Toroidal moments in the physics of electromagnetic and weak interactions

V. M. Dubovik

Joint Institute for Nuclear Research, Dubna

L. A. Tosunyan

Scientific-Research Institute of Nuclear Physics, Moscow State University

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The analytic structure and geometrical meaning of multipole expansions for systems containing convection and induction currents are illustrated in the framework of classical electrodynamics. The main attention is devoted to the expansion of the transverse electric part of the current, in which the authors have identified a third family of multipole parameters—toroidal moments and their radii. The electromagnetic properties of the simplest toroidal systems are analyzed. The connection between multipole expansions and the methods of invariant parametrizations of relativistic matrix elements of a current operator is emphasized. Some consequences and effects of the existence of these moments are pointed out in the theory of atomic and nuclear transitions. A review is made of the results achieved on the extraction of toroidal moments of nuclear transitions through nuclear conversion penetration parameters, the possibilities of extracting information about toroidal giant resonances, etc. The results are given of the study of anapoles (toroidal dipoles) in the framework of the problem of parity violation by the electroweak interactions in atomic, nuclear, quasinuclear, and quark systems.

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INTRODUCTION

History

At the end of the sixties, Shirokov made a series of well-known studies on the finding of finite-dimensional unitary representations of the general Lorentz group. On the basis of these studies, he posed the problem of creating general methods of invariant parametrization of relativistic matrix elements of local operators of arbitrary tensor or spinor dimension, specified in the Heisenberg picture and acting on the states of free particles. This problem was solved by Shirokov and his prematurely deceased student A. A. Cheshkov in Ref. 1.

The identity of the laws of nature in all inertial frames of reference means that the states of physical systems and the processes of their interaction are determined by parameters that do not depend on the choice of the frame; they are invariant parameters. From the group-theoretical point of view, any relativistic parametrization is the choosing of a representation of the Poincaré group of the class $P^{s=0}_{\pm\varkappa}$ (where s is the spin, and $\varkappa$ the mass) from the direct product of the representations with respect to which the state vectors and the operator itself (or product of operators) transform. In other words, the matrix element of any operator can be reduced in such a way that its value is specified by a finite number of invariant functions, called form factors.

For the invariant parametrization of a relativistic matrix element, one can, expanding it in one of the nonrelativistic ways in a kinematically distinguished Lorentz frame (in the center-of-mass system, in the Breit system, etc.), then go over to an arbitrary system (the laboratory system) by means of an appropriate Lorentz transformation (boost) $D^s_{mm'}$:

$$|\varkappa, \mathbf{p}, s, m\rangle = D^s_{mm'}(\mathbf{p}, \mathbf{u})\,|\varkappa, \mathbf{p}', s, m'\rangle.$$

Here, $\mathbf{u}$ is the velocity of the Lorentz transformation, and $D^s_{mm'}(\mathbf{p}, \mathbf{u})$ is the matrix of the relativistic rotation of the spin s, which has projection m or m' onto the quantization axis and depends not only on $\mathbf{u}$ but also on the momentum of the particle.

This kinematic effect is known in classical relativity theory as relativistic top precession. Its existence was pointed out for the first time by Møller in 1945. In 1954, this effect was described in relativistic quantum field theory by Shirokov,[2] who obtained in explicit form the rotation matrix for particles with spin $\frac{1}{2}$.

In individual cases, an invariant parametrization can be implemented in a manifestly covariant form by using a relativistic generalization of the spin concept[3] introduced in covariant form by Shirokov in 1951.[4] It was in this way that Shirokov and Cheshkov in 1963 parametrized the matrix element of an electromagnetic current diagonal with respect to the mass and spin and found a third family of form factors different from the charge and magnetic form factors for a particle with arbitrary spin.[1)]

Later, in Refs. 6 and 7, Cheshkov and one of the present authors discovered and studied classical analogs of these form factors—multipole toroidal distributions (moments) of convection currents—and established their duality to induced electric moments[8–12] and Zel'dovich's anapole.[5a] An anapole was introduced for the first time in 1957 in connection with the discovery of parity nonconservation in weak interactions as a third dipole of an elementary point particle with spin $\frac{1}{2}$ interacting with a current: $\mathscr{L} \sim a\sigma \cdot \mathbf{J}^{\text{ext}}$. The existence of a third dipole was also pointed out in other studies on relativistic parametrizations of the electromagnetic vertex for particles with higher spins (see, for example, Ref. 5b). However, it was only in Ref. 5a that

[1)]For spins 1/2 and 3/2, such form factors were introduced earlier in Refs. 5a and 5b, respectively, using bispinor representations.

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estimates were made of it, classical analogs pointed out, and a name—anapole—given to it that has taken root in the literature. As a consequence of the existence of the new electroweak parameter, various parity-violating effects were predicted in atomic spectra, in lepton–hadron scattering, etc. Thus, the problem of parity violation stimulated interest in the study of what is actually an intrinsic (nontransition) moment in the series of induced electric moments.[8]

Later, in connection with the discovery of CP nonconservation, the possibilities of measuring a toroidal quadrupole in electron–dueteron scattering were investigated.[13,14]

Recently, mechanisms of violation of discrete symmetries have been constructed on the basis of unified gauge theories in the quark–lepton sector. Since the existence of intermediate bosons is apparently confirmed experimentally, the toroidal dipole moment of, say, the electron is something no more exotic than the anomalous magnetic moment $\alpha/2\pi$, as can be seen from the diagrams (see, for example, Ref. 15). Thus, the expression "anapole" of an electron or a quark becomes something entirely familiar to a physicist working in the field of elementary particles, and anapole contributions to various effects are discussed in connection with ongoing experiments.[9,16,17]

Recently, as a result of increased experimental accuracy, it has become necessary to take into account the toroidal-moment contributions to electric dipole transitions. In Ref. 18, it was shown that the toroidal contributions can be important in the calculation of hindered electric transitions in nuclei, which are convenient for studying parity nonconservation in weak NN interactions. It has also been found that toroidal transition dipoles of nuclei are associated with the so-called penetration coefficients in the theory of internal conversion of nuclear γ rays.[19a] Experimental values of toroidal dipoles for $E1$ transitions in some nuclei are now known.[19b]

Appreciable contributions to transition probabilities are made by toroidal dipoles in strongly bound systems such as quasinuclear or quarkonium systems. For the example of a quasinuclear system we recently showed[20] how the toroidal part can be separated from the total electric moment by means of selection rules.

Thus, the development of the method of parametrization of the relativistic amplitudes of quantum theory and the discovery of violations of discrete symmetries have reacted back on classical electrodynamics, stimulating a complete investigation of the structure of the multipole expansion of electromagnetic currents. A review covering both the classical and quantum aspects of the problem of toroidal moments was published[7] in 1974. In the field of classical electrodynamics, the present authors obtained during 1975–1976 a number of new expressions to do with toroidal moments; together with other investigators, we also found new manifestations of toroidal moments in the physics of nuclei, particles, and subparticles. In the present review, an attempt is made to summarize these results. We hope that our exposition will help to give a better understanding of the cumbersome multipole expansions through the systematic interpretation of them in the language of geometrical images, in particular the expressions for toroidal distributions, and that this will facilitate their use. In the review it is shown, basically qualitatively, how one should use the general physical formalism of multipole parametrizations in different branches of physics, beginning with classical electrodynamics and ending with the theory of subparticles. The technique of multipole expansions in both classical and quantum theory with allowance for toroidal distributions has been developed in detail in Ref. 7.

The existence of toroidal moments is unavoidable

We recall (see Ref. 7) that the need to introduce toroidal moments already arises in the framework of ordinary classical electrodynamics. We shall show how the existence of toroidal moments can be deduced from general considerations.

Suppose there is an arbitrary system described by a charge density $\rho(\mathbf{r},t)$ and current density $\mathbf{j}(\mathbf{r},t)$. We set ourselves the task of describing the electromagnetic properties of this system by means of a set of parameters. We choose some complete system of functions, for example, spherical harmonics, and expand $\rho(\mathbf{r},t)$ with respect to them. We obtain the well-known series of multipole moments: the total charge of the system, the charge dipole, quadrupole, etc.

Each of the components of the current $\mathbf{j}(\mathbf{r},t)$ can be expanded similarly. However, the current conservation condition $\partial_\mu j_\mu(\mathbf{r},t)=0$ means that only three components of $j_\mu(\mathbf{r},t)$ are independent. Indeed, it is well known (see, for example, Refs. 8 and 10–12) that a multipole expansion leads to three families of multipole moments: charge Q_{lm}, magnetic M_{lm}, and transverse electric E_{lm} moments.

We now note that the multipole expansion of the current density is not a formal mathematical operation; it has a transparent geometrical meaning. To each multipole moment there corresponds a system of charges or currents possessing only this moment (at least, in a first approximation). For example, a planar frame with current and a charge dipole are described by a magnetic dipole moment and a charge dipole moment, respectively. Transverse electric moments also correspond to definite current distributions. Let us attempt to find geometrical images for them. We consider the simplest example—a toroidal constant current (see Fig. 1). Such a current has axial symmetry and therefore must be characterized by a vector pointing along the

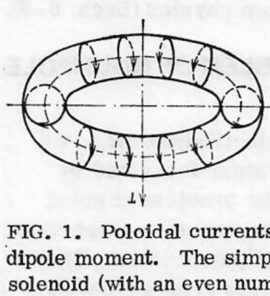

FIG. 1. Poloidal currents on a torus determine the toroidal dipole moment. The simplest model of this is an ordinary solenoid (with an even number of windings!) bent into a torus.

axis of the torus, i.e., by a dipole moment. Since $\rho \equiv 0$ for the given configuration, and the magnetic moments of the windings of the torus close in the symmetry plane of the torus, both the charge dipole moment and the magnetic dipole moment of the torus are equal to zero. Thus, the toroidal current can be characterized by the transverse electric dipole moment alone. However, the transverse electric moments E_{lm} are usually equated[10,12] in the long-wave approximation to $\dot{Q}_{lm}: E_{lm} = \dot{Q}_{lm}$, where the dot denotes differentiation with respect to the time. At the same time it is assumed that in the general case equality of the charge and transverse electric moments holds up to relativistic corrections. In reality, everything is more complicated.

Let us consider, for example, the static case ($\omega = 0$). Then, if we find the characteristics of a constant current on a torus, using the expression $E_{lm} = \dot{Q}_{lm}$, we automatically obtain zero. Moreover, in the general case one can directly say that this relation is in principle invalid even in the long-wave approximation, since it leaves from the three independent multipole families Q_{lm}, M_{lm}, and E_{lm} only two: Q_{lm} and M_{lm}.

Thus, the structure and properties of transverse electric moments requires detailed study in the framework of classical electrodynamics.

We return to the discussion of currents on a torus (see Fig. 1). The proposed configuration can be regarded as convection currents of free charges producing a circular magnetic field, or one can assume that we have here a solenoid bent into a torus (with an even number of windings to ensure that there is no component of the current "along" the torus, which would lead to its having a magnetic moment!). However, one could regard the currents on the torus as induction currents, i.e., as produced by magnetization of a medium "within" the torus. It is interesting that precisely this "magnetic" part of the toroidal moments was already introduced by Blatt and Weisskopf in their study of electromagnetic transitions in nuclei and was called induced electric moments.[8,11] Note that on the basis of the expressions for the induced electric moments it is rather difficult to recover the form of the convection part. Such a problem must be solved by establishing the connection between the relativistic (anapole) parametrization and the classical toroidal parametrization (see Refs. 7 and 9 and the commentaries in Sec. 8). In this review, following the plan of our preceding paper of Ref. 7, we solve the general problem of multipole expansion in the framework of classical electrodynamics (see Secs. 1–4), we discuss the properties of a classical toroidal dipole (Sec. 5), and we point out manifestations of toroidal form factors in quantum physics (Secs. 6–9).

