Interaction of particles and fields in classical theory

A. K. Pogrebkov and M. K. Polivanov

V. A. Steklov Mathematics Institute, USSR Academy of Sciences, Moscow

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A scheme is proposed for describing the interaction of classical relativistic particles and fields based on the dynamics of the singularities of solutions of essentially nonlinear equations of field theory. On the basis of physical considerations, a description of such equations is obtained together with the types of their singular solutions, and the proposed scheme is treated in detail for the case of two-dimensional space–time.

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INTRODUCTION

In this paper, we propose a possible scheme for describing the interactions of relativistic particles and fields in classical theory. Before we formulate our problem, we recall the main difficulties of the standard approach to this problem. A moving (charged) particle creates and radiates a certain field, which is usually determined as the solution of a linear equation with a source associated with the world line of the particle. For example, for the scalar case in four-dimensional space–time

$$\Box\varphi(x)=\lambda\int_{-\infty}^{+\infty} ds\,\delta^{(4)}(x-y(s)), \tag{1}$$

where $y(s)$ is the world line of the particle and s is the proper time. Equation (1) determines the field for a given world line of the particle. Therefore, to obtain a description of the interaction of fields and particles it is necessary to close the system, i.e., by analogy with Lorentz's electron theory to augment (1) with the equations of motion of the particle in the field. Consider the action

$$S=\frac{1}{2}\int d^4x\,\partial^\mu\varphi(x)\,\partial_\mu\varphi(x)+\lambda\int ds\varphi(y(s))-m\int ds. \tag{2}$$

The Euler–Lagrange equations for this action with respect to φ give (1), and with respect to $y(s)$

$$\ddot{y}^\mu(s)=\lambda\frac{\dot{y}^\mu(s)\,(\dot{y}(s)\,\partial\varphi(y(s)))-\partial^\mu\varphi(y(s))}{m-\lambda\varphi(y(s))}. \tag{3}$$

Equations (1) and (3) form a closed system and could pretend to the description of the interaction of fields and particles. However, the solutions of Eq. (1) are singular for $x\in\{y(s)\,|\,-\infty<s<\infty\}$. Namely, at these points it is necessary to take φ and $\partial\varphi$ on the right-hand side of (3). Thus, the right-hand side of (3) is infinite. This fact does not depend on the choice of the action (2) and is known in the literature as the self-interaction effect (infinite self-mass). In reality, this means that we have not obtained the required description. Numerous attempts to overcome this difficulty were made in the first half of this century by, for example, Poincaré, Planck, and Wheeler and Feynman. However, the consequences of these attempts were the loss of either finiteness of the energy, or Poincaré invariance, or causality. To this day, a closed theory of the interaction of classical particles and fields does not exist. By analogy with electrodynamics, one can specify the motion of the particle (if one is not interested in the field) by Eq. (3), and regard $\varphi(t,x)$ as a sum of an external field, a "friction" field, etc. However, the exact equation contains derivatives of $y(s)$ of infinitely high order. Therefore, the elimination of the field variables in an appropriate modification of (3) also does not solve the problem.

At the same time, it is obvious that the main shortcoming of the standard description of the interaction of particles and fields is that the field variables and the coordinates of particles occur in it on an equal footing. Then the natural physical requirement of locality leads to infinity of the "force" acting on a particle [as in (3)]: The particle radiates a field that becomes infinite at the point at which the particle is situated, and this field influences the particle. But it can already be seen from what we have said that this "equal-footing" description of the fields and particles is redundant. For if we have specified a field with a singularity on some (timelike) curve (or curves), every such curve can be interpreted as the world line of a particle. Actually, the experimental detection of a particle always reduces to measurement of the radiated field and its characteristics. Thus, the question reduces to whether there exist closed field equations admitting solutions with nontrivial dynamics of the singularities. If yes, then we can dispense with the need to introduce into the theory particles independent of the field variables. The entire information about the particles will be contained exclusively in the fixing of the singularities of the initial data of the Cauchy problem.

It is a widely held opinion, based on the results for linear and quasilinear equations and for some not too strong singularities that singularities of the initial data propagate along characteristics. It is obvious that such dynamics of the singularities does not advance us, and we must expect equations of any real interest to be essentially nonlinear. Therefore, it is not surprising that we shall rely on the ideas and methods developed in recent years in connection with the investigation of such equations. Indeed, for a number of well-known equations there exist special localized regular solutions (solitons). Such solutions, at least asymptotically, can be interpreted as particles. Moreover, for a nonlinear equation for one scalar field in action–angle variables we find both field and discrete variables, which it is natural to associate with particles. Here there arises a picture close to the one which we wish to construct. It is not yet completely satisfactory, since ordinary regular solitons exist only in the time asymptotic behaviors. Our task must obviously consist of

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constructing solutions localized at all instants of time.

Thus, the first problem solved in this paper is the following. If there is a moving particle, it is possible, on the basis of reasonable general assumptions such as causality, Poincaré invariance, etc., to find the equations that the field which it radiates must satisfy?

To this question a positive answer is given. Indeed, under the additional assumption that the field is lightlike we find Poincaré-invariant equations for the scalar fields created by a particle. The form of these equations depends on the dimension of space–time. A common feature of them is conformal invariance, which is a consequence of the additional assumption. Some equations are also written down for vector fields.

We arrive at the following self-consistent picture. A moving particle creates a field that satisfies a certain nonlinear Poincaré-invariant equation, and the particle itself is a singularity of the same field, i.e., it is associated with the singular solution of the nonlinear equation obtained in this manner. We emphasize the Poincaré invariance of these equations, which, in particular, means independence of the world line of the particle. This rules out equations with any terms of source type and leads uniquely to closed field systems.

