Solitons and numerical experiments

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The properties of one- and two-dimensional solitons are discussed. Particular attention is paid to qualitative results obtained by computer simulation. Where possible, a brief description of the computational method is given.

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INTRODUCTION

It should be said at the start that it was the computer that, about 25 years ago, created the Fermi-Pasta-Ulam problem and then discovered solitons. As a result of a numerical experiment on the dynamics of nonlinear waves of the Korteweg-de Vries equation (KdV equation) there appeared the concept of solitons (Zabusky) as solitary waves that emerge from an interaction without changing their shape or velocity. Somewhat earlier, similar effects had been discovered in the framework of the sine-Gordon equation in the experiments of Perring and Skyrme, though for very different objects.

It is interesting to note that "two-soliton" solutions (bions) were found analytically ten years earlier (Seager *et al.*). In 1970, Ooyama and Saitô discovered solitons on the Toda lattice, thereby approaching the Fermi-Pasta-Ulam problem. Finally, in 1971 solitons were found in the framework of the Schrödinger equation with cubic nonlinearity (S3) (Yajima and Outi). All references can be found in the review of Scott *et al.*[1] and below in the text.

As a result, the computer, having laid the foundations of an entirely new direction in the theory of nonlinear partial differential equations, withdrew to the background. There began the period of discovery and investigation of completely integrable Hamiltonian systems and the related inverse scattering method, Hirota's method, and the method of Bäcklund transformations. The development and formalization of this method, which took the form of an international "competition," showed that integrable equations can be generated in an almost unlimited number. As a result, it came to seem that, if not all, at least the majority of Lagrangian systems are completely integrable.

The first conflict with such views was again the achievement of the computer. In 1974 there was discovered at Dubna an inelastic interaction of Langmuir solitons in a plasma and also for solitons of the "improved" variants of the Boussinesq, KdV, Higgs, and Klein-Gordon equations.[2] It was found that something like a "small" change of the equation is sufficient to make it nonintegrable. Moreover, it was found that certain specific properties of integrability disappear on the transition from flat (x, t) geometry to spherically (or cylindrically) symmetryc (r, t) geometry.

There arose the concept of *nearly* integrable systems, for which completely integrable equations can be taken as a zeroth approximation and an investigation constructed in terms of the deviation from complete integrability. Moreover, the deviation from a completely integrable equation plays the part of the coupling constant of modern theories.

The investigation is "simplest" when such a deviation can be separated in the form of a right-hand side with a small parameter. Then the method of successive approximation can be used. We emphasize that such an approach must be used with all possible care, since it can lead to "solutions" that do not satisfy the original equation.

In addition, even in this case it is possible to have soliton solutions fundamentally different from the well-known solutions of completely integrable equations. Sometimes, it is not possible to separate such deviation in a pure form [for example, for the Boussinesq (Bq) equation φ is replaced by φ_{xxtt}], and the proximity of the Higgs and Klein-Gordon equations to the sine-Gordon (SG) equation is even more nontrivial.

We emphasize here that numerical experiments are now one of the most powerful tools for investigating Hamiltonian systems; this applies especially to the question of whether a given system is integrable. The elastic interaction of solitons makes it possible to answer this question in the affirmative (we recall the KdV, S3, and SG equations). The discovery of inelasticity of the interaction of solitons renders the search for the consequences of integrability, in particular many-soliton formulas, pointless.[1)] Nevertheless, the idea of proximity of an investigated system to an integrable system did help in the discovery of pulsating solitons—bions—both in flat (x, t) geometry and in spherically symmetric (r, t) geometry in the framework of the Klein-Gordon, Higgs, and SG equations. Note that whereas in the flat case one can still find an approximate analytic solution for pulsons (pulsating solitons) in (r, t) geometry, the very discovery as well as the investigation of the properties of pulsons are entirely associated with computers (hitherto, the standard analytic methods have been without effect; the absence of a small parameter and the essential nonlinearity were to blame).[2]

The transformation of the "physical" system describing the interaction of Langmuir (plasma) and ion-acoustic waves in a plasma from a "strongly nonintegrable"

[1)]This assertion is true for systems with internal symmetry not higher than *SU* (2).

system at low velocities of the colliding solitons (which may coalesce) into a completely integrable system in the limit $v \to 1$ was first followed on a computer. It is here worth noting that in the nonlinear description of ion-acoustic waves (for example, by means of the Boussinesq equation) the integrability disappears. A system of two integrable equations was found to be nonintegrable.

As a result of the work of many investigators in the whole world the properties of solitons in a flat (x, t) world have been fairly well studied. The time came for the transition to more realistic and complicated multidimensional worlds. This transition—as one could have expected—was nontrivial.

On the transition from one to several spatial variables the question of soliton stability moved to the fore [in the flat (x, t) case, it was only in the case of the Klein-Gordon and Boussinesq equations that unstable solitons were found]. In recent years, many studies, especially in plasma theory, have been devoted to the investigation of the stability of planar (x, t) solitons in the direction perpendicular to their motion. We note here that the solitons of two well-known integrable equations, the KdV and S3 equations, were found to be opposites in this sense: The KdV solitons are stable, which ultimately made it possible to solve the two-dimensional problem; the S3 solitons are unstable, which indicated once more the existence of the collapse of Langmuir waves in a plasma.

In 1963–1964, Derrick and Hobart proved the following theorem: In a space of more than one dimension, there do not exist stable stationary solitonlike solutions in the framework of ordinary relativistically invariant nonlinear theories (without internal symmetries and differential interactions).

Here, we must define what we shall understand by the expression *soliton in a non-one-dimensional space.* Our definition of a soliton is suggested by the well-known concept of a quasiparticle solution[2]: A soliton is a solution of some nonlinear hyperbolic equation that possesses finite energy, momentum, "charge," and the "correct" asymptotic behavior at infinity.

The Derrick-Hobart theorem actually states that the Hamiltonian of a system in the stationary case, $H[\varphi] = \text{const}$, considered as a functional of the field defines a surface that cannot be a valley; it is either a hill or, in the best case, a saddle in the functional space. Therefore, to stabilize the system additional constraints (integrals of the motion) are required. Two ways are possible:

1) to look for conservation laws associated with nontrivial topology of the solution (Faddeev, Polyakov, 't Hooft, and others);

2) to go over to theories with internal global or local gauge symmetries.

Here, we shall not be concerned with the first approach. In the second case, the soliton ensures that the Hamiltonian has a conditional extremum, which may lead to stability of the soliton. Stability questions, and also the static properties of solitons, have already been studied in the framework of various scalar and also with the inclusion of color spinor (quark) fields (see Refs. 2 and 3).

Soliton dynamics in multidimensional worlds is as yet entirely the domain of the computer. Thus, the computer has discovered pulsons—bions in a spherical three-dimensional space—and has studied their properties. But this is only the first step. The real dynamics, in which the particular properties of solitons are fully manifested, is in their interaction. There have already been several studies of the interaction dynamics of two-dimensional[3] solitons. The obtained results are very impressive and, which is particularly important, the investigations have a very broad geographical spread (England, Poland, the Soviet Union, the United States, Japan). Below, we shall consider these investigations in more detail.

In the present review, we consider mainly the qualitatively new nonlinear phenomena that have been discovered and studied by means of the computer. We have here the finding of localized solutions of nonintegrable nonlinear wave equations (especially in more than two spatial dimensions), and also the investigation of their stability and interaction dynamics.

We study these phenomena by taking the examples of the simplest models that arise in the various branches of modern physics, from optics and hydrodynamics to field theory and elementary particles.

As we have already said, the currently widely known concept of a soliton arose from the interaction of solitary waves in the framework of the KdV equation. Below, we shall adopt a wider definition, i.e., by a soliton we shall understand a solitary wave propagating in a nearly integrable system, this being especially appropriate in view of the fact that in recent years completely integrable systems with nontrivial dynamics (interaction) of solitons have been discovered and investigated. We have in mind the studies of Calogero and de Gasperis (boomerons and so forth), V. E. Zakharov, and A. V. Mikhaĭlov (decoupled solitons). Even for such systems, the old definition is not suitable.

In non-one-dimensional geometries our definition will be very close to the concept of a particlelike solution, which is well known to specialists in field theory.

We consider the properties of planar solitons, i.e., solitons with a plane wave front. We investigate their form, stability, and interaction, and also bound states of solitons. We investigate non-one-dimensional solitons. We study their stability, the dynamics of their formation and interaction, and bound states (resonances). In conclusion, we discuss some new directions of further development that are now appearing.

[2]Here, we must note the difference between a soliton, which is a solution of a hyperbolic equation, and an instanton, which is a solution of an elliptic equation.

[3]Three-dimensional solitons are evidently as yet too complicated for computers.

1. PLANAR SOLITONS

Equations and the Applications. One of the first equations whose solutions were investigated numerically and revealed soliton properties was the KdV equation, which we write in the following "general" form:

$$\varphi_t + \varphi_x + \alpha\varphi^\nu\varphi_x + \varphi_{xxx} = 0. \qquad (1)$$

So far, it has been established that for $\nu=1$ and 2 this equation is completely integrable, and its solitons interact elastically. For $\nu \geqslant 3$, Eq. (1) loses this property, though it remains very "close" to an integrable equation.[4] Solitonlike solutions of (1) can be readily obtained[4)]:

$$\left.\begin{aligned}&\varphi_s = \left(A\,\mathrm{sech}\left[\sqrt{\frac{\alpha}{2}}\frac{A\nu}{\sqrt{(\nu+1)(\nu+2)}}(x-vt-x_0)\right]\right)^{2/\nu};\\ &v = 2\alpha A^2/[(\nu+1)(\nu+2)]+1.\end{aligned}\right\} \qquad (2)$$

Equation (1) describes a wide spectrum of physical phenomena in systems in which small oscillations have an acoustic spectrum, i.e., $\omega(k)\to k$ as $k\to 0$. However, it has an unpleasant feature—the nonlinear waves can move only in the direction of the positive x axis, hence the epithet "right" KdV. Indeed, in the derivation of the KdV equation the symmetric operator $\partial_t^2 - \partial_x^2$ is replaced by $(\partial_t - \partial_x)(\partial_t + \partial_x) \approx -2\partial_x(\partial_t + \partial_x)$.[5)] The Boussinesq equation (1872)

$$(\partial_t^2 - \partial_x^2 - \partial_x^4)\varphi - \partial_x^2\varphi^2 = 0, \qquad (3)$$

is free of this defect and by means of the above procedure, one integration, and a change of variables can be readily reduced to the form

$$(\partial_t + \partial_x + \partial_x^3)\varphi + \partial_x\varphi^2 = \mathrm{const}.$$

When solutions bounded in space, i.e., solutions with finite energy, are considered, this equation is identical to the right KdV equation, since the constant is equal to zero on this class of solutions.

Equation (3) is also integrable,[5] and for it Hirota obtained many-soliton formulas,[42] which were used to calculate the interaction of the solitons.

