Parastatistics and internal symmetries

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Possible generalizations of the statistics of identical particles, parastatistics, and the corresponding theories of parafields are reviewed. The connection between parastatistics and cases of degeneracy of particles with respect to an internal degree of freedom is discussed, and also the restrictions that are imposed on internal symmetries if they are given a parafield description. In particular, the difference between the hypotheses of "three-color" quarks and "paraquarks" is discussed.

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INTRODUCTION

Definition: *A generalized statistics of identical particles is defined as a statistics for which the number of particles in a symmetric or antisymmetric state cannot exceed some given integer p, which is called the order of the statistics.* Obviously, if $p = 1$, then this definition covers ordinary Fermi and Bose statistics. But if $p > 1$, then one speaks of *para-Fermi* and *para-Bose* statistics, respectively, or simply *parastatistics*.

The possible existence of statistics of identical particles intermediate between Fermi and Bose statistics was discussed long ago[1,2] (see Refs. 3–5 on the behavior of a "*paragas*"). On this subject, Pauli wrote: "The fact that quantum mechanics yields *more* states than actually occur in nature (and all of them equally possible) is still a puzzle..." Dirac also noted that: "Other more complicated kinds of symmetry are possible mathematically, but do not apply to any known particles" (Ref. 7, p. 211). Such authoritative statements emphasize the importance of the problem of the theoretical admissibility but at the same time actual absence of realization in nature of intermediate (or para) statistics.

Many authors have attempted to prove the impossibility of the existence of parastatistics.[8-19] Careful study of these attempts showed[2,20-23] that usually they are based on a formulation of the principle of indistinguishability which is more narrow than is required by the definition of identity of particles; this is the case in Refs. 9 and 10 and also in the recent papers of Refs. 18 and 19. Alternatively, there may be an incorrect understanding of the cluster law for paraparticles,[13] as was clarified in Ref. 20.

Finally, Haag and collaborators[24] proved rigorously the consistency of parastatistics with the fundamental axioms of quantum theory. They obtained a classification of parastatistics identical to the classification of Hartle, Stolt, and Taylor,[20-22] whose point of departure was the cluster properties of the wave functions of paraparticles. According to this classification, the various parastatistics are above all divided into finite and infinite statistics. To an *infinite* statistics there correspond arbitrary Young diagrams with respect to which the indices of the particles are symmetrized (Fig. 1a). Infinite parastatistics have been little studied, and it has not yet been established whether there exists one such statistics[24] or a set of them.[21] In what follows, we shall not consider them.

Finite parastatistics correspond to Young diagrams with a limited number of columns for para-Fermi statistics or rows for para-Bose statistics (Figs. 1b and 1c).

It is natural to ask this question: Can parastatistics of identical particles (first-quantized theory) be associated with some thoery of a quantized field (second-quantized theory)? It is well known that for ordinary Fermi and Bose statistics the entire information about the permutation properties of the wave functions of the particles is contained in the commutation rules for the particle creation and annihilation operators.

The only attempt at the direct construction of analogous operators for particles described by multidimensional Young representations was made by Okayama.[25] However, an error in his construction was subsequently pointed out.[26] After this, many authors discussed the connection between first- and second-quantized theories of paraparticles (see Refs. 20–22 and 27–31), but they did not have a constructive program. (Note, however, the important result of Refs. 20–22 concerning the one-to-one correspondence of these theories.)

One can, however, avoid the complicated direct construction of a second-quantized theory in accordance with a first-quantized theory and solve this problem indirectly. For this it is necessary to determine the restrictions that the creation and annihilation operators must satisfy and then establish all the schemes of second quantization that satisfy these requirements. This is the approach we shall follow in this review. Fundamental in this will be Bogolyubov's method[32] of direct quantization of the density matrix, extended in Ref. 33 to the general case of parastatistics. It is found that the schemes of *paraquantization* of fields proposed by Green[34] and Volkov[35] in a generalization of field quantization are schemes of this kind.

It was established subsequently that there is an intimate connection between paraquantization and the Lie

FIG. 1. Young diagrams corresponding to parastatistics: a) infinite; b) para-Fermi statistics; c) para-Bose statistics.

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algebras of the classical orthogonal groups $SO(2K)$ and $SO(2K+1)$ and the symplectic group $Sp(2K)$ when the dimension—the number of all possible single-particle states—tends to infinity, $K\to\infty$.[26 36-42] It proved possible recently to develop a quantization scheme corresponding to the Lie algebras of the unitary groups $SU(2K)$ and $SU(2K+1)$.[43,44] Thus, it has proved possible to associate a definite scheme of generalized quantization with every Lie algebra of the classical groups and thus to give these schemes a definite mathematical basis.

Thus, parastatistics and a corresponding theory of parafields are well founded in the sense of consistency with the fundamental postulates of quantum theory.[1)] But then why are there in nature no objects that can be described by such a remarkable theory? The answer is remarkably simple: *In fact, such a theory describes cases of exact degeneracy of particles with respect to an internal coordinate.* Moreover, if in nature one encounters an exact degeneracy of particles with respect to some internal coordinate, it is more logical not to distinguish these particles by an internal index, with which it is impossible to associate any observable, but rather to regard such particles as identical but satisfying a parastatistics whose order is equal to the degeneracy number. It may also be expected that such a description is a first approximation for broken symmetries, when a small perturbation lifts the degeneracy. These conclusions will be illustrated for the examples of parastatistics of second and third order when used to describe internal quantum numbers such as isospin and strangeness.

Further, it is found that the parafield approach to the description of internal symmetries imposes on them very stringent restrictions. Strictly speaking, in such an approach one cannot even speak of internal symmetries (parafields do not have indices); one can only speak of a correspondence between the results of the theory formulated in the language of parafields and the results of the theory of ordinary fields satisfying a definite internal symmetry. It is in these restrictions that the heuristic value of the parafield approach may reside. Particularly stringent restrictions arise in the formulation of the analogs of gauge theories. It is found that *in a parafield theory one can construct a unique copy of the Yang–Mills Lagrangian, which corresponds to the gauge symmetry* $SO(3)$! In this connection, we discuss at the end of the review the difference between the descriptions of quarks as *"three-color"* fermions and as *parafermions* of third order, and also the possibility of extending the paraquantization scheme with a view to including in it the color symmetry $SU(3)$.

1. THEOREM ON THE CONNECTION BETWEEN PARASTATISTICS AND PARAFIELDS

A system of n particles is completely characterized by a density matrix $\rho(k_1,\ldots,k_n;k_1',\ldots,k_n';t)$, a complex function of $2n$ variables $k_1,\ldots,k_n$; $k_1',\ldots,k_n'$ and the time t with real positive-definite diagonal elements. By k_i we here understand a complete set of quantum numbers characterizing the single-particle state, for example, the momentum and spin projection of particle i. For simplicity, we shall assume that the spectrum of these quantum numbers is discrete.

According to Bogolyubov (Ref. 32, p. 303), the principle of indistinguishability of identical particles consists in the requirement of *symmetry of the density matrix* with respect to all permutations $\mathcal{P}$ of the particle indices:

$$\rho(k_{\mathcal{P}_1},\ldots,k_{\mathcal{P}_n};k'_{\mathcal{P}_1},\ldots,k'_{\mathcal{P}_n})=\rho(k_1,\ldots,k_n;k_1',\ldots,k_n') \qquad (1)$$

(here and in all that follows, we omit the argument t). Bogolyubov emphasized the correspondence between this requirement and the symmetry property of the classical distribution function for identical particles and called it the *"classical symmetry"* of the density matrix in contrast to the more specific *"quantum symmetry"* of wave functions or, which is the same thing, symmetry of the density matrix with respect to the *"primary"* $k_1,\ldots,k_n$ and *"secondary"* $k_1',\ldots,k_n'$ arguments separately.

For second quantization, it is necessary to define a space of occupation numbers. However, because of the absence in the general case of any symmetry of the density matrix with respect to the primary arguments alone or with respect to the secondary arguments alone we cannot introduce occupation numbers for each of these sets of arguments separately. But this can be done for the two sets together. One can define the number n_{ij} of particles occupying the given state $k^{(i)}$ among all the primary states $k_1,\ldots,k_n$ and the state $k^{(j)}$ among all the secondary states $k_1',\ldots,k_n'$. We shall call the numbers n_{ij} the occupation numbers in the *"double state* $(k^{(i)},k^{(j)})$*,"* and the space of such numbers the *"space of double occupation numbers."*[33] We can now characterize the state of the system by specifying the density matrix in this space:

$$f\{n_{ij}\},\ \sum_{n_{ii}=1}^{\infty}f\{n_{ii}\}=1. \qquad (2)$$

Here, $\{n_{ij}\}$ means the set of numbers $n_{11},n_{12},\ldots,n_{21}$, $n_{22},\ldots$. The diagonal density matrix $f\{n_{ii}\}$ represents the probability of finding n_{11} particles in the state $k^{(1)}$, n_{22} particles in the state $k^{(2)}$, and so forth. For ordinary Fermi and Bose statistics,

$$f\{n_{ij}\}=\overline{\chi}(N_j')\chi(N_i), \qquad (3)$$

where

$$N_i=\sum_{j=1}^{\infty}n_{ij},\quad N_j'=\sum_{i=1}^{\infty}n_{ij} \qquad (4)$$

and the bar denotes the complex conjugate. (For Fermi statistics, of course, $N_i,N_j',n_{ij}\leqslant 1$.) In the general case, (3) does not hold and the specification of the numbers (4) does not completely determine the states of the complete system.

The space of double occupation numbers is a Hilbert space with the scalar product

$$\langle f\mid g\rangle=\sum_{n_{ij}=1}^{\infty}\overline{f}\{n_{ij}\}g\{n_{ij}\}=\langle\overline{g\mid f}\rangle. \qquad (5)$$

[1)] A paraquantization scheme is also presented in great detail in the recently published book: Y. Ohnuki and S. Kamefuchi, Quantum Field Theory and Parastatistics, University of Tokyo Press, Springer, Berlin (1982).

Note that the vectors in it are themselves density matrices.[45] In it, one can introduce basis vectors with fixed double numbers:

$$|n^0_{ij}\rangle = \prod_{i,\,j=1}^{\infty} \delta_{n_{ij} n^0_{ij}}. \tag{6}$$

One can now define *operators of transition* of a particle from one primary state s to another primary state r without any transitions in the secondary states:

$$N_{rs} \mid n^0_{ij}\rangle = \sum_{q=1}^{\infty} [n^0_{sq}(n^0_{rq} - \delta_{rs} + 1)]^{1/2} \mid n^0_{rq} + 1,\, n^0_{sq} - 1\rangle. \tag{7}$$

The operator $N_{rr} \equiv N_r$ is the operator of the total number of particles in the primary state r:

$$N_r \mid n^0_{ij}\rangle = \Big(\sum_{q=1}^{\infty} n^0_{rq}\Big) \mid n^0_{ij}\rangle. \tag{8}$$

The operator N^+_{rs}, which is the Hermitian-conjugate of the operator N_{rs} in the sense of the scalar product (5), is identical to N_{sr}:

$$N^+_{rs} = N_{sr}. \tag{9}$$

For the commutator of two operators (7) we have

$$[N_{ij},\, N_{rs}] = \delta_{jr} N_{is} - \delta_{is} N_{rj}. \tag{10}$$

In this relation, we can readily recognize the Lie algebra of the generators of the unitary group $SU(K)$ in the space of all possible single-particle states with dimension $K \to \infty$.

Thus, our result is as follows: *In any generalization of quantization compatible with the principle of the indistinguishability of identical particles the necessary conditions for the operators of shifts of the states are* (9) *and* (10). Note that these relations were obtained by Bogolyubov when quantizing the density matrix in Bose and Fermi statistics (Ref. 32, p. 333). Our analysis, based on the introduction of the concept of the space of double occupation numbers, showed that they must be satisfied for any generalization of the statistics of identical particles.[33]

We now turn to the direct construction of operators of creation and annihilation of particles in, for example, the state r (in what follows, all the arguments will apply to the primary states in the density matrix, but they can be repeated in exactly the same way for the secondary states). Here, we must formulate a number of propositions.

PROPOSITION 1. *The transition operator N_{ij} can be represented as a product of two operators,*

$$N_{ij} = a^{-1} [b^+_i,\, b_j]_\varepsilon + c_{ij}, \tag{11}$$

where a, ε, c_{ij} are numbers and the brackets are defined as

$$[b^+_i,\, b_j]_\varepsilon \equiv b^+_i b_j + \varepsilon b_j b^+_i. \tag{12}$$

By virtue of (9), the numbers a and ε must be real and

$$\bar{c}_{ij} = c_{ji}. \tag{13}$$

The coefficient a is arbitrary, and in all that follows we shall set $a = 2$.

PROPOSITION 2. *The operator b_i reduces the number of particles in state i by unity*, i.e., we have the following relation with the particle-number operator:

$$[N_r,\, b_i] = -\delta_{ri} b_i. \tag{14}$$

It also follows from this that b^+_i is a creation operator:

$$[N_r,\, b^+_i]_- = \delta_{ri} b^+_i. \tag{15}$$

PROPOSITION 3. *The nonsingular transformations*

$$b'_r = \sum_j u_{rj} b_j, \quad (b'_r)^+ = \sum_j \bar{u}_{rj} b^+_j, \tag{16}$$

which do not change the relations (9), (10), *and* (14), *are admissible.*

Substitution of (16) in (10) leads to a condition of unitarity of these transformations

$$\sum_m u_{jm} \bar{u}_{im} = \delta_{ij} \tag{17}$$

and a condition for the constants:

$$\sum_{m,\,n} \bar{u}_{im} u_{jn} c_{mn} = c_{ij}. \tag{18}$$

This last condition must be satisfied for arbitrary unitary transformations (17), and therefore its solution will be

$$c_{mn} = c\delta_{mn}, \tag{19}$$

where c is some real constant. Note that invariance of the theory with respect to the unitary transformations (16) was postulated in Ref. 46. We have succeeded in deriving the unitarity condition of these transformations from the original relations (10).

Substitution of (11) in (14) gives

$$[[b^+_r,\, b_r]_\varepsilon,\, b_j]_- = -2\delta_{rj} b_j. \tag{20}$$

Performing on the operators the infinitesimal transformation[46]

$$b'_r = b_r + \sum_j \omega_{rj} b_j, \quad \bar{\omega}_{rj} = -\omega_{jr} \tag{21}$$

and equating the small quantities of first order in ω on both sides of (20), we arrive at the *fundamental trilinear relation* for the creation and annihilation operators:

$$[[b^+_r,\, b_s]_\varepsilon,\, b_j]_- = -2\delta_{rj} b_s. \tag{22}$$

Hermitian conjugation gives

$$[[b^+_r,\, b_s]_\varepsilon,\, b^+_j]_- = 2\delta_{sj} b_r. \tag{23}$$

Using the identity

$$[[A, B]_\varepsilon,\, [C, D]_\eta]_- = [[[A, B]_\varepsilon,\, C]_-,\, D]_\eta + \eta\, [[[A, B]_\varepsilon,\, D]_-,\, C]_{1/\eta}, \tag{24}$$

we can readily verify that (22) and (23) transform (10) into an identity, i.e., they are also sufficient conditions [with allowance for (11) and (19)].

