Interaction of two particles in an external field

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For two quantum-mechanical particles in an external field a decomposition is obtained for the integral scattering operator in which all the pole (principal) singularities corresponding to the formation of bound states in independent subsystems (bound states of the first and of the second particle in the field and of the two particles) are separated explicitly. For the unknown functions of this decomposition (the components of the integral scattering operator) a system of integral equations with kernels that have only integrable singularities is obtained. The connection between the elements of the *S* matrix and the components of the total scattering operator is established. The spectral properties of the total Green's function of the system of two particles in the field are investigated. A method is proposed for generalizing this spectral analysis to the case of a problem of three bodies with arbitrary masses; this method makes it possible to describe the bound states and resonances that arise in the system.

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INTRODUCTION

In the present paper, we propose a method for solving the problem of the scattering of two nonrelativistic particles in an external field. Investigation of this quantum-mechanical problem is interesting not only because it has great applied significance in different branches of nuclear and atomic physics, quantum chemistry, etc., but also because this problem is the simplest many-particle process that possesses all the basic features of multichannel scattering in problems of $N \geqslant 3$ bodies when there are bound states embedded in the continuum.

The method of solving the problem of the interaction of two particles in an external field proposed in the present paper is based on the theory of the integral equations for the scattering amplitudes of the particles in the system. In recent years, this theory has been developed[1)] in both the time-independent and the time-dependent formulation for the following reasons. In contrast to the traditional method of describing many-particle problems on the basis of the differential Schrödinger equation, when it is first necessary to determine explicitly the vectors of all asymptotic states of the system, in the integral approach it is sufficient to know the nature of the singularities of the kernel of the scattering operator of the particles in the momentum representation and to rearrange the perturbation series (time-dependent or stationary) into a system of integral equations in such a way that the known singularities of the total scattering operator are represented explicitly in the kernels and free terms of these equations. Here, we have in mind the poles (principal singularities) corresponding to the formation of bound subsystems in the complete system of particles and the δ-function singularities corresponding to the law of conservation of momentum and energy in the independent subsystems.

Experience shows that such a rearrangement of the perturbation series (see Refs. 5, 7, 9, 10, 13, and 14) makes it possible to obtain equations with only dynamical singularities and, therefore, to solve the problem numerically. The problem we are considering of the interaction of two particles in an external field is not a special case of a three-body problem, since the total scattering operators in these problems have singularities of different types. Indeed, in the considered problem the operator can, first, have poles that correspond to bound states embedded in the continuum when the interaction between the particles is switched off. Second, in the problem of two particles in a field, as compared with the problem of three bodies with finite masses, there is a new type of unbound process corresponding to independent scattering of the particles by the external field. The scattering operator of this process contains two δ functions due to the energy conservation laws for each particle. Further, the singular scattering amplitudes of the unbound process we have just mentioned do not separate explicitly on the transition to the limit of infinite mass of one of the particles in the free term and in the kernels of the integral equations of the problem of three bodies with finite masses.

Therefore, to obtain integral equations that admit numerical solution for the problem of two bodies in a field it is necessary to rearrange perturbation theory by an appropriate method. In Ref. 6, a system of integral equations for the two-body problem was derived in the framework of time-dependent perturbation theory, and it was shown that the obtained equations are equivalent to the corresponding equations in stationary perturbation theory.

In the present paper, we consider the equations obtained in time-independent perturbation theory, since this approach is more traditional in the theory of many-particle scattering, although the principle of rearranging the Lippmann–Schwinger equation is taken completely from the method of rearranging the series of four-dimensional perturbation theory.

1. HAMILTONIANS AND ASYMPTOTIC STATES OF THE CHANNELS IN THE PROBLEM OF TWO-PARTICLE SCATTERING IN AN EXTERNAL FIELD

Formulation of the problem. We formulate here the conditions under which we solve the problem of the interaction of two quantum-mechanical particles in an external field. We define the Hamiltonian of the problem in the form

$$H = H_0 + V; \quad H_0 = H_{01} + H_{02}; \quad V = V_1 + V_2 + V_{12}. \qquad (1)$$

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Here, the operator H_{0i} denotes the kinetic energy of particle i with mass m_i. The operators V_α ($\alpha=1,2$) are the interaction potentials of particles 1 and 2 with the external field, and the operator V_{12} is the potential of the interaction of particles 1 and 2 with each other. The properties of these operators in the momentum representation are given below. We assume that particles 1 and 2 are nonrelativistic and are described by momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ or, in the coordinate representation, by vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively. Then the potentials V_α ($\alpha=1,2$) are functions of only r_α, while the two-particle potential V_{12} is a function of $\mathbf{r}_{12}=\mathbf{r}_1-\mathbf{r}_2$, the vector of the relative distance between the particles. The Hamiltonian H acts on the Hilbert space $\mathscr{G}$ of functions square-integrable with respect to the measure $d\mathbf{p}_1 d\mathbf{p}_2$.

The action of the operator H on an arbitrary function $f(\mathbf{p}_1\mathbf{p}_2)$ in $\mathscr{G}$ has the form

$$Hf(\mathbf{p}_1\mathbf{p}_2)=(p_1^2/2m_1)f(\mathbf{p}_1\mathbf{p}_2)+(p_2^2/2m_2)f(\mathbf{p}_1\mathbf{p}_2)+(V_1+V_2+V_{12})f(\mathbf{p}_1\mathbf{p}_2). \quad (2)$$

The operators V_α ($\alpha=1,2,12$) in (2) are integral operators:

$$\left.\begin{aligned}V_1f(\mathbf{p}_1\mathbf{p}_2)&=\int d\mathbf{q}_1V_1(\mathbf{p}_1-\mathbf{q}_1)f(\mathbf{q}_1\mathbf{p}_2);\\ V_2f(\mathbf{p}_1\mathbf{p}_2)&=\int d\mathbf{q}_2v_2(\mathbf{p}_2-\mathbf{q}_2)f(\mathbf{p}_1\mathbf{q}_2);\\ V_{12}f(\mathbf{p}_1\mathbf{p}_2)&=\int d\mathbf{q}_{12}\int d\mathbf{Q}_{12}v_{12}(\mathbf{p}_{12}-\mathbf{q}_{12})\,\delta(\mathscr{P}_{12}-\mathbf{Q}_{12})f(\mathbf{q}_1\mathbf{q}_2).\end{aligned}\right\} \quad (3)$$

Here, $v_\alpha(\mathbf{p}_\alpha)$ are the Fourier transforms of the functions $v_\alpha(\mathbf{r}_\alpha)$ ($\alpha=1,2,12$); the momenta $\mathbf{q}_{12}$ and $\mathbf{Q}_{12}$ are defined by

$$\mathbf{q}_{12}=(m_2\mathbf{q}_1-m_1\mathbf{q}_2)/M;\quad \mathbf{Q}_{12}=\mathbf{q}_1+\mathbf{q}_2;\quad M=m_1+m_2. \quad (4)$$

In all that follows, we shall assume that the functions $v_\alpha(\mathbf{p}_\alpha)$ satisfy the following conditions: 1) reality, $v(\mathbf{p})=v(-\mathbf{p})^*$; 2) boundedness and decrease, $|v(\mathbf{p})|\leqslant c(1+|\mathbf{p}|)^{-1-\theta_0}$ ($\theta_0>1/2$); 3) smoothness, $|v(\mathbf{p}+\mathbf{h})-v(\mathbf{p})|\leqslant c(1+|\mathbf{p}|)^{-1-\theta_0}|\mathbf{h}|\mu_0$ ($|\mathbf{h}|\leqslant 1$, $\mu_0>0$); 4) smoothness and boundedness of the derivative,

$$\left|\frac{d}{d\mathbf{p}}v(\mathbf{p})\right|\leqslant c(1+|\mathbf{p}|)^{-1-\theta_0};\quad \left|\frac{d}{d\mathbf{p}}[v(\mathbf{p}+h)-v(\mathbf{p})]\right|\leqslant c(1+|\mathbf{p}|)^{\theta_0-1},|\bar{\mathbf{h}}|^{\mu_0}(|\bar{\mathbf{h}}|\leqslant 1,\ \mu_0>0).$$

We now define the Hamiltonians h_α ($\alpha=1,2,12$):

$$\left.\begin{aligned}h_i&=H_{0i}+V_i,\ i=1,\ 2;\\ h_{12}&=h_{012}+V_{12}.\end{aligned}\right\} \quad (5)$$

The operator h_{012} in (5) has the meaning of the free Hamiltonian of the internal motion in the subsystem of particles 1 and 2. Note that the Hamiltonian H_0 can be represented in the form

$$H_0=h_{012}+H_{012}, \quad (6)$$

where H_{012} is the Hamiltonian of the free motion of the center of mass of the pair of particles 1 and 2, and, therefore,

$$h_{012}f(\mathbf{p}_1\mathbf{p}_2)=(p_{12}^2/2\mu_{12})f(\mathbf{p}_1\mathbf{p}_2);\quad H_{012}f(\mathbf{p}_1\mathbf{p}_2)=(\mathscr{P}_{12}^2/2M)f(\mathbf{p}_1\mathbf{p}_2);\quad \mu_{12}=m_1m_2/M. \quad (7)$$

For the class of potentials $\mathfrak{Y}$ satisfying conditions 1–4, the following property is proved in Refs. 1–3 and 11. The discrete spectrum (5) of the operator H and h_α ($\alpha=1,2,12$) has only a finite number of negative eigenvalues, these being of finite multiplicity. It is this property that makes it possible to study the structure of the singularities of the T matrix of the considered problem.

Conditions 1–4 in the coordinate representation correspond to the fact that the potential $v(\mathbf{r})$ decreases as $r^{-2-\varepsilon}$ with $\varepsilon>0$. Therefore, $v(\mathbf{r})$ is a square-integrable function, and we can define on $\mathscr{G}$ the scattering operator $T(z)$ by means of the relations[12]

$$\left.\begin{aligned}T(z)&=V+VG(z)V;\ T(z)\\ &=V+VG_0(z)T(z),\end{aligned}\right\} \quad (8)$$

where $G(z)=(z-H)^{-1}$ is the total Green's function of the system (the resolvent of the operator H), $G_0(z)=(z-H_0)^{-1}$ is the free Green's function of the system, and $z=E+i0$, where E is the energy of the system.

We also define the operators

$$t_\alpha(z)=V_\alpha+V_\alpha g_\alpha(z)V_\alpha;\ t_\alpha(z)=V_\alpha+V_\alpha g_{0\alpha}(z)t_\alpha(z), \quad (9)$$

where $g_\alpha(z)=(z-h_\alpha)^{-1}$ is the resolvent of the operator h_α, and $g_{0\alpha}(z)=(z-H_{0\alpha})^{-1}$ ($\alpha=1,2$) and $g_{012}(z)=(z-h_{012})^{-1}$ and the corresponding free Green's functions.

System of integral equations for the scattering amplitudes of two particles in an external field. We consider the Lippmann–Schwinger equation for the scattering operator of the two particles in the field $T(z)$:

$$T(z)=(V_1+V_2+V_{12})+(V_1+V_2+V_{12})G_0(z)T(z). \quad (10)$$

Besides the singularity corresponding to formation of a bound state of the two particles in the external field at a negative total energy $z\in\sigma_{\text{disc}}(H)$, the operator $T(z)$ must obviously have all the singularities that the scattering operators have in the independent subsystems possible in the considered case, i.e., in the subsystem of the two particles in the absence of the field, $t_{12}(z)$, and in the subsystem of independent scattering of the particles by the field, $N_{1,2}(z)$.

We rearrange Eq. (10), separating the above singularities of the operators explicitly. For this, we represent the operator in the form of two terms:

$$T(z)=T_{1,2}(z)+T_{12}(z), \quad (11)$$

where

$$T_{1,2}(z)=(V_1+V_2)+(V_1+V_2)G_0(z)T(z); \quad (12)$$

$$T_{12}(z)=V_{12}+V_{12}G_0(z)T(z). \quad (13)$$

The representation (11) makes it possible to obtain as kernels of the equations all possible scattering operators of the independent subsystems in the given problem and, therefore, to separate explicitly all possible singularities. Indeed, if Eq. (12) is multiplied from the left by the operator $[1-(V_1+V_2)]G_0(z)^{-1}$ and Eq. (13) from the left by the operator $[1-V_{12}G_0(z)]^{-1}$, then for the operators $T_{1,2}(z)$ and $T_{12}(z)$ we obtain the system of equations

$$\left.\begin{aligned}T_{1,2}(z)&=N_{1,2}(z)+N_{1,2}(z)G_0(z)T_{12}(z);\\ T_{12}(z)&=t_{12}(z)+t_{12}(z)G_0(z)T_{1,2}(z).\end{aligned}\right\} \quad (14)$$

We transform the system (14), separating the amplitudes of the unbound processes. For this, we introduce the operators $L_{1,2}(z)$ and $L_{12}(z)$, which are related to

$T_{1,2}(z)$ and $T_{12}(z)$ by

$$\left.\begin{array}{l} L_{1,2}(z)=T_{1,2}(z)-N_{1,2}(z); \\ L_{12}(z)=T_{12}(z)-t_{12}(z). \end{array}\right\} \quad (15)$$

Then for $L_{1,2}(z)$ and $L_{12}(z)$ we have the system of equations

$$\left.\begin{array}{l} L_{1,2}(z)=N_{1,2}(z)G_0(z)t_{12}(z_{12})+N_{1,2}(z)G_0(z)L_{12}(z); \\ L_{12}(z)=t_{12}(z_{12})G_0(z)N_{1,2}(z)+t_{12}(z_{12})G_0(z)L_{1,2}(z). \end{array}\right\} \quad (16)$$

To transform the system (16) in what follows, it is necessary to have a spectral decomposition of the operators $N_{1,2}(z)$ and $t_{12}(z)$. The properties of the operator $t_{12}(z)$ and of the operators $t_1(z)$ and $t_2(z)$ defined in (9) have been well studied in Refs. 4 and 5, and therefore we give them below without proof.

Properties of the operator $t_\alpha(z)$. The conditions 1–4 on the potentials make it possible to represent each of the operators $t_\alpha(z)$ as the sum of a singular function and a nonsingular function, the singular part of the operator $t_\alpha(z)$ having the pole nature of the singularities that arise because of the existence of the discrete spectrum of the Hamiltonian h_α. In what follows, we shall assume that the discrete spectrum $\sigma_{\rm disc}(h_\alpha)$ of the operator h_α consists of one negative eignevalue of multiplicity one. Then for $t_\alpha(z)$, we have

$$t_\alpha(z)=|\varphi_\alpha\rangle\langle\varphi_\alpha|/(z+\varkappa_\alpha^2)+\hat{t}_\alpha(z), \quad (17)$$

where $-\varkappa_\alpha^2\in\sigma_{\rm disc}(h_\alpha)$, $|\varphi_\alpha\rangle=V_\alpha|\psi_\alpha\rangle$, and $|\psi_\alpha\rangle$ is the eigenfunction of the discrete spectrum of h_α: $h_\alpha|\psi_\alpha\rangle=-\varkappa_\alpha^2|\psi_\alpha\rangle$. The operator $\hat{t}_\alpha(z)$ in (17) does not have singularities; more precisely, the function $\hat{t}_\alpha(\mathbf{p}_\alpha,\mathbf{p}'_\alpha,z)$ is a Hölder function of its variables.[5]

We now formulate a number of properties of the scattering wave operators $\Omega_\alpha^\pm$. The operators $\Omega_\alpha^\pm$, which act on the corresponding Hilbert spaces of the functions $f_\alpha(\mathbf{p}_\alpha)$ ($\alpha=1,2,12$), are defined as strong limits in the form

$$\left.\begin{array}{l} \Omega_\alpha^\pm=S-\lim\limits_{t\to\mp\infty}\exp(ih_\alpha t)\exp(-iH_{0\alpha}t), \quad \alpha=1,\ 2 \\ \Omega_{12}^\pm=S-\lim\limits_{t\to\mp\infty}\exp(ih_{12}t)\exp(-ih_{012}t). \end{array}\right\} \quad (18)$$

For interaction potentials in the class $\mathfrak{Y}$, the operators $\Omega_\alpha^\pm$ ($\alpha=1,2,12$) satisfy the condition of asymptotic completeness. Namely, an arbitrary function f_α in the corresponding Hilbert space has the representation

$$f_\alpha=\Omega_\alpha^\pm f_\alpha^\pm+f_d, \quad (19)$$

where $f_\alpha^\pm=(\Omega_\alpha^\pm)^+f_\alpha, f_d=P_d f_\alpha$; P_d is the projector onto the discrete spectrum of the operator h_α: $P_d=|\psi_d\rangle\langle\psi_d|$. In addition, the operators $\Omega_\alpha^\pm$ are characterized by the following properties[5]:

a) the operator $\Omega_\alpha^\pm$ is orthogonal to P_d:

$$(\Omega_\alpha^\pm)^+P_d=0; \quad (20)$$

b) the operator $\Omega_\alpha^\pm$ is isometric:

$$(\Omega_\alpha^\pm)^+\Omega_\alpha^\pm=I_\alpha; \quad (21)$$

c) for any bounded function $\varphi(x)$, $-\infty<x<\infty$,

$$\left.\begin{array}{l} \varphi(h_\alpha)\Omega_\alpha^\pm=\Omega_\alpha^\pm\varphi(H_{0\alpha}), \quad \alpha=1,\ 2; \\ \varphi(h_{12})\Omega_{12}^\pm=\Omega_{12}^\pm\varphi(h_{012}). \end{array}\right\} \quad (22)$$

Property c) is a consequence of the well-known relations for the Møller operators:

$$\left.\begin{array}{l} h_\alpha\Omega_\alpha^\pm=\Omega_\alpha^\pm H_{0\alpha}, \quad \alpha=1,\ 2; \\ h_{12}\Omega_{12}^\pm=\Omega_{12}^\pm h_{012}; \end{array}\right\} \quad (23)$$

d) the operators $\Omega_\alpha^\pm$ are related to the operator $t_\alpha(z)$ by

$$\langle\mathbf{p}_\alpha|\Omega_\alpha^\pm|\mathbf{p}_\alpha^0\rangle=\delta(\mathbf{p}_\alpha-\mathbf{p}_\alpha^0)+(\mathbf{p}_\alpha^{02}/2m_\alpha-\mathbf{p}_\alpha^2/2m_\alpha\pm i0)^{-1} \times t_\alpha(\mathbf{p}_\alpha,\ \mathbf{p}_\alpha^0,\ p_\alpha^{02}/2m_\alpha\pm i0) \quad (24)$$

[for $\alpha=12$ in (24), m_α must be replaced by μ_{12}].

Scattering of two noninteracting subsystems of particles. To investigate the properties of the operator $N_{1,2}$, it is necessary to consider the Hamiltonian

$$H_{1,2}=h_1+h_2=H_0+V_1+V_2, \quad (25)$$

which is defined on $\mathscr{G}$.

For the operators in (25), the folowing commutation properties are characteristic:

$$[h_1,\ h_2]=0; \quad [H_{01},\ H_{02}]=0. \quad (26)$$

As will be seen from what follows, the commutation properties (26) make it possible to construct in explicit form the Green's function $G_{1,2}(z)=(z-H_{1,2})^{-1}$ and the corresponding scattering operator $N_{1,2}(z)$, defined by means of one of the relations

$$\left.\begin{array}{l} N_{1,2}(z)=(V_1+V_2)+(V_1+V_2)G_{1,2}(z)(V_1+V_2); \\ G_{1,2}(z)=G_0(z)+G_0(z)N_{1,2}(z)G_0(z); \\ N_{1,2}(z)=(V_1+V_2)+(V_1+V_2)G_0(z)N_{1,2}(z). \end{array}\right\} \quad (27)$$

The commutation relations (26) have the consequence that the evolution operator of such a system is equal to the product of the evolution operators of each of the independent subsystems:

$$\exp(-iH_{1,2}t)=\exp(-ih_1t)\exp(-ih_2t). \quad (28)$$

The system with the Hamiltonian (25) has three scattering channels, which we shall denote by γ ($\gamma=0,1,2$); they correspond to the channel Hamiltonians $H_0=H_{01}+H_{02}$ for $\gamma=0$, $H_1=h_1+H_{02}$ for $\gamma=1$, and $H_2=h_2+H_{01}$ for $\gamma=2$. In addition, the system with the Hamiltonian (17) has a localized state at total energy $E=-\varkappa_1^2-\varkappa_2^2,-\varkappa_i^2$ of the system, this being the energy of the bound state in the subsystem with the Hamiltonian h_i. In accordance with (28), the Møller scattering operator for the channel $\gamma=0$ can be represented in the form

$$W_0^\pm f=S-\lim_{t\to\mp\infty}\exp(iH_{1,2}t)\exp(-iH_0t) =S-\lim_{t\to\mp\infty}[\exp(ih_1t)\exp(-iH_{01}t)] \times[\exp(ih_2t)\exp(-iH_{02}t)]f=(\Omega_1^\pm\otimes\Omega_2^\pm)f. \quad (29)$$

The tensor product in (29) means that

$$\langle\mathbf{p}_1\mathbf{p}_2|W_0^\pm|\mathbf{p}_1^0\mathbf{p}_2^0\rangle=\langle\mathbf{p}_1|\Omega_1^\pm|\mathbf{p}_1^0\rangle\langle\mathbf{p}_2|\Omega_2^\pm|\mathbf{p}_2^0\rangle. \quad (30)$$

The remaining wave operators $W_1^\pm$ and $W_2^\pm$ corresponding to the channels 1 and 2 are

$$W_2^\pm=\Omega_1^\pm\otimes P_2; \quad W_1^\pm=P_1\otimes\Omega_1^\pm. \quad (31)$$

The asymptotic completeness of the wave operators $\Omega_i^\pm$ ($i=1,2$) (27) means that any function in $\mathscr{G}$ admits a representation of the form

$$f=W_0^\pm f_0^\pm+W_1^\pm f_1^\pm+W_2^\pm f_2^\pm+f_d, \quad (32)$$

where

$$f_\gamma^\pm=(W_\gamma^\pm)f(\gamma=0,\ 1,\ 2);\ f_d=(P_1\otimes P_2)f. \quad (33)$$

From the properties of the single-channel wave operators (20)–(23) we deduce the following properties of the operators $W_\gamma^\pm$ $(\gamma=0,1,2)$:

$$\left.\begin{array}{l}(W_\gamma^\pm)^+W_{\gamma'}^\pm=I\times\delta_{\gamma\gamma'};\quad (W_\gamma^\pm)^+(P_1\otimes P_2)=0;\\ H_{1,2}W_0^\pm=W_0^\pm H_0;\quad H_{1,2}W_i^\pm=W_i^\pm(H_{0j}-\varkappa_i^2),\end{array}\right\} \quad (34)$$

where $i\neq j$, with $i,j=1,2$.