Further, we must mention the nonlinear Schrödinger equation

$$i\psi_t + \psi_{xx} + \beta\,|\psi|^\nu\,\psi = 0 \qquad (4)$$

or, in more general form,

$$i\psi_t + \psi_{xx} + \Phi(|\psi|)\psi = 0. \qquad (5)$$

Solitonlike solutions of (4) have a form similar to (2):

$$\psi_s = A_0\exp\left\{i\left[\left(\frac{v^2}{4}+\frac{2\beta}{\nu+2}A_0^\nu\right)t + \frac{v}{2}(x-vt)-\vartheta_0\right]\right\}\mathrm{sech}^{2/\nu}\left[\frac{\nu}{2}A_0^{\nu/2}\sqrt{\frac{2\beta}{\nu+2}}(x-vt-x_0)\right]. \qquad (6)$$

For $\nu=2$, Eq. (4), or S3, is completely integrable,[6] and its envelope solitary waves are true solitons. For other values of ν, the solutions of (4) are only quasi-solitons. Equations (4) and (5) also describe a large class of physical phenomena in systems in which the spectrum of small oscillations has a gap Δ and a dispersion quadratic in k.

The reader can find details about the physical applications of these equations together with the relativistically invariant Klein-Gordon equation

$$(\Box \mp m^2)\psi + \beta\psi\Phi(|\psi|) = 0; \quad \Box = \partial_t^2 - \partial_x^2 \qquad (7)$$

and the sine-Gordon equation

$$\Box\varphi + \sin\varphi = 0 \qquad (8)$$

in the reviews of Refs. 1, 2, and 7–11.

In plasma theory one also encounters Eq. (5), in which $\Phi(|\psi|)$ satisfies one of the following equations:

$$\Box\Phi = \partial_x^2(|\psi|^2) \qquad (9)$$

[Zakharov approximation (1972, Ref. 12)],

$$\Phi_t - \Phi_x + \beta(\Phi^2)_x + \alpha\Phi_{xxx} = -(|\psi|^2)_x \qquad (10)$$

[Nishikawa-Hojo-Mima-Ikezi approximation (1974, Ref. 13)], and

$$\Box\Phi - \beta(\Phi^2)_{xx} - \alpha\Phi_{xxxx} = (|\psi|^2)_{xx} \qquad (11)$$

(approximation obtained by the present author in 1974 in Ref. 14).

The solitonlike solutions have the form

$$\left.\begin{aligned}&\psi_s = A\exp\left\{i\left(\frac{v}{2}x-\Omega t-\vartheta\right)\right\}\mathrm{sech}\left[\frac{A\gamma}{\sqrt{2}}(x-vt-x_0)\right];\\ &\Phi = -\gamma^2|\psi_s|^2;\quad \gamma^{-2}=1-v^2;\quad \Omega = v^2/4-\gamma^2A^2/2\end{aligned}\right\} \qquad (12)$$

for the system (5), (9), and

$$\left.\begin{aligned}&\psi_s = A\exp\left\{i\left(\frac{v}{2}x-\Omega t-\vartheta\right)\right\}\mathrm{th}\,\xi\,\mathrm{sech}\,\xi;\\ &\xi = \sqrt{-\lambda}(x-vt-x_0);\\ &\Phi_s = 6\lambda\,\mathrm{sech}^2\xi;\quad A^2 = 48\lambda^2\varepsilon;\quad \Omega = v^2/4-\lambda\end{aligned}\right\} \qquad (13)$$

for the systems (5), (10), and (5), (11) with

$$\lambda = (-3/20\,\varepsilon\gamma^2),\quad \beta = 3\alpha = \varepsilon.$$

It follows from (10) that the system (5), (9) gives a fairly good description of slow, $v<1$, solitons. In the region $v\to 1$, it is necessary to use the approximation (10) or (11).

Apart from the equations mentioned above and systems of them, there frequently arises in nonlinear optics an equation that is a multiple of the sine-Gordon equation:

$$\Box\varphi = \sum_{m=1}^{n}\frac{m}{n}\sin\left(\frac{m}{n}\varphi\right), \qquad (14)$$

which reduces to the ordinary SG equation for $n=1$.

Solitonlike solutions of this equation were investigated numerically in detail in Ref. 18, and for $n=2$ [the so-called double sine-Gordon (DSG) equation] in Refs. 19 and 20.

We give two further equations frequently encountered in the literature:

$$\varphi_t + \varphi_x + \varphi^\nu\varphi_x - \varphi_{xxt} = 0, \qquad (15)$$

for $\nu=1$; it was proposed by Petegrine[21] for the de-

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4) *Translator's Note.* The Russian notation for the trigonometric, inverse trigonometric, hyperbolic trigonometric functions, etc., is retained here and throughout the article in the displayed equations.

5) The equation for waves moving to the left, the "left" KdV equation, can be written down similarly.

scription of tidal waves[6]:

$$\varphi_s = \left\{A\,\mathrm{sech}\left(\frac{A\nu}{\sqrt{2}}\,[(\nu+1)(\nu+2)+2A^2]^{-1/2}(x-vt-x_0)\right)\right\}^{2/\nu};$$
$$v = 1+2A^2/(\nu+1)(\nu+2), \qquad (16)$$

and the Benjamin-Ono equation[22,23]

$$\varphi_t + 6\varphi\varphi_x + H\varphi_{xx} = 0, \qquad (17)$$

where H is the Hilbert operator,

$$Hf(x) \equiv \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - x}\, d\xi,$$

used to describe internal waves in layered fluids.

As was shown in Refs. 17 and 24, its solution

$$\varphi_s = (2/3)\, v/[1 + v^2 (x - vt)^2] \qquad (18)$$

is a true soliton.

Some variants of the above equations and equations that reduce to them will be considered below.

In conclusion, we note that a large number of equations possessing soliton (and quasisoliton) solutions arise in classical and quantum two-dimensional (x, t) field theories. In these theories, the fields are invariant with respect to the Lorentz (or Poincaré) group, and also with respect to a certain internal (isotopic) symmetry group, and satisfy various commutation relations.

Almost all the results have been obtained here by means of analytic methods of investigation: spectral transformation (inverse scattering technique), field-theoretical and group-theoretical methods, and also the methods of differential geometry.

In this field, computer experiments are only beginning to make their first timid steps.

Dynamics of the formation and interaction of planar solitons. This question has been most fully and systematically studied in the framework of completely integrable equations, for which it is frequently possible to solve the Cauchy problem analytically by means of the inverse method.[7] We mention once more that the investigations in this direction were initiated by computer experiments on the Fermi-Pasta-Ulam problem and solitons of the KdV, sine-Gordon, and S3 equations. We recall that, by analogy with mechanics, an equation is said to be integrable if it possesses an infinite set of (additive) integrals of the motion. These integrals are constructed from powers of the field function and its derivatives and must be in involution. The first results obtained in the investigation of completely integrable systems led to the conclusion that in their framework the interaction of solitary waves reduces solely to a shift in position and phase, the shape and velocity of the waves being left unchanged. It is this that explains the name "soliton" as a stable object like a particle. This was the case as long as fields with symmetry group not higher than $U(1)$ were considered.

The inclusion of higher symmetries, for example, $SU(2)$, led to the discovery in 1976 of very original objects, namely, solitons that come back again, and were called "boomerons," and solitons whose center of gravity oscillates about a certain position, "trappons."[26] Finally, at the beginning of 1978, there appeared a paper in which it was shown that in the framework of an integrable system (model of the principal chiral field) on the group $SU(N)$ it is possible to have decay and mutual transformation of solitons when $N \geq 3$.[27] It was thus shown that the original definition of the soliton concept is too narrow even for the set of integrable systems. We note that beginning in 1976 there were discovered and investigated, in addition to the well-known sine-Gordon equation, several completely integrable field-theoretical models. These include in particular the massive Thirring model[28] [classical spinor field with self-interaction $\gamma^\nu\psi(\bar\psi\gamma_\nu\psi)$], the complex generalization of the sine-Gordon equation,[29] and various others.[30]

We now turn to numerical experiments on soliton dynamics. Apart from the many-soliton formulas, the dynamics of the formation of solitons from arbitrary initial packets even in integrable systems is known only asymptotically as $t \to \infty$, this being the case *a fortiori* in the framework of the perturbed equations.

First, some words about methods of numerical investigation. In the review paper "Numerical investigations of solitons" at the Oxford Symposium on Nonlinear (Soliton) Structure and Dynamics in the Condensed State (June, 1978),[31] Eilbeck noted that in the field of partial differential equations numerical analysis had remained as much an art as a science. For any given equation, the question of the "best" numerical method is the most complicated. This is due to numerous factors, in particular, the required accuracy of the calculations, limitation of time and computer memory (power), word length, etc. Moreover, it is very rare to find in the literature a comparison of the numerical algorithms of different authors and a thorough investigation of each given algorithm. This applies especially to nonlinear equations. Thus, a vast amount of work is needed both on the theoretical investigation of the employed methods and on the analysis of their practical application. Below, we shall only briefly dwell on the methods of calculation.

Usually, two main approaches are employed to reduce the problem to a finite number of parameters: approximation of functions and approximation of difference schemes.

In the first case, the exact solution $y(x, t)$ is approximated by means of an approximate expression defined on a finite subspace, for example,

$$y(x, t) \approx \bar{y}(x, t) = \sum_{i=1}^{n} C_i(t)\, y_i(x), \qquad (19)$$

where $y_i(x)$ are basis functions chosen in a certain way (for more detail, see Ref. 31 and the literature quoted there).

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[6]It should be noted that in my review of Ref. 2 there is an unfortunate mistake on p. 17. The reference should be to Peregrine's paper and not Lee's review.

[7]During recent years, a huge number of papers (more than 600) have been published on this question. There are also very extensive reviews (Refs. 1, 4, 7, and 25) and a book (Ref. 15).

It is more common to use the second approach in the investigation of partial differential equations, i.e., the exact solution $y(x, t)$ is approximated by a set of values y_m^n defined at the sites of a rectangular grid on the xt plane (or in the xyt cube) with step h and τ, respectively, i.e., $x_m = hm, t_n = \tau n$. Different schemes are used to determine the time derivatives—explicit and implicit. There is an entire science associated with such a method (we here refer only to Ref. 32). Nevertheless, for the investigation of *nonlinear* partial differential equations ready-made prescriptions still do not exist. We merely note that the quality of the calculation sometimes depends strongly on the form of approximation of the nonlinear term, namely, on whether its values are taken at the points x_m, i.e., $F(y_m^n)$, in the form of a half-sum at symmetric points, i.e., $F(\frac{1}{2}[y_{m+1}^n + y_{m-1}^n])$ with respect to x or $F(\frac{1}{2}[y_m^{n+1} + y_m^{n-1}])$ with respect to t, etc.[31]

We emphasize in conclusion that all the employed methods must satisfy and condition of stability, i.e., $|y(x_m, t_n) - y_m^n| \leq N$ as $n \to \infty$. This condition together with the condition of accuracy of the approximation usually imposes restrictions on the relationship between the steps h and τ and on their values. Also very promising is the approach in which one uses a Fourier expansion of the unknown solution with respect to the coordinate, with subsequent solution of the ordinary differential equations by means of known difference (leap-frog) schemes. After an inverse Fourier transformation (one usually employs the algorithm of the so-called *fast Fourier transformation*[33]) the required result is obtained. We mention here only three papers in which this method has been successfully used: Refs. 34–36. Sometimes, a modified mixture of these approaches is used.