All the information about the particle is then contained in the initial data for the field equations. The corresponding class of singular initial data is described.

In the simplest case of two-dimensional space–time ($n=2$) the obtained equation is Liouville's equation. Singular solutions of this equation have been studied in detail in a number of earlier papers.[1-4] Liouville's equation can be solved exactly by virtue of its rich symmetry, and the analysis of its solutions can be carried through to a detailed picture of the motion of the singularities (particles). Here, we establish a connection between the fields and particles by means of these solutions.

In the light of the results given in this paper, the distinguished role of Liouville's equation becomes clear from a new point of view. Another interesting point is that the generalization of Liouville's equation to higher dimensions does not reduce to the addition of derivatives to the d' Alembertain. Instead, equations are obtained whose form changes with changing dimension, but they can at the same time be regarded as a direct generalization of Liouville's equation for two dimensions. It is interesting to note that for the four-dimensional case a scalar Poincaré-invariant equation is not obtained. In fact, in this case we arrive at d'Alembert's equation with the standard source term.

In the first part of the paper, we consider the transition from a particle to a field. All the necessary definitions and the formulation of the problem are given. Then, following a paper of one of the authors,[5] we give the solution of this problem, i.e., Poincaré-invariant equations for the field. Then, by means of a conformal transformation we construct a solution with two particles. At the end of this part we formulate the Cauchy problem for the field equation with the description of the class of singular initial data.

The second part is devoted to the two-dimensional case, which illustrates the outlined program. On the basis of Refs. 1–4, we give a brief description of solutions of Liouville's equation containing many singularities, and we discuss the dynamics of relativistic particles corresponding to them. We also consider questions related to the superposition principle in nonlinear dynamics and the possible occurrence of "massive" fields (in a sense that is made precise). Examples are considered that give some partial answers to these questions. These are the sinh–Gordon equation and the dynamics of its singular solutions, and also a brief discussion of the singular solutions of the $(e^{2\varphi}-e^{-\varphi})$ equation.

1. FIELDS RADIATED BY A PARTICLE AND SOLUTIONS OF NONLINEAR POINCARÉ-INVARIANT EQUATIONS. FORMULATION OF THE CAUCHY PROBLEM WITH SINGULARITIES

1. Lightlike scalar field radiated by a particle

We consider a particle in an n-dimensional Minkowski space $\mathscr{M}_n$. Let

$$x^\mu = y^\mu(s), \quad \mu = 0, 1, \ldots, n-1 \tag{4}$$

be the given world line of this particle. Here, $y^\mu(s)$ is a function of the proper time s, i.e.,

$$\dot{y}^2 \equiv 1, \quad \dot{y}^0 > 0. \tag{5}$$

The dot denotes the derivative with respect to s, $\dot{y}^2 = \dot{y}^\mu \dot{y}_\mu$, and the timelike metric tensor is $g^{00} = -g^{11} = \ldots = -g^{n-1,n-1} = 1$.

Suppose the particle radiates a certain field which is invariant with respect to transformations in the Poincaré group of the coordinate system in $\mathscr{M}_n$. Let us say more precisely what we mean by this. "The particle radiates a scalar field" means that for every point $x \in \mathscr{M}_n$ there is defined a function f that depends functionally on the world line: $f = f(x|y)$. As a consequence of causality, $f(x|y)$ can depend only on the part of the world line that lies either in the forward light cone with tip at the point x (advanced field) or in the backward light cone with the same tip (retarded field). We begin with the retarded field, and, following Ref. 5, we shall assume that the field depends on the behavior of $y(s)$ only at the point of intersection of the world line with the backward cone and does not depend on $y(s)$ within the cone. For obvious reasons, we shall call such a field *lightlike*.

Because the world line is timelike, a point of intersection of it with the backward cone always exists and is unique, i.e., the equation

$$[x - y(s)]^2 = 0 \tag{6}$$

uniquely determines s as a function of the point x: $s = s(x)$ under the condition that the scalar product

$$r = (x - y(s(x)))\, \dot{y}(s(x)) \tag{7}$$

is non-negative (retardation). Thus,

$$f = f(x, y(s), \dot{y}(s), \ddot{y}(s), \dots)|_{s=s(x)}, \tag{8}$$

where f on the right-hand side depends on an arbitrary but finite number of derivatives of $y(x)$. A dependence on an inifinite number of derivatives would amount to an actual dependence of the field on the world line within the cone, which for the time being we preclude.

Further, Poincaré invariance of f means that it can depend only on scalar products of the vectors $x-y$, $\dot{y}$, $\ddot{y}$, etc. In accordance with what we discussed in the Introduction, we now consider the properties of f as a function of x alone. Accordingly, we denote

$$\varphi(x) = f(x-y(s), \dot{y}(s), \ddot{y}(s), \dots)|_{s=s(x)}, \tag{9}$$

where $\varphi(x)$ is the field in which we are interested. We emphasize that φ as a function of x alone [explicitly and implicity through $s=s(x)$], i.e., for fixed world line, is invariant neither with respect to translations nor with respect to Lorentz transformations.

2. Poincaré-invariant equations for the radiation fields of particles

Does the lightlike field $\varphi(x)$ radiated by the particle satisfy some Poincaré-invariant second-order differential equation? What is the explicit form of φ and this equation? We repeat that these Poincaré-invariant equations cannot depend on the world like of the particle generating the field and, in particular, cannot contain any terms of the type of a source.