1. FORMULATION OF THE PROBLEM OF MULTIPOLE EXPANSION

In the framework of classical electrodynamics, we consider an arbitrary bounded system described by charge density $\rho(\mathbf{r}, t)$. We pose the problem of going over to a description of the electromagnetic properties of the system by means of a set of parameters. The practical gain from such a transition arises from the possibility of separating one or a few of the most important parameters determining integrated properties of the system (for example, in the long-wave approximation or in view of its particular geometrical properties).

We use the following formal device (see, for example, Refs. 21 and 22). In accordance with the definition of the δ function, we rewrite $\rho(\mathbf{r}, t)$ tautologically in the form

$$\rho(\mathbf{r}, t) = \int \rho(\xi, t)\, \delta(\xi - \mathbf{r})\, d^3\xi, \tag{1}$$

and we replace the function $\delta(\xi - \mathbf{r})$ by its formal Taylor expansion

$$\delta(\xi - \mathbf{r}) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \underbrace{\xi_i \ldots \xi_k}_{l} \partial_i \ldots \partial_k \delta(\mathbf{r}). \tag{2}$$

We obtain the expression

$$\rho(\mathbf{r}, t) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} A^{(l)}_{i\ldots k} \partial_i \ldots \partial_k \delta(\mathbf{r}), \tag{3}$$

which represents $\rho(\mathbf{r}, t)$ in the form of a sum of δ-functional charge sources characterized by the parameters

$$A^{(l)}_{i\ldots k}(t) = \int \rho(\xi, t)\, \xi_i \ldots \xi_k\, d^3\xi, \tag{4}$$

where the integration is extended over the complete space. The quantities $A^{(l)}_{i\ldots k}$ are called moments, and in our case of a function $\rho(\xi, t)$ bounded in space they are defined for all l. In the framework of electrodynamics, we see that $A^{(0)}$ is the total charge of the system, $A^{(1)}_i$ is its charge (Coulomb) dipole, etc. However, to find the higher irreducible (multipole) moments—it is more convenient to use them—we must decompose the tensors $\xi_i \ldots \xi_k$ with respect to irreducible representations of the rotation group. Then on the transformation of $A^{(l)}_{i\ldots k}$ into irreducible tensors further series of parameters arise in the sum (3): with weight $l = 0$,

$$A_{ik}\delta_{ik},\; A_{iabk}\delta_{ia}\delta_{bk},\; \ldots; \tag{5}$$

with weight $l = 1$,

$$A_{iak}\delta_{ak},\; A_{iabck}\delta_{ab}\delta_{ck},\; \ldots \tag{6}$$

etc.

It is not difficult to separate from $A^{(l)}(t)$ the multipole moments of the lowest ranks ($l = 0, 1, 2$), but for $l > 2$ this task becomes a very laborious procedure, and it cannot be implemented in general form. Therefore, for a multipole expansion it is necessary to use directly a complete set of suitable basis functions such that the expansion coefficients are irreducible tensors (multipole form factors; for example, in the Fourier-conjugate space of wave vectors **k**). Further, if necessary, one can go over to δ-type functions (decomposition into point sources). The numerical coefficients then obtained are the moments and mean radii of the multipole distributions. In this way one solves the problem of parametrizing the electromagnetic properties of a system described by a given 4-current.

We emphasize that multipole form factors dependent on the wave number **k** (the Fourier transforms of the multipole distributions) must already be introduced in classical electrodynamics if we wish to describe the properties of a system fully. The use of only moments may be insufficient in the presence of high multipoles even in the long-wave limit. Let us consider, for example, the problem of radiation. The radiation intensity is determined by the value of the form factors for $k = \sqrt{\mathbf{k}^2} = \omega$. In the long-wave approximation, we replace them by their values at $k = 0$, i.e., by the moments. Suppose that for the given system $\mu_i = 0$, $\mu_{ij} = 0$ and only $\mu_{ijk} \neq 0$ (here, μ_i is the magnetic dipole moment, μ_{ij} is the magnetic quadrupole moment, etc.). Then it is also necessary to consider the mean-square radius of the magnetic dipole distribution

$$\overline{\rho_i^2} = \frac{1}{2} \int \xi^2 [\xi \times \mathbf{j}]_i \, d^3\xi, \tag{7}$$

which, in general, has the same order as the octupole moment μ_{ijk} $(\sim e(v/c)a^3)$. Similarly, in a scattering problem one must take into account not only the charge quadrupole moment but also the mean-square radius of the charge.

Generalizing, we can say that the complete parametrization of the electromagnetic properties of a system presupposes two expansions in powers of k: one associated with the representation of the charge and current densities in the form of a set of irreducible tensor functions of k^2, and another associated with expansion of each of these functions in series in powers of k^2. The multipole parametrization of, for example, the Fourier transform of the charge density is given in Table I.

In Table I the row with $n = 0$ reproduces (without coefficients) the series expansion of the charge density with respect to moments. The column corresponding to each term of this row gives the series of mean $2n$-power radii of the given 2^l total distribution (for brevity, we shall in what follows say simply the radius of given moment of weight l). The radius characterizes the degree of spatial diffuseness of this moment. It is readily seen that for the charge they are defined as

$$\overline{r^{2n}} = \int \rho(\xi, t) \xi^{2n} d^3\xi, \quad n = 1, 2, \ldots. \tag{8}$$

The diffuseness of the dipole moment of the system is characterized by the radii

$$\overline{r_i^{2n}} = \int \rho(\xi, t) \xi_i \xi^{2n} d^3\xi, \quad n = 1, 2, \ldots. \tag{9}$$

Similar expressions can be written down for all the other multipole moments.

From the expressions (8), (9) and others like them it can be established that q_l have the order ea^l, and the mean radii are $\overline{r_l^{2n}} \sim ea^{2n+l}$ (a is the linear dimension of the system). Generally speaking, Table I is not restricted in either the vertical or the horizontal direction. Note that in quantum theory Table I does stop in the horizontal direction because of the kinematic properties of the spin, so that the introduction of form factors in this case has meaning independently of the value of the dimensionless parameter ka. To avoid confusion, we mention here that in electromagnetism the concept of induced moments is also used. Such moments reflect the influence of an external field on the properties of the system. We demonstrate this for the example of a widely used characteristic—the polarizability. If the configuration of the external electric fields is sufficiently complicated and these fields are strong, they are capable of stretching (exciting) the charges of the system, and new dynamic quadrupole characteristics of the electromagnetic properties of the system arise. The simplest, the polarizability tensor p_{ik}, determines the magnitude of the induced dipole $d_i = p_{ik}E_k$. It enters the Lagrangian quadratically in the field, since $\mathscr{H} = -d_iE_i = -p_{ik}E_iE_k$, whereas the tensor characteristic of second rank, the ordinary quadrupole, interacts with the derivative of the field: $\mathscr{H} = -Q_{ik}\nabla_i E_k$. Thus, with allowance for nonlinear interactions additional series of multipole dynamical characteristics of the system arise.

TABLE I. Multipole parametrization of the Fourier transform of the charge density $\rho(k, t)$.

n	$l=0$	$l=1$	$l=2$	$l=3$
0	q_0	kq_1	k^2q_2	...
1	$k^2\overline{r_0^2}$	$k^3\overline{r_1^2}$	$k^4\overline{r_2^2}$	...
2	$k^4\overline{r_0^4}$	$k^5\overline{r_1^4}$	$k^6\overline{r_2^4}$	...
3	...	...	...	...

Thus, we have shown how to characterize the properties of a spatially distributed system of charges by multipole moments and their mean $2n$-power radii. Note also that the multipole moment of weight l is sometimes called the 2^l total moment, since a dipole is a system of two charges (poles), a quadrupole is a system of four, etc. However, there is no need to appeal to charges in the construction of higher multipoles. In the case when there are no charges, as, for example, in magnetostatics, and only dipoles exist, all the higher moments can be constructed from dipoles. This last remark must be borne in mind in the following pages.

2. EXPANSION OF CURRENT DENSITIES

Suppose a system is described by not only a charge density but also a current density $\mathbf{j}(\mathbf{r}, t)$. Just as we expanded the density $\rho(\mathbf{r}, t)$, we can make a multipole expansion of the spatial components of the current, for example, of the three scalar functions $j_x(\mathbf{r}, t)$, $j_y(\mathbf{r}, t)$, $j_z(\mathbf{r}, t)$. We then obtain a further three families of moments and their radii (three further tables of parameters similar to the one in Table I). In fact, we are in the framework of electrodynamics and must take into account the current conservation law[2)]

$$\operatorname{div} \mathbf{j} = -\dot{\rho}. \tag{10}$$

It is convenient to realize this connection in terms of the multipole expansion by using Helmholtz's theorem (see, for example, Ref. 23, Vol. 1 or Ref. 24). This states that any vector field (singled-valued, continuous, and vanishing at infinity) can be decomposed into a sum of a potential field and a solenoidal field. Using this theorem, we can represent $\mathbf{j}$ in the form

$$\mathbf{j} = \mathbf{j}_{\parallel} + \mathbf{j}_{\perp}, \tag{11}$$

2) Here and below, the dot above a symbol denotes the time derivative.

where $\mathbf{j}_{\parallel} = \nabla\varphi$, i.e., curl $\mathbf{j}_{\parallel} = 0$, and $\mathbf{j}_{\perp}$ = curl f, the vector f satisfying the condition

$$\operatorname{div} \mathbf{f} = 0. \tag{12}$$

Using the decomposition (11), we rewrite the condition (10) in the form

$$-\dot{\rho} = \operatorname{div} \mathbf{j}_{\parallel}. \tag{13}$$

It follows from this, first, that the expansion of the longitudinal part $\mathbf{j}_{\parallel}$ of the current (i.e., the expansion parameters of the scalar function φ introduced above) is determined by the time derivatives of the multipole moments and their radii characterizing the charge density $\rho(\mathbf{r}, t)$; second, with allowance for the relation (12) there remain two independent components of the vector f, their expansion determining $\mathbf{j}_{\perp}$. Since there are no more connections of the type (10) and (12),[3] we conclude that $\mathbf{j}_{\perp}$ is characterized by two families of moments, which are independent of the charge moments. Indeed, it is possible to have cases when there are no charge moments at all, for example, for $\rho = 0$ or $\dot{\rho} = 0$, which does not affect the components of the vector $\mathbf{j}_{\perp}$.

Obviously, it will now be convenient for us to make a further decomposition, by means of certain criteria, of the solenoidal field f into a sum of two fields, each of them specified by the action of differential operators on scalar fields. We are helped in this by a theorem given by Chandrasenkhar in his book of Ref. 25 (see also Ref. 26). The theorem states that any solenoidal field can be represented as a sum of two vectors,

$$\mathbf{f}(\mathbf{r}) = i\mathbf{L}\psi(r, \theta, \varphi) + i\,\mathrm{curl}(\mathbf{L}\chi(r, \theta, \varphi)), \tag{14}$$

where $\mathbf{L} = -i[\mathbf{r}\times\nabla]$. The first vector in (14) describes the toroidal currents (the currents flowing along the "parallels" on the torus), and the second the poloidal currents (those flowing along the "meridians" on the torus).[4] Note that the functions ψ and χ for a fixed property of f under reflections are one a scalar and the other a pseudoscalar.

It can be learned from the book of Ref. 23 (Vol. 2) that a similar decomposition was already used by Neumann in problems of mathematical physics; he extended Helmholtz's theorem as follows:

$$\mathbf{j} = \nabla\varphi + \mathrm{curl}\,(\mathbf{r}\psi(\mathbf{r})) + \mathrm{curl\ curl}(\mathbf{r}\,\chi(\mathbf{r})). \tag{15}$$

We now introduce a definition of current moments similar to the definition of charge moments:

$$B^{(l)}_{ij\ldots k} = \frac{(-1)^{l-1}}{(l-1)!}\int j_i \underbrace{\xi_j \ldots \xi_k}_{l-1} d^3\xi. \tag{16}$$

Taking into account the properties of longitudinality (13) and transversality (14) formulated above, we can further decompose the tensors $B^{(l)}_{ij\ldots k}$ into irreducible tensors, separating the various multipole moments and radii. Obviously, the tensor $B^{(2)}_{ij}$ can be decomposed into a symmetric part, which is related by Eq. (13) to the time derivatives of the charge quadrupole moment Q_{ij} and the mean-square radius $\overline{r_q^2}$ of the charge, and an antisymmetric part, which we can characterize by an axial vector, called the magnetic moment μ_i. The definitions of these parameters are well known. The next tensor, of third rank, $B^{(3)}_{ijk}$ can obviously be reduced in accordance with the scheme

$$1\times(2+0) = (3+1)+2+1. \tag{17}$$

It can be seen that the representation of weight $l = 1$ is extracted twice from $B^{(3)}_{ijk}$.