As we have already noted, some of the first nonlinear equations whose numerical investigation revealed unusual properties were the Korteweg-de Vries equation, the sine-Gordon equation, and the finite-difference analog of the Boussinesq equation (or nonlinear string) (Fermi-Pasta-Ulam, 1955):

$$d^2 y_m/dt^2 = f(y_{m+1} - y_m) - f(y_m - y_{m-1}), \tag{20}$$

where $f(y) = y + \alpha y^2$ or $f(y) = y + \beta y^3$.[8)]

As was subsequently found, the Fermi-Pasta-Ulam problem is intimately related to the proximity of the considered system to an integrable system and, hence, to solitons (see, for example, Ref. 1). Therefore, in what follows we shall restrict ourselves to discussing the properties of solitons.

From the point of view of numerical experiments, the dynamics of the formation of solitons and solitonlike solutions from identical initial conditions φ_0 is virtually the same in the framework of the KdV equation and in Eqs. (1) and (13), which are close to it. Depending on the integral $\int_{-\infty}^{\infty} \varphi_0 dx$, there are formed one, two, etc., solitons and a certain oscillator tail. It was this property of equations of the KdV type together with the same form of the dispersion as $k \to 0$ which led to the idea that they are all integrable. As a result, many efforts of analytic and computational nature were expended in vain to prove this fact. The decay of the initial condition into solitons is a characteristic feature of any quasi-integrable system.

The picture changes qualitatively for the interaction of solitons. The KdV and modified KdV solitons, being true solitons, only undergo a shift in their position,[37] whereas the solitons of the remaining KdV-like equations, i.e., Eq. (1) for $\nu \neq 1$ and 2 and (13), interact inelastically (see Refs. 35 and 38). The inelasticity of the interaction is usually small, but increases with increasing ν, and for $\nu = 4$ becomes manifest. This can also be very clearly followed in the example of Eq. (13) for $\nu = 1$ and 2. The equation

$$\varphi_t + \varphi_x + \varphi\varphi_x - \varphi_{xxt} = 0, \tag{21}$$

which is also sometimes called the PBBM or RLW equation,[9)] was apparently one of the first to destroy the illusion of integravility. A difference scheme for it was investigated by Eilbeck and McGuire[40] and by Abdulloev, Bogolubsky, and the present author.[38]

It is well known that in the numerical integration of the KdV equation one encounters a fairly stringent relationship between the steps of the grid in the t and x directions: $\tau \ll ch^3$, or, more precisely, $\tau/h^3 \ll 1/(4 + h^2|\varphi_0|$ or $\tau/h^3 \leq 1/|2 - h^2\varphi_0|$) (for more detail, see Ref. 31). It may be said that it is no easy task to reveal fine details in the numerical investigation of the KdV equation. For the investigation of the RLW equation (21) the situation is better, and the condition of stability of the difference schemes becomes $\tau \leq h$ (Ref. 38).[10)] Nevertheless, in this case too there are awkward problems. For example, after the exact (analytic) single-soliton solution

$$\varphi_s = A\,\mathrm{ch}^{-2}\{\sqrt{A/[2(2A+3)]}\,(x - [1 + (2/3)A]\,t - x_0)\} \tag{22}$$

has been fed into the computer, it undergoes a slight (at the level of fractions of a percent) change—the soliton "breathes" and even appears to radiate weakly during a short time, after which the picture stabilizes. By virtue of this property of the RLW equation, which was not understood by the authors of Ref. 38, it was necessary to invent a fairly complicated computational procedure to reveal a subtle effect such as inelasticity of the soliton interaction. Here, it should be said that encouragement for the authors of Ref. 38 was given by the results obtained by analogous modification of the Boussinesq equation (6).

As we have already noted above, the interaction of solitons in the framework of the Boussinesq equation has been investigated numerically only by means of the two-soliton formula.[42] Direct calculations on the basis

[8)]Equation (20) can be reduced for $f(y) = y + \alpha y^2 + \beta y^3$ and by transition to the continuum analog, $h \to 0$, to the KdV or Boussinesq equation (see, for example, Ref. 7).

[9)]In the first case, the name derives from the initials of the surnames of the authors: Peregrine,[21] Benjamin, Bona, and Mahony[39]; in the second case, it corresponds to the name regularized long-wave equation.[31]

[10)]It is possible that the name of the equation is associated with this property. This equation was also used to simulate tsunamis.[41]

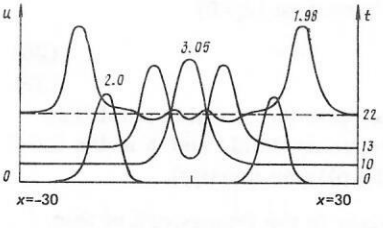

FIG. 1.

of the Boussinesq equation are virtually impossible because of the instability of the linear spectrum $\omega^2 = k^2(1-k^2)$. This forced us to modify the equation in such a way as to make the linear spectrum agree in the limit $k \to 0$ with the Boussinesq spectrum but at the same time be stable. Ion-acoustic waves in a plasma have such a spectrum, $\omega^2 = k^2(1+k^2)$, which corresponds to the equation

$$(\Box - \partial_t^2\partial_x^2)\varphi - \partial_x^2\varphi^2 = 0,$$

which we proposed should be called the improved Boussinesq (IBq) equation.[11)]

As a result of the numerical experiment of Ref. 44 it was shown that the IBq solitons interact inelastically (Fig. 1). The computational algorithm was also discussed in Ref. 44. Despite the undoubted similarity of the IBq and RLW equations, the degree of inelasticity when their solitons interact is very different. This is due to the fact that the RLW solitons overhaul each other, whereas the IBq solitons can have head-on collisions. Numerical investigation of the latter is preferable by virtue of their symmetry (economy of memory). In reality, as was already noted in 1975 in Refs. 45 and 46, head-on collisions of solitons lead in the framework of various equations to larger inelasticity in interactions at the same soliton amplitudes than is the case for single-direction collisions. Therefore, it is predominantly head-on collisions that are responsible for ergodization in a nonintegrable system (for ion-acoustic waves in a plasma[45] and for Langmuir waves[46]).

As a result of numerical experiments on KdV-like equations, it was found that the inelasticity of the interaction of quasisolitons increases with increasing amplitude of the quasisolitons and increasing degree of nonlinearity $\nu > 1$ in the equation [for Eq. (19) with $\nu = 2$, 4, see, for example, Ref. 35]. Bona, Pritchard, and Scott[47] developed a very refined computational algorithm,[47] which enabled them to study in detail the interaction dynamics of RLW solitons. Their results agree with the results obtained earlier.[38,48] A typical picture of the interaction of large-amplitude solitons is shown in Fig. 2.[47]

Sometimes, complex modifications of the KdV equation are considered (see, for example, Refs. 35 and 49)[12)]:

$$\psi_t + |\psi|^2\psi_x + \psi_{xxx} = 0; \tag{23}$$

$$\psi_t + (|\psi|^2\psi)_x + \psi_{xxx} = 0. \tag{24}$$

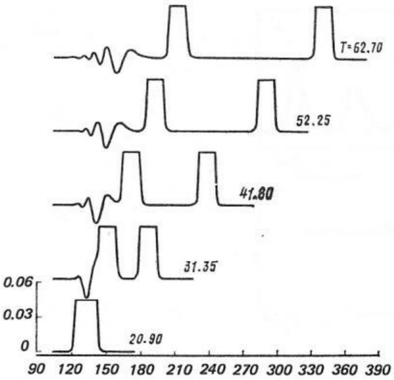

FIG. 2.

The first of these equations is integrable, as was shown by Hirota[51] in 1972 by means of the inverse method. As was illustrated in Refs. 35 and 49, the second is nonintegrable, and the decay of an initial condition and the interaction of the quasisolitons have a rather complicated inelastic nature.

We mention one further KdV-like equation, which is usually called the Benjamin-Ono equation. It is Eq. (17):

$$\varphi_t + 6\varphi\varphi_x + \frac{P}{\pi}\int_{-\infty}^{\infty}\frac{\varphi_{\xi\xi}}{\xi - x}\,d\xi = 0$$

and it is used to describe waves in a layered fluid (internal waves).[22] The numerical experiments of Meiss and Pereira[52] revealed a soliton nature of the behavior of the traveling waves ("lorentzians") of the form

$$\varphi_s = (2/3)\,v/\{1 + [v(x - vt)]^2\}$$

in collisions and in the decay of an initial packet (many-soliton solutions were found by Joseph[53]).[13)]

Numerical investigations of soliton phenomena in the framework of the nonlinear Schrödinger equation with a self-consistent potential were made for the first time by Yajima and Outi in 1971.[54] They found that the envelope solitary waves of the Schrödinger equation with cubic nonlinearity, equation S3, interact elastically with one another. This fact was explained by Zakharov and Shabat,[6,55] who showed that equation S3 describes an integrable system, and the solutions of (6) for $\nu = 2$ are true envelope solitons. Here, we shall merely note that the time of formation of an S3 soliton (or solitons) from an initial packet depends strongly on the form of the packet and can be very long if solitons of sufficiently small amplitude are formed during the decay of the initial condition. In one of the first experiments[56] it did not prove possible to observe the formation of a soliton from an initial packet of Gaussian form. A possible explanation of this fact (and its difference from KdV-like equations) was given in Ref. 2. Fig. 3a shows the maximal amplitude φ_{max} of a packet as a function of the time.

The fact that an initial packet does not decay into

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11)We note that it was also known to Boussinesq.[43]

12)Equations of this kind arise, for example, in plasma theory.[50]

13)The problem was completely solved in Ref. 24, in which the integrability of Eq. (17) as a limiting case of a "nonlinear intermediate long-wave equation" was proved.

