0)!\,(p+q_0+n-1)!\,(p'+q_0+n-1)!}\right]^{1/2}$$
$$\times (p_0+q_0+n-1)\,\frac{1}{z!}\,(q-q_0+z)!\,(q'-q_0+z)!\,(p_0+q_0+n-2-z)!,$$
$$z = q_0 + \sum_{i=1}^{n}(\sigma_i - \alpha_i),$$

can be used for orthonormalization of the recursion expressions for the isofactors. Here, $[\dot{0}, -q]$ and p denote, respectively, the contravariant and covariant symmetric irreducible representations of U_n; $[p_0, 0, -q_0]$ denotes mixed tensor irreducible representations of the class 1, $\bar{1}$.

Matrix Elements of Powers of Generators of the Group Sp_4 (SO_5)

Among the ten infinitesimal generators of the group SO_5 (Sp_4), six are generators of the subgroup $SU_2\times SU_2$, and the remainder form a bispinor of rank $(\frac{1}{2}, \frac{1}{2})$ of the group $SU_2\times SU_2$ and can be expressed in terms of the generators of the group SU_4 (cf. Refs. 87 and 88):

$$\left.\begin{matrix} T_{++} = -E_{14} - E_{32}, & T_{--} = E_{41} + E_{23}, \\ T_{+-} = E_{13} - E_{42}, & T_{-+} = E_{31} - E_{24}. \end{matrix}\right\} \tag{27}$$

Powers of these operators are extremal components of irreducible tensor operators of the group $SU_2\times SU_2$, and therefore their matrox elements factorize into two CG coefficients of the group SU_2 and a submatrix element (see Ref. 88):

$$\left\langle \begin{matrix}\langle K\Lambda\rangle \\ I'J'\end{matrix} \middle\| T^{2\alpha} \middle\| \begin{matrix}\langle K\Lambda\rangle \\ IJ\end{matrix}\right\rangle = [(2I+1)\,(2J+1)]^{1/2}$$
$$\times \frac{(J+J'-\alpha)!\,\nabla(\alpha II')\,\nabla(\alpha JJ')\,\nabla(K-J,\ I,\ \Lambda)\,\nabla(K-J',\ I',\ \Lambda)}{E\,(K+J,\ I,\ \Lambda)\,E\,(K+J',\ I',\ \Lambda)} \tag{28}$$
$$\times \sum_{i,\,j} \frac{(-1)^{J-I'+\alpha+i-j}\,(2i+1)\,E^2\,(K+j,\ i,\ \Lambda)}{(2j+1)!\,(\alpha+J+J'-2j)!\,\nabla^2(K-j,i,\Lambda)\,\nabla^2(j-J,i,I)\,\nabla^2(j-J',i,I')}.$$

Here,

$$\nabla(abc) = \left[\frac{(a+b-c)!\,(a-b+c)!\,(a+b+c+1)!}{(b+c-a)!}\right]^{1/2}; \tag{29}$$

$$E\,(a,\ b,\ c) = [(a-b-c)!\ (a-b+c+1)!\ \ (a+b-c+1)!$$
$$\times (a+b+c+2)!]^{1/2}. \tag{30}$$

Applying matrix elements of powers of the generators T_{ij}, we can find representations of finite transformations of one-parameter nilpotent and other subgroups of the groups $Sp(4, C)$ and $SO(5, C)$ and their real forms (cf. the case of the group $GL(n, C)$ in Refs. 95 and 20).

Unfortunately, for Sp_4 one cannot in such a simple form as (16) for U_n write down the nonextremal components of the tensor operators of the subgroup $SU_2\times SU_2$. Attempts to find matrix elements of general form for powers of the generators of the orthoginal groups SO_n for $n>5$ have also not resulted in success (cf. Ref. 101). It should be noted that in the expressions for the matrix elements of finite transformations of one-parameter subgroups of the groups SO_n in the form of integrals,[102,103] and also in the expressions given in Refs. 20 and 104, there is no reflection of the symmetry properties of these elements of Regge type, which are discussed in Ref. 101 and make it possible to permute the parameters of the irreducible representations of the groups SO_n and SO_{n-2}, $m_{ni}\leftrightarrow m_{n-2,i-1}$, in the Gel'fand–Tsetlin schemes.[105] The well-known analytic expressions for the matrix elements of irreducible representations of finite transformations of the groups SO_3, SO_4, and SO_5 are discussed in Ref. 106.

Weight-lowering operators of the group Sp_4 (SO_5) in the form of polynomials of ordered generators are given in Ref. 88, where they are used to construct special isofactors of this group.

Semistrected Isofactors of the Group Sp_4 (SO_5).

These isofactors are necessary constructive elements of the most general isofactors of this group, but they also play an important part in the WR calculus of various compact groups.

Semistretched isofactors of the first kind, whose parameters satisfy the condition $K_3+\Lambda_3=K_1+\Lambda_1+K_2+\Lambda_2$, are proportional to the $9j$ coefficients of the group SU_2 (Ref. 74):

$$\left[\begin{matrix}\langle K_1\Lambda_1\rangle & \langle K_2\Lambda_2\rangle & \langle K_3\Lambda_3\rangle \\ I_1J_1 & I_2J_2 & I_3J_3\end{matrix}\right] = (-1)^{I_1+I_2-I_3}$$
$$\times [(2K_3-2\Lambda_3+1)\,(2I_1+1)\,(2I_2+1)\,(2J_1+1)\,(2J_2+1)]^{1/2} \tag{31}$$
$$\times {}_3\prod_{a}{}_{1,2}\,\frac{[(2K_a+1)!\,(2\Lambda_a)!\,(2K_a+2\Lambda_a+2)!]^{1/2}}{E\,(K_a+\Lambda_a,\ I_a,\ J_a)}$$
$$\times \left\{\begin{matrix} K_1-\Lambda_1 & K_2-\Lambda_2 & K_3-\Lambda_3 \\ I_1 & I_2 & I_3 \\ J_1 & J_2 & J_3\end{matrix}\right\}.$$

Here and in what follows, we use the notation

$${}_3\prod_{a}{}_{1,2}\,X_a = \frac{X_1X_2}{X_3}, \tag{32}$$

where X_a is any function of the parameters with indices a.

By means of the relation (31) and the above lowering operators for the weight of the irreducible representations of the group Sp_4 (SO_5) an optimal expression was obtained in Ref. 88 for the $9j$ coefficients in the form of a triple sum, which was established later in the framework of the theory of angular momentum in Ref. 17 (Eq. (32.10)). By analytic continuation of this formula expressions were found for the first time in the form of finite triple series for all cases of the isofactors of the Lorentz group $SL(2, C)$ $(SO(3, 1))$ that reduce direct products of irreducible unitary representations of both the principal and the complemntary series in an SU_2 (SO_3) basis (Ref. 107).[4)]

Semistretched isofactors of the second kind, whose parameters satisfy the condition $K_3=K_1+K_2$, can be expressed in the following form[88]:

[4)]This result is still optimal after the appearance of Ref. 108, in which there is a more rigorous derivation of the normalization of the isofactors but the sum is a fourfold one.

$$\begin{bmatrix} \langle K_1\Lambda_1\rangle & \langle K_2\Lambda_2\rangle & \langle K_3\Lambda_3\rangle \\ I_1J_1 & I_2J_2 & I_3J_3 \end{bmatrix} = (-1)^{\Lambda_1+\Lambda_2-\Lambda_3}$$

$$\times [(2\Lambda_3+1)(2I_1+1)(2I_2+1)(2J_1+1)(2J_2+1)]^{1/2}$$

$$\times {}_{3}\prod_{a}{}_{1,2} [(2K_a-2\Lambda_a)!\,(2K_a+1)!\,(2K_a+2\Lambda_a+2)!]^{1/2}$$

$$\times \left[\begin{array}{c|ccc} K_1 & \Lambda_1 & I_1 & J_1 \\ K_2 & \Lambda_2 & I_2 & J_2 \\ K_3 & \Lambda_3 & I_3 & J_3 \end{array}\right], \qquad (33)$$

where

$$\left[\begin{array}{c|ccc} k_1 & j_{11} & j_{12} & j_{13} \\ k_2 & j_{21} & j_{22} & j_{23} \\ k_3=k_1+k_2 & j_{31} & j_{32} & j_{33} \end{array}\right] = \frac{(\sum_{i=1}^{3} j_{3i}-k_3)!}{\prod_{i=1}^{3} \nabla(j_{3i},\, j_{1i},\, j_{2i})}$$

$$\times {}_{3}\prod_{a}{}_{1,2} [E(k_a+j_{a1},\, j_{a2},\, j_{a3})\,\nabla(k_a-j_{a1},\, j_{a2},\, j_{a3})]^{-1}$$