Indeed, the decomposition of the tensor $j_i\xi_j\xi_k$ can be written in the form

$$j_i\xi_j\xi_k = \frac{1}{3}\left[j_i\xi_j\xi_k + j_k\xi_i\xi_j + j_j\xi_k\xi_i - \frac{3}{5}(\delta_{ij}\theta_k + \delta_{ik}\theta_j + \delta_{jk}\theta_i)\right] - \frac{1}{3}(\varepsilon_{ijl}\mu_{kl} + \varepsilon_{ikl}\mu_{jl})$$
$$-\frac{1}{6}(\delta_{ij}\lambda_k + \delta_{ik}\lambda_j - 2\delta_{jk}\lambda_i) + \frac{1}{5}(\delta_{ij}\theta_k + \delta_{ik}\theta_j + \delta_{jk}\theta_i). \tag{18}$$

Here, the expression in the square brackets, which is a completely symmetric tensor of third rank with the trace subtracted, is readily seen to determine the time derivative of the charge octupole moment $\dot{Q}_{ijk}$ (see Sec. 3); μ_{ik} is the magnetic quadrupole moment, equal to $\mu_{ik} = \mu_i\xi_k + \mu_k\xi_i$, where $\mu_i = \frac{1}{2}\varepsilon_{ijk}\xi_j j_k$; $\theta_i = 2\xi_i(\boldsymbol{\xi}\mathbf{j}) + \xi^2 j_i$ determines the time derivative of the mean-square radius of the dipole; and $\lambda_i = \xi_i(\boldsymbol{\xi}\mathbf{j}) - \xi^2 j_i$. Thus, the formula for reducing $j_i\xi_j\xi_k$ has been realized and we do indeed find in the decomposition of this tensor two vector components θ_i and λ_i.[5] It is readily verified that $\int \theta_i d^3\xi \equiv \dot{\overline{r_i^2}}$ vanishes on the substitution in it of the definition of the transverse current, and thus it occurs only in the longitudinal (Coulomb) part of the current, which is related by the condition of its conservation div $\mathbf{j}_{\parallel} = -\dot{\rho}$ to the expansion of the density of the charge distribution. The quantity $\int \lambda_i d^3\xi$ is neither longitudinal nor transverse and, therefore, must be decomposed into transverse and longitudinal parts. We now recall that, with allowance for Eqs. (4), (6), and (9), the mean-square radius must, in accordance with its definition, occur in the term of the expansion (16) containing the operator $\delta_{jk}\nabla_j\nabla_k \equiv \Delta$. Therefore, in the decomposition (18) we can retain the term containing θ_i only for δ_{jk}, and, therefore, the corresponding sums of the first two combinations must automatically give the definition of a third transverse vector:

$$\tau_i = \frac{1}{5}\theta_i - \frac{1}{6}\lambda_i = \frac{1}{10}[(\mathbf{j}\boldsymbol{\xi})\xi_i - 2\xi^2 j_i]. \tag{19}$$

The vector part of the reduction (18) now takes a multipole nature:

$$\delta_{ij}\tau_k + \delta_{ik}\tau_j - 2\delta_{jk}\tau_i + 3\delta_{jk}\theta_i. \tag{20}$$

It can be seen directly from its form that the obtained third vector determines one further (a third) dipole

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[3]Note that these constraints are differential; they are transformed into algebraic constraints by going over to the space of wave vectors k, where $\mathbf{j}_{\parallel}(\mathbf{k})$ is in fact collinear with k, and $\mathbf{j}_{\perp}(\mathbf{k})$ is orthogonal to it. In the coordinate space, accordingly,

$$\mathbf{j}_{\perp} = \mathbf{j} - \nabla\int\frac{\dot{\rho}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'.$$

[4]This can be readily established by going over to toroidal instead of spherical coordinates.

[5]In the recent paper of Friar and Fallieros,[27] the transverse electric part of the current is decomposed in accordance with such a prescription. It is asserted that such a decomposition is convenient for considering the problem of exchange currents. This appears to us questionable, since θ_i and λ_i contain mutually compensating longitudinal parts, as we wrote to Friar in a private communication.

moment and cannot be called a mean-square radius, i.e., a correction for retardation. It is convenient (see Sec. 3) to introduce the definition of the moment associated with τ_i as follows:

$$\mathbf{T}=\frac{1}{10}\int[\boldsymbol{\xi}(\boldsymbol{\xi}\mathbf{j})-2\xi^2\mathbf{j}]\,d^3\xi. \tag{21}$$

It is easy to show that the currents corresponding to point dipoles, namely, $\mathbf{j}=\mathbf{d}\delta(\boldsymbol{\xi})$, which corresponds to an elementary linear current, and $\mathbf{j}=\operatorname{curl}\mathbf{M}\delta(\boldsymbol{\xi})$, which corresponds to an elementary induction current (a ring with circular current compressed to a point), lead on substitution in the definition (21) to $\mathbf{T}=0$. The definition (18) is naturally transformed into an indentity on the substitution in (18) of the current

$$\mathbf{j}=\operatorname{curl}\operatorname{curl}\mathbf{T}\delta(\boldsymbol{\xi}). \tag{22}$$

If we consider the current $\mathbf{j}=\operatorname{curl}\mathbf{M}\delta(\boldsymbol{\xi})$ on a torus, this current flows along its parallels. The current (22) flows on the torus along the meridians (see Fig. 1).

Thus, the geometrical model of $\mathbf{T}$ is a torus with poloidal currents flowing on it. The dipole $\mathbf{T}$ is directed along the axis of axial symmetry of the torus as indicated in Fig. 1.

3. MULTIPOLE EXPANSION OF THE ENERGY OF THE INTERACTION OF A SYSTEM OF POINT CHARGES WITH AN EXTERNAL ELECTROMAGNETIC FIELD

We make the multipole expansion directly in the Lagrangian

$$\mathscr{L}=\int(\mathbf{jA}-\rho\varphi)\,d^3r \tag{23}$$

of the interaction with the external field of a system of point charged particles described by current and charge densities

$$\left.\begin{aligned}\mathbf{j}&=\sum_\alpha e^{(\alpha)}\dot{\boldsymbol{\xi}}^{(\alpha)}\delta(\mathbf{r}-\boldsymbol{\xi}^{(\alpha)})\\&=\sum_\alpha e^{(\alpha)}\dot{\boldsymbol{\xi}}^{(\alpha)}\exp(\boldsymbol{\xi}^{(\alpha)}\nabla)\,\delta(\mathbf{r});\\ \rho&=\sum_\alpha e^{(\alpha)}\exp(\boldsymbol{\xi}^{(\alpha)}\nabla)\,\delta(\mathbf{r}).\end{aligned}\right\} \tag{24}$$

Here, $\boldsymbol{\xi}^{(\alpha)}$ is the vector of the charged particle α, and for simplicity we shall assume $\sum_\alpha \mathbf{e}^{(\alpha)}=0$.

If we expand the exponential in a Taylor series, integrate by parts, and transfer the differentiation to the potentials $\mathbf{A}$ and φ, then after simple manipulations we can obtain

$$\mathscr{L}=\sum_\alpha\sum_{n=0}^{\infty}\frac{e^{(\alpha)}}{(n+1)!}(\boldsymbol{\xi}^{(\alpha)}\nabla)^n\times\left[\boldsymbol{\xi}^{(\alpha)}\mathbf{E}(0)+2\frac{n+1}{n+2}\boldsymbol{\mu}\mathbf{B}(0)\right], \tag{25}$$

where the fields (and their derivatives) are specified at the point $\mathbf{r}=0$ because of the $\delta(\mathbf{r})$ in the integrand of (23), and the magnetic dipole moment is $\boldsymbol{\mu}=\frac{1}{2}(\boldsymbol{\xi}\times\dot{\boldsymbol{\xi}})$. In the expression (25), the terms beginning with the quadrupole term have nonvanishing traces. If we want only the 2^l pole to contribute to the transition $0\to l$, we must make its trace vanish. In addition, a term with the magnetic field, beginning with $n=1$, contains for each n terms with opposite parities. Decomposing the product $(\boldsymbol{\xi}\cdot\nabla)(\boldsymbol{\mu}\cdot\mathbf{H})$ into symmetric and antisymmetric parts, separating the dipole terms, and shifting the sum, we reduce (25) to the form

$$\begin{aligned}\mathscr{L}=&\sum_\alpha e^{(\alpha)}[\boldsymbol{\xi}^{(\alpha)}\mathbf{E}+\boldsymbol{\mu}^{(\alpha)}\mathbf{B}]\\&+\sum_\alpha\sum_{n=0}^{\infty}\frac{e^{(\alpha)}}{(n+2)!}(\boldsymbol{\xi}^{(\alpha)}\nabla)^n\Big\{(\boldsymbol{\xi}^{(\alpha)}\nabla)(\boldsymbol{\xi}^{(\alpha)}\mathbf{E})\\&+\frac{n+2}{n+3}[(\boldsymbol{\xi}^{(\alpha)}\nabla)(\boldsymbol{\mu}^{(\alpha)}\mathbf{B})+(\boldsymbol{\mu}^{(\alpha)}\nabla)(\boldsymbol{\xi}^{(\alpha)}\mathbf{B})]\\&\quad+\frac{n+2}{n+3}(\boldsymbol{\xi}^{(\alpha)}\times\boldsymbol{\mu}^{(\alpha)})\operatorname{curl}\mathbf{B}\Big\}.\end{aligned} \tag{26}$$

The first term in the curly brackets corresponds to charge multipoles, the second, which interacts with $\mathbf{B}$, corresponds to magnetic multipoles, and both begin with quadrupoles. The third term, combined with the time derivatives of the traces of the charge moments, will lead to toroidal moments. We demonstrate this for the example $n=0$. For simplicity, we shall assume that $\mathbf{E}$ is a purely solenoidal field. For $n=0$, the last term in (26) is (we omit the index α)

$$\frac{1}{2}\cdot\frac{2}{3}(\boldsymbol{\xi}\times\boldsymbol{\mu})\operatorname{curl}\mathbf{B}=\frac{1}{6}[\boldsymbol{\xi}\times(\boldsymbol{\xi}\times\dot{\boldsymbol{\xi}})]\operatorname{curl}\mathbf{B}. \tag{27}$$

The octupole electric term is ($n=2$)

$$\frac{1}{6}\xi_i\xi_j\xi_k\nabla_i\nabla_jE_k. \tag{28}$$

From it we must subtract the trace, which is equal to

$$\frac{\xi^2}{5}(\xi_i\delta_{jk}+\xi_j\delta_{ik}+\xi_k\delta_{ij})\nabla_i\nabla_jE_k. \tag{29}$$

We subtract it from the octupole and, referring to Sec. 2, we recall that a contribution to the transverse part of the current will be made by only its last term

$$\frac{1}{5}\xi^2\boldsymbol{\xi}\Delta\mathbf{E}. \tag{30}$$

We transform this term, using the wave equation $\Delta\mathbf{E}=\ddot{\mathbf{E}}$, and we then transfer one derivative with respect to the time and, using the constraint $\dot{\mathbf{E}}=\operatorname{curl}\mathbf{B}$, reduce it to the form

$$-\frac{1}{5}\frac{d}{dt}(\xi^2\boldsymbol{\xi})\operatorname{curl}\mathbf{B}. \tag{31}$$

Adding (27) and (31), we obtain the dipole toroidal term in the Lagrangian:

$$-\frac{1}{10}[\boldsymbol{\xi}(\boldsymbol{\xi}\dot{\boldsymbol{\xi}})-2\xi^2\dot{\boldsymbol{\xi}}]\operatorname{curl}\mathbf{B}. \tag{32}$$

It follows from its form that the toroidal dipole can interact either with $\operatorname{curl}\mathbf{B}$, or with $\dot{\mathbf{E}}$, or with the external current $\mathbf{j}^{\text{ext}}$.