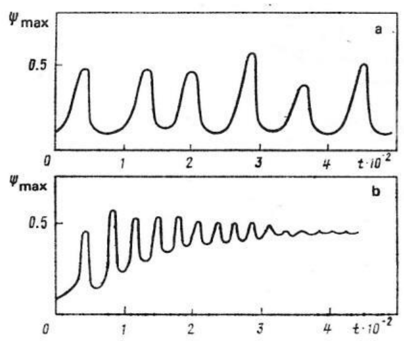

FIG. 3.

solitons for such a long time can be attributed to the formation from the packet of a bound state of solitons—a bion (for more details, see below). The solution of the initial-value problem, i.e., the investigation of the decay of various initial packets in the framework of equation S3 by means of the inverse scattering technique, and also numerical experiments, are described in detail by Satsuma and Yajima.[57]

For what follows, we introduce some additional concepts, which are well known from classical field theory and, in application to soliton phenomena, are described in detail in Ref. 2. Here, we shall merely give a brief summary:

1) the overwhelming majority of the equations that admit soliton or solitonlike solutions can be obtained as the solution of some extremal problem;

2) this means, by virtue of Noether's theorem, that certain conservation laws will correspond to the symmetries of the Lagrangian; in particular, there are conservation laws for the "particle number" (charge), the energy, the momentum, the angular momentum of the field, and so forth.

We now turn to the nonlinear Schrödinger equation. It is used in different fields of physics. Besides the well-known applications in plasma physics and nonlinear optics, we also mention here the theory of waves on deep water, density waves in spiral galaxies, one-dimensional spin systems, and the theory of excitons in one-dimensional molecular chains and biopolymers. Questions relating to the derivation of Schrödinger equations, their limits of applicability, and so forth can be found in Refs. 58–60.

The nonlinear Schrödinger equation has been most fully investigated (including numerically) in connection with the description of Langmuir (plasma) waves and turbulence in plasmas. Therefore, we shall here dwell briefly on the investigation of Langmuir solitons (in other fields of physics in which equations of such type are encountered it is frequently sufficient to change merely the terminology and notation).

Thus, equation S3,

$$i\psi_t + \psi_{xx} + \alpha |\psi|^2 \psi = 0, \tag{25}$$

has a countable set of integrals of the motion and is completely integrable. In the theory of Langmuir turbulence, it arises as the "quasistatic limit" in the framework of Zakharov's system ($\alpha > 0$)

$$i\psi_t + \psi_{xx} - \Phi\psi = 0; \tag{26}$$

$$\Box\Phi = \partial_x^2 |\psi|^2 \tag{27}$$

in the limit $\partial_t^2\Phi \to 0$ (here, ψ is the complex amplitude of the Langmuir high-frequency field, and Φ is the low-frequency variation of the plasma density).

The initial-value problem in the framework of this system was, like the interaction of two colliding solitons, investigated for the first time in Ref. 56. In numerical experiments, it was found that identical initial packets behave quite differently in the framework of equation S3 and in the system (26)–(27). In Figs. 3a and 3b we give for comparison the results of such calculations; Fig. 3a corresponds to the dynamics of an S3 packet, and Fig. 3b (Ref. 61) to Zakharov's system. In the second case, the process of formation of the quasisoliton takes place fairly rapidly, and the excess energy and momentum are carried away by Φ density waves. A similar situation is observed when quasisolitons interact. The quasisolitons of the system (26)–(27) exhibit a rich picture of interactions between themselves and acoustic pulses of various shapes. These effects were studied in great detail in Refs. 46, 61, and 34.

In Refs. 46 and 61, elementary interaction events of solitons and acoustic pulses were simulated numerically in the framework of the system (26)–(27). On the basis of this work, an approximate kinetic equation for the solitons was written down in Ref. 46 and its solution, which describes the distribution of the solitons with respect to their width, was obtained.[14)] It also follows from the results of Ref. 61 that for given amplitude of the colliding solitons the inelasticity of their interaction is smaller, the larger their relative velocity Δv. In the limit $\Delta v \to 2$ ($\Delta v = v_1 - v_2, v_1 = -v_2 \to 1$), the inelasticity becomes negligibly small. This result suggested to the two Japanese theoreticians Yajima and Oikawa that the transonic system (26)–(27) could be completely integrable, and this they proved brilliantly in Ref. 63. For this, it is necessary to go over from Eq. (27) to its right or left unidirectional variant, i.e., to make the substitution $\Box \to -2\partial/\partial x(\partial_1 + \partial_x)$ and integrate over x. Setting the constant of integration equal to zero by virtue of the soliton conditions at infinity, we obtain

$$(\partial_t + \partial_x)\Phi = -\partial_x |\psi|^2. \tag{28}$$

The system (26), (28) was found to be integrable.[63] Thus, we here encounter a very interesting fact: A certain system can, when a physical parameter is varied (in the considered case, the soliton velocity), be transformed from a strongly nonintegrable system (at small v) to a nearly integrable system ($v \to 1$), so that by means of a very simple approximation an integrable analog of it can be obtained.[15)] There is something similar in the case of non-one-dimensional solitons as

[14)]Note that in these studies no investigation was made into the influence of the phase relations on the interaction of the solitons.

[15)]We emphasize that the original system (26)–(27) also remains nonintegrable in the region $v \to 1$.

well. In the calculations in the framework of the system (26)–(27), a check was made on the conservation of the quantities

$$\left.\begin{aligned} S_1 &= \int_{-\infty}^{\infty} |\psi|^2\,dx; \\ S_2 &= \int_{-\infty}^{\infty} \{i(\psi\psi_x^* - \psi^*\psi_x) + 2\Phi u\}\,dx; \\ S_3 &= \int_{-\infty}^{\infty} \left\{|\psi_x|^2 + \Phi|\psi|^2 + \frac{1}{2}\Phi^2 + \frac{1}{2}u^2\right\}dx; \end{aligned}\right\} \quad (29)$$

$$\Phi_t + u_x = 0, \quad (30)$$

which are, respectively, the particle number, the momentum, and the energy of the system. In accordance with the conjecture made in Ref. 64, for this system there exists only one, $N = n - 2 = 1$, exact soliton solution (n is the number of conservation laws).

In Ref. 34, a detailed and systematic analysis of soliton phenomena as applied to plasma theory was made, namely, in the framework of the system (26)–(27), its modifications with allowance for Landau damping of the Langmuir and ion-acoustic waves, and also in the framework of the model of particles of finite size. We compare the results for different plasma parameters and soliton amplitudes and velocities.

A detailed comparison of the various codes used in the calculations can be found, for example, in the original papers of Ref. 66. We note here also that a major numerical investigation of Langmuir turbulence in a plasma in the presence of a pump field or an electromagnetic field was made by Californian groups (see, for example, Ref. 67).

To end our discussion of "Schrödinger" quasisolitons, we mention here a very interesting, and in one sense paradoxical, result. As we already know, transonic solitons in the limit $(1 - v^2) \to m_e/m_i$ must be described by the system (5), (10), or (11).

In Ref. 68, an attempt was made to investigate the interaction of two colliding two-hump solitons in the framework of the system consisting of the Schrödinger and IBq equations:

$$i\psi_t + \psi_{xx} - \psi\Phi = 0; \quad \left(\Box - \frac{\varepsilon}{3}\partial_x^2\partial_t^2\right)\Phi - \varepsilon\partial_x^2(\Phi^2) = \partial_x^2(|\psi|^2).$$

It can be seen from the results in Fig. 4 that, in contrast to the solitons of Zakharov's system, the two-hump transonic solitons interact strongly with each other as well as with the acoustic pulses.

The IBq equation is not integrable, and it was therefore very interesting to investigate the unidirectional interaction of solitons in the framework of the Nishidawa-Hojo-Mima-Ikezi system [Eqs. (10) and (5)]. Without the terms $\beta(\Phi^2)_x$ and $\alpha\Phi_{xxx}$, this system not only goes over into the integrable Yajima-Oikawa system, but Eq. (10), like Eq. (28) without the right-hand side, is also integrable. Is the Nishikawa-Hojo-Mima-Ikezi system, which consists of two coupled integrable equations, integrable? The results of Ref. 68 appear to give a disappointing answer to this question, but a final answer has still not yet been obtained.

If in Eq. (20) the function $f(y_n)$ is substituted in the form

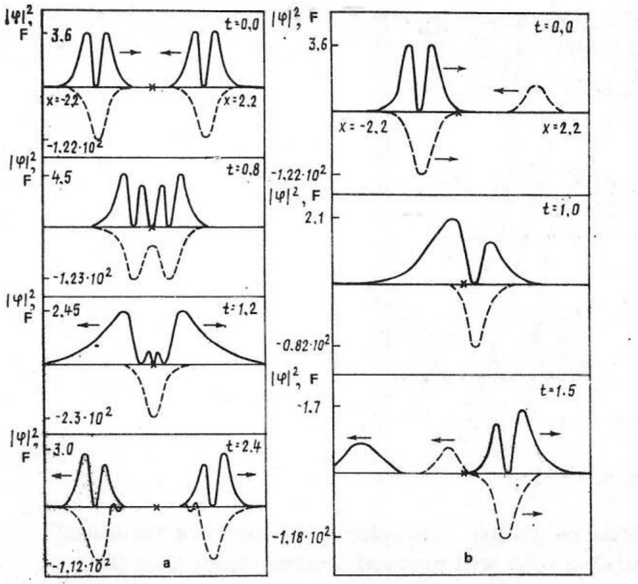

FIG. 4.

$$f(y_n) = \exp y_n, \quad (31)$$

we arrive at an integrable system—the Toda chain.[69] In Ref. 69, it was also shown that the system

$$\frac{d}{dt}y_n = y_n(y_{n+1} - y_{n-1}) \quad (32)$$

is integrable.

Systems of the form (32) arise in definite approximations in the study of the structure of the spectra of Langmuir oscillations in a weakly turbulent plasma, the interaction of populations in biology (predator-prey approximation), induced Compton scattering, and many other induced effects.

Equation (32) was obtained from an integro-differential equation of the form

$$\partial_t y(x, t) = y(x, t)\int_0^{\infty} dx' W(x, x')\,y(x', t), \quad (33)$$

when the kernel $W(x, x') = -W(x', x)$, which is proportional to the transition probability density, has resonance denominators corresponding to certain decay processes. Thus, for Langmuir spectra we have the processes $l \to l + s$, for photons $t \to t + l$ or $t \to t + s$, etc. At the same time, the kernel $W(x, x')$ degenerates into $W(x, x') = W_0[\delta(x - x' + \varkappa) - \delta(x - x' - \varkappa)]$, and Eq. (33) into (32).

If the decay processes are forbidden by the conservation laws $\omega_1 + \omega_2 \neq \omega_3$, $k_1 + k_2 = k_3$, the kernel $W(x, x')$ for induced scattering of Langmuir and electromagnetic waves by plasma particles takes the form

$$W(x, x') = \frac{d}{dx}G(x - x') \equiv \frac{W_0}{\sqrt{\pi}\Delta}\frac{\partial}{\partial x}\exp\left[-\frac{(x - x')^2}{\Delta^2}\right] \quad (34)$$

(see, for example, Refs. 71 and 72, where expressions for W_0, Δ, and $\varkappa$ in terms of the plasma parameters can also be found).

