Space–time description of motion and the Hamiltonian approach to the dynamics of relativistic particles

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Fiz. Elem. Chastits At. Yadra 14, 1092–1111 (September–October 1983)

Recent results obtained in the relativistic mechanics of directly interacting point particles are reviewed. A parametrization-independent formulation of the problem is given in terms of projective tangent spaces and second-order differential systems. There is a discussion of the relationship between the general space–time picture and the Hamiltonian approach with constraints, including the theorem on the gauge dependence of the world lines in the space of canonical coordinates. In the two-particle case noncanonical variables of position and time are constructed (by perturbation theory) with parametrization-independent trajectories. The application of the canonical Hamiltonian approach to the two-body problem in the general theory of relativity is also reviewed.

PACS numbers: 11.80.Cr, 03.30. + p, 04.20.Cv

INTRODUCTION

Although 75 years have passed since the creation of the special theory of relativity and, simultaneously, the formulation of the mechanics of one relativistic particle in an external field, it is only in recent years that the hitherto disconnected attempts to construct the relativistic mechanics of a system of several directly interacting particles have come to be seen as different aspects of a common consistent picture. For example, the so-called no-interaction theorem (see, for example, the collection of Ref. 15, the monograph of Ref. 19, and the papers of Refs. 12, 16, 20, 26, and 27, in which the obtained results are discussed from a modern point of view) has been interpreted by many as an indication that it is altogether impossible to construct a relativistic theory of interacting particles without the participation of a field, which implements the idea of short-range interaction. although the authors of the theorem actually pointed out that it leads merely to the necessity of considering noncanonical coordinates.

Caution is here appropriate. The picture of charged particles interacting through the electromagnetic field, which has infinitely many degrees of freedom, is a better approximation to reality than the description of the interaction of such particles by means of an effective potential. However, there is a class of phenomena (including both the bound-state problem and Coulomb scattering) in which radiation effects are negligibly small and for which one is justified in using a simpler "quasipotential" approach in both the nonrelativistic and the relativistic case.[1)] (The difference between these two cases is manifested in the relativistic retardation effect, which is coded by the energy dependence of the potential.) Such an approximation can be combined with exact Lorentz (or Galilean) invariance of the problem. For more than two centuries we have worked with the mathematically self-consistent (invariant with respect to the Galileo group) Newtonian mechanics of a finite number of point particles. In recent years, there has been completed the construction of a no less consistent relativistic particle mechanics which can serve, above all, as a classical basis of all existing calculations of the fine structure and Lamb shift of the spectrum of hydrogenlike atoms and positronium.

It is not at all simple to follow the history of the creation of the relativistic mechanics of directly ininteracting point particles. The number of papers on this subject—especially in the postwar period—is huge (see, for example, the bibliography in the Trieste Lectures of Ref. 38; several monographs have already been devoted to this subject—see, for example, Refs. 2, 14, and 19). In the forties, this problem occupied Dirac[6] and Wheeler and Feynman.[41] In the fifties, it attracted the attention of Shirokov.[2)] Papers in the sixties (see, for example, Ref. 15) put in doubt the applicability of the canonical Hamiltonian approach to the relativistic problem of interacting particles. The way out of these difficulties considered in Sec. 2 (see also Refs. 36–48) is based on the Hamiltonian formalism with constraints developed by Dirac *et al.*[7,10,13] and on the idea of noncanonical coordinates. Other approaches leading to a similar result can be found in the papers of Sokolov,[30-32] Bel *et al.*,[3] Droz-Vincent,[8] Sazdjian,[27,28] and others.[12,34] (In all cases, we cite late papers of the quoted authors, in which one can find references to their earlier publications.)

The first section of the present paper contains a compressed exposition of the parametrization-independent formulation of the laws of motion of relativistic particles developed in Ref. 21. In Sec. 2, we discuss the connection between this space–time picture and the theorem[20] on the gauge dependence of the world lines in the space of canonical coordinates and we consider the

[1)]The sources of the quasipotential approach can already be found in studies made in the thirties (see, for example, Ref. 11 and the literature quoted there). However, the systematic development and application of this method began with Logunov and Tavkhelidze's paper of Ref. 17. Our exposition in Sec. 2 is related to the variant of this approach given in Refs. 25 and 35, in which the reader can also find a more complete bibliography.

[2)]Shirokov's contribution to this subject is not exhausted by his publications of Ref. 29. By his competent and well-meaning criticism, in particular as leader of the seminar on quantum field theory at the V. A. Steklov Mathematics Institute, he supported a number of studies in this direction and helped in their improvement (including Refs. 20, 30, 31, and 36, which are discussed below).

 0090-4759/83/050458-09$04.40

application of the canonical Hamiltonian approach to the two-body problem in the general theory of relativity.

1. SPACE-TIME FORMULATION OF THE RELATIVISTIC DYNAMICS OF PARTICLES WITHOUT A DISTINGUISHED EVOLUTION PARAMETER

A. Some results from differential geometry. The concept of a differential system

We begin by recalling some concepts from differential geometry that we shall need later.

Let M be a smooth (C^∞) real D-dimensional manifold; a k-dimensional *differential system* (also called a *distribution*) on $M (1 \leqslant k \leqslant D)$ is defined by specifying for each $x \in M$ a k-dimensional subspace $\sigma(x)$ of the tangent space T_xM satisfying the following condition of smoothness. For every $x \in M$ there exists a neighborhood $U \ni x$ and k smooth vector fields $X_1, \ldots, X_k$ on U that span $\sigma(y)$ for any $y \in U$.

A connected immersed[3)] k-dimensional submanifold S of the manifold M is called an *integral manifold* for the differential system σ if for any $x \in S$

$$T_xS = \sigma(x). \tag{1}$$

There exists not more than one maximal integral submanifold passing through a given point. The differential system σ is said to be integrable if there exists an integral manifold of it passing through every point of M. In accordance with the classical *Frobenius theorem* a system σ is integrable if and only if it is *involutive*, i.e., if for every pair of vector fields $X_1(x)$, $X_2(x)$ belonging to σ (for $x \in M$) the commutator $[X_1, X_2]$ also belongs to σ. The space of all (maximal) integral manifolds of an involutive differential system σ defines a smooth k-dimensional fibration on M.