Applying this method of expansion with respect to point sources and decomposing the tensors $\xi_i\xi_j\ldots\xi_k$ into irreducible tensors, we can obtain the following expressions for the charge and current parts of the energy W of the interaction of an arbitrary electromagnetic system with an external electromagnetic field. For W^{ch}, we have

$$\begin{aligned}W^{\text{ch}}=&\int\rho\varphi d^3x=q\varphi+(\mathbf{d}\nabla)\varphi+Q_{ij}\nabla_i\nabla_j\varphi\\&+\overline{r_q^{(2)}}\Delta\varphi/6+Q_{ijk}\nabla_i\nabla_j\nabla_k\varphi+(\overline{\mathbf{r}_d^{(2)}}\nabla)\Delta\varphi/10+\ldots,\end{aligned} \tag{33}$$

where $q=\int\rho d^3\xi$ is the total charge of the system, $\mathbf{d}=\int\rho\boldsymbol{\xi}d^3\xi$ is the charge dipole moment, $Q_{ij}=\frac{1}{2}\int(\xi_i\xi_j-(\xi^2/3)\delta_{ij})\rho d^3\xi$ is the charge quadrupole moment,

$$Q_{ijk}=\frac{1}{6}\int\Big[\xi_i\xi_j\xi_k-\frac{1}{5}\xi^2(\xi_i\delta_{jk}+\xi_j\delta_{ik}+\xi_k\delta_{ij})\Big]\rho\,d^3\xi$$

is the charge octupole moment, $\overline{\mathbf{r}_q^{(2)}}$ and $\overline{\mathbf{r}_d^{(2)}}$ are the mean-square radii of the charge and the dipole: $\mathbf{r}_q^{(2)}=\int\xi^2\rho d^3\xi$ and

$$\overline{r_d^{(2)}}=\int\xi^2\boldsymbol{\xi}\rho\,d^3\xi,\ \text{and}\ \Delta\varphi=-\rho^{\text{ext}}.$$

Proceeding in the same way, we can obtain an expansion of $W^{\rm cur}$ in the form[7]

$$
\begin{aligned}
W^{\rm cur} &= -\int \mathbf{j}\mathbf{A}\,d^3x = \dot{\mathbf{d}}\mathbf{A} - \boldsymbol{\mu}\,[\nabla\mathbf{A}] \\
&+ \tfrac{1}{2}\dot{Q}_{ij}(\nabla_i A_j + \nabla_j A_i) + \dot{\overline{r_q^{(2)}}}(\nabla\mathbf{A})/6 \\
&+ \dot{Q}_{ijk}\nabla_i\nabla_j A_k - \mu_{ij}\nabla_i[\nabla\mathbf{A}]_j \\
&+ (\dot{\overline{\mathbf{r}_d^{(2)}}}\nabla)(\nabla\mathbf{A})/10 - \mathbf{T}[\nabla[\nabla\mathbf{A}]] - \ldots,
\end{aligned}
\tag{34}
$$

where $\boldsymbol{\mu} = \frac{1}{2}[\xi\dot{\xi}]$ is the magnetic dipole moment of the system, $\mu_{ij} = \frac{1}{3}\{[\xi\dot{\xi}]_i\xi_j + [\xi\dot{\xi}]_j\xi_i\}$ is the magnetic quadrupole moment, $\mathbf{T} = \frac{1}{10}\{\xi(\xi\dot{\xi}) - 2\xi^2\dot{\xi}\}$ is the toroidal dipole moment, $\mathbf{d} = \xi$; $\overline{\mathbf{r}_d^{(2)}} = 2\xi(\xi\dot{\xi}) + \xi^2\dot{\xi}$, etc.

It follows from the expressions of this section and from (34) that the toroidal dipole moment is the only characteristic having an interaction energy from which one can directly measure the displacement current. In practice, it could be measured on the basis of the torque of a small toroidal coil placed between the plates of a capacitor. We may also mention that this system is an electric motor of a third type (by the first two, we mean a magnetic dipole rotating in a magnetic field and an electric dipole rotating in an electric field).

4. DERIVATION OF MULTIPOLE EXPANSIONS IN A CANONICAL BASIS

Thus, the problem of expanding $\rho(\mathbf{x}, t)$ and $\mathbf{j}(\mathbf{x}, t)$ with respect to multipole moments and their radii has been reduced to the expansion of three scalar functions $\rho(\varphi)$, ψ, and χ. A system of basis functions suitable for this is provided by the solutions of the scalar wave equation of Helmholtz:

$$F_{lm\,k}(\mathbf{r}) = f_l(kr)\,Y_{lm}(\mathbf{n}), \quad \mathbf{n} = \frac{\mathbf{r}}{|\mathbf{r}|}, \tag{35}$$

where $Y_{lm}(\mathbf{n})$ are ordinary spherical harmonics, and $f_l(kr)$ are spherical Bessel functions:

$$f_l(kr) = (2\pi)^{3/2}\, i^l I_{l+1/2}(kr)/\sqrt{kr}. \tag{36}$$

For example, the expansion of the charge density $\rho(\mathbf{r}, t)$ then takes the form

$$
\begin{aligned}
\rho(\mathbf{r}, t) = (2\pi)^{-3}\int_0^\infty \sum_{l,m} (-ik)^l \frac{\sqrt{4\pi(2l+1)}}{(2l+1)!!} \\
\times F_{lm\,k}(\mathbf{r})\, Q_{lm}(k^2, t)\, k^2\, dk.
\end{aligned}
\tag{37}
$$

Because of the orthonormality of the functions $F_{lmk}(\mathbf{r})$, the definition of $Q_{lm}(k^2, t)$ follows from this directly.

The expansion of the charge multipole distribution (the form factor) $Q_{lm}(k^2, t)$ in a Taylor series in k^2 gives the 2^l total charge moment $Q_{lm}(0, t)$ and its mean $2n$-power radii $\overline{r_{lm}^{(2n)}}(0, t)$. We shall not here give their definitions or the definitions of the magnetic form factors $M_{lm}(k^2, t)$ [expansion of the function ψ in (15)] and the transverse electric form factors $E_{lm}(k^2, t)$ (expansion of the function χ). The derivation and definitions of these are given in Ref. 7.

Let us consider what happens if we substitute in (37) the expansion of $Q_{lm}(k^2, t)$ in a series in powers of $\mathbf{k}^2$. Bearing in mind that the $f_l(kr)$ contain $\sin kr$ and $\cos kr$, the integration over k in (37) does indeed give an expansion of the density $\rho(\mathbf{r}, t)$ in a series in δ-type functions (point sources), but now on an irreducible basis. Thus, the problem of multipole parametrization has been solved (cf. Sec. 1).

We recall that the transverse electric moments are the sum of two independent terms (for the explanation of this fact, see Ref. 7),

$$E_{lm}(\mathbf{k}^2, t) = \dot{Q}_{lm}(0, t) + \mathbf{k}^2 T_{lm}(\mathbf{k}^2, t), \tag{38}$$

where $\dot{Q}_{lm}(0, t)$ are associated with the expansion of $\dot{\rho}(\mathbf{j}_\parallel)$, i.e., the longitudinal-scalar part of the current, and are determined in the form

$$\dot{Q}_{lm}(0, t) = \sqrt{4\pi l}\int \mathbf{j}(\mathbf{r}, t)\, r^{l-1}\mathbf{Y}^*_{ll-1m}(\mathbf{n})\, d^3r. \tag{39}$$

We shall not give the definitions of the vector spherical function $\mathbf{Y}_{ll-1m}(\mathbf{n})$ introduced here, or of the vectors $\mathbf{Y}_{ll+1m}(\mathbf{n})$ and $\mathbf{Y}_{llm}(\mathbf{n})$. They can be found, for example, in the books of Ref. 28 and also in Ref. 7. Using Olsson's formulas, which are given in Ref. 28a, we can readily establish that the vector $r^{l-1}\mathbf{Y}_{ll-1m}(\mathbf{n})$ has remarkable properties. It is simultaneously longitudinal,

$$\operatorname{curl} r^{l-1}\mathbf{Y}_{ll-1m}(\mathbf{n}) = 0, \tag{40a}$$

and transverse,

$$\operatorname{div} r^{l-1}\mathbf{Y}_{ll-1m}(\mathbf{n}) = 0, \tag{40b}$$

i.e., it is degenerate with respect to these properties, and it is also poloidal, i.e., it can be represented in the form[6)]

$$r^{l-1}\mathbf{Y}_{ll-1m}(\mathbf{n}) = i\operatorname{curl}\hat{\mathbf{L}}\, r^l\, Y_{lm}(\mathbf{n}), \quad \hat{\mathbf{L}} = -i\,[\mathbf{r}\nabla]. \tag{41}$$

It is for this reason that in $\dot{Q}_{lm}(\mathbf{k}^2, t)$ and in $E_{lm}(\mathbf{k}^2, t)$, which contain in their definitions $\mathbf{Y}_{ll-1m}$ as the lowest harmonic, the moments $\dot{Q}_{lm}(0, t)$ are distinguished in the long-wave limit $|\mathbf{k}| \to 0$!

We give the definitions of the toroidal distributions[7)] and moments (for the derivation, see Ref. 7):

$$
\begin{aligned}
T_{lm}(\mathbf{k}^2, t) = \frac{i(2l+1)!!}{(-ikr)^{l+2}\sqrt{4\pi(2l+1)(l+1/l)}} \\
\times \int \mathbf{j}\operatorname{curl}\left[f_l^*(kr) - \frac{(2\pi)^{3/2}}{(2l+1)!!}(-ikr)\right]\mathbf{Y}^*_{llm}(\mathbf{n})\, d^3r;
\end{aligned}
\tag{42a}
$$

$$
\begin{aligned}
T_{lm}(0, t) &= -\frac{\sqrt{4\pi l}}{2(2l+1)}\int \mathbf{j} r^{l+1}\left\{\mathbf{Y}^*_{ll-1m}(\mathbf{n}) + \frac{2\sqrt{l/l+1}}{2l+3}\mathbf{Y}^*_{ll+1m}(\mathbf{n})\right\} d^3r \\
&= \sqrt{\frac{4\pi}{2l+1}}\frac{i}{2(l+1)(2l+3)}\int \mathbf{j}\operatorname{curl}\{r^{l+2}\mathbf{L}Y^*_{lm}(\mathbf{n})\}\, d^3r.
\end{aligned}
\tag{42b}
$$

It can be seen from the definition (42b) that $\mathbf{j}(\mathbf{r}, t)$ is indeed contracted with a poloidal vector, while the expression for $T_{1m}(0, t)$, transformed to the orthogonal basis, is identical to the definition of $\mathbf{T}$ in the form (21).

The toroidal moments T_{lm} contribute to the radiation of El type, like $\dot{Q}_{lm}$, but higher by two orders of magnitude with respect to kr. The corresponding general expressions for the vector potential of the radiation are given in Ref. 7. We note here a very important circumstance, namely, by comparing the definitions of $\dot{Q}_{lm}(0, t)$ (39) and $T_{lm}(0, t)$ (42b) we see that the second ("redundant") term in the first definition in (42b) of the toroidal moments does not permit us to regard them as mean radii of the charge moments $\dot{Q}_{lm}(0, t)$!

[6)]The uniqueness of this nonsingular vector can be proved directly by solving simultaneously the differential equations (40a) and (40b).

[7)]The second variant of Eq. (35a) in Ref. 7 is incorrect.