$$\times \sum_{z_1,\, z_2,\, z_3} \prod_{i=1}^{3} (-1)^{z_i} \frac{(2j_{1i}-z_i)!\,(j_{3i}-j_{1i}+j_{2i}+z_i)!}{z_i!\,(j_{1i}+j_{2i}-j_{3i}-z_i)!}$$

$$\times \left\{\left[\sum_{b=1}^{3}(j_{1b}-z_b)-k_1\right]!\left[\sum_{b=1}^{3}(j_{3b}-j_{1b}+z_b)-k_2\right]!\right\}^{-1} \qquad (34a)$$

$$= \frac{(-1)^{k_1-j_{11}-j_{12}-j_{23}+j_{33}}(k_2-j_{21}-j_{22}+j_{23})!\ {}_{3}\prod_{a}{}_{1,2}\nabla(j_{2a},\, j_{1a},\, j_{3a})}{{}_{3}\prod_{a}{}_{1,2} E(k_a+j_{a1},\, j_{a2},\, j_{a3})\,\nabla(k_a-j_{a1},\, j_{a2},\, j_{a3})}$$

$$\times \sum_{z_1,\, z_2,\, z_3} \frac{(-1)^{z_3}(j_{13}+j_{23}-j_{33}+z_3)!\,(2j_{33}-z_3)!}{z_3!\,(j_{13}-j_{23}+j_{33}-z_3)!\left[\sum_{b=1}^{2}(j_{3b}-j_{2b})+j_{23}-j_{33}-k_1+\sum_{i=1}^{3} z_i\right]!}$$

$$\times \frac{1}{(k_3-j_{31}-j_{32}+j_{33}-\sum_{i=1}^{3} z_i)!} \prod_{i=1}^{2} \frac{(j_{1i}-j_{2i}+j_{3i}+z_i)!}{z_i!\,(j_{1i}+j_{2i}-j_{3i}-z_i)!\,(2j_{3i}+1+z_i)!}. \qquad (34b)$$

The entity represented by (34) has a higher symmetry than the isofactors of the group SO_5. Permutations of the last three columns do not change its values, and permutations of the first two rows give only a phase factor:

$$(-1)^{\sum_{i=1}^{3}(j_{1i}+j_{2i}-j_{3i})}.$$

The expression (34b), which was obtained in Refs. 55 and 56 from (34a) by permutation of the IR parameters, is less symmetric but simplifies more in certain limiting cases.

Stretched isofactors of the group SO_5 (Sp_4) with parameters satisfying the condition $K_3=K_1+K_2$, $\Lambda_3=\Lambda_1+\Lambda_2$ can be calculated in accordance with both (31) and (33). Application of elements of the group of permutations of the IR parameters[74] also makes it possible to find nonstandard semistretched isofactors with parameters satisfying definite linear dependences (for example, $K_3-\Lambda_3=K_1-\Lambda_1+K_2+\Lambda_2$ or $\Lambda_3=K_2+\Lambda_1$; see Ref. 56).

Isofactors of Symmetric Irreducible Representations of the Orthogonal Groups

The CG coefficients of the groups SO_n in a canonical basis (for $n\geqslant 5$), when the coupled and resulting irreducible representations are symmetric, were first considered in Refs. 55, 109, and 110. In Refs. 109 and 110, they appear as the expansion coefficients of products of two hyperspherical functions (see Ref. 111, Chap. 9) with respect to the same functions. The relation (4), a special case of which was used in Ref. 55, and Eqs. (33) and (34) make it possible to obtain the simplest expressions for normalized special isofactors of both a semicanonical and a canonical basis of an irreducible representation of SO_n:

$$\left[\begin{array}{c|ccc} SO_n & l_1 & l_2 & l_3 \\ SO_{n'}+SO_{n''} & l_1',\, l_1'' & l_2',\, l_2'' & l_3',\, l_3'' \end{array}\right]$$

$$= (-1)^{\frac{1}{2}\sum_{i=1}^{3}(l_i-l_i'-l_i'')} N\left[\left(\sum_{i=1}^{3} l_i+n-2\right)!!\right]^{-1}$$

$$\times \prod_{i=1}^{3} [(l_i-l_i'-l_i'')!!\,(l_i-l_i'+l_i''+n''-2)!!\,(l_i+l_i'-l_i'' \qquad (35a)$$

$$+n'-2)!!\,(l_i+l_i'+l_i''+n-4)!!]^{1/2}$$

$$\times \sum_{z_1,\, z_2,\, z_3} \prod_{i=1}^{3} \frac{(-2)^{-z_i}[(2l_i''+n''-2+2z_i)!!]^{-1}}{z_i!\,(l_i-l_i'-l_i''-2z_i)!!\,(l_i+l_i'-l_i''+n'-2-2z_i)!!}$$

$$\times \left[\sum_{i=1}^{3}(l_i''+z_i)+n''-2\right]!!\left[\sum_{i=1}^{3}(l_i-l_i''-z_i)+n'-2\right]!!$$

$$= (-1)^{\frac{1}{2}(l_1'+l_2'-l_3')} N\,(l_1''+l_2''-l_3'')!!$$

$$\times {}_{3}\prod_{a}{}_{1,2}\left[\frac{(l_a-l_a'-l_a'')!!\,(l_a+l_a'-l_a''+n'-2)!!}{(l_a-l_a'+l_a''+n''-2)!!\,(l_a+l_a'+l_a''+n-4)!!}\right]^{1/2}$$

$$\times \sum_{z_1,\, z_2,\, z_3} \frac{(-2)^{-z_3}(l_3-l_3'-l_3''+2z_3)!!\,(l_3+l_3'-l_3''+n'-2+2z_3)!!}{z_3!\,(2l_3+n-2+2z_3)!!\left[l_3-l_3''+\sum_{i=1}^{2}(l_i''-l_i)+2\sum_{i=1}^{3} z_i\right]!!}$$

$$\times \frac{1}{(l_1+l_2-l_3-2\sum_{i=1}^{3} z_i)!!} \prod_{i=1}^{2} \frac{2^{-z_i}(2l_i+n-4-2z_i)!!}{z_i!\,(l_i-l_i'-l_i''-2z_i)!!\,(l_i+l_i'-l_i''+n'-2-2z_i)!!}, \qquad (35b)$$

where

$$N = \left[\frac{2^3(2l_1'+n'-2)(2l_2'+n'-2)(2l_1''+n''-2)(2l_2''+n''-2)(2l_3+n-2)}{(l_3'+n'-3)!\,(l_3''+n''-3)!\,l_3!}\right.$$

$$\left.\times \frac{(n-4)!!\,l_3'!\,l_3''!\,(l_3+n-3)!}{(n'-4)!!\,(n''-4)!!}\right]^{1/2} \frac{D_n(l_1l_2l_3)}{D_{n'}(l_1'l_2'l_3')\,D_{n''}(l_1''l_2''l_3'')}; \qquad (36)$$

$$D_n(l_1l_2l_3) = \left[\frac{(l_1+l_2-l_3)!!\,(l_3-l_1+l_2)!!}{(l_1+l_2-l_3+n-4)!!\,(l_3-l_1+l_2+n-4)!!}\right.$$

$$\left.\times \frac{(l_1-l_2+l_3)!!\,(l_1+l_2+l_3+n-2)!!}{(l_1-l_2+l_3+n-4)!!\,(l_1+l_2+l_3+2n-6)!!}\right]^{1/2}; \qquad (37)$$

$l_i-l_i'-l_i''$, $l_1+l_2+l_3$ — are integers.

The expression (35b), obtained by analytic continuation of Eq. (34b), is less symmetric than (35a) but has stronger and more varied conditions restricting the sums. One sum vanishes under the condition $l_1''+l_2''=l_3''$, and in the stretched case ($l_1+l_2=l_3$) all three sums vanish. In the case of the reduction $SO_n\supset SO_{n-2}+SO_2$, the condition $l_1''+l_2''=l_3''$ is always satisfied, and Eq. (35b) after elimination of divergences by a passage to the limit [replacement of the factors that depend on the parameters n'' and l_i'' in (36) by $1/\sqrt{2}$] makes it possible to express the considered isofactor as a double sum.

We obtain the isofactors of the canonical basis of the group SO_n after substitution of the parameters δ_1, δ_2, δ_3, which are equal to 0 or 1, in place of l_1'', l_2'', l_3'', respectively. From the conditions $l_i-l_i'-\delta_i=0 \bmod 2$, $\delta_1+\delta_2+\delta_3=0 \bmod 2$, it can be seen that the following parameter sets are possible: $\delta_1=\delta_2=\delta_3=0$; $\delta_1=\delta_3=1$, $\delta_2=0$; $\delta_1=0$, $\delta_2=\delta_3=1$; $\delta_1=\delta_2=1$, $\delta_3=0$.