It was shown in Ref. 69 that Eq. (32) with the boundary conditions $y_n \to$ const, $n \to \pm\infty$ has the KdV equation as its

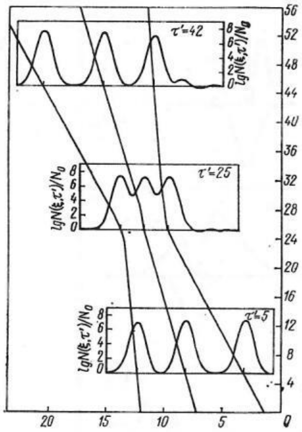

FIG. 5.

continuum analog. Therefore, the decay of the initial condition $y_n(0)$ will proceed in accordance with the well-known KdV scheme.

By means of numerical experiments, Montes and his collaborators showed that Eq. (33) with the kernel (34) describes a system close to an integrable (KdV?) system. In Ref. 72, Montes investigated the problem of the decay of an initial condition (spectrum) $y(x, 0)$ and showed that from a "narrow" packet $\delta x \leqslant \Delta$ one quasisoliton is formed after a stage of satellite transformation has been passed through.[16)]

A "broad" initial spectrum ($\delta x > \Delta$) decays into a set of quasisolitons distributed in accordance with their amplitude (or, which is the same thing, velocity). In Ref. 74, Montes *et al.* showed that the previously found solutions are indeed quasisolitons, as was the case for the RLW equation.

We note here that if solitons of not too different amplitudes approach each other, their interaction recalls satellite transfer, i.e., they remain well separated, which is similar to the interaction of KdV solitons. In triple interaction of quasisolitons, the departure from integrability is very appreciable, and a fourth soliton of small amplitude is produced; this process is shown in Fig. 5.[74] Similar experiments were made by a different French group, who investigated the interaction of quasisolitons in the framework of perturbed variants of Eq. (32),

$$\frac{d}{dt} y_n = y_n (y_{n+1} - y_{n-1}) + \mu (y_{n+1} + y_{n-1} - 2y_n) \text{ (Ref. 75)},$$

and the KdV equation (KdV–Burgers equation),

$$y_t + 6 (y^2)_x + y_{xxx} = \mu y_{xx} \text{ (Ref. 76)}.$$

In these cases, as one would expect, the departure from integrability is manifested in the appearance of oscillator tails in the process of evolution of the quasisolitons and during their interaction and is illustrated in Figs. 6 and 7 (see the section on structural stability of solitons).

The formation and interaction of planar solitons in the framework of nonlinear relativistically invariant equations were studied in Refs. 20 and 78–80. A discussion of possible difference schemes for investigating such equations can be found in Ref. 31.

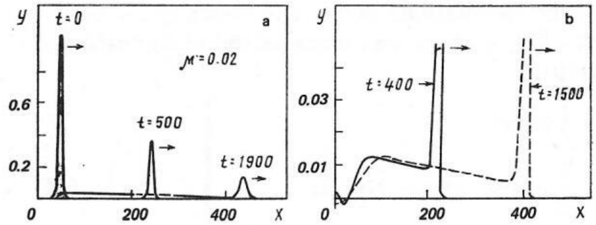

FIG. 6.

Usually, the investigated equation has the form

$$\Box y = F(y), \tag{35}$$

where $F(y)$ is either a polynomial[20,79,80] or a series of sines.[18,19,20,78] It is interesting to note here that Eq. (35) has an infinite set of conservation laws if

$$\frac{d^2}{dy^2} F(y) = F(y)$$

(see, for example, Ref. 81) or $F(y) = \exp y + \exp(-2y)$.[70]

In contrast to the nonrelativistic equations considered above, equations of the form (35) have not only bell solitons but also kink solitons, i.e., solutions representing the transition from one asymptotic state to another that is not equal to the first.[17)]

The very wide physical applications of the so-called sine-Gordon equation are well known. A large number of original studies and several reviews have been devoted to them. We mention here only two of the studies: Refs. 10 and 82. It was in the framework of this first confirmation was obtained of the existence of quasisolitons in real physical systems 100 years after the observations of Scott-Russell. We are referring to the self-induced transparency that occurs when ultrashort laser pulses propagate through a two-level system,[18)] the "quantization" of the magnetic flux in a Josephson transmission line, etc.

Since the sine-Gordon equation $-\Box y = \sin y$ is completely integrable,[83] we shall not dwell here on the investigation of the properties of its solutions. We merely mention that, as for the KdV and S3 equations, an arbitrary initial pulse decays in the framework of the sine-Gordon equation into a set of solitons. Frequently, it is more convenient to follow, not the function $y(x, t)$ itself, but its derivatives y_x or y_t. There are beautiful films taken by an English soliton group which demonstrate different types of interaction of solitons and quasisolitons and also their bound states (bions) (some of them were shown at the soliton symposia

[16)] We note that for spherically symmetric packets this result was obtained earlier in Ref. 73, though its proper significance was not recognized there.

[17)] Kinklike solutions can also appear in the framework of the S3 equation (25) for $\alpha < 0$.

[18)] Note that self-induced transparency, like KdV and Fermi–Pasta–Ulam solitons, "was discovered from an analysis of numerical solutions of the equations which describe optical pulse propagation" (Lamb in the review of Ref. 62, which can be recommended for beginners in the study of this phenomenon).

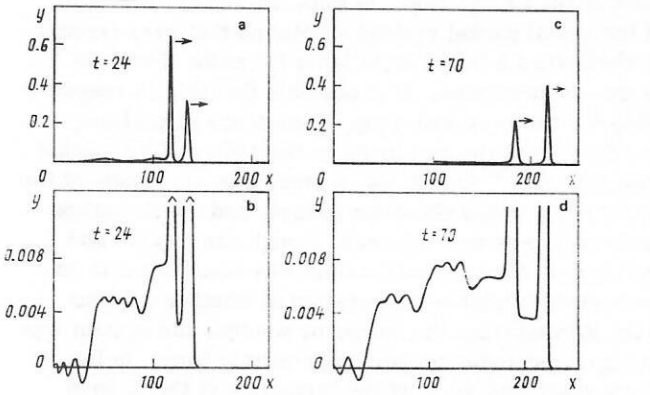

FIG. 7.

in 1977 and 1979 in Poland and in 1980 in Crete). Figure 8 (Ref. 31) shows the interaction of two bions.

Much computational work was done by a Manchester group investigating solutions of multiple sine-Gordon equations. In particular, the double sine-Gordon (DSG)

$$\Box y = \mp[\sin y + (1/2)\lambda \sin y/2],$$

which arises in the investigation of self-induced transparency in systems with double degeneracy, was studied in Refs. 18, 20, and 84.[19)] The authors were able to show that the double sine-Gordon equation describes a system that is very nearly integrable: Initial pulses decay under certain conditions into a set of solitons whose interaction is virtually elastic at sufficiently high velocities $v > 0.4$. When $v < 0.4$, the inelasticity becomes appreciable, and when $v \approx 0.1$ a kink-antikink bound state (bion) is formed.[20]

Thus, in these studies, despite the existing pessimistic predictions, it was shown that in the framework of a multiple sine-Gordon equation there also exist quasisoliton solutions, decay of an initial pulse into them, and, as a consequence, self-induced transparency. We emphasize that we here again encounter synergetic use of the computer in theoretical investigations leading to qualitatively new conclusions.

Equation (35) with

$$F(y) = \begin{cases} -y + y^3 \ (\varphi^4_-); & (36) \\ y - y^3 \ (\varphi^4_+), & (37) \end{cases}$$

or the so-called *phi-four theory* (φ^4), has been investigated fairly fully for both real and complex functions.[2,20,79,80] Equations (35) with right-hand side (36) and (37) can be obtained from a Lagrangian in which the potential $U(y)$ is a polynomial of fourth degree.

We also investigated quasisoliton solutions in a more general formulation when

$$U(y) \propto \sum_{n=1}^{m} C_n y^n \text{ (see Ref. 2, p. 50).}$$

These studies revealed in all clarity the inelasticity of the φ^4 interaction of quasisolitons, which leads to the

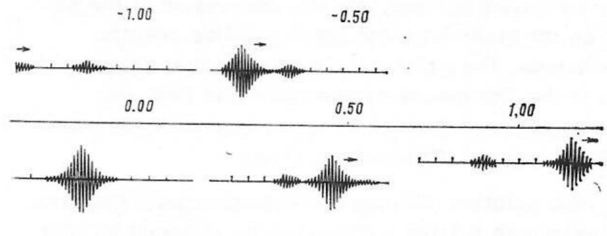

FIG. 8.

possibility of formation of long-lived bound states of them.

The equation of the φ^4_- theory has as quasisoliton solutions kinks and antikinks,

$$y(x, t) = \pm \operatorname{th} \frac{\gamma}{2}(x - vt - x_0), \quad \gamma = (1 - v^2)^{-1/2},$$

between which the interaction is attractive.

The equation of the φ^4_+ theory, like the S3 equation, can have sech and tanh quasisolitons as solutions; the sech solutions are stable only for a complex function $y = a(x)\exp(-i\omega t)$ when $\omega > 2^{-1/2}$. The inelasticity of the interaction of these quasisolitons is smaller, the higher their relative velocity.

In conclusion, we note that the computer investigation of the φ^4_+ quasisolitons apparently gave the first indication of their instability.

Bound states of solitons. Very interesting objects that arise in the theory of solitons are their bound states. If one soliton or quasisoliton can be regarded as a certain bound state of an infinite (in the classical case) number of constituents,[2] it is natural to expect that the solitons themselves can form bound states.

Such states were apparently described for the first time in Ref. 77, in which a bisoliton solution (kink-antikink) was found analytically for the sine-Gordon equation (below, as in Ref. 2, we shall, following Caudrey, Eilbeck, and Gibbon,[82] call such solutions beions). The solution has the form

$$\varphi_b = 4 \operatorname{arctg}\{\sqrt{1/\omega^2 - 1} \operatorname{sech} \zeta' \sin(\zeta'' + \delta)\}, \tag{38}$$

where $\zeta' = \gamma\sqrt{1-\omega^2}(x - vt - x_0)$, $\zeta'' = \gamma\omega(vx - t)$, $\gamma^{-2} = 1 - v^2$.

In the frame in which $v = 0$, this solution is a solitary oscillating hump. In the laboratory system, in which $v \neq 0$, there is high-frequency filling of the hump (as in the case of the solitons of the S3 equation) associated with the appearance of an x dependence in the argument of the sine. Before we study in detail the structure and dynamics of the bound states of solitons in the framework of the nonlinear Klein-Gordon equation

$$\Box\varphi = F(\varphi), \tag{39}$$

we shall say a few words about the possibility of such solutions in KdV-like models. To the best of my knowledge, bound states have not so far been found in KdV-like models. That they do not exist for the integrable KdV and modified KdV equations follows directly from the solution of the inverse scattering problem, since only one soliton corresponds to each discrete level. With regard to nonintegrable KdV-like equations, the

[19)]In Ref. 18, the breakup of an initial pulse into quasisolitons was also investigated in the framework of the triple sine-Gordon equation.

numerical investigations into the interaction of the solitons so far made have not given positive results. Nevertheless, the question of the existence of bound states in the framework of nonintegrable KdV-like equations can be regarded as open, especially for complex functions and high nonlinearities ν.