[We recall that a family $\mathscr{F} = \{S_\alpha, \alpha \in A\}$ of linearly connected subsets S_α of a manifold M is called a k-dimensional (smooth) fibration if

1) $\alpha\beta \in A,\ \alpha \neq \beta \Rightarrow S_\alpha \cap S_\beta = \varnothing$; 2) $\bigcup_{\alpha \in A} S_\alpha = M$;

3) for every $x \in M$ there exists a *chart*, i.e., a neighborhood $U \ni x$ and a mapping φ from U to an open subset of D-dimensional Euclidean space such that the linearly connected components of the set $\varphi(S_\alpha \cap U (\neq \varnothing))$ have the form $\{(x^1, \ldots, x^2) \in \varphi(U);\ x^{k+1} = C^{k+1}, \ldots, x^D = C^D\}$]. It follows from the definition of a fibration that for every sufficiently small neighborhood $U \subset M$, if $S_\alpha \cap U \neq \varnothing$, then the linearly connected components $S_\alpha \cap U$ are k-dimensional submanifolds of M and S_α ($\alpha \in A$) are immersed submanifolds. Every fibration defines an involutive differential system such that $\sigma(x) = T_xS_\alpha$ for all $x \in S_\alpha$, $\alpha \in A$.

The correspondence between involutive differential systems and fibrations is analogous to the correspondence between vector fields (first-order differential operators) and flows. In the first case, the fibers (the analog of a flow trajectory) are determined without distinguishing an "evolution" parameter. In the consideration of a relativistic system, this is an advantage, since in this case there is no priviliged choice of the time parameter. However, classical mechanics (both relativistic and nonrelativistic) deals with second-order differential equations. Thus, we need the more general concept of a second-order differential system introduced in Refs. 1 and 21. Here we give only a special case of this concept used in the considered formulation of relativistic particle dynamics.

B. One-dimensional second-order differential system. Single-particle dynamics

As a preliminary to the introduction of the concept of a one-dimensional second-order differential system we note that every straight line $\sigma(x)$ can be regarded as a point in the projective space $P(T_xM)$. The space of all smooth sections in the projective tangent bundle $P(TM) \xrightarrow{\pi} M$ (with standard fiber $P_{D-1} = P(\mathrm{R}^D)$) is canonically isomorphic to the space of all one-dimensional differential systems on M, and we shall use for them the same notation $\tilde{\sigma}$.

Let M be a D-dimensional manifold. The section

$$\sigma\colon P(TM) \to P(TP(TM)) \tag{2}$$

is called a *(one dimensional) second-order differential system on* M if

$$\tilde{\pi}(\sigma(u)) = u \quad \text{for all} \quad u \in P(TM), \tag{3}$$

where $\tilde{\pi}$ is the tangent mapping corresponding to the projection π (in the bundle $P(TM) \xrightarrow{\pi} M$).

We use the following proposition (see Ref. 21, Theorem 1): Every second-order differential system σ on M defines an integrable (one-dimensional) fibration F_σ on $P(TM)$.

Having in mind application to relativistic particle dynamics, we assume that the manifold M (which we shall identify with the physical space-time for $D=4$) has a *local causual structure*. This means that there is given a (smooth) field of cones on the tangent bundle TM (i.e., a *light cone* in every T_xM) or a *conformal class*[4)] of *pseudo-Riemannian metrics* g with signature $(- + \ldots +)$.

A *curve* (i.e., a one-dimensional submanifold) on M is said to be *timelike* (respectively, *isotropic*) if the tangent vector to it at every point is timelike (respectively, isotropic).

[3)]It is here important that the topology of the immersed manifold $S \subset M$ may differ from the topology of the induced manifold M. As a standard reference on differential geometry to become acquainted with the concepts employed here we recommend the following monographs: B. A. Dubrovin, S. P. Novikov, and A. T. Fomenko, Sovremennaya geometriya. Metody i prilozheniya (Modern Geometry. Methods and Applications), Nauka, Moscow (1979) and S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Vol. 1 and 2, Interscience, New York (1963) (Russian translation published by Nauka, Moscow (1981)).

[4)]The metric tensors $g_{\mu\nu}(x)$ and $\tilde{g}_{\mu\nu}(x)$ belong to the same conformal class if there exists a positive function $\Omega(x)$ such that $\tilde{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x)$.

A single-particle relativistic dynamics is defined as a one-dimensional differential system σ on M whose integral curves are timelike or isotropic. The dynamics defines a (one-dimensional) *phase fibration* $\mathscr{F}_\sigma=\{\tilde{l}_\alpha, \alpha\in A\}$; through every point of the space $P(TM)$ there passes a unique fiber $\tilde{l}_\alpha$ whose projection $l_\alpha=\pi(l_\alpha)$ can be identified with the (timelike or isotropic) *world line* of the particle. Thus, for the given differential system σ the world line l is determined by the *initial conditions*, i.e., by the point $x\in M$ and the direction in T_xM [i.e., by the point in $P(T_xM)$].

We shall show that this parametrization-independent definition includes Newton's equations for a particle in an external field.

Let $(x^\mu)=(x^0, x^1, x^2, x^3)$ be local coordinates on M and (x^μ, x^ν) be the corresponding coordinates in TM. Assuming $\dot{x}\neq 0$ (which is certainly the case for timelike and luminal velocities), we shall use $[\dot{x}^\nu]$ as homogeneous coordinates of the point in the three-dimensional fiber of the fiber space $P(TM)$. Let $u=(u^i)=(u^1, u^2, u^3)$ be independent local coordinates on $P(\mathbf{R}^4)$ $(\equiv P_3)$, and let $(x^\mu, u^i[\dot{x}^\mu])\equiv(x^\mu, u^i)$ be the corresponding coordinates in the seven-dimensional "state space" $P(TM)$. (A possible choice of the functions $u^i[\dot{x}^\mu]$, equally well suited for timelike and isotropic world lines, is $\mathbf{u}=\dot{\mathbf{x}}/\dot{x}^0(=\mathbf{v})$, and a convenient choice for timelike velocities is $\mathbf{u}=\dot{\mathbf{x}}/\sqrt{-\dot{x}^2}=((\dot{x}^0)^2-\dot{\mathbf{x}}^2)^{-1/2}\dot{\mathbf{x}})$. Similarly, $(x^\mu, u^i; [\dot{x}^\nu, u^j])$ parametrizes an open set in $P(TP(TM))$ (by points in $\mathbf{R}^7\otimes P(\mathbf{R}^7)$). The section $\sigma\{P(TM)\to P(TP(TM))\}$ can be specified in local coordinates by the formula