As follows from (38), the toroidal multipole moments $T_{lm}(t)$ have the form of small corrections as $k \to 0$ to the charge multipole moments $\dot{Q}_{lm}(0,t)$ of the same rank. However, in contrast to true corrections such as $\overline{Q_{lm}^{(2n)}}(t)$, which vanish if $Q_{lm}(-k^2, t) \equiv 0$, the toroidal moments, being independent parameters, do not vanish in this case. Moreover, the toroidal moments are in general nonzero if $\rho(\mathbf{r},t) \equiv 0$ or $\dot{\rho}(\mathbf{r},t) \equiv 0$. In other words, there are systems for which the representative of the family $T_{lm}(t)$ is the principal or only term of the multipole expansion (see the Introduction and Ref. 7). Therefore, multipole expansions[10,11,22] in which one sets $E_{lm}(k^2) = \dot{Q}_{lm}(k^2)$ are invalid even in the long-wave approximation. Treatment of the radiation of such systems by means of them in the long-wave approximation does not take into account the radiation effect itself. The neglect of $T_{lm}(t)$ compared with $\dot{Q}_{lm}(t)$ is analogous to the neglect of the higher multipole moments compared with the lower, which, of course, is valid only if the system has the lower moments.

It can be seen from (38) that the structure of the transverse electric form factors $E_{lm}(\mathbf{k}^2, t)$ does not permit their normalization in the usual manner, as magnetic and charge form factors in such a way that the long-wave approximation as k tends to zero gives static multipole moments.

We also give the well-known definition of the magnetic moments:

$$M_{lm}(0, t) = -\mathrm{i}\sqrt{\frac{4\pi l}{(2l-1)(l-1)}}\int \mathbf{j}(\mathbf{r}, t)\, r^l \mathbf{Y}_{llm}^*(\mathbf{n})\, d^3r$$
$$= \frac{1}{l-1}\sqrt{\frac{4\pi}{2l-1}}\int [\mathbf{r}\mathbf{j}]\, \nabla r^l Y_{lm}^*(\mathbf{n})\, d^3r. \qquad (43)$$

We shall need it later.

To conclude this section, we must point out that if, following Ref. 29, we were to restrict ourselves to an expansion of only div **j** and curl **j** with respect to the regular solutions $r^l Y_{lm}(\mathbf{n})$ of the Laplace equation, we would immediately lose the toroidal moments and the mean radii of all distributions. The point is that this system of functions is not closed,

$$\sum_{l,m} r^l Y_{lm}^*(\mathbf{n})\, r'^l Y_{lm}(\mathbf{n}') \neq \delta(\mathbf{r}-\mathbf{r}'), \qquad (44)$$

and, thus, we cannot "expand" $\delta(\mathbf{r}-\mathbf{r}')$ with respect to this basis in the expression (1). Therefore, the assertion made in Ref. 29 that the specification of the moments is equivalent to specifying the functions ρ and **j** is clearly false. For example, for a spherical capacitor all moments are equal to zero but the mean radii of the charge distribution are nonzero. Only the expansion of the densities ρ and **j** with respect to all moments and radii—apparently first made in Ref. 7—gives the complete parametrization of the electromagnetic 4-current.

We now suppose that the medium contains neither magnetic nor electric charges, and is described by the distribution of the macroscopic magnetization $\mathfrak{M}(\mathbf{r}, t)$. It has been known since Ampère's time that instead of $\mathfrak{M}$ one can introduce effective currents, known as induction currents, with density

$$\mathbf{j}_{\mathfrak{M}}(\mathbf{r}, t) = \mathrm{curl}\, \mathfrak{M}(\mathbf{r}, t). \quad (c=1). \qquad (45)$$

Note that div $\mathbf{j}_{\mathfrak{M}} = 0$. This indicates the absence in the medium of free magnetic charges and presupposes that the magnetization of the medium arises, say, as a result of a distribution of elementary (microscopic) magnetic dipoles.

What do we obtain if we replace **j** by $\mathbf{j}_{\mathfrak{M}} = \mathrm{curl}\, \mathfrak{M}$ in the definitions of the current multipole moments $\dot{Q}_{lm}$, T_{lm}, and M_{lm} [in Eqs. (39), (42b), and (43)]? First of all, it is obvious that all the $\dot{Q}_{lm}$ vanish identically by virtue of the condition (45), since the solenoidal current makes the integral (39) vanish. We now introduce the effective density of the magnetization distribution:

$$-\rho_{\mathfrak{M}} = \mathrm{div}\, \mathfrak{M}. \qquad (46)$$

If we now substitute $\mathbf{j}_{\mathfrak{M}}$ in (43), we readily see that after the corresponding integration by parts and the use of (46) the definition of the magnetic moments M_{lm} is transformed in its form into the definition of the moments Q_{lm}, in which the substitution $\rho \to \rho_{\mathfrak{M}}$ is made. However, this does not lead to a dual-symmetric scheme of multipole expansions, since div $\mathbf{j}_{\mathfrak{M}} \neq -\rho_{\mathfrak{M}}$, and therefore, as in ordinary electrodynamics, $M_{00} = 0$.

The situation with regard to T_{lm} is curious. Substituting (45) in the definition (42b), we can readily transform the latter to the form

$$T_{lm}^{\mathrm{ind}}(t) = -\mathrm{i}\sqrt{\frac{4\pi l}{(2l+1)(l+1)}}\int r^l\, \mathbf{Y}_{llm}^*(\mathbf{n})\, \mathfrak{M}(\mathbf{r}, t)\, d^3r. \qquad (47)$$

The form of this expression is identical to the definition of M_{lm} from the convection currents if in it the substitution $\mathbf{j} \to \mathfrak{M}$ is made. It follows from this, for example, that

$$\mathbf{T}^{\mathrm{ind}} = \frac{1}{2}\int [\mathbf{r}\mathfrak{M}]\, d^3r. \qquad (48)$$

It is readily seen that for such an origin the geometrical image of a toroidal dipole is simply a closed circular chain consisting of a succession of elementary magnets one after another. Their internal magnetic field has the same annular configuration as the field in a solenoid bent into a torus. In contrast to T_{lm}^{ch}, the moments T_{lm}^{ind} were already introduced by Blatt and Weisskopf.[8] They are usually denoted by the symbol Q'_{lm} and are called induced electric moments.[11]

All these remarks are rather trivial, but they help the geometrical understanding of the cumbersome, inconveniently formulated definitions of multipole form factors and moments given in books and reviews.

5. ELECTROMAGNETIC PROPERTIES OF STATIC AND VARIABLE TOROIDAL DIPOLES

Some of the properties of the dipole moment produced by poloidal currents on a torus are discussed in Ref. 7. We formulate them briefly here. First of all, a toroidal dipole does not have external static fields. Indeed, the magnetic fields of diametrically opposite elementary annular currents flowing on a torus compensate each other far from it (i.e., in the point approximation). Thus, the magnetic field produced by a toroid is entirely concentrated within the "coil" and the toroid can interact only with an external current. If a current line passes through a point at which there is an elementary toroidal dipole, the dipole will be oriented along the current (or, in an electromagnetic radiation field, along the displacement currents $\dot{\mathbf{D}}$).

Since the simplest model of a toroidal dipole is an ordinary solenoidal coil bent into a torus, we can readily find what is the toroidal dipole moment of a torus with small radius r and large R on the surface of which poloidal currents with density j flow. Instead of direct integration in Eq. (21), we use the analogy with the calculation of the magnetic moment of a thin ring, in which volume currents of density j flow. By means of the appropriate expressions, taken, for example, from Ref. 30, we can readily show that the magnetic moment of the closed current is

$$\mathbf{M} = js\mathbf{S} = I\mathbf{S}, \tag{49}$$

where **S** is the vector magnitude of the surface spanned by the contour of the conductor with section s, and I is the linear current flowing through it. Obviously, by virtue of the geometrical analogy, we can write down a similar expression for the toroidal dipole moment T of the toroidal solenoid merely by replacing I in (49) by the magnetic flux Φ within the solenoid, i.e.,

$$\mathbf{T} = \Phi\mathbf{S} = Bs\mathbf{S}. \tag{50}$$

The magnetic field B in the toroidal solenoid can be readily calculated (see, for example, Ref. 12):

$$B = \frac{2NI_0}{R}, \tag{51}$$

where R is the large radius of the solenoid, and I_0 is the current in its coil. Bearing in mind that $NI_0 = I$ is the total poloidal current on the surface of the solenoid, $s = \pi r^2$, $S = \pi R^2$, and that the volume of the torus is $V = 2\pi^2 R r^2$, we readily find that the static toroidal dipole moment of the toroidal coil is

$$T = JV. \tag{52}$$

This expression could be helpful as an approximation in macroscopic electrodynamics, for example, to estimate the global interaction of the magnetosphere of a planet with currents like those in the solar wind, or in the electrodynamics of neutron stars.

We now consider how the radiation problem is related to expansion of the current with respect to multipole moments. We recall that the multipole expansion arose as a procedure for finding parameters of a system of charges and currents that determine the radiation field of the system in the wave ("far") zone. Because of the transversality of electromagnetic waves, the vector potential $\mathbf{A}(\mathbf{r},t)$ far from the sources satisfies the subsidiary condition $\operatorname{div}\mathbf{A} = 0$. It follows from this in accordance with (15) that multipole radiation fields[8)] can be of only two types: electric $\mathbf{A}^{el}_{lm}$ and magnetic $\mathbf{A}^{m}_{lm}$. They differ in their parity properties under spatial reflections. However, a difficulty arises: It would seem that a transverse field $\mathbf{A}_\perp$ should arise only as a result of the transverse part of the current $\mathbf{j}_\perp$, which is associated with the magnetic and, as we have found, toroidal moments. We know, or, rather, have become accustomed to the fact that a radiation field also arises as a result of changes in $\rho(\mathbf{r},t)$, for example, by virtue of $\ddot{\mathbf{d}}(t)$. This problem was solved in its time in classical electrodynamics in an indirect way by Hertz. In the Lorentz gauge, $-\dot{\varphi} = \operatorname{div}\mathbf{A}$, it is assumed that the field φ is associated with an auxiliary dipole **Z** (see, for example, Refs. 22 and 23):

$$\varphi = \operatorname{div}\mathbf{Z}\ \text{(Hertz's ansatz)}. \tag{53}$$

Then the scalar wave equation for $\dot{\varphi}$ can be rewritten in the form

$$\left(\Delta - \frac{\partial^2}{\partial t^2}\right)\dot{\varphi} = -\dot{\rho} = \left(\Delta - \frac{\partial^2}{\partial t^2}\right)\operatorname{div}\dot{\mathbf{Z}} = \operatorname{div}\mathbf{j}, \tag{54}$$

and the Lorentz condition leads to the constraint

$$\operatorname{div}\mathbf{A} = -\dot{\varphi} = -\operatorname{div}\dot{\mathbf{Z}}. \tag{55}$$

Using (54) to find $\operatorname{div}\mathbf{Z}$ in the form of the well-known wave solution of the scalar equation, and then "canceling" the operation "div" on both the left- and right-hand sides, we obtain

$$-\dot{\mathbf{Z}}(\mathbf{R}) = \mathbf{A}(\mathbf{R}) = \int \frac{\mathbf{j}\left(\mathbf{r},\, t - \frac{R}{c}\right)}{|\mathbf{R}-\mathbf{r}|}\, d^3r \simeq \frac{1}{R_0}\int \mathbf{j}\left(\mathbf{r},\, t - \frac{R}{c}\right) d^3r \to \frac{\dot{\mathbf{d}}\left(t - \frac{R_0}{c}\right)}{R_0} \tag{56}$$

Since $\dot{\mathbf{d}}$ in this approximation is the characteristic of a point dipole situated, say, at the origin, $\operatorname{div}\mathbf{d}\,(t - R_0/c) = 0$, and we have indeed found the vector potential of the radiation field. This is also indicated by its decrease as $1/R$ with the distance.