As is shown in Ref. 5, the condition that φ satisfies such equations determines the function $\varphi(x)$ uniquely [up to a transformation of the field variable $\varphi(x) \to F(\varphi(x))$] for any dimension n of space-time:

$$\varphi(x) = \begin{cases} -\ln r^2, & n=2; \\ r^{-(n-2)/2}, & n \geqslant 3, \end{cases} \tag{10}$$

where r is determined by Eq. (7) with $s=s(x)$. The corresponding differential equations for these φ have the form

$$\partial^2\varphi = \begin{cases} -2\exp\varphi, & n=2; \\ \frac{(n-2)(n-4)}{2}\varphi^{(n+2)/(n-2)}, & n\geqslant 3,\ n\neq 4. \end{cases} \tag{11}$$

We make some comments with regard to these equations:

a) The functions $\varphi(x)$ defined by Eqs. (10) are singular for $r=0$, so that Eqs. (11) are to be understood in the sense of generalized functions. Below, we shall give an explicit description of the singularities of φ, and then it will be readily seen that for each field x^0 all the considered functions $(\varphi, \partial^2\varphi, -2e^{\varphi}, \varphi^{(n+2)/(n-2)})$ are defined as generalized functions (see Ref. 7), and Eqs. (11) are satisfied in the same sense.

b) Among Eqs. (11) there is no equation for $n=4$. Formally, if $n=4$ we have $\partial^2\varphi=0$. But in fact $\partial^2\varphi = 4\pi\int_{-\infty}^{\infty} ds\,\delta^{(4)}(x-y(s))$, since for $n=4$ the function φ in accordance with (10) is precisely the convolution of the standard retarded Green's function of the d'Alembert equation with such a right-hand side. Note that φ in the solution (10) depends only on r, which in accordance with (8) and (9) corresponds to a field f that depends only on the vectors $x-y$ and $\dot{y}$ [r in this case is, by virtue of (5) and (6), a unique nontrivial scalar]. The only modification of (10) and (11) associated with a general case of dependence in (8) and (9) is the field $\varphi(x) = \mathrm{const}(\partial^2)^{n/2-1}r^{-1}$ for all even $n \geqslant 6$, which is a solution of the same (up to a constant factor) d'Alembert equations with δ function given above for $n=4$. We do not consider these equations, since they are not Poincaré-invariant: $\varphi(\Lambda x + a)$ satisfies the equation with world line $\Lambda^{-1}[y(s)-a]$. Obviously, this effect does not occur in Eqs. (11), since they do not contain sources.

c) For $n=2$, we have Liouville's equation. For $n\geqslant 3$, we obtain conformally invariant equations for a scalar field. That we arrive at conformally invariant equations is obviously due to our choice of a lightlike nature of the radiated field. Below, dropping this requirement, we show that there are quite different equations (for example, the sinh–Gordon equation) having analogous solutions.

d) On the substitution

$$\chi = \begin{cases} \varphi, & n=2; \\ \frac{2}{n-2}\ln\varphi^2, & n\geqslant 3 \end{cases} \tag{12}$$

Eqs. (11) go over into

$$\partial^2\chi + \frac{n-2}{4}(\partial\chi)^2 = (n-4)\exp\chi. \tag{13}$$

These are well-known equations describing the metric $g^{\mu\nu}\exp\chi$ of conformally flat pseudo-Riemannian spaces with constant scalar curvature (zero for $n=4$). Any other changes of the field variable φ are possible, and we need additional physical information (behavior at the singularity point, behavior at infinity, etc.) in order to choose the correct representative for the field radiated by the particle. The form (11) appears natural since the corresponding Lagrangians are

$$\mathcal{L} = \begin{cases} \frac{1}{2}(\partial\varphi)^2 - 2\exp\varphi, & n=2; \\ \frac{1}{2}(\partial\varphi)^2 + \frac{(n-2)^2(n-4)}{8n}\varphi^{\frac{2n}{n-2}}, & n\geqslant 3,\ n\neq 4. \end{cases}$$

and any change in φ leads to multiplication of the kinetic term by some function of the new field variable. At the same time, φ in (10) for $n\geqslant 3$ is positive, so that we must consider only positive solutions of (11). This is inconvenient when one is considering the Cauchy problem for such equations, so that it may be helpful to solve Eq. (13) for real χ and determine φ by the formula that is the inverse of (12): $\varphi = \exp\{(n-2)/4\chi\}$.

e) One can also introduce advanced field radiated by the particle. The only difference is that the proper time s will be determined as a function of x by means of the condition $r \leqslant 0$ [cf. (7)]. The advanced fields are solutions of the same equations (11). But it is easy to show that $\varphi(x)$ cannot be a nontrivial function of retarded and advanced fields (again for $n\neq 4$). Only for $n=2$ one can, instead of retarded or advanced fields, consider fields that propagate with velocity +1 or −1. These last also satisfy Liouville's equation.

f) As was noted in Ref. 5, it is easy to modify the entire scheme for a vector lightlike field radiated by the particle. Two types of solution were considered in Ref. 5.

i) The field

$$A=\frac{\dot{y}}{r}\bigg|_{s=s(x)}, \tag{14}$$

which satisfies the equation

$$\partial^2 A^\mu=(n-4)\{(A\partial)A^\mu+A^2A^\mu\},\quad n\geqslant 2 \tag{15}$$

with the subsidiary condition (constraint)

$$\partial A=0.$$

For $n=4$, we again have d'Alembert's equation and once more, regarding (15) in the sense of generalized functions, we find that a source is to be introduced if and only if $n=4$. The field (14) for $n=4$ is of course simply the Liénard–Wiechert potential.