$$\sigma:(x^\mu,\ u^i)\to(x^\mu,\ u^i;\ [S^\nu(x,\ u),\ F^j(x,\ u)]).$$

It follows from the condition (3) that S^ν must be proportional to the 4-velocity:

$$S\ (x,\ u)=\lambda\,(x,\ u)\,\dot{x}^\nu \quad\text{or}\quad \sigma:(x^\mu,\ u^i)\to(x^\mu,\ u^i;\ [\lambda\,(x,\ u)\,\dot{x}^\nu,\ F^j\,(x,\ u)]). \tag{4}$$

Let $\tilde{l}$ be an integral manifold of the system σ in $P(TM)$, so that

$$T\,(x,\ u)\,\tilde{l}=\sigma\,(x,\ u) \quad \text{for all} \quad (x,\ u)\in\tilde{l}. \tag{5}$$

To establish the correspondence with the ordinary treatment of the single-particle problem, we introduce the "time parameter" τ, which specifies a local parametrization $(x^\mu(\tau), u^i(\tau))$ of the curve $\tilde{l}$. It follows from (4) and (5) that

$$\left[\frac{dx^\mu}{d\tau},\ \frac{du^i}{d\tau}\right]=[\lambda\dot{x}^\mu,\ F^i]. \tag{6}$$

(It is assumed that the 7-vector on the left-hand side does not vanish, and equality of the square brackets means that the corresponding straight lines coincide.) From (6) there follows an equation of Newtonian type:

$$m\,(\tau)\,\frac{du^i}{d\tau}=F^i\left(u^i=u^i\left[\frac{dx^\mu}{d\tau}\right]\right). \tag{7}$$

The factor $m(\tau)$ depends on the choice of the time parameters (and on the definition of the "force" F). In the special case of timelike motion, taking the proper time as τ, we can rewrite (7) in the covariant 4-dimensional form

$$m\,\frac{du^\mu}{d\tau}=F^\mu, \tag{8}$$

where $u^2+1=0=u\,du/d\tau=uF(=\eta_{\mu\nu}u^\mu F^\nu=\mathbf{u}\mathbf{F}-u^0F^0)$.

We conclude the discussion of the single-particle dynamics with some remarks.

1. The approach we have presented gives only the possibility of determining the ratio F/m (even after the choice of the time parameter has been fixed). A *mass* m can be defined only if there is an independent definition of the force F. For example, for a particle with charge e in an external electromagnetic field $F^{\mu\nu}(x)$ $(=-F^{\nu\mu}(x))$ we can set

$$F^\mu\,(x,\ u)=eu_\nu F^{\mu\nu}\,(x)\ (\text{for}\ \ u^2=\mathbf{u}^2-u_0^2=-1). \tag{9}$$

Then the coefficient m in Eq. (8) defines the mass of the charged particle. It is also possible to give a theoretical definition of the mass of a free particle in terms of the associated representation of the Poincaré group (see below, Sec. 2C). We emphasize, however, that only the procedure of "weighting" different particles in a given force field as described here gives an experimental value of the mass.

2. One usually considers dynamics in which the type of the world line does not change: it is either timelike or isotropic. One says that the velocity of a massive particle cannot reach the velocity of light during a finite time interval. A second-order differential system parametrized by the proper time τ ensures fulfillment of this condition if the force F in (8) is bounded.

3. The assumption of a timelike or isotropic nature of the world lines makes it possible to replace the state space $P(TM)$ by the space of rays $P(T_>M)$, where $T_>M$ consists (locally) of all pairs $(x, \dot{x})$ for which $\dot{x}^0\geq|\dot{\mathbf{x}}|$, $\dot{x}^0>0$.

C. *N*-particle dynamics. Consequences of Poincaré invariance

Let M be a (4-dimensional) *causal manifold* (i.e., a smooth manifold equipped with a local causal structure as defined in the previous subsection). We define an *N-particle relativistic dynamics* by means of the commutative diagram

$$\begin{array}{ccc} [P\,(TP\,(T_>M))]^N & \xrightarrow{\tilde{\pi}} & [P\,(T_>M)]^N \\ \sigma\ \uparrow\downarrow & & \downarrow\ \pi \\ [P\,(T_>M)]^N & \xrightarrow{\pi} & M^N, \end{array} \tag{10}$$

where σ is an involutive section, and $T_>M$ was defined in the third remark at the end of the previous subsection. [We assume that σ in the diagram (10) lies in the domain of definition of the projective tangent mapping $\tilde{\pi}$.] The restriction σ_k of the differential system σ to $P(T_>M_k)$, where k is the number of the particle, satisfies all the requirements of a single-particle dynamics. The external force F_k acting on particle k will, in general, be a function of the coordinates and velocities of all the particles.

We now turn to the consideration of dynamics with symmetry.

Let G be a group of transformations of the manifold

M whose action $(V_g, \tilde{V}_g; g\in G)$ on the pair $(P(T_>M), P(TP(T_>M)))$ is a *bundle homomorphism*. In other words, if π is a projection in the bundle $P(TP)T_>M)) \to P(T_>M)$, then $\pi\cdot \tilde{V}_g = V_g\cdot\pi$. We shall say that the group G is a symmetry group of the differential system σ (regarded as a section on the given bundle) of

$$(\tilde{V}_g\cdot\sigma\cdot V_g)\,(y)=\sigma\,(y)\quad \text{for all}\quad g\in G,\ y\in P\,(T_>M). \tag{11}$$

This concept can be readily transfered to the N-particle case.