The expression (11) with unfixed number ε had the most general bilinear form. If one sets $\varepsilon = \pm 1$, then the relations (22) and (23) are transformed into the relations postulated by Green.[34] For them the following theorem was proved.

THEOREM 1 (Greenberg and Messiah[47]). *If there exists a unique vacuum state $|0\rangle$ such that*

$$b_r \mid 0\rangle = 0 \quad \text{for all} \quad r, \tag{25}$$

then

$$b_s b^+_r \mid 0\rangle = p\delta_{sr} \mid 0\rangle, \tag{26}$$

and, further, it follows from the condition of positive definiteness of the norm of the state vectors in the Fock *space constructed over the vector* $|0\rangle$ *that the number* p *must be a positive integer. It determines the maximal number of particles in a symmetric* ($\varepsilon = -1$) *or antisymmetric* ($\varepsilon = +1$) *state.* In other words, Greenberg and Messiah proved the *sufficiency* of Green paraquantization for the description of parastatistics of finite order p. The question of the *necessity* of such quantization was studied in Ref. 48, in which the following theorem was proved.

THEOREM 2. *If* $\varepsilon \neq 0$, *then from the condition of positive definiteness of the norm of the state vectors in the Fock space and the requirement that the number of particles in either a symmetric or an antisymmetric state cannot exceed a given number* $p \geqslant 2$ (the condition of parastatistics) *it follows that* $\varepsilon = -1$ *and* $\varepsilon = +1$, respectively.

Thus, if $\varepsilon \neq 0$, then Green paraquantization is also necessary for the description of parastatistics, and this establishes a one-to-one connection between parastatistics and Green parafields.

Still uninvestigated is the case $\varepsilon = 0$ that we rejected. For it, the relations (22) and (23) are insufficient (for example, for the calculation of the norm of state vectors in the Fock space) and must be augmented by some further relations. For this reason, it is still unclear what type of statistics this case corresponds to. One can only conjecture that it corresponds to the infinite statistics mentioned in the Introduction, but this conjecture has not been proved at all. For this reason, the assertion of the necessity of paraquantization cannot be regarded as definitively proved and must be taken with reservation for the exception $\varepsilon = 0$.

In the Fock space constructed over $|0\rangle$ by the action on it of all possible polynomials of creation operators the relations

$$[[b_r, b_s]_\varepsilon, b_j]_- = 0, \quad [[b_r^+, b_s^+]_\varepsilon, b_j^+]_- = 0 \tag{27}$$

for $\varepsilon = \pm 1$ are automatically satisfied. (Here, it is appropriate to recall that for ordinary, for example, Fermi statistics the relation $[b_r, b_s]_+ = 0$ in the Fock space is a consequence of the relations

$$[b_r, b_s^+]_+ = \delta_{rs}.$$

Other relations can be obtained from (22), (23), and (27) by using the generalized Jacobi identity

$$\alpha [[A, B]_\varepsilon, C]_\eta = -\varepsilon\eta [[A, C]_{-\alpha/\varepsilon}, B]_{-\alpha/\eta} + \alpha^2 [[B, C]_{\varepsilon\eta/\alpha}, A]_{1/\alpha}, \tag{28}$$

where α, η, and ε are arbitrary nonvanishing numbers. Thus, setting $\alpha = \eta = -1$, we obtain

$$[[b_r, b_s]_\varepsilon, b_j^+]_- = 2\varepsilon\delta_{rj} b_s + 2\delta_{js} b_r. \tag{29}$$

It is convenient to write all the Green relations in a unified *symbolic* form. We denote the creation operators by placing the state symbol as a superscript:

$$b_r^+ \equiv b^r. \tag{30}$$

We define the tensors

$$g_{hl} = g^{hl} = 0, \quad g_h^l = -\varepsilon g^l_h = \delta_{hl}. \tag{31}$$

Then all of Green's relations can be written in the form

$$[[b_\rho, b_\sigma]_\varepsilon, b_\tau]_- = 2\varepsilon g_{\rho\tau} b_\sigma + 2 g_{\sigma\tau} b_\rho, \tag{32}$$

it being assumed that any of the indices ρ, σ, τ can be either a subscript or a superscript. The tensors (31) have the property

$$g_{\rho\sigma} = -\varepsilon g_{\sigma\rho}. \tag{33}$$

It is readily verified that the ordinary relations

$$[b_\rho, b_\sigma]_{-\varepsilon} = g_{\rho\sigma} \tag{34}$$

transform Green's relations (32) into an identity, i.e., are a solution of them. However, this is not a unique solution of Green's relations, and this emphasizes once more that Fermi and Bose statistics are not the only statistics allowed by the principle of indistinguishability of identical particles.

2. RELATIONS FOR PARASTATISTICS OF A GIVEN ORDER

For parastatistics of a given order p additional commutation relations must be satisfied, these following from the restrictions imposed on the spectrum of the particle-number operator in the given state:

$$N_r = \frac{1}{2}[b_r^+, b_r]_\varepsilon - \frac{\varepsilon p}{2}, \quad \varepsilon = \pm 1. \tag{35}$$

Bracken and Green[40] proposed a consistent method for obtaining all such relations. In it, one considers a sequence of tensors, symmetric for para-Fermi statistics ($\varepsilon = -1$) and antisymmetric for para-Bose statistics ($\varepsilon = +1$):

$$h \equiv 1, 2, \quad h_\rho \equiv b_\rho;$$

$$h_{\rho\sigma} = [h_\rho, b_\sigma]_{-\varepsilon} - 2h g_{\rho\sigma}; \tag{36a}$$

$$h_{\rho\sigma\tau} = [h_{\rho\sigma}, b_\tau]_+ - 2(h_\rho g_{\sigma\tau} - \varepsilon h_\sigma g_{\rho\tau} - h_\tau g_{\rho\sigma}); \tag{36b}$$

$$\begin{aligned} h_{\rho\sigma\tau\nu} &= [h_{\rho\sigma\tau}, b_\nu]_{-\varepsilon} \\ &- 2(h_{\rho\sigma} g_{\tau\nu} - \varepsilon h_{\rho\tau} g_{\sigma\nu} - h_{\sigma\tau} g_{\rho\nu} + h_{\rho\nu} g_{\sigma\tau} - \varepsilon h_{\sigma\nu} g_{\rho\tau} + h_{\tau\nu} g_{\rho\sigma}) \\ &+ 4h(g_{\rho\sigma} g_{\tau\nu} - \varepsilon g_{\rho\tau} g_{\sigma\nu} + g_{\rho\nu} g_{\sigma\tau}) \end{aligned} \tag{36c}$$

etc. The indicated ($-\varepsilon$) symmetry of these tensors for all indices apart from the last two follows from their sequential construction. The proof for the last two indices is by induction, invoking the Green relations (32). For parastatistics of order p the tensor of order $p+1$ must vanish. Thus, the vanishing for $p = 1$ of the expression (36a) leads to the ordinary relations (34). For $p = 2$ with allowance for (32) we obtain

$$\frac{1}{2} h_{\rho\sigma\tau} = b_\rho b_\sigma b_\tau - \varepsilon b_\tau b_\sigma b_\rho - 2g_{\rho\sigma} b_\tau - 2g_{\sigma\tau} b_\rho = 0. \tag{37}$$

Quantization with such commutation relations was considered for the first time by Green[34] and Volkov.[35] In Refs. 35 and 49 an analogy between these relations for $\varepsilon = -1$ and the Kemmer–Duffin algebra was noted. For $p = 3$,

$$\begin{aligned} \frac{1}{2} h_{\rho\sigma\tau\nu} &= [b_\rho b_\sigma b_\tau - \varepsilon b_\tau b_\sigma b_\rho, b_\nu]_{-\varepsilon} - 3g_{\rho\sigma}[b_\tau, b_\nu]_{-\varepsilon} + \varepsilon g_{\rho\tau}[b_\sigma, b_\nu]_{-\varepsilon} \\ &- g_{\rho\nu}[b_\sigma, b_\tau]_{-\varepsilon} - 3g_{\sigma\tau}[b_\rho, b_\nu]_{-\varepsilon} + \varepsilon g_{\sigma\nu}[b_\rho, b_\tau]_{-\varepsilon} - g_{\tau\nu}[b_\rho, b_\sigma]_{-\varepsilon} \\ &+ 3(g_{\rho\sigma} g_{\tau\nu} - \varepsilon g_{\rho\tau} g_{\sigma\nu} + g_{\rho\nu} g_{\sigma\tau}) = 0. \end{aligned} \tag{38}$$

These relations were first derived by Kamefuchi and Takahashi[26] and Scharfstein.[50]

The relations (37) transform the general Green relations (32) into identities. Therefore, they are not only necessary but also sufficient, and in the construction of parastatistics of second order one can proceed directly from them. However, already for $p = 3$ the quadrilinear relations (38) are necessary but not sufficient. Indeed, one can verify that they satisfy the ordinary relations (34), and therefore they describe both the $p = 3$ and $p = 1$ cases. To distinguish these cases, it is necessary and sufficient to set $p = 3$ in the condition (26).[26] But there is then no need to use (38), since the general relations (32) are sufficient for construction of a theory with the fixed condition (26). We note that this is sufficient even for the construction of ordinary statistics when $p = 1$.[51]

3. LIE-ALGEBRA APPROACH TO FIELD QUANTIZATION. UNIQUANTIZATION

Many authors[26,36–42] have noted the intimate connection between the Green relations (32) and the Lie algebras of the orthogonal and symplectic groups. Indeed, for the bilinear combinations

$$A_{\rho\sigma}=\frac{1}{2}[b_\rho, b_\sigma]_\varepsilon, \quad A_{\rho\sigma}=\varepsilon A_{\sigma\rho} \tag{39}$$

the commutator is

$$[A_{\rho\sigma}, A_{\tau\nu}]_-=\varepsilon g_{\rho\tau}A_{\sigma\nu}+g_{\sigma\tau}A_{\rho\nu}+g_{\rho\nu}A_{\sigma\tau}+\varepsilon g_{\sigma\nu}A_{\rho\tau}. \tag{40}$$

Here, we have used the identity (24) for $\varepsilon = \eta = \pm1$ and (32). In the algebra (40), it is not difficult to recognize the Lie algebra of the orthogonal group $SO(2K)$ for $\varepsilon = -1$ and of the symplectic group $Sp(2K)$ for $\varepsilon = +1$ in a four-dimensional space of indices when there are first K subscripts and then K superscripts and $K \to \infty$.[40]

Instead of (39), one can write out in detail the three combinations[26]

$$N_{ij}=\frac{1}{2}[b_i^\dagger, b_j]_\varepsilon=N_{ji}^\dagger; \tag{41a}$$

$$M_{ij}=\frac{1}{2}[b_i, b_j]_\varepsilon; \tag{41b}$$

$$L_{ij}=M_{ji}^\dagger=\frac{1}{2}[b_i^\dagger, b_j^\dagger]_\varepsilon \tag{41c}$$

and the algebra (40) in the form

$$[N_{ij}, N_{rs}]_- = \delta_{jr}N_{is} - \delta_{is}N_{rj}; \tag{42a}$$

$$[M_{ij}, N_{rs}]_- = \delta_{jr}M_{is} + \varepsilon\delta_{ir}M_{js}; \tag{42b}$$

$$[L_{ij}, N_{rs}]_- = -\delta_{js}L_{ir} - \varepsilon\delta_{is}L_{jr}; \tag{42c}$$

$$[L_{ij}, M_{rs}]_- = -\varepsilon\delta_{jr}N_{is} - \delta_{js}N_{ir} - \delta_{ir}N_{js} - \varepsilon\delta_{is}N_{jr}; \tag{42d}$$

$$[L_{ij}, L_{rs}]_- = [M_{ij}, M_{rs}]_- = 0. \tag{42e}$$

The algebra (42a) for the operators N_{ij} is the Lie algebra of the unitary group $SU(K)$ and in accordance with (10) is a consequence of the principle of indistinguishability of identical particles for systems with a constant number of such particles. To consider systems with a variable number of particles, we needed to introduce operators of creation and annihilation of particles, and then the algebra of bilinear operators (41) was extended to the Lie algebras $SO(2K)$ and $Sp(2K)$. Thus, the origins of the Lie-algebra approach to field quantization are to be found in the very principle of indistinguishability of identical particles.[33]

It is natural to consider the possibility of a quantization scheme related to the Lie algebra of the unitary group $SU(2K)$. Such quantization was proposed in Refs. 43 and 44. Below, we briefly present the scheme considered in Ref. 44 and called "uniquantization." Uniquantization can be constructed in such a way that either para-Fermi or para-Bose quantization is a subalgebra of it. In addition to the operators b_ρ satisfying these subalgebras (32), we introduce a new set of operators c_ρ (c_r or $c^r \equiv c_r^+$) satisfying the same relations

$$[[c_\rho, c_\sigma]_\varepsilon, c_\tau]_-=2\varepsilon g_{\rho\tau}c_\sigma+2g_{\sigma\tau}c_\rho \tag{43}$$

and the following relations with b_ρ:

$$[[b_\rho, c_\sigma]_\varepsilon, b_\tau]_-=-4g_{\rho\sigma}c_\tau+2\varepsilon g_{\rho\tau}c_\sigma-2g_{\sigma\tau}c_\rho; \tag{44a}$$

$$[[c_\rho, b_\sigma]_\varepsilon, c_\tau]_-=-4g_{\rho\sigma}b_\tau+2\varepsilon g_{\rho\tau}b_\sigma-2g_{\sigma\tau}b_\rho; \tag{44b}$$

$$[b_\rho, c_\sigma]_\varepsilon=-[c_\rho, b_\sigma]_\varepsilon; \tag{44c}$$

$$[b_\rho, b_\sigma]_\varepsilon=[c_\rho, c_\sigma]_\varepsilon. \tag{44d}$$

Note that by virtue of (44d) the operators (39) or (41) can with equal success be expressed in terms of either the b or the c operators. Further, we introduce the bilinear combinations

$$\widetilde{A}_{\rho\sigma}=\frac{1}{2}[b_\rho, c_\sigma]_\varepsilon=-\varepsilon\widetilde{A}_{\sigma\rho}. \tag{45}$$

For them we have

$$[\widetilde{A}_{\rho\sigma}, \widetilde{A}_{\tau\nu}]_-=\varepsilon g_{\rho\tau}A_{\sigma\nu}-g_{\sigma\tau}A_{\rho\nu}-g_{\rho\nu}A_{\sigma\tau}+\varepsilon g_{\sigma\nu}A_{\rho\tau}; \tag{46a}$$

$$[\widetilde{A}_{\rho\sigma}, A_{\tau\nu}]_-=-\varepsilon g_{\rho\tau}\widetilde{A}_{\sigma\nu}+g_{\sigma\tau}\widetilde{A}_{\rho\nu}-g_{\rho\nu}\widetilde{A}_{\sigma\tau}+\varepsilon g_{\sigma\nu}\widetilde{A}_{\rho\tau}. \tag{46b}$$

These relations complete the subalgebra (40) to the complete Lie algebra of $SU(2K)$. It is interesting that in this algebra there is besides the particle-number operator

$$N=\sum_s\left(\frac{1}{2}[b_s^+, b_s]_\varepsilon-\frac{\varepsilon p}{2}\right) \tag{47}$$

the operator

$$\widetilde{N}=\frac{i}{2(2K+1)}\sum_{s=1}^{K}([b_s^+, c_s]_\varepsilon+\text{const}), \tag{48}$$

which commutes with it and has the properties

$$\left.\begin{aligned}&[b_\rho+ic_\rho, \widetilde{N}]_-=b_\rho+ic_\rho;\\ &[b_\rho-ic_\rho, \widetilde{N}]_-=-b_\rho+ic_\rho.\end{aligned}\right\} \tag{49}$$

The states of particles in such a theory will be characterized not only by their number but also by a certain "charge" $\widetilde{N}$.