ii) The dimension $n=4$ was an exceptional case above. But for a vector field it can be included in the general scheme by setting

$$A=\frac{x-y}{r^2}-\frac{\dot{y}}{r}\bigg|_{s=s(x)}. \tag{16}$$

In the sense of generalized functions, such a vector field satisfies the equation

$$\partial^2 A^\mu+2A^2A^\mu=0\quad (n=4) \tag{17}$$

with the subsidiary condition

$$\partial A+A^2=0.$$

3. Construction of two-particle solution by means of a conformal transformation

We consider a special solution of Eqs. (11) corresponding to uniform rectilinear motion of the particle. Then

$$y=us, \tag{18}$$

where u is a constant vector, and by virtue of (5)

$$r=\sqrt{(xu)^2-x^2}. \tag{19}$$

Solving (6) in this special case, we find $s(x)=xu-\sqrt{(xu)^2-x^2}$, so that by virtue of (7)

$$u^2=1. \tag{20}$$

If $x_\perp=x-(xu)u$, then $r=\sqrt{-x_\perp^2}$. In particular, if $u=(1,0)$, i.e., if the particle is at rest, then $r=\sqrt{\mathbf{x}^2}$. Thus, r is the distance between the point x and the world line of the particle. The distance r vanishes when $-x_\perp^2=0$: $x_\perp$ is a spacelike vector, so that $r=0$ if and only if $x_\perp=0$. Substituting this r in (10) for the corresponding space–time dimension n, we obtain the solution of Eqs. (11) describing the field radiated by the freely moving particle. We now use the conformal invariance to construct a new solution on the basis of the one we have been discussing.

Suppose $n\geqslant 3$ and let $x\to {}'x$ be a conformal transformation, i.e., $\partial x/\partial' x^\mu\, \partial x/\partial' x^\nu=\omega^{-2}(x)g_{\mu\nu}$, where ω is a real scalar function. Suppose $\varphi(x)$ satisfies (11); then ${}'\varphi(x)=[\omega(x)]^{(n-2)/2}\varphi('x(x))$ also satisfies (11) (see Ref. 8). We choose $\varphi(x)$ as indicated above with $u=(1,0,\ldots,0)$ (particle at rest):

$$\varphi(x)=(\mathbf{x}^2)^{-\frac{n-2}{4}} \tag{21}$$

and consider the special conformal transformation

$${}'x=\frac{x-bx^2}{1-2bx+b^2x^2},$$

where b is a fixed vector. Then instead of (21) we obtain the new solution

$${}'\varphi(x)=(\mathbf{x}-\mathbf{b}x^2)^{-\frac{n-2}{4}}. \tag{22}$$

This solution is also singular, but its singularities form two lines

$$\mathbf{x}_\pm=-\frac{\mathbf{b}}{2\mathbf{b}^2}[1\pm\sqrt{4b^2(x^0)^2+1}] \tag{23}$$

instead of one (the t axis) in the solution (21). Without loss of generality, we set $b=(-a,0,\ldots,0)$, $a>0$. We obtain two lines in the (x^0,x^1) plane (Fig. 1). The solution (22) describes two particles whose world lines can be identified with the singularities of this solution. It corresponds to the repulsion of two (asymptotically) massless particles.

Thus, Eqs. (11), which were constructed by means of single-particle solutions, also have two-particle solutions. It is natural to assume that they also have many-particle solutions. This is so for the two-dimensional case ($n=2$). Here [see (10)] for a particle at rest we have the solution of (11)

$$\varphi(x)=\ln(x^1)^2, \tag{24}$$

since $r=(\mathbf{x}^2)^{1/2}=|x^1|$. For $n=2$, the conformal group is infinite-dimensional, and the confomal invariance of Liouville's equation means that if $\varphi(x,t)$ is a solution, then

$$\varphi\left(\frac{A(x+t)+B(x-t)}{2},\ \frac{A(x+t)-B(x-t)}{2}\right)+\ln A'(x+t)B'(x-t)$$

is too. The two monotonic functions A and B parametrize the conformal transformation. On the basis of the solution (24) and using this conformal transformation, we obtain a solution of Liouville's equation that depends on two arbitrary ($A'>0$, $B'>0$) real functions,

$$\varphi(x,t)=\ln\frac{4A'(x+t)B'(x-t)}{[A(x+t)+B(x-t)]^2}. \tag{25}$$

This is precisely Liouville's representation[9] for the general regular solution of the equation. As is shown in Refs. 3 and 4 and will be discussed below, A and B can be determined in such a way that (25) really does describe solutions with an arbitrary number of singularity lines.

4. Formulation of the Cauchy problem for the equations (11)

We have shown that there is in principle a possibility of investigating some many-particle mechanical systems

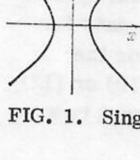

FIG. 1. Singularity lines of the two-singularity solution.

in the framework of a field approach if we consider singular solutions of nonlinear equations. It is natural to parametrize the solutions of all differential equations by specifying the Cauchy data. Of course, in our case these initial data must be singular. In addition, they must contain information about the behavior of the singularity lines of the corresponding solutions. Therefore, to implement the program outlined for describing classical particles by means of singularities of the solutions of nonlinear field equations we must: i) identify the class of initial data that lead to the necessary solutions; ii) prove an existence and uniqueness theorem for the solution of the Cauchy problem in this class. The second problem is very complicated. There are no such theorems for Eqs. (11) (for $n \geq 3$), nor for Eqs. (15) and (17) even for regular solutions. Therefore, in what follows we shall consider only Liouville's equation ($n=2$), for which such a theorem was proved in Ref. 2. We now consider the class of initial data.