The largest symmetry group that arises in the given approach is the conformal group, which is locally isomorphic to the (connected) pseudo-orthogonal group $SO_0(D, 2)$. For a single-particle system, it follows from conformal symmetry that the particle must move with the velocity of light (i.e., $\dot{x}^2=\dot{\mathbf{x}}^2-\dot{x}_0^2=0$). The largest group of space–time symmetry for a particle with fixed positive mass is the Poincaré group, whose connected component (including the identity transformation) will be denoted by $\mathscr{P}_+^{\uparrow}$.

We illustrate the consequences of Poincaré invariance for the example of a single-particle system in D-dimensional space–time, assuming the standard (affine) action of the group $\mathscr{P}_+^{\uparrow}$ on M, for which the transformation V_g of the manifold $P(T_>M)$ has the form

$$V(a,\ \Lambda)\,(x,\ [\dot{x}])\to(\Lambda x+a,\ [\Lambda\dot{x}]). \tag{12}$$

We consider an open [(2D-1)-dimensional] submanifold $P(T_+M)$ of the manifold $P(T_>M)$ defined as the set of pairs $(x, [\dot{x}])$ for which the 4-velocity $\dot{x}$ is a positive timelike vector: $\dot{x}^0>|\dot{\mathbf{x}}|$. This submanifold is a homogeneous space of the group $\mathscr{P}_+^{\uparrow}$ with stability subgroup of a point ("little point") isomorphic to $SO(D-1)$. For space–time of dimension $D>2$, this little group acts nontrivially on the acceleration and, therefore, on the forces, leaving no nonzero $(D-1)$-vector invariant. Therefore, for $D\geqslant 3$ the condition of Poincaré invariance of the single-particle dynamics leads to free motion ($F=0$). For $D=2$, however, the little to free motion ($F=0$). For $D=2$, however, the little group $SO(1)$ is itself trivial and does not impose any conditions on the force, so that the above argument breaks down. Indeed, in two-dimensional space–time there exists a nontrivial Poincaré-invariant single-particle dynamics. Using the proper time as variable, we can write down the $\mathscr{P}_+^{\uparrow}$-invariant equation

$$\dot{u}_\mu=k\varepsilon_{\mu\nu}u^\nu,\ \text{where}\ u^2+1=0,\ \varepsilon_{10}=-\varepsilon_{01}\,(=\varepsilon^{01})=1. \tag{13}$$

Its solution (for the velocity) satisfying the initial condition $V\equiv(u/u_0)_{\tau=0}=\tanh\alpha$, is[5)]

$$u^0=\operatorname{ch}(\alpha+k\tau),\ \ \mathbf{u}=\operatorname{sh}(\alpha+k\tau)\ (\mathbf{u}=u^1). \tag{14a}$$

The corresponding world line is a branch of the hyperbola (with isotropic asymptotes)

$$\left(\mathbf{x}-\mathbf{x}\,(t=0)+\frac{\operatorname{ch}\alpha}{k}\right)^2-\left(t+\frac{\operatorname{sh}\alpha}{k}\right)^2=\frac{1}{k^2}. \tag{14b}$$

[5)]*Translator's Note*. The Russian notation for the trigonometric, inverse trigonometric, hyperbolic trigonometric functions, etc., is retained here and throughout the article in the displayed equations.

Note, however, that the system (13) is noninvariant with respect to spatial reflection. In fact, there does not exist a nontrivial (smooth) single-particle dynamical system invariant with respect to the orthochronous Poincaré group $\mathscr{P}^{\uparrow}$. (The condition of smoothness is essential. Without it, for example, the system with the equation of motion

$$\dot{u}_\mu=k\varepsilon_{\mu\nu}u^\nu\,\operatorname{sign}\mathbf{u},\ \ k>0, \tag{15}$$

is a $\mathscr{P}_+^{\uparrow}$-invariant dynamical system with nonstraight world lines.)

Returning to higher dimensions of space–time (including the real case $D=4$), we assume that there are nontrivial Poincaré-invariant N-particle dynamics for $N\geqslant 2$. The basis of this is the circumstance that the little group of almost all points of the manifold $[P(T_>M)]^N$ is trivial. (This plausible assertion will be justified from the point of view of the Hamiltonian formalism in Sec. 2C.)

We omit here the discussion of spin (the dynamics of a particle with spin in an external field was studied recently in Ref. 39[6)]; the phase space of a classical particle with spin is described in Sec. 3 in the second reference of Ref. 38).

2. HAMILTONIAN APPROACH WITH CONSTRAINTS

A. Phase space of a relativistic particle in an external field

In the previous section, we have a general treatment of the space–time formulation of relativistic particle dynamics. In this connection, we eliminated all concepts that we do not regard as strictly necessary for the formulation of the problem.

In contrast, in attempting to construct realistic examples of systems of relativistic particles one must use all available means. Especially fruitful in the study of particle dynamics (both classical and quantum) is the Hamiltonian formalism in phase space (including system with constraints).

In the case of one spinless particle (in an external field) we begin with the 8-dimensional *extended phase space* $\Gamma=T^*M$, where M in the general case is a four-dimensional pseudo-Riemannian manifold, and T^*M is the tangent bundle on M; for given $x\in M$, the elements T_x^*M are defined as linear functions on the tangent space T_xM (or, equivalently, as 1-forms of the form $f_\mu(x)dx^\mu$); Γ is a symplectic manifold with canonical 2-form $\omega=dx\wedge dp(=dx^\mu\wedge dp_\mu)$.