The bion solution (38) has two characteristic features that distinguish it from such solutions in nonintegrable systems and in a space of more than one dimension. First, the state described by it cannot be obtained as a result of interaction of kinks and antikinks, since it follows from the integrability of the sine-Gordon equation that radiation is not produced when solitons interact. Second, because there is no radiation, the lifetime of the bion (38) is infinite. Finally, the interaction of sine-Gordon bions is elastic, which is very clearly demonstrated by Fig. 8 (Ref. 31).

Recently, papers have been published which draw attention to the great importance that solutions of bion type can have in different one-dimensional physical models of the condensed state. Here, we mention only Refs. 11 and 85–87. It is necessary to emphasize the important difference between kink-soliton solutions and their bound states, namely, bions (and, as we shall see below, quasibions, or pulsons) have, like envelope solitons, an internal oscillator ("rotational") degree of freedom, and, as a consequence, their rest mass $M_b = 2M\sqrt{1-\omega^2}$ in the classical limit varies continuously from 0 to $2M$ as a function of ω, which recalls S3 solitons. (Here, $M=8$ is the rest mass of a kink or antikink in dimensionless variables.) The analogy between SG bions and envelope solitons of the S3 equation is much deeper than might appear at the first glance, as was noted in Refs. 86a and 86c. Indeed, at small amplitudes $1/\omega^2 - 1 \ll 1$, $\tan^{-1}x \approx x$ and (38) goes over to accuracy $O(x)$ into the corresponding expression for a Klein-Gordon envelope soliton, $\varphi_b \approx 2\sqrt{2}\,\mathrm{Im}\,\psi_{\mathrm{KG}}$ or $\mathrm{Re}\,\psi_{\mathrm{KG}}$, depending on the phase δ, and for $v \ll 1$ we obtain an appropriately transformed S3 soliton, i.e., $\varphi_b \approx 2\sqrt{2}\,\mathrm{Re}\,\psi_{Sa}\exp(-it)$.

This feature of bions has the consequence that (as in systems described by the Schrödinger and Klein-Gordon equations) application of an external field to the system can result in an effect in which energy is accumulated in bion excitations even when the amplitude of the external field is below the threshold for the production of a kink-antikink pair, i.e., $E \ll 2M$ (for envelope solitons, see Ref. 46). As a result, the discovery of long-lived bound states of solitons and the investigation of their properties is one of the most important branches of soliton theory (this applies particularly to non-one-dimensional models; see below). For nonintegrable models with polynomial or logarithmic Lagrangians (in particular, the φ^4 theory) and also the multiple sine-Gordon equation, exact analytic solutions describing bound states have not been obtained. Here, as in the study of the interaction dynamics of solitons, numerical experiments are basic.

Indications of the possible existence of bound states were already obtained in the early numerical investigations of the dynamics of initial packets in the framework of the S3 equation. In Refs. 56 and 57, evolution of the initial packet yielded a solution that very strongly resembles a bound state from the point of view of modern conceptions. It is possible that this is responsible for the very long time of existence of nonlinear oscillations of the amplitude in the center of the packet (see Fig. 3a).[56] In Ref. 54, a study was also made of the decay of an initial Gaussian packet, and the formation of three humps was followed, though, as Yajima and Outi noted, the calculation time was insufficient to answer unambiguously the question of whether solitons were formed from the humps or whether the system was in a quasiperiodic regime for a certain time. In Ref. 57, a study was made of the formation of the S3 bion

$$\psi_b = 4\exp(-it/2)\,[\mathrm{ch}\,3x + 3\,\mathrm{ch}\,x\exp(-4it)]\,(\mathrm{ch}\,4x + 4\,\mathrm{ch}\,2x + 3\cos 4t)$$

from the initial packet $\psi_0 = 2\,\mathrm{sech}\,x$. Unfortunately, only the initial stage in the evolution of the packet was presented.

A bion was found beyond all doubt in the numerical investigation of the interaction of kinks in the Higgs model (φ_-^4 theory) made by Kudryavtsev.[79] These results were later confirmed in Refs. 2, 20, and 80.

We consider them in more detail. In the numerical experiment in Ref. 79, a study was made of the interaction of kinks and antikinks for the equation

$$(\Box - 1 + \varphi^2)\,\varphi = 0. \tag{40}$$

An initial state describing a kink and antikink that approach each other from infinity,

$$\left.\begin{aligned} \varphi(x,0) &= \mathrm{th}\,\frac{\gamma}{\sqrt{2}}(x - x_0) - \mathrm{th}\,\frac{\gamma}{\sqrt{2}}(x + x_0) - 1,\\ \varphi_t(x,0) &= -\frac{\gamma}{\sqrt{2}}\,v\left[\mathrm{sech}^2\frac{\gamma}{\sqrt{2}}(x - x_0) + \mathrm{sech}^2\frac{\gamma}{\sqrt{2}}(x + x_0)\right], \end{aligned}\right\} \tag{41}$$

leads with exponential accuracy $[\propto \exp(-4x_0)]$ to a solution of Eq. (40) in the form

$$\varphi = \mathrm{th}\,\frac{\gamma}{\sqrt{2}}(x - vt - x_0) - \mathrm{th}\,\frac{\gamma}{\sqrt{2}}(x + vt + x_0) - 1.$$

Feeding it into the computer, we obtain the following picture of the interaction.[20)] The final state is found to depend on the velocity v of the colliding quasisolitons. If the velocity v is greater than a certain critical v_{cr}, the kink and antikink repel each other, losing a certain fraction of their kinetic energy on radiation. This fraction is smaller, the larger $v > v_{\mathrm{cr}}$, so that in the relativistic region $v \to 1$ the collision becomes quasielastic. For $v < v_{\mathrm{cr}}$, the interaction picture changes qualitatively,[21)] and a solution that oscillates in time and describes the bound state of a kink and antikink is formed. The mass defect, which is $\Delta M = 2E_k - M_b$, is radiated in the form of linear waves in a vacuum distinguished by the initial conditions. The energy E_k of a kink or antikink can be estimated in accordance with the standard formula

[20)] Below, we summarize the results of all the papers just quoted.

[21)] In Ref. 20, a certain small "resonance" region Δv near $v \approx 0.3 > v_{\mathrm{cr}} \approx 0.2$ in which bound states are also formed was discovered. This effect, which is not yet understood, must lead to the appearance of short-lived bions that decay into kinks and antikinks ("resonances").

$$E_k = \frac{1}{2}\int_{-\infty}^{\infty}\left[\varphi_t^2 + \varphi_x^2 + \frac{1}{2}(\varphi^2-1)^2\right]dx = \frac{2\sqrt{2}}{3}\gamma \equiv M_k\gamma,$$

and the mass M_b of the resulting bion is less than the mass $2M_k$ of two kinks. As a result of numerical experiments, we obtain v_{cr} of order 0.2 ± 0.01.[80] After a brief relaxation, the oscillations of the field $\varphi(0,t)$ in the bion take on an exceptional regularity, remarkably similar to the graph of the dependence of the amplitude $\psi(0,t)$ in the S3 experiment (see Fig. 3a). In contrast to the S3 equation, Eq. (40) is nonintegrable, and therefore the bion formed by the collision of the kinks continues to radiate small-amplitude waves, losing its energy. The lifetime of a large-amplitude bion is very appreciable, $T \approx 2\cdot 10^3/\omega$,[80] increasing with decreasing amplitude.

These results showed that the existence of bound states is not a privilege of the integrable S3 and SG equations but that they can be formed by the interaction of quasisolitons in the framework of other equations too.

It can be assumed that nearly integrable systems must have solutions describing long-lived bound states. This conjecture was also verified in numerical experiments. In Ref. 80, a bound state of two kinks and one antikink was found for Eq. (40); it was called a *triton*. In Ref. 20, a bion was found in the framework of the DSG equation ($v_{cr} \approx 0.1$). The pulsations of the field in a triton are entirely regular and are accompanied by appreciable radiation, so that its lifetime is shorter than a bion's and is of the order of hundreds of periods of oscillations at the main frequency; the decay rate with respect to the energy is $\delta_T \approx (3 \pm 0.1) \times 10^{-2}$. Finally, we note that a triton is formed as a result of a triple collision of kinks, and therefore the probability of its occurrence is much lower than that of a bion, and accordingly the contribution of such states to various physical processes is small. This cannot be said of the bion states, whose contribution may be decisive in the consideration of an "ideal gas" of kinks at low temperature $T_k < T_{pt} = M_k v_{cr}^2 = (2\sqrt{2}/3)\cdot 0.2^2 = 0.04$ for the φ_-^4 theory. The temperature T_{pt} can be regarded as the Curie point for a phase transition that takes place with a change in the topology. This transition can also be interpreted as a transition from a gas of "charged" particles, i.e., kinks (and antikinks), to a gas of "neutral dipoles," i.e., bions.[22)]

In this connection, it becomes very important to study the structure and dynamics of bions, their formation, stability, and interaction in the framework of different models. Particularly interesting here could be the formation of stable, i.e., long-lived, bound states from unstable quasisolitons. There are already the first indications of such a possibility.

Stability of solitons. First, we shall define what we understand by stability of solitonlike solutions. As we have already said above, such solutions describe extremal properties of a certain nonlinear system. One can speak of two types of stability of a considered system: 1) with respect to perturbation of the initial data; 2) with respect to perturbation of the evolution equation that governs the behavior of the system (structural stability).

In the first case, the question has been investigated fairly fully. There exist various methods and approaches, linear and nonlinear. In the linear approximation, the stability problem usually reduces to study of the eigenvalue spectrum of a linearized evolution equation; in the nonlinear approach, to the investigation of Lyapunov inequalities (see, for example, Refs. 1, 2, and 89).

In recent years, there have been many studies of structural stability in the framework of various evolution equations for different forms of perturbation. Attempts were made to construct a general theory of perturbation of solitons by means of Green's functions and spectral transformation, i.e., by transition from the configuration space (x,t) to the space of the scattering data, on the basis of the well-known two-time formalism.[90-96] However, the developed method was suitable only for a fairly restricted class of perturbation functionals.[59] Despite the entirely impressive results obtained in this direction, one cannot regard the investigations as completed, especially in the case of the structural stability of systems that are *nearly* integrable.