Note also that the relations (44) cannot be satisfied by the ordinary relations (34) for the b and c operators. They can be satisfied by the Green–Volkov relations (37) for the b operators, and the c operators can be expressed in the form

$$c_\rho=\pm i(-1)^N b_\rho. \tag{50}$$

In this case, the operator $\widetilde{N}$ in (48) can be expressed solely in terms of b operators:

$$\widetilde{N}=\frac{\pm(-1)^N}{2(2K+1)}\sum_{s=1}^{K}([b_s^+, b_s]_{-\varepsilon}+\text{const}). \tag{51}$$

One can also show directly that when (37) is satisfied the commutator and anticommutator of b_s^+ and b_s commute with one another.

Finally, we note the following. In the case of para-Fermi statistics $\varepsilon = -1$, and the operators b_ρ themselves can be included in the algebra (40) if (39) is augmented by the operators

$$A_{0\lambda}=-A_{\lambda 0}=\frac{1}{\sqrt{2}}b_\lambda, \quad A_{00}=0. \tag{52}$$

Then the algebra (40) is extended to the Lie algebra of $SO(2K+1)$.[36] Similarly, the Lie algebra (46) of $SU(2K)$

is extended to $SU(2K+1)$ when it is augmented by the operators

$$\widetilde{A}_{0\sigma}=\widetilde{A}_{\sigma 0}=\frac{1}{\sqrt{2}}c_{\sigma}. \tag{53}$$

At the same time, the diagonal operator $\tilde{A}_{00}$ satisfies

$$\widetilde{A}_{00}\equiv\frac{1}{\sqrt{2}}c_0=-\sum_{\mu=1}^{2K}\widetilde{A}_{\mu\mu}, \tag{54}$$

which guarantees the condition for the group to be special.

Thus, with the Lie algebras of all the classical groups we can associate a definite scheme of generalized quantization:

$$\left.\begin{matrix}SO(2K)\\ SO(2K+1)\end{matrix}\right\}\rightarrow\text{para-Fermi quantization},\ \varepsilon=-1;$$

Note how para-Bose quantization is distinguished. In this scheme, the creation and annihilation operators themselves cannot be included in the Lie algebra of the group $Sp(2K)$ as can be done for the other schemes by going over to a space of an odd number of dimensions. However, it can also be done for para-Bose quantization if one goes over to Lie superalgebras. In this case, the para-Bose algebra is isomorphic to the simple classical orthosymplectic superalgebra $osp(1, 2K)$.[52]

4. PARA-FERMI STATISTICS AND THE THEORY OF HIGHER SPINS

Thus, in the case of para-Fermi statistics we have the Lie algebra $SO(2K+1)$. It can be rewritten in terms of rotation operators. For this, we go over to the self-adjoint operators[36]

$$\beta_{2j-1}=\frac{1}{2}(b_j-b_j^{\dagger}); \tag{55a}$$

$$\beta_{2j}=\frac{i}{2}(b_j-b_j^{\dagger}),\ j=1,\ \dots,\ K. \tag{55b}$$

They satisfy the relations

$$[[\beta_{\mu},\beta_{\nu}]_-,\beta_{\varkappa}]_-=-\delta_{\varkappa\mu}\beta_{\nu}+\delta_{\varkappa\nu}\beta_{\mu}. \tag{56}$$

The indices μ, ν, $\varkappa$ now range over the even set of values $1, 2, \dots, 2K\rightarrow\infty$. We define the operators

$$J_{\lambda 0}=-J_{0\lambda}=\beta_{\lambda},\ \beta_0\equiv J_{00}=0; \tag{57}$$

$$J_{\mu\nu}=i\,[\beta_{\mu},\beta_{\nu}]_-. \tag{58}$$

Using the identity (24) for $\varepsilon=\eta=-1$ and (56)–(58), we obtain the algebra

$$[J_{\mu\nu},J_{\varkappa\lambda}]_-=-i\delta_{\mu\varkappa}J_{\nu\lambda}-i\delta_{\nu\lambda}J_{\mu\varkappa}+i\delta_{\mu\lambda}J_{\nu\varkappa}+i\delta_{\nu\varkappa}J_{\mu\lambda}. \tag{59}$$

The indices μ, ν, $\varkappa$, λ now range over the odd set of values $0, 1, 2, \dots, 2K\rightarrow\infty$. The algebra (59) is the algebra of the self-adjoint generators $J_{\mu\nu}$ of the rotations of the group $SO(2K+1)$. In this sense, para-Fermi quantization is equivalent to the Rao–Bhabha theory of higher spins.

Exactly the same construction can be made for uniquantization corresponding to the group $SU(2K+1)$.[44] The operators

$$\zeta_{2j-1}=\frac{1}{2}(c_j+c_j^{\dagger}); \tag{60a}$$

$$\zeta_{2j}=\frac{i}{2}(c_j-c_j^{\dagger}), \tag{60b}$$

like the β operators, satisfy the relations

$$[[\zeta_{\mu},\zeta_{\nu}]_-,\zeta_{\varkappa}]_-=-\delta_{\varkappa\mu}\zeta_{\nu}+\delta_{\varkappa\nu}\zeta_{\mu} \tag{61}$$

and have with the β operators the relations

$$[[\zeta_{\mu},\beta_{\nu}]_-,\zeta_{\varkappa}]_-=-2\delta_{\mu\nu}\beta_{\varkappa}-\delta_{\varkappa\nu}\beta_{\mu}-\delta_{\varkappa\mu}\beta_{\nu}; \tag{62a}$$

$$[[\beta_{\mu},\zeta_{\nu}]_-,\beta_{\varkappa}]_-=-2\delta_{\mu\nu}\zeta_{\varkappa}-\delta_{\varkappa\nu}\zeta_{\mu}-\delta_{\varkappa\mu}\zeta_{\nu}; \tag{62b}$$

$$[\beta_{\mu},\beta_{\nu}]_-=[\zeta_{\mu},\zeta_{\nu}]_-; \tag{62c}$$

$$[\zeta_{\mu},\beta_{\nu}]_-=[\zeta_{\nu},\beta_{\mu}]_-. \tag{62d}$$

Further, we introduce besides (57) and (58) the operators

$$\widetilde{J}_{0\lambda}=\widetilde{J}_{\lambda 0}=\zeta_{\lambda}; \tag{63}$$

$$\widetilde{J}_{\mu\nu}=i\,[\zeta_{\mu},\beta_{\nu}]_-+\delta_{\mu\nu}\zeta_0=\widetilde{J}_{\nu\mu}, \tag{64}$$

where

$$\widetilde{J}_{00}\equiv\zeta_0=-\frac{i}{2K+1}\sum_{\lambda=1}^{2K}[\zeta_{\lambda},\beta_{\lambda}]. \tag{65}$$

The condition

$$\sum_{\mu=1}^{2K}\widetilde{J}_{\mu\mu}+\widetilde{J}_{00}=0 \tag{66}$$

for being special is guaranteed by (65). Note that the relations (61) and (62) are satisfied automatically for ζ_0 because they are for the other ζ_{λ}. Therefore, in them the indices μ, ν, $\varkappa$ can be assumed to take the values $0, 1, 2, \dots, 2K$. Besides (59), we now obtain for $J_{\mu\nu}$ and $\tilde{J}_{\varkappa\lambda}$ the relations

$$[\widetilde{J}_{\mu\nu},\widetilde{J}_{\varkappa\lambda}]_-=-i\delta_{\nu\varkappa}J_{\mu\lambda}-i\delta_{\mu\lambda}J_{\nu\varkappa}-i\delta_{\mu\varkappa}J_{\nu\lambda}-i\delta_{\nu\lambda}J_{\mu\varkappa}; \tag{67a}$$

$$[J_{\mu\nu},\widetilde{J}_{\varkappa\lambda}]_-=i\delta_{\nu\varkappa}\widetilde{J}_{\mu\lambda}-i\delta_{\mu\lambda}\widetilde{J}_{\nu\varkappa}-i\delta_{\mu\varkappa}\widetilde{J}_{\nu\lambda}+i\delta_{\nu\lambda}\widetilde{J}_{\mu\varkappa}. \tag{67b}$$

The relations (59) and (67) are the algebra of self-adjoint generators of the group $SU(2K+1)$. It is interesting that the operators N and $\tilde{N}$ defined by (47) and (48) can be expressed as sums of nondiagonal and diagonal generators:

$$N=\sum_{j=1}^{K}J_{2j2j-1}+\text{const}; \tag{68}$$

$$\widetilde{N}=\frac{1}{2}\zeta_0+\text{const}=-\frac{1}{2}\sum_{\mu=1}^{2K}\widetilde{J}_{\mu\mu}+\text{const}. \tag{69}$$

5. GREEN'S ANSATZ AND INTERNAL SYMMETRIES

Green[34] found for his relations (32) a simple solution, which later became known as "Green's ansatz." It is as follows. One considers a set of p ordinary Fermi or Bose operators having, however, anomalous[53] mutual commutation relations:

$$[a_{\rho}^{A},a_{\sigma}^{B}]_{\varepsilon_{AB}}=\delta_{AB}g_{\rho\sigma},\ \ \varepsilon_{AB}=\varepsilon(1-2\delta_{AB}), \tag{70}$$

where $A, B=1, \dots, p$. Here, we again use the symbolic notation in which the indices ρ and σ may denote an operator of annihilation a_r^A or creation $a_r^{A+}=a^{Ar}$, and the tensor $g_{\rho\sigma}$ is determined by (31). It is now readily verified that the sum

$$b_{\rho}=\sum_{A=1}^{p}a_{\rho}^{A} \tag{71}$$

satisfies Green's relations (32). Note that this solution is defined up to arbitrary phase factors in front of the components. The sums

$$b_r=\sum_{A=1}^{p}\exp(i\varphi_A)\,a_r^A,\ \ b_r^{+}=\sum_{A=1}^{p}\exp(-i\varphi_A)\,a_r^{A+} \tag{72}$$

are also solutions of Green's relations. It is also readily verified that

$$\sum_{\mathscr{P}}(-1)^{\eta(\mathscr{P})}b^{+}_{\mathscr{P}r_1},\ \ldots,\ b^{+}_{\mathscr{P}r_n}=0\ \text{for}\ n>p. \qquad (73)$$

where the sum is taken over all permutations $\mathscr{P}$ of the states $r_1,\ldots,r_n$ and $\eta(\ )$ are their parities. Thus, Green's ansatz (71) satisfies all the requirements of the Greenberg–Messiah theorem (see Sec. 2), and therefore any other representation of the paraoperators satisfying Green's relations and these requirements will be equivalent to it. Therefore, all the representations (72) are equivalent to it. It is merely to be noted that both the Greenberg–Messiah theorem itself and this conclusion that all representations of paraoperators are equivalent to Green's ansatz have been proved only for free parafields (see Refs. 13, 47, and 54, p. 368).

It would seem that the expression of the paraoperators in terms of a system of ordinary Fermi or Bose operators means that paraquantization is equivalent to ordinary quantization in the presence of a certain additional internal degree of freedom that takes p values. In a certain sense this is indeed so, but one must clearly understand the restrictions that the hidden nature of their parafield representation imposes on the internal symmetries.

First of all, it must be pointed out that in Green's ansatz (71) the indices of its components are *unobservable*, since they always occur in a symmetric manner, and using paraoperators it is impossible to form any operator that could act on them differently. All that one can expect is that they are associated with some exact symmetry, for which it is no longer meaningful to distinguish internal states. The discovery of such an internal symmetry in paraquantization will be the subject of the further exposition. Green's ansatz will play the part of a mathematically convenient, but by no means obligatory, formalism.