As early as the forties, Dirac[5] noted that the dynamics of a charged relativistic particle in an external electromagnetic field A_μ (in Minkowski space) can be defined by a Hamiltonian constraint of the form

$$H_{em}=\frac{\lambda}{2}\,[m^2+(p-eA)^2]\approx 0\ \ ((p-eA)^2=(\mathbf{p}-e\mathbf{A})^2-(p^0-eA^0)^2), \tag{16}$$

where $\lambda(>0)$ is a *Lagrange multiplier* associated with the choice of the time parameter. The symbol of *weak*

[6)]Note that the authors of Ref. 39 use the expression "chronometric" (instead of reparametrization) invariance.

equality $\approx$ indicates that when the Poisson brackets are calculated x and p must be regarded as independent variables; the constraint (16) must be taken into account only after the differentiations have been performed. For example, the 4-velocity of the particle can be calculated from the "Hamiltonian" (16) by means of the standard formula

$$\dot{x}=\{x,\ H_{em}\}=\lambda\,(p-eA) \tag{17}$$

(where we have used the canonical commutation relations $\{x^\mu, p_\nu\}=\delta^\mu_\nu$). From the condition $\dot{x}^0 \geqslant |\dot{\mathbf{x}}|$(or $(x,\dot{x})\in T_> M$) there follows the restriction

$$p^0-eA^0 \geqslant |\,\mathbf{p}-e\mathbf{A}\,| \tag{18}$$

on the variables of the phase space [for given $A_\mu(x)$]. The choice $\lambda=1/m$ of the Lagrange multiplier corresponds to the proper-time variable [which follows from comparison of (17) and (16)].

As second example, we may mention the motion of a particle with mass m in an external gravitational field. In accordance with the general theory of relativity, it can be described as motion along a geodesic in a pseudo-Riemannian space with metric tensor $g_{\mu\nu}(x)$; this motion can be specified by means of the Hamiltonian constraint

$$H_{GR}=\frac{\lambda}{2}\left(g^{\mu\nu}(x)\,p_\mu p_\nu+\frac{1}{6}R\,(x)+m^2\right)\approx 0. \tag{19}$$

The term $R/6$ [where $R=R(x)$ is the scalar curvature] ensures conformal invariance of the limit $m=0$ (this term was first introduced by Penrose[23] in the context of quantum theory). The condition $-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\geqslant 0$ (which expresses the fact that the 4-velocity is timelike or isotropic) leads by virtue of (19) to the inequality $m^2+1/6R(x)\geqslant 0$.

In the general case, a single-particle Hamiltonian dynamics is determined by specifying a seven-dimensional submanifold $\mathcal{M}$—a generalized single-particle mass shell—in the eight-dimensional (extended) phase space, which admits the canonical form

$$H^{\mathrm{can}}=h\,(\mathbf{p},\ x)-p^0=0, \tag{20}$$

where $h\to\sqrt{m^2+\mathbf{p}^2}$ for vanishing external field.

To establish the correspondence between the (single-particle) system defined by means of the Hamiltonian constraint $H(p,x)$ and the second-order differential system of the type considered in Sec. 1, it is simplest to introduce the Lagrangian $L(x,\dot{x})$ by means of the Legendre transformation

$$L\,(x,\ \dot{x})=p\dot{x}-H\,(p,\ x), \tag{21}$$

where $p=p(x,\dot{x})$ is determined from the Hamiltonian equation $\dot{x}=\partial H/\partial p$ (for the given Lagrange multiplier). In particular, for the examples (16) and (19) we find

$$L_{em}=\frac{\dot{x}^2}{2\lambda}-\frac{\lambda}{2}\,m^2+eA\dot{x},\ \ L_{GR}=\frac{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}{2\lambda}-\frac{\lambda}{2}\left(m^2+\frac{R}{6}\right). \tag{22}$$

If $(m^2+1/6R)>0$, the Lagrange multiplier can be eliminated by using the constraint $\partial L/\partial\lambda=0$, which leads to a manifestly reparametrized-invariant Lagrangian:

$$L_{em}=-m\sqrt{-\dot{x}^2}+eA\dot{x},\ L_{RG}=-\sqrt{-\left(m^2+\frac{R}{6}\right)g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}. \tag{23}$$

[The Euler–Lagrange equations for L_{em} are a parametrization-independent generalization of Eqs. (8) and (9).]

B. Gauge dependence of the world lines of interacting particles in the space of canonical coordinates

We turn to the study of a Poincaré-invariant N-particle system in the framework of the Dirac Hamiltonian formulation with constraints.[7,10,13] Following Refs. 20 and 36–38, we begin with the extended phase space $\Gamma^N=\Gamma_1 x\ldots x\Gamma_N$, $\Gamma_k=T^*M_k$, where M_k is the (flat) Minkowski space of the canonical coordinates q_k^μ of particle k. The space Γ^N is equipped with the symplectic form

$$\omega=\sum_{k=1}^{N}dq_k\wedge dp_k\ (dq_k\wedge dp_k\equiv dq_k^\mu\wedge dp_{k\mu}) \tag{24}$$

and the standard diagonal action of the Poincaré group.

The *generalized N-particle mass shell* is defined as a $7N$-dimensional connected Poincaré-invariant submanifold $\mathcal{M}$ of the space Γ^N satisfying the following conditions.

1. In every frame of reference, the surface $\mathcal{M}$ can be represented locally by N *canonical equations* of the form

$$\left.\begin{aligned}&\varphi_k^{\mathrm{can}}=h_k\,(\mathbf{p}_1,\ \ldots,\ \mathbf{p}_N;\ q_{12},\ \ldots,\ q_{N-1N})-p_k^0=0;\\ &q_{lm}=q_l-q_m;\ k\,(l,\ m)=1,\ \ldots,\ N\end{aligned}\right\} \tag{25}$$

so that the total energy is positive:

$$(0<)\ P^0\geqslant|\,\mathbf{P}\,|\quad\text{for}\quad P=p_1+\cdots+p_N. \tag{26}$$

2. Let $\mathrm{Ker}(|\omega|_\mu)$ be the set of all vectors tangent to $\mathcal{M}$ for which the restriction to $\mathcal{M}$ of the symplectic form (24), $|\omega|\ \mathcal{M}$, vanishes. If $\mathcal{M}$ is defined by the local equations $\varphi_k=0$, $k=1,\ldots,N$, then we require that $\mathrm{Ker}(|\omega|\mathcal{M})$ be generated by the Liouville operators

$$L_{\varphi_k}=\sum_{l=1}^{N}\left(\frac{\partial\varphi_k}{\partial p_l}\frac{\partial}{\partial q_l}-\frac{\partial\varphi_k}{\partial q_l}\frac{\partial}{\partial p_l}\right).$$

We assume further that the fibration

$$\mathcal{M}\to\Gamma_*\equiv\mathcal{M}/\mathrm{Ker}\,(\omega|_{\mathcal{M}}) \tag{27}$$

is a (locally trivial) fiber bundle. From this there follows the consistency condition

$$\{\varphi_k,\ \varphi_l\}\approx 0,\ \text{i.e.,}\ \sum_{m=1}^{N}\left(\frac{\partial\varphi_k}{\partial q_m}\frac{\partial\varphi_l}{\partial p_m}-\frac{\partial\varphi_k}{\partial p_m}\frac{\partial\varphi_l}{\partial q_m}\right)\Big|_{\mathcal{M}}=0. \tag{28}$$

To have a standard scattering theory, we must require fulfillment of the following condition.