We interpreted the solution with one singularity line as the field radiated by one particle. It is natural to interpret a solution with several singularity lines as the field radiated by a corresponding number of particles, i.e., as a kind of superposition. Although this superposition is nonlinear, physical intuition suggests that in the neighborhood of some particle the singularity of the field is determined by the contribution of precisely this particle, the remaining particles making only a regular contribution in this neighborhood. Then (10) can be used to determine the type of singularity. Since we are interested in the initial conditions, we fix

$$x^0 = t$$

and consider the behavior with respect to the variable $\mathbf{x}$. Suppose the point $(t, \mathbf{z})$ belongs to the world line (4), i.e., $r|_{x=(t,\mathbf{z})}=0$. Then it follows from (7) that for $\mathbf{x}$ near $\mathbf{z}$

$$r = \sqrt{(\mathbf{x}-\mathbf{z})^2 + \frac{((\mathbf{x}-\mathbf{z})\mathbf{v})^2}{1-\mathbf{v}^2}} + O(|\mathbf{x}-\mathbf{z}|^2);$$

$$\partial^0 r = -\frac{(\mathbf{x}-\mathbf{z})\mathbf{v}}{1-\mathbf{v}^2}\frac{1}{\sqrt{(\mathbf{x}-\mathbf{z})^2 + \frac{((\mathbf{x}-\mathbf{z})\mathbf{v})^2}{1-\mathbf{v}^2}}} + O(|\mathbf{x}-\mathbf{z}|).$$

We have here introduced the vector $\mathbf{v}$ of the particle velocity at the point $(t, \mathbf{z})$:

$$\dot{y}(s(t, \mathbf{z})) = \left(\frac{1}{\sqrt{1-\mathbf{v}^2}}, \frac{\mathbf{v}}{\sqrt{1-\mathbf{v}^2}}\right).$$

Thus, for $\chi = -\ln r^2$ and arbitrary n [see (10) and (12)], we obtain

$$\chi(t, \mathbf{x}) = -\ln\left[(\mathbf{x}-\mathbf{z})^2 + \frac{((\mathbf{x}-\mathbf{z})\mathbf{v})^2}{1-\mathbf{v}^2}\right] + O(|\mathbf{x}-\mathbf{z}|);$$

$$\partial^0\chi(t, \mathbf{x}) = \frac{2(\mathbf{x}-\mathbf{z})\mathbf{v}}{(\mathbf{x}-\mathbf{z})^2(1-\mathbf{v}^2) + ((\mathbf{x}-\mathbf{z})\mathbf{v})^2} + O(1). \tag{26}$$

Using (12) once more, we can readily obtain the singularities of φ: for $n=2$, $\varphi=\chi$, and for $n>2$ the strongest singularity of φ is $|\mathbf{x}-\mathbf{z}|^{-n-2}$. In accordance with the discussion, it is natural to consider initial data with a behavior of this type near each singularity point $\mathbf{z}_i$ with its velocity $\mathbf{v}_i$. We emphasize that the initial data for the particles are included in the initial data for the field. In the same way, using the solutions (10) or (12), we can find the asymptotic behavior with respect to $\mathbf{x}$ of the introduced fields. It is easy to see that $\chi(t, \mathbf{x})$ increases logarithmically in asymptotia, while $\partial^0\chi(t, \mathbf{x})$ decreases as $1/|\mathbf{x}|$. This completes the description of the initial data for Eqs. (11) or (13). In the same way one can find the class of initial data for the vector fields satisfying Eqs. (15) and (17).

2. Singular solutions of some two-dimensional models

1. Liouville's Equation

In this section, we describe the properties of the singular solutions and show how the general scheme works in these cases. We begin with Liouville's equation, i.e., with out scheme for $n=2$. It is convenient to change the notation somewhat, namely, instead of $x=(x^0, x^1)$ we shall write $(x^0, x^1)=(t, x)$. In this notation, the first of Eqs. (11) can be written in the form

$$\varphi_{tt} - \varphi_{xx} + 2\exp\varphi = 0. \tag{27}$$

We consider the Cauchy problem from this equation:

$$\varphi(0, x) = \phi(x); \quad \varphi_t(0, x) = \pi(x), \tag{28}$$

where $\phi(x)$ and $\pi(x)$ are given functions. These functions are singular, and their singularities are determined by (26). The properties of these functions are as follows: $\phi(x)$ is twice differentiable, and $\pi(x)$ is once differentiable everywhere in $-\infty < x < \infty$ except for a finite set of singularity points $\{x_j\}_{j=1}^N$. To each singularity point x_j there corresponds some neighborhood U_j, smooth functions $f_j(x)$, $g_j(x)$, and a real parameter v_j, $|v_j|<1$, such that for every $x \in U_j$

$$\left.\begin{aligned} \phi(x) &= -\ln\frac{(x-x_j)^2}{(1-v_j^2)} + (x-x_j)f_j(x); \\ \pi(x) &= v_j\left(\frac{2}{x-x_j} + f_j(x)\right) + (x-x_j)g_j(x). \end{aligned}\right\} \tag{29}$$

In accordance with the discussion following (26), we consider in this way a solution with N singularities. The function $\phi(x)$ behaves at spatial infinity logarithmically, while $\pi(x)$ decreases as $1/x$ [because of (12), $\varphi=\chi$ in this case]. Just such a Cauchy problem with singularities was considered in our earlier studies. In Ref. 2, one of us proved an existence and uniqueness theorem for the solution of this problem. In Refs. 1 and 3, the general properties of these solutions were studied. It was established there that all such solutions can be described by means of Liouville's representation (25), where A and B have the general form

$$\left.\begin{aligned} A(\xi) &= I(\xi) + \alpha + \sum_{j=1}^{N_A}\frac{c_j}{x_j - I(\xi)}, \\ B(\eta) &= J(\eta) + \beta + \sum_{j=1}^{N_B}\frac{d_j}{z_j - J(\eta)}. \end{aligned}\right\} \tag{30}$$