In the first three cases directly, and in the fourth after application of the symmetry property [which can be seen from (35a)], Eq. (35b) makes it possible to express all the isofactors of the canonical basis of symmetric irreducible representations of the orthogonal groups in the form of double sums.[5)]

Neither from (35b) nor (35a) does there follow the expression proved in Refs. 110 and 55 for the special isofactor

$$\begin{bmatrix} SO_n \\ SO_{n-1} \end{bmatrix}\!\!\begin{array}{|ccc} l_1 & l_2 & l_2 \\ 0 & 0 & 0 \end{array}\!\!\Bigg] = \frac{1}{D_n\,(l_1 l_2 l_3)} \left[\frac{(n-3)!!\, l_1!\, l_2!\, 2^{-n+3}}{(n-4)!!\,(l_1+n-3)!\,(l_2+n-3)!} \right]^{1/2} \qquad (38)$$

($n \geqslant 4$; in the case $n=3$ a phase factor appears; cf. Eq. (15.10) in Ref. 17), which is important for the theory of hyperspherical functions.

The relation (5) and the methods developed for Sp_4 (SO_5) with allowance for the correspondence between the parameters of the irreducible representations of the chains $SO_5 \supset SO_4$ and $Sp_4 \supset SU_2 \times SU_2$,

$$L_1 = K + \Lambda,\ L_2 = K - \Lambda,\ L_1' = I + J,\ L_2' = I - J, \qquad (39)$$

made it possible in Ref. 56 to find the most genereal isofactors of the CG coefficients that realize vector addition of the canonical bases of the symmetric irreducible representations of the orthogonal groups SO_n.

Proportionality of Some Isofactors of the Unitary and Symplectic Groups to Special Isofactors of the Orthogonal Groups

In Refs. 110 and 112 there is a study of special isofactors of the CG coefficients of the groups U_n, these being used as expansion coefficients of products of hyperspherical functions of the group SU_n realized on the factor space SU_n/SU_{n-1}. It is shown in Ref. 113 that such isofactors of the mixed tensor irreducible representations of the class 1, $\bar{1}$ of the group U_n are proportional to the isofactors of the chain $SO_{2n} \supset SO_{2n-2} + SO_2$ represented by (35b). Further, it is shown in Ref. 69 that the special isofactors corresponding to $U_n \supset U_{n'} \dot{+} U_{n''}$ and $SO_{2n} \supset SO_{2n'} \dot{+} SO_{2n''}$.

It is also shown in Ref. 69 that the special isofactors of the irreducible representations of class 2 of the symplectic groups that correspond to $Sp_{2m} \supset Sp_{2m'} \dot{+} Sp_{2m''}$ are proportional to products of the isofactors of the chain $SO_{4m} \supset SO_{4m'} \dot{+} SO_{4m''}$ represented by Eq. (35) and $9j$ coefficients of the group SU_2. In particular, Eq. (31) is generalized in a definite sense.

3. BIORTHOGONAL SYSTEMS OF CG COEFFICIENTS

As we mentioned in the Introduction, the appearance of multiple irreducible representations in the decomposition of a direct product of irreducible representations of groups that are not simply reducible leads to nonorthogonality of the analytically constructed CG coefficients with respect to the indices ρ. The basic operations of the Wigner–Racah algebra (including the decomposition of matrix elements of irreducible tensor operators in accordance with the Wigner–Eckart theorem) can be realized analytically if one knows biorthogonal systems of isofactors of CG coefficients, i.e., sets of dual isofactors satisfying the biorthogonality condition

$$\sum_{\bar{\rho},\,\mu_{(1)},\,\mu_{(2)}} \begin{bmatrix} \lambda_{(1)} & \lambda_{(2)} & {}_{\rho}\lambda \\ \mu_{(1)} & \mu_{(2)} & {}^{\bar{\rho}}\mu \end{bmatrix} \begin{bmatrix} \lambda_{(1)} & \lambda_{(2)} & {}^{\rho'}\lambda' \\ \mu_{(1)} & \mu_{(2)} & {}_{\bar{\rho}}\mu \end{bmatrix} = \delta_{\lambda\lambda'}\delta_{\rho\rho'} \qquad (40a)$$

and the completeness condition

$$\sum_{\lambda,\,\rho} \begin{bmatrix} \lambda_{(1)} & \lambda_{(2)} & {}_{\rho}\lambda \\ \mu_{(1)} & \mu_{(2)} & {}^{\bar{\rho}}\mu \end{bmatrix} \begin{bmatrix} \lambda_{(1)} & \lambda_{(2)} & {}^{\rho}\lambda \\ \mu'_{(1)} & \mu'_{(2)} & {}_{\bar{\rho}'}\mu \end{bmatrix} = \delta_{\mu_{(1)}\mu'_{(1)}}\delta_{\mu_{(2)}\mu'_{(2)}}\delta_{\overline{\rho\rho}'}. \qquad (40b)$$

Duality in the sense of biorthogonality is reflected naturally in two approaches to CG coefficients. On the one hand, the fundamental definition of CG coefficients as elements of the matrix that reduces the direct product of two irreducible representations makes it possible to express the sums over the indices ρ of products of pairs of these quantities (bilinear forms) in the form of integrals over the group of products of elements of the D matrices of three irreducible representations (see Ref. 20) or by means of projection operators of the Lie algebras.[30-32] In such a case, individual CG coefficients can be found by imposing subsidiary conditions.

On the other hand, the alternative definition of CG coefficients as the coefficients of vector addition of any basis functions of to irreducible representations and the use of the explicit form of the representations of the infinitesimal operators of the group make it possible to write down recursion relations connecting the CG coefficients with the same set of parameters of the irreducible representations of the group itself and the index ρ. The constructed system of difference equations makes it possible to express any CG coefficient (with fixed ρ) in terms of its values in a definite region of power r_0, and as ρ varies the boundary conditions of this region change accordingly.

If as a complete set of nonorthonormalized isofactors we choose, for example, linearly independent ones among linear combinations of the isofactors

$$\begin{bmatrix} \lambda_{(1)} & \lambda_{(2)} & {}^{+,-,}{}_{\rho}\lambda \\ \mu_{(1)} & \mu_{(2)} & {}^{\bar{\rho}}\mu \end{bmatrix} = \sum_{\rho'} \begin{bmatrix} \lambda_{(1)} & \lambda_{(2)} & {}^{\rho'}\lambda \\ \mu_{(1)\max} & \mu_{(2)\min} & {}_{\bar{\rho}'}\mu' \end{bmatrix} \begin{bmatrix} \lambda_{(1)} & \lambda_{(2)} & {}_{\rho'}\lambda \\ \mu_{(1)} & \mu_{(2)} & {}^{\bar{\rho}}\mu \end{bmatrix} \qquad (41)$$

with one-to-one correspondence between the values of the index ρ and the set of parameters $\bar{\rho}'$, μ' of the auxiliary isofactor (the expansion coefficient), the dual isofactors will satisfy the boundary conditions

$$\begin{bmatrix} \lambda_{(1)} & \lambda_{(2)} & {}^{+,-,\rho}\lambda \\ \mu_{(1)} & \mu_{(2)} & {}_{\bar{\rho}}\mu \end{bmatrix} = \delta_{\bar{\rho}\mu,\,\rho'}, \qquad (42)$$

if $\mu_{(1)} = \mu_{(1)\max}$, $\mu_{(2)} = \mu_{(2)\min}$, $\bar{\rho}$, μ take the same values for which the linear combinations (41) are linearly independent (cf. the SU_3 case[82]). The symbols + and − here indicate fixed extremal basis characteristics of the auxiliary isofactor.[6)] It is readily seen that on the expansion of the linearly dependent isofactors—bilinear forms of the type (41)—with respect to a complete system there appear as expansion coefficients dual isofactors whose basis characteristics take the values that appeared in (41) instead of the set $\mu_{(1)\max}$, $\mu_{(2)\min}$, $\bar{\rho}'$, μ'

[5)]In particular, there follows from (35b) the simplest expression [see Eq. (32.14) in Ref. 17] for a $9j$ coefficient with two equal rows. [It is necessary there to correct the factor $(2j_3+1+z)!!$ to $(2j_3+1+z)!$.]

[6)]The corresponding signs in Ref. 82 have the opposite meaning; for example, the sign + indicates the state of minimal weight.

corresponding to the complete system.