3. *Separability*, or the *cluster-decomposition property* means physically that clusters of particles separated by a large spacelike interval do not interact. In particular, if the constraints (25) are defined globally, then

$$\lim_{\mathbf{q}_h\to\infty}h_k=\sqrt{m_k^2+\mathbf{p}_k^2}, \tag{29}$$

where m_k is the mass of particle k. (In fact, one needs a somewhat stronger asymptotic condition which ensures also that the derivatives of $h_k-\sqrt{m_k^2+\mathbf{p}_k^2}$ tend to zero.)

To show that the above definition is nontivial, we write down a class of two-particle constraints that includes realistic interactions Φ and satisfies conditions 1–3:

$$\varphi_k = \frac{1}{2}(m_k^2 + p_k^2) + \Phi(r, w), \quad k=1, 2, \tag{30}$$

where

$$w = \sqrt{-P^2}\,(P = p_1 + p_2), \quad q_\perp = q - \frac{1}{P^2}(Pq)\,P, \quad q = q_1 - q_2, \quad r = |q_\perp|; \tag{31}$$

we assume that the system (30) can be solved for the energies p_k^0 of the particles and that $\lim_{r\to\infty}\Phi = 0$ ($\lim \partial\Phi/\partial w = \lim \partial\Phi/\partial r = 0$). The consistency condition (28) follows from the equation

$$\{\varphi_1, \varphi_2\} = -P\frac{\partial\Phi}{\partial q} = -\frac{\partial\Phi}{\partial r}P\frac{\partial r}{\partial q} = -\frac{1}{r}\frac{\partial\Phi}{\partial r}Pq_\perp = 0.$$

The condition of integrability contained in 2 and the condition of positivity of the energy (26) impose certain additional restrictions on the interaction Φ.

It is not possible to write down simply in a closed form N-particle ($N \geq 3$) interactions. However, their existence can be established either by Sokolov's method[31] (using classical wave operators[32]—see Sec. 6 of the second reference in Ref. 38) or by means of the (local) existence theorem for a system of partial differential equations (see Refs. 4 and 28).

The Hamiltonian H of the system of N particles is defined as a linear combination of the constraints φ_k^{can} with positive (variable) coefficients ("Lagrange multipliers"). Thus, every trajectory on the phase space belongs to some (N-dimensional) fiber $F \subset \mathscr{M}$ of the bundle (27). A question arises: under what conditions are the space-time world lines independent of the choice of the Lagrange multipliers (or, as we shall say, *gauge-invariant*)? The answer to this question is given by the following negative result, which is formulated for the example of the two-particle case.

THEOREM.[20] Let $\mathscr{M}$ be a generalized two-particle mass shell satisfying the conditions 1 and 2 formulated above (the separability condition 3 is not needed here). The projection $\pi_k(F)$ of every two-dimensional fiber $F \subset \mathscr{M}$ onto the Minkowski space M_k of particle k [$\pi_k(q_1, p_1; q_2 p_2) = q_k$, $k = 1, 2$] is a one-dimensional submanifold of M_k only if the trajectories in the space of the variables q_k^μ are straight lines.

The precise relationship between this result and the earlier "no-interaction theorems" is discussed in the second reference of Ref. 20. A similar result was obtained (independently) in the framework of the Lagrangian approach (see Ref. 12).

C. Noncanonical position variables with gauge-invariant world lines

The negative result of the theorem can be circumvented by the assumption that the physical position variables are (noncanonical)[7)] functions of the canonical coordinates and momenta satisfying

$$\{x_k, \varphi_l^{\text{can}}\} = 0 \text{ for } k \neq l. \tag{32}$$

We sketch the construction of the physical coordinates x_k in the two-particle case (cf. Ref. 27).

First, we assume that x_k are Lorentz 4-vectors which can be written in the form

$$x_k = q_k + a_k\hat{P} + b_k\hat{r} + c_k\pi, \quad k=1, 2, \tag{33}$$

where $\hat{P}$ and $\hat{r}$ are unit vectors in the directions of P and $q_\perp$ ($\hat{r}^2 = 1 = -\hat{P}^2$, $q_\perp = r\hat{r}$, $P = w\hat{P}$), and π is the projection of the relative momentum

$$p = \mu_2 p_1 - \mu_1 p_2, \text{ where } \mu_1 + \mu_2 = 1, \ \mu_1 - \mu_2 = \frac{m_1^2 - m_2^2}{w^2}, \tag{34a}$$

which [for φ_k given by Eq. (30)] satisfies the relation

$$\varphi_1 - \varphi_2 = pP \approx 0, \tag{34b}$$

onto the plane orthogonal to $\hat{r}$

$$\pi = p - p_r\hat{r}, \text{ where } p_r = \hat{r}p. \tag{35}$$

By assumption, the coefficients a_k, b_k, and c_k can depend only on the Poincaré-invariant variables r, p_r, π^2, w and

$$\chi = \hat{P}q/w. \tag{36}$$

The condition of invariance of the world line

$$\{x_1, \varphi_2\} = 0 = \{x_2, \varphi_1\} \tag{37}$$

[for φ_k given by Eq. (30)] leads to the following equations for the coefficients of the decomposition (33);

$$\left(L_R - \frac{\partial}{\partial\chi_k}\right)a_k = \Phi_w \left(\equiv \frac{\partial}{\partial w}\Phi(r, w)\right); \tag{38a}$$

$$b_k = \left[r\left(\frac{\partial}{\partial\chi_k} - L_R\right) + p_r\right]c_k; \tag{38b}$$

$$\frac{\pi^2}{r}c_k + \left(\frac{\partial}{\partial\chi_k} - L_R\right)b_k = \chi\Phi_r \left(\equiv \chi\frac{\partial\Phi}{\partial r}\right). \tag{38c}$$