In these expressions, ξ and η are cosine variables:

$$\xi = (x+t)/2; \quad \eta = (x-t)/2. \tag{31}$$

$\alpha, \beta, c_j, d_j, y_j, z_j$ are constants such that c_j, $d_j > 0$, $y_1 < \ldots < y_{N_A}$, $z_1 < \ldots < z_{N_B}$, and I and J are thrice continuously differentiable real functions for $-\infty < \xi < \infty$, $-\infty < \eta < \infty$ with the monotonicity property $I'(\xi) > 0$, $J'(\eta) < 0$ and with linear behavior as $|\xi| \to \infty$ and $|\eta| \to \infty$, respectively. To determine all the indicated constants and the functions I and J in terms of the initial data, it is necessary to solve the Schrödinger equation for zero energy with regular potential given in Ref. 3. It is easy to show that the solution (25) after substitution of (30) does not have singularities apart from those determined by the equation

$$A(\xi) + B(\eta) = 0. \tag{32}$$

The solutions of this equation form N smooth timelike lines in the (x, t) plane without intersections (Fig. 2), and

$$N = N_A + N_B + 1. \tag{33}$$

It is natural to interpret this picture of singularities in terms of particles, so that the solution of Liouville's equation with definite N describes the motion of N interacting classical particles. To obtain the equations of motion of these particles, we note that the representation (30) suggests the presence of a distinguished subclass of solutions, for which

$$I(\xi) = c\xi, \quad J(\eta) = d\eta; \quad c, d > 0. \tag{34}$$

In Refs. 3 and 4, we called such solutions purely singular. These are the simplest solutions with given N. In order to obtain from this subclass the complete class, we must use the conformal transformation $\xi \to I(\xi)$, $\eta \to J(\eta)$, which by virtue of (33) does not change N. A remarkable property of the subclass of purely singular solutions is that it is Poincaré-invariant. Indeed, translations and Lorentz boosts can be compensated by a change of the parameters in (30) and (34). The general properties of dynamical systems corresponding to purely singular solutions are considered in Ref. 4, in which it is shown that they are relativistic, completely integrable systems. The behavior of the world lines of the particles of these systems is the same as in Fig. 2. In the limit $|t| \to \infty$, we find N_A massless particles with velocities -1, N_B massless particles with velocities -1, and one massive particle with velocity v, $|v| < 1$. As a result of the interaction, the particles exchange not only velocities but also masses. We consider in more detail the following simplest examples:

N=1. From (30) and (32)–(34) there follows an equation for the singularity line: $x = q + vt$, $-1 < v < 1$, so that the purely singular solution describes a free relativistic particle with nonzero mass. Its equation of motion is $\ddot{x} = 0$ (the dot here denotes differentiation with respect to t), and its Lagrangian is

$$\mathscr{L} = \text{const}\sqrt{-\xi\eta} = -\text{const}\sqrt{1 - \dot{x}^2}$$

It is found from this that the general solution with $N=1$ describes the motion of the particle in an "external" potential with

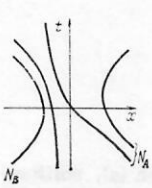

FIG. 2. General behavior of the singularity lines of the solution of Liouville's equation for $N_A = 2$, $N_B = 1$.

$$\mathscr{L} = -\text{const}\,(I'(\xi)\, J'(\eta))^{1/2}\sqrt{1 - \dot{x}^2}.$$

N=2. Here the equations of motion of the particles are

$$\ddot{x}_i = 2\,\frac{1 - |\dot{x}_1 + \dot{x}_2| + \dot{x}_1\dot{x}_2}{x_i - x_j}, \quad (i, j) = (1, 2), (2, 1),$$

and the Lagrangian has the form

$$\mathscr{L} = -\frac{1}{|x_1 - x_2|}\sqrt{1 - |\dot{x}_1 + \dot{x}_2| + \dot{x}_1\dot{x}_2 - \text{const}\,|}\sqrt{1 - (1 - |\dot{x}_1 + \dot{x}_2|)^2}.$$

Making again the conformal transformation $\xi \to I(\xi)$, $\eta \to J(\eta)$, we find the equations and the Lagrangian that describe the motion of the singularities of the general solution with $N=2$. It is again natural to interpret it as motion of the same system in an external field. Thus, we can regard the general singular solution of Liouville's equation as the nonlinear superposition of an external field and the field radiated by moving particles.

2. Singular solutions of the Sinh-Gordon equation

In the first part of this paper, we considered only lightlike fields. An important point was that the field at the point x was determined by just one point on the world line of the particle [see Eq. (6)]. This was the point of intersection of the world line with the surface of the backward light cone with tip at the point x. Because of this, the considered fields are in a certain sense massless, which is confirmed by their conformal invariance. In order to include "massive" fields, we must regard the field at a point at the superposition of fields reaching this point from all points of the world line within the corresponding cone; such a superposition must necessarily be nonlinear. This is a very complicated problem. However, there is a way around the difficulty. From the point of view of physics, the behavior of the field near the particle should not depend on the mass of this field. In addition, the singular solutions we have constructed could have a singularity on an arbitrary timelike curve, the type of the singularity being conserved in the dynamics. Bearing this in mind, we consider the following problem (for simplicity, in two-dimensional space–time).