Isofactors satisfying boundary conditions of the type (42), i. e., with upper indices of multiple irreducible representations, are very conveninet (even more so than orthonormalized ones) when the Wigner–Eckart theorem is used: To separate submatrix elements or to find elements of a recoupling matrix with lower indices of multiple irreducible representations, it is sufficient to choose the corresponding extremal basis characteristics in the matrix element or in a combination of an odd number of CG coefficients of the group, i. e., the business is almost as simple as in the case of simply reducible groups, for example, SU_2 (cf. Eqs. (28.10) and (31.26) of Ref. 17). In this way, all matrix elements can be decomposed with respect to products of their values in an exyremal region of power r_0 (r_0 is the multiplicity of the irreducible representation), isofactors satisfying standard boundary values, and CG coefficients of a subgroup.

Isofactors of both dual classes for the unitary groups and of the second class for the group SO_5 (Sp_4) could be expressed recursively as combinations of simpler (for the second-rank groups SU_3 and SO_5, fundamentally) isofactors or linear combinations. Essentially, the structure of these recursion expressions reflects the structure of the generating invariants used by Shelepin and Karasev[24,25] as well as operations with tensors preformed in decomposition of a direct product.[34,37]

Recursion Expressions for Bilinear Forms of Isofactors of the Unitarity Groups

These expressions were considered in Refs. 100, 114, and 82. As an example, we consider the isofactors that reduce the direct product of a general α and a two-row $\varepsilon=[\varepsilon_1\varepsilon_2]$ irreducible representation, including as a special case the most general isofactors of the group SU_3:

$$\sum_{\rho'}\begin{bmatrix}\alpha & \varepsilon & {}^{\rho'}\lambda\\ \alpha & \varepsilon_2 & \mu'\end{bmatrix}\begin{bmatrix}\alpha & \varepsilon & {}_{\rho'}\lambda\\ \beta & \gamma & {}^{\bar\rho}\mu\end{bmatrix}
=\left[\frac{(\varepsilon_1+1)!\,\mathfrak{M}(\mu')}{(\varepsilon_1-\varepsilon_2+1)\,\mathfrak{M}(\lambda)}\right]^{1/2}\sum_{s,\,s',\,\nu}\begin{bmatrix}\varepsilon_2 & \varepsilon_1 & \varepsilon\\ s' & s & \gamma\end{bmatrix}\begin{bmatrix}\alpha & \varepsilon_2 & \mu'\\ \beta & s' & \nu\end{bmatrix}\begin{bmatrix}\mu' & \varepsilon_1 & \lambda\\ \nu & s & \mu\end{bmatrix}
\times\langle\beta\ s'(\nu)\,s;\ \mu|\beta,\ s's(\gamma);\ {}^{\bar\rho}\mu\rangle. \tag{43}$$

Without loss of generality, we have chosen in Eq. (43) an irreducible representation α of the group U_n with Young diagram containing not more than $n-1$ rows. The element of the recoupling matrix of the U_n irreducible representation that appears on the left-hand side is replaced in accordance with (12) by a special isofactor. On the right-hand side there appears an element of the recoupling matrix of an irreducible representation of the subgroup U_{n-1}, and the individual isofactors are calculated in accordance with (22).

A complete set of isofactors is realized of the intermediate irreducible representation μ' varies in the region bounded by the inequalities

$$\sum_{i=k}^{n}\lambda_i-\sum_{i=k}^{n-1}\mu_i'\geqslant\sum_{i=k-1}^{n-1}(\mu_i'-\alpha_i)\ (k=2,\ 3,\ \ldots,\ n-1) \tag{44}$$

in addition to the natural conditions $\lambda_i\geqslant\mu_i'\geqslant\lambda_{i+1}$, $\alpha_{i-1}\geqslant\mu_i'\geqslant\alpha_i$. It is these values of μ' that appear on decomposition in accordance with Littlewood's rules[34-37] for the direct product of irreducible representations, $\lambda^*\times\varepsilon\to\alpha^*$, i. e., to each value there corresponds a separate linearly independent generating invariant.[24] [The "inversion" of Littlewood's rules is due to the effectiveness for the given problem of the relation (12).]

On the other hand, the same Young diagram μ' characterizes the irreducible representation of the subgroup U_{n-1}, which is an illustration of the correspondence, established by Biedenharn and his collaborators[79,114] (see Ref. 115), between the indices of multiple irreducible representations in the decomposition of a direct product and the basis characteristics of one of the irreducible representations.

A special case of (43) (see Refs. 100 and 82), like the corresponding expressions obtained by integration over the group[20] or by means of projection operators[31,32] gives expressions for nonorthonormalized isofactors of the group SU_3 containing six sums each. But for the separation from (43) of orthonormalized isofactors, the most convenient formula is derived from (25) and (12):

$$\sum_{\rho}\begin{bmatrix}\alpha & \varepsilon & {}_{\rho}\lambda\\ \alpha & \varepsilon_2 & \mu'\end{bmatrix}\begin{bmatrix}\alpha & \varepsilon & {}^{\rho}\lambda\\ \alpha & \varepsilon_2 & \mu\end{bmatrix}=W_{n(n)}\left(\begin{matrix}\mu_{1,0} & \mu'_{1,0}\\ \alpha_{1,0} & \lambda_{0,-1}\end{matrix}\middle| f_{s2}\right), \tag{45}$$

where

$$f_{s2}(\sigma)=(-1)^{\sum_{i=1}^{n-1}\mu_i}\frac{(\varepsilon_1-\varepsilon_2)!\,(\varepsilon_1+1)!}{\mathfrak{M}(\lambda)}\frac{\left[\sum_{i=1}^{n-1}(\mu_i-\sigma_i)\right]!\prod_{i=1}^{n-1}(\sigma_i-i+n-1)!}{\left[\sum_{i=1}^{n-1}(\mu_i-\sigma_i)+\varepsilon_1-\varepsilon_2+1\right]!}.$$

General recursion expressions for bilinear forms of isofactors of the unitary groups are given in Ref. 114, where there is also established a correspondence between the indices of multiple irreducible representations associated with generating invariants[24,25] and the indices that appear when CG coefficients are constructed by means of integration over the group.[20]

Isofactors of the Unitary Groups Satisfying Boundary Conditions

Such isofactors can be expressed in their most elementary form as follows ($\alpha_n=0$):

$$\begin{bmatrix}\alpha & [\gamma,\ 0,\ -q] & {}^{\rho,+,+}[\lambda,\ -h]\\ \beta & [\delta,\ -r] & {}_{\bar\rho,+,+}\mu\end{bmatrix}=\begin{bmatrix}\alpha & [\dot{0},\ -q] & [\beta',\ -h]\\ \beta' & 0 & \beta'\end{bmatrix}^{-1}
\times\sum_{\bar\rho',\delta',r',\bar\gamma}\begin{bmatrix}\alpha & [\dot{0},\ -q] & [\beta',\ -h]\\ \beta & [\dot{0},\ -r'] & \delta'\end{bmatrix}\begin{bmatrix}[\beta',\ -h] & \gamma & {}^{\rho',+,+}[\lambda,\ -h]\\ \delta' & \bar\gamma & {}_{\bar\rho',+,+}\mu\end{bmatrix}
\times\begin{bmatrix}\gamma & [\dot{0},\ -q] & [\gamma,\ 0,\ -q]\\ \bar\gamma & [\dot{0},\ -r'] & [\delta,\ -r]\end{bmatrix}\langle\beta\,[\dot{0},\ -r']\,(\delta')\,\bar\gamma;\ {}_{\bar\rho',+,+}\mu|
\beta,\ \bar\gamma\,[\dot{0},\ -r']\,([\delta,\ -r]);\ {}_{\bar\rho,+,+}\mu\rangle. \tag{46}$$

(A special case of this expression for SU_3 is obtained in Ref. 82). In the given case, it is more convenient to denote some of the irreducible representations as mixed tensors. Equation (46) holds, provided $h\geqslant 0$. Almost all the isofactors on the right-hand side can, with allowance for the symmetry properties, be expressed in accordance with Eqs. (22), except for one, which after translation of the parameters to the region of covariant irreducible representations can be replaced in accordance with Eq. (12) by an element of the recoupling matrix of the irreducible representations of the subgroup U_{n-1}. The index ρ is uniquely correlated with the values taken by the parameters of the irreducible representation β' and the index ρ' of the auxiliary isofactor.