It follows that in the point (long-wave) limit, which corresponds to the transition to approximate equality in (56), the moment $\dot{\mathbf{d}}$ naturally satisfies the properties of both longitudinality and transversality, since it loses its dependence on the vector **R**. Therefore, it can arise as a result of transition to the long-wave approximation in the multipole expansions of both $\mathbf{j}_\parallel$ and $\mathbf{j}_\perp$. This appears trivially if one goes over to the space of the wave vectors **k**. Then the conditions of longitudinality and transversality, $\mathbf{k}\cdot\mathbf{j}(\mathbf{k},t) = 0$ and $\mathbf{k}\times\mathbf{j}(\mathbf{k},t) = 0$, are satisfied simultaneously when $|\mathbf{k}| \to 0$. These, however, are very crude arguments and are suitable only for a dipole **d**. Moreover, this explanation is formal, since in concrete problems $|\mathbf{k}|$ is fixed and as a small parameter one can take only a dimensionless quantity, for example, kr_0, where r_0 is the characteristic linear dimension of the electromagnetic system. Therefore, we need a genuine relationship between $\mathbf{j}_\perp$ and $\mathbf{j}_\parallel$ in the coordinate space. Such a relationship was already proposed in 1959 in Ref. 31, and the justification and discussion of it still continues (see, for example, Refs. 32 and 33 and also Ref. 7).

That such a relation must exist follows from the very meaning of an expansion with respect to moments. We recall that moments are numerical (tensor) characteristics of an electromagnetic system that arise when its geometrical dimensions are reduced to zero and the magnitudes of its charges and currents are made infinite. At the same time, the densities of the charge and current distributions tend to infinity [cf. (3)] and the condition on the boundary of the system Σ, say $\mathbf{j}_\Sigma = 0$,

[8)]These should not be confused with the multipole moments in the expansion of a current (for details, see the expansions of A in Ref. 7 (Secs. 7 and 8) and in the monographs of Refs. 10 and 11).

can be satisfied only by equating the components

$$\mathbf{j}_\perp = -\mathbf{j}_\parallel \tag{57}$$

in such a way that, as before, $\mathbf{j}_\Sigma = 0$.

The condition (57) leads to equality of the lowest moments of $\mathbf{j}_\perp$ and $\mathbf{j}_\parallel$ (see, for example, the corresponding derivations in Refs. 31 or 32), and, thus, time derivatives of the charge moments appear in the expansion of $\mathbf{j}_\perp$. However, the toroidal moments and mean $2n$-power radii of scalar-longitudinal distributions, which are sensitive to the details of the structure of the system within the surface Σ, are different for $\mathbf{j}_\parallel$ and $\mathbf{j}_\perp$, since within the system the condition (57) is not satisfied. We refer those wishing for more rigorous discussions of the relation (57) to Refs. 32 and 33.

It is easy to work out what are the properties of a toroidal dipole antenna. Since it is a closed system, the electric field (or rather the displacement current $\dot{\mathbf{D}}$) can be separated from it only when there are very abrupt changes of the current in it. Otherwise a quasi-steady process is established and the field will be drawn (return) into the winding of the torus because of its self-induction.

We obtain the radiation intensity of the toroidal dipole antenna by means of the expression that determines the $E1$ radiation with allowance for the existence of $\mathbf{T}$:

$$\mathbf{E} = \mathbf{d} + \omega^2\mathbf{T} \quad (c = 1). \tag{58}$$

It is well known that the power (intensity) of the radiation of a charge dipole $\mathbf{d}$ (more precisely, of a linear antenna of length $|\mathbf{d}|$ with current $I = I_0 \sin\omega t$ that decreases linearly at its ends; see, for example, Ref. 11, p. 301 of the Russian translation) in the long-wave approximation, $\omega d \ll 1$, is

$$P_d \sim |\ddot{\mathbf{d}}|^2 \sim \omega^4 q^2 l^2 = \frac{q^2\omega^2}{\lambda^2} l^2 = I_0 \left(\frac{l}{\lambda}\right)^2, \tag{59}$$

where the current I_0 is related to the change in the effective charges in the antenna, $I_0 = dq/dt$, and $\lambda \sim 1/\omega$ is the wavelength emitted by the antenna.

It follows formally from the form of the expressions (58) and (59) that the radiation power of the toroidal dipole T can be estimated as follows:

$$P_T \sim \omega^2\omega^4 T^2 \sim \frac{1}{\lambda^6} I_0 (Rr^2)^2 = I_0 \left(\frac{Rr^2}{\lambda^3}\right)^2, \tag{60}$$

where we have used (52) for the static moment $\mathbf{T}$. Thus, a toroidal dipole is in principle a two-parameter antenna, i.e., an antenna that depends on two of its geometrical parameters. We note that a toroidal dipole could actually be an effective antenna only in the short-wave regime (for $\omega R \gg 1$) because of the remark with which we began our discussion of the radiation of $\mathbf{T}$.

The angular distribution of the emitters d and T (which have parallel orientation) is naturally the same. The exact expressions for the vector potentials of the radiation by the point sources d and T are given above. They justify the dependences of P_d and P_T on the radiation frequency ω used in (59) and (60).

We also give here expressions for the field intensities of the radiation of an arbitrary system, accurate to the terms in kr, including the toroidal dipole contribution:

$$H_i = \frac{1}{R_0}\Big[\ddot{d}_i + \dddot{Q}_{ik}n_k + [\ddot{\mathbf{M}}\mathbf{n}]_i + \dddot{Q}_{ijk}n_j n_k + \dddot{T}_i + \varepsilon_{ijk}\dddot{M}_{jl}n_k n_l\Big], \quad \mathbf{E} = [\mathbf{n} \times \mathbf{H}], \quad \mathbf{n} = \frac{\mathbf{R}_0}{|\mathbf{R}_0|} \tag{61}$$

and also the expression for the radiation intensity to terms $\sim(kr)^5$:

$$I = \frac{2}{3}(\ddot{\mathbf{d}})^2 + \frac{1}{20}\dddot{Q}_{ik}^2 - 2\ddot{\mathbf{d}}\,\dddot{\mathbf{T}} + \frac{2}{3}\ddot{\mathbf{M}}^2. \tag{62}$$

In such form, these expressions were obtained recently[34] on the basis of the invariance of the moments with respect to Galilean displacements of the origin (without reference to the geometrical image of $\mathbf{T}_1$).

If a radiation source is constructed in such a way that it does not have charge and magnetic dipole and quadrupole moments, the dominant radiation of such a source will be radiation of $E1$ type generated by its toroidal dipole moment. A torus with alternating current has such a property.

6. TOROIDAL MOMENTS AND ELECTROMAGNETIC TRANSITIONS IN ATOMS

In classical electrodynamics, a definite, geometrically simple configuration of charges or currents can be described approximately by a moment of multipolarity l. The currents of electromagnetic transitions in quantum systems have a rather complicated structure, and in their multipole expansion all moments usually appear. Therefore in practical calculations of transition probabilities it is customary to go over to the long-wave approximation and take into account the contributions of only the lowest moments.

Since the contributions T_{lm} to transitions of electric type contain an additional factor $(kr)^2$ compared with the contributions $\dot{Q}_{lm}$, the former can be ignored in the majority of cases. They become important when, for example, the long-wave approximation becomes a rough approximation ($kr \lesssim 1$). Such is the situation for x-ray transitions in heavy nuclei. If for simplicity we consider a $1S \to 2P$ transition in a hydrogenlike ion, then with allowance for the contribution of the toroidal dipole the transition probability is

$$W_{2P1S} \sim |\ddot{\mathbf{d}}|\left(1 + \frac{\hbar\omega_{1S2P}}{mc^2}\right) = |\ddot{\mathbf{d}}|^2\left(1 + \frac{Z^2\alpha^2}{3}\right). \tag{63}$$

It can be seen from this that the correction for the toroidal dipole moment (and its radii) can reach tens of percent for heavy elements. These corrections become even more important for transitions to higher levels or for ionization processes [when $\hbar\omega$ is replaced in (63) by the ionization potential I], and also for many-photon ionization processes (when $n\hbar\omega \simeq I$, where n is the number of quasistationary levels produced in the atom by an external alternating electric field). Similar corrections must be taken into account in the problem of the splitting of levels in an intense electric field (Stark effect in an alternating field). This effect was considered for the first time by Blokhintsev in 1933,[35] and the possibility of its experimental study appeared only with the creation of sufficiently powerful single-mode lasers.

It is well known that the additional energy of atoms in homogeneous electric, $\mathbf{E}$, and magnetic, $\mathbf{B}$, fields,

$$\Delta W_{\text{el}} = -\mathbf{d}\cdot\mathbf{E}; \tag{64}$$

$$\Delta W_{\rm mag} = - \mathbf{M}\cdot\mathbf{B}, \qquad (65)$$

leads to splitting of spectral lines (Stark and Zeeman effects). Similarly, the existence of the additional energy of atoms in a homogeneous external current, which is

$$\Delta W_{\rm cur} = - \mu_0 \mathbf{T}\,(\mathbf{j} + \dot{\mathbf{D}}), \qquad (66)$$

also leads to splitting of spectral lines. The expression (66) contains in addition to the external conduction current j the external displacement current $\dot{\mathbf{D}}$, which interacts with the toroidal dipole moment T like j.

For example, the atoms and ions of a plasma in a gas-discharge tube interact in this manner with the current flowing in the plasma.

As we noted earlier, the contact interaction of an atom (or ion) with a current in a plasma can in principle be particularized and then expressed in terms of the Liénard–Wiechert potentials describing the field of the current carriers. A similar interaction (with the displacement current) arises when atoms or ions enter a rapidly varying electric field (rapid flight through a capacitor, powerful high-frequency electromagnetic field).

A toroidal effect due to the inhomogeneity and nonstationarity of the field arises almost always with the Stark and Zeeman effects, although its presence and magnitude do not depend on the latter.

The splitting of the spectral lines as a result of the "toroidal" interaction, like the Stark effect, is in general an effect of second order in perturbation theory, and it is only for hydrogen (and hydrogenlike ions) that there is a first-order effect due to the degeneracy of the spectrum with respect to the orbital quantum number. It is interesting to compare the toroidal effect with the Stark and Zeeman effects for, for example, the transition from the first excited state of the hydrogen atom to the ground state. The splitting of the spectral lines in these cases can be estimated by means of the expressions

$$\Delta W_{\rm el} = 3ea_0E;\ \Delta W_{\rm mag} = \mu_0 B;\ \Delta W_{\rm cur} = \mu_0 a_0 j, \qquad (67)$$

where a_0 is the Bohr radius, and μ_0 is the Bohr magneton. It is not difficult to propose an idea for an experiment to measure the toroidal effect, but the actual experiments may be difficult, since the effect arises on the background of the Stark or Zeeman effects.

The splittings due to toroidal moments arise only in transitions in which the parity changes. If one takes into account the violation of P invariance due to the presence of weak interactions, then in the first order in the constant G a splitting of the terms of all atoms arises.[36] However, because the coupling constant of the weak interactions is small, this effect is much smaller than the second-order effects mentioned above (by a factor α/G).

It should be noted that the corrections from the transition toroidal moment may be large in, for example, the calculation of exciton transitions in semiconductors, in which the electron mass becomes an effective mass and may be reduced by an order of magnitude or more.