Discussing the connection between paraquantization and degeneracy with respect to an internal coordinate, Greenberg and Messiah[47] wrote[2]: "Some readers may suspect that the equation for the b_k in terms of the a_k^A may introduce (a) some composite structure or (b) degeneracy into the description of the *individual* particles annihilated by the b_k (or created by the b_k^+). That (a) does not occur should be clear since Green's ansatz is linear, while in contrast a compound structure would require a multiplicative relation. Neither does (b) occur since the state $b_k^+|0\rangle$ is nondegenerate, in contrast to a particle with a hidden degree of freedom. We emphasize that Green's ansatz is only a mathematical device and that the a_k^A and a_k^{A+} by themselves have no physical significance." It is not possible to dissent from the last assertion. However, the claims (a) and (b) are not entirely accurate. In Refs. 55 and 56 there are examples for which paraquantization arises because of the complicated structure of objects, as, for example, for pairs of nucleons in nuclear shells, and in this case the index A acquires the clear physical meaning of the projection of the angular momentum of a nucleon. With regard to (b), it can also be noted that the presence of a hidden degree of freedom is expressed, not in the degeneracy of single-particle states (in the case of degeneracy they are indistinguishable), but in the degeneracy of *many-particle* states, which is precisely what happens in paraquantization. Moreover, in the framework of paraquantization one can construct operators that have different eigenvalues in these states and thus lift the degeneracy. In this way, one can seek a correspondence between paraquantization and "horizontal symmetries" of the type of isospin and $SU(3)$.[41, 57–60]

6. PARAFIELDS

For the investigation of the algebraic structure of paraquantization it was convenient to use a discrete space of particle states. We now turn to the field formulation in space and time $x(x_0, \mathbf{x})$. We consider free fields of any spin, but we shall write all expressions as if we were dealing with Dirac fields $\psi(x)$ and $\overline{\psi}(x)=\psi^{+}(x)\gamma_0$. The free fields can be expanded with respect to negative- and positive-frequency solutions:

$$\psi(x)=\sum_{k}\{b_{k-}\varphi_k^{(-)}(x)+b_{k+}\varphi_k^{(+)}(x)\}. \qquad (74)$$

The signs ∓ indicate whether the operator $b_{k\pm}$ corresponds to particles or antiparticles, and in what follows we include them in the state index k. From Green's relations (32) there follow the relations for the parafield:

$$[[\psi(x),\psi(y)]_{\varepsilon},\psi(z)]_{-}=2\varepsilon\{\psi(x),\psi(z)\}_{-\varepsilon}\psi(y)-2\{\psi(y),\psi(z)\}_{-\varepsilon}\psi(x). \qquad (75)$$

By the field ψ one can here understand the field itself as well as its adjoint $\overline{\psi}$, and the curly brackets on the right-hand side of (75), the so-called Volkov symbols,[35] have $(-\varepsilon)$ symmetry and the values

$$\{\psi(x),\overline{\psi}(y)\}_{-\varepsilon}=(-\varepsilon)\{\overline{\psi}(y)\ \psi(x)\}_{-\varepsilon}=-\mathrm{i}S(x-y); \qquad (76a)$$

$$\{\psi(x),\psi(y)\}_{-\varepsilon}=\{\overline{\psi}(x),\overline{\psi}(y)\}_{-\varepsilon}=0, \qquad (76b)$$

where $S(x)$ is the singular commutation function corresponding to the given field. The parafield relations for parastatistics of given order can be formulated similarly. In particular, for parastatistics of second order we have in accordance with (37)

$$\psi(x)\psi(y)\psi(z)-\varepsilon\psi(z)\psi(y)\psi(x)=2\{\psi(x),\psi(y)\}_{-\varepsilon}\psi(z)+2\{\psi(y),\psi(z)\}_{-\varepsilon}\psi(x). \qquad (77)$$

Green[34] postulated the relations (75) and (77). Subsequently, they were derived by a number of authors (see Refs. 35, 50, and 61–63) on the basis of Schwinger's variational principle with generalization of the commutation properties of the field variation. In an axiomatic formulation, they should be replaced by equal-time relations for Heisenberg fields.[13]

Green's ansatz[34 47] makes it possible to express a parafield in terms of ordinary fields with anomalous mutual commutation relations:

$$\psi(x)=\sum_{A=1}^{p}\psi^{A}(x); \qquad (78)$$

$$\left.\begin{aligned}&[\psi^{A}(x),\overline{\psi}^{B}(y)]_{\varepsilon_{AB}}=-\mathrm{i}\delta_{AB}S(x-y);\\&[\psi^{A}(x),\psi^{B}(y)]_{\varepsilon_{AB}}=[\overline{\psi}^{A}(x),\overline{\psi}^{B}(y)]_{\varepsilon_{AB}}=0,\end{aligned}\right\} \qquad (79)$$

where $\varepsilon_{AB}=\varepsilon(1-2\delta_{AB})$. Note that the ansatz (78) is de-

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[2] We have changed the notation for the operators used in Ref. 47 to the notation that we follow in the present review: $a\rightleftarrows b$.

fined up to a phase transformation of each of its components. For free fields it follows from the Greenberg–Messiah theorem that all other representations of the parafields are equivalent to Green's ansatz. In the case of interacting fields, this has not been proved.

Green's ansatz is convenient for investigating the local properties of the interactions of parafields,[47,64] but it is not necessary for consistent construction of a theory of a parafield. McCarthy and Volkov[3,35] demonstrated the possibility of calculating S-matrix elements using the paracommutation relations (77) themselves.

All the foregoing treatment given in the discrete representation can be reproduced in the continuous x space. In the unipole theory of Ref. 44, one classical field is associated with two quantum parafields formed from b and c operators, respectively.

7. THEOREM ON THE CONNECTION BETWEEN PARAFIELDS AND A DEGENERATE SET OF ORDINARY FIELDS

Suppose we are given a set of different fields

$$\phi_1(x),\ \dots,\ \phi_m(x),$$

some of which may be ordinary Fermi or Bose fields but among which there are parafields satisfying the equal-time Green relations (75). The fields may have diverse natures, i.e., the scalar, spinor, vector, etc., and, in addition, for brevity we shall not distinguish the fields and the adjoint fields $\phi_i^*(x)$. The *normal* case is characterized by all *Bose* fields commuting with all fields, all *Fermi* fields commuting with all Bose-like (Bose and para-Bose) fields and anticommuting with all Fermi-like (Fermi and para-Fermi) fields, all *para-Bose* fields having para-Bose relations with the para-Bose and para-Fermi fields of given order and commuting with all the remaining fields, and all *para-Fermi* fields having para-Fermi relations with the para-Fermi fields and para-Bose relations with the para-Bose fields of given order and commuting with all Bose-like fields and anticommuting with all Fermi-like fields of other orders.[47] It can be shown that this case contains minimal restrictions, whereas in other, anomalous cases additional restrictions arise—superselection rules.[53]

Our main assumption will be that of the possibility of representing any parafield by Green's ansatz (78) and of its equivalence to all other representations of the parafield.[65] As we have pointed out, this has not been proved for Heisenberg parafields, and we adopt it only as a plausible hypothesis. Further, we shall use the formalism of Wightman functions of parafields, which can now be expressed in terms of the Wightman functions of the components of Green's ansatz:

$$\langle\phi_1(x_1)\dots\phi_m(x_m)\rangle_0=\sum_{\substack{A_1=1,\dots,p_1\\ A_m=1,\dots,p_m}}\langle\phi_1^{A_1}(x_1)\dots\phi_m^{Am}(x_m)\rangle_0. \tag{80}$$

Of these we require the fulfillment of the ordinary and axiomatic formulation of Lorentz invariance and the spectral condition (see, for example, Ref. 66, p. 682 of the Russian translation). Then the cluster properties hold and Araki's theorem can be used[53]: If the products of the field operators C_{I} and C_{II} are such that C_{I} and $L(\lambda a,1)C_{\mathrm{II}}L(\lambda a,1)^{-1}$ anticommute for a Lorentz displacement along a spacelike vector a for sufficiently large λ, then

$$\langle C_I\rangle_0\ \langle C_{II}\rangle_0=0. \tag{81}$$

In the normal case, there arise just two restrictions, which are necessary and sufficient for the Wightman function of the components of Green's ansatz to be nonzero: 1) the number of fermionlike fields in it must be even (this requirement also follows from Lorentz invariance); 2) the numbers of components of the parafields of given order must have the same parity.

To derive the first rule, it is necessary to consider the commutation of two identical field products in regions far from each other. If the number of Fermi-like fields in them is odd, they anticommute and in accordance with (81) the Wightman functions of them vanish.

To derive the second rule, it is necessary to consider the commutation of two products, the one obtained from the other by interchanging any two Green components, $A\rightleftharpoons B$, referring to parafields of the given order. If the parities of the numbers of these components are different, then the products anticommute, and we again have (81), and therefore one of the Wightman functions vanishes. But since in the theory of a parafield all the components enter symmetrically, the other Wightman function must also vanish (otherwise it will distinguish the A and B components).[3]

According to the second rule, one can decompose the complete space of the vectors

$$\phi_1^{A_1}(x_1)\dots\phi_m^{Am}(x_m)|0\rangle \tag{82}$$

into two orthogonal subspaces containing even and odd numbers of the fields A and B. One can then define a Lorentz-invariant Klein operator $\hat{q}(A,B)$, which takes the value +1 on the first of them and −1 on the second. Obviously, it anticommutes with $\phi^A(x)$ and $\phi^B(x)$, but commutes with all the other fields. In addition, it has the properties

$$\hat{q}^2(A,B)=1,\ \hat{q}^+(A,\ B)=\hat{q}(A,B). \tag{83}$$

If we now make a Klein transformation[67] and define the new fields

$$\begin{gathered}\Phi^A(x)=\hat{q}(A,B)\phi^A(x),\quad \Phi^B(x)=\phi^B(x),\\ \Phi^C(x)=\hat{q}(A,B)\phi^C(x),\ \text{etc.},\end{gathered} \tag{84}$$

then for the fields $\boldsymbol{\Phi}^A$ and Φ^B the mutual commutation relations become normal both between them and between them and the remaining components. Further, one can decompose into such pairs the remaining components and make the same transformation for them. For example, for fourth order

$$\begin{gathered}\Phi^A(x)=\hat{q}(A,B)\,\phi^A(x),\ \Phi^B(x)=\phi^B(x),\\ \Phi^C(x)=q(C,D)\hat{q}(A,B)\,\phi^C(x),\ \Phi^D(x)=\hat{q}(A,B)\,\phi^D(x).\end{gathered} \tag{85}$$

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[3] One could think that in accordance with the second rule for the scalar field $\varphi(x)$ we shall have $\langle\varphi^A(x)\rangle_0=0$ and, therefore, $\langle\varphi(x)\rangle_0=0$. In fact, one can have $\langle\varphi(x)\rangle_0=\text{const}\neq0$, since the cluster properties do not apply for a constant value. It is also obvious that this deviation does not influence the following arguments, which apply to many-point Wightman functions.

In the same way one can reduce to ordinary fields any set of Green components.

The second rule strongly limits the class of nonvanishing Wightman functions (80) for parafields. Only two structures are nonzero: these can be expressed in terms of the Green components and, accordingly, normal fields in the form

$$\sum_{A=1}^{p} \langle \dots \phi_i^A(x)\, \phi_j^A(y) \dots \rangle_0 = \sum_{A=1}^{p} \langle \dots \Phi_i^A(x)\, \Phi_j^A(y) \dots \rangle_0; \tag{86}$$

$$\sum_{A_1 \neq \dots \neq A_p = 1}^{p} \langle \dots \phi_1^{A_1}(x_1) \dots \phi_p^{A_p}(x_p) \dots \rangle_0$$
$$= \sum_{A, \dots, A_p = 1}^{p} \varepsilon_{A_1 \dots A_p} \langle \dots \Phi_1^{A_1}(x_1) \dots \Phi_p^{A_p}(x_p) \dots \rangle_0, \tag{87}$$

where $\varepsilon_{A_1 \dots A_p}$ is the completely antisymmetric tensor and these equations are defined up to an overall sign. These structures can be expressed in terms of parafields in the form of corresponding sequences of commutators and anticommutators. The restrictions completely correspond to the analogous requirements imposed on the interaction Hamiltonian of parafields by the locality condition.[47] We needed to use the Wightman formalism only for the consistent introduction of the Klein operator.

It is readily seen that the right-hand sides of (86) and (87) have $SO(p)$ symmetry. If, however, (86) always contains products of Hermitian-conjugate fields $\phi_i^{A*}(x)\phi_j^A(y)$, and (87), conversely, contains either only the fields $\phi_i^A(x)$ or only the fields $\phi_i^{A*}(x)$, then these functions also have $SU(p)$ symmetry. Thus, the following theorem holds.

THEOREM. *In the case of fulfillment of the condition* (80), *Lorentz invariance, and the spectral condition any theory of parafields is equivalent to the theory of p-fold degenerate sets of ordinary fields satisfying $SO(p)$ symmetry in the general case and $SU(p)$ symmetry for a restricted choice of the admissible Wightman functions.* This theorem was proved in Ref. 68, and then, in a somewhat different formulation, in Refs. 69 and 70 (see also the review of Ref. 71).

8. IRREDUCIBLE REPRESENTATIONS OF PARAOPERATORS IN THE SPACE OF GREEN'S ANSATZ

The Fock space $\mathscr{A}$ of Green's ansatz is obtained by applying to the vacuum vector $|0\rangle$ all possible polynomials in the operators $a_k^A, k = 1, \dots, K \to \infty, A = 1, \dots, p$. It is reducible with respect to the algebra of the operators b_k and b_k^+ defined by the sums (71). We now describe the procedure for separating irreducible representations of these operators from the large space $\mathscr{A}$.[41, 51, 59]

In the space $\mathscr{A}$ we seek vectors

$$|k_1, \dots, k_f\rangle = \sum_{A_1, \dots, A_f = 1}^{p} y_{A_1 \dots A_f} a_{k_1}^{A_1 +} \dots a_{k_f}^{A_f +} |0\rangle, \tag{88}$$

which satisfy the condition

$$b_r \mid k_1, \dots, k_f \rangle = 0 \text{ for any } r. \tag{89}$$

We shall call these the *lowest* vectors. They include the vacuum vector $|0\rangle$ itself. The condition (89) imposes on the coefficients in (88) the restrictions

$$\left.\begin{array}{l} \sum\limits_{A_1=1}^{p} y_{A_1 A_2 \dots A_f} = 0; \\ \sum\limits_{A_2=1}^{p} \varepsilon_{A_1 A_2} y_{A_1 A_2 \dots A_f} = 0; \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \sum\limits_{A_f}^{p} \varepsilon_{A_1 A_f} \dots \varepsilon_{A_{f-1} A_f} y_{A_1 A_2 \dots A_f} = 0, \end{array}\right\} \tag{90}$$

where $\varepsilon_{AB} = \varepsilon(1 - 2\delta_{AB})$. From these equations we deduce the following properties of the lowest vectors:

$$\begin{array}{c} b_r b_s^+ \mid k_1, \dots, k_f \rangle = p\delta_{rs} \mid k_1, \dots, k_f \rangle \\ + 2\varepsilon\delta_{rk_1} |s, k_2, \dots, k_f\rangle + \dots + 2\varepsilon\delta_{rk_f} \mid k_1, \dots, k_{f-1}, s\rangle, \end{array} \tag{91}$$

and a vector symmetric for para-Fermi statistics ($\varepsilon = -1$) and antisymmetric for para-Bose statistics ($\varepsilon = +1$) with respect to all states $r_1, \dots, r_n, k_1, \dots, k_f$,

$$\sum_{\mathscr{P}} (-\varepsilon)^{\eta(\mathscr{P})} b_{\mathscr{P} r_1}^+ \dots b_{\mathscr{P} r_n}^+ | \mathscr{P} k_1, \dots, \mathscr{P} k_f \rangle = 0, \tag{92a}$$

if

$$2f + n \geqslant p + 1. \tag{92b}$$

The last condition, in particular, means that for $p = 2$ and $p = 3$ the lowest vectors ($n = 0$) must possess ε symmetry (antisymmetric for para-Fermi statistics and symmetric for para-Bose). In the general case, the symmetry of the lowest vectors is higher than the symmetry of the parastatistics itself: On Fermi (or Bose) symmetric lowest vectors one can construct para-Fermi (or para-Bose) statistics of second and third orders; on lowest vectors possessing the symmetry of parastatistics of second and third orders one can construct parastatistics of fourth, fifth, and sixth, seventh orders, respectively, etc. In this sense, one can speak of a *series* of parastatistics.[51]

Note that the conditions (89), (91), and (92) completely determine the properties of the lowest vectors. One could postulate these or equivalent conditions without invoking Green's ansatz.[51]

The basis vectors of every irreducible representation can be obtained by applying polynomials in b_r^+ to the lowest vectors with a fixed number of arguments f. Thus, the Fock representation is constructed on the vacuum vector. Fock-like representations are constructed on the remaining lowest vectors. When the lowest vectors (88) are separated from $\mathscr{A}$, several solutions of the system (90) may be obtained. Then we shall have several equivalent irreducible representations determined by the same conditions for the lowest vectors.