The behavior of the operators $W_\gamma^\pm$ $(\gamma=0,1,2)$ leads to the following result.[13]

The Green's function $G_{1,2}(E+i\tau)$ for the Hamiltonian $H_{1,2}$, which is the sum of the commuting Hamiltonians of the independent subsystems, is

$$G_{1,2}(E+\mathrm{i}\tau)=\int_{-\infty}^{\infty}\frac{d\varepsilon}{-2\pi\mathrm{i}}g_1(\varepsilon+\mathrm{i}\tau_1)\otimes g_2(E-\varepsilon+\mathrm{i}\tau_2),$$
$$(\tau_1,\ \tau_2>0,\ \tau_1+\tau_2=\tau), \quad (35)$$

where $g_i(z_i)$ is the Green's function of the independent subsystem with number i.

Proof. The relation (35) is equivalent to

$$G_{1,2}(E+\mathrm{i}\tau)f=\int_{-\infty}^{\infty}\frac{d\varepsilon}{-2\pi\mathrm{i}}g_1(\varepsilon+\mathrm{i}\tau_1)\otimes g_2(E-\varepsilon+\mathrm{i}\tau_2)f \quad (36)$$

for any function f in $\mathscr{G}$. Since the function f has the representation (32), for (36) to hold it is sufficient if

$$G_{1,2}(E+\mathrm{i}\tau)W_\gamma^\pm=\int_{-\infty}^{\infty}\frac{d\varepsilon}{-2\pi\mathrm{i}}g_1(\varepsilon+\mathrm{i}\tau_i)\otimes g_2(E-\varepsilon+\mathrm{i}\tau_2)W_\gamma^\pm \quad (37)$$

for $\gamma=0,1,2$ and

$$G_{1,2}(E+\mathrm{i}\tau)(P_1\otimes P_2)$$
$$=\int_{-\infty}^{\infty}\frac{d\varepsilon}{-2\pi\mathrm{i}}g_1(\varepsilon+\mathrm{i}\tau_1)\otimes g_2(E-\varepsilon+\mathrm{i}\tau_2)(P_1\otimes P_2). \quad (38)$$

We consider, for example, the case $\gamma=0$. Using the relations (23) and (34), we transform (37) to

$$W_0^\pm G_0(E+\mathrm{i}\tau)=W_0^\pm\int_{-\infty}^{\infty}\frac{d\varepsilon}{-2\pi\mathrm{i}}g_{01}(\varepsilon+\mathrm{i}\tau_1)\otimes g_{02}(E-\varepsilon+\mathrm{i}\tau_2). \quad (39)$$

We multiply from the left by $(W_0^\pm)^+$ and, using the fact that the operator $W_0^\pm$ is isometric, we obtain a relation in the momentum representation in the form

$$\frac{1}{E+\mathrm{i}\tau-p_1^2/2m_1-p_2^2/2m_2}=$$
$$=\int_{-\infty}^{\infty}\frac{d\varepsilon}{-2\pi\mathrm{i}}\frac{1}{\varepsilon-p_1^2/2m_1+\mathrm{i}\tau_1}\frac{1}{E-\varepsilon-p_2^2/2m_2+\mathrm{i}\tau_2}, \quad (40)$$

which, obviously, is an identity by virtue of the residue theorem. Therefore, the relations (37) and (39) are satisfied. The case of other $\gamma=1,2$ and the relation (38) are studied similarly. Thus, the representation (35) is valid.

We now obtain an expression for the scattering operator $N_{1,2}(z)$ of two independent subsystems. To this end, we use the second of the relations (27):

$$G_{1,2}(z)=G_0(z)+G_0(z)N_{1,2}(z)G_0(z). \quad (41)$$

Since

$$g_i(z_i)=g_{0i}(z_i)+g_{0i}(z_i)t_i(z_i)g_{0i}(z_i),\ i=1,2, \quad (42)$$

combining (35) and (42), we obtain

$$G_{1,2}(E+\mathrm{i}\tau)=\int_{-\infty}^{\infty}\frac{d\varepsilon}{-2\pi\mathrm{i}}[g_{01}(\varepsilon+\mathrm{i}\tau_1)+g_{01}(\varepsilon+\mathrm{i}\tau_1)t_1(\varepsilon+\mathrm{i}\tau_1)g_{01}(\varepsilon+\mathrm{i}\tau_1)]$$
$$\otimes[g_{02}(E-\varepsilon+\mathrm{i}\tau_2)+g_{02}(E-\varepsilon+\mathrm{i}\tau_2)t_2(E-\varepsilon+\mathrm{i}\tau_2)g_{02}(E-\varepsilon+\mathrm{i}\tau_2)]$$
$$=I_1+I_2+I_3+I_4. \quad (43)$$

The operators I_i in (43) correspond to the four possible products in the integrand. As a consequence, from the relation (40) we can find the expression

$$I_1=\int_{-\infty}^{\infty}\frac{d\varepsilon}{-2\pi\mathrm{i}}g_{01}(\varepsilon+\mathrm{i}\tau_1)\otimes g_{02}(E-\varepsilon+\mathrm{i}\tau_2)=G_0(E+\mathrm{i}\tau). \quad (44)$$

By the operators I_2 and I_3 we denote the products in (43) that include one t matrix. For example, for I_2 we have

$$I_2=\int_{-\infty}^{\infty}\frac{d\varepsilon}{-2\pi\mathrm{i}}g_{01}(\varepsilon+\mathrm{i}\tau_1)\otimes$$
$$\otimes g_{02}(E-\varepsilon+\mathrm{i}\tau_2)t_2(E-\varepsilon+\mathrm{i}\tau_2)g_{02}(E-\varepsilon+\mathrm{i}\tau_2). \quad (45)$$

Using the relation (42), we transform the integral (45) to the form

$$I_2=\int_{-\infty}^{\infty}\frac{d\varepsilon}{-2\pi\mathrm{i}}g_{01}(\varepsilon+\mathrm{i}\tau_1)\otimes[g_2(E-\varepsilon+\mathrm{i}\tau_2)-g_{02}(E-\varepsilon+\mathrm{i}\tau_2)]$$
$$=\int_{-\infty}^{\infty}\frac{d\varepsilon}{-2\pi\mathrm{i}}g_{01}(\varepsilon+\mathrm{i}\tau_1)\otimes g_2(E-\varepsilon+\mathrm{i}\tau_2)-G_0(E+\mathrm{i}\tau). \quad (46)$$

Further, using the device employed in the proof of (35), we can readily establish

$$\int_{-\infty}^{\infty}\frac{d\varepsilon}{-2\pi\mathrm{i}}g_{01}(\varepsilon+\mathrm{i}\tau_1)\otimes g_2(E-\varepsilon+\mathrm{i}\tau_2)=G_2(E+\mathrm{i}\tau), \quad (47)$$

where $G_2(z)=(z-H_{01}-h_2)^{-1}$. Now, substituting (47) in (46) and expressing the operator $G_2(z)$ in terms of the scattering matrix $t_2(z)$, we obtain

$$I_2=G_0(E+\mathrm{i}\tau)t_2(E-H_{01}+\mathrm{i}\tau)G_0(E+\mathrm{i}\tau) \quad (48)$$

or, in the momentum representation,

$$\langle\bar{p}_1\bar{p}_2\mid I_2\mid\bar{p}_1^0\bar{p}_2^0\rangle=\frac{\delta(\bar{p}_1-\bar{p}_1^0)\,t_2(\bar{p}_2,\ \bar{p}_2^0,\ E-p_1^{02}/2m_1+\mathrm{i}\tau)}{(E-p_1^2/2m_1-p_2^2/2m_2+\mathrm{i}\tau)(E-p_1^{02}/2m_1-p_2^{02}/2m_2+\mathrm{i}\tau)}. \quad (49)$$

Similarly, for I_3

$$I_3=G_0(E+\mathrm{i}\tau)t_1(E-H_{02}+\mathrm{i}\tau)G_0(E+\mathrm{i}\tau). \quad (50)$$

We transform the remaining integral I_4 by means of the identity

$$g_{01}(\varepsilon+\mathrm{i}\tau_1)\otimes g_{02}(E-\varepsilon+\mathrm{i}\tau_2)$$
$$=G_0(E+\mathrm{i}\tau)[g_{01}(\varepsilon+\mathrm{i}\tau_1)+g_{02}(E-\varepsilon+\mathrm{i}\tau_2)]$$
$$=[g_{01}(\varepsilon+\mathrm{i}\tau)+g_{02}(E-\varepsilon+\mathrm{i}\tau_2)]G_0(E+\mathrm{i}\tau) \quad (51)$$

to the form

$$I_4=\int_{-\infty}^{\infty}\frac{d\varepsilon}{-2\pi\mathrm{i}}g_{01}(\varepsilon+\mathrm{i}\tau_1)t_1(\varepsilon+\mathrm{i}\tau_1)g_{01}(\varepsilon+\mathrm{i}\tau_1)$$
$$\otimes g_{02}(E-\varepsilon+\mathrm{i}\tau_2)t_2(E-\varepsilon+\mathrm{i}\tau_2)g_{02}(E-\varepsilon+\mathrm{i}\tau_2)$$
$$=G_0(E+\mathrm{i}\tau)\mathscr{F}_{1\otimes2}(E+\mathrm{i}\tau)G_0(E+\mathrm{i}\tau),$$

where the operator $\mathscr{F}_{1\ 2}(E+i\tau)$ denotes the integral

$$\mathscr{F}_{1\otimes2}(E+\mathrm{i}\tau)=\int_{-\infty}^{\infty}\frac{d\varepsilon}{-2\pi\mathrm{i}}[g_{01}(\varepsilon+\mathrm{i}\tau_1)+g_{02}(E-\varepsilon+\mathrm{i}\tau_2)]$$
$$\times t_1(\varepsilon+\mathrm{i}\tau_1)t_2(E-\varepsilon+\mathrm{i}\tau_2)[g_{01}(\varepsilon+\mathrm{i}\tau_1)+g_{02}(E-\varepsilon+\mathrm{i}\tau_2)],$$
$$(\tau_1,\ \tau_2>0,\ \tau_1+\tau_2=\tau). \quad (52)$$

Substituting the sum of the integrals I_i $(i=1,2,3,4)$ in

(41), we obtain an expression for the operator

$$N_{1,2}(z)=t_1(z_1)+t_2(z_2)+\mathscr{F}_{1\otimes 2}(z), \tag{53}$$

where $z_1=z-H_{02}$, $z_2=z-H_{01}$, and the operator $\mathscr{F}_{1\otimes 2}(z)$ is defined by (52).

We now turn to the study of the S matrix of the independent subsystems: $S_{1\otimes 2}$. The expression for it has the form

$$(S_{1\otimes 2})_{\gamma\gamma'}=(W_\gamma^-)^+W_{\gamma'}^+. \tag{54}$$

We consider in detail the case $\gamma=\gamma'=0$. Then

$$S_{1\otimes 2}=(W_0^-)W_0^+=(\Omega_1^-\otimes\Omega_2^-)^+(\Omega_1^+\otimes\Omega_2^+)=(\Omega_1^{-+}\Omega_1^+)\otimes(\Omega_2^{-+}\Omega_2^+)=S_1\otimes S_2 \tag{55}$$

or, in the momentum representation,

$$\begin{aligned}\langle\overline{\mathbf{p}}_1\overline{\mathbf{p}}_2\mid S_{1\otimes 2}\mid\overline{\mathbf{p}}_1^0\overline{\mathbf{p}}_2^0\rangle&=\delta(\overline{\mathbf{p}}_1-\overline{\mathbf{p}}_1^0)\,\delta(\overline{\mathbf{p}}_2-\overline{\mathbf{p}}_2^0)\\&-2\pi i\delta(\overline{\mathbf{p}}_2-\overline{\mathbf{p}}_2^0)\,t_1\left(\overline{\mathbf{p}}_1,\overline{\mathbf{p}}_1^0,\frac{p_1^{02}}{2m_1}+i0\right)\delta(p_1^2/2m_1-p_1^{02}/2m_1)\\&-2\pi i\delta(p_2^2/2m_2-p_2^{02}/2m_2)\,\delta(\overline{\mathbf{p}}_1-\overline{\mathbf{p}}_1^0)\,t_2(\overline{\mathbf{p}}_2,\overline{\mathbf{p}}_2^0,p_2^{02}/2m_2+i0)\\&+(2\pi i)^2\delta(p_1^2/2m_1-p_1^{02}/2m_1)\,\delta(p_2^2/2m_2-p_2^{02}/2m_2)\,t_1(\overline{\mathbf{p}}_1,\overline{\mathbf{p}}_1^0,p_1^{02}/2m_1+i0)\\&\times t_2(\overline{\mathbf{p}}_2,\overline{\mathbf{p}}_2^0,p_2^{02}/2m_2+i0).\end{aligned} \tag{56}$$

The same result (56) follows from the representation of the S matrix in terms of the operator $N_{1,2}(z)$:

$$\begin{aligned}\langle\overline{\mathbf{p}}_1\overline{\mathbf{p}}_2\mid S_{1\otimes 2}\mid\overline{\mathbf{p}}_1^0\overline{\mathbf{p}}_2^0\rangle&=\delta(\overline{\mathbf{p}}_1-\overline{\mathbf{p}}_1^0)\,\delta(\overline{\mathbf{p}}_2-\overline{\mathbf{p}}_2^0)\\&-2\pi i\delta(p_1^2/2m_1+p_2^2/2m_2-E_0)\,\langle\overline{\mathbf{p}}_1\overline{\mathbf{p}}_2\mid N_{1,2}(E_0+i0)\mid\overline{\mathbf{p}}_1^0\overline{\mathbf{p}}_2^0\rangle,\end{aligned} \tag{57}$$

where $E_0=p_1^{02}/2m_1+p_2^{02}/2m_2$. Indeed, the terms containing only one t matrix in (56) and (57) are identical, and therefore it is necessary to prove the equality of the terms quadratic in t. Setting $\tau_1=\tau_2=\tau$ in (52), we obtain on the energy shell

$$\begin{aligned}\langle\overline{\mathbf{p}}_1\overline{\mathbf{p}}_2\mid\mathscr{F}_{1\otimes 2}(E_0+2i\tau)\mid\overline{\mathbf{p}}_1^0\overline{\mathbf{p}}_2^0\rangle&=\int_{-\infty}^{\infty}\frac{d\varepsilon}{-2\pi i}\left[\frac{-2i\tau}{(\varepsilon-p_1^2/2m_1)^2+\tau^2}\right]\\&\times t_1(\overline{\mathbf{p}}_1,\overline{\mathbf{p}}_1^0,\varepsilon+i\tau)\,t_2(\overline{\mathbf{p}}_2,\overline{\mathbf{p}}_2^0,E-\varepsilon+i\tau)\left[\frac{-2i\tau}{(\varepsilon-p_1^{02}/2m_1)^2+\tau^2}\right],\\&(p_1^2/2m_1+p_2^2/2m_2=p_1^{02}/2m_1+p_2^{02}/2m_2).\end{aligned} \tag{58}$$

Going to the limit $\tau\to 0$ in (58) and noting that the terms in the square brackets are equal to δ functions,

$$[-2\pi i\delta(\varepsilon-p_1^2/2m_1)]\,[-2\pi i\delta(\varepsilon-p_1^{02}/2m_1)],$$

we obtain after integration over ε

$$\begin{aligned}&\langle\overline{\mathbf{p}}_1\overline{\mathbf{p}}_2\mid\mathscr{F}_{1\otimes 2}(E_0+i0)\mid\overline{\mathbf{p}}_1^0\overline{\mathbf{p}}_2^0\rangle\\&=-2\pi i\delta(p_1^2/2m_1-p_1^{02}/2m_1)\,t_1(\overline{\mathbf{p}}_1,\overline{\mathbf{p}}_1^0,p_1^{02}/2m_1+i0)\\&\times t_2(\overline{\mathbf{p}}_2,\overline{\mathbf{p}}_2^0,p_2^{02}/2m_2+i0),\quad(p_1^2/2m_1+p_2^2/2m_2=p_1^{02}/2m_1+p_2^{02}/2m_2).\end{aligned} \tag{59}$$

Obviously, after multiplication of (59) by $-2\pi i\delta(p_1^2/2m_1+p_2^2/2m_2-E_0)$ we obtain the quadratic term in (56). Therefore, the representations (56) and (57) are identical.

Investigation of the analytic properties of the kernel of the operator $\mathscr{F}_{1\otimes 2}(z)$. It follows from (17) and (52) that the kernel of the operator $\mathscr{F}_{1\otimes 2}(z)$,

$$\begin{aligned}&\langle\mathbf{p}_1\mathbf{p}_2\mid\mathscr{F}_{1\otimes 2}(z)\mid\mathbf{p}_1^0\mathbf{p}_2^0\rangle\\&=\int_{-\infty}^{\infty}\frac{d\varepsilon}{-2\pi i}\left(\frac{1}{\varepsilon-p_1^2/2m_1+i\tau_1}+\frac{1}{E-\varepsilon-p_2^2/2m_2+i\tau_2}\right)\\&\times\left[\frac{\varphi_1(\overline{p}_1)\varphi_1^*(\overline{p}_1^0)}{\varepsilon+\varkappa_1^2+i\tau_1}+\hat{t}_1(\overline{p}_1,\overline{p}_1^0,\varepsilon+i\tau_1)\right]\\&\times\left[\frac{\varphi_2(\overline{p}_2)\varphi_2^*(\overline{p}_2^0)}{E-\varepsilon+\varkappa_2^2+i\tau_2}+\hat{t}_2(\overline{p}_2,\overline{p}_2^0,E-\varepsilon+i\tau_2)\right]\\&\times\left(\frac{1}{\varepsilon-p_1^{02}/2m_1+i\tau_1}+\frac{1}{E-\varepsilon-p_2^{02}/2m_2+i\tau_2}\right),\end{aligned} \tag{60}$$

can be represented in the form

$$\begin{aligned}&\langle\mathbf{p}_1\mathbf{p}_2\mid\mathscr{F}_{1\otimes 2}(z)\mid\mathbf{p}_1^0\mathbf{p}_2^0\rangle\\&=\frac{\varphi_1(\overline{p}_1)}{z+\varkappa_1^2-p_2^2/2m_2}\left[M_1^2(\overline{p}_1,\overline{p}_1^0,z)\frac{\varphi_2^*(\overline{p}_2^0)}{z+\varkappa_2^2-p_1^{02}/2m_1}\right.\\&\left.+M_2^2(\overline{p}_2,\overline{p}_2^0,z)\frac{\varphi_1^*(\overline{p}_1^0)}{z+\varkappa_1^2-p_2^{02}/2m_2}+M_3^2(\overline{p}_2,\overline{p}_1^0,\overline{p}_2^0,z)\right]\\&+\frac{\varphi_2(\overline{p}_2)}{z+\varkappa_2^2-p_1^2/2m_1}\left[M_1^1(\overline{p}_1,\overline{p}_1^0,z)\frac{\varphi_2^*(\overline{p}_2^0)}{z+\varkappa_2^2-p_1^{02}/2m_1}\right.\\&\left.+M_2^1(\overline{p}_1,\overline{p}_2^0,z)\frac{\varphi_1^*(\overline{p}_1^0)}{z+\varkappa_1^2-p_2^{02}/2m_2}+M_3^1(\overline{p}_1,\overline{p}_1^0,\overline{p}_2^0,z)\right]\\&+\left[M_1^3(\overline{p}_1\overline{p}_2,\overline{p}_1^0,z)\frac{\varphi_2^*(\overline{p}_2^0)}{z+\varkappa_2^2-p_1^{02}/2m_1}\right.\\&\left.+M_2^3(\overline{p}_1\overline{p}_2,\overline{p}_2^0,z)\frac{\varphi_1^*(\overline{p}_1^0)}{z+\varkappa_1^2-p_2^{02}/2m_2}+M_3^3(\overline{p}_1,\overline{p}_2,\overline{p}_1^0\overline{p}_2^0,z)\right].\end{aligned} \tag{61}$$

Below, we give a derivation of the spectral decomposition (61). We transform the singular denominators in (60) by means of the identity

$$1/ab=(1/a+1/b)/(a+b), \tag{62}$$

where a and b are, for example, $\varepsilon+\varkappa_1^2+i\tau_1$ and $E-\varepsilon-p_2^2/2m_2+i\tau_2$, etc. Then if in the first stage we do not consider the singularities of the Green's functions standing to the right of the scattering amplitudes in the integral (60), for the operator $\mathscr{F}_{1\otimes 2}(z)$ we obtain a representation of the form

$$\mathscr{F}_{1\otimes 2}(z)=\frac{\mid\varphi_1\rangle}{z+\varkappa_1^2-H_{01}}\mathscr{K}_2(z)+\frac{\mid\varphi_2\rangle}{z+\varkappa_2^2-H_{02}}\mathscr{K}_1(z)+\mathscr{K}_3(z) \tag{63}$$

or, in matrix form,

$$\begin{aligned}\langle\overline{p}_1\overline{p}_2\mid\mathscr{F}_{1\otimes 2}(z)\mid\overline{p}_1^0\overline{p}_2^0\rangle&=\frac{\varphi_1(\overline{p}_1)\,\mathscr{K}_2(\overline{p}_2,\overline{p}_1^0\overline{p}_2^0,z)}{z+\varkappa_1^2-p_2^2/2m_2}\\&+\frac{\varphi_2(\overline{p}_2)\,\mathscr{K}_1(\overline{p}_1,\overline{p}_1^0\overline{p}_2^0,z)}{z+\varkappa_2^2-p_1^2/2m_1}+\mathscr{K}_3(\overline{p}_1\overline{p}_2,\overline{p}_1^0\overline{p}_2^0,z).\end{aligned} \tag{64}$$