Suppose the field $\varphi(t, x)$ satisfies the scalar equation

$$\varphi_{tt} - \varphi_{xx} + U'(\varphi) = 0, \tag{35}$$

where $U' = dU/d\varphi$. We shall assume that the following three conditions are satisfied:

1. $\varphi_t(t, x), \varphi_x(t, x), U(\varphi) \to 0$ as $|x| \to \infty$, so that the corresponding integrals for P^μ and the Lorentz boost converge at spatial infinity; $U(\varphi) \geqslant 0$, i.e., in the regular case the energy is non-negative.

2. There exists a stationary solution $\varphi_0(x)$ which becomes $+\infty$ at the point $x = 0$.

3. For every timelike curve $x = q(t)$ there exists a solution $\varphi(t, x)$ that becomes $+\infty$ on this curve, the singular behavior of this solution with respect to x at each fixed time t near the point $q(t)$ being determined by the corresponding Poincaré transformation of the function $\varphi_0(x)$. In other words,

$$\varphi(t, x) = \varphi_0\left(\frac{x - q(t)}{\sqrt{1 - \dot{q}(t)^2}}\right) + R(t, x), \tag{36}$$

where R is a twice continuosuly differentiable function. What form must the corresponding $U(\varphi)$ have? By fairly lengthy but straightforward calculations, substituting (36) in (35) and requiring cancellation of the singular terms, we find that to satisfy requirements 1–3 the potential $U(\varphi)$ must be asymptotically the Liouville potential:

$$U(\varphi) = 2e^{\varphi}V(\varphi), \tag{37a}$$

where

$$V(+\infty) = 1,\ V'(+\infty) = 0,\ V''(+\infty) = 0,\ \ldots, \tag{37b}$$

this holding, of course, up to the substitution $U(\varphi) \to c_1 U(c_2\varphi)$ where the constants c_1 and c_2 are positive. Moreover, it can be shown that the singularity of φ_0, and hence of φ by virtue of (36), is logarithmic and precisely the same as that described by Eq. (26).

This means that besides the "massless" equations (11) [for φ or, in accordance with (13), for χ] equations close in the above sense to (11) [or (13)] must possess singular particlelike solutions when the field becomes infinite. The singularities of the fields are the same as in the massless case.

The condition (37) is only necessary for the fulfillment of the requirements 1–3. Without going into the sufficient conditions, we turn to the consideration of examples for which these conditions are certainly satisfied. The simplest of them not identical to Liouville's equation is the sinh–Gordon equation[1)]:

$$\varphi_{tt} - \varphi_{xx} + 4\,\mathrm{sh}\,\varphi = 0. \tag{38}$$

Indeed, suppose $\varphi \to \pm\infty$. Then (35) goes over into

$$(\pm\varphi)_{tt} - (\pm\varphi)_{xx} - 2\exp(\pm\varphi) = 0.$$

This suggests that the sinh-Gordon equation has solutions that go over in the limit into singular solutions of Liouville's equation with one obvious difference: If φ is a solution of Eq. (38), then so is $-\varphi$. Hence, for such an equation one must consider the same Cauchy problem with singularities as is described by Eqs. (28) and (29), though the right-hand side for $\phi(x)$ and for $\pi(x)$ in the representation (29) must be multiplied by the parameter $s_j = \pm 1$, which indicates the sign of the singularity at the point x_j. Another difference relates to the asymptotic behavior of the initial data. Since $\varphi = 0$ satisfies the sinh–Gordon equation, it is natural to choose $\phi(x)$ and $\pi(x)$ such that they tend to zero at spatial infinity.

For Eq. (38) there is no analog of Liouville's representation (25), but it can be solved by the inverse scattering technique. We have not succeeded in proving an existence and uniqueness theorem for a global solution with singularities, but a special class of solutions analogous to the purely singular solutions of Liouville's equation was constructed in Ref. 6. These solutions are none other than singular solitons [it is well known that Eq. (39) does not have regular soliton solutions]. They again have timelike singularity lines, which can also be interpreted as the world lines of relativistic particles with direct interaction (for more details, see Ref. 6). A solution with one singularity again describes a free particle, but already a solution with two singularities leads to a dynamics more interesting than in the case of Liouville's equation (Fig. 3). Figure 3a shows the singularity lines for the two-soliton solution. We have here two massive particles with repulsion. The signs of the two singularity lines are here the same: $s_1 = s_2$. Figure 3b shows the singularity lines for the soliton–antisoliton solution. This is a case of attraction of two particles, which intersect with the velocity of light. In this case, $s_1 = -s_2$. For a special choice of the velocity parameter for $s_1 = -s_2$ one can obtain a different case (see Fig. 3c), whch corresponds to a breather solution, which is periodic in time. It describes two particles pulsating about a common center, i.e., a bound state. This picture shows that the sign of the singularity can be understood as the spin of the corresponding particle. The equations of motion for such a system, and also an example of a three-particle soliton–breather system, are considered in Ref. 6.

3. Singular solutions for the ($e^{2\varphi} - e^{-\varphi}$)–Gordon equation

It is well known that the equation

$$\varphi_{tt} - \varphi_{xx} + \frac{4}{3}(e^{2\varphi} - e^{-\varphi}) = 0 \tag{39}$$

is also completely integrable and also tends to Liouville's equation when $\varphi \to \pm\infty$. The symmetry $\varphi \to -\varphi$ is here absent and the limit equations differ somewhat.