Choosing $\mu=\lambda$ in (46), one can show that the special case of the U_n isofactor is proportional to an element of the recoupling matrix of the irreducible representations of U_{n-1} (in particular, the special isofactor of SU_3 is proportional to a $6j$ coefficient of SU_2; see Ref. 82):

$$\begin{bmatrix} \alpha\,[\gamma,\,0,\,-q] & {}^{\rho,+,+}[\lambda,\,-h] \\ \beta\,[\delta,\,-r] & {}_{\bar{\rho},+,+}\lambda \end{bmatrix} = (-1)^{r'}\Big[\frac{(q-r)!\,d_{n-1}(\beta)}{q!} \times \prod_{j=1}^{n-1}\frac{(\delta_j+q+n-1-j)!}{(\gamma_j+q+n-1-j)!}\Big]^{1/2}\frac{S_{n,\,n-1}(\alpha;\,\beta')}{S_{n,\,n-1}(\alpha;\,\beta)\,S_{n-1,\,n-1}(\beta;\,\beta')} \tag{47}$$
$$\times\langle\beta\,[\dot{0},\,-r']\,(\beta')\,\gamma;\,{}^{\rho',+,+}\lambda\,|\,\beta,\,\gamma\,[\dot{0},\,-r']\,([\delta,\,-r]);\,{}_{\bar{\rho},+,+}\lambda\rangle,$$

where $r'=\sum_{i=1}^{n-1}(\beta_i-\beta_i')=q-h+\sum_{i=1}^{n-1}(\beta_i-\alpha_i)$. In particular, Eq. (47) is transformed into

$$\delta_{\bar{\rho}\rho'}\delta_{\beta\beta'},$$

when the irreducible representation $[\delta,-r]$ is maximal, i.e., $\delta=\gamma$, $r=0$. In this way we can show that the expressions (46) and (47) satisfy boundary conditions that are related to (42) by a definite permutation of the parameters. The expression (47) is valid not only in the region $h\geqslant 0$ but also in the region $\lambda_{n-1}+q\geqslant\alpha_1$. The expression (47) also has a certain meaning in the region $h<0$, $\lambda_{n-1}+q<\alpha_1$, in which it can be called a pseudoisofactor. A pseudoisofactor is the expansion coefficient of isofactors with respect to their boundary values in a region of greater power than the multiplicity of the irreducible representation in the r_0 region. Pseudoisofactors do not play an independent part in the WR calculus (for example, they cannot be orthonormalized), but they can be used in a generalized Wigner–Eckart theorem if one knows the boundary values of the matrix elements in the complete hyperplane of boundary basis characteristics and not only in its minimal region of power r_0 distingusihed by conditions of the type (44) discussed above. To expand isofactors with respect to pseudoisofactors, it is necessary to find the boundary values of the isofactors in the complete hyperplane using other considerations. For example, bilinear forms of isofactors more general than (41) can be constructed as follows. The first set of isofactors of the type (41) is taken to be overcomplete. Then instead of dual isofactors one takes pseudoisofactors, and pairwise products of both quantities are summed over a complete hyperplane of boundary values covering a subregion isomorphic to the region of the indices ρ. We may point out that the problem of making the boundary conditions more precise is rather common for problems of analytic continuation of WR functions satisfying boundary conditions.

In the case of SU_3, the corresponding boundary isofactors are found (see Refs. 82 and 117) using their symmetry properties. A transformation in the space of the indices ρ by triangular matrices makes it possible to reduce the corresponding boundary isofactors to standard form [of the type (42)] and to find the remaining (i.e., dependent) boundary values.[117] In general, it is expedient to choose (from the six possible types) a type of the boundary conditions such that the power of the corresponding boundary hyperplane is equal to r_0 or exceeds it minimally. Unfortunately, for SU_4 it is already not always possible to make such a choice of the boundary conditions enabling one to use the expression (46) transformed using the symmetry properties of the isofactors. In such a case, one must use combinations of isofactors of other types. The obtained expressions are rather cumbersome, and therefore we note that the problem does not appear as long as one of the irreducible representations of the group U_n belongs to the class 1, $\bar{1}$, for example, in Eq. (46) $\gamma=\dot{p}$ (with allowance for the symmetries, this case also covers all isofactors of the group SU_3). Equation (26) makes it possible to write down an expression for the overlap matrices of the isofactors of the following type (cf. the SU_3 case in Ref. 117):

$$\sum_{\beta,\,\bar{p},\,\bar{q},\,\bar{\rho}}\begin{bmatrix} \alpha\,[p,\,\dot{0},\,-q] & {}^{\beta',+,+}\lambda \\ \beta\,[\bar{p},\,\dot{0},\,-q] & {}_{\bar{\rho}}\mu \end{bmatrix}\begin{bmatrix} \alpha\,[p,\,\dot{0},\,-q] & {}^{\mu',+,+}\lambda \\ \beta\,[\bar{p},\,0,\,-\bar{q}] & {}^{\bar{\rho}}\mu \end{bmatrix} = W_{n(n)}\left(\begin{matrix}\alpha_{0,0} & \lambda_{1,0}\\ \beta'_{1,0} & \mu'_{1,0}\end{matrix}\middle|\, f_{m2}\right), \tag{48}$$

where

$$f_{m2}(\sigma)=(-1)^{q+\sum_{i=1}^{n}\alpha_i-\lambda_n}\frac{[d_{n-1}(\beta')\,d_{n-1}(\mu')]^{1/2}}{q!\,(p+q+n-2)!}\prod_{i=1}^{n}(\alpha_i-\lambda_n+n-i)!$$
$$\times\frac{z!\,(p+q+n-2-z)!}{\prod_{i=1}^{n-1}(\sigma_i-\lambda_n+n-i)!},\quad z=\sum_{i=1}^{n-1}(\sigma_i-\mu_i').$$

Equation (48) holds, provided $\alpha_n\geqslant\lambda_n$.

Finally, applying operators of the type (16) (cf. the SU_3 case in Ref. 117), we obtain a recursion expression of a different class for the most general isofactors of the group U_n:

$$\begin{bmatrix} \alpha\cdot\gamma & {}^{\rho}\lambda \\ \beta & \delta & {}_{\bar{\rho}}\mu \end{bmatrix} = \left\langle\begin{matrix}\lambda\\ \mu_{\max}\end{matrix}\right\|E_{n-1,\,n}^{\;p+q}\left\|\begin{matrix}\lambda\\ \mu\end{matrix}\right\rangle^{-1}$$
$$\times\sum_{\beta',\,\delta',\,\bar{\rho}'}\frac{(p+q)!}{p!\,q!}\left\langle\begin{matrix}\alpha\\ \beta'\end{matrix}\right\|E_{n-1,\,n}^{\;p}\left\|\begin{matrix}\alpha\\ \beta\end{matrix}\right\rangle\left\langle\begin{matrix}\gamma\\ \delta'\end{matrix}\right.E_{n-1,\,n}^{\;q}\left\|\begin{matrix}\gamma\\ \delta\end{matrix}\right\rangle$$
$$\times\langle\beta\delta\,({}_{\bar{\rho}}\mu);\;pq\,(p+q);\;\mu_{\max}\,|\,\beta p\,(\beta');\;\delta q\,(\delta');\;{}^{\bar{\rho}'}\mu_{\max}\rangle$$
$$\times\begin{bmatrix} \alpha & \gamma & {}^{\rho}\lambda \\ \beta' & \delta' & {}_{\bar{\rho}'}\mu_{\max} \end{bmatrix}, \tag{49}$$

where

$$p=\sum_{i=1}^{n-1}(\beta_i'-\beta_i);\quad q=\sum_{i=1}^{n-1}(\delta_i'-\delta_i);\quad p+q=\sum_{i=1}^{n-1}(\lambda_i-\mu_i).$$

On the right-hand side of (49), an element of the recoupling matrix of the irreducible representations of the subgroup U_{n-1} has appeared. The special isofactor on the right-hand side can be calculated using Eq. (47) or in any other way.

In this way, in the framework of the Wigner–Racah algebra of the subgroup U_{n-1} all isofactors of the group U_n can be expanded with respect to their values in a definite boundary hyperplane. In Ref. 32, a start was made on the realization of an analogous program for bilinear forms of the isofactors.

Although for the SU_3 isofactors both Eq. (46) and the combination of Eqs. (49) and (47) also have six sums each, these expressions have advantages over (43) and the other expressions which we have mentioned for bilinear forms of isofactors. For example, the special cases of them needed to construct isofactors of a noncanonical projection basis have a much simpler form (see Ref. 82).