Let us consider the physical meaning of the lowest vectors. Although they satisfy (89), they correspond not to degenerate vacuum states but to states with the number f of particles in the states $k_1, \dots, k_f$. Applying to them the particle-number operator (47), we obtain by virtue of (89) and (91)

$$N \mid k_1, \dots, k_f \rangle = f \mid k_1, \dots, k_f \rangle. \tag{93}$$

Thus, we have no fundamental arguments for rejecting all irreducible representations of a para-algebra of given order except the Fock representation.

From the mathematical point of view, Green's ansatz is anaologous in the case of para-Fermi statistics to representation of the operators of higher spins in the form of sums of mutually commuting Dirac γ matrices. The described procedure for constructing irreducible representations of a para-algebra corresponds to the addition of half spins to a definite spin. It is known that in this way one can construct any higher spin, but, however, this does not mean that it must always be regarded as a composite.

Representations of paraoperators directly in matrix form were obtained for para-Bose operators in Ref. 39 and for para-Fermi operators in Ref. 42.

9. ISOSPIN AND IRREDUCIBLE REPRESENTATIONS OF PARASTATISTICS OF SECOND ORDER

As a first example, we consider the parastatistics of second order defined by the relations (37). The paraoperator can be represented as a sum of two commuting (anticommuting) fermion (boson) operators:

$$b_\rho = a_\rho^1 + a_\rho^2. \tag{94}$$

We introduce the auxiliary operator [it also satisfies (37)]

$$\tilde{b}_\rho = a_\rho^1 - a_\rho^2. \tag{95}$$

For b_ρ and $\tilde{b}_\rho$ we have the bilinear relations

$$\left.\begin{array}{l} \tilde{b}_\rho b_\sigma = \varepsilon \tilde{b}_\sigma b_\rho, \; b_\rho \tilde{b}_\sigma = \varepsilon b_\sigma \tilde{b}_\rho; \\ b_\rho b_\sigma = \varepsilon \tilde{b}_\sigma \tilde{b}_\rho - 2g_{\rho\sigma} \end{array}\right\} \tag{96}$$

The lowest vectors have a simple construction[57]:

$$|k_1, \ldots, k_j\rangle = \tilde{b}_{k_1}^+ b_{k_2}^+ \tilde{b}_{k_3}^+ b_{k_4}^+ \ldots |0\rangle. \tag{97}$$

Using (96), we can readily verify that they satisfy the relations

$$b_r |k_1, \ldots, k_j\rangle = 0; \tag{98}$$

$$b_r b_s^+ |k_1, \ldots, k_j\rangle = 2\delta_{rs} |k_1, \ldots, k_j + \\ + 2\varepsilon\delta_{rk_1} |s, k_2, \ldots, k_j\rangle + \ldots + 2\varepsilon\delta_{rk_j} |k_1, \ldots, k_{j-1}, s\rangle; \tag{99}$$

$$|\ldots, k_i, \ldots, k_j, \ldots\rangle = \varepsilon |\ldots, k_j, \ldots, k_i, \ldots\rangle; \tag{100}$$

$$b_r^+ |k_1, \ldots, k_j\rangle = \varepsilon b_{r_1}^+ |r, k_2, \ldots, k_j\rangle. \tag{101}$$

We construct the operators

$$N = \sum_k \frac{1}{2} ([b_k^+, b_k]_\varepsilon - 2\varepsilon); \tag{102}$$

$$I_1 = \sqrt{-\varepsilon} \sum_k \frac{1}{4} [\tilde{b}_k^+, b_k]_-; \tag{103}$$

$$I_2 = \sqrt{-\varepsilon} \sum_k \frac{i}{4} [\tilde{b}_k^+, b_k]_+; \tag{104}$$

$$I_3 = \sum_k \frac{1}{4} ([b_k^+, b_k]_{-\varepsilon} + 2\varepsilon); \tag{105}$$

$$I^2 = I_1^2 + I_2^2 + I_3^2 = \sum_{k,l} \frac{1}{4} b_k b_l^+ b_k^+ b_l + I_3 (I_3 - 1). \tag{106}$$

By means of (96), we can show that the operators I_i ($i = 1, 2, 3$) satisfy the *isospin* algebra

$$[I_i, I_j]_- = i\varepsilon_{ijh} I_h. \tag{107}$$

and the operators N and I^2 commute with them and with one another. Every state of the paraparticles can now be characterized by three quantum numbers: $|N, I, I_3\rangle$. Moreover, the eigenstates I are obtained by symmetrization with respect to the Young diagrams. The normalized states are given in Table I. Each column in this table is an irreducible representation of the algebra of the paraoperators, whose action on it is determined by (37) and the conditions (98)–(101). Obviously, this action does not take one out of the given column, and transitions between states belonging to different columns cannot be realized by means of these operators. The change in the states of the lowest vectors of a given irreducible representation is realized by means of (99). For example,

TABLE I. States $|N, I, I_3\rangle$ of paraparticles satisfying parastatistics of *second order*.

Irreducible representations of the algebra of paraoperators ↓

$\mathscr{B}_0$	$\mathscr{B}_1$	$\mathscr{B}_2$
$\|0, 0, 0\rangle = \|0\rangle$	–	–
$\|1, 1/2, +1/2\rangle = \frac{1}{\sqrt{2}} b_{r_1}^+ \|0\rangle$	$\|1, 1/2, -1/2\rangle =$ $= \frac{1}{\sqrt{2}} \|r_1\rangle$	–
$\|2, 0, 0\rangle =$ $= \frac{(b_{r_1}^+ b_{r_2}^+ - \varepsilon b_{r_2}^+ b_{r_1}^+) \|0\rangle}{2\sqrt{2} (1 - \varepsilon\delta_{r_1 r_2})^{1/2}}$	–	–
$\|2, 1, 0 =$ $= \frac{(b_{r_1}^+ b_{r_2}^+ + \varepsilon b_{r_2}^+ b_{r_1}^+) \|0\rangle}{2\sqrt{2} (1 + \varepsilon\delta_{r_1 r_2})^{1/2}}$	$\|2, 1, +1\rangle =$ $= \frac{b_{r_1}^+ \|r_2\rangle}{2 (1 + \varepsilon\delta_{r_1 r_2})^{1/2}}$	$\|2, 1, -1\rangle =$ $= \frac{\|r_1, r_2\rangle}{2 (1 + \varepsilon\delta_{r_1 r_2})^{1/2}}$

→ Irreducible representations of isospin

$$b_{k_1} b_r^+ |k_1\rangle = 2\varepsilon |r\rangle \quad \text{for} \quad r \neq k_1.$$

It can be shown that in this way we have decomposed the complete space of Green's ansatz into irreducible parts[57]:

$$\mathscr{A} = \mathscr{A}_1 \otimes \mathscr{A}_2 = \bigoplus_{f \in \{0, 1, 2, \ldots\}} \mathscr{B}_f, \tag{108}$$

where $\mathscr{A}_1$ and $\mathscr{A}_2$ are the Fock spaces of the representations of the a^1 and a^2 operators, and $\mathscr{B}_f$ are the irreducible representations of the algebra of paraoperators constructed on the basis of the lowest vectors with number f of indices. In the given case, equivalent representations do not arise.

We note that in each column of an irreducible representation there are only two values of I_3:

$$I_3 = \begin{cases} -f/2 & \text{for even} \quad n, \\ (f+1)/2 & \text{for odd} \quad n, \end{cases} \tag{109}$$

where n is the number of operators b_r^+ acting on the lowest vector with number f of indices. Thus, degenerate states with the same number of particles forming a given isomultiplet I belong to different irreducible representations of the algebra of paraoperators.

It must be emphasized that the operators N, I, I_3 themselves can be formed in accordance with (102), (105), and (106) only from the paraoperators b themselves. To form the operators I_1 and I_2, it is necessary to introduce the auxiliary operators $\tilde{b}$. Thus, it may be concluded that the classification of the states of paraparticles with respect to N, I, I_3 can be done in the framework of the algebra of paraoperators itself, whereas the formulation of the isospin algebra and the consideration of transitions within isomultiplets require us to go beyond this algebra.

The successful classification of the states of paraparticles with respect to the isospin makes one wonder whether they could be directly associated with the states of ordinary fermions or bosons of two species. Such a transformation can be achieved by means of the *σ-matrix representation of paraoperators*.[57]

Suppose p_r, p_r^+, n_r, n_r^+ are operators satisfying ordinary, including normal, mutual commutation relations:

$$\left.\begin{array}{l}[p_r,\, p_s^+]_{-\varepsilon}=\delta_{rs},\ [p_r,\, p_s]_{-\varepsilon}=[p_r^+,\, p_s^+]_{-\varepsilon}=0;\\ [n_r,\, n_s^+]_{-\varepsilon}=\delta_{rs},\ [n_r,\, n_s]_{-\varepsilon}=[n_r^+,\, n_s^+]_{-\varepsilon}=0;\\ [p_r,\, n_s^+]_{-\varepsilon}=[p_r^+,\, n_s]_{-\varepsilon}=[p_r,\, n_s]_{-\varepsilon}=[p_r^+,\, n_s^+]_{-\varepsilon}=0.\end{array}\right\} \tag{110}$$

Paraoperators can be expressed as sums of direct products of these operators with Pauli matrices,

$$\left.\begin{array}{l}b_r=\sqrt{2}\,(\sigma_-\otimes p_r+\sigma_+\otimes n_r);\\ b_r^+=\sqrt{2}\,(\sigma_+\otimes p_r^++\sigma_-\otimes n_r^+),\end{array}\right\} \tag{111}$$

and the auxiliary operators

$$\left.\begin{array}{l}\tilde{b}_r=\sqrt{2}\,(\sigma_+\otimes p_r+\sigma_-\otimes n_r);\\ \tilde{b}_r^+=\sqrt{2}\,(\sigma_-\otimes p_r^++\sigma_+\otimes n_r^+).\end{array}\right\} \tag{112}$$

where

$$\sigma_+=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad \sigma_-=\sigma_+^+=\begin{pmatrix}0&0\\1&0\end{pmatrix}. \tag{113}$$

It is readily verified that (111) [like (112)] satisfy the Green–Volkov relations (37), and together with (112) the algebra (96).

The vacuum vector also has a direct-product structure:

$$\Phi_0=\xi_0\otimes|0\rangle, \tag{114}$$

where $|0\rangle$ is the vacuum vector for the p and n operators, and ξ_0 is chosen in the form

$$\xi_0=\begin{pmatrix}0\\1\end{pmatrix}, \tag{115}$$

so that

$$\sigma_-\xi_0=0,\ \ \sigma_+\xi_0=\begin{pmatrix}1\\0\end{pmatrix},\ \sigma_-\sigma_+\xi_0=\xi_0. \tag{116}$$

The remaining lowest vectors (97) now have the form

$$\Phi_{k_1\ldots k_f}=2^{f/2}\begin{cases}p_{k_1}^+\ldots p_{k_f}^+\Phi_0 & \text{for even}\quad f,\\ n_{k_1}^+\ldots n_{k_f}^+\sigma_+\Phi_0 & \text{for odd}\quad f.\end{cases} \tag{117}$$

Any vector of paraparticles can now be expressed in terms of the vectors for p and n particles:

$$\begin{array}{l}\dfrac{1}{2^{n+f}}\,b_{r_1}^+\ldots b_{r_{2l}}^+\begin{cases}\Phi^{2q}_{k_1\ldots k_{2q}}\\ \Phi^{2q+1}_{k_1\ldots k_{2q+1}}\end{cases}\\ =\begin{array}{l}n_{r_1}^+p_{r_2}^+\ldots n_{r_{2l-1}}^+p_{r_{2l}}^+p_{k_1}^+\ldots p_{k_{2q}}^+\Phi_0;\\ p_{r_1}^+n_{r_2}^+\ldots p_{r_{2l-1}}^+n_{r_{2l}}^+n_{k_1}^+\ldots n_{k_{2q+1}}^+\sigma_+\Phi_0;\end{array}\\ \dfrac{1}{2^{n+f}}\,b_{r_1}^+\ldots b_{r_{2l+1}}^+\begin{cases}\Phi^{2q}_{k_1\ldots k_{2q}}\\ \Phi^{2q+1}_{k_1\ldots k_{2q+1}}\end{cases}\\ =\begin{array}{l}p_{r_1}^+n_{r_2}^+p_{r_3}^+\ldots n_{r_{2l}}^+p_{r_{2l+1}}^+p_{k_1}^+\ldots p_{k_{2q}}^+\sigma_+\Phi_0;\\ n_{r_1}^+p_{r_2}^+n_{r_3}^+\ldots p_{r_{2l}}^+n_{r_{2l+1}}^+n_{k_1}^+\ldots n_{k_{2q+1}}^+\Phi_0,\end{array}\end{array} \tag{118}$$

where n is the number of operators b^+, and f is the number of indices of the lowest vector.