As $\varphi \to +\infty$, we have

$$\varphi_{tt} - \varphi_{xx} + \frac{4}{3}e^{2\varphi} = 0,$$

and as $\varphi \to -\infty$

$$(-\varphi)_{tt} - (-\varphi)_{xx} + \frac{4}{3}e^{(-\varphi)} = 0.$$

Therefore, it is necessary to renormalize φ, x, and t in order to find possible singularities of the initial data by means of (26). Thus, we find two types of singularity for $\phi(x) = \varphi(0, x) = (\chi(0, x))$:

$$\phi(x) = -\ln\frac{2|x|}{\sqrt{3}\sqrt{1 - v^2}} + O(x) \quad \text{as} \quad \phi \to +\infty;$$

$$\phi(x) = \ln\frac{2x^2}{3\sqrt{1 - v^2}} + O(x) \quad \text{as} \quad \phi \to -\infty.$$

The expressions for $\pi(x) = \varphi_t(0, x)$ are obtained in the same way. The corresponding (stationary) solutions with one singularity, i.e., single-soliton solutions,

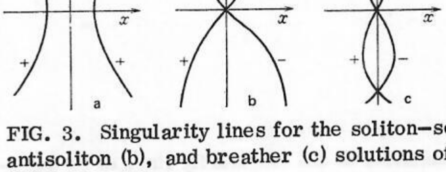

FIG. 3. Singularity lines for the soliton–soliton (a), soliton–antisoliton (b), and breather (c) solutions of the sinh–Gordon equation.

[1)] *Translator's Note.* The Russian notation for the trigonometric, inverse trigonometric, hyperbolic trigonometric functions, etc., is retained here and throughout the article in the displayed equations.

are

$$\varphi(t,\ x)=\ln\frac{(\sqrt{3}\,\mathrm{ch}\,x+\mathrm{sh}\,|\,x\,|)^2}{2\,\mathrm{sh}\,|\,x\,|\,(\sqrt{3}\,\mathrm{ch}\,x+2\,\mathrm{sh}\,|\,x\,|)},$$

$$\varphi(t,\ x)=\ln\frac{\mathrm{sh}^2\,x}{\mathrm{ch}^2\,x+1/2}.$$

Here, there is no symmetry between solutions with positive and negative singularity. This means that the corresponding particles have masses that depend on the spin, so that the dynamical systems corresponding to the solutions with many singularities can be regraded as analogous systems related to the sinh–Gordon equation but without mass degeneracy.

3. CONCLUSIONS

Beginning with the very general idea of a lightlike Poincaré-invariant field radiated by a particle, we have established unique Poincaré-invariant equations for such fields. We have considered a possibility of constructing many-particle solutions and the choice of the class of singular initial data for the corresponding Cauchy problem. These results indicate that the possibility of describing a system of Poincaré-invariant particles with a direct interaction between them (generally speaking, in some external field) by means of causual Poincaré-invariant (covariant) equations. We have considered the implementation of this program for two-dimensional space–time. We have also touched on some generalizations in which the radiated field is not lightlike, i.e., when the field has mass.

We emphasize that all the considered equations for the field are quite independent of the parameters of the particles. As always in the inverse-scattering method, these parameters arise only in action–angle variables, when discrete variables correspond to the particles and the continuous variables describe the external field.

We note that the canonical energy-momentum tensors obtained by means of Noether's theorem have nonintegrable singularities for all the considered equations. However, using the well-known arbitrariness in the choice of these tensors, we can introduce them in such a way that they are regular for all $x\in\mathscr{M}_n$. For example for Liouville's equation such a tensor has the form

$$T^{\mu\nu}=T^{\mu\nu}_{\mathrm{can}}-2\varepsilon^{\mu\lambda}\partial_\lambda\varepsilon^{\nu\sigma}\partial_\sigma\varphi,$$

where $\varepsilon^{00}=\varepsilon^{11}=0$, $\varepsilon^{01}=-\varepsilon^{10}=1$. This is precisely the tracless tensor corresponding to the dilatation invariance of Eq. (27). Regular tensors can be constructed for all of Eqs. (11) or (13). The regular tensors for the sinh–Gordon and $(e^{2\varphi}=e^{-\varphi})$–Gordon equations are

$$T^{\mu\nu}=T^{\mu\nu}_{\mathrm{can}}-2\varepsilon^{\mu\lambda}\partial_\lambda\varepsilon^{\nu\sigma}\partial_\sigma\ln\mathrm{ch}\,\varphi\ \text{for (38)};$$

$$T^{\mu\nu}=T^{\mu\nu}_{\mathrm{can}}-\varepsilon^{\mu\lambda}\partial_\lambda\varepsilon^{\nu\sigma}\partial_\sigma\ln(e^{\varphi}+e^{-2\varphi})\ \text{for (39)}.$$

At the same time, we obtain finite energy and finite momentum for the singular fields, but the Hamiltonians cease to be positive definite.

The idea of regarding the singularity lines of the solutions of nonlinear equations as the world lines of particles is not in itself new. For example, in Refs. 10 and 11 the singularity lines for the Korteweg–de Vries equation were studied, and in Ref. 12 they were studied for some other equations. However, in the quoted papers only solutions of special rational form were considered:

$$\varphi(t,\ x)=\sum_{j=1}^{N}\frac{r_j(t)}{x-x_j(t)}.$$

For this reason, the resulting particle dynamics was noninvariant and included constraints as well as the equations of motion. We have succeeded in avoiding these difficulties, since we have not used any special ansatz for our solutions, and the type of singularities of the initial data has been dictated by general considerations.

We frequently discussed the ideas developed in this paper, at the time when they first came to us, with the now deceased Yurii Mikhailovich Shirokov, and his interest in them and well-meaning criticism were for us very valuable. In concluding now this paper, we feel particularly acutely the heavy loss that we have all suffered through his untimely death.

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