The kernels of the operators $\mathscr{K}_i(z)$, $i=1,2,3$, in (63) are given as follows:

$$\left.\begin{aligned}&\mathscr{K}_1(\overline{p}_1,\overline{p}_1^0\overline{p}_2^0,z)=\int_{-\infty}^{\infty}\frac{d\varepsilon}{-2\pi i}\left(\frac{1}{\varepsilon-p_1^2/2m_1+i\tau_1}+\frac{1}{E-\varepsilon+\varkappa_2^2+i\tau_2}\right)\\&\times t_1(\overline{p}_1,\overline{p}_1^0,\varepsilon+i\tau_1)\,\varphi_2^*(\overline{p}_2^0)\\&\times\left(\frac{1}{\varepsilon-p_1^{02}/2m_1+i\tau_1}+\frac{1}{E-\varepsilon-p_2^{02}/2m_2+i\tau_2}\right);\\&\mathscr{K}_2(\overline{p}_2,\overline{p}_1^0\overline{p}_2^0,z)=\int_{-\infty}^{\infty}\frac{d\varepsilon}{-2\pi i}\left(\frac{1}{E-\varepsilon-p_2^2/2m_2+i\tau_2}+\frac{1}{\varepsilon+\varkappa_1^2+i\tau_1}\right)\\&\times t_2(\overline{p}_2,\overline{p}_2^0,E-\varepsilon+i\tau_2)\,\varphi_1^*(\overline{p}_1^0)\\&\times\left(\frac{1}{\varepsilon-p_1^{02}/2m_1+i\tau_1}+\frac{1}{E-\varepsilon-p_2^{02}/2m_2+i\tau_2}\right);\\&\mathscr{K}_3(\overline{p}_1\overline{p}_2,\overline{p}_1^0\overline{p}_2^0,z)\\&=\int_{-\infty}^{\infty}\frac{d\varepsilon}{-2\pi i}\left[\frac{t_1(\overline{p}_1,\overline{p}_1^0,\varepsilon+i\tau_1)\,\hat{t}_2(\overline{p}_2,\overline{p}_2^0,E-\varepsilon+i\tau_2)}{\varepsilon-p_1^2/2m_1+i\tau_1}\right.\\&\left.+\frac{\hat{t}_1(\overline{p}_1,\overline{p}_1^0,\varepsilon+i\tau_1)\,t_2(\overline{p}_2,\overline{p}_2^0,E-\varepsilon+i\tau_2)}{E-\varepsilon-p_2^2/2m_2+i\tau_2}\right]\\&\times\left(\frac{1}{\varepsilon-p_1^{02}/2m_1+i\tau_1}+\frac{1}{E-\varepsilon-p_2^{02}/2m_2+i\tau_2}\right).\end{aligned}\right\} \tag{65}$$

The expressions for the kernels $\mathscr{K}_i(\cdot,\overline{p}_1^0\overline{p}_2^0,z)$ are transformed in the same way as the expression for $\langle\mathbf{p}_1\mathbf{p}_2\mid\mathscr{F}_{1\otimes 2}(z)\mid\overline{\mathbf{p}}_1^0\overline{\mathbf{p}}_2^0\rangle$. As a result, the decomposition for kernels of the type $\mathscr{K}$ takes the form

$$\begin{aligned}\mathscr{K}_i(\cdot,\overline{p}_1^0\overline{p}_2^0,z)&=M_1^i(\cdot,\overline{p}_1^0,z)\frac{\varphi_2^*(\overline{p}_2^0)}{z+\varkappa_2^2-p_1^{02}/2m_1}\\&+M_2^i(\cdot,\overline{p}_2^0,z)\frac{\varphi_1^*(\overline{p}_1^0)}{z+\varkappa_1^2-p_2^{02}/2m_2}+M_3^i(\cdot,\overline{p}_1^0\overline{p}_2^0,z).\end{aligned} \tag{66}$$

The explicit form of the functions $M_j^i(z)$, which in what follows we shall call the components of $\mathscr{F}_{1\otimes 2}(z)$, will be given below. In calculating the matrix elements of the operators $M_j^i(z)$, we shall use the following properties of the function $\hat{t}(\bar{\mathbf{p}}, \bar{\mathbf{p}}^0, z)$.[5] The function $\hat{t}(\mathbf{p}, \mathbf{p}^0, z)$ is a sum of the form

$$t(\bar{\mathbf{p}}, \bar{\mathbf{p}}^0, z) = V(\bar{\mathbf{p}} - \bar{\mathbf{p}}^0) + \tau(\bar{\mathbf{p}}, \bar{\mathbf{p}}^0, z), \tag{67}$$

where

$$\tau(\bar{\mathbf{p}}, \bar{\mathbf{p}}^0, z) = \int d\bar{\mathbf{q}} \frac{t(\bar{\mathbf{p}}, \bar{\mathbf{q}}, q^2/2m + i0)\, t(\bar{\mathbf{q}}, \bar{\mathbf{p}}^0, q^2/2m - i0)}{z - q^2/2m}. \tag{68}$$

We consider a typical integral with the function $\tau(\bar{\mathbf{p}}, \bar{\mathbf{p}}^0, z)$:

$$I(z) = \int_{-\infty}^{\infty} \frac{d\varepsilon}{-2\pi i} \frac{\tau(\bar{\mathbf{p}}, \bar{\mathbf{p}}^0, \varepsilon + i\tau)}{\varepsilon - z}. \tag{69}$$

For $\operatorname{Im} z \neq 0$, $\tau > 0$, the integral $I(z)$ can be calculated by means of the representation (69):

$$\begin{aligned} I(z) &= \int_{-\infty}^{\infty} \frac{d\varepsilon}{-2\pi i} \frac{1}{\varepsilon - z} \\ &\times \int d\bar{\mathbf{q}} \frac{t(\bar{\mathbf{p}}, \bar{\mathbf{q}}, q^2/2m + i0)\, t(\bar{\mathbf{q}}, \bar{\mathbf{p}}^0, q^2/2m - i0)}{\varepsilon - q^2/2m + i\tau} \\ &= \int d\bar{\mathbf{q}}\, t(\bar{\mathbf{p}}, \bar{\mathbf{q}}, q^2/2m + i0)\, t(\bar{\mathbf{q}}, \bar{\mathbf{p}}^0, q^2/2m - i0) \\ &\times \int_{-\infty}^{\infty} \frac{d\varepsilon}{-2\pi i} \frac{1}{(\varepsilon - z)(\varepsilon + i\tau - q^2/2m)}. \end{aligned} \tag{70}$$

In this expression, we have changed the order of integration, which is valid for all finite τ and $\operatorname{Im} z$. From (70), we have

$$I(z) = \begin{cases} 0, & \operatorname{Im} z < 0, \\ -\tau(\bar{p}, \bar{p}^0, z + i\tau), & \operatorname{Im} z > 0. \end{cases} \tag{71}$$

Calculating by means of (71) the integrals for the kernels of the operators $M_j^i(z)$, we obtain the result

$$\left.\begin{aligned} & M_1^1(\bar{\mathbf{p}}_1, \bar{\mathbf{p}}_1^0, z) = V_1(\bar{\mathbf{p}}_1 - \bar{\mathbf{p}}_1^0) \\ & + \int_{-\infty}^{\infty} \frac{d\varepsilon}{-2\pi i} \left[\frac{\varphi_1(\bar{\mathbf{p}}_1)\varphi_1^*(\bar{\mathbf{p}}_1^0)}{\varepsilon + \varkappa_1^2 + i\tau_1} + \tau_1(\bar{\mathbf{p}}_1, \bar{\mathbf{p}}_1^0, \varepsilon + i\tau_1) \right] \\ & \times \left(\frac{1}{\varepsilon - p_1^{02}/2m_1 + i\tau_1} + \frac{1}{E - \varepsilon + \varkappa_2^2 + i\tau_2} \right) \\ & = V_1(\bar{\mathbf{p}}_1 - \bar{\mathbf{p}}_1^0) + \frac{\varphi_1(\bar{\mathbf{p}}_1)\varphi_1^*(\bar{\mathbf{p}}_1^0)}{z + \varkappa_1^2 + \varkappa_2^2} \\ & + \tau_1(\bar{\mathbf{p}}_1, \bar{\mathbf{p}}_1^0, z + \varkappa_2^2) = t_1(\bar{\mathbf{p}}_1, \bar{\mathbf{p}}_1^0, z + \varkappa_2^2); \\ & M_2^1(\bar{\mathbf{p}}_1, \bar{\mathbf{p}}_2^0, z) \\ & = \int_{-\infty}^{\infty} \frac{d\varepsilon}{-2\pi i} \left(\frac{1}{\varepsilon - p_1^2/2m_1 + i\tau_1} + \frac{1}{E - \varepsilon + \varkappa_2^2 + i\tau_2} \right) \\ & \times \varphi_1(\bar{\mathbf{p}}_1)\varphi_2^*(\bar{\mathbf{p}}_2^0) \left(\frac{1}{\varepsilon + \varkappa_1^2 + i\tau_1} + \frac{1}{E - \varepsilon - p_2^{02}/2m_2 + i\tau_2} \right) \\ & = \frac{\varphi_1(\bar{\mathbf{p}}_1)\varphi_2^*(\bar{\mathbf{p}}_2^0)}{z + \varkappa_1^2 + \varkappa_2^2} + \frac{\varphi_1(\bar{\mathbf{p}}_1)\varphi_2^*(\bar{\mathbf{p}}_2^0)}{z - p_1^2/2m_1 - p_2^{02}/2m_2}; \\ & M_3^1(\bar{\mathbf{p}}_1, \bar{\mathbf{p}}_1^0\bar{\mathbf{p}}_2^0, z) \\ & = \int_{-\infty}^{\infty} \frac{d\varepsilon}{-2\pi i} \frac{t_1(\bar{\mathbf{p}}_1, \bar{\mathbf{p}}_1^0, \varepsilon + i\tau_1)\varphi_2^*(\bar{\mathbf{p}}_2^0)}{(\varepsilon - p_1^2/2m_1 + i\tau_1)(\varepsilon - p_2^{02}/2m_2 + i\tau_2)} \\ & + \int_{-\infty}^{\infty} \frac{d\varepsilon}{-2\pi i} \left(\frac{1}{\varepsilon - p_1^2/2m_1 + i\tau_1} + \frac{1}{E - \varepsilon + \varkappa_2^2 + i\tau_2} \right) \\ & \times \frac{\hat{t}_1(\bar{\mathbf{p}}_1, \bar{\mathbf{p}}_1^0, \varepsilon + i\tau_1)\varphi_2^*(\bar{\mathbf{p}}_2^0)}{E - \varepsilon - p_2^{02}/2m_2 + i\tau_2} = \frac{\hat{t}_1(\bar{\mathbf{p}}_1, \bar{\mathbf{p}}_1^0, z - p_2^{02}/2m_2)\varphi_2^*(\bar{\mathbf{p}}_2^0)}{z - p_1^2/2m_1 - p_2^{02}/2m_2} \\ & + [\tau_1(\bar{\mathbf{p}}_1, \bar{\mathbf{p}}_1^0, z + \varkappa_2^2) - \tau_1(\bar{\mathbf{p}}_1, \bar{\mathbf{p}}_1^0, z - p_2^{02}/2m_2)] \frac{\varphi_2^*(\bar{\mathbf{p}}_2^0)}{-\varkappa_2^2 - p_2^{02}/2m_2}. \end{aligned}\right\} \tag{72}$$

The expressions for the kernels of the operators $M_j^2(z)$ differ from (72) by replacement of the indices 1 by 2 and vice versa:

$$\left.\begin{aligned} & M_1^2(\bar{\mathbf{p}}_2, \bar{\mathbf{p}}_1^0, z) = \frac{\varphi_2(\bar{\mathbf{p}}_2)\varphi_1^*(\bar{\mathbf{p}}_1^0)}{z + \varkappa_1^2 + \varkappa_2^2} + \frac{\bar{\varphi}_2(\bar{\mathbf{p}}^2)\varphi_1^*(\mathbf{p}_1^0)}{z - p_1^{02}/2m_1 - p_2^2/2m_2}; \\ & M_2^2(\bar{\mathbf{p}}_2, \bar{\mathbf{p}}_2^0, z) = t_2(\bar{\mathbf{p}}_2\, \bar{\mathbf{p}}_2^0, z + \varkappa_1^2); \\ & M_3^2(\bar{\mathbf{p}}_2, \bar{\mathbf{p}}_1^0\bar{\mathbf{p}}_2^0, z) = \frac{\hat{t}^2(\bar{\mathbf{p}}_2^0, \bar{\mathbf{p}}_2^0, z - p_1^{02}/2m_1)\varphi_1^*(\bar{\mathbf{p}}_1^0)}{z - p_2^2/2m_2 - p_1^{02}/2m_1} \\ & + \left[\tau_2(\bar{\mathbf{p}}_2, \bar{\mathbf{p}}_2^0, z + \varkappa_1^2) - \tau_2\left(\bar{\mathbf{p}}_2, \bar{\mathbf{p}}_2^0, z - \frac{p_1^{02}}{2m_1}\right) \right] \frac{\varphi_1^*(\bar{\mathbf{p}}_1^0)}{-\varkappa_1^2 - p_1^{02}/2m_1}. \end{aligned}\right\} \tag{73}$$

The kernels of the operators $M_1^0(z)$ and $M_2^3(z)$ have a form analogous to $M_3^1(z)$ and $M_3^2(z)$ by virtue of the symmetry of the expression for $\langle \mathbf{p}_1\mathbf{p}_2 | \mathscr{F}_{1\otimes 2}(z) | p_1^0\mathbf{p}_2^0 \rangle$ (60) with respect to transposition of the "in" and "out" states:

$$\left.\begin{aligned} & M_1^3(\bar{\mathbf{p}}_1\bar{\mathbf{p}}_2, \bar{\mathbf{p}}_1^0, z) = \frac{\varphi_2(\bar{\mathbf{p}}_2)\hat{t}_1(\bar{\mathbf{p}}_1, \bar{\mathbf{p}}_1^0, z - p_2^2/2m_2)}{z - p_2^2/2m_2 - p_1^{02}/2m_1} \\ & + [\tau_1(\bar{\mathbf{p}}_1, \bar{\mathbf{p}}_1^0, z + \varkappa_2^2) - \tau_1(\bar{\mathbf{p}}_1, \bar{\mathbf{p}}_1^0, z - p_2^2/2m_2)] \frac{\varphi_2(\bar{\mathbf{p}}_2)}{-\varkappa_2^2 - p_2^2/2m_2}; \\ & M_2^3(\bar{\mathbf{p}}_1\bar{\mathbf{p}}_2, \bar{\mathbf{p}}_2^0, z) = \frac{\varphi_1(\bar{\mathbf{p}}_1)\hat{t}_2(\bar{\mathbf{p}}_2, \bar{\mathbf{p}}_2^0, z - p_1^2/2m_1)}{z - p_1^2/2m_1 - p_2^{02}/2m_2} \\ & + \left[\tau_2(\bar{\mathbf{p}}_2, \bar{\mathbf{p}}_2^0, z + \varkappa_1^2) - \tau_2\left(\bar{\mathbf{p}}_2, \bar{\mathbf{p}}_2^0, z - \frac{p_1^2}{2m_1}\right) \right] \frac{\varphi_1(p_1)}{-\varphi_1^2 - p_1^2/2m_1}. \end{aligned}\right\} \tag{74}$$

Further, the expression for the operator $M_3^3(z)$ has the form

$$M_3^3(z) = A_1(z) + A_2(z) + A_3(z), \tag{75}$$

where

$$\left.\begin{aligned} & A_1(z) = \int_{-\infty}^{\infty} \frac{d\varepsilon}{-2\pi i} \frac{\hat{t}_2(\bar{\mathbf{p}}_2, \bar{\mathbf{p}}_2^0, E - \varepsilon + i\tau_2)\varphi_1(\bar{\mathbf{p}}_1)\varphi_1^*(\bar{\mathbf{p}}_1^0)}{(\varepsilon - p_1^2/2m_1 + i\tau_1)(\varepsilon + \varkappa_1^2 + i\tau_1)(\varepsilon - p_1^{02}/2m_1 + i\tau_1)}; \\ & A_2(z) = \int_{-\infty}^{\infty} \frac{d\varepsilon}{-2\pi i} \\ & \qquad \times \frac{\hat{t}_1(\bar{\mathbf{p}}_1, \bar{\mathbf{p}}_1^0, \varepsilon + i\tau_1)\varphi_2(\bar{p}_2)\varphi_2^*(\bar{\mathbf{p}}_2^0)}{(E - \varepsilon - p_2^2/2m_2 + i\tau_2)(E - \varepsilon + \varkappa_2^2 + i\tau)(E - \varepsilon - p_2^{02}/2m_2 + i\tau_2)}; \\ & A_3(z) = \int_{-\infty}^{\infty} \frac{d\varepsilon}{-2\pi i} \left(\frac{1}{\varepsilon - p_1^2/2m_1 + i\tau} + \frac{1}{E - \varepsilon + p_2^2/2m_2 + i\tau_2} \right) \\ & \qquad \times \hat{t}_1(\bar{\mathbf{p}}_1, \bar{\mathbf{p}}_1^0, \varepsilon + i\tau_1)\hat{t}_2(\bar{\mathbf{p}}_2, \bar{\mathbf{p}}_2^0, E - \varepsilon + i\tau_2) \\ & \qquad \times \left(\frac{1}{\varepsilon - p_1^{02}/2m_1 + i\tau_1} + \frac{1}{E - \varepsilon - p_2^{02}/2m_2 + i\tau_2} \right). \end{aligned}\right\} \tag{76}$$

The integrals A_1 and A_2 can be readily calculated by means of (34):

$$A_1(\bar{\mathbf{p}}_1\bar{\mathbf{p}}_2, \bar{\mathbf{p}}_1^0\bar{\mathbf{p}}_2^0, z) = \varphi_1(\bar{\mathbf{p}}_1)\varphi_1^*(\bar{\mathbf{p}}_1^0) \frac{\Phi_1(\bar{\mathbf{p}}_2, \bar{\mathbf{p}}_2^0; \mathbf{p}_1^2, \tau_1) - \Phi_1(\bar{\mathbf{p}}_2, \bar{\mathbf{p}}_2^0; \mathbf{p}_1^{02}, \tau_1)}{p_1^2 - p_1^{02}},$$

where

$$\begin{aligned} \Phi_1(\bar{\mathbf{p}}_2, \bar{\mathbf{p}}_2^0; q_1^2, \tau_1) &= 2m_1 \int_{-\infty}^{\infty} \frac{d\varepsilon}{-2\pi i} \frac{\hat{t}_2(\bar{\mathbf{p}}_2, \bar{\mathbf{p}}_2^0, E - \varepsilon + i\tau_2)}{(\varepsilon + \varkappa_1^2 + i\tau_1)(\varepsilon - q_1^2/2m_1 + i\tau_1)} \\ &= 2m_1 \frac{\tau_2(\bar{\mathbf{p}}_2, \bar{\mathbf{p}}_2^0, z + \varkappa_1^2) - \tau_2(\bar{\mathbf{p}}_2, \bar{\mathbf{p}}_2^0, z - q_1^2/2m_1)}{-\varkappa_1^2 - q_1^2/2m_1}. \end{aligned} \tag{77}$$

Since $\tau(\bar{\mathbf{p}}, \bar{\mathbf{p}}^0, z)$ is a Hölder function of its variables,[5] in the limit $\tau_1, \tau_2 \to 0$ the function $A_1(\bar{\mathbf{p}}_1, \bar{\mathbf{p}}_2, \bar{\mathbf{p}}_1^0\bar{\mathbf{p}}_2^0, z)$ has in the region $p_1 \approx p_1^0$ a secondary singularity of the form $|p_1 - p_1^0|^{-1+\nu}$, $0 < \nu < 1/2$. The expression for A_2 is obtained from A_1 by replacement of the indices 1 and 2:

$$\left.\begin{aligned} & A_2(\bar{\mathbf{p}}_1\bar{\mathbf{p}}_2, \bar{\mathbf{p}}_1^0\bar{\mathbf{p}}_2^0, z) \\ & = \varphi_2(\bar{\mathbf{p}}_2)\varphi_2^*(\bar{\mathbf{p}}_2^0) \frac{\Phi_2(\bar{\mathbf{p}}_1, \bar{\mathbf{p}}_1^0; p_2^{02}, \tau_2) - \Phi_2(\bar{\mathbf{p}}_1, \bar{\mathbf{p}}_1^0; p_2^{02}, \tau_2)}{p_2^2 - p_2^{02}}; \\ & \Phi_2(\bar{\mathbf{p}}_1, \bar{\mathbf{p}}_1^0; q_2^2, \tau_2) \\ & = 2m_2 \frac{\tau_1(\bar{\mathbf{p}}_1, \bar{\mathbf{p}}_1^0; z + \varkappa_2^2) - \tau_1(\bar{\mathbf{p}}_1, \bar{\mathbf{p}}_1^0, z - q_2^2/2m_2)}{-\varkappa_2^2 - q_2^2/2m_2}. \end{aligned}\right\} \tag{78}$$

Since the operator $\hat{t}(z)$ is the sum of two terms, $\hat{t}(z) = V + \tau(z)$,

$$A_3(z) = B_1(z) + B_2(z) + B_3(z) + B_4(z). \tag{79}$$

The term $B_1(z)$ in (79) is generated by the product $V_1 V_2$:

$$B_1(z)=V_1(\overline{\mathbf{p}}_1-\overline{\mathbf{p}}_1^0)\,V_2(\overline{\mathbf{p}}_2-\overline{\mathbf{p}}_2^0)\times\left(\frac{1}{z-p_1^2/2m_1-p_2^{02}/2m_2}+\frac{1}{z-p_2^2/2m_2-p_1^{02}/2m_1}\right), \quad (80)$$

and $B_2(z)$ and $B_3(z)$ by the product of the potential and the operator $\tau(z)$:

$$\begin{aligned}B_2(z)&=\int_{-\infty}^{\infty}\frac{d\varepsilon}{-2\pi i}\left(\frac{1}{\varepsilon-p_1^2/2m_1+i\tau_1}+\frac{1}{E-\varepsilon-p_2^2/2m_2+i\tau_2}\right)\\&\times V_1(\overline{\mathbf{p}}_1-\overline{\mathbf{p}}_1^0)\,\tau_2(\overline{\mathbf{p}}_2,\overline{\mathbf{p}}_2^0,E-\varepsilon+i\tau_2)\\&\times\left(\frac{1}{\varepsilon-p_1^{02}/2m_1+i\tau_1}+\frac{1}{E-\varepsilon-p_2^{02}/2m_2+i\tau_2}\right)\\&=V_1(\overline{\mathbf{p}}_1-\overline{\mathbf{p}}_1^0)\left[\frac{\tau_2(\overline{\mathbf{p}}_2,\overline{\mathbf{p}}_2^0,z-p_1^2/2m_1)}{z-p_1^2/2m_1-p_2^{02}/2m_2}+\frac{\tau_2(\overline{\mathbf{p}}_2,\overline{\mathbf{p}}_2^0,z-p_1^{02}/2m_1)}{z-p_1^{02}/2m_1-p_2^2/2m_2}\right.\\&\left.+\frac{\tau_2(\overline{\mathbf{p}}^2,\overline{\mathbf{p}}_2^0,z-p_1^2/2m_1)-\tau_2(\overline{\mathbf{p}}_2,\overline{\mathbf{p}}_2^0,z-p_1^{02}/2m_1)}{p_1^2/2m_1-p_1^{02}/2m_1}\right]. \quad (81)\end{aligned}$$

The case of $B_3(z)$ is treated similarly. Therefore, the kernels of the operators $B_2(z)$ and $B_3(z)$ have singularities of the form $(z-p^2/2m-p^{02}/2m)^{-1}$ and $|p-p^0|^{-1+\nu}$. Finally, the kernel of the operator $B_4(z)$ has the form

$$\begin{aligned}B_4(z)&=\int_{-\infty}^{\infty}\frac{d\varepsilon}{-2\pi i}\left(\frac{1}{\varepsilon-p_1^2/2m_1+i\tau_1}+\frac{1}{E-\varepsilon-p_2^2/2m_2+i\tau}\right)\\&\times\tau_1(\overline{\mathbf{p}}_1,\overline{\mathbf{p}}_1^0,\varepsilon+i\tau_1)\,\tau_2(\overline{\mathbf{p}}_2,\overline{\mathbf{p}}_2^0,E-\varepsilon+i\tau_2)\\&\times\left(\frac{1}{\varepsilon-p_1^{02}/2m_1+i\tau_1}+\frac{1}{E-\varepsilon-p_2^{02}/2m_2+i\tau_2}\right). \quad (82)\end{aligned}$$

Separating the singular denominators in (82) by means of the identity (62), we obtain

$$B_4(z)=\frac{D(\xi_1)-D(\xi_1^0)}{\xi_1-\xi_1^0}+\frac{D(\xi_1)-D(\xi_2^0)}{\xi_2^0-\xi_1}+\frac{D(\xi_1^0)-D(\xi_2)}{\xi_2-\xi_1^0}+\frac{D(\xi_2)-D(\xi_2^0)}{\xi_2-\xi_2^0}, \quad (83)$$

where

$$\begin{aligned}&\xi_1=p_1^2/2m_1-i\tau_1;\ \xi_1^0=p_1^{02}/2m_1-i\tau_1;\ \xi_2=E-p_2^2/2m_2+i\tau_2;\ \xi_2^0=E-p_2^{02}/2m_2+i\tau_2;\\&D(\overline{\mathbf{p}}_1\overline{\mathbf{p}}_2,\overline{\mathbf{p}}_1^0\overline{\mathbf{p}}_2^0,\xi)=\int_{-\infty}^{\infty}\frac{d\varepsilon}{-2\pi i}\frac{\tau_1(\overline{\mathbf{p}}_1,\overline{\mathbf{p}}_1^0,\varepsilon+i\tau_1)\,\tau_2(\overline{\mathbf{p}}_2,\overline{\mathbf{p}}_2^0,E-\varepsilon+i\tau_2)}{\varepsilon-\xi}. \quad (84)\end{aligned}$$

In accordance with the lemma on singular integrals,[5,14] the function $D(\xi)$ is a Hölder function of its variables in the region $\mathrm{Im}\,\xi\geqslant 0$ ($\mathrm{Im}\,\xi\leqslant 0$). Therefore, the term $B_4(z)$ has the same secondary singularities as $B_2(z)$.