It is also important to take into account the higher multipoles when one is finding the influence of the internal field on the spectra of molecular crystals.[37]

To conclude this section, we note that it is the toroidal dipole moment that gives the well-known "retardation correction" (see, for example, Ref. 38) to the Thomas–Reiche–Kuhn sum rule. This was pointed out in Ref. 39. The first correction to the matrix element of the dipole operator arises if one takes into account in the expansion of $\exp(ikr)$ not only the unity but also the term $(kr)^2$. For example, if the photon propagates along the z axis and is polarized along the y axis, the retardation corrections to the oscillator strength f_{0n} are

$$f^R_{0n} = -\frac{2m\omega}{\hbar c^2}\,(yz^2)_{0n}\,(y)_{0n}. \qquad (68)$$

For the ground S state the corrections to the sum of the oscillator strengths can be found by means of the expression

$$\sum_n f^R_{0n} = -\frac{1}{mc^2}\left(z^2\frac{\partial^2 V}{\partial y^2}\right)_{00} = -\frac{1}{15mc^2}\left(r^2\frac{d^2V}{dr^2} + 4r\frac{dV}{dr}\right)_{00}, \qquad (69)$$

where $V(r)$ is any potential that commutes with the radius vector r. It is known[38,40] that it is precisely this term that interferes destructively with the contribution of the charge quadrupole. It was shown in Ref. 39 that the correction (69) arises because of the interference of the charge and toroidal moments [cf. the interference term in Eq. (62)]. However, besides the convection part of the toroidal moment, corrections to the sum rule can also arise from its induction part. The complete expression for the operator of the toroidal dipole moment is given in Ref. 39:

$$\mathbf{T} = \frac{i}{5}\,r^2\nabla + \frac{k^2}{5}\left[\mathbf{r}\times\left(g_l\mathbf{L} + \frac{5}{2}g_\sigma\boldsymbol{\sigma}\right)\right]. \qquad (70)$$

It should be pointed out that in some papers the orbital part of this operator was introduced by analogy with its "induced" spin part, as, for example, in Ref. 40, since the general definition (21) of the toroidal dipole moment was not known. To conclude this section, we also note that the contribution of the toroidal dipole moment can be important as well in more complicated sum rules such as those considered in Ref. 41.

7. CONTRIBUTION OF TOROIDAL MOMENTS TO THE PROBABILITIES OF ELECTRIC TRANSITIONS IN NUCLEI

The replacement of $E_l(k^2)$ by $kQ_l(k^2)$, which is frequently practiced in the calculation of nuclear transitions (Siegert's theorem), may not give the necessary accuracy in a number of cases. It is then necessary to take into account the higher terms in the expansion of $E_l(k^2)$, i.e., in the first place $T_l(k^2)$. Such a situation can arise when γ transitions in heavy nuclei are considered.

It is well known that in heavy nuclei the intensities of some transitions are strongly suppressed by virtue of structural features that participate in transitions of states. For example, in the transition ^{175}Lu $(\frac{9}{2}^-\frac{9}{2} \to \frac{7}{2}^+\frac{7}{2})$ the intensity of the $E1$ multipole is suppressed so much that the contribution of the multipole next in seniority, $M2$, is anomalously large (see, for example, Ref. 42), being about 20% (calculations[18] in Nilsson's model give satisfactory agreement with experiment for the probability of the moment M_2, whereas the probability of the moment Q_1 in this model exceeds its experimental value by two orders). It follows from the form of the expressions (61) and (62) and from the comment made after them that in the calculation of hindered $E1$ transitions it is necessary to take into account the contribution of the moment T_1.[9)] Then the probability of the multipole transition $E1$ can be estimated by means of the formula

$$E1 \sim \dot{Q}_1^2 + 2k^2\dot{Q}_1 T_1 + k^4 T_1^2, \tag{71}$$

and the contribution of the last two terms for the transition in ^{175}Lu reaches almost 40%.[18] Even if the $M2$ contribution in the transition were only 1%, the contribution of the toroidal part would be about 10%. From this a rule can be deduced: If in the tables of electromagnetic transitions in nuclei there are transitions in which the $M2$ contribution is greater than or of order 1%, the contribution associated with the moment T_1 must be taken into account in the calculation of such transitions. Note, however, that a T_1 contribution may also exist in transitions in which $M2$ is forbidden by the selection rules, for example, in the transition $^{19}\mathrm{F}\ (\frac{1}{2}^- \to \frac{1}{2}^+)$.

We note that the toroidal currents of nuclear transitions have in fact long been studied in the phenomenon of internal conversion of γ rays. Indeed, conversion is based on the interaction of two currents—an electron current, with known properties, and the studied transition current of the nucleus. As we already know, it is the toroidal part of the parametrized current that interacts with the external current, in this case the electron current. The nuclear penetration parameters introduced in Ref. 19a are essentially identical to our definition of toroidal moments. Recently, in Ref. 19b, toroidal transition moments of some nuclei were extracted from experimental data on internal conversion. In the units (nuclear magneton) · F they are (in absolute magnitude) as follows[19b]: 4.0 ∓ 1.0, 3.0 ∓ 0.4, 4.5 ∓ 0.5, 2.1 ∓ 0.3, 1.6 ∓ 0.4 for the transitions ^{177}Lu (150 keV), ^{175}Lu (396 keV), ^{181}Ta (6.21 keV), ^{177}Hf (321 keV), ^{233}Pa (84.3 keV), respectively, and less for other nuclei. The experimental values of the contributions of the toroidal distributions to the probability of γ emission are, respectively, $(10 \mp 2)\%$, $(-17 \mp 2)\%$, $(0.20 \mp 0.02)\%$, $(-18 \mp 2)\%$, $(1.9 \mp 0.3)\%$.

We now recall that the study of hindered transitions helped to prove the existence of weak nucleon–nucleon interactions (see, for example, the review of Ref. 43). The point is that in a hindered transition with regular (allowed) multipole $E1$ there is effectively mixed a small irregular multipole $\tilde{M}1$ ($\sim G_F$) due to P-odd nucleon–nucleon interactions. At the same time, as is well known, the γ ray emitted in the transition is circularly polarized. The coefficient of polarization P_γ takes in this case, with allowance for the contribution of the toroidal moment, the form

$$P_\gamma = \frac{2}{1+\left|\frac{M_2}{\dot{Q}_1 + k^2 T_1}\right|^2} \frac{\tilde{M}_1}{\dot{Q}_1 + k^2 T_1}. \tag{72}$$

We see that in principle, if $\dot{Q}_1$ and T_1 have opposite signs, their relative magnitude will determine the sign of P_γ, which determines whether the γ ray will have left- or right-handed circular polarization. In a comparison of the experimental values of P_γ with theoretical values, only $\tilde{M}_1$ in (72) is calculated, while the values of $|M_2/E_1|^2$ and $\sqrt{|\dot{Q}_1 + k^2 T_1|^2}$ are usually taken from (independent) experiments. However, this conceals a certain danger. One should make a theoretical calculation of all the moments on the right-hand side of (72). First, this would make it possible to determine the theoretical sign of P_γ. Second, nuclear models for the calculation of electromagnetic transitions could, for example, give reasonable agreement with experiment for moments whose definitions include spin operators but bad agreement for moments that do not depend on the spin ($\dot{Q}_1 \sim \mathbf{r}$). Therefore, to be sure that in calculating $\tilde{M}_1$ we do not make a mistake in the nuclear part, we must choose a nuclear model that gives satisfactory values of $|M_2/\dot{Q}_1 + k^2 T_1|^2$ and $|\dot{Q}_1 + k^2 T_1|^2$.

It follows from the form of the expression $E(k^2) = -i\omega Q_1 + k^2 T_1$ that in scattering of electrons by nuclei at small excitations ω but large transfers k^2 a situation may arise in which it is necessary to take into account the toroidal moments of the considered transition.[44]

We introduce single-particle operators of an $E1$ transition in the form

$$\mathbf{E}_1 = -i\omega\,\mathbf{Q}_1 + k^2\,\mathbf{T}_1 = -i\omega\mathbf{r} + \mu_0 k^2 \left\{ i\frac{r^2}{5}\nabla + \mathbf{r}\times\left[\frac{g_l\mathbf{L}}{5} + \frac{g_\sigma\boldsymbol{\sigma}}{2}\right]\right\} \tag{73}$$

and the single-particle operator of a magnetic quadrupole transition

$$M_{2m} = \mu_0 k^2 \left(g_\sigma\boldsymbol{\sigma} + \frac{2}{3} g_l \mathbf{L}\right) \nabla (r^2 Y^*_{2m}) \tag{74}$$

in the standard notation (see, for example, Ref. 8 or Ref. 38).

Because T_1 and M_2 have the same order in kr, the numerical values of their contributions may be about the same.

We now note that because of the existence of the toroidal moments the structure of the electron–nucleus scattering cross section changes. In the case of a $3/2^- \to 1/2^+$ transition, the cross section takes the form

$$d\sigma = d\sigma_{\mathrm{Mott}} \frac{k^2}{2} \left\{ Q_1^2 + \frac{2}{5} k^2 \left(\frac{1}{2} + \tan^2\theta/2\right) \left(T_1^2 + \frac{3}{5} M_2^2\right)\right\}. \tag{75}$$

It can be seen from (75) that in the given case the traditional opinion about the possibility of separating the contributions of the charge and magnetic moments on the basis of the angular distributions is incorrect. If we ignore the existence of T_1, then attempts to estimate M_2 from the magnitude of the cross section (75) at large values of $\tan^2(\theta/2)$ would lead to an overestimation of M_2 by a factor of about 2. In Ref. 44, there is an analysis of all possibilities (complete experiment) of actually making a separate measurement of the moments of the $3/2^- \to 1/2^+$ transition. Also calculated are all the form factors of the $3/2^- \to 1/2^+$ transition of the ^{9}Be nucleus in the framework of the shell model. We note that a situation analogous to the one we have considered oc-

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[9)] This conclusion can also be drawn by comparing the more perspicuous expressions (6) and (7) of Ref. 18 (because of the similarity of the structure and the same order in kr of the operators $\hat{T}_1$ and $\hat{M}_2$).

curs in the case of electromagnetic decays of particles with γ-ray conversion (for example, $N^{**}_{1520} \to pe^-e^+$). The effect is actually strengthened here because of the large value of the parameter $kr_0 \sim (M_{N^{**}} - m_p)/(M_{N^{**}} + m_p)$.

Naturally, toroidal moments will contribute to giant electric multipole resonances. Estimates of these contributions were made in the pioneering study of Ref. 45 by means of a hydrodynamic model of the nucleus, in which it has a torsion modulus to describe the resistance to shear displacements. In such a system, there are solenoidal toroidal vibrations, giving a toroidal dipole resonance in the region of energies $(50-70)A^{1/3}$ MeV. The study of solenoidal and also spin waves in nuclei would be extremely helpful, for example, for establishing the isospin structure of nuclear forces.

8. DIPOLE RADIATION OF A COMPOSITE QUANTUM SYSTEM AND THE POSSIBILITY OF SEPARATING THE CONTRIBUTIONS OF THE CHARGE AND TOROIDAL DIPOLE MOMENTS

For a system of two particles with equal masses, for example, NN or $N\bar{N}$, the operators $\dot{\mathbf{Q}}$, $\mathbf{T}$, and $\mathbf{M}$ in the center-of-mass system are described as follows[20]:

$$\left.\begin{aligned}
&\dot{Q}_1 = -\frac{1}{M_N}(e_1 - e_2)\,\mathbf{p};\\
&\mathbf{T} = \frac{1}{10M_N}(e_1 - e_2)\frac{\mathbf{r}^2}{4}\mathbf{p} - \mu_0\frac{1}{4}\mathbf{r}\\
&\times\left[\frac{\mu_1-\mu_2}{2}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2) + \frac{\mu_1+\mu_2}{2}(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2) + \frac{2}{5}\,\frac{g_1-g_2}{2}(\mathbf{L}_1+\mathbf{L}_2)\right];\\
&\mathbf{M} = \mu_0\left[\frac{\mu_1+\mu_2}{2}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2) + \frac{\mu_1-\mu_2}{2}(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2) + \frac{g_1+g_2}{2}(\mathbf{L}_1+\mathbf{L}_2)\right].
\end{aligned}\right\} \quad (76)$$

Here, $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and $\mathbf{p} = \mathbf{p}_1 - \mathbf{p}_2/2$ are the relative coordinate and momentum; μ_i and g_i are the spin and orbital magnetic moments of particle i, measured in nuclear magnetons μ_0.