Here

$$\chi_1 = -\frac{\chi}{\mu_2}, \quad \chi_2 = \frac{\chi}{\mu_1}, \quad \mu_{1,2} = \frac{1}{2}\left(1 \pm \frac{m_1^2 - m_2^2}{w^2}\right) \tag{39}$$

while L_R is the Liouville operator corresponding to the one-dimensional "Hamiltonian" $R = 1/2\,p_r^2 + \Phi$:

$$L_R = p_r\frac{\partial}{\partial r} - \Phi_r\frac{\partial}{\partial p_r}. \tag{40}$$

These equations lead to a Cauchy problem with respect to the variable χ, and we specify the initial conditions

$$a_k = b_k = c_k = 0 \quad \text{for} \quad \chi_k = 0 \tag{41}$$

(in other words, we require that the physical coordinate coincide with the canonical ones for $\chi = 0$). We write down the first few terms in the expansion with respect to χ_k of the (unique) solution of Eqs. (38)–(41):

$$a_k = \sum_{n=1}^{\infty}\frac{\chi_k^n}{n!}L_R^{n-1}\Phi_w = \chi_k\Phi_w + \frac{\chi_k^2}{2!}p_r\Phi_{wr} \tag{42}$$

$$+ \frac{\chi_k^3}{3!}(p_r^2\Phi_{wr^2} - \Phi_r\Phi_{wr}) + O(\chi_k^4); \tag{43}$$

$$b_1 = -\mu_2 b(\chi_1;\ r,\ p_r,\ \pi^2), \quad b_2 = \mu_1 b(\chi_2;\ r;\ p_r,\ \pi^2);$$
$$c_1 = -\mu_2 c(\chi_1;\ r,\ p_r,\ \pi^2), \quad c_2 = \mu_1 c(\chi_1;\ r,\ p_r,\ \pi^2), \tag{44}$$

where

---

7)The idea of using noncanonical coordinates (noncommuting position operators in the quantum case) to circumvent the "no-interaction theorems" arose long ago; see, for example, Refs. 8, 15, 22, and 27 (the literature quoted in them).

$$c=\chi_k^3\left\{\frac{1}{6}+\frac{\chi_k L_R}{12}+\frac{\chi_k^2}{40}\left[\frac{1}{3r}\left(\Phi_r-\frac{\pi^2}{r}\right)-L_R^2\right]\right\}\frac{1}{r}\Phi_r+O(\chi_k^6); \quad (45a)$$

$$b=\left[r\left(\frac{\partial}{\partial\chi_k}-L_R\right)+p_r\right]c=\frac{1}{2}\chi_k^2\left(1+\frac{\chi_k}{3}p_r\frac{\partial}{\partial r}\right)\Phi_r-$$
$$+\frac{\chi_k^4}{4!}\left[\Phi_r\left(1+4r\frac{\partial}{\partial r}\right)+p_r^2\left(1-4r\frac{\partial}{\partial r}\right)\frac{\partial}{\partial r}-\frac{\pi^2}{r}\right]\frac{\Phi_r}{r}+O(\chi_k^5). \quad (45b)$$

The existence of a class of interactions Φ for which the above expansions converge and the physical coordinates are well defined is in no doubt. We assume that in this way one can prove the existence of interacting Poincaré- and reparametrization-invariant two-particle systems (in accordance with the general remark made at the end of Sec. 1C).

D. Gauge invariance of the S matrix

The foregoing analysis shows that although the physical coordinates x_k (in terms of which the world lines of the particles are gauge-invariant) evidently exist, we know how to find them only in the framework of perturbation theory, and the obtained expressions are too cumbersome for concrete applications to relativistic systems. Therefore, it is interesting to find direct applications of a much simpler original scheme, using only the canonical variables p and q.

It turns out that although the trajectories in the q space depend on the gauge (i.e., on the choice of the surface of equal times), the asymptotic properties of the N-particle motion (for example, the S matrix if it exists) are gauge-invariant (see Ref. 20). Simplifying somewhat, we can say that in the case of scattering the world lines of the particles have straight asymptotic behavior, and straight lines do not depend on the gauge. In the two-particle case, there is an elementary proof of this based on gauge invariance with respect to the coordinate $q_\perp$ (31), which follows from the equation $\{q_\perp, pP\}=0$ (see Sec. 2E of Ref. 20). We outline here a more general argument (also given in Ref. 20). which applies to the N-particle case.

The scattering problem is characterized by a pair of Hamiltonians: the "free Hamiltonian" H_0, which describes the asymptotic system of noninteracting particles, and the total Hamiltonian H; those are such that the classical wave operators

$$W_\pm=W_\pm(H, H_0)=s\lim_{t\to\pm\infty} e^{tL_H}e^{-tL_{H_0}},$$
$$\text{where } L_f=\frac{\partial f}{\partial p}\frac{\partial}{\partial q}-\frac{\partial f}{\partial q}\frac{\partial}{\partial p} \quad (46)$$

exist [the strong limits are defined with respect to the $\mathscr{L}_1$ norm in the phase space (see Ref. 32)]. For every such system there exists a (classical) scattering operator

$$S=W_+^*W_-. \quad (47)$$

The reparametrization invariance of the wave operators and of the S matrix follows from (and is essentially expressed by) the Birman–Kato invariance principle[32]: For every smooth monotonically increasing function $F(\xi)(F'(\xi)>0)$

$$W_\pm(H, H_0)=W_\pm(F(H), F(H_0)). \quad (48)$$

In the two-particle case, gauge invariance of the operator S is obtained as a consequence of (48) by setting

$$H_0=h_0-P^0,\ h_0=\sqrt{m_1^2+\mathbf{p}_1^2}+\sqrt{m_2^2+\mathbf{p}_2^2},\ H=h-P^0, \quad (49a)$$

where $h=h_1+h_2$ is determined from the canonical form (25) of the constraints (30), and

$$F(H)=\lambda(p_k, q)H+\mu(p_k, q)P_P\ (P_p\approx 0$$
$$\approx\{pP, H\}=\{pP, H_0\}). \quad (49b)$$