We mention here that there exist rigorous results on the structural stability of Hamiltonian systems with a finite number of degrees of freedom, namely, there is the so-called *Kolmogorov-Arnol'd-Moser theorem*. The main assertion of this theorem is that when a perturbation is included in an integrable system the greater part (finite in measure) of the structure of its phase space is preserved if the conditions of linear independence (irrationality) of the frequencies of the unperturbed system are satisfied. This means that in the case of small perturbations most of the solutions remain quasiperiodic functions of the normal coordinates (action, angle). However, the remaining part of the structure of the phase space, where the conditions of the theorem are not satisfied, is broken up, and the phase trajectories of the system can wander over the whole of this part, filling it densely (so-called *Arnol'd diffusion*). Thus, some of the orbits (quasiperiodic solutions) of the unperturbed system become unstable and wander over the entire accessible region of the phase space. The stability or instability of a state of the perturbed system depends on the region of phase space in which the unperturbed system is situated (for more detail, see Ref. 98). The question of the applicability of this result fo infinite systems is open. Particularly complicated is the question of the stability of the motion of an integrable system under the influence of a non-Hamiltonian perturbation.

From the point of view of computational science, two

[22)]Thus, at low temperatures the statistics of a system, for example, in the model of Krumhansl and Schrieffer,[88] can be determined by bion excitations. This means that the excitation frequency must be determined by the temperature by means of the approximate formula $\omega \approx \sqrt{2[1-(9/32)T^2]} = \sqrt{2[1-(T/2M_k)^2]}$.

problems can be investigated in the framework of a single approach—the initial-value problem. In the first case, one studies the evolution described by the unperturbed evolution equation of a perturbed or unperturbed (the computer itself introduced perturbations) initial state, which is specified in the form of the soliton soliton solution whose stability is being investigated.

In the second case, the evolution of an analogous initial state is subject to the perturbed equation. In both cases, the investigated solutions can depend on some slow time as on a parameter. In the first case, we shall understand by "stable solutions" those for which the initial perturbations do not grow with the time during the evolution of the initial state. Note that in accordance with this definition we must also regard as "stable" weakly radiating solitonlike solutions that do not break up under the influence of initial perturbations.

By "structurally stable solutions" it is natural to understand solutions that preserve their form for a fairly long time. The meaning of *fairly long* is determined by the time scale of the physical processes taking place in the considered system. We note here that some of the unperturbed solutions will be broken up in this case very rapidly and can be replaced by entirely new types of solution, especially for perturbations of non-Hamiltonian type.

We give below some examples of numerical investigations in both directions. The choice of the examples is to a large degree arbitrary and makes no pretence to completeness.

In the flat (x, t) case, the majority of the investigated systems have stable soliton or solitonlike solutions at least with respect to perturbations that do not change the symmetry of the system, i.e., depend on x and t.[23)] The stability (longitudinal) of true solitons follows, as a rule, from integrability of the corresponding S3, KdV, modified KdV, and SG equations (see, for example, Ref. 6). A unique, but very instructive, counterexample was given by Berryman in Ref. 99. He considered the longitudinal stability of solitons in the Boussinesq model and showed that these solitons are unstable with respect to infinitesimal perturbations of plane-wave type. And although this instability evidently arises because of the "band" dispersion $\omega^2 = k^2(1-k^2)$ of the Boussinesq equation, this example shows that one must exercise a certain caution with regard to the established opinion that integrability of a system necessarily entails stability of the soliton and, in particular, many-soliton solutions.

Indeed, a second counter example concerns the instability of a bound state of two solitons (a bion) in the framework of the entirely respectable (in the sense of the dispersion) S3 equation. As was found by Satsuma and Yajima[57] in a numerical experiment, the bion

$$\varphi(x, t) = 4A \frac{\operatorname{ch} 3Ax + 3 \operatorname{ch} Ax \exp(4iA^2 t)}{\operatorname{ch} 4Ax + 4 \operatorname{ch} 2Ax + 3 \cos 4A^2 t} \exp\left(i \frac{A^2}{2} t\right)$$

decays into its constituent solitons under the influence of an initial perturbation with asymmetric imaginary part.[24)]

Unstable solitonlike solutions appear in the framework of the nonlinear Klein-Gordon equation. For the equation of the φ_+^4 theory of the field $F(\varphi) = (1 - |\varphi|^2)\varphi$ the possible instability of solitonlike solutions was already pointed out by Zastavenko in Ref. 100, in which he also determined the instability region for complex solutions of the form $\varphi = \chi(x)\exp(-i\,\omega t), 0 \leq \omega^2 \leq 1/2$. In the computer investigation of such solutions the instability is manifested very clearly, especially in the limit $\omega \to 0$, i.e., in the case of a real field function. Even for the very small perturbations introduced by the computer, the solitonlike solutions break up during times of the order of a few units, usually leading to a singularity of χ.

For Lagrangian equations describing complex fields there exists in addition to the ordinary energy-momentum conservation laws (and angular momentum when $d \geqslant 2$) a further "charge" conservation law, $(d/dt)Q = 0$, the charge for scalar fields being given by

$$Q = -\frac{i}{2} \int (\varphi_t^* \varphi - \varphi^* \varphi_t)\, dx. \tag{42}$$

Using the variational principle and this equation, one can prove a theorem (Q theorem) that formulates sufficient conditions for stability of scalar complex solitonlike solutions (see Ref. 2, p. 94, and also Ref. 3). The stability region of these solutions is determined by the inequality

$$\frac{\omega}{Q} \frac{dQ}{d\omega} < 0. \tag{43}$$

We apply, for example, this condition to the complex Schrödinger and Klein-Gordon equations with nonlinearity of the form $\varphi|\varphi|^\nu$ (see, for example, Ref. 101):

$$\left.\begin{array}{l} i\varphi_t + \varphi_{xx} + |\varphi|^\nu \varphi = 0; \ (1) \\ (\Box + 1)\varphi - |\varphi|^\nu \varphi = 0. \ (2) \end{array}\right\} \tag{44}$$

In the first case, it is not the charge Q but the normalization of the wave function φ that is conserved,

$$\frac{d}{dt} \int |\varphi|^2\, dx \equiv dS/dt = 0, \tag{45}$$

and instead of (43) we have $dS/d\omega < 0$, with $\omega < 0$.[25)] After the substitution $\varphi(x, t) = \chi(x)\exp(-i\,\omega t)$, we obtain from (44)

$$-\chi_{xx} + \varkappa^2 \chi - \chi^{\nu+1} = 0; \ \varkappa^2 = \left\{ \begin{array}{ll} -\omega & (1); \\ 1 - \omega^2 & (2). \end{array} \right. \tag{46}$$

One can verify that the conditions for applicability of the Q theorem for Eq. (46) are satisfied. It remains to find the dependence of Q and S on ω. We find the dependence by means of the scale transformation $x \to \varkappa^{-1}\xi$, $\chi \to \varkappa^{2/\nu}\eta$, under which (46) takes a form that is free of $\varkappa$: $-\eta_{\xi\xi} + \eta - \eta^{\nu+1} = 0$. Therefore $S = \int \chi^2\, dx$

[23)]We shall say that such perturbations are longitudinal, in contrast to transverse perturbations which break the initial symmetry and depend, for example, on x, y, and t.

[24)]We emphasize that the S3 bion does not have a mass defect, i.e., $E_b = E_{S_1} + E_{S_2}$, so that in perturbed S3 systems it is unstable. The fact that a bion is formed from an initial packet and lives for a long time (!?) in the framework of the S3 equation is surprising rather than part of a pattern and can probably be attributed to higher integrals.

[25)]The condition $\omega < 0$ is necessary for the existence of solitonlike solutions (cf. p. 33 of Ref. 2).

$= \varkappa^{(4-\nu)/\nu} \int \eta^2 d\xi \equiv \varkappa^{(4-\nu)/\nu} C(\nu)$ and $Q = \omega S = \omega \varkappa^{(4-\nu)/\nu} C(\nu)$. Calculating $dS/d\omega = -dS/d\varkappa^2$ and $dQ/d\omega$, we find the corresponding stability regions:

(1) $\nu < 4$, $\omega \equiv -\varkappa^2$ — arbitrary;

(2) $1 > \omega^2 > \nu/4$.

Thus, in both cases stable solitonlike solutions exist for $\nu < 4$.

A more interesting situation arises in the investigation of transverse stability of solitons and solitonlike solutions. Here, we in fact make a transition to three-dimensional systems (two spatial and a time coordinate). Depending on the form of the physical system, this transition may vary. The transformation of the KdV-like equations (1) usually leads to a "generalized" Kadomtsev-Petviashvili equation:

$$\varphi_{tx} + [1/(\nu+1)](\varphi^{\nu+1})_{xx} + \varphi_{xxxx} + \alpha\varphi_{yy}/2 = 0.$$

For equations of Schrödinger and Klein-Gordon type, the operator ∂_x^2 may be replaced by $\partial_+^2 = \partial_x^2 + \partial_y^2$ or $\partial_-^2 = \partial_x^2 - \partial_y^2$, i.e.,

$$(\partial_t^2 - \partial_+^2)\varphi = F(\varphi) \quad (47)$$

or

$$(\partial_t^2 - \partial_-^2)\varphi = F(\varphi). \quad (48)$$

From the point of view of computational science, the most thorough and systematic study has been made of the stability of planar solitons in plasma physics described by the Schrödinger equation with a self-consistent potential[34,102,103]:

$$\nabla(i\partial_t\varphi - \nabla\nabla\varphi - \Phi\varphi) = 0;\ \Box\Phi = \nabla|\varphi|^2; \quad \nabla \times \varphi = 0;\ \varphi = \{\varphi_x, \varphi_y\}; \quad (49)$$

for $\varphi_x \gg \varphi_y$, $i\partial_t\varphi_x + \partial_+^2\varphi_x + |\varphi_x|^2\varphi_x = 0$, and also[102,104]

$$i\partial_t\varphi - \partial_-^2\varphi - 2|\varphi|^2\varphi = 0. \quad (50)$$

Omitting the details of the physical formulation of the problems, which can be found in Refs. 34 and 102–104 and in the following section, we describe qualitatively the behavior of planar solitons under transverse perturbations in the framework of these equations.

It is found that planar solitons are unstable in both cases. Moreover, numerous investigations in recent years (including numerical investigations) have shown that two forms of planar solitons have transverse stability—the KdV and SG solitons—as does one form of solitonlike solution—the kinks of the Higgs equation (φ_-^4 theory). Of these, only the KdV soliton is not topological. All the remaining nontopological (bell-shaped) solitons and solitonlike solutions, including the Schrödinger and Klein-Gordon examples, are unstable.

Finally, we say a few words about structural stability of solitons and solitonlike solutions (soliton perturbation theory). As we have already said, despite the considerable amount of work already done in this direction, this problem is still far from full clarification in both the practical and conceptual aspects for systems with infinitely many degrees of freedom. Here, I merely give references to papers with which I am acquainted and also, as an example, brief results of some numerical investigations.

Naturally, among the first investigations in this direction were those of KdV and S3 solitons with allowance for the effects of weak linear[90,105] and nonlinear[65] dissipation. In Ref. 90, Bogolyubov's two-time formalism was used to obtain equations describing the variation in time of the parameters of KdV solitons under the assumption that their shape (functional dependence) does not change for different models of the damping.[26)] In Refs. 65 and 105, a numerical investigation was made of the influence of dissipation on S3 solitons.