Complete Systems of Isofactors of the Group SO_5 (Sp_4)

Such systems are represented in Ref. 118 in the form of combinations of three simply reaching isofactors that realize pairwise vector additions of the auxiliary irreducible representations $\langle k_1\lambda_1\rangle$, $\langle k_2\lambda_2\rangle$, $\langle k_3\lambda_3\rangle$ and the $6j$ coefficients of the subgroup SU_2. An expression equivalent to the one found in Ref. 119 is obtained when auxiliary semistretched isofactors of the first kind are chosen; a second expression is obtained when the auxiliary isofactors are semistretched isofactors of the second kind. It can be shown that complete systems of isofactors are obtained if one chooses at least one auxiliary isofactor of stretched type pr (in the case of expressions of the second type if $\Lambda_1+\Lambda_2+\Lambda_3$ is not an integer) the closest (to it) semistretched isofactor is the first kind.

Isofactors satisfying the boundary conditions can be expanded by means of triangular matrices (see Ref. 120) with respect to each of the complete systems of isofactors mentioned above. The boundary region of basis characteristics has a form not completely analogous to the case of the unitary groups, and is located in the hyperplane.

$$I_1+J_1=K_1+\Lambda_1,\ I_2+J_2=K_2+\Lambda_2,\ I_3+J_3=K_3+\Lambda_3$$

and is correlated with the region of variation of the parameters s_1, s_2, s_3 used to find the multiplicity of the irreducible representation $\langle K_3\Lambda_3\rangle$ in the decomposition of $\langle K_1\Lambda_1\rangle\times\langle K_2\Lambda_2\rangle$. This multiplicity is equal to the number of integral solutions of the system of six inequalities

$$\begin{cases} s_j-s_i\leqslant K_l-\Lambda_l+K_i-\Lambda_i-K_j+\Lambda_j; \\ s_j+s_l-s_i\leqslant 2\Lambda_j+2\Lambda_l-2\Lambda_i \end{cases} \tag{50}$$

($s_1\geqslant 0,\ s_2\geqslant 0,\ s_3\geqslant 0$; i, j, l are permutations of the numbers 1, 2, 3) subject to the subsidiary conditions of integrality of $\Lambda_1+\Lambda_2+\Lambda_3-\frac{1}{2}(s_1+s_2+s_3)$ and the vanishing of at least one of the numbers s_1, s_2, s_3.

4. DUAL NONCANONICAL BASES

In physical applications of the theory of representations of Lie groups, especially in nuclear theory, an important problem is that of constructing noncanonical bases of representations corresponding to reductions of the type $SU_3\supset SO_3$, $SU_n\supset SO_n$, $U_n\subset O_n$ (unitary-scheme model), $SU_4\supset SU_2\times SU_2$ (supermultiplet model), $Sp_4\supset U_2$ (five-dimensional quasispin model), $SO_5\supset SO_3$ (description of quadrupole excitations), $SO_7\supset G_2\supset SO_3$ (classification of states of the f shell), and the determination of matrix elements of different operators in these bases. As we have already said, the corresponding methods were developed initially for $SU_3\supset SO_3$ (see the reviews of Refs. 21 and 81). Different types of bases, to each of which there correspond different ways of separating multiple irreducible representations of the subgroup, i.e., different additional characteristics of the basis states, enable one to describe different model interactions with more or less simplicity (cf. Refs. 121, 122 and 123–126).

The concept of biorthogonal systems makes it possible to find a certain system in all the diversity of the noncanonical bases, to find coefficients of expansion with respect to nonorthogonal systems of functions, and also to simplfiy the construction of group invariants and other operations of Wigner–Racah algebras.

A biorthogonal system is formed by two complete sets of dual functions a^ω, a_ω, of a noncanonical basis characterized by the same purely group sets of indices (i.e., irreducible representations of the group and its subgroup) and satisfying the condition

$$\langle a^\omega | b_{\omega'}\rangle=\delta_{\omega\omega'}, \tag{51}$$

where the number of values taken by the indices ω and ω' is equal to the multiplicity of the irreducible representation of the subgroup. Then any function in the same subspace can be expanded in the form

$$f=\sum_{\omega'}\langle b_{\omega'}|f\rangle a^\omega=\sum_{\omega}\langle a^\omega|f\rangle\, b_\omega. \tag{52}$$

It is readily seen that the matrix elements in the dual bases are connected by the relations of Hermitian conjugation.

It can be seen that the metric tensor of the first basis is identical to the overlapmatrix of the second (dual) basis. In the presence of functions of both dual bases, invariants (scalars) of the group can be readily constructed by using the operation of complex conjugation (contragredience):

$$\sum_{\omega\mu}|a_\mu^\omega\rangle^*\,|b_{\omega\mu}\rangle, \tag{53}$$

where μ is a set of purely group basis characteristics of the states of the given irreducible representation of the group.

The use of dual bases is expedient when one is solving eigenvalue problems for tensor operators acting on nonorthonormalized basis functions. If one uses the matrix elements of an operator between functions of the same basis, it is necessary to solve a generalized secular equation with allowance for the overlap integrals [see, for example, Ref. 112, Eqs. (5.2)–(5.5)]. An ordinary secular equation appears when the operator is represented by an expansion with respect to functions of the same basis on which it acts. In this situation, Eq. (52) can be used.

It is shown in Ref. 81 that there are dual bases among the known realizations of a noncanonical basis for the reduction $SU_3\supset SO_3$. We recall the basic properties of these realizations.

The *projection basis*, constructed for $SU_3\subset SO_3$ for the first time by Elliott,[1] is characterized by "hidden" projection of the moment K, which is connected in its origin to the special form of the functions of the canonical basis. For $SU_4 \supset SU_2 \times SU_2$, the projection basis was first constructed by Draayer.[127] The simplest way of constructing the projection basis for $SU_3 \supset SO_3$ was developed by Asherova and Smirnov,[121,122] who used the projection operators of the subalgebra SO_3 in the form of Refs. 91 and 92. Later, this method was developed for $Sp_4 \supset U_2$,[78,128] for $SU_4 \supset SU_2 \times SU_2$,[129-131] and for $SO_5 \supset SO_3$.[132] Expressions for the coefficients of the expansion of the projection bases with respect to the functions of the canonical bases and the overlap integrals are simplified by the use of the matrix elements of powers of the generators of the groups and the methods of transformation of standard sums (see Refs. 77, 78, 81, 129, and 131).

One of the advantages of projection bases is that on the basis of the commutation relations between the projection and tensor operators (formulated in their most general form in Ref. 32) one can express rather elegantly the action of the group generators (see Refs. 1, 122, 128, and 130). For example, in the $SU_3 \supset SO_3$ case we obtain the expansion somewhat simpler than in Refs. 10 and 122:

$$Q_m \left| \begin{matrix} (\lambda\mu)_{E^+} \\ KLM \end{matrix} \right\rangle = \sum_{L', K', n} \frac{2L+1}{2L'+1} \begin{bmatrix} L & 2 & L' \\ M & m & M' \end{bmatrix} \begin{bmatrix} L & 2 & L' \\ K & n & K' \end{bmatrix} \times b^{(\lambda\mu)}_{LK;\, L'K'} \left| \begin{matrix} (\lambda\mu)_{E^+} \\ K'L'M' \end{matrix} \right\rangle, \tag{54}$$

where

$$b^{(\lambda\mu)}_{LK;\, L'K'} = \delta_{KK'} \frac{1}{\sqrt{6}} \left[2\lambda + \mu + 3 + \frac{1}{2} (L'-L)(L'+L+1) \right] - \delta_{K',\, K+2} \frac{1}{4} [(\mu - K)(\mu + K + 2)]^{1/2} - \delta_{K',\, K-2} \frac{1}{4} [(\mu + K)(\mu - K + 2)]^{1/2}.$$

(In this section, we denote the irreducible representations of the groups SU_n by the differences of the lengths of the Young diagrams.)

Further, to construct the isofactors that realize the vector addition of functions of a noncanonical (necessarily projection) basis to functions of the projection basis (see Refs. 81, 121, 127, and 133), it is sufficient to use fairly simple special cases of isofactors of the canonical basis (including the cases presented in Sec. 3 of the present paper). For example, in Ref. 75 isofactors are found for the chain $SU_4 \supset SU_2 \times SU_2$ that reduce the direct product of two symmetric irreducible representations, namely, the expression for them in Ref. 66 is simplified to a threefold sum. The relation (6a) makes it possible to use this expression as well for calculating the analogous isofactors of the chain $SU_n \supset SO_n$; in particular, for $n=3$ one obtains (apart from the normalization) the expression for the isofactor E^- of Elliott's basis,[7)] and this demonstrates the interconnection of the classification of multiple irreducible representations of the subgroups of SO_n in the irreducible representations of class 2 of the groups SU_n.