If the N-particle dynamics for $N\geqslant 3$ is defined by means of the relativistic addition of interactions,[4,30,31] then for it one can also prove that the scattering operator is independent of the choice of the parametrization. And in the case of finite motion (bound state) the asymptotic ("adiabatic") invariance, in the first place the quantum-mechanical energy levels[25,35] and the parameters of the classical ellipse[37,38] (and its relativistic procession), also do not depend on the gauge and can be found in the framework of the canonical Hamiltonian approach (without requiring knowledge of the precise form of the physical coordinates). We give here only one example of the (classical) application of the canonical formalism with constraints: the two-body problem in the general theory of relativity.[18]

E. Application of the quasipotential approach to the theory of gravitation

The gravitational interaction of two massive bodies is the most recent example of a relativistic two-particle problem that reduces to the study of the motion of one effective particle in an external field. It takes from the earlier treatment of relativistic systems in flat space[35-37] the concept of the relativistic reduced mass m_w and the energy E of the effective particle:

$$m_w=\frac{m_1m_2}{w},\ E=\sqrt{m_w^2+b^2(w)}=\frac{w^2-m_1^2-m_2^2}{2w}, \quad (50)$$

where

$$w=\sqrt{-P^2}\,(P=p_1+p_2),\quad b^2(w)=\frac{w^4-2(m_1^2+m_2^2)w^2+(m_1^2-m_2^2)^2}{4w^2} \quad (51)$$

[$b^2(w)$ is the value of the square of the relative momentum (34a) on the mass shell]. Further, our treatment of the gravitational two-body problem is based on postulating a constraint of the type (19) with m replaced by m_w and $p_0=-E$. The metric tensor $g_{\mu\nu}$ can be calculated (in a suitable gauge) from the diagram of one-graviton exchange of the linearized variant of the quantum theory of gravitation (see Ref. 18). The Hamiltonian constraint then obtained has the form

$$H=\frac{1}{2}\left[m_w^2-\left(1-\frac{r_t}{r}\right)^{-1}p_0^2+\left(1-\frac{r_s}{r}\right)p_r^2+\frac{1}{r^2}\left(p_\theta^2+\frac{1}{\sin^2\theta}p_\varphi^2\right)\right]\approx 0, \quad (52)$$

where

$$r_t=2Gw\left[1-4\frac{b^2}{m_w^2}\left(2\frac{E}{w}-3\frac{b^2}{w^2}\right)\right],\ r_s=$$
$$=2Gw\left[1+\left(8\frac{E}{w}-12\frac{b^2}{w^2}\right)\frac{E^2}{m_w^2}\right] \quad (53)$$

(G is the Newtonian gravitational constant). The metric that occurs in the constraint (53) can be regarded as a relativistic two-particle generalization of the Schwarzschild metric. The resulting two-particle motion is a planar motion; the trajectory of the relative motion can be described in polar coordinates in the form

$$\frac{l}{r}=1+\varepsilon\left[\cos\eta\varphi+\varepsilon\frac{r_s}{2l}(1-|\sin\eta\varphi|)^2\right]+O\left(\frac{r_s^2}{l^2}\right), \tag{54a}$$

where

$$\left.\begin{aligned} l&=\frac{2J^2}{r_t m_w^2-r_s b^2}+O(r_s)\left(\frac{r_s}{l}\ll 1\right);\quad \varepsilon^2=\\ &=1+\frac{4r_t}{l}+\frac{l^2b^2}{J^2}\left(1+\frac{3r_t+r_s}{l}\right);\\ \eta&=1-\frac{3}{2}\frac{r_t}{l}+\frac{1}{2l}(r_t-r_s).\end{aligned}\right\} \tag{54b}$$

In the test-body approximation [i.e., for $m_1m_2 \ll (m_1+m_2)^2$] and for low velocities [i.e., for

$$1-\frac{w}{m_1+m_2}+O\left(G\frac{m_1+m_2}{J^2}\right)\ll 1, \tag{55}$$

where J is the total angular momentum of the system] Eqs. (54) reproduce the classical result of Einstein, Infeld, and Hoffmann[9] (see also Ref. 14). These equations, obtained in a completely covariant calculation (without using expansions of the $1/c$ type) must have an advantage over the results of other approaches in the region of weak fields and relativistic velocities.

3. CONCLUSIONS

In conclusion, we discuss some key points.

1. The general space–time description of the motion of relativistic point particles in terms of a parametrization–independent second-order differential system of Newtonian type leaves room for a nontrivial Poincaré-invariant interaction.

2. The relativistic Hamiltonian approach with constraints in the extended phase space makes it possible to write down realistic two-particle interactions. However, the world lines of the particles in the space of canonical coordinates depend on the choice of the surface of equal times through the corresponding Lagrange multipliers in the Hamiltonian constraint (we say that they are *gauge dependent*).

3. The connection between the Hamiltonian approach and the parametrization-independent space-time formulation (Sec. 1) is established by introducing noncanonical coordinates and momenta satisfying a system of partial differential equations (which ensure gauge invariance of the world lines in the physical space). For suitable initial conditions, the physical coordinates x_k^μ (in the two-particle case $k=1, 2$) can be specified in the form of series in powers of the variable Pq (where $P=p_1+p_2$, $q=q_1-q_2$, q_k are canonical coordinates), the coefficients being polynomials in the interaction function Φ and its derivatives (they vanish when $\Phi=0$).

4. Canonical coordinates can be used directly to obtain asymptoic results, whose gauge invariance has been proved. As an illustration of this fact, we have considered the two-body problem in the general theory of relativity, in which not only the parameters of the nonrelativistic ellipse but also its relativistic precession are independent of the gauge.

We thank A. K. Pogrebkov for reading the draft and for helpful comments.

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Translated by Julian B. Barbour