We note that all the secondary singularities disappear when $E<0$. Indeed, for $E<0$ and $\tau\to+0$ there can only be secondary singularities of the form $|p-p^0|^{-1+\nu}$. We consider the typical integral that for $E<0$ generates this singularity:

$$I=\int_{-\infty}^{\infty}\frac{d\varepsilon}{-2\pi i}\frac{\tau_1(\overline{\mathbf{p}}_1,\overline{\mathbf{p}}_1^0,\varepsilon+i\tau_1)\,\tau_2(\overline{\mathbf{p}}_2,\overline{\mathbf{p}}_2^0,E-\varepsilon+i\tau_2)}{(\varepsilon-p_1^2/2m_1+i\tau_1)(\varepsilon-p_1^{02}/2m_1+i\tau_1)}. \quad (85)$$

In (85), we substitute $\tau_2(\overline{\mathbf{p}}_2,\overline{\mathbf{p}}_2^0,E-\varepsilon+i\tau_2)$ in the form (68). Then, proceeding exactly as in the proof of (71), we obtain

$$\begin{aligned}I&=\int_{-\infty}^{\infty}\frac{d\varepsilon}{-2\pi i}\frac{\tau_1(\overline{\mathbf{p}}_1,\overline{\mathbf{p}}_1^0,\varepsilon+i\tau_1)}{(\varepsilon-p_1^2/2m_1+i\tau_1)(\varepsilon-p_1^{02}/2m_1+i\tau_1)}\\&\times\int d\overline{\mathbf{q}}_2\frac{t_2(\overline{\mathbf{p}}_2,\overline{\mathbf{q}}_2,q_2^2/2m_2+i0)\,t_2(\overline{\mathbf{q}}_2,\overline{\mathbf{p}}_2^0,q_2^2/2m_2-i0)}{E-\varepsilon-q_2^2/2m_2+i\tau_2}\\&=\int d\overline{\mathbf{q}}_2\frac{\tau_1(\overline{\mathbf{p}}_1,\overline{\mathbf{p}}_1^0,z-q_2^2/2m_2)\,t_2(\overline{\mathbf{p}}_2,\overline{\mathbf{q}}_2,q_2^2/2m_2+i0)\,t_2(\overline{\mathbf{q}}_2,\overline{\mathbf{p}}_2^0,q_2^2/2m_2-i0)}{(z-p_1^2/2m_1-q_2^2/2m_2)(z-p_1^{02}/2m_1-q_2^2/2m_2)}, \quad (86)\end{aligned}$$

from which it can be clearly seen that for $E<0$ and $\tau=0$ the integrand in (85) does not have singularities.

Finally, the kernel of the operator $\mathscr{F}_{1\otimes 2}(z)$ has the form (61). The components of the operator $\mathscr{F}_{1\otimes 2}(z)M_j^i(z), i,j=1,2,3$, for $E>0$ and $\tau=0$ have secondary singularities of the form $(z-p_i^2/2m_i-p_j^{02}/2m_j)^{-1}$, $i,j=1,2, i\neq j$, and $|\mathbf{p}_i-\mathbf{p}_i^0|^{-1+\gamma}$. For $E<0$ and $\tau=0$, there is a singularity only with respect to the total energy $(z+\varkappa_1^2+\varkappa_2^2)^{-1}$ corresponding to the state of the discrete spectrum of $H_{1,2}$.

2. INTEGRAL EQUATIONS FOR COMPONENTS OF THE SCATTERING OPERATOR OF THE SYSTEM OF TWO PARTICLES IN THE EXTERNAL FIELD

System of integral equations for the components. The representations for the operators $t_\alpha(z)$ ($\alpha=1,2,12$) and $\mathscr{F}_{1\otimes 2}(z)$ of the form (25) and (61), (63), (66) make it possible to separate explicitly in the kernels and free terms of the system (16) the principal singularities and go over to a system of integral equations containing only secondary singularities. We first obtain integral equations for the components of the operator $T(z)$ under the condition that the total energy E of the system is not equal to $(-\varkappa_1^2-\varkappa_2^2)$. We shall consider the case $E=-\varkappa_1^2-\varkappa_2^2$ later by itself. From the expressions for the operators $t_\alpha(z)$ ($\alpha=1,2,12$) and $\mathscr{F}_{1\otimes 2}(z)$ it follows that the operators $L_{1,2}(z)$ and $L_{12}(z)$ have representations of the form

$$\left.\begin{aligned}L_{1,2}(z)&=u_{1,2}(z)+\frac{|\varphi_1\rangle}{z+\varkappa_1^2-H_{02}}v_2(z)+\frac{|\varphi_2\rangle}{z+\varkappa_2^2-H_{01}}v_1(z);\\L_{12}(z)&=u_{12}(z)+\frac{|\varphi_{12}\rangle}{z+\varkappa_{12}^2-H_{012}}v_{12}(z),\end{aligned}\right\} \quad (87)$$

where the operators $u_{1,2}(z), u_{12}(z), v_1(z), v_2(z), v_{12}(z)$ are given by

$$\left.\begin{aligned}u_{1,2}(z)&=[\hat{t}_1(z_1)+\hat{t}_2(z_2)+\mathscr{K}_3(z)]\,G_0(z)\,t_{12}(z_{12})\\&+[\hat{t}_1(z_1)+\hat{t}_2(z_2)+\mathscr{K}_3(z)]\,G_0(z)\,L_{12}(z);\\u_{12}(z)&=\hat{t}_{12}(z_{12})\,G_0(z)\,[t_1(z_1)+t_2(z_2)+\mathscr{F}_{1\otimes 2}(z)]\\&+\hat{t}_{12}(z_{12})\,G_0(z)\,L_{1,2}(z);\\v_1(z)&=[\langle\varphi_2|+\mathscr{K}_1(z)]\,G_0(z)\,t_{12}(z_{12})\\&+[\langle\varphi_2|+\mathscr{K}_1(z)]\,G_0(z)\,L_{12}(z);\\v_2(z)&=[\langle\varphi_1|+\mathscr{K}_2(z)]\,G_0(z)\,t_{12}(z_{12})\\&+[\langle\varphi_1|+\mathscr{K}_2(z)]\,G_0(z)\,L_{12}(z);\\v_{12}(z)&=\langle\varphi_{12}|G_0(z)\,[t_1(z_1)+t_2(z_2)+\mathscr{F}_{1\otimes 2}(z)]\\&+\langle\varphi_{12}|G_0(z)\,L_{1,2}(z).\end{aligned}\right\} \quad (88)$$

In turn, each of the operators $u(z)$ and $v(z)$ in the expressions (87) and (88) can be represented in factorized form. For example, for the operator $v_{12}(z)$ we have

$$\begin{aligned}v_{12}(z)&=\omega_{12}^0(z)+\omega_{12,12}(z)\langle\varphi_{12}|/(z+\varkappa_{12}^2-H_{012})\\&+\omega_{12,1}(z)\langle\varphi_2|/(z+\varkappa_2^2-H_{01})+\omega_{12,2}(z)\langle\varphi_1|/(z+\varkappa_1^2-H_{02}). \quad (89)\end{aligned}$$

A similar representation exists for the remaining operators of the type $u(z)$ and $v(z)$. We shall call the operators $v_\alpha(z)$ ($\alpha=1,2,12$) and $\omega_{\alpha,\beta}(z)$ in (87)–(89), and also the operators $u_{(1,2),\alpha}(z)$ and $u_{12,\alpha}(z)$ for $\alpha=1,2,12$ and the operators $u_{1,2}^0(z), u_{12}^0(z)$ and $\omega_\alpha^0(z)$, the components of the total scattering T matrix. From the representations (87)–(89), there follow systems of equations for the components. It is obvious that all these systems differ in the free terms but have the same kernel. We write down one of the systems, for example, the one for the components $u_{(1,2),1}(z), u_{12,1}(z), \omega_{\alpha,1}(z)$ ($\alpha=1,2,12$):

$$
\left.\begin{aligned}
&u_{(1,2),1}(z)=[\hat{t}_1(z_1)+\hat{t}_2(z_2)+\mathscr{K}_3(z)]\\
&\times G_0(z)\left[u_{12,1}(z)+\frac{|\varphi_{12}\rangle}{z+\varkappa_{12}^2-H_{012}}\,\omega_{12,1}(z)\right];\\
&u_{12,1}(z)=\hat{t}_{12}(z_{12})\,G_0(z)\,[|\varphi_2\rangle+B_1(z)]+\hat{t}_{12}(z_{12})\\
&\times G_0(z)\left[u_{(1,2),1}(z)+\frac{|\varphi_1\rangle}{z+\varkappa_1^2-H_{02}}\,\omega_{2,1}(z)\right.\\
&\left.+\frac{|\varphi_2\rangle}{z+\varkappa_2^2-H_{01}}\,\omega_{1,1}(z)\right];\\
&\omega_{1,1}(z)=[\langle\varphi_2|+\mathscr{K}_1(z)]\,G_0(z)\left[u_{12,1}(z)\right.\\
&\left.+\frac{|\varphi_{12}\rangle}{z+\varkappa_{12}^2-H_{012}}\,\omega_{12,1}(z)\right];\\
&\omega_{2,1}(z)=[\langle\varphi_1|+\mathscr{K}_2(z)]\,G_0(z)\left[u_{12,1}(z)\right.\\
&\left.+\frac{|\varphi_{12}\rangle}{z+\varkappa_{12}^2-H_{012}}\,\omega_{12,1}(z)\right];\\
&\omega_{12,1}(z)=\langle\varphi_{12}|G_0(z)|\varphi_1\rangle+\langle\varphi_{12}|G_0(z)\,B_1(z)\\
&+\langle\varphi_{12}|G_0(z)\left[u_{(1,2),1}(z)+\frac{|\varphi_1\rangle}{z+\varkappa_1^2-H_{02}}\,\omega_{2,1}(z)\right.\\
&\left.+\frac{|\varphi_2\rangle}{z+\varkappa_2^2-H_{01}}\,\omega_{1,1}(z)\right].
\end{aligned}\right\}\qquad(90)
$$

We now turn to the study of the integral equations for the components of the total T matrix. We write down, for example, the integral analog of the system (90):

$$
\left.\begin{aligned}
&u_{(1,2),1}(\bar{\mathbf{p}}_1,\ \bar{\mathbf{p}}_2,\ \mathbf{p}_1^0,\ z)\\
&=\int d\bar{\mathbf{p}}_1'\int d\bar{\mathbf{p}}_2'\,[\delta(\bar{\mathbf{p}}_2-\bar{\mathbf{p}}_2')\,\hat{t}_1(\bar{\mathbf{p}}_1,\ \bar{\mathbf{p}}_1',\ z-p_2^2/2m_2)\\
&+\delta(\bar{\mathbf{p}}_1-\bar{\mathbf{p}}_1')\,\hat{t}_2(\bar{\mathbf{p}}_2,\ \bar{\mathbf{p}}_2',\ z-p_1^2/2m_1)\\
&+\mathscr{K}_3(\bar{\mathbf{p}}_1\bar{\mathbf{p}}_2,\ \bar{\mathbf{p}}_1'\bar{\mathbf{p}}_2',\ z)]\,\frac{1}{z-p_1'^2/2m_1-p_2'^2/2m_2}\\
&\times\left[u_{12,1}\left(\bar{\mathbf{p}}_1'+\bar{\mathbf{p}}_2',\ \frac{m_2\bar{\mathbf{p}}_1'-m_1\bar{\mathbf{p}}_2'}{M},\ \bar{\mathbf{p}}_1^0,\ z\right)\right.\\
&\left.+\frac{\varphi_{12}((m_2\bar{\mathbf{p}}_1'-m_1\bar{\mathbf{p}}_2')/M)\,\omega_{12,1}(\bar{\mathbf{p}}_1'+\bar{\mathbf{p}}_2',\ \bar{\mathbf{p}}_1^0,\ z)}{z+\varkappa_{12}^2-(\bar{\mathbf{p}}_1'+\bar{\mathbf{p}}_2')^2/2M}\right];\\
&u_{12,1}(\bar{\mathscr{P}}_{12}\bar{\mathbf{p}}_{12},\ \bar{\mathbf{p}}_1^0,\ z)\\
&=\frac{\hat{t}_{12}(\bar{\mathbf{p}}_{12},\ \bar{\mathbf{p}}_1^0-(m_1/M)\,\bar{\mathscr{P}}_{12},\ z-\mathscr{P}_{12}^2/2M)\,\varphi_2(\bar{\mathscr{P}}_{12}-\bar{\mathbf{p}}_1^0)}{z-\mathbf{p}_1^{02}/2m_1-(\bar{\mathscr{P}}_{12}-\bar{\mathbf{p}}_1^0)^2/2m_2}\\
&+\int d\mathbf{p}_1'\,\frac{\hat{t}_{12}(\bar{\mathbf{p}}_{12},\ \bar{\mathbf{p}}_1'-(m_1/M)\,\bar{\mathscr{P}}_{12},\ z-\mathscr{P}_{12}^2/2M)}{z-p_1'^2/2m_1-(\bar{\mathscr{P}}_{12}-\bar{p}_1')^2/2m_2}\\
&\times[B_1(\bar{\mathbf{p}}_1',\bar{\mathscr{P}}_{12}-\bar{\mathbf{p}}_1',\bar{\mathbf{p}}_1^0,z)+u_{(1,2),1}(\bar{\mathbf{p}}_1',\bar{\mathscr{P}}_{12}-\bar{\mathbf{p}}_1',\bar{\mathbf{p}}_1^0,z)\\
&+\frac{\varphi_1(\bar{p}_1')\,\omega_2(\bar{\mathscr{P}}_{12}-\bar{\mathbf{p}}_1',\ \bar{\mathbf{p}}_1^0,\ z)}{z+\varkappa_1^2-(\bar{\mathscr{P}}_{12}-\bar{\mathbf{p}}_1')^2/2m_2}\\
&+\frac{\varphi_2(\bar{\mathscr{P}}_{12}-\bar{\mathbf{p}}_1')\,\omega_{1,1}(\bar{\mathbf{p}}_1',\ \bar{\mathbf{p}}_1^0,\ z)}{z+\varkappa_2^2-p_1'^2/2m_1}];\\
&\omega_{1,1}(\bar{\mathbf{p}}_1,\ \bar{\mathbf{p}}_1^0,\ z)\\
&=\int d\bar{\mathbf{p}}_1'\int d\bar{\mathbf{p}}_2'\,\frac{\varphi_2^*(\bar{\mathbf{p}}_2')\,\delta(\bar{\mathbf{p}}_1-\bar{\mathbf{p}}_1')+\mathscr{K}_1(\bar{\mathbf{p}}_1,\ \bar{\mathbf{p}}_1'\bar{\mathbf{p}}_2'z)}{z-p_1'^2/2m_1-p_2'^2/2m_2}\\
&\times\left[\frac{\varphi_{12}((m_2\bar{\mathbf{p}}_1'-m_1\bar{\mathbf{p}}_2')/M)\,\omega_{12,1}(\bar{\mathbf{p}}_1'+\bar{\mathbf{p}}_2',\ \bar{\mathbf{p}}_1^0,\ z)}{z+\varkappa_{12}^2-(\bar{\mathbf{p}}_1'+\bar{\mathbf{p}}_2')^2/2M}\right.\\
&\left.+u_{12,1}\left(\bar{\mathbf{p}}_1'+\bar{\mathbf{p}}_2',\ \frac{m_2\bar{\mathbf{p}}_1'-m_1\bar{\mathbf{p}}_2'}{M},\ \bar{\mathbf{p}}_1^0,\ z\right)\right];\\
&\omega_{2,1}(\bar{\mathbf{p}}_2,\ \bar{\mathbf{p}}_1^0,\ z)\\
&=\int d\bar{\mathbf{p}}_1'\int d\bar{\mathbf{p}}_2'\,\frac{\varphi_1^*(\bar{\mathbf{p}}_1')\,\delta(\bar{\mathbf{p}}_2-\bar{\mathbf{p}}_2')+\mathscr{K}_2(\bar{\mathbf{p}}_2,\ \bar{\mathbf{p}}_1'\bar{\mathbf{p}}_2',\ z)}{z-p_1'^2/2m_1-p_2'^2/2m_2}\\
&\times\left[\frac{\varphi_{12}(m_2\bar{\mathbf{p}}_1'-m_1\bar{\mathbf{p}}_2')/M)\,\omega_{12,1}(\bar{\mathbf{p}}_1'+\bar{\mathbf{p}}_2',\ \bar{\mathbf{p}}_1^0,\ z)}{z+\varkappa_{12}^2-(\bar{\mathbf{p}}_1'+\bar{\mathbf{p}}_2')^2/2M}\right.\\
&\left.+u_{12,1}\left(\bar{\mathbf{p}}_1'+\bar{\mathbf{p}}_2',\ \frac{m_2\bar{\mathbf{p}}_1'-m_1\bar{\mathbf{p}}_2'}{M},\ \bar{\mathbf{p}}_1^0,\ z\right)\right];\\
&\omega_{12,1}(\bar{\mathscr{P}}_{12},\ \bar{\mathbf{p}}_1^0,\ z)=\frac{\varphi_{12}^*(\bar{\mathbf{p}}_1^0-(m_1/M)\,\bar{\mathscr{P}}_{12})\,\varphi_2(\bar{\mathscr{P}}_{12}-\bar{\mathbf{p}}_1^0)}{z-p_1^{02}/2m_1-(\bar{\mathscr{P}}_{12}-\bar{\mathbf{p}}_1^0)^2/2m_2}\\
&+\int d\bar{\mathbf{p}}_1'\,\frac{\varphi_{12}^*(\bar{\mathbf{p}}_1'-(m_1/M)\,\bar{\mathscr{P}}_{12})}{z-\mathbf{p}_1'^2/2m_1-(\bar{\mathscr{P}}_{12}-\bar{\mathbf{p}}_1')^2/2m_2}\\
&\times\left[B_1(\bar{\mathbf{p}}_1',\bar{\mathscr{P}}_{12}-\bar{\mathbf{p}}_1',\bar{\mathbf{p}}_1^0,z)+u_{(1,2),1}(\bar{\mathbf{p}}_1',\bar{\mathscr{P}}_{12}-\bar{\mathbf{p}}_1',\bar{\mathbf{p}}_1^0,z)\right.\\
&+\frac{\varphi_2(\bar{\mathscr{P}}_{12}-\bar{\mathbf{p}}_1')\,\omega_{1,1}(\bar{\mathbf{p}}_1',\ \bar{\mathbf{p}}_1^0,\ z)}{z+\varkappa_2^2-p_1'^2/2m_1}\\
&\left.+\frac{\varphi_1(\bar{\mathbf{p}}_1')\,\omega_{2,1}(\bar{\mathscr{P}}_{12}-\bar{\mathbf{p}}_1',\ \bar{\mathbf{p}}_1^0,\ z)}{z+\varkappa_1^2-(\bar{\mathscr{P}}_{12}-\bar{\mathbf{p}}_1')^2/2m_2}\right].
\end{aligned}\right\}\qquad(91)
$$

The operator $B_1(z)$ in the system (91) has the form

$$B_1(z)=M_1^3(z)+\frac{|\varphi_1\rangle}{z+\varkappa_1^2-H_{02}}\,M_1^2(z)+\frac{|\varphi_2\rangle}{z+\varkappa_2^2-H_{01}}\,M_1^1(z);\qquad(92)$$

$$B_1(z^*)=\mathscr{K}_1^+(z).\qquad(93)$$

Analysis of the properties of the integral equations for the components of the scattering operator. We write the system of integral equations (91) for the components of the scattering operator of the problem of two particles in an external field in the matrix form

$$\mathscr{T}(z)=\mathscr{T}_0(z)+\tilde{\mathscr{B}}(z)\,\mathscr{T}(z),\qquad(94)$$

where $\mathscr{T}(z)$ is the vector with components $u_{(1,2),1}(z)$, $u_{12,1}(z), w_{1,1}(z), w_{2,1}(z), w_{21,1}(z)$. Our task is to investigate the analytic properties of the kernel $\mathscr{B}(z)$ of the system of equations for the components of the total T matrix, as was done in Ref. 5.