The operators (102)–(105) are transformed to

$$N=\sum_k\mathcal{N}_k^+\tau_0\mathcal{N}_k\otimes 1; \tag{119a}$$

$$I_1=\sum_k(\mathcal{N}_k^+\tau_{(3+\varepsilon)/2}\mathcal{N}_k)\otimes\left(\frac{1-\varepsilon}{2}+\frac{1+\varepsilon}{2}\sigma_3\right); \tag{119b}$$

$$I_2=(-\varepsilon)\sum_k(\mathcal{N}_k^+\tau_{(3-\varepsilon)/2}\mathcal{N}_k)\otimes\left(\frac{1+\varepsilon}{2}+\frac{1-\varepsilon}{2}\sigma_3\right); \tag{119c}$$

$$I_3=\sum_k(\mathcal{N}_k^+\tau_3\mathcal{N}_k)\otimes\sigma_3, \tag{119d}$$

where $\mathcal{N}_k$ is the column

$$\mathcal{N}_k=\begin{bmatrix}p_k\\n_k\end{bmatrix}, \tag{120}$$

on which there act the isospin matrices

$$\tau_0=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \ \tau_1=\begin{pmatrix}0&1/2\\1/2&0\end{pmatrix},\ \ \tau_2=\begin{pmatrix}0&-i/2\\i/2&0\end{pmatrix},\ \ \tau_3=\begin{pmatrix}1/2&0\\0&-1/2\end{pmatrix}.$$

The presence in I_3 of the matrix σ_3 indicates that the p and n particles have isospin $\frac{1}{2}$ and $-\frac{1}{2}$ alternately, depending on whether the total number of particles is even or odd, since

$$\sigma_3\xi_0=-\xi_0,\quad \sigma_3\,(\sigma_+\xi_0)=\sigma_+\xi_0. \tag{121}$$

It is obvious that such renotation does not play any role.

The new interpretation explains naturally why some of the state vectors vanish for $\varepsilon=-1$, which was previously regarded as a paradox of paraquantization.[27,28] Indeed, for $\varepsilon=-1$ we obtain from (37)

$$b_r^+b_s^+b_t^+=-b_t^+b_s^+b_r^+$$

and the vector

$$\frac{1}{2\sqrt{2}}\,b_r^+b_s^+b_t^+|0\rangle=p_r^+n_s^+p_t^+\sigma_+\Phi_0$$

vanishes for $r=t$. But this is necessary to ensure that two p fermions are not in the same state.

We now go over to the x space with the aim of determining the symmetry type of the wave functions on the class of which Green–Volkov quantization is defined. We do this concretely for parafermions satisfying the Dirac equation. The general state vector has the form[58]

$$\begin{array}{l}|\Psi\rangle=\Psi^{(0)}|0\rangle+\ldots+\int d\sigma^{\mu_1}(x_1)\ldots\\ \ldots\int d\sigma^{\mu_N}(x_N)\left\{\sum\limits_{n=0}^{N}\mathscr{C}(n,\,f)\,(\overline{\psi^{(+)}}(x_1)\,\gamma_{\mu_1})\ldots\right.\\ \ldots(\overline{\psi^{(+)}}(x_n)\,\gamma_{\mu_n})|x_{n+1},\,\ldots,\,x_{n+f}\rangle\,\gamma_{\mu_{n+1}}\ldots\gamma_{\mu_{n+f}}\\ \left.\times\Psi^{(n,\,f)}(x_1,\,\ldots,\,x_n;\,x_{n+1},\,\ldots,\,x_{n+f})\right\}+\ldots,\end{array} \tag{122}$$

where $N=n+f$, and the integrations are performed over spacelike surfaces σ (Ref. 66, p. 225 of the Russian translation). The normalization factors are

$$\mathscr{C}(2l,\,N-2l)=\mathscr{C}(2l+1,\,N-2l-1)=[2^N(N-l)!\,l!]^{-1/2}. \tag{123}$$

The lowest vectors have a form analogous to (97),

$$|x_1,\,\ldots,\,x_f\rangle=\overline{\tilde{\psi}^{(+)}}(x_1)\,\overline{\psi^{(+)}}(x_2)\,\overline{\tilde{\psi}^{(+)}}(x_3)\ldots|0\rangle, \tag{124}$$

and the operators are determined by Green's ansatz:

$$\psi(x)=\psi^1(x)+\psi^2(x); \tag{125a}$$

$$\tilde{\psi}(x)=\psi^1(x)-\psi^2(x). \tag{125b}$$

The amplitudes of the vector (122) can be written in the form

$$\begin{array}{l}\Psi^{(2l,\,f)}\begin{pmatrix}x_1&&x_3&\ldots&x_{2l-1}&&&\\&x_2&&x_4&&x_{2l};&x_{2l+1}&\ldots\ x_{2l+f}\end{pmatrix}\\ =[2^{2l+f}(l-f)!\,l!]^{-1/2}\langle x_{2l+f},\,\ldots,\,x_{2l+1}|\psi^{(+)}(x_{2l})\ldots\psi^{(+)}(x_1)|\Psi\rangle;\end{array} \tag{126a}$$

$$\begin{array}{l}\Psi^{(2l+1,\,f)}\begin{pmatrix}x_1&&x_3\ \ldots\ x_{2l+1};&x_{2l+2}\ \ldots\ x_{2l+1+f}\\&x_2&x_{2l}&\end{pmatrix}\\ =[2^{2l+f+1}(l-f+1)!\,l!]^{-1/2}\langle x_{2l+1+f},\,\ldots,x_{2l+2}|\psi^{(+)}(x_{2l+1})\ldots\psi^{(-)}(x_1)|\Psi\rangle.\end{array} \tag{126b}$$

These amplitudes are antisymmetric with respect to the arguments in one row, but they have no symmetry with respect to arguments in different rows. (For parabosons, antisymmetry is to be replaced by symmetry.) The equal-time amplitudes ($x_{10}=x_{20}=\ldots$) determine the probability of finding particles with coordinates $\mathbf{x}_1,\ldots,\mathbf{x}_N$. The symmetry of the wave functions (126) shows that they describe two species of fermions, whose arguments are in the even and odd positions[49] and are continued as the arguments of the lowest vectors after the semicolon.[58] By means of the σ transformation

$$\left.\begin{aligned}\psi(x)&=\sqrt{2}\{\sigma_-\otimes p(x)+\sigma_+\otimes n(x)\};\\ \overline{\psi}(x)&=\sqrt{2}\{\sigma_+\otimes\overline{p}(x)+\sigma_-\otimes\overline{n}(x)\};\end{aligned}\right\}\tag{127a}$$

$$\left.\begin{aligned}\tilde{\psi}(x)&=\sqrt{2}\{\sigma_+\otimes p(x)+\sigma_-\otimes n(x);\\ \overline{\tilde{\tilde{\psi}}}(x)&=\sqrt{2}\{\sigma_-\otimes\overline{p}(x)+\sigma_+\otimes\overline{n}(x)\}\end{aligned}\right\}\tag{127b}$$

one can realize explicitly such a transition to ordinary fermion fields $p(x)$ and $n(x)$. Below, we shall call them "*nucleons*." We emphasize, however, that the wave functions (127a) describe identical particles, these arising as the excitation quanta of the single parafield $\psi(x)$ but in different even–odd internal states. Our σ transformation shows merely what theory of fermions of two species is equivalent to the original theory of identical parafermions.

We now consider the possibility of formulating different interactions in the theory of a parafermion field of second order. In contrast to an ordinary theory, there are two independent currents in the given theory:

$$j_\mu(x)=\frac{1}{2}[\overline{\psi}(x),\ \gamma_\mu\psi(x)]_-=\overline{p}(x)\gamma_\mu p(x)+\overline{n}(x)\gamma_\mu n(x);\tag{128}$$

$$j'_\mu(x)=\frac{1}{2}[\overline{\psi}(x),\ \gamma_\mu\psi(x)]_+=\{\overline{p}(x)\gamma_\mu p(x)-\overline{n}(x)\gamma_\mu n(x)\}\otimes\sigma_3.\tag{129}$$

Here and in all that follows we understand in all such field products the normal product in the sense of subtraction of the vacuum expectation values:

$$j_\mu(x)\rightarrow j_\mu(x)-\langle j_\mu(x)\rangle_0\tag{130}$$

etc.

Both currents (128) and (129) are local, viz., at points x and y separated by a spacelike interval all the currents $j_\mu(x)$, $j_\mu(y)$, $j'_\mu(x)$, $j'_\mu(y)$ commute with one another [which is a consequence of the relations (77)].

The interaction including the current (128) will be the same for all states of the paraparticles, and it may be called "strong." On its basis, one could create a kind of composite Fermi–Yang model for mesons. The "electromagnetic interaction" of the parafield with the electromagnetic field $\mathscr{A}_\mu(x)$ contains a combination of the currents (128) and (129):

$$\mathscr{H}_{\mathrm{em}}(x)=\mathscr{A}^\mu(x)\,j_\mu^{\mathrm{em}}(x);\tag{131}$$

$$\begin{aligned}j_\mu^{\mathrm{em}}(x)&=\frac{e}{2}\,\overline{\psi}(x)\gamma_\mu\psi(x)\\ &=e\left\{\overline{p}(x)\gamma_\mu p(x)\otimes\frac{1+\sigma_3}{2}+\overline{n}(x)\gamma_\mu n(x)\otimes\frac{1-\sigma_3}{2}\right\}.\end{aligned}\tag{132}$$

The factors $(1\pm\sigma_3)/2$ appear because of the formal renotation of the charged particles: For odd total number, the p particles will be charged; for even, the n particles.

A "weak" interaction can be introduced by assuming the existence of one further, "lepton" para-Fermi field $\lambda(x)$ of second order, having with the "nucleon" field $\psi(x)$ the Green–Volkov mutual paracommutation relations (77) (of course, the mutual Volkov symbols for them vanish). In the σ representation, it has the same form as the original field:

$$\left.\begin{aligned}\lambda(x&=\sqrt{2}\{\sigma_-\otimes\nu(x)+\sigma_+\otimes e(x)\};\\ \overline{\lambda}(x)&=\sqrt{2}\{\sigma_+\otimes\overline{\nu}(x)+\sigma_-\otimes\overline{e}(x)\}.\end{aligned}\right\}\tag{133}$$

At the same time, all the fields p, n, ν, and e anticommute with one another. A local weak interaction can now be written in the form (the γ matrices, which are now unimportant, are omitted)

$$\begin{aligned}\mathscr{H}_{\mathrm{w}}(x)&=\frac{1}{4}\{\overline{\psi}(x)\overline{\lambda}(x)\psi(x)\lambda(x)+\overline{\lambda}(x)\overline{\psi}(x)\lambda(x)\psi(x)\}\\ &=\overline{p}(x)\overline{e}(x)n(x)\nu(x)+\text{h.c.},\end{aligned}\tag{134}$$

Thus, in the theory of the parafermion field of second order we have succeeded not only in classifying all states of the parafermions with respect to the isospin but also in formulating local "strong, electromagnetic, and weak" interactions. Moreover, one need consider only a single Fock representation, since in a sufficiently large system of particles one can always separate a subsystem with any values of I and I_3.

10. STRANGENESS AND PARA-FERMI STATISTICS OF THIRD ORDER

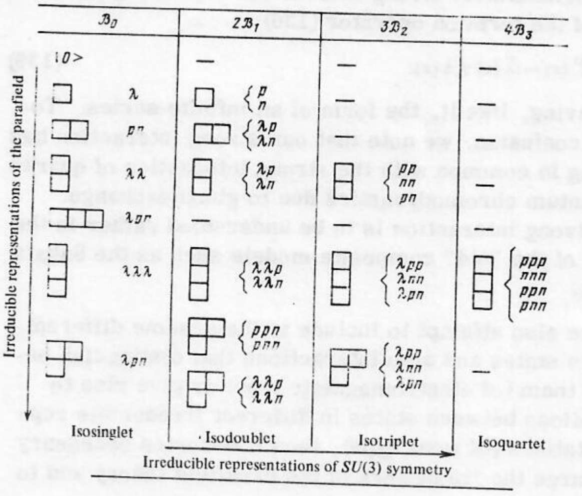
For the case $\varepsilon=-1$ and $p=3$ the decomposition of the large space $\mathscr{A}$ of Green's ansatz into irreducible representations of the algebra of the paraoperators leads to the appearance of *equivalent* representations:

$$\mathscr{A}=\mathscr{A}_1\otimes\mathscr{A}_2\otimes\mathscr{A}_3=\bigoplus_{f\in\{0,1,2,\ldots\}}(f+1)\mathscr{B}_f,\tag{135}$$

where the factor $f+1$ indicates the number of them.

Using the symmetry properties of the state vectors, including the lowest vectors, we obtain the symmetrized states of paraparticles represented by the Young diagrams in Table II.[59] In associating these states with the states of ordinary fermions of three species—"*quarks*" p, n, λ—we adopt the following classification. We shall assume that states belonging to different irreducible representations are states with different

TABLE II. Irreducible representations of para-Fermi statistics of *third order* and quark content of the states of the paraparticles.



"strangeness" quantum number. We shall interpret states of equivalent irreducible representations as different states of one isomultiplet with the same strangeness. The states of one irreducible representation can also be distinguished by means of strangeness if the vectors corresponding to them have different symmetries. In Table II, the quark content is indicated to the right of the corresponding state. We see that the adopted classification is identical to the classification of the physical $SU(3)$ and its isospin subgroup.

In a parafield theory proper without Green's ansatz there is no point in considering equivalent representations. In it, every state in a given irreducible representation of the parafield corresponds to an entire isomultiplet of ordinary fermions, viz., states in the Fock space $\mathscr{B}_0$ correspond to *isosinglets*, states in $\mathscr{B}_1$ to *isodoublets*, etc. In other words, we have a classification of the states with respect to the factor group $SU(3)/SU(2)$.

It can be shown that from a parafermion field one can form *one fermion* operator corresponding to the strange quark. However, it has the form of an infinite series[59]:

$$\begin{aligned}\lambda^+(\mathrm{x}) = \frac{1}{\sqrt{3}}\psi^+(\mathrm{x}) + \int d^3\mathrm{y}\,\{a_1\psi^+(\mathrm{x})\,\psi^+(\mathrm{y})\,\psi(\mathrm{y})\\ + a_2\psi^+(\mathrm{y})\,\psi^+(\mathrm{x})\,\psi(\mathrm{y}) + a_3\psi^+(\mathrm{x})\,(:\psi(\mathrm{y})\,\psi^+(\mathrm{y}):)\\ + a_4\psi^+(\mathrm{y})\,(:\psi(\mathrm{y})\,\psi^+(\mathrm{x}):)\} + \ldots,\end{aligned} \tag{136}$$

where

$$\left.\begin{aligned} a_1 = \frac{1}{4} - \frac{1}{2\sqrt{3}} + \frac{1}{12\sqrt{5}}, \qquad a_2 = -\frac{1}{12} - \frac{1}{12\sqrt{5}},\\ a_3 = -\frac{1}{4} + \frac{1}{2\sqrt{3}} - \frac{1}{4\sqrt{5}}, \quad a_4 = -\frac{1}{4} + \frac{1}{4\sqrt{5}}.\end{aligned}\right\} \tag{137}$$

(For the single-level problem it was shown in Ref. 56 that for any odd order one fermion operator can always be constructed from parafermion operators, whereas for any even order this cannot be done.)