Characteristic complications follows from the overcompleteness of the projection bases. Thus, when the group generators act on the basis, linearly dependent functions also appear. One of the methods of expanding the excess functions is based on the use of the commutation relations of powers of the generators (Refs. 10. 128, 130, and 132), but in our view a more universal method is based on using an auxiliary (so-called stretched) basis.

A *stretched basis* for $SU_3 \supset SO_3$ and $SU_4 \supset SU_2 \times SU_2$ was proposed by Sharp *et al.*[134,135] A stretched basis of the irreducible representations of $SU_3 \supset SO_3$ (cf. Ref. 81) can be constructed by vector addition of normalized symmetric covariant and contravariant tensors by means of CG coefficients of the subgroup SO_3 and isofactors satisfying the boundary conditions

$$\begin{bmatrix} (\lambda 0) & (0\mu) & (\lambda\mu)_S \\ l_1 & l_2 & (l_{10} l_{20}) L \end{bmatrix} = \delta_{l_1 l_{10}} (l_{10} + l_{20} = L + \delta) \tag{55}$$

in the region $l_1 + l_2 = L + \delta$ ($\delta = 0$ or 1, $\lambda + \mu - L - \delta = 0$ mod 2). These isofactors are found to be proportional to CG coefficients of the group SU_2 with multiple 1/4 parameters (Eqs. (3.7) of Ref. 81), which has a natural explanation (see Ref. 125) on the basis of the relation (6b) and the construction of a stretched basis for $SU_4 \supset SU_2 \times SU_2$. Special isofactors reducing the IR direct product $(\lambda 00) \times (00\mu) \to (\lambda 0 \mu)$ in the case of the last chain are proportional to the CG coefficients of the group SU_2 (Ref. 75) and take boundary values in the region of parameters $j_1 + j_2 = S \geqslant T$ proportional to $\delta_{j_1 j_{10}} (j_{10} + j_{20} = S = K_S,\ j_{10} - j_{20} = K_T)$. Therefore, the functions of the supermultiplet projection basis (numbered by the parameters $K_S = S$, K_T) and the supermultiplet stretched basis (numbered by the parameters j_{10}, j_{20}) of the irreducible representations of the class 1, $\bar{1}$ of the group SU_4 differ only in the normalization. Proceeding from this, it was possible in Ref. 75 to find isovectors of a supermultiplet basis of SU_4 reducing the IR direct product $(\lambda + q_1, 0, 0) \times (0, 0, \mu + q) \to (\lambda, 0\ \mu)$, and in Ref. 77 the most general expression for them was simplified to threefold sums. In the same way the simplest expressions for isofactors of the same type corresponding to the reduction $SU_n \supset SO_n$ were obtained.

The boundary values of definite isofactors of a projection basis are the expansion coefficients of the functions of the projection basis of the most general irreducible representations with respect to functions of a stretched basis and form triangular matrices. In Ref. 134 (see Refs. 21 and 81) for $SU_3 \supset SO_3$ and in Ref. 130 for $SU_4 \supset SU_2 \times SU_2$ it was also possible to find the inverse matrices, i.e., to expand the stretched basis with respect to a complete set of functions of the projection basis. This solved the problem of the expansion of the excess projection functions.

To understand the position of a stretched basis among the other noncanonical bases, we give an alternative definition of a polynomial basis.

A *polynomial basis* for $SU_3 \supset SO_3$ was proposed by Bargmann and Moshinsky[2] (see also Refs. 21, 62, 123, and 136). The best known may of constructing bases of this type is in the form of products of elementary

[7)]We recall that there exist two variants of the projection basis for both $SU_3 \supset SO_3$ and $SU_4 \supset SU_2 \times SU_2$, these corresponding to covariant and contravariant tensors (see Refs. 1, 81, and 127).

blocks. By this method one can directly obtain the basis states of some irreducible representation of U_n, these being the states of highest weight of a definite irreducible representation of a subgroup G' (for example, SO_n or SO_3). Thus, in Ref. 137 this method is used to construct bases of irreducible representations of the class 1 for the reduction $SO_5 \supset SO_3$, and in Ref. 138 for symmetric irreducible representations of the group SU_6 reduced to the maximal subgroup SU_3, which leaves the six-dimensional irreducible representation of the group SU_6 irreducible.

If one wished to find polynomial functions of an irreducible representation of a longer chain $U_n \supset G' \supset G''$ one can use weight-lowering operators and projection operators of the subgroup G'', but in a number of cases it is expedient to use the alternative construction of a polynomial basis proposed for $SU_3 \supset SO_3$ in Ref. 81.

We take special matrix elements of finite transformations of a subgroup of the general linear group. Suppose the first indices correspond to the states of highest weight of the canonical basis, while the second indices are labeled by the irreducible representations of some noncanonical chain of subgroups. The products of such matrix elements form the basis of an irreducible representation of the group whose highest weight is equal to the sum of the highest weights of the factors. If we realize vector addition by CG coefficients of a subgroup (for states of weight higher relative to the subgroup, this operation is trivial), the obtained states will have the corresponding basis characteristics. Then in this way we obtain in the case $SU_3 \supset SO_3$

$$\begin{aligned} D^{(\lambda\mu)}_{\max,\,(l_{10}l_{20})LM}(g) &= \sum_{\omega} \begin{bmatrix} (\lambda 0) & (0\mu) & (\lambda\mu) \\ l_{10} & l_{20} & \omega L \end{bmatrix} D^{(\lambda\mu)}_{\max,\,\omega LM}(g) \\ &= \sum_{m_1,\,m_2} \begin{bmatrix} l_{10} & l_{20} & L \\ m_1 & m_2 & M \end{bmatrix} D^{(\lambda 0)}_{\max,\,l_{10}m_1}(g)\, D^{(0\mu)}_{\max,\,l_{20}m_2}(g). \end{aligned} \tag{56}$$

Among the functions (56) those for which the parameters l_{10}, l_{20} satisfy the condition $l_{10}+l_{20}=L+\delta$ are linearly independent. It is readily seen that isofactors satisfying the condition (55) are also expansion coefficients of the excess functions of the type (56). Equation (56) also enables us to write down the coefficients of expansion of our variant of the polynomial basis (which differs from the one proposed in Ref. 2 by a simple factor) with respect to functions of the canonical basis:

$$\begin{aligned} \left\langle \begin{matrix} (\lambda\mu)_B \\ (l_{10}l_{20})\,LM \end{matrix} \middle| \begin{matrix} (\lambda\mu) \\ YI\,\tfrac{1}{2}\,M \end{matrix} \right\rangle = \sum_{\substack{m_1+m_2=M \\ i_2-i_1=\mathrm{const}}} \begin{bmatrix} l_{10} & l_{20} & L \\ m_1 & m_2 & M \end{bmatrix} \\ \times \left\langle \begin{matrix} (\lambda 0) \\ l_{10}m_1 \end{matrix} \middle| \begin{matrix} (\lambda 0) \\ i_1,\,\tfrac{1}{2}\,m_1 \end{matrix} \right\rangle \left\langle \begin{matrix} (0\mu) \\ l_{20}m_2 \end{matrix} \middle| \begin{matrix} (0\mu) \\ i_2,\,\tfrac{1}{2}\,m_2 \end{matrix} \right\rangle \\ \times \begin{bmatrix} (\lambda 0) & (0\mu) & (\lambda\mu) \\ y_1 i_1 & y_2 i_2 & YI \end{bmatrix} \begin{bmatrix} i_1 & i_2 & I \\ \tfrac{1}{2}\,m_1 & \tfrac{1}{2}\,m_2 & \tfrac{1}{2}\,M \end{bmatrix}. \end{aligned} \tag{57}$$

What we have obtained is identical to a coefficient of expansion of the functions of the canonical basis with respect to functions of the stretched basis (see Ref. 21). Thus, we see that the stretched and polynomial bases can be obtained by mutually invertible transformations of an orthonormal basis, i.e., these bases are dual. The coefficients of expansions of the functions of the polynomial basis with respect to the functions of the stretched basis are integrals of the overlapping of the polynomials basis and can be expressed as bilinear forms of special isofactors.