We consider the properties of powers of the kernel $\mathscr{B}(z)$. We write the N-th power of this kernel in the form

$$[\tilde{\mathscr{B}}(z)]^N=\tilde{\mathscr{D}}^N(z)\,G_0(z).\qquad(95)$$

The elements of the matrix $\mathscr{D}^N(z)$ in the momentum representation, i.e., the functions $\langle\mathbf{p}|\tilde{\mathscr{D}}_{mn}^N(z)|\mathbf{p}'\rangle$ [$\mathbf{p}$ and $\mathbf{p}'$ is either one of the momenta $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_{12}, \mathscr{P}_{12}$), or one of the sets $(\mathbf{p}_1\mathbf{p}_2)$, $(\mathbf{p}_{12}, \mathscr{P}_{12})$, and similarly for the primed quantities], have principal singularities with respect to the momenta $\mathbf{p}'$. Therefore, the system of equations (94) and the matrices $\tilde{\mathscr{D}}^N(z)$ have meaning only in terms of the components. It follows from the property of the kernels of the system (94) that for the operators $\tilde{\mathscr{D}}_{mn}^N(z)$ one of the following decompositions into components holds:

$$
\left.\begin{aligned}
&\tilde{\mathscr{D}}_{mn}^N(z)=[\tilde{\mathscr{D}}_{mn}^N(z)]_0+[\tilde{\mathscr{D}}_{mn}^N(z)]_{12}\,\frac{\langle\varphi_{12}|}{z+\varkappa_{12}^2-H_{012}};\\
&\tilde{\mathscr{D}}_{mn}^N(z)=[\tilde{\mathscr{D}}_{mn}^N(z)]_0+[\tilde{\mathscr{D}}_{mn}^N(z)]_1\,\frac{\langle\varphi_2|}{z+\varkappa_2^2-H_{01}}\\
&+[\tilde{\mathscr{D}}_{mn}(z)]_2\,\frac{\langle\varphi_1|}{z+\varkappa_1^2-H_{02}}.
\end{aligned}\right\}\qquad(96)
$$

Using the integral representations found above, for the components we can show that when $N\geqslant 5$, $\langle\mathbf{p}|[\tilde{\mathscr{D}}_{mn}^N(z)]_\alpha\|\mathbf{p}_\alpha'\rangle$ ($\alpha=0, 1, 2, 12$) are Hölder functions of their variables. This means that the system of equations (94) is a Fredholm system. Using the method for analyzing a Fredholm system of equations of the three-body problem,[5,15] we can show that the system of equations (94) has a unique solution, and the corresponding system of homogeneous equations for $E\neq-\varkappa_1^2-\varkappa_2^2$ and $E\notin\delta_{\rm disc}(H)$ has only trivial solutions.

3. CONNECTION BETWEEN THE COMPONENTS OF THE T MATRIX AND THE MØLLER OPERATORS AND THE ELEMENTS OF THE S MATRIX FOR THE PROBLEM OF TWO PARTICLES IN AN EXTERNAL FIELD

The components of the T matrix constructed in accordance with the initial channels. Besides the system of equations for the "out" components of the T matrix constructed in accordance with the channels in the final state of the system, we introduce "in" components of the T matrix corresponding to the channels in the initial state. These systems must be equivalent to each other and differ by the operation of Hermitian conjugation.

We define the operators $T^{\text{in}}_{\alpha}(z)$ ($\alpha=1, 2, 12$) by

$$T^{\text{in}}_{\alpha}(z)=V_{\alpha}+VG(z)V_{\alpha}, \tag{97}$$

and then for $T^{\text{in}}_{\alpha}(z)$ we have the system of equations

$$\left.\begin{aligned} T^{\text{in}}_{1,2}(z)&=N_{1,2}(z)+T^{\text{in}}_{12}(z)G_0(z)N_{1,2}(z);\\ T^{\text{in}}_{12}(z)&=t_{12}(z_{12})+T^{\text{in}}_{1,2}(z)G_0(z)t_{12}(z).\end{aligned}\right\} \tag{98}$$

The analytic properties of the operators $N_{1,2}(z)$ and $t_{12}(z)$ studied above enable us to write down for the operators $T^{\text{in}}_{\alpha}(z)$ the decompositions

$$\left.\begin{aligned} T^{\text{in}}_{1,2}(z)&=t_1(z_1)+t_2(z_2)+\eta_{1,2}(z)+\zeta_2(z)<\varphi_1|/(z+\varkappa_1^2-H_{02})\\ &+\zeta_1(z)<\varphi_2|/(z+\varkappa_2^2-H_{01});\\ T^{\text{in}}_{12}(z)&=t_{12}(z_{12})+\eta_{12}(z)+\zeta_{12}(z)<\varphi_{12}|/(z+\varkappa_{12}^2-H_{021}).\end{aligned}\right\} \tag{99}$$

Since the operators $T_{\alpha}(z)$, for which there exists a decomposition into components of the form (87), are related to the operators $T^{\text{in}}_{\alpha}(z)$ by

$$T^{+}_{\alpha}(z)=T^{\text{in}}_{\alpha}(z^*), \tag{100}$$

the components $u_{\alpha}(z)$ and $v_{\alpha}(z)$ are related to $\zeta_{\alpha}(z)$ and $\eta_{\alpha}(z)$ by

$$\left.\begin{aligned} &[v_1(z)+\mathscr{K}_1(z)]^+=\zeta_1(z^*);\ [v_2(z)+\mathscr{K}_2(z)]^+=\zeta_2(z^*);\\ &[u_{1,2}(z)+\mathscr{K}_3(z)]^+=\eta_{1,2}(z);\ v^{+}_{12}(z)=\zeta_{12}(z^*);\ \ u^{+}_{12}(z)=\eta_{12}(z^*).\end{aligned}\right\} \tag{101}$$

In the momentum representation, the operator equations (101) have the form

$$\left.\begin{aligned} &\zeta^*_1(\mathbf{p}^0_1,\ \mathbf{p}^0_2;\ \mathbf{p}_1,\ z^*)=v_1(\mathbf{p}_1,\ \mathbf{p}^0_1\mathbf{p}^0_2,\ z)+\mathscr{K}_1(\mathbf{p}_1,\ \mathbf{p}^0_1\mathbf{p}^0_2,\ z);\\ &\zeta^*_2(\mathbf{p}^0_1\mathbf{p}^0_2,\ \mathbf{p}_2,\ z^*)=v_2(\mathbf{p}_2,\ \mathbf{p}^0_1\mathbf{p}^0_2,\ z)+\mathscr{K}_2(\mathbf{p}_2,\ \mathbf{p}^0_1\mathbf{p}^0_2,\ z);\\ &\zeta^*_{12}(\mathbf{p}^0_1\mathbf{p}^0_2,\ \mathscr{P}_{12},\ z^*)=v_{12}(\mathscr{P}_{12},\ \mathbf{p}^0_1\mathbf{p}^0_2,\ z);\\ &\eta^*_{1,2}(\mathbf{p}^0_1\mathbf{p}^0_2,\ \mathbf{p}_1\mathbf{p}_2,\ z^*)=u_{1,2}(\mathbf{p}_1\mathbf{p}_2,\ \mathbf{p}^0_1\mathbf{p}^0_2,\ z)+\mathscr{K}_3(\mathbf{p}_1\mathbf{p}_2,\ \mathbf{p}^0_1\mathbf{p}^0_2,\ z);\\ &\eta^*_{12}(\mathbf{p}^0_1\mathbf{p}^0_2,\ \mathscr{P}_{12}\mathbf{p}_{12},\ z^*)=u_{12}(\mathscr{P}_{12}\mathbf{p}_{12},\ \mathbf{p}^0_1\mathbf{p}^0_2,\ z).\end{aligned}\right\} \tag{102}$$

It follows that, using the solutions of the system of equations (91) and the compositions (102), we can construct the "in" components of the T matrix for all z, except the points of the discrete spectrum of H and the point corresponding to the localized state of the system.

In the considered problem, we define the space of asymptotic states H_{as}, on which the Møller operators $\Omega^{\pm}_{\alpha}$ and the S matrices are defined. As before, we shall assume that the potentials V_{α} ($\alpha=1, 2, 12$) belong to the class $\mathfrak{Y}$ and that the Hamiltonians h_{α} ($\alpha=1, 2, 12$) have a discrete spectrum consisting of one simple negative eigenvalue. In this case, the space H_{as} is a direct sum of the form[8]

$$\mathscr{H}_{\text{ac}}=\mathscr{G}^0\oplus\mathscr{G}^1\oplus\mathscr{G}^2\oplus\mathscr{G}^{12}. \tag{103}$$

Here, $\mathscr{G}^0$ is the space of square-integrable functions $f(\bar{p}_1\bar{p}_2)$ of two variables; $\mathscr{G}^{\alpha}$ ($\alpha=1, 2, 12$) are spaces of square-integrable functions of one variable. The wave operator $\Omega^{\pm}$, which maps from H_{as} to $\mathscr{G}$, is defined by

$$\Omega^{\pm}f=\Omega^{\pm}_0 f+\Omega^{\pm}_1 f_1+\Omega^{\pm}_2 f_2+\Omega^{\pm}_{12}f_{12}, \tag{104}$$

where the state f in H_{as} is equal to $\{f_0, f_1, f_2, f_{12}\}$. In accordance with the results of time-dependent scattering theory, the wave operators $\Omega^{\pm}_{\alpha}$, which map from $\mathscr{G}_{\alpha}$ to $\mathscr{G}$, are

$$\left.\begin{aligned} &\Omega^{\pm}_0|f_0\rangle=|f_0\rangle+\int d\bar{\mathbf{p}}_1\int d\bar{\mathbf{p}}_2 G(z^{\pm}_0)(V_1+V_2+V_{12})\,|\bar{\mathbf{p}}_1\bar{\mathbf{p}}_2\rangle f_0(\bar{\mathbf{p}}_1\bar{\mathbf{p}}_2);\\ &z^{\pm}=p_1^2/2m_1+p_2^2/2m_2\pm \mathrm{i}0;\ f_0(\bar{\mathbf{p}}_1\bar{\mathbf{p}}_2)=\langle\bar{\mathbf{p}}_1\bar{\mathbf{p}}_2|f_0\rangle;\\ &\Omega^{\pm}_1|f_1\rangle=|f_1\rangle|\psi_2\rangle+\int d\bar{\mathbf{p}}_1 G_1(z^{\pm}_1)(V_1+V_{12})|\bar{\mathbf{p}}_1\psi_2\rangle f_1(\bar{\mathbf{p}}_1);\\ &z^{\pm}_1=p_1^2/2m_1-\varkappa_2^2\pm \mathrm{i}0;\quad f_1(\mathbf{p}_1)=\langle\bar{\mathbf{p}}_1|f_1\rangle;\\ &\Omega^{\pm}_2|f_2\rangle=|f_2\rangle|\psi_1\rangle+\int d\bar{\mathbf{p}}_2 G(z^{\pm}_2)(V_2+V_{12})|\psi_1\bar{\mathbf{p}}_2\rangle f_2(\bar{\mathbf{p}}_2);\\ &z^{\pm}_2=p_2^2/2m_2-\varkappa_1^2\pm \mathrm{i}0;\ f_2(\mathbf{p}_2)=\langle\mathbf{p}_2|f_2\rangle;\\ &\Omega^{\pm}_{12}|f_{12}\rangle=|\psi_{12}\rangle|f_{12}\rangle+\int d\mathscr{P}_{12}G(z^{\pm}_{12})(V_1+V_2)\,|\psi_{12}\mathscr{P}_{12}\rangle f_{12}(\mathscr{P}_{12});\\ &z^{\pm}_{12}=\mathscr{P}^2_{12}/2M_{12}-\varkappa_{12}^2\pm \mathrm{i}0;\quad f_{12}(\mathscr{P}_{12})=\langle\mathscr{P}_{12}|f_{12}\rangle.\end{aligned}\right\} \tag{105}$$

We shall show that the operators $\Omega^{\pm}_{\alpha}$ can be expressed in terms of the "in" components of the total T matrix. We introduce an additional condition: The discrete spectrum of H lies strictly below all $-\varkappa^2_{\alpha}$ ($\alpha=1, 2, 12$). The fulfillment of this condition means that for all $z^{\pm}_{\alpha}$ ($\alpha=1, 2, 12$) the components of the operator $T(z)$ can be determined uniquely from the system of equations (91). The case $E=-\varkappa_1^2-\varkappa_2^2$ is not considered here.

The fact that $\sigma_{\text{disc}}(H)$ lies below all $-\varkappa^2_{\alpha}$ makes it possible to prove the following proposition.

The operator $\Omega^{\pm}_0$ is related to the operator $T(E\pm 0)$ and the operators $\Omega^{\pm}_{\alpha}$, with components $\zeta_{\alpha}(E\pm \mathrm{i}0)$, by means of the relations

$$\begin{aligned} \langle\bar{\mathbf{k}}_1\bar{\mathbf{k}}_2\,|\Omega^{\pm}_0|\,f_0\rangle&=f(\bar{\mathbf{k}}_1\bar{\mathbf{k}}_2)\\ &+\int d\mathbf{p}_1\int d\mathbf{p}_2\frac{T(\bar{\mathbf{k}}_1\bar{\mathbf{k}}_2,\ \bar{\mathbf{p}}_1\bar{\mathbf{p}}_2,\ z^{\pm}_0)\,f_0(\bar{\mathbf{p}}_1\bar{\mathbf{p}}_2)}{z^{\pm}_0-k_1^2/2m_1-k_2^2/2m_2}; \end{aligned}\tag{106}$$

$$\begin{aligned} \langle\bar{\mathbf{k}}_1\bar{\mathbf{k}}_2\,|\Omega^{\pm}_1|\,f_1\rangle&=f_1(\bar{\mathbf{k}}_1)\,\psi_2(\bar{\mathbf{k}}_2)\\ &+\int d\bar{\mathbf{p}}_1\frac{\zeta_1(\bar{\mathbf{k}}_1\bar{\mathbf{k}}_2,\ \bar{\mathbf{p}}_1,\ z^{\pm}_1)\,f_1(\bar{\mathbf{p}}_1)}{z^{\pm}_1-k_1^2/2m_1-k_2^2/2m_2}; \end{aligned}\tag{107}$$

$$\begin{aligned} \langle\bar{\mathbf{k}}_1\bar{\mathbf{k}}_2\,|\Omega^{\pm}_2|\,f_2\rangle&=\psi_1(\bar{\mathbf{k}}_1)\,f_2(\bar{\mathbf{k}}_2)\\ &+\int d\bar{\mathbf{p}}_2\frac{\zeta_2(\bar{\mathbf{k}}_1\bar{\mathbf{k}}_2,\ \bar{p}_2,\ z^{\pm}_2)\,f_2(\bar{\mathbf{p}}_2)}{z^{\pm}_2-k_1^2/2m_1-k_2^2/2m_2}; \end{aligned}\tag{108}$$

$$\begin{aligned} \langle\bar{\mathbf{k}}_1\bar{\mathbf{k}}_2\,|\Omega^{\pm}_{12}|\,f_{12}\rangle&=\psi_{12}(\bar{\mathbf{k}}_{12})\,f_{12}(\mathscr{K}_{12})+\int d\mathscr{P}_{12}\zeta_{12}(\bar{\mathbf{k}}_1\bar{\mathbf{k}}_2,\ \mathscr{P}_{12},\ z^{\pm}_{12})\\ &\times[z^{\pm}_{12}-k_1^2/2m_1-k_2^2/2m_1]^{-1}\,f^*_{12}(\mathscr{P}_{12}). \end{aligned}\tag{109}$$

Here, the parameters $z^{\pm}_{\alpha}$ ($\alpha=0, 1, 2, 12$) are the same as in Eqs. (105), and the operators $\zeta_{\alpha}(z)$ ($\alpha=1, 2, 12$) are defined in (99). By virtue of the conditions imposed on the discrete spectrum of H, the systems of equations (91), (94), and (98) are uniquely solvable, and the definitions (106)–(109) are correct. We shall prove the equivalence of representations of the form (105) and (106)–(109) for the Møller operators.

The equivalence of (105) and (106) for $\Omega^{\pm}_0$ is obvious, since $G(z)V=G_0(z)T(z)$. We now consider Eq. (107). From the definition for $\zeta_1(z)$ of the form (99) we have

$$\zeta_1(z)=B_1(z)+T^{\text{in}}_{12}(z)\,G_0(z)\,[\,|\,\varphi_2>+B_1(z)], \tag{110}$$

where the operator $B_1(z)$ is the same as in (91) and (92). In Eq. (110) we set $z=z^{\pm}_1$ and calculate $\zeta_1(\bar{k}_1\bar{k}_2, p_1 z^{\pm}_1)$ for $z^{\pm}_1=p_1^2/2m_1-\varkappa_2^2\pm \mathrm{i}0$:

$$\begin{aligned} \zeta_1(\mathbf{k}_1\mathbf{k}_2,\ p_1,\ z^{\pm}_1)&=\langle\mathbf{k}_1\mathbf{k}_2\,|B_1(z^{\pm}_1)|\,\mathbf{p}_1\rangle\\ &+\langle\mathbf{k}_1\mathbf{k}_2\,|T^{\text{in}}_{12}(z^{\pm}_1)\,G_0(z^{\pm}_1)|\,\varphi_2\mathbf{p}_1\rangle\\ &+\langle\mathbf{k}_1\mathbf{k}_2\,|T^{\text{in}}_{12}(z^{\pm}_{18})\,G_0(z^{\pm}_1)\,B_1(z^{\pm}_1)|\,\mathbf{p}_1\rangle\\ &=\mathscr{K}^*_1(\mathbf{p}_1,\ \mathbf{k}_1,\ \mathbf{k}_2,\ z^{\pm}_1)+\langle\mathbf{k}_1\mathbf{k}_2\,|T^{\text{in}}_{12}(z^{\pm}_1)|\,\psi_2\mathbf{p}_1\rangle\\ &+\langle\mathbf{k}_1\mathbf{k}_2\,|T^{\text{in}}_{12}(z^{\pm}_1)\,G_0(z^{\pm}_1)\,B_1(z^{\pm}_1)|\,\mathbf{p}_1\rangle. \end{aligned}\tag{111}$$

Here, we have used (92) and (93) and the definition of the vector $|\varphi\rangle$ by means of the corresponding wave function. We calculate the function $\mathscr{K}_1(\mathbf{p}_1, \mathbf{k}_1\mathbf{k}_2, z^{\pm}_1)$. In accordance with Eq. (65),

$$\mathscr{K}_1(\bar{\mathbf{p}}_1, \bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, p_1^2/2m_1 \pm i\tau - \varkappa_2^2)$$
$$= \int_{-\infty}^{\infty} \frac{d\varepsilon}{\mp 2\pi i} \left(\frac{1}{E-\varepsilon+\varkappa_2^2 \pm i\tau_2} + \frac{1}{\varepsilon - p_1^2/2m_1 \pm i\tau_1} \right)$$
$$\times t_1(\mathbf{p}_1, \mathbf{k}_1, \varepsilon \pm i\tau_1)\, \varphi_2^*(\mathbf{k}_2) \left(\frac{1}{\varepsilon - k_1^2/2m_1 \pm i\tau_1} + \frac{1}{E-\varepsilon-k_2^2/2m_2 \pm i\tau_2} \right). \quad (112)$$

In (112), we set $\tau_1 = \tau_2$ and multiply this equation by $(z_1 - k_1^2/2m_1 - k_2^2/2m_2)^{-1}$. Noting that in the limit $\tau \to 0$

$$\left(\frac{1}{E-\varepsilon+\varkappa_2^2 \pm i\tau} + \frac{1}{\varepsilon - p_1^2/2m_1 \pm i\tau} \right)$$
$$= \left(\frac{1}{\varepsilon - p_1^2/2m_1 \mp i\tau} + \frac{1}{p_1^2/2m_1 - \varepsilon \pm i\tau} \right)$$
$$= \frac{\mp 2i\tau}{(\varepsilon - p_1^2/2m_1)^2 + \tau^2} \to \mp 2\pi i \delta(\varepsilon - p_1^2/2m_1),$$

we obtain

$$\frac{\zeta_1(\mathbf{k}_1\mathbf{k}_2, \mathbf{p}_1, z_1^{\pm})}{z_1^{\pm} - k_1^2/2m_1 - k_2^2/2m_2} = \frac{t_1(\mathbf{k}_1, \mathbf{p}_1, p_1^2/2m_1 \pm i0)\, \psi_2(\mathbf{k}_2)}{p_1^2/2m_1 - k_1^2/2m_1 \pm i0}$$
$$+ \frac{\langle \bar{\mathbf{k}}_1 \bar{\mathbf{k}}_2 \,|\, T_{12}^{\text{in}}(z_1^{\pm}) \,|\, \psi_1^{\pm}\psi_2 \rangle}{z_1^{\pm} - k_1^2/2m_1 - k_2^2/2m_2}, \quad (113)$$

where $\langle \mathbf{k} | \psi^{\pm} \rangle = \delta(\mathbf{k}-\mathbf{p}) + (p^2/2m - k^2/2m \pm i0)^{-1} t(\mathbf{k}, \mathbf{p}, p^2/2m \pm i0)$. We now transform the integral term in the expression (105) for $\Omega_1^{\pm}$. For all $\text{Im}\, z \neq 0$, we have

$$G(z) = G_{1,2}(z) + G(z)\, V_{12} G_{1,2}(z), \quad (114)$$

whence

$$G(z_1^{\pm})(V_1 + V_{12})\, |\mathbf{p}_1\psi_2\rangle$$
$$= g_{01}(p_1^2/2m_1 \pm i0)\, t_1(p_1^2/2m_1 \pm i0) |\, \mathbf{p}_1\rangle\psi_2\rangle + G(z_1^{\pm})\, V_{12} |\, \mathbf{p}_1\psi_2\rangle$$
$$+ G(z_1^{\pm})\, V_{12} g_{01}(p_1^2/2m_1 \pm i0)\, t_1(p_1^2/2m_1 \pm i0) |\mathbf{p}_1\psi_2\rangle$$
$$= g_{01}(p_1^2/2m_1 \pm i0)\, t_1(p_1^2/2m_1 \pm i0) |\, \mathbf{p}_1\rangle\psi_2\rangle$$
$$+ G_0(z_1^{\pm})\, T_{12}^{\text{in}}(z_1^{\pm}) |\psi_1^{\pm}\psi_2\rangle. \quad (115)$$

Comparing (113) and (115), we conclude that the representation (107) of $\Omega_1^{\pm}$ in terms of the component $\zeta_1(z_1^{\pm})$ is valid. The proof of (108) for the operator $\Omega_2^{\pm}$ is similar. It remains to investigate the representation for $\Omega_{12}^{\pm}$. We consider the equation

$$\zeta_{12}(z_{12}^{\pm}) |\mathscr{P}_{12}\rangle = T_{1,2}^{\text{in}}(z_{12}^{\pm})\, G_0(z_{12}^{\pm}) |\varphi_{12}\mathscr{P}_{12}\rangle, \quad (116)$$

where $T_{12}^{\text{in}}(z)$ is determined from the system (98). Combining (97) and (116), we arrive at the proof of (109).