Every state in an irreducible representation in Table II can now be characterized by two quantum numbers: the total number of particles N or the "baryon charge" $B = N/3$ and the "hypercharge"

$$Y = B - \int d^3x\lambda^+(x)\,\lambda(x). \tag{138}$$

In the framework of the theory of the parafield proper one can formulate only a *strong interaction*, which includes a commutator current of the type (128), and also an *intermediately strong* interaction containing the current of the fermion operator (136)

$$j_\mu^{(\lambda)}(x) = :\overline{\lambda}(x)\,\gamma_\mu\lambda(x): \tag{139}$$

and having, like it, the form of an infinite series. To avoid confusion, we note that our strong interaction has nothing in common with the strong interaction of quarks in quantum chromodynamics due to gluon exchange. Our strong interaction is to be understood rather in the spirit of the "old" composite models such as the Sakata model.

If we also attempt to include in the scheme different isospin states and also interactions that distinguish between them (of electromagnetic type) or give rise to transitions between states in different irreducible representations (of weak type), then it becomes necessary to enlarge the framework of the parafield theory and to regard the large space of Green's ansatz itself as a physical space, preserving, however, its parafield structure in the form of the decomposition into irreducible representations of the parafield.[41, 59] In this case, it is necessary to consider the theory of not one but *three* parafields

$$\left.\begin{aligned}\psi(x) = \psi^{(1)}(x) + \psi^{(2)}(x) + \psi^{(3)}(x);\\ \psi'(x) = k\psi^{(1)}(x) + \overline{k}\psi^{(2)}(x) + \psi^{(3)}(x);\\ \psi''(x) = \overline{k}\psi^{(1)}(x) + k\psi^{(2)}(x) + \psi^{(3)}(x),\end{aligned}\right\} \tag{140}$$

where $k = \exp(2\pi i/3)$, $\overline{k} = \exp(-2\pi i/3)$.

It is possible that the new interpretation of the internal symmetries based on the generalized algebra of Green's ansatz may help in establishing their physical essence. In this connection, it is appropriate to recall the analogy with the subspaces in the theory of the addition of higher spins. The addition of three $\frac{1}{2}$ spins leads to two equivalent representations with spin $\frac{1}{2}$ and one representation with spin $3/2$. In the considered space $\mathscr{A}(135)$ they correspond to two states p and n from equivalent representations $\mathscr{B}_1$ and a state λ from $\mathscr{B}_0$.

In Refs. 41 and 72, definite results on the classification of states for any order were obtained.

11. PARASTATISTICS AND GAUGE SYMMETRIES

In Secs. 9 and 10, we considered the symmetry of states of paraparticles belonging to different irreducible representations. We now consider the symmetry hidden in a parafield itself and the possibility of constructing a gauge theory on its basis. We again use Green's ansatz (78). However, we have no right to make direct transformations on each of its components (apart from phase transformations on each of them) because of their mutual anomalous relations (79). Therefore, our approach will be to construct analogs or "copies" of gauge symmetries in the theory of parafields.[73]

We consider local Yukawa-type "current × field" interactions:

$$\mathscr{H}_{\mathrm{int}}(x) = j_\mu(x)\,\mathscr{B}^\mu(x). \tag{141}$$

In the theory of a free Dirac parafield there are two independent conserved currents in the form of a commutator and anticommutator[47, 74]:

$$j_\mu(x) = \frac{1}{2}[\overline{\psi}(x),\ \gamma_\mu\psi(x)]_- = \sum_{A=1}^{p}\overline{\psi}_A(x)\,\gamma_\mu\psi_A(x); \tag{142}$$

$$j'_\mu(x) = \frac{1}{2}[\overline{\psi}(x), \gamma_\mu\psi(x)]_+ = \sum_{A\neq B=1}^{p}\overline{\psi}_A(x)\,\gamma_\mu\psi_B(x). \tag{143}$$

Above, we assume normal products of the type (130). It is readily seen that the current (142) is always local, whereas the locality properties of the current (143) depend on the order of the parafield. It is local for $p = 2$:

$$j'_\mu = \overline{\psi}_1\gamma_\mu\psi_2 + \text{h.c.}, \tag{144}$$

For $p = 3$, it has paraboson properties, since it consists of three anticommuting (for spacelike separation) terms:

$$j'_\mu = \overline{\psi}_1\gamma_\mu\psi_2 + \overline{\psi}_1\gamma_\mu\psi_3 + \overline{\psi}_2\gamma_\mu\psi_3 + \text{h.c.} \tag{145}$$

For $p = 4$, it also has these properties:

$$j'_\mu = (\overline{\psi}_1\gamma_\mu\psi_2 + \overline{\psi}_3\gamma_\mu\psi_4) + (\overline{\psi}_1\gamma_\mu\psi_3 + \overline{\psi}_2\gamma_\mu\psi_4) + (\overline{\psi}_2\gamma_\mu\psi_3 + \overline{\psi}_1\gamma_\mu\psi_4) + \text{h.c.} \tag{146}$$

However, for $p \geqslant 5$ it no longer has even definite paralocal properties.

The local current (142) can interact only with an ordinary boson field, which we call "electromagnetic" and denote by $\mathscr{A}_\mu(x)$:

$$\mathscr{H}_{\text{em}}(x) = j_\mu(x)\mathscr{A}^\mu(x). \tag{147}$$

Only for $p = 3$ can one form a local interaction of the current (143) with a paraboson field $\mathscr{B}_\mu(x)$ of third order[47]:

$$\mathscr{H}(x) = \frac{1}{2}[j'_\mu(x), \mathscr{B}^\mu(x)]_+ = \sum_{A\neq B\neq C\neq A=1}^{3} \overline{\psi}_A(x)\gamma_\mu\psi_B(x)\mathscr{B}^\mu_C(x). \tag{148}$$

The components of $\mathscr{B}^\mu(x)$ satisfy the relations

$$[\mathscr{B}^\mu_A(x), \mathscr{B}^\nu_B(y)]_{\varepsilon_{AB}} = -i\delta_{AB}D^{\mu\nu}(x-y), \quad \varepsilon_{AB} = 1 - 2\delta_{AB} \tag{149}$$

and have with the components of $\psi(x)$ mutual relations of paraboson type:

$$[\psi_A(x), \mathscr{B}^\mu_B(y)]_{\varepsilon_{AB}} = [\overline{\psi}_A(x), \mathscr{B}^\mu_B(y)]_{\varepsilon_{AB}} = 0. \tag{150}$$

In the remaining cases, we cannot do this without going beyond the bounds of a consistent theory of parafields, since interacting parafields must satisfy Green's mutual trilinear relations and their orders must be equal.[47] (We do not consider the trivial case when such fields satisfy mutual fermion or boson relations. In this case, they can, of course, have different orders, but from them it is no longer possible to construct a local interaction.)

Thus, the third order is distinguished, and it is only in this case that one can expect to be able to construct a Lagrangian of Yang–Mills type. This can in fact be done by introducing the nonlinear paraboson field

$$\mathscr{F}_{\mu\nu}(x) = \partial_\mu\mathscr{B}_\nu(x) - \partial_\nu\mathscr{B}_\mu(x) + \frac{ig}{2}[\mathscr{B}_\mu(x), \mathscr{B}_\nu(x)]_-. \tag{151}$$

It can be rewritten in terms of Green components:

$$\mathscr{F}_{\mu\nu}(x) = \sum_{A=1}^{3} \mathscr{F}^A_{\mu\nu}(x); \tag{152}$$

$$\mathscr{F}^A_{\mu\nu} = \partial_\mu\mathscr{B}^A_\nu - \partial_\nu\mathscr{B}^A_\mu + ig\sum_{A\neq B\neq C\neq A} \mathscr{B}^B_\mu\mathscr{B}^C_\nu. \tag{153}$$

A copy of the Yang–Mills Lagrangian will be[73]

$$\begin{aligned}\mathscr{L}(x) &= -\frac{1}{8}[\mathscr{F}_{\mu\nu}(x), \mathscr{F}^{\mu\nu}(x)]_+ + \frac{1}{2}[\overline{\psi}(x), (i\gamma_\mu\partial^\mu - m)\psi(x)]_- \\ &+ \frac{g}{2}[j'_\mu(x), \mathscr{B}^\mu(x)]_+ = -\frac{1}{4}\sum_{A=1}^{3}\mathscr{F}^A_{\mu\nu}\mathscr{F}^{\mu\nu A} + \sum_{A=1}^{3}\overline{\psi}^A(i\gamma_\mu\partial^\mu - m)\psi^A \\ &+ g\sum_{A\neq B\neq C\neq A=1}^{3}\overline{\psi}^A\gamma_\mu\psi^B\mathscr{B}^{\mu C}.\end{aligned} \tag{154}$$

A difference from the present Yang–Mills Lagrangian is in the anomalous relations between the Green components. However, as we know from Sec. 7, these relations can be replaced by normal relations without affecting the physical consequences of the theory. For the free fields, the Klein operator can be expressed in the explicit form

$$\mathscr{K} = \exp\{i\pi(N_2 + N_3)\}, \tag{155}$$

where N_2 and N_3 are the operators of the numbers of particles with indices 2 and 3, including fermions and bosons. We now replace the components of Green's ansatz by ordinary fields:

$$\Psi_1 = \psi_1\mathscr{K}, \ \Psi_2 = -i\psi_2\mathscr{K}, \ \Psi_3 = \psi_3; \tag{156}$$

$$B^1_\mu = \mathscr{B}^1_\mu\mathscr{K}, \quad B^2_\mu = -i\mathscr{B}^2_\mu\mathscr{K}, \quad B^3_\mu = \mathscr{B}^3_\mu. \tag{157}$$

The relations between Ψ_A and B^A_μ ($A = 1, 2, 3$) are normal, i.e., all the Bose fields commute, and all the Fermi fields anticommute and commute with the Bose fields. The Lagrangian (154) can be rewritten identically in the form

$$\mathscr{L}(x) = -\frac{1}{4}\mathbf{F}^2_{\mu\nu}(x) + \overline{\mathbf{\Psi}}(x)(i\gamma_\mu\partial^\mu - m)\mathbf{\Psi}(x) - ig\mathbf{B}_\mu(x)\cdot[\overline{\mathbf{\Psi}}(x)\times\gamma_\mu\mathbf{\Psi}(x)], \tag{158}$$

where × denotes the vector product of the vectors

$$\mathbf{\Psi} = (\Psi_1, \Psi_2, \Psi_3), \ \mathbf{B} = (B_1, B_2, B_3) \tag{159}$$

and

$$\mathbf{F}_{\mu\nu} = \partial_\mu\mathbf{B}_\nu - \partial_\nu\mathbf{B}_\mu + g\mathbf{B}_\mu\times\mathbf{B}_\nu. \tag{160}$$

The Lagrangian (158) is the Yang–Mills Lagrangian of the gauge symmetry $SO(3)$. We have found that in the theory of parafields one can formulate only this Lagrangian (an indication of such a possibility can also be found in Ref. 71) and it is unique!

12. HYPERCOMPLEX REPRESENTATION OF PARAFIELDS

In the theory of a parafield one can also directly formulate a *gauge principle* by using, not Green's ansatz, but a different, equivalent representation of parafields by means of *hypercomplex* numbers.[62,75,76]

A parafield can be represented in the form of a linear combination of ordinary Fermi or Bose fields with normal commutation relations

$$\psi(x) = \sum_{A=1}^{p}\alpha_A\Psi_A(x) \tag{161}$$

with coefficients α_A that are certain algebraic quantities—hypernumbers. Green's relations (75) are satisfied if and only if these hypernumbers satisfy the Clifford algebra[76]

$$[\alpha_A, \alpha_B]_+ = 2\delta_{AB}, \ \alpha^+_A = \alpha_A. \tag{162}$$

The numbers α_A play the part of Klein operators. Further one can require invariance of the theory with respect to local phase transformations of the field (161),

$$\psi'(x) = \exp\{i\phi(x)\}\psi(x)\exp\{-i\phi(x)\}, \tag{163}$$

where $\phi(x)$ is an arbitrary hypernumber in the Clifford algebra, introduce a vector gauge hyperboson field, and obtain a corresponding gauge Lagrangian.[76] It is found that this Lagrangian is a hyperscalar only for $p = 2, 3, 4$. In the cases $p = 2$ and 4, the orders of the para-Fermi and para-Bose fields are not equal, and we are outside the framework of parafield theory proper. It is only for $p = 3$ that, once again, we obtain a consistent theory. In this case, the algebra of Clifford numbers is isomorphic to the algebra of *quaternions*. Therefore, the gauge symmetry $SO(3)$ arises in an entirely natural way. We give this demonstration in more detail.[77]

A Dirac para-Fermi field can be represented by the sum

$$\psi(x)=\sum_{A=1}^{3}\Psi_A(x)\,e_A,\quad \bar{\psi}(x)=-\sum_{A=1}^{3}\bar{\Psi}_A(x)\,e_A, \tag{164}$$

where e_A are quaternions satisfying the algebra

$$e_Ae_B=-\delta_{AB}+\varepsilon_{ABC}e_A,\quad \bar{e}_A=-e_A, \tag{165}$$

and $\Psi_A(x)$ are normal Fermi fields. Further, we require invariance of the theory with respect to the transformations

$$\psi'(x)=\exp\{\varphi(x)\}\,\psi(x)\exp\{-\varphi(x)\}, \tag{166}$$

where $\varphi(x)$ is an arbitrary phase:

$$\varphi(x)=\sum_{A=1}^{3}\varphi^A(x)\,e_A. \tag{167}$$

We introduce the paraboson vector gauge field

$$\mathscr{B}_\mu(x)=\sum_{A=1}^{3}B_\mu^A e_A, \tag{168}$$

where $B_\mu^A(x)$ are normal Bose fields, and we define the covariant derivative

$$D_\mu(x)=\partial_\mu+\frac{g}{2}\mathscr{B}_\mu(x). \tag{169}$$

The covariant derivative must always occur in the commutator with the field on which it acts. Then

$$[D'_\mu(x),\,\psi'(x)]=\exp\{\varphi(x)\}\,[D_\mu(x),\,\psi(x)]_-\exp\{-\varphi(x)\}, \tag{170}$$

if $\mathscr{B}_\mu(x)$ transforms in accordance with the law

$$\begin{aligned}\mathscr{B}'_\mu(x)&=\exp\{\varphi(x)\}\,\mathscr{B}_\mu(x)\exp\{-\varphi(x)\}\\&+\frac{2}{g}\exp\{\varphi(x)\}\,\partial_\mu\exp\{-\varphi(x)\}.\end{aligned} \tag{171}$$

Thus, we arrive at the gauge Lagrangian

$$\begin{aligned}\mathscr{L}(x)=&-\frac{1}{4}\mathscr{F}^2_{\mu\nu}(x)+\frac{1}{2}[\bar{\psi}(x),\,(i\gamma_\mu\partial^\mu-m)\,\psi(x)]_-\\&+\frac{ig}{4}[\bar{\psi}(x),\,[\mathscr{B}_\mu(x),\,\gamma_\mu\psi(x)]_-]_-,\end{aligned} \tag{172}$$

where

$$-i\mathscr{F}_{\mu\nu}(x)=\partial_\mu\mathscr{B}_\nu(x)-\partial_\nu\mathscr{B}_\mu(x)+\frac{g}{2}[\mathscr{B}_\mu(x),\,\mathscr{B}_\nu(x)]_-. \tag{173}$$

After substitution of (164) and (168) in (172) and (173) we obtain the Lagrangian (158).