To illustrate the possibility of separating the contributions of the charge and toroidal dipole moments, we consider the specific hadronic system $\bar{p}n$ ($T=1$, $T_3=-1$), in which there is a transition of electric type ${}^{33}P_1 \to {}^{31}S_0$.[20] In this case,

$$\begin{aligned}
&e_1 - e_2 = -e;\quad \mu_1 + \mu_2 = -\mu_p + \mu_n;\\
&\mu_1 - \mu_2 = -\mu_p - \mu_n;\quad g_1 + g_2 = -1.
\end{aligned} \quad (77)$$

Since the operator of the charge dipole does not depend on the spin [see the expression (76)] in the transition we consider (a spin-flip transition: $S_i = 1 \to S_f = 0$), the charge dipole automatically drops out. Thus, we have an $E1$ transition that takes place solely as a result of the operator of the toroidal dipole moment. Analyzing further the spin selection rules, we see that (for the same reason) the matrix element of the considered transition also does not contain the radial and orbital (the operator $[\mathbf{r}\times(\mathbf{L}_1 + \mathbf{L}_2)]$) parts of the toroidal moment. In addition, in the given case, $S_i = 1 \to S_f = 0$, the operator of the total spin $\sigma_1 + \sigma_2$ also drops out. Thus, in the $E1$ transition ${}^{33}P_1 \to {}^{31}S_0$ ($\Delta S = 1$) only the operator $\sigma_1 - \sigma_2$ survives, i.e., the transition takes place solely as a result of the spin part of the operator of the toroidal moment.

Thus, it follows from this example that the sum rules (in the given case with respect to the spin) can separate the contributions of the charge and toroidal moments (in the very simple example considered, a contribution of the charge dipole is simply absent and only part of the toroidal moment is effective).

The contributions of the toroidal dipole moments to electric transitions can also be appreciable in systems of quarkonium type, for example, in $E1$ transitions in charmonium: $\Psi'(3684\ \mathrm{MeV}) \to \chi_J + \gamma\ ({}^3S_1 \to {}^3P_J)$. We have obtained estimates giving simple oscillator functions giving contributions to the transition probabilities up to 10%. This is evidently a lower bound, since the potential that confines the c quarks in charmonium binds them more strongly than the harmonic potential.

The potential mechanism of parity nonconservation in hadronic systems makes it possible to estimate the corresponding effects in the radiation of a quasinuclear system. For example, because of the mixing by weak interactions of an $M1$ transition with the $E1$ transition ${}^{33}P_1 \to {}^{31}S_0$, which takes place because of the toroidal dipole, the circular polarization of the photon may reach $P_\gamma \sim 10^{-6}$.

By means of (76) and parity-violating potentials one can estimate the nuclear toroidal dipole moment of the deuteron. The result agrees with the estimate given in Ref. 9. We note that it is more convenient to use the expression (76) for the toroidal dipole than the definition $\mathbf{a} \sim \int r^2 \mathbf{j} d^3r$ of the "anapole," since the contribution of the longitudinal part of the current is automatically eliminated in (76), as in (21).

9. TOROIDAL MOMENTS AND VIOLATIONS OF DISCRETE SYMMETRIES

In relativistic quantum theory, the electromagnetic properties of a particle are given by the matrix elements of the operator of the 4-current. These matrix elements can be represented in terms of a limited number of form factors—phenomenological invariant functions of scalar variables that do not depend on the arbitrariness in the choice of the coordinate system. In special frames, this problem can be solved in an implicitly covariant form by means of the formalism of multipole expansions (see Ref. 7, Part II). Thus, for a particle with spin s in local field theory there arise three sets of form factors, including toroidal ones.

The law of transformation of the current operator under spatial and time reflections (P and T) imposes additional sum rules and restrictions on the form factors (in the general case, they are formulated in Ref. 7). Whereas the current in classical electrodynamics can, in general, have any multipole moment, in the quantum case (since the operators of the multipole moments are spin operators) this is not so. For example, the vector of the dipole moment of a point particle may be directed solely along the spin. The spin and magnetic dipole moment are pseudovectors, while the charge and toroidal dipole moments are true vectors. Therefore, under the condition of conservation of spatial parity the particle can possess only one dipole form factor—the magnetic—and its charge and toroidal dipole form factors are identically equal to zero.

The toroidal form factors have unusual selection rules with respect to the transformations of the discrete symmetries, distinguishing them from the charge and magnetic form factors. For a point particle one cannot have any charge form factors of odd ranks or any magnetic form factors of even ranks simultaneously with respect to both P and T. Toroidal moments of odd ranks are absent in the presence of P invariance, while in the presence of T invariance those of even ranks are absent. If we now remember that weak interactions, which violate P invariance, contribute to the form factors of particles, all particles with nonzero spin must in general have small (in magnitude) toroidal moments of odd ranks. Even toroidal moments may exist as a manifestation of the mechanism of violation of T invariance, which has not yet been precisely established. Thus, measurement of the toroidal form factors would be interesting for the study of the nature of the violation of discrete symmetries. For this reason, it was proposed in 1965 to make experiments to measure the toroidal quadrupole moment of the deuteron.[13,14] However, the experiments following this at Stanford[46] did not achieve the necessary degree of accuracy to discover the expected effect.[13] This experiment should evidently be repeated at the present level of accuracy in connection with the appearance of indications of violation of T invariance in nuclear reactions.

As we have already noted, following the discovery of parity violation in weak interactions in 1957 Zel'dovich pointed out[5] that particles with spin $\frac{1}{2}$ must possess not only a magnetic moment but a further dipole characteristic. It was called the anapole and introduced into the electromagnetic current as follows:

$$j_\mu \sim e\bar{u}(\mathbf{p}_1)\, a_\mu \gamma_5 G^{(a)}(q^2)\, u(\mathbf{p}_1), \tag{78}$$

where

$$a_\mu = \frac{1}{\varkappa^2}(q^2\gamma_\mu - \hat{q}q_\mu),\ \ q = p_2 - p_1,\ \ P = p_1 + p_2,\ \ \varkappa = \varkappa_1 + \varkappa_2.$$

In addition to this, a form factor is introduced in the transition (with respect to the parity) current and called the dipole electric (charge) form factor:

$$j_\mu \sim e\bar{u}(\mathbf{p}_2)\,\frac{1}{\varkappa}\sigma_{\mu\nu}q_\nu\gamma_5 G^{(\mathrm{d})}(q^2)\, u(\mathbf{p}_1). \tag{79}$$

It is easy to show[7] that, in general, none of these form factors corresponds to a definite multipole distribution. Indeed, in the proper frame (which is determined by the absence of a Doppler shift and an emitted photon[7]) the charge dipole moment found from this current is

$$d \sim \frac{e}{\varkappa}\left[G^{(\mathrm{d})}(\Delta\varkappa^2) + \frac{\Delta\varkappa}{\varkappa}G^{(a)}(\Delta\varkappa^2)\right]. \tag{80}$$

It can be seen from (80) that only in the static case ($\varkappa_1 - \varkappa_2 \equiv \Delta\varkappa = 0$) can $G^{(\mathrm{d})}(0)$ be called a charge dipole moment. However, relativistic form factors are not actually measured quantities on the mass shell.

We recall that for Dirac particles the following identity holds:

$$\bar{u}_2(\Delta\varkappa\sigma_{\mu\nu}q_\nu + a_\mu\varkappa - R_\mu\gamma_5)\gamma_5 u_1 = 0, \quad R_\mu = i\varepsilon_{\mu\nu\lambda\rho}P_\nu q_\lambda\gamma_\rho. \tag{81}$$

It follows from this that an alternative parametrization can contain in place of a_μ the vector R_μ, and, as we see, it is only for $\Delta\varkappa = 0$ that these vectors are identical. It was shown in Ref. 7 that the parametrization corresponding to a multipole parametrization can be made by means of the vector R_μ, like the Sachs multipole parametrization for the case of a vector current by means of $R_\mu\gamma_5$. In the proper frame, this parametrization has a rather complicated form.[7]

We restrict ourselves here to a discussion of the main conclusion of Ref. 7 relating to the properties of the anapole. In the general case, it is not in fact a multipole quantity. It belongs to the class of so-called nonradiating sources,[47] for which the Fourier transform of the distribution density in the space of wave vectors vanishes on the light cone (in this case, because of the factor q_μ^2). The fields produced by nonradiating sources are equal to zero at least in the order $1/r$, where r is the distance from the source to the point of observation. A source corresponding to parametrization of the current by means of the vector R_μ does not contain such suppressions. Moreover, the combination $d_1\sigma_{\mu\nu}q_\nu\gamma_5 + T_1R_\mu$, in which we retain for the radiation problem by means of (81) only the radiating part $(d_1 + \Delta\varkappa T_1)\sigma$, has the structure of transverse electric moments (see Ref. 7).

In modern gauge theories combining electromagnetic and weak interactions, a toroidal dipole (an anapole) arises as a radiative correction due to virtual weak interactions. A calculation of the toroidal dipole form factor of a lepton in the Weinberg–Salam model in the lowest (third) order of perturbation theory was begun in Ref. 15a and by our diploma student A. A. Chepkasov. However, it has not been possible to complete the calculations because of the presence of divergent integrals in the unitary gauge. The point is that, despite the renormalizability of the theory, the divergences from the triangle diagrams that determine the anapole cancel in the unitary gauge only together with the divergences from the diagrams of "box" type. Thus, only estimates of the scale of magnitude of the anapole were made.[15]

In the 't Hooft–Feynman gauge, the contributions of the triangle diagrams are finite. In Ref. 16, the quark anapole moment is estimated in the single-loop approximation with logarithmic accuracy (corrections $\sim\alpha\ln M_W^2/q^2$). The finite and power-law contributions are omitted, and there is no discussion of the gauge invariance of the sum of the amplitudes that determine the magnitude of the anapole. The renormalization-group formalism is used to find the gluon corrections to this quantity in the single-loop approximation, which change the values of $G^{(a)}(q^2)$ by several times. On the basis of these calculations, the anapole corrections to the parity-violating effects in the atomic experiments of Ref. 9 are estimated in Ref. 16.

Interesting in this situation are attempts to deduce the anapole moments of leptons and quarks directly from polarization experiments in electron–nucleon and electron–deuteron scattering. With fitting of the experimental values for the asymmetry of inelastic scattering of longitudinally polarized electrons by unpolarized targets[17a] by the expressions obtained in the single-photon approximation with allowance for the electron anapole, the estimate $a_e = 6\times10^{-12}$ is obtained. The "natural" estimate is $a_e \sim G_F m_e^2 = 2\times10^{-12}$. The anapole moments of quarks have been estimated[17b] by analyzing the asymmetry of deep inelastic scattering of polarized electrons by unpolarized nucleons in the framework of the quark–parton model; the result is $a_u = -1.4\times10^{-4}$, $a_d = 0.5\times10^{-4}$ (u and d are the quark flavors). The presence of the anapole moments of electrons and quarks must be taken into account in the analysis of high-energy polarization experiments and the effects of parity nonconservation in atoms.

CONCLUSIONS

We have listed the physics problems in which the concept of toroidal moments is used explicitly or implicitly. Note that in problems of classical electromagnetism in which the configurations of the fields and currents can be nearly toroidal the toroidal characteristics of objects may be helpful for estimating the magnitude of global effects. For example, the configuration of the fields that sustain the plasmosphere of a pulsar is evidently similar to that produced in thermonuclear facilities of the tokamak type. On the basis of this analogy, a theory of the structure and dynamics of a pulsar plasmosphere was developed.[48] Since from the point of view of a distant observer both a tokamak and a pulsar plasmosphere are toroids, their global multipole characteristics include toroidal ones. Analogous parameters can evidently be introduced in the study of the electrodynamics of toroidal black holes and other cosmic objects.

Toroidal moments find an application as approximation parameters in biophysical problems in which the influence of electromagnetic fields on biological objects is studied. This is so because toroidal configurations are conservative systems shielded maximally from electric and magnetic induction.

In microphysics the most fundamental direction is the study of the toroidal characteristics of particles and subparticles in the framework of gauge theories. Here, transition moments may help to elucidate the "isotope-spin" structure of the Hamiltonians. Study of the transition toroidal characteristics of point particles is helpful for elucidating the mechanisms of violation of discrete symmetries. Thus, besides the dipole an interesting question is that of the magnitude of the toroidal quadrupole of quarkonium, which is determined by the scale of the violation of CP invariance.

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