Elements of the $\mathscr{S}$ matrix in the problem of two particles in a field. We now establish the connection between the components of the T matrix and the elements $S_{\alpha\beta}$ of the S matrix. We write the components in the form

$$\zeta_\alpha(z) = \zeta_\alpha^0(z) + \frac{|\varphi_1\rangle}{z+\varkappa_1^2 - H_{02}} \zeta_{2,\alpha}(z)$$
$$+ \frac{|\varphi_2\rangle}{z+\varkappa_2^2 - H_{01}} \zeta_{1,\alpha}(z) + \frac{|\varphi_{12}\rangle}{z+\varkappa_{12}^2 + H_{012}} \zeta_{12,\alpha}. \quad (117)$$

We shall show that the elements $S_{\alpha\beta}$ of the S matrix can be expressed in terms of the components $\zeta_{\alpha\beta}(z)$ ($\alpha, \beta = 1, 2, 13$) by means of the relations

$$\langle \mathbf{p}_\alpha | S_{\alpha\beta} | \mathbf{p}_\beta' \rangle = \delta_{\alpha\beta}\delta(\mathbf{p}_\alpha - \mathbf{p}_\beta')$$
$$- 2\pi i \delta(E_\alpha - E_\beta)\, \zeta_{\alpha,\beta}(\mathbf{p}_\alpha, \mathbf{p}_\beta', E_\beta \pm i0), \quad (118)$$

where $E_\beta \pm i0 = z_\beta^{\pm}$, and similarly for $z_\alpha^{\pm}$. If the final reaction channel corresponds to breakup, i.e., the channel Hamiltonian is H_0 and $\alpha = 0$, then

$$\langle \mathbf{p}_1\mathbf{p}_2 | S_{0\beta} | \mathbf{p}_\beta' \rangle$$
$$= -2\pi i \delta(p_1^2/2m_1 + p_2^2/2m_2 - E_\beta)\, \zeta_\beta(\mathbf{p}_1\mathbf{p}_2, \mathbf{p}_\beta', E_\beta \pm i0). \quad (119)$$

We note that in (118) and (119) we have $\mathbf{p}_\beta' = \mathscr{P}_{12}'$, $\mathbf{p}_\alpha = \mathscr{P}_{12}$ for $\alpha, \beta = 12$.

Equations (118) and (119) are proved in the same way as Eqs. (106)–(109). Indeed, in accordance with Ref. 8,

$$\langle \mathbf{p}_1\mathbf{p}_2 | S_{0\beta} | \mathbf{p}_\beta' \rangle = -2\pi i \left(\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - E_\beta \right)$$
$$\times \langle \mathbf{p}_1\mathbf{p}_2 | T^{0\beta}(E_\beta + i0) | \psi_\beta\mathbf{p}_\beta' \rangle, \quad (120)$$

where the operator $T^{0\beta}(z)$ is defined in the form

$$T^{0\beta}(z) = V^\beta + VG(z)\, V^\beta;\; V^\beta = V - V_\beta. \quad (121)$$

Since $G(z) = G_0(z) + G_0(z) VG(z)$,

$$G(z)\, V^\beta = G_0(z)\, T^{0\beta}(z). \quad (122)$$

From this equation, and also from comparison of Eqs. (105), (121), and (107)–(109) we obtain the required result, since

$$\langle \mathbf{p}_1\mathbf{p}_2 | T^{0\beta}(z_\beta^{\pm}) | \psi_\beta\mathbf{p}_\beta' \rangle = \zeta_\beta(\mathbf{p}_1\mathbf{p}_2, \mathbf{p}_\beta', z_\beta^{\pm}). \quad (123)$$

We now prove the validity of (118). As in the previous case, for α, β ($\alpha, \beta = 12$) p_α and p_β' are $\mathscr{P}_{12}$ and $\mathscr{P}_{12}'$. The matrix elements of $S_{\alpha,\beta}$ for $\alpha, \beta = 1, 2, 12$ have the form[8]

$$\langle \mathbf{p}_\alpha | S_{\alpha,\beta} | \mathbf{p}_\beta' \rangle = \delta_{\alpha\beta}(\mathbf{p}_\alpha - \mathbf{p}_\beta')$$
$$- 2\pi i \delta(E_\alpha - E_\beta) \langle \mathbf{p}_\alpha\psi_\alpha | T^{\alpha\beta}(z_\beta^{\pm}) | \psi_\beta\mathbf{p}_\beta' \rangle, \quad (124)$$

where $T^{\alpha\beta}(z) = V^\beta + V^\alpha G(z) V^\beta$. To calculate the components $\zeta_{\alpha,\beta}(z)$ we use Eqs. (117), (123), (62), and (63) and the decomposition of the operators $T_\alpha(z)$ into components. For the operators $T^{0\beta}(z)$ we have a representation of the form

$$T^{0\beta}(z) = V^\beta + VG(z)\, V^\beta = V^\beta + [T_{1,2}(z) + T_{1^\circ}(z)]\, G_0(z)\, V^\beta, \quad (125)$$

and therefore for the components of $T^{0\beta}(z)$ we obtain

$$\zeta_{12,\beta}(\mathscr{P}_{12}, \mathbf{p}_\beta', z_\beta^{\pm}) = \langle \varphi_{12}\mathscr{P}_{12} | G_0(z_\beta^{\pm})\, V^\beta | \psi_\beta\mathbf{p}_\beta' \rangle$$
$$+ \langle \varphi_{12}\mathscr{P}_{12} | G_0(z_\beta^{\pm})(V_1 + V_2)\, G(z_\beta^{\pm})\, V^\beta | \psi_\beta\mathbf{p}_\beta' \rangle; \quad (126)$$
$$\zeta_{1,\beta}(\mathbf{p}_1, \mathbf{p}_\beta', z_\beta^{\pm}) = \langle \varphi_2\mathbf{p}_1 | G_0(z_\beta^{\pm})\, V^\beta | \psi_\beta\mathbf{p}_\beta' \rangle$$
$$+ \langle \varphi_2\mathbf{p}_1 | G_0(z_\beta^{\pm})\, V_{12} G(z_\beta^{\pm})\, V^\beta | \psi_\beta\mathbf{p}_\beta' \rangle$$
$$+ \langle \mathbf{p}_1 | \mathscr{K}_1(z_\beta^{\pm})\, G_0(z_\beta^{\pm})\, V^\beta | \psi_\beta\mathbf{p}_\beta' \rangle + \langle \mathbf{p}_1 | \mathscr{K}_1(z_\beta^{\pm})\, G_0(z_\beta^{\pm}) V_{12} G\, z_\beta^{\pm} V^\beta | \psi_\beta\mathbf{p}_\beta' \rangle. \quad (127)$$

The expression (126) on the energy shell reduces to

$$\zeta_{12,\beta}(\mathscr{P}_{12}, \mathbf{p}_\beta', z_\beta^{\pm}) = \langle \psi_{12}\mathscr{P}_{12} \,|\, T^{12,\beta}(z_\beta^{\pm}) \,|\, \psi_\beta\mathbf{p}_\beta' \rangle. \quad (128)$$

The expression (127) for $E_\beta = p_1^2/2m_1 - \varkappa_2^2$ admits in accordance with (112) and (113) a representation of the form

$$\zeta_{1,\beta}(\mathbf{p}_1, \mathbf{p}_\beta', E_\beta \pm i0) = \langle \psi_1 - \psi_2 \,|\, V^\beta + V_{12} G(z_\beta^{\pm})\, V^\beta \,|\, \psi_\beta\mathbf{p}_\beta' \rangle, \quad (129)$$

where $\langle \mathbf{k} | \psi^- \rangle = \delta(\mathbf{k}-\mathbf{p}) + (p^2/2m - k^2/2m - i0)^{-1} t(\mathbf{k}, \mathbf{p}, p^2/2m - i0)$. The relation (129) is a well-known variant of the expression of the scattering amplitude with a distorted wave of particle 1 separated in the final state.[8] This completes the proof of the validity of the representations (118) and (119).

4. BOUND STATES AND RESONANCES IN THE SYSTEM OF TWO PARTICLES IN AN EXTERNAL FIELD AND IN A SYSTEM OF THREE PARTICLES OF FINITE MASS

Bound states and resonances in the system of two particles in an external field. Hitherto, we have assumed throughout that the energy of the system is not equal to the energy $-E_0$ of the state of the discrete spectrum of the Hamiltonian $H_{1,2}$. We now consider the case when E is near $-E_0$, and estimate the influence of the states in $\sigma_{\text{disc}}(H_{1,2})$ on the properties of the spectrum of the operator H and, therefore, on the properties of the operator $T(z)$. To make the investigation more general, we shall assume that the sets $\sigma_{\text{disc}}(h_1)$ and $\sigma_{\text{disc}}(h_2)$ contain $n_1 \geqslant 1$ and $n \geqslant 1$ elements, some of the values $(-\varkappa_1^2 - \varkappa_2^2) \in \sigma_{\text{disc}}(H_{1,2})$ belonging to the continuum of the system $\sigma_c(H) = (\Sigma, \infty)$. Here, Σ is defined as follows[16]:

$$\Sigma = \min_{\alpha} \{-\varkappa_\alpha^2\}, \quad \alpha = 1, 2, 12, \tag{130}$$

where the minimum is calculated over all states of the discrete spectrum of the Hamiltonians h_1, h_2, h_{12} (5).

To describe the influence of the states in $\sigma_{\text{disc}}(H_{1,2})$ on the spectrum of H, we use the Lippmann–Schwinger equation relating the Green's functions $G(z) = (z - H)^{-1}$ and $G_{1,2}(z) = (z - H_{1,2})^{-1}$ of the operators H and $H_{1,2}$, respectively:

$$G(z) = G_{1,2}(z) + G_{12}(z) V_{12} G(z). \tag{131}$$

We represent the Green's function $G_{1,2}(z)$ in the form

$$G_{1,2}(z) = \frac{|\psi_{1,2}\rangle\langle\psi_{1,2}|}{z + E_0} + \hat{G}_{1,2}(z), \tag{132}$$

where we have separated the contribution of the state in $\sigma_{\text{disc}}(H_{1,2})$ with energy $-E_0$ and wave function $|\psi_{1,2}\rangle = |\psi_2\rangle|\psi_2\rangle$. The operator $\hat{G}_{1,2}(z)$ in (132) is given by

$$\hat{G}_{1,2}(z) = M G_{1,2}(z), \tag{133}$$

where

$$M = 1 - \Lambda; \; \Lambda = |\psi_{1,2}\rangle\langle\psi_{1,2}|. \tag{134}$$

The operators M and Λ in (134) are orthogonal projectors, so that $M = M^+ = M^2$ and $\Lambda = \Lambda^+ = \Lambda^2$. Proceeding from the representation of the kernel of Eq. (131) in the form of the two terms (132), we write $G(z)$ as follows:

$$\left.\begin{aligned} &G(z) = G_1(z) + G_2(z);\\ &G_1(z) = \frac{|\psi_{1,2}\rangle\psi_{1,2}|}{z + E_0}[1 + V_{12}G(z)];\\ &G_2(z) = \hat{G}_{1,2}(z)[1 + V_{12}G(z)]. \end{aligned}\right\} \tag{135}$$

From (135), we obtain a system of equations for $G_1(z)$ and $G_2(z)$:

$$\left.\begin{aligned} &G_1(z) = \frac{|\psi_{1,2}\rangle\langle\psi_{1,2}|}{z + E_0}[1 + V_{12}G_1(z) + V_{12}G_2(z)];\\ &G_2(z) = \hat{G}_{1,2}(z)[1 + V_{12}G_1(z) + V_{12}G_2(z)]. \end{aligned}\right\} \tag{136}$$

Assuming the existence of the operator

$$R(z) = [1 - \hat{G}_{1,2}(z) V_{12}]^{-1} \hat{G}_{1,2}(z), \tag{137}$$

we find from (136) a representation of the form

$$G_2(z) = R(z) + R(z) V_{12} G_1(z) \tag{138}$$

and

$$G_1(z) = |\psi_{1,2}\rangle \frac{1}{z + E_0 - b(z)} \langle\psi_{1,2}|[1 + V_{12}R(z)]. \tag{139}$$

The function $b(z)$ in (139) has the form

$$b(z) = \langle\psi_{1,2}| V_{12} + V_{12} R(z) V_{12} |\psi_{1,2}\rangle. \tag{140}$$

Summing (138) and (139), we obtain an expression for $G(z)$ in terms of $R(z)$:

$$G(z) = [1 + R(z) V_{12}] |\psi_{1,2}\rangle\langle\psi_{1,2}| [1 + V_{12}R(z)]/[z + E_0 - b(z)] + R(z). \tag{141}$$

It follows from this that

$$\langle\psi_{1,2}| G(z) |\psi_{1,2}\rangle = 1/[z + E_0 - b(z)]. \tag{142}$$

We now analyze the conditions under which the operator $R(z)$ exists. It follows from (137) that the operator $R(z)$ is the solution of an equation of the form

$$R(z) = \hat{G}_{1,2}(z) + \hat{G}_{1,2}(z) V_{12} R(z) \tag{143}$$

defined in the subspace $M\mathcal{G}$ ($\mathcal{G}$ is the Hilbert state space of the system), and, therefore,

$$\begin{aligned} &(zM - MHM) R(z) = M(z - H_{1,2}) [\hat{G}_{1,2}(z)\\ &+ \hat{G}_{1,2}(z) V_{12} R(z)] - MV_{12} MR(z) = M(z - H_{1,2}) G_{1,2}\\ &\times [M + MV_{12} R(z)] - MV_{12} MR(z) = M^2\\ &+ MV_{12} MR(z) - MV_{12} MR(z) = M. \end{aligned}$$

Hence

$$R(z) = (zM - MHM)^{-1} M. \tag{144}$$

Thus, the operator $R(z)$ is the resolvent of the operator MHM in the subspace of M and, therefore, exists.[17]

The representation (140) of $b(z)$ in terms of the resolvent of a self-adjoint operator makes it possible to conclude that $b(E + i0)$ has a negative imaginary part. This result can be obtained from the spectral decomposition of the operator $R(E + i0)$ by applying the Sochocki formula:

$$\operatorname{Im} b(E + i0) = -\pi \sum_{\alpha}\!\!\!\!\!\!\int \; |\langle\psi_{1,2}| V_{12} |\psi_\alpha\rangle|^2 \delta(E - E_\alpha) \leqslant 0, \tag{145}$$

where $|\psi_\alpha\rangle$ is a complete set of eigenfunctions of the Hamiltonian MHM, i.e., $\sum\!\!\!\!\!\!\int_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| = M$.

It follows from (141) that the existence of the discrete spectrum of the Hamiltonian $H_{1,2}$ leads to the appearance in the total Green's function of the system of a term with the characteristic factor $[z + E_0 - b(z)]^{-1}$. A similar result holds for the scattering matrix $T(z)$. We analyze the dependence of the spectral properties of H on the behavior of this factor. It follows from (141) that the vanishing of the factor $z + E_0 - b(z)$ or of the real part of this factor at certain values $z = E + i0$ signifies the existence of a bound state or resonance in the system with the Hamiltonian H. We consider some cases.

Case 1. Discrete spectrum of the operator H. Suppose the equation

$$E + E_0 - b(E + i0) = 0 \tag{146}$$

has a root E_d satisfying the condition $E_d < \Sigma$. In this case $E_d \notin \sigma_c(H)$, and, therefore, $b(E)$ in some neighborhood of the point E_d is real. It follows from the Hilbert identity for the resolvent of a self-adjoint operator that $b(z)$ satisfies

$$b(z_1) - b(z_2) = (z_2 - z_1) < \psi_{1,2} | V_{12} R(z_1) R(z_2) V_{12} |\psi_{1,2}\rangle. \tag{147}$$

Setting $z_1 = z$ and $z_2 = E_d$ in (147), we find that in a neighborhood of the point E_d the Green's functions $G(z)$ behave as follows:

$$(z - E_d)^{-1} [1 + \langle\psi_{1,2}| V_{12} R(z) R(E_d) V_{12}] |\psi_{1,2}\rangle. \tag{148}$$

Since the quantity in the square brackets in (148) exists for $z = E_d$ and is equal to $1 + a^2$, where a is some real number, the function $G(z)$ has at $z = E_d$ a simple pole of the form $(z-E_d)^{-1}$. This means that $E_d \in \sigma_{disc}(H)$, i.e., the point E_d is the energy of the bound state of the system with the Hamiltonian H, i.e., E_d is an eigenvalue of the operator H. In accordance with (141), the corresponding wave eigenfunction has the form

$$|\psi_d> = (1+a^2)^{-1/2}\,[1+R(E_d)\,V_{12}]\,|\psi_{1,2}>. \qquad (149)$$

In (149), we have written out explicitly the normalization of the function $|\psi_d\rangle$. One can show that $|\psi_d\rangle$ is indeed an eigenfunction of the Hamiltonian H of the system with energy E_d by means of the following substitution in the Schrödinger equation:

$$\begin{aligned}(1+a^2)^{1/2}(E_d-H)\,|\psi_d> &= (E_d-H_{1,2})\,[1+R(E_d)\,V_{12}]\,|\psi_{1,2}>\\ &-(1+a^2)^{1/2}\,V_{12}\,|\psi_d> = (E_d+E_0)\,|\psi_{1,2}>\\ &+(E_d-H_{1,2})\,G_{1,2}(E_d)\,[M+MV_{12}\,R(E_d)]\,V_{12}\,|\psi_{1,2}>\\ &-(1+a^2)^{1/2}\,V_{12}\,|\psi_d> = (E_d+E_0)\,|\psi_{1,2}>\\ &+(1-|\psi_{1,2}><\psi_{1,2}|)\,V_{12}\,[1+R(E_d)\,V_{12}]\,|\psi_{1,2}>\\ &-(1+a^2)^{1/2}V_{12}\,|\psi_d> = [E_d+E_0-b(E_d)]\,|\psi_{1,2}> = 0. \qquad (150)\end{aligned}$$

It can be seen from the foregoing that in the neighborhood of the point E_d the first term in (141) is the contribution to the spectral decomposition of the function $G(z)$ corresponding to the bound state of the system with energy E_d and eigenfunction $|\psi_d\rangle$ of the form (149).

Case 2. Resonances in the case of small V_{12} in the system of two particles in an external field.

Suppose the potential V_{12} is small, i.e., $V_{12}=\beta V_{12}^0$, where β is a small parameter. In this case, the factor $[z+E_0-b(z)]^{-1}$ can be written approximately in the form

$$(E-E_R+i\Gamma/2)^{-1}, \qquad (151)$$

where

$$E_R = -E_0+\beta\,\langle\psi_{1,2}\,|\,V_{12}^0\,\psi_{1,2}\rangle,\ E_R>\Sigma;$$
$$\Gamma = -2\,\mathrm{Im}\,b\,(-E_0+i0)\geqslant 0. \qquad (152)$$

Therefore, in this situation we have an ordinary Breit-Wigner resonance.

Case 3. Resonances in the system of two particles in an external field for arbitrary V_{12}.

Suppose the equation

$$E+E_0-\mathrm{Re}\,b\,(E+i0)=0 \qquad (153)$$

has a root E_R such that $E_R>\Sigma$. Then in the neighborhood of the point E_R the argument can be expressed by $[E+E_0-b(E+i0)]^{-1}$ and changes abruptly by π and, therefore, E_R is the energy of a resonance state of the system.[8,18] To describe the shape of the resonance curve, we assume that the imaginary part of the function $b(E+i0)$ varies slowly in the neighborhood of the point E_R. Writing the real part of the function $b(E+i0)$ in the neighborhood of the point E_R in the form

$$\mathrm{Re}\,[b\,(E+i0)-b\,(E_R+i0)] = (E-E_R)\,\lambda\,(E) \qquad (154)$$

and setting $\lambda(E)\approx\lambda(E_R)=-\lambda_0$, we obtain

$$[E+E_0-b\,(E+i0)]^{-1} = (1+\lambda_0)^{-1}\,[E-E_R+i\Gamma_R/2], \qquad (155)$$

where Γ_R is defined by the expression

$$\Gamma_R = \Gamma\,(E_R)/(1+\lambda_0);\ \Gamma\,(E_R) = -2\mathrm{Im}\,b\,(E_R+i0). \qquad (156)$$

For small V_{12}, the function $\lambda(E)$ has second order in the parameter β, and Eqs. (155) and (156) go over into (151) and (152). It must be emphasized that the shape of the resonance curve (155) holds only when the function $\Gamma(E)$ varies slowly in the neighborhood of the point of resonance. This condition is violated when the point E_R is near to Σ or to one of the $-\varkappa_\alpha^2$ defining the threshold energies in the system. In this case, the shape of the resonance may differ strongly from (155), and (156) becomes invalid. In addition, in the derivation of Eq. (155) it was assumed that the function $b(E+i0)$ has a continuous first derivative. This assumption is justified, since in the class of potentials $\mathfrak{Y}$ the components of the operator $A(z)=V_{12}+V_{12}R(z)V_{12}$ are continuous functions of E for $E\neq-\varkappa_\alpha$.

In accordance with (140) and (147), the half-width of the resonance can be represented in the form

$$\begin{aligned}\Gamma &= b\,(E_R-i0)-b\,(E_R+i0)\\ &= \lim_{\tau\to+0} 2i\tau\,\langle\psi_{1,2}\,|\,V_{12}R\,(E-i\tau)\,R\,(E+i\tau)\,V_{12}\,|\,\psi_{1,2}\rangle. \qquad (157)\end{aligned}$$

Using the results of Refs. 5 and 17, we can show that the function Γ, defined in the form (157), is an additive quantity with respect to all threshold singularities that lie above the point E_R. Therefore,

$$\Gamma = \Gamma_0+\Gamma_1+\Gamma_2+\Gamma_{12}, \qquad (158)$$

where Γ_i are the partial half-widths of decay through channel i.

Reduction of the three-particle Hamiltonian. The analysis of the properties of the system of two particles in a field proposed above can be used to describe the bound states and resonances in a system of three particles interacting with one another through short-range potentials. For this, it is necessary to reduce the three-particle potential to the form of the Hamiltonian of two particles in an external field, using, for example, the method proposed in Ref. 26. The essence of the transformation is as follows.