Attempts have been made[4)] to extend the algebra of quaternions to the algebra of *octonions* with a view to including in the theory hyper(para)fields of the $SU(3)$ gauge symmetry.[78] The algebra of octonions was specified in the *split* basis

$$u_0=\frac{1}{2}(1+ie_7),\quad u_0^*=\frac{1}{2}(1-ie_7);$$

$$u_n=\frac{1}{2}(e_n+ie_{n+3}),\quad u_n^*=\frac{1}{2}(e_n-ie_{n+3}),\quad n=1,\,2,\,3$$

and had the form

$$u_0^2=u_0,\quad u_0u_0^*=0; \tag{174a}$$

$$u_0u_n=u_nu_0^*=u_n,\quad u_nu_0=u_0^*u_n=0: \tag{174b}$$

$$u_lu_m=\varepsilon_{lmn}u_n^*,\quad u_mu_n^*=-\delta_{mn}u_0 \tag{174c}$$

plus the complex-conjugate relations. The Dirac octonion field was assumed to "*transverse*," i.e., to contain only the units u_n:

$$\psi(x)=\sum_{n=1}^{3}q^n(x)\,u_n, \tag{175}$$

where $q^n(x)$ are ordinary fermion Dirac fields. For it, a Dirac equation was assumed in the form

$$i\gamma_\mu\partial^\mu\psi=m\psi-ig\gamma_\mu\sum_{n=1}^{3}(M_\mu^n\psi)\,u_n^*, \tag{176}$$

where M_μ^n are three transverse octonion gauge vector meson fields ($n=1,2,3$):

$$M_\mu^n=\sum_{m=1}^{3}B_\mu^{mn}u_m,\quad \sum_{n=1}^{3}M_\mu^nu_n^*=0\ \text{or}\ \sum_{n=1}^{3}B_\mu^{nn}=0 \tag{177}$$

or nine real gauge boson fields B_μ^{mn} with one condition. The vacuum vector has the form

$$\Omega=u_0\,|\,0\rangle, \tag{178}$$

where $|0\rangle$ is the vacuum for the fermion q^n and boson B_μ^{mn} fields from (175) and (177). We see that Eq. (176) includes directly the octonion units u_n^* and it cannot be written down solely in terms of the fields (175) and (177).

13. QUARKS: FERMIONS WITH COLOR OR PARAFERMIONS?

Initially the problem of quark statistics was discussed in connection with the difficulties of applying the quark model to baryon spectroscopy. In conflict with the Pauli principle, it was necessary to place three identical quarks in a symmetric state. A natural solution of this problem was first given by Bogolyubov, Struminskiĭ, and Tavkhelidze[79] and independently by Han and Nambu[80] (see also Refs. 81 and 82) and consisted of the assumption that there exists for quarks a new internal quantum number taking three values and with respect to which the antisymmetrization required by the Pauli principle can be performed. Subsequently, this quantum number was called "color" and the color gauge symmetry $SU(3)$ was associated with it.[83,84] This now provides the basis of the dynamical theory of the strong interactions of quarks and gluons—"quantum chromodynamics." At the present time there are serious grounds for believing that the color symmetry is *perfect* and not broken in any way. Hitherto, we knew of the existence of two absolutely exact symmetries in nature: the permutational symmetry of identical particles (the Pauli principle is absolute) and the $U(1)$ symmetry of electromagnetic interactions (the law of conservation of the electric charge). To these there may now be added a third symmetry—the $SU(3)$ color symmetry.

In this connection, it is appropriate to recall *another* solution to the problem of quark statistics, which was proposed by Greenberg[85] and was that para-Fermi statistics of third order should be applied to quarks. Initially, this assumption did not at all presuppose the presence of any internal symmetry. But, as we have seen, if this approach is developed systematically in the framework of parafield theory, we arrive at a nonlinear Lagrangian equivalent to the Yang–Mills Lagrangian of $SO(3)$ gauge symmetry. It is exceptionally interesting that in this theory the third order is distinguished, which could be the origin of the "three colors" of the quarks. Further, such a symmetry, contained in the parafield in hidden form, could not be broken. But it was not $SU(3)$ but only $SO(3)$ symmetry!

What would be changed if the color symmetry of

[4)] A parafield representation by means of a complex Clifford algebra is used for this purpose in a recent preprint: O. W. Greenberg and K. I. Macrae, "Locally gauge-invariant formulation of parastatistics," University of Maryland (1982).

quarks were $SO(3)$? This question was discussed in the review of Ref. 86 and in Ref. 73. The condition of asymptotic freedom would in this case restrict the number of different quarks species to two! One could reconcile this condition with the experimental discovery of already five species ("flavors") of quarks u, d, s, c, b by arranging them in "generations" (u,d), (c,s), $(b,t?)$ and assuming that to each generation there is coupled just its own triplet of gluons. But then there would be no interaction of quarks from different generations transmitted by gluons.

Another problem would be associated with the reality of $SO(3)$: It would not distinguish quarks from antiquarks and its singlets would be not only mesons and baryons constructed, as usual, from a quark and an antiquark and three quarks, respectively, but also diquarks q^2 and analogs of baryons with one quark replaced by an antiquark, $q^2\bar{q}$, and even bound states qg of a quark and a gluon. Infrared instability would not forbid the existence of such objects in the free state. They would all have fractional charges.

It is rather to be expected that $SO(3)$ is an exact subgroup of broken $SU(3)$, restored, however, at short distances.[87] In this case, the first difficulty with the condition of asymptotic freedom is eliminated, since in asymptotia all the eight gluons are put on an equal footing. However, there remains the second problem of the possible existence of diquark and other such states. It is interesting that this scheme was invoked, without reference to parastatistics, in Ref. 87 to explain the persistent observations in Millikan-type experiments[88] of residual charges $\pm(1/3)e$.

One can simply *extend* the paraquantization scheme, assuming that the *complete* large space of Green's ansatz is the physical space. Such a scheme will, of course, be equivalent to a three-dimensional color space but with the restrictions due to the anomalous relations between the Green components. Moreover, in it one can preserve the parafield structure by considering three mutually coupled parafields (140) instead of one. As gluon fields one must then consider three complex paraboson vector fields of third order:

$$\left.\begin{aligned}&\mathscr{B}_\mu(x)=\mathscr{B}^1_\mu(x)+\mathscr{B}^2_\mu(x)+\mathscr{B}^3_\mu(x);\\&\mathscr{B}'_\mu(x)=k\mathscr{B}^1_\mu(x)+\bar{k}\mathscr{B}^2_\mu(x)+\mathscr{B}^3_\mu(x);\\&\mathscr{B}''_\mu(x)=\bar{k}\mathscr{B}^1_\mu(x)+k\mathscr{B}^2_\mu(x)+\mathscr{B}^3_\mu(x),\end{aligned}\right\}\tag{179}$$

whose components satisfy the relations

$$\left.\begin{aligned}&[\mathscr{B}^\mu_A(x),\mathscr{B}^{\nu+}_B(y)]_{\varepsilon_{AB}}=-i\delta_{AB}D^{\mu\nu}(x-y);\\&[\mathscr{B}^\mu_A(x),\mathscr{B}^\nu_B(y)]_{\varepsilon_{AB}}=[\mathscr{B}^{\mu+}_A(x),\mathscr{B}^{\nu+}_B(y)]_{\varepsilon_{AB}}=0\end{aligned}\right\}\tag{180}$$

and with the components of the parafermion fields have the relations

$$[\mathscr{B}^\mu_A(x),\psi_B(y)]_{\varepsilon_{AB}}=0\tag{181}$$

etc., where $\varepsilon_{AB}=1-2\delta_{AB}$, and there is also one complex boson field $\mathscr{H}_\mu(x)$ that has normal relations with all fields. The Lagrangian can be written in the form

$$\begin{aligned}\mathscr{L}(x)=&-\frac{1}{4}[\mathscr{F}^+_{\mu\nu}(x),\mathscr{F}^{\mu\nu}(x)]_+-\frac{1}{2}\mathscr{H}_{\mu\nu}(x)\mathscr{H}^{\mu\nu+}(x)\\&+\frac{1}{2}[\bar{\psi}(x),(i\gamma^\mu\partial_\mu-m)\psi(x)]_-\\&+\frac{g}{4\sqrt{2}}(-i[\mathscr{B}_\mu(x),J^\mu(x)]_++\text{h.c.})+\frac{g}{\sqrt{2}}(I^\mu(x)\mathscr{H}^+_\mu(x)+\text{h.c.}),\end{aligned}\tag{182}$$

where the tensors of the gluon fields are

$$\mathscr{F}_{\mu\nu}=\partial_\mu\mathscr{B}_\nu+\frac{g}{2\sqrt{2}}\mathscr{B}^+_\mu\mathscr{B}^+_\nu+\frac{ig}{\sqrt{2}}(\mathscr{H}^+_\mu\mathscr{B}'_\nu+\mathscr{H}_\mu\mathscr{B}''_\nu)-(\mu\rightleftarrows\nu);\tag{183}$$

$$\mathscr{H}_{\mu\nu}=\partial_\mu\mathscr{H}_\nu+\frac{ig}{2\sqrt{2}}[\mathscr{B}_\mu,\mathscr{B}''^+_\nu]_+-(\mu\rightleftarrows\nu)\tag{184}$$

and the quark currents are

$$J_\mu=\frac{1}{2}[\bar{\psi},\gamma_\mu\psi]_++\frac{i}{2\sqrt{3}}([\bar{\psi}',\gamma_\mu\psi']_+-[\bar{\psi}'',\gamma_\mu\psi'']_+);\tag{185}$$

$$I_\mu=-\frac{i}{2\sqrt{3}}[\bar{\psi},\gamma_\mu\psi']_-.\tag{186}$$

If the Klein transformation (156), (157) is performed on the Green components, the Lagrangian (182) is transformed into the Yang–Mills Lagrangian of the gauge $SU(3)$:

$$\mathscr{L}=-\frac{1}{4}\sum_{a=1}^{8}F^a_{\mu\nu}F^{\mu\nu a}+\bar{\Psi}\left(i\gamma^\mu\partial_\mu+\frac{g}{2}\gamma^\mu\sum_{a=1}^{8}G^a_\mu\lambda^a-m\right)\Psi,\tag{187}$$

where

$$F^a_{\mu\nu}=\partial_\mu G^a_\nu-\partial_\nu G^a_\mu+g\sum_{b,\,c=1}^{8}f^{abc}G^b_\mu G^c_\nu.\tag{188}$$

With the ordinary gluon fields G^a_μ there are associated the components of the complex fields

$$\left.\begin{aligned}&B^A_\mu=(b^A_\mu+i\tilde{b}^A_\mu)/\sqrt{2}\text{ and }\mathscr{H}_\mu=(h_\mu+i\tilde{h}_\mu)/\sqrt{2};\\&G^1_\mu=b^3_\mu,\quad G^2_\mu=\tilde{b}^3_\mu,\quad G^3_\mu=h_\mu,\quad G^4_\mu=b^2_\mu;\\&G^5_\mu=-\tilde{b}^2_\mu,\quad G^6_\mu=b^1_\mu,\quad G^7_\mu=\tilde{b}^1_\mu,\quad G^8_\mu=\tilde{h}_\mu.\end{aligned}\right\}\tag{189}$$

If we set $\mathscr{H}_\mu=0$, and the fields $\mathscr{B}^A_\mu$ are assumed to be imaginary, we return to the Lagrangian (154), which is equivalent to the Yang–Mills Lagrangian of $SO(3)$ symmetry.

It is interesting that in such a theory all states belonging to a non-Fock representation will be dynamically forbidden because of infrared instability (if, of course, it can be proved at all), since none of them are singlets. Nonsinglet states in the Fock space will also be forbidden.

CONCLUSIONS

Our analysis has shown that:

1. The principle of indistinguishability of identical particles does not prohibit the existence of parastatistics. From the mathematical point of view, it corresponds to the Lie-algebra approach to the quantization of fields in which each Lie algebra of the classical groups is associated with a definite generalized quantization scheme.

2. Irreducible representations of the algebra of paraoperators are constructed in complete analogy with the theory of composite higher spins. The states in different irreducible representations form multiplets of internal symmetries of the type of isospin $SU(2)$ for $p=2$, $SU(3)/SU(2)$ for $p=3$, etc. The formulation of the symmetries themselves requires, however, that we go beyond the framework of parafield theory proper and go over to the large space of Green's ansatz. It is then possible to construct a "composite model" of internal symmetries by analogy with the composite model of higher spins. The advantages of such an approach to the description of "horizontal symmetries" have not yet been elucidated.

3. From the logical point of view, exact degeneracy should be described by parastatistics and not by the introduction of fictitious internal spaces, unless, of course, the latter have some dynamical basis of the type of a composite structure. Here, it may be appropriate to make an analogy with second quantization, which can be regarded as the most consistent way of describing identical particles as the excitation quanta of one field, since there is then no need to introduce indices of identical particles that have no direct meaning.

We have seen that in the formulation of Lagrangians of Yang–Mills type in parafield theory the third order is uniquely distinguished, which could provide a basis for understanding why there are *three colors* of the quarks. But *the gauge symmetry is then only* $SO(3)$! Inclusion in the treatment of $SU(3)$ gauge symmetry *is possible only if we go beyond the framework of the theory of one parafield* and extend it to the theory of *three* mutually coupled fields.

It can be concluded from what we have said that so far the parafield approach has not brought any noticeable advantages in the description of internal symmetries. Nevertheless, the restrictions that arise in such an approach inspire the hope that its further development may cast light on the origin of physical symmetries.

It was a great pleasure for me to participate in the present collection devoted to the memory of my teacher Yuriĭ Mikhaĭlovich Shirokov, whose favorable attitude to the study of the problem discussed here of the statistics of identical particles was for me always an inestimable support.

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