A basis equivalent to the polynomial basis can also be represented in the form of linear combinations of orthonormalized functions with special isofactors as coefficients. For example, for the construction one can use the special isofactors (see Ref. 81)

$$\begin{aligned} &\sum_{\omega} \begin{bmatrix} (\lambda+\mu,\,0) & (\mu 0) & \lambda\mu \\ l_1 & l_2 & \omega L \end{bmatrix} \begin{bmatrix} (\lambda 0) & (0\mu) & (\lambda\mu) \\ l_{10} & l_{20} & \omega L \end{bmatrix} \\ &= \left[\frac{(\lambda+1)(\mu+1)(2l_{20}+1)(2l_1+1)}{\lambda+\mu+1} \right]^{\frac{1}{2}} \sum_{l'} (-1)^{l'+l_2+l_{10}+L} \begin{Bmatrix} L & l_2 & l_1 \\ l' & l_{10} & l_{20} \end{Bmatrix} \\ &\quad \times \begin{bmatrix} (\mu 0) & (\mu 0) & (0\mu) \\ l' & l_2 & l_{20} \end{bmatrix} \begin{bmatrix} (\lambda 0) & (\mu 0) & (\lambda+\mu,\,0) \\ l_{10} & l' & l_1 \end{bmatrix}. \end{aligned} \tag{58}$$

Equation (58) (like its generalization, Eq. (3.6) of Ref. 75, for $SU_4 \supset SU_2 \times SU_2$) follows from the uniqueness of a definite recoupling matrix of the irreducible representations of SU_3 (respectively, SU_4). The special cases of (58) for $l_1+\Delta=L+l_2$ ($\Delta=0$ or 1, $\lambda-L-\Delta=0$ mod 2), which after rather lengthy transformations can be expressed without sums (see Ref. 81), form a triangular matrix and are the expansion coefficients of the projection basis introduced in Ref. 21.

An antistretched basis (see Refs. 21 and 81) can be constructed by means of isofactors that realize the vector addition of the bases of the symmetric irreducible representations $(\lambda+\mu,0)$ and $(\mu 0)$ and satisfy special boundary conditions. The use of this basis and its contravariant made it possible to find much simpler expressions for the overlap integrals of the fundamental bases.[81,76]

The isofactors (for definite values of the parameters, pseudofactors; see Sec. 3) of the antistretched basis were obtained in Ref. 81 by analytic continuation of the isofactors of the stretched basis. The use of them and the boundary bilinear form (58) mentioned above made it possible to represent the same bilinear form (58) in a form more convenient for analytic manipulations. Further application to it of permutations of the parameters of the irreducible representations of the groups SU_3 and SO_3 using the formulas for transformation and summation of generalized hypergeometric functions of a common argument made it possible to find optimal expressions for some other coefficients of mutual expansion, these being bilinear forms of isofactors, and ultimately to derive much simpler expressions in the form of double sums for the overlap matrices and the metric tensors of the stretched and polynomial bases (see Refs. 77, 81, and 125).

In Ref. 76, it was found to be effective to make a formal representation of the overlap integral of the projection basis in the form

$$\left\langle \begin{matrix} E^- \\ K \end{matrix} \middle| \begin{matrix} E^- \\ K' \end{matrix} \right\rangle = \sum_{\bar{l}_1,\, l_{10}} \left\langle \begin{matrix} E^- \\ K \end{matrix} \middle| \begin{matrix} \overline{A}^+ \\ \bar{l}_1 \end{matrix} \right\rangle \left\langle \begin{matrix} A^+ \\ \bar{l}_1 \end{matrix} \middle| \begin{matrix} S \\ l_{10} \end{matrix} \right\rangle \left\langle \begin{matrix} B \\ l_{10} \end{matrix} \middle| \begin{matrix} E^- \\ K' \end{matrix} \right\rangle, \tag{59}$$

where $\overline{A}^+$, A^+, and S, B are pairs of dual bases. Each of the three sums in (59) (one of them is contained in the overlap integral $\langle E^{-}|\overline{A}^{+}\rangle$; see Ref. 76) contains no more terms than the multiplicity of the irreducible representation of the group. It is interesting that it is simpler to find the blocks in (59) for irreducible representations of the class 2 of the group SU_4 and continue the obtained expression for $SU_n \supset SO_n$.

It should be noted that the projection and polynomial bases for complementary groups change places. Thus, by analytic continuation of the matrix elements of the projection operators of five-dimensional quasispin (i.e., of the subgroup U_2 embedded in the group Sp_4; see Refs. 78 and 128) expressions were found in Ref. 75 for bilinear forms of the special isofactors of $SU_4 \supset SU_2 \times SU_2$ and $SU_n \supset SP_n$. In Ref. 75, it was also shown that polynomial bases for $SU_n \supset SO_n$ can be constructed by means of projection operators of the complementary group.[8)] On the other hand, it was shown in Ref. 78 that the expansion coefficients of the five-dimensional quasispin basis dual to the projection basis can be found by analytic continuation in accordance with Eq. (7) of the stretched isofactors of the group $SU_4 \supset SU_2 \times SU_2$ obtained in Ref. 77. In this case, the expression itself was found to be much simpler than in the case of the projection basis.

The general principles of construction of dual bases can be formulated as follows. We consider the overlap integrals between different realizations of noncanonical bases, these usually having the meaning of boundary values of bilinear forms of transformation coefficients. If these integrals form a triangular matrix, one can usually find the explicit form of the inverse matrix too, which makes it possible to expand the functions of a basis dual to one of the bases with respect to the basis functions of the second of the considered bases.

Thus, the overlap integrals between the projection and polynomial bases are equivalent to the above-mentioned boundary values of the isofactors of the projection basis as well as to the special values of the coefficients of expansion of the polynomial basis with respect to the functions of the canonical basis (for $SU_3 \supset SO_3$ represented by Eq. (57); see Ref. 81).

The analytic form of the matrices that are the inverse of the triangular matrices can usually be readily found if the index ρ is one-dimensional (see Refs. 81 and 82).[9)] The author has also considered several examples of biorthogonal systems with two-dimensional indices of multiple irreducible representations.[120,130] Thus, using bases of irreducible representations of $SU_4 \supset SU_2 \times SU_2$ in Ref. 130 he found a biorthogonal system formed by CG coefficients of the group SU_2 and certain renormalized analytic continuations of them characteristic of $SU(1,1)$. Generally speaking, the methods of analytic continuation helpful in the investigation of dual bases are not exhausted by those discussed in the present review.

CONCLUSIONS

In this review, we have become acquainted with certain more or less definitive results of the Wigner–Racah calculus of compact Lie groups that are not simply reducible. In recent years, it has proved possible to advance significantly in the investigation of the interrelations between the representation spaces of all representations of groups of second and third rank, and also the spaces of one-parameter and two-parameter irreducible representations of compact Lie groups. One-dimensional or two-dimensional spaces of indices of multiple itreduible representations are characteristic of the problems solved in analytic (and, quite often, optimal) form. In connection with applications to nuclear theory, investigations of representations of three-parameter irreducible representations are very topical. For further advance in this region, the following empirical observations may be helpful:

1. *Principle of Asymmetry*. As a rule, the optimal (containing the smallest number of sums and the smallest number of terms in these sums) expressions for the transformation coefficients (except for the fundamental isofactors and some other WR functions of the unitary groups) do not reflect the symmetry of these quantities. The indices of multiple irreducible representations also in general do not possess the maximal possible symmetry of the isofactors or other transformation coefficients.

2. *The Principle of the Conservation of Difficulties*. The simplification of the algorithms for calculating WR functions satisfying boundary conditions is usually associated with an impossibility of common algorithms for all values of the parameters of not only the WR function but also the overlap integrals, whereas the bilinear forms of the WR functions and their special cases representing overlap integrals can be calculated using a common, albeit very complicated, algorithm. For example, the expressions for the overlap integrals of isofactors of the first (difference) type diverge on the appearance of the parameters characteristic of pseudo-isofactors, this being due to the degeneracy of the overlap matrix of the dual isofactors.

From these principles it follows that it is important to take into account as well the ordinary symmetry properties of the transformation coefficients.

We have not cinsidered methods and problems of the WR calculus of higher groups such as the graphical methods of representing complicated transformation coefficients (recoupling matrices and matrices of transformation of the reduction schemes on subgroups) and the construction of different complete sets of operators making it possible to distinguish multiple irreducible representations. Of great interest too are the questions of the asymptotic behavior of the WR functions for large

8)It should be noted that it is expedient to use the projection operators of the algebra of the complementary group of the subgroup SO_n instead of the laborious replacement of simple boson creation operators by zero-trace operators in the construction of the polynomial basis in the case of a chain of the type $SU_n \supset SO_n \supset G''$ (cf. the $SU_5 \supset SO_5 \supset SO_3$ in Ref. 137), since these projection operators do not depend on the basis characteristics of the given irreducible representation of the subgroup SO_n and do not change them.

9)In particular, mutually inverse relations are considered in Ref. 139.

values of the parameters of the irreducible representations and ranks of the groups, the position of the obtained results in the theory of special functions, and the possibilities of analytic continuation and generalization of these results for irreducible representations of the principal (continuous) series of the noncompact groups.

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