The Hamiltonian H of the system of three particles with finite masses in the momentum representation in the variables $\mathbf{p}_{ij}$ has the form[5]

$$\left.\begin{aligned}&H=H_0+V;\quad H_0=p_{13}^2/2\mu_{13}+p_{23}^2/2\mu_{23}+p_{13}p_{23}/m_3;\\ &V=V_{13}+V_{23}+V_{12}.\end{aligned}\right\} \qquad (159)$$

In (159), H_0 is the kinetic-energy operator if we take as independent variables the momenta $\mathbf{p}_{13}$ and $\mathbf{p}_{23}$ of the relative motion, V_{ij} is the interaction potential of pair ij, and the kernel of the operator V_{ij} can also be expressed by means of the variables $\mathbf{p}_{13}$ and $\mathbf{p}_{23}$.[5] Writing (159) in the form

$$H=H_{13}+H_{23}+(V_{12}+\mathbf{p}_{13}\mathbf{p}_{23}/m_3);\quad H_{ij}=\frac{p_{ij}^2}{2\mu_{ij}}+V_{ij}(\mathbf{r}_{ij}), \qquad (160)$$

we find that the Hamiltonian of the three-particle system differs from the Hamiltonian of the two-particle system in the force field by the presence of the term $\mathbf{p}_{13}/\mathbf{p}_{23}/m_3$. This term is equal to zero when the mass of the third particle is equal to infinity.

We apply to the operator H a unitary transformation of the form $\exp(iX)$, where

$$X=\alpha\,(\mathbf{p}_{13}\,\mathbf{r}_{23}+\mathbf{p}_{23}\,\mathbf{r}_{13}), \qquad (161)$$

and the parameter α is given by the expression[2)]

$$\begin{aligned} \operatorname{th} 2\alpha &= 2\mu_{13}\mu_{23}/[m_3(\mu_{13}+\mu_{23})] \\ &= 2\mu_{12}/(2\mu_{12}+m_3) < 1. \end{aligned} \quad (162)$$

Then the coordinates and momenta of the particle transform as follows:

$$\left.\begin{aligned} \mathbf{r}'_{13} &= \exp(iX)\,\mathbf{r}_{13}\exp(-iX) = \mathbf{r}_{13}\operatorname{ch}\alpha + \mathbf{r}_{23}\operatorname{sh}\alpha; \\ \mathbf{r}'_{23} &= \exp(iX)\,\mathbf{r}_{23}\exp(-iX) = \mathbf{r}_{23}\operatorname{ch}\alpha + \mathbf{r}_{13}\operatorname{sh}\alpha; \\ \mathbf{p}'_{13} &= \exp(iX)\,\mathbf{p}_{13}\exp(-iX) = \mathbf{p}_{13}\operatorname{ch}\alpha - \mathbf{p}_{23}\operatorname{sh}\alpha; \\ \mathbf{p}'_{23} &= \exp(iX)\,\mathbf{p}_{23}\exp(-iX) = \mathbf{p}_{23}\operatorname{ch}\alpha - \mathbf{p}_{13}\operatorname{sh}\alpha; \\ \mathbf{r}'_{12} &= \exp(iX)\,\mathbf{r}_{12}\exp(-iX) = \exp(-\alpha)\,\mathbf{r}_{12}. \end{aligned}\right\} \quad (163)$$

Further, the transformed Hamiltonian

$$H_X = \exp(iX)\,H\exp(-iX) \quad (164)$$

has the form

$$H_X = (k_1 + V_1) + (k_2 + V_2) + V_{12} + W. \quad (165)$$

Here, we have used the notation

$$\left.\begin{aligned} & k_1 = p_{13}^2/2\mu'_{13}; \quad k_2 = p_{23}^2/2\mu'_{23}; \quad V_1 = V_{13}(\mathbf{r}_{13}); \quad V_2 = V_{23}(\mathbf{r}_{23}); \\ & V_{12} = V_{12}(\exp(-\alpha)\,\mathbf{r}_{12}); \\ & W = [V_{13}(\mathbf{r}_{13}\operatorname{ch}\alpha + \mathbf{r}_{23}\operatorname{sh}\alpha) - V_{13}(\mathbf{r}_{13})] \\ & + [V_{23}(\mathbf{r}_{23}\operatorname{ch}\alpha + \mathbf{r}_{13}\operatorname{sh}\alpha) - V_{23}(\mathbf{r}_{23})]. \end{aligned}\right\} \quad (166)$$

The reduced masses μ'_{13} and μ'_{23} in (166) are given by

$$\left.\begin{aligned} \frac{1}{\mu'_{13}} &= \frac{1}{m_1}\operatorname{ch}^2\alpha + \frac{1}{m_2}\operatorname{sh}^2\alpha + \frac{1}{m_3}(\operatorname{ch}\alpha - \operatorname{sh}\alpha)^2; \\ \frac{1}{\mu'_{23}} &= \frac{1}{m_2}\operatorname{ch}^2\alpha + \frac{1}{m_1}\operatorname{sh}^2\alpha + \frac{1}{m_3}(\operatorname{ch}\alpha - \operatorname{sh}\alpha)^2. \end{aligned}\right\} \quad (167)$$

It follows from (165) and (166) that the operator H_X has the form of the Hamiltonian of a system of two particles in an external field in which the many-particle force W is present. The Hamiltonian of the independent subsystems in this case has the form

$$H_{X1,2} = h_{X1} + h_{X2}; \; h_{Xi} = k_i + V_i. \quad (168)$$

For the system with the Hamiltonian H_X (165), the channel Hamiltonians are

$$\left.\begin{aligned} & H_{X1} = k_1 + k_2 + V_1; \quad H_{X1}\varphi_1 = E\varphi_1; \\ & H_{X2} = k_1 + k_2 + V_2; \quad H_{X2}\varphi_2 = E\varphi_2; \\ & H_{X12} = k_1 + k_2 + \bar{V}_{12}; \quad H_{X12}\varphi_{12} = E\varphi_{12}; \\ & H_{X0} = k_1 + k_2; \quad H_{X0}\varphi_0 = E\varphi_0. \end{aligned}\right\} \quad (169)$$

To each asymptotic state of the original Hamiltonian H (159) there corresponds an asymptotic state of the Hamiltonian H_X described by one of the states $\varphi_1, \varphi_2, \varphi_{12}, \varphi_0$ (169), and for the corresponding scattering states we have the following relations[18]:

$$\Psi_\alpha^\pm(E) = \exp(-iX)\,\Psi_{X\alpha}^\pm(E), \quad (170)$$

where $\Psi_\alpha(E)$ is the wave function of channel α for the Hamiltonian H; $\Psi_{X\alpha}(E)$ is the wave function of the same channel for the Hamiltonian H_X corresponding to the asymptotic state φ_α. It follows from (170) that the elements of the S matrix for the systems with the Hamiltonians H and H_X are equal:

$$S_{\alpha\beta} = \langle \psi_\alpha^-(E_\alpha) \mid \psi_\beta^+(E_\beta)\rangle = \langle \Psi_{X\alpha}^-(E_\alpha) \mid \Psi_{X\beta}^+(E_\beta)\rangle = S_{X\alpha,\,\beta}. \quad (171)$$

We characterize the spectral properties of the Hamiltonians of the internal motion in the channels (169), i.e., the Hamiltonians h_{X1}, h_{X2}, and h_{X12} where the last has the form

$$\left.\begin{aligned} & h_{X12} = p_{12}^2/2\mu'_{12} + V_{12}(\exp(-\alpha)\,\mathbf{r}_{12}); \\ & \mu'_{12} = \mu'_{13}\mu'_{23}/(\mu'_{13} + \mu'_{23}); \quad \mathbf{p}_{12} = (\mathbf{p}_{13}\mu'_{23} - \mathbf{p}_{23}\mu'_{13})/(\mu'_{13} + \mu'_{23}). \end{aligned}\right\} \quad (172)$$

We begin with the operator $h_{X1} = k_1 + V_1$. We note that the interaction potential V_1 has not changed under this unitary transformation, but there has been a formal change in the reduced mass of the particles (1 and 3) in the considered channel. Indeed, for μ'_{13} we have the relation

$$\mu'_{13} = \mu_{13}(1 + \Delta)/[1 + (1/2)(1 - \mu_{13}/\mu_{23})\,\Delta] > \mu_{13}, \quad (173)$$

where $\Delta = \cosh 2\alpha - 1 > 0$. Therefore, under the given unitary transformation there has been an effective increase in the mass of the channel by $\beta > 1$ times, which is equivalent to an increase by β times in the force of the potential V_1. The upshot is that the Hamiltonian h_{X1} has the form $h_0 + \beta V_1$. If the potential V_1 were attractive in at least some region of the coordinate space, then in accordance with Ref. 16 the eigenvalues $E(\beta)$ of the Hamiltonian $h_0 + \beta V$ decrease monotonically, and their number increases monotonically with increasing parameter β. The case of the operator h_{X2} is treated similarly. It follows from the foregoing that the Hamiltonian $H_{X1,2}$ can have a discrete spectrum even if in the original Hamiltonian the potentials V_{13} and V_{23} in channels 13 and 23 do not form bound states.

The spectrum in channel 12 is determined from the equation

$$h_{X12}\varphi_{12} = -\varkappa_{X12}^2\varphi_{12}. \quad (174)$$

The reduced mass μ'_{12} of channel 12 in (172) is equal to $\exp(-2\alpha)\,\mu_{12}$. By the change of variables $\mathbf{r}'_{12} = \exp(-\alpha)\mathbf{r}_{12}$, Eq. (174) can be reduced to the form

$$h_{12}\varphi_{12}(\mathbf{r}'_{12}) = -\varkappa_{X12}^2\varphi_{12}(\mathbf{r}'_{12}),$$

whence $\varkappa_{12}^2 = \varkappa_{X12}^2$, i.e., the spectrum in channel 12 is not changed by the considered unitary transformation. With this we conclude the brief review of the properties of the unitary transformation (164) that reduces the three-body problem to the problem of two particles in a force field.

Resonances and bound states in the three-body system. The unitary transformation described above makes it possible to extend the obtained results to the case of the three-body problem.

Suppose the Hamiltonian H (159) is such that on reduction with respect to one of the particles the Hamiltonian $H_{X1,2}$ has a nonempty discrete spectrum, and let $|\psi_{X1,2}\rangle$ be one of the states of $\sigma_{\mathrm{disc}}(H_{X1,2})$ with energy $-E_0$. Then, repeating the arguments that led us to (141), we find that in the considered system there exist states of the discrete spectrum and resonances, which are determined by the roots of the equation

$$E = -E_0 + \operatorname{Re} b(E + i0), \quad (175)$$

where

$$\left.\begin{aligned} & b(z) = \langle \psi_{X1,2} \mid U_X + U_X R_X(z)\,U_X \mid \Psi_{X1,2}\rangle; \\ & U_X = \bar{V}_{12} + W; \quad R_X(z) = (zM_X - M_X H_X M_X)^{-1} M_X; \\ & M_X = 1 - \Lambda_X; \quad \Lambda_X = |\psi_{X1,2}\rangle\langle\psi_{X1,2}|. \end{aligned}\right\} \quad (176)$$

Equation (176) contains the resolvent $R_X(z)$, which corresponds to the Hamiltonian $M_X H_X M_X$, this including a many-particle force. To give meaning to the formal expression for $R_X(z)$, we apply to this operator the inverse transformation $\exp(-iX)$. We denote

$$\Lambda = \exp(-iX)\,\Lambda_X \exp(iX);\; M = \exp(-iX)\,M_X \exp(iX). \quad (177)$$

Obviously, Λ and M are orthogonal projectors. Then from the identity for $R_X(z)$ of the form

$$(zM_X - M_X H_X M_X)\,R_X(z) = M_X \quad (178)$$

we obtain

$$(zM - MHM)\exp(-iX)\,R_X(z)\exp(iX) = M,$$

whence

$$\exp(-iX)\,R_X(z)\exp(iX) = R(z) = (zM - MHM)^{-1}M. \quad (179)$$

From the last equation, we conclude that the resolvents $R_X(z)$ and $R(z)$ exist, and for $b(z)$ we have an expression of the form

$$b(z) = \langle\psi_{1,2}\,|\,U + UR(z)\,U\,|\,\psi_{1,2}\rangle, \quad (180)$$

where

$$\left.\begin{aligned} &|\psi_{1,2}\rangle = \exp(-iX)\,|\psi_{X1,2}\rangle;\\ &U = \exp(-iX)\,U_{\bar{X}}\exp(iX)\\ &= V_{12}(\bar{r}_{12}) + [V_{13}(\bar{r}_{13}) - V_{13}(\bar{r}_{13}\,\mathrm{ch}\,\alpha - \bar{r}_{23}\,\mathrm{sh}\,\alpha)]\\ &+ [V_{23}(\bar{r}_{23}) - (V_{23}\bar{r}_{23}\,\mathrm{ch}\,\alpha - \bar{r}_{13}\,\mathrm{sh}\,\alpha)]. \end{aligned}\right\} \quad (181)$$

Further, since the Hamiltonian H_X differs from the ordinary Hamiltonian of a system of two particles in an external field by the replacement of V_{12} by $\overline{V}_{12} + W$, we obtain from (141) the following representation for the resolvent of the operator $H_X G_X(z)$:

$$G_X(z) = \frac{[1 + R_X(z)\,U_X]\,|\,\psi_{X1,2}\rangle\langle\psi_{X1,2}\,|\,[1 + U_X R_X(z)]}{z + E_0 - b(z)} + R_X(z). \quad (182)$$

Since

$$G(z) = \exp(-iX)\,G_X\exp(iX),$$

we find from (182) that the Green's function of the original Hamiltonian H of the three-body problem (159) has the form

$$G(z) = \{[1 + R(z)\,U]\,|\,\psi_{1,2}\rangle\langle\psi_{1,2}\,|\,[1 + UR(z)]\}/[z + E_0 - b(z)] + R(z). \quad (183)$$

We denote by Σ the point at which the continuum $\sigma_c(H) = (\Sigma, \infty)$ of the Hamiltonian H begins:

$$\Sigma = \min_{\alpha}\{-\varkappa_\alpha^2\},\quad \alpha = 12,\,13,\,23,$$

and $-\varkappa_\alpha^2$ are the energies of the bound states in the channels, i.e., $-\varkappa_\alpha^2 \in \sigma_{\mathrm{disc}}(h_\alpha)$.

If the equation

$$E = -E_0 + \mathrm{Re}\,b(E + i0) \quad (184)$$

has a root E_d such that $E_d < \Sigma$, then the point E_d is the energy of a bound state in the system. Indeed, in this case the function $b(E)$ exists and is real in some neighborhood of the point E_d, and the Green's function $G(z)$ (183) has in this neighborhood a simple pole of the form $(z - E_d)^{-1}$. It follows from Ref. 5 that $E_d \in \sigma_{\mathrm{disc}}(H)$, and the corresponding wave function $|\psi_d\rangle$ has the form

$$|\psi_d\rangle = (1 + a^2)^{-1/2}\,[1 + R(E_d)\,U]\,|\psi_{1,2}\rangle, \quad (185)$$

where $a^2 = \langle\psi_{1,2}|UR(E_d)R(E_d)U|\psi_{1,2}\rangle$. The fact that $|\psi_d\rangle$ satisfies the equation $H|\psi_d\rangle = E_d|\psi_d\rangle$ can be verified by direct calculation in complete analogy with (150). Note that by virtue of the definition (181) the function $|\psi_{1,2}\rangle$ satisfies the equation

$$[E_0 + H_0 + V_{13}(\mathbf{r}_{13}\,\mathrm{ch}\,\alpha - \mathbf{r}_{23}\,\mathrm{sh}\,\alpha) + V_{23}(\mathbf{r}_{23}\,\mathrm{ch}\,\alpha - \mathbf{r}_{13}\,\mathrm{sh}\,\alpha)]\,\psi_{1,2}(\mathbf{r}_{13}, \mathbf{r}_{23}) = 0. \quad (186)$$

By a change of variables of the form

$$\mathbf{r}_1 = \mathbf{r}_{13}\,\mathrm{ch}\,\alpha - \mathbf{r}_{23}\,\mathrm{sh}\,\alpha;\; \mathbf{r}_2 = \mathbf{r}_{23}\,\mathrm{ch}\,\alpha - \mathbf{r}_{13}\,\mathrm{sh}\,\alpha$$

Eq. (186) is transformed to the equation

$$[E_0 - \Delta_{r_1}/2\mu'_{13} - \Delta_{r_2}/2\mu'_{23} + V_{13}(\mathbf{r}_1) + V_{23}(\mathbf{r}_2)]\,\psi_{1,2}(\mathbf{r}_1\,\mathbf{r}_2) = 0.$$

Hence, the function $\psi_{1,2}(\mathbf{r}_{13}, \mathbf{r}_{23})$ is given by

$$\psi_{1,2}(\mathbf{r}_{13}, \mathbf{r}_{23}) = \psi_{X1}(\mathbf{r}_{12}\,\mathrm{ch}\,\alpha - \mathbf{r}_{23}\,\mathrm{sh}\,\alpha)\,\psi_{X2}(\mathbf{r}_{23}\,\mathrm{ch}\,\alpha - \mathbf{r}_{13}\,\mathrm{sh}\,\alpha). \quad (187)$$

The roots of Eq. (184) lying above the point Σ describe the resonance states in the three-body system.

On the number of bound states and resonances in the three-particle system. We shall analyze the conditions under which the equation

$$E + E_0 - \mathrm{Re}\,b(E + i0) = 0 \quad (188)$$

has solutions. We recall that for the function

$$f(z) = z + E_0 - b(z) \quad (189)$$

there exists a representation in terms of the Green's function $G(z)$ of the original Hamiltonian of the form

$$G(z) = \frac{[1 + R(z)\,U]\,|\,\psi_{1,2}\rangle\langle\psi_{1,2}\,|\,[UR(z) + 1]}{z + E_0 - b(z)} + R(z). \quad (190)$$

Hence, by virtue of the properties of the operator $R(z)$,

$$\langle\psi_{1,2}\,|\,G(z)\,|\,\psi_{1,2}\rangle = [z + E_0 - b(z)]^{-1}, \quad (191)$$

and Eq. (188) reduces to the form

$$\mathrm{Re}\,[\langle\psi_{1,2}\,|\,G(E + i0)\,|\,\psi_{1,2}\rangle]^{-1} = 0. \quad (192)$$

We now show that Eqs. (188) and (192) reproduce all the points of $\sigma_{\mathrm{disc}}(H)$ whose eigenvectors are nonorthogonal to $|\psi_{1,2}\rangle$. If the Hamiltonian H has a bound state with energy E_d and wave function $|\psi_d\rangle$ such that $\psi_{1,2}|\psi_d\rangle = 0$, then E_d is a root of Eq. (192), since the matrix element $\langle\psi_{1,2}|G(E + i0)|\psi_{1,2}\rangle$ tends to infinity as $E \to E_d$. As was established earlier, the function

$$|\psi_d\rangle = C\,[1 + R(E_d)\,U]\,|\psi_{1,2}\rangle \quad (193)$$

is an eigenfunction for H. If the function $|\psi_d\rangle$ is to be square-integrable, it is necessary that $E_d < \Sigma$ and that the resolvent $R(z)$ should exist for $z = E_d$. The condition $E_d < \Sigma$ means that the system does not have bound states embedded in the continuum. The existence of the resolvent $R(z)$ at the point E_d under the condition on $\langle\psi_{1,2}|\psi_d\rangle$ can be established by means of the expression for $R(z)$ in terms of $G(z)$ obtained from (190):

$$R(z) = G(z) - G(z)\,|\,\psi_{1,2}\rangle\langle\psi_{1,2}\,|\,G(z)/\langle\psi_{1,2}\,|\,G(z)\,|\,\psi_{1,2}\rangle. \quad (194)$$

It follows from (194) that $R(E_d)$ is nonsingular for $E_d < \Sigma$, and $\|\psi_d\| < \infty$.

Thus, Eq. (188) does indeed reproduce all the states of the discrete spectrum that are nonorthogonal to $|\psi_{1,2}\rangle$. Now suppose that all $|\psi_d\rangle$ are orthogonal to $|\psi_{1,2}\rangle$ or $\sigma_{\mathrm{disc}}(H)$. We rewrite Eq. (192) in the form

$$\mathrm{Re}\,\langle\psi_{1,2}\,|\,G(E + i0)\,|\,\psi_{1,2}\rangle = 0. \quad (195)$$

Expanding $|\psi_{1,2}\rangle$ with respect to the eigenfunctions of the operator H,

$$|\psi_{1,2}\rangle = \sum_\alpha \int dE_\alpha C_\alpha(E_\alpha) |\psi_\alpha(E_\alpha)\rangle, \quad (196)$$

we reduce Eq. (195) to the form

$$\sum_\alpha \mathcal{P} \int dE_\alpha \frac{|C_\alpha(E_\alpha)|^2}{E-E_\alpha} = 0. \quad (197)$$

In (196) and (197) we have borne in mind that all the $|\psi_d\rangle$ are orthogonal to $|\psi_{1,2}\rangle$ or, which in the given case is equivalent to the condition on $\sigma_{disc}(H)$, are absent. The function

$$\Phi(E) = \sum_\alpha \mathcal{P} \int dE_\alpha |C_\alpha(E_\alpha)|^2/(E-E_\alpha)$$

is a continuous function of E. For $E<\Sigma$, the function $\Phi(E)$ is strictly negative, and for sufficiently large positive E the function $\Phi(E)$ is strictly positive, since the functions $C_\alpha(E_\alpha)$ decrease rapidly with increasing E_α. By continuity, Eq. (197) has a root, and it lies in the continuum. Therefore, in the system there exists at least one resonance.

Summarizing the results obtained above, we find that if the Hamiltonian H is such that the corresponding Hamiltonian $H_{1,2}$ or $H_{X1,2}$ has a discrete spectrum, then each state in $\sigma_{disc}(H_{1,2})$ or $\sigma_{disc}(H_{X1,2})$ generates either a bound state in $\sigma_{disc}(H)$ or a resonance. This fact can be expressed by the inequality

$$N_R + N_d \geqslant 1, \quad (198)$$

where N_d is the number of states in $\sigma_{disc}(H)$, and N_R is the number of resonances described by Eq. (1). If the number of these equations is equal to N [here, N is the number of states in $\sigma_{disc}(H_{1,2})$ or $\sigma_{disc}(H_{X1,2})$], then

$$N_R + N_d \geqslant N. \quad (199)$$

We now apply the inequalities (198) and (199) to systems of three nucleons. To be specific, we consider the *pnn* system. Since the interaction potentials between nucleons depend on the spins, it is necessary to consider two cases corresponding to the different spin configurations in the system of three nucleons.

Case 1. $S=3/2$.

Suppose the spin of the *pnn* system of three nucleons is 3/2. In this case, making a reduction with respect to the photon, we obtain

$$H_{X1,2} = K_1 + K_2 + V_{pn}^t + V_{pn}^t, \quad (200)$$

where V_{pn}^t is the triplet nucleon–nucleon potential. Since V_{pn}^t had a bound state in the channels 13 and 23 in the original Hamiltonian H, it follows from the results given above that the Hamiltonians $h_{X1} = K_1 + V_{pn}^t$, $h_{X2} = K_2 + V_{pn}^t$, and $H_{X1,2}$ have discrete spectra. Therefore, for $S=3/2$ Eq. (199) has at least one root. Since the spin of the bound state in the *pnn* system, i.e., $-H^3$, is equal to 1/2, this root must correspond to a resonance.

Case 2. $S=1/2$.

If the spin of the *pnn* system is 1/2, then after a unitary transformation of the form (193) we obtain two types of Hamiltonian $H_{X1,2}$:

$$H_{X1,2}^1 = K_1 + K_2 + V_{pn}^t + V_{nn}^S; \quad (201)$$

$$H_{X1,2}^2 = K_1 + K_2 + V_{pn}^S + V_{nn}^S. \quad (202)$$

The Hamiltonians $H_{X1,2}^1$ and $H_{X1,2}^2$ have a nonempty discrete spectrum, or there exists a bound state of the Hamiltonian $h_X^S = K + V^S$. As we pointed out above, as a result of the transformation (193) states of the discrete spectrum of h_X^S exist if they do for the Hamiltonian $-\Delta/2\mu + \beta V$, where $\beta = \mu'_{13}/\mu_{13} = \cosh 2\alpha = \sqrt{4/3}$. From this point of view, all phenomenological singlet nucleon–nucleon potentials, which we shall assume are charge-independent, can be divided into two classes. Potentials in the first class are such that when the strength of the potential is increased by $\beta = \sqrt{4/3}$ times they do not transform a virtual state into a bound state. In this class, for example, we have the potential in the form of a rectangular well with parameters $R = 2.583$ F, $V_0 = 102\ 276 \times 0.8893/R^2$ MeV used in Ref. 19 to seek resonances in the system of three neutrons. Potentials in the second class are such that the Hamiltonian $-\Delta/2\mu + \beta V^S$ has a bound state. To this class there belongs the Yamaguchi potential

$$V(\mathbf{k}, \mathbf{k}^0) = -(\lambda/2\mu)(\beta^2 + k^2)^{-1}(\beta^2 + k^{02})^{-1}$$

with parameters $\lambda^S = 0.291$ F^{-3}, $\beta^S = 1.4487$ F^{-1}. As is well known, the condition for the existence of bound state in the Yamaguchi potential is

$$\pi\sqrt{\lambda/\beta} - \beta > 0. \quad (203)$$

The inequality (203) is not satisfied for the sets of parameters (λ^S, β^S) but is satisfied for the set $(\sqrt{4/3}\lambda^S, \beta^S)$.

Thus, potentials in the second class, in contrast to potentials in the first, lead to the existence of a discrete spectrum of the Hamiltonians (201) and (202), and, therefore, in this case the condition $N_R + N_d \geqslant 2$ holds.

The results of our treatment can be formulated as follows: In the approximation of the three-body problem with two-body interactions in the *pnn* system of three nucleons three nucleons there always exists at least one resonance with $S=3/2$. The same is true of the *ppn* system. If the singlet potential V_{NN}^S belongs to the second class, then in the state with $S=1/2$ the *pnn* and *ppn* systems also have at least one resonance state, or, more precisely, $N_R + N_d \geqslant 2$, and there exists a resonance in the system of three neutrons. But if the real nucleon–nucleon potential (in the state $S=0$) belongs to the first class, then there is no equation (188), and the question of the existence of a resonance in the three-nucleon system remains open. Thus, our method makes it possible to establish sufficient conditions for the existence of bound states and resonances in a system of three particles and two particles in a force field. This condition consists of the existence of a discrete spectrum of the Hamiltonians $H_{X1,2}$ and $H_{1,2}$, and the reduction for $H_{X1,2}$ can be made with respect to any of the particles.

On scattering theory for particles whose interaction has a long-range nature. In a number of cases, the description of the interaction of several particles requires one to take into account not only short- but also long-range forces. It is well known that the direct generalization of the results of scattering theory to these processes is not valid and that they must be analyzed differently. In the present section, we shall propose one of the methods of describing systems of scattering particles with long-range forces; it is based on the time-dependent theory proposed by Dollard[20] and developed in Refs. 16, 21, and 25.

Following Refs. 20–25, we define the channel wave operators in the form

$$\Omega^{\pm}_{\alpha D}=S-\lim_{t\mp\infty}\exp(\mathrm{i}Ht)\,U_{\alpha D}(t). \tag{204}$$

Here, H is the energy operator of the system defined in the space $\mathcal{G}=\mathcal{L}^2(R^{3N-3})$, $U_{\alpha D}(t)$ is the evolution operator of the system in the channel α,

$$U_{\alpha D}(t)=\exp[-\mathrm{i}H_\alpha t-\mathrm{i}\varphi^{\pm}_{\alpha}(t)]\,P_\alpha; \tag{205}$$

$H_\alpha=H_{0\alpha}+V_\alpha$ is the energy operator in the channel α, $P_\alpha=|\psi_\alpha\rangle\langle\psi_\alpha|$, where $|\psi_\alpha\rangle$ is the wave function of the bound state of the fragments in channel α, and $H_\alpha|\psi_\alpha \mathbf{p}_\alpha\rangle=E_\alpha|\psi_\alpha\ \mathbf{p}_\alpha\rangle$, $|\mathbf{p}_\alpha\rangle$ is the wave function of free motion of the fragments in the channel α. The behavior of the function $\varphi^{\pm}_{\alpha}(t)$ as $t\to\mp\infty$ was considered in Refs. 16 and 22 and for the Coulomb potential it is

$$\varphi^{\pm}_{\alpha}(t)\xrightarrow[t\to\mp\infty]{}\operatorname{sign}t\,\eta_\alpha(\hat{p}_\alpha)\ln|t|. \tag{206}$$

Here, $\hat{p}_\alpha$ are the momentum operators of the relative motion of the bound fragments in channel α and in the momentum representation $\eta_\alpha(p_\alpha)$ is defined by

$$\eta_\alpha(p_\alpha)=\sum_{\gamma,\,\nu\in\alpha}\frac{z_\gamma z_\nu e^2\mu_{\gamma\nu}}{p_{\gamma\nu}}, \tag{207}$$

where (γ,ν) is the pair of fragments in channel α; z_γ and z_ν are the charges of the fragments; $\mu_{\gamma\nu}$ and $p_{\gamma\nu}$ are the reduced mass and the momentum of the relative motion of the fragments (γ,ν). As is shown in Ref. 16, for the operators $\Omega^{\pm}_{\alpha D}$ we have

$$(\Omega^{\pm}_{\alpha D})^{+}\,\Omega^{\pm}_{\alpha D}=P_\alpha;\quad H\Omega^{\pm}_{\alpha D}=\Omega^{\pm}_{\alpha D}H_\alpha.$$

In the considered method of analysis of systems with long-range potentials, we can obtain from the condition (204) an equation for the wave function of the system and an expression for the elements of the S matrix. We define the wave function of the system, in which the action develops from channel α, in the form

$$|\psi^{D}_{\alpha}(E_\alpha\pm \mathrm{i}0)\rangle=\lim_{t\to\mp\infty}\exp(\mathrm{i}Ht)\exp[-\mathrm{i}H_\alpha t-\mathrm{i}\varphi^{\pm}_{\alpha}(t)]\,|\psi_\alpha,\,p_\alpha\rangle. \tag{208}$$

In accordance with the arguments given in Ref. 16 (Chap. 11), we shall assume that the function $\varphi^{\pm}_{\alpha}(t)$ is continuous, has a piecewise continuous derivative, and $\varphi^{\pm}_{\alpha}(0)=0$. Then from (208) we have

$$|\psi^{D}_{\alpha}(E_\alpha\pm \mathrm{i}0)\rangle-\mathrm{i}\int_0^t ds\exp[\mathrm{i}H_\alpha S+\mathrm{i}\varphi_\alpha(S)]$$
$$\times\Big[H-H_\alpha-\frac{d\varphi^{\pm}_{\alpha}(s)}{ds}\exp(-\mathrm{i}Hs)\,|\psi^{D}_{\alpha}(E_\alpha\pm \mathrm{i}0)\rangle\xrightarrow[t\to\mp\infty]{}|\psi_\alpha,\,p_\alpha\rangle. \tag{209}$$

Since the integral in (209) converges in the norm, by introducing a cutoff factor in the integrand and noting that the functions $|\psi_\alpha(E)\rangle$ are eigenfunctions for H, we obtain the result

$$|\psi^{D}_{\alpha}(E_\alpha\pm \mathrm{i}0)\rangle=|\psi_\alpha,\,p_\alpha\rangle+[\mathrm{i}K^{\pm}_{1\alpha}V^{\alpha}-\mathrm{i}K^{\pm}_{2\alpha}]\,|\psi^{D}_{\alpha}(E_\alpha\pm \mathrm{i}0)\rangle, \tag{210}$$

where

$$\left.\begin{aligned}K^{\pm}_{1\alpha}&=\lim_{\varepsilon\to 0}\int_0^{\mp\infty}dt\exp[-u^{\pm}_{\alpha}t+\mathrm{i}\varphi^{\pm}_{\alpha}(t)];\\ K^{\pm}_{2\alpha}&=\lim_{\varepsilon\to 0}\int_0^{\mp\infty}dt\exp[-u^{\pm}_{\alpha}t+\mathrm{i}\varphi^{\pm}_{\alpha}(t)]\frac{d\varphi^{\pm}_{\alpha}(t)}{dt};\end{aligned}\right\} \tag{211}$$

$$u^{\pm}_{\alpha}=\mathrm{i}(E_\alpha-H_\alpha\pm \mathrm{i}\varepsilon)=\mathrm{i}G^{-1}_{\alpha}(E_\alpha\pm \mathrm{i}\varepsilon). \tag{212}$$

Integrating by parts the expression for the operator $K^{\pm}_{1\alpha}$, we arrive at the following relation between $K^{\pm}_{1\alpha}$ and $K^{\pm}_{2\alpha}$:

$$K^{\pm}_{1\alpha}=\frac{1}{\mathrm{i}}G_\alpha(E_\alpha\pm \mathrm{i}0)[1+\mathrm{i}K^{\pm}_{2\alpha}]=\frac{1}{\mathrm{i}}[1+\mathrm{i}K^{\pm}_{2\alpha}]\,G_\alpha(E_\alpha\pm \mathrm{i}0). \tag{213}$$

Further, combining Eq. (210) and the relation (213), we obtain the required equation of Lippmann–Schwinger type:

$$|\psi^{D}_{\alpha}(E_\alpha\pm \mathrm{i}0)\rangle=|\psi^{\pm}_{\alpha D}\rangle+G_\alpha(E_\alpha\pm \mathrm{i}0)\,V^{\alpha}\,|\psi^{D}_{\alpha}(E_\alpha\pm \mathrm{i}0)\rangle, \tag{214}$$

where

$$\begin{aligned}|\psi^{\pm}_{\alpha D}\rangle=\Phi^{\pm}_{\alpha}(p_\alpha)\,|\psi_\alpha,\,\mathbf{p}_\alpha\rangle&=\lim_{\varepsilon\to 0}[1+\mathrm{i}K^{\pm}_{2\alpha}(E_\alpha\pm \mathrm{i}\varepsilon)]^{-1}\,|\psi_\alpha,\,\mathbf{p}_\alpha\rangle\\ &=\lim_{\varepsilon\to 0}[u^{\pm}_{\alpha}(\varepsilon)\,K_{1\alpha}(E_\alpha\pm \mathrm{i}\varepsilon)^{-1}\,|\psi_\alpha,\,\mathbf{p}_\alpha\rangle.\end{aligned} \tag{215}$$

In the absence of long-range forces between the fragments moving in channel α, Eq. (214) is the ordinary Lippmann–Schwinger equation, since $V^{\alpha}_{D}=0$ and $K^{\pm}_{2\alpha}=0$.

We now find the expression for the elements of the multichannel S matrix in the case of long-range forces. From Eq. (214) and the well-known resolvent identity

$$G(z)=G_\alpha(z)+G_\alpha(z)\,V^{\alpha}G(z)$$

we have

$$|\psi^{D}_{\alpha}(E_\alpha\pm \mathrm{i}0)\rangle=[1+G(E_\alpha\pm \mathrm{i}0)\,V^{\alpha}]\,|\psi_\alpha,\,\mathbf{p}_\alpha\rangle\,\Phi^{\pm}_{\alpha}(p_\alpha); \tag{216}$$

$$\begin{aligned}|\psi^{D}_{\alpha}(E_\alpha+\mathrm{i}0)\rangle&=|\psi^{D}_{\alpha}(E_\alpha-\mathrm{i}0)\rangle\,[\Phi^{-}_{\alpha}(p_\alpha)]^{-1}\,\Phi^{+}_{\alpha}(p_\alpha)\\ &+[G(E_\alpha+\mathrm{i}0)-G(E_\alpha-\mathrm{i}0)]\,V^{\alpha}\,|\psi_\alpha,\,\mathbf{p}_\alpha\rangle\,\Phi^{+}_{\alpha}(p_\alpha).\end{aligned} \tag{217}$$

Since for the elements of the S matrix there is a representation of the form

$$\langle\mathbf{p}_\alpha\,|\,S_{\alpha\beta}\,|\,\mathbf{p}_\beta\rangle=\langle\psi_\alpha(E_\alpha-\mathrm{i}0)\,|\,\psi_\beta(E_\beta+\mathrm{i}0)\rangle, \tag{218}$$

substituting (216) and (217) in (218), and also assuming that $\alpha\neq\beta$, we obtain

$$\begin{aligned}\langle\mathbf{p}_\alpha\,|\,S_{\alpha\beta}\,|\,\mathbf{p}_\beta\rangle&=\Big(\frac{1}{E_\beta-E_\alpha+\mathrm{i}0}-\frac{1}{E_\beta-E_\alpha-\mathrm{i}0}\Big)\,t^{D}_{\alpha\beta}(\mathbf{p}_\alpha,\,\mathbf{p}_\beta,\,E_\alpha+\mathrm{i}0)\\ &=-2\pi\mathrm{i}\,\delta(E_\alpha-E_\beta)\,t^{D}_{\alpha\beta}(\mathbf{p}_\alpha,\,\mathbf{p}_\beta,\,E_\alpha+\mathrm{i}0),\end{aligned} \tag{219}$$

where

$$\begin{aligned}t^{D}_{\alpha\beta}(\mathbf{p}_\alpha,\,\mathbf{p}_\beta,\,E_\alpha+\mathrm{i}0)&=[\Phi^{-}_{\alpha}(p_\alpha)]^{*}\,\langle\mathbf{p}_\alpha\,|\,T^{D}_{\alpha\beta}(E_\alpha+\mathrm{i}0)\,|\,\mathbf{p}_\beta\rangle\,\Phi^{+}_{\beta}(p_\beta)\\ &=\lim_{\varepsilon\to 0}[1+\mathrm{i}K^{-}_{2\alpha}(E_\alpha+\mathrm{i}\varepsilon)]^{-1}\,\langle\mathbf{p}_\alpha\,|\,T^{D}_{\alpha\beta}(E_\alpha+\mathrm{i}\varepsilon)\,|\,\mathbf{p}_\beta\rangle\\ &\quad\times[1+\mathrm{i}K^{+}_{2\beta}(E_\alpha+\mathrm{i}\varepsilon)]^{-1}.\end{aligned} \tag{220}$$

The operator $T^{D}_{\alpha\beta}(z)$ in (220) is defined by

$$T^{D}_{\alpha\beta}(z)=\langle\psi_\alpha\,|\,V^{\beta}+V^{\alpha}G(z)\,V^{\beta}\,|\,\psi_\beta\rangle. \tag{221}$$

In the case of Coulomb interaction between the particles we must define the function $\Phi^{\pm}_{\alpha}(p_\alpha)$ in order to write down Eq. (214) and the expression (218). For this, we choose the function $\varphi^{\pm}_{\alpha}(t)$ in the form

$$\varphi^{\pm}_{\alpha}(t)=\begin{cases}0, & |t|\leqslant 1;\\ \operatorname{sign}t\,\eta_\alpha(p_\alpha)\ln|t|, & |t|\geqslant 1,\end{cases} \tag{222}$$

where η_α is defined by Eq. (207). We consider, for example, the function $\Phi^{-}_{\alpha}(p_\alpha)$. In accordance with (215), it is expressed by an integral of the form

$$I(p)=p\int_0^\infty dt \exp[-pt+i\varphi_\alpha^-(t)]=1-\exp(-p)+\frac{p\Gamma(1+i\eta_\alpha p)}{p^{1+i\eta_\alpha}}. \quad (223)$$

Following Ref. 27, we let the modulus of the parameter p in (223) tend to zero, and then

$$I(p)\underset{|p|\to 0}{\longrightarrow} p^{-i\eta_\alpha}\Gamma(1+i\eta_\alpha). \quad (224)$$

Hence, for the function $\Phi_\alpha^\pm(p_\alpha)$ we have a representation of the form

$$\Phi^\mp(p_\alpha)=(\mp i\varepsilon)^{\pm i\eta_\alpha}\exp[-(\pi/2)\eta_\alpha(p_\alpha)]/\Gamma(1\pm i\eta_\alpha). \quad (225)$$

Substituting the expression (225) in Eq. (215) and the relation (220), we obtain the following results for the description of the scattering of Coulomb particles:

$$\begin{gathered}|\psi_\alpha^c(E_\alpha\pm i0)\rangle=(\pm i\varepsilon)^{\mp i\eta_\alpha}\frac{\exp[-(\pi/2)\eta_\alpha(p_\alpha)]}{\Gamma(1\mp i\eta_\alpha)}|\psi_\alpha,\mathbf{p}_\alpha\rangle\\ +G_\alpha(E_\alpha\pm i0)V_\alpha|\psi_\alpha^c(E_\alpha\pm i0)\rangle;\\ t_{\alpha\beta}^c(\mathbf{p}_\alpha,\mathbf{p}_\beta,E_\alpha\pm i0)=F_\alpha(p_\alpha)F_\beta(p_\beta)\\ \times\lim_{\varepsilon\to 0}(\pm i\varepsilon)^{\mp i\eta_\alpha(p_\alpha)}T_{\alpha\beta}^c(\mathbf{p}_\alpha,\mathbf{p}_\beta,E_\alpha+i\varepsilon)(\pm i\varepsilon)^{\mp i\eta_\beta(p_\beta)};\\ F_\alpha(p_\alpha)=\exp[-(\pi/2)\eta_\alpha(p_\alpha)]/\Gamma(1\mp i\eta_\alpha).\end{gathered}$$

These results for Coulomb scattering agree with the results obtained by other methods in Refs. 28 and 29.

1)An exposition of various aspects of integral equations in the theory of many-particle scattering can be found, for example, in Refs. 1–10.

2)Translator's Note. The Russian notation for the trigonometric, inverse trigonometric, hyperbolic trigonometric functions, etc., is retained here and throughout the article in the displayed equations.

1B. Simon, Helv. Phys. Acta 43, 607 (1970).
2D. R. Yafaev, Funksional: Analiz 6, 103 (1972).
3W. Hunziker, Helv. Phys. Acta 39, 451 (1966).
4A. Ya. Povzner, Dokl. Akad. Nauk SSSR 104, No. 3 (1955).
5L. D. Faddeev, Tr. Mat. Inst. Akad. Nauk SSSR 69 (1963).
6V. V. Komarov, A. M. Popova, and V. L. Shablov, J. Math. Phys. 24, 854 (1980).
7E. W. Schmid and H. Ziegelmann, The Quantum-Mechanical Three-Body Problem, Pergamon Press, Oxford (1974) [Russian translation published by Nauka, Moscow (1979)]
8J. R. Taylor, Scattering Theory: the Quantum Theory of Nonrelativistic Scattering, Wiley, New York (1972) [Russian translation published by Mir, Moscow (1975)].
9V. V. Komarov, V. L. Shablov, A. M. Popova, and T. A. Osborn, Phys. Rev. C 22, 976 (1980).
10T. A. Osborn and K. L. Kowalski, Ann. Phys. (N. Y.) 68, 361 (1971).
11K. Jörgens and J. Weidmann, Spectral Properties of Hamilton Operators, Springer, Berlin (1973) [Russian translation published by Mir, Moscow (1976)].
12W. Hunziker, Mathematical Theory of Multiparticle Systems. Lectures in Theoretical Physics, Barut and Britten, New York (1968).
13N. M. Hugenholz, Physica (Utrecht) 23, 481 (1957).
14N. I. Muskhelishvili, Singulyarnye integral'nye uravneniya, Nauka, Moscow (1962); English translation: Singular Integral Equations, Groningen (1972).
15S. Albeverio, W. Hunziker, W. Schneider, and R. Schrader, Helv. Phys. Acta 40, 745 (1967).
16M. Reed and B. Simon, Methods of Modern Mathematical Physics, Vol. 4, Academic Press, New York (1978) [Russian translation published by Mir, Moscow (1982)].
17T. Kato, Perturbation Theory of Linear Operators, Springer, Berlin (1966) [Russian translation published by Mir, Moscow (1972)].
18R. G. Newton, Scattering Theory of Waves and Particles, McGraw-Hill, New York (1966) [Russian translation published by Mir, Moscow (1969)].
19A. M. Badalyan *et al.*, Yad. Fiz. 20, 1147 (1974) [Sov. J. Nucl. Phys. 20, 602 (1975)].
20J. D. Dollard, J. Math. Phys. 5, 729 (1964).
21W. O. Amrein, P. A. Martin, and B. Misra, Helv. Phys. Acta 43, 313 (1970).
22V. S. Buslaev and V. B. Matveev, Teor. Mat. Fiz. 2, 367 (1970).
23C. Chandler and A. G. Gibson, J. Math. Phys. 15, 291 (1974).
24E. Prugovecki and J. Zorbas, Nucl. Phys. A213, 541 (1973).
25V. Enss, Commun. Math. Phys. 65, 151 (1979).
26S. Sunakawa, Kvantovaya teoriya rasseyaniya (Quantum Theory of Scattering: Russian translation from the Japanese), Mir, Moscow (1979).
27A. Erdélyi *et al.* (eds.), Higher Transcendental Functions, Vols. 1 and 2 (California Institute of Technology), H. Bateman MS Project, McGraw-Hill, New York (1953, 1954) [Russian translation published by Nauka, Moscow (1973, 1974)].
28Z. Baizer, in: Few Body Nuclear Physics (eds. G. Pisent, V. Vanzani, and L. Fonda), IAEA, Vienna (1978), p. 365.
29A. M. Veselova, Teor. Mat. Fiz. 13, 368 (1972).

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