These first investigations already showed that the behavior of a soliton, namely, the preservation of its shape through a balance of the effects of dispersion and nonlinearity, depends very strongly on the form of the perturbing term. This was brought out particularly clearly in Ref. 106, in which Pereira studied the behavior of S3 solitons under the influence of power-law damping of the form $\gamma \propto \varepsilon k^b$ (k is the wave number). He succeeded in showing that only for $b=2$, which is related to the scaling properties of the S3 equation, does the soliton have virtually no change of shape with the time. In the remaining cases when $b \neq 2$ the evolution of the soliton is accompanied by a change of shape, which is greater, the larger is the coefficient ε in the damping term.

Thus, for $b=4$, 3, and 2 the inequalities $\varepsilon \lesssim 0.01$, 0.03, and 0.2 must be satisfied if the change in the shape of the soliton is to be negligible. For the KdV equation, the situation is even more complicated. Even in the case of weak perturbations of the KdV or S3 equations the solitons usually acquire tails (see, for example, Refs. 75, 76, 107, and 65) or they may altogether change their shape.[65,97] We consider in more detail the very interesting results obtained in this direction by the Japanese group in Ref. 65. They studied the influence of nonlinear damping on the dynamics of the S3 solitons of sech and tanh types, i.e., they studied the initial-value problem

$$F(\varphi, \varepsilon) \equiv i\varphi_t + \frac{1}{2}\varphi_{xx} + \alpha|\varphi|^2\varphi + \varepsilon\varphi P\int_{-\infty}^{\infty} \frac{|\varphi(x', t)|^2}{x - x'}dx' = 0; \quad (51)$$

$$\psi(x, 0) = f(x) \quad (52)$$

(ε is a constant, and $\alpha = \pm 1$).

The function $f(x)$ was chosen in the form of a soliton, a bound state of two solitons, or a periodic wave. For $\alpha = 1$, the modulation instability results in the formation of sech solitons, while for $\alpha = -1$ the soliton solutions of $F(\varphi, 0) = 0$ are kinks of tanh type.

As the numerical investigations showed, the presence of the integral term (the perturbation) in (51) leads to very important qualitative effects, namely, for one soliton $f(x) = A\,\mathrm{sech}(Ax)$ to its motion in the positive direction of the x axis and to the appearance of a tail on the trailing edge (Fig. 9), though these effects are smaller, the smaller ε; for the soliton bound state $f(x) = 2A\,\mathrm{sech}\,Ax$ the perturbation leads to its decay into the constituent solitons. Evidently, this effect is associ-

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[26)]Subsequently, this method was also widely used in other studies (see, for example, Ref. 94).

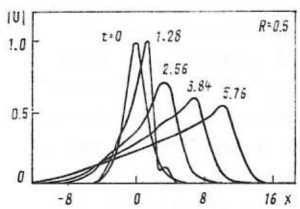


FIG. 9.

ated with the asymmetry with respect to x of the perturbation in (51). In fact, an analogous decay of a bound state was observed earlier[57] even in the framework of the unperturbed integrable S3 equation following superposition on a bion of an initial perturbation with asymmetric imaginary part (unfortunately, the amplitude of the perturbation was large).

Nonlinear Landau damping has a even stronger influence on kind solitons ($\alpha=-1$), $\varphi_k = A\tanh(Ax)\exp[i(vx-\omega t)]$. Depending on the direction of motion of the kink, its amplitude either decreases ($v>0$).

Accordingly, the periodic waves $\varphi(x,0)=1+A\cos\pi x$ decay asymmetrically into a series of: 1) damped sech solitons for $\alpha=1$; 2) growing tanh solitons for $\alpha=-1$.

Another series of studies that we must consider here was devoted to the structural stability of SG solitons and was made by a Cornell group in Refs. 85a and 92 (theory) and Ref. 108 (numerical experiment). In this series, the following initial-value problem was studied:

$$\Box\varphi+\omega_0^2(x)\sin\varphi+AF(x)=0;\quad \varphi(x,0)=f(x). \tag{53}$$

Here, the dependence of the eigenfrequency ω_0 of the system on x can simulate the presence in the system of impurities (inhomogeneities), and also macroscopic structure of the system (lamination and so forth). The last term in (53) simulates the external influences (fields, currents, etc.). In Refs. 85a and 92 it was predicted analytically, and in Ref. 108 confirmed numerically that the SG kink solitons behave in the presence of sufficiently small perturbations like Newtonian particles with *internal structure* in external fields, namely, they can be accelerated (or decelerated), can radiate, and slightly change their structure, which is accompanied by transition radiation; they may be trapped in a certain region of space and then ejected from it by the removal of one of the barriers. The appearance of radiation in the model (53), and also in its discrete analog even when $\partial\omega_0/\partial x=0$, must strongly influence the structure and, especially, the dynamics of bound states, i.e., bions, their formation, lifetime, and decay. All these results are very interesting (and I would even say impressive) from the point of view of applications in solid-state physics. Here, we should also mention Ref. 109.

In conclusion, we note that work in this direction continues very actively and, although there are still many complicated problems to be solved, it may be hoped that in the next years a fairly good understanding of the problems of stability (especially structural) of solitons will be achieved. This will make it possible to turn to a systematic experimental study of soliton phenomena a different branches of physics.

2. NON-ONE-DIMENSIONAL SOLITONS

Stability of planar solitons. In systems of more than one spatial dimension, it is possible to have solitons with different symmetries: plane, cylindrical, spherical, etc. The simplest and most natural generalizations of one-dimensional solitons are planar solitons, for which the field function φ depends only on the coordinate in the direction of their motion, for example, the z axis, so that their wave front is parallel to the xoy plane. It is readily seen that such solitons are indeed particular solutions of the corresponding multidimensional equations by virtue of $\partial_x\varphi=\partial_y\varphi=0$.

Naturally, the question of the stability of such solutions in the presence of transverse (i.e., dependent on x and y) perturbations arises. From our point of view, this means that it is necessary to solve the initial-value problem

$$\left.\begin{array}{l} M[\varphi]=0;\\ \varphi=\varphi_s(z,0)+\delta\varphi(r_\perp,z,0),\end{array}\right\} \tag{54}$$

where M is the nonlinear differential operator corresponding to a particular nonlinear partial differential equation, $\varphi_s(z,0)$ is a solution of this equation in the form of a planar soliton, and $\delta\varphi(r_\perp,z,0)$ is a perturbation. It was in such a formulation that a solution was found to the problem of the transverse stability of planar Langmuir solitons and the collapse of Langmuir waves in a series of analytic and computational studies.[34,102-104] In this case, the operator M is determined by the system of equations (49). We illustrate the results of these investigations by taking the example of Ref. 34. In their calculations, Pereira, Sudan, and Denavit used a two-dimensional algorithm based on Fourier transformation of the system (49) in the presence of periodic boundary conditions with respect to y and z and a spatial grid of 32×32 points.

The initial function φ, the electric field intensity, was chosen in the form of a planar soliton solution, and on the potential Φ, the particle-number density, the perturbations

$$\Phi(z,y,0)=-(2A^2\,\mathrm{sech}^2\,Az)\,(1+2\varepsilon\cos k_y y)$$

were imposed, so that neither the electric field nor the derivative Φ_t was perturbed at the initial time. As one would expect, the planar soliton is unstable for a sufficiently large amplitude or for long wavelengths $l_y=2\pi/k_y$ of the perturbation. The stability region of the planar soliton is determined by the relation $k_y\geqslant\alpha A$, where α is a numerical coefficient, of order unit in the experiments of Ref. 34. After passing through the short stage of matched adjustment of the electric field, the unstable solitons enter the collapse stage, in which the amplitude φ increases unboundedly (at least in the case of the considered model). Figure 10 shows the successive stages in the development of the instability of a planar soliton at rest for $l_y=16$, $A=2$, $v=0$, and $\varepsilon=0.1$ (Ref. 34) (similar results were also obtained in Ref. 103). Pereira, Sudan, and Denavit also found a connec-

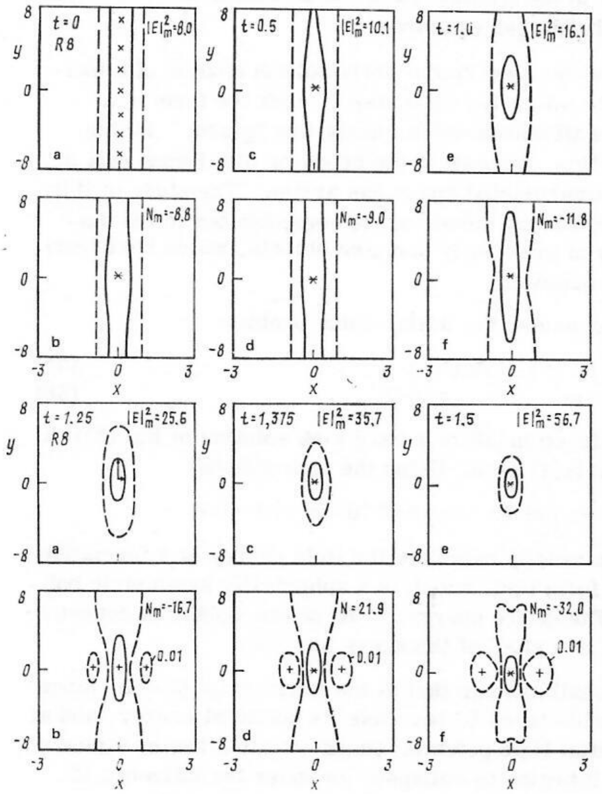


FIG. 10.

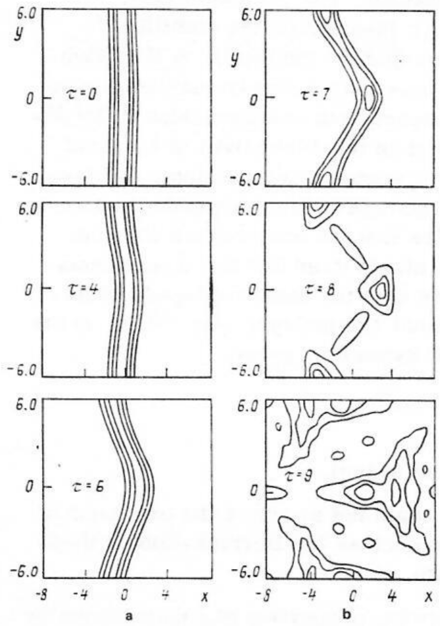


FIG. 11.

tion between the group velocity of a moving soliton and the local energy density $|\varphi|^2$ that arises in transient processes of the type of collapse or Landau damping.[27)]

The second example concerns the evolution of a planar soliton in the framework of the "asymmetric" S3 equation

$$i\varphi_t + \partial_-^2\varphi + |\varphi|^2\varphi = 0.$$

Equations of this type and also somewhat more complicated equations, for example,

$$2i\varphi_t - \partial_-^2\varphi = \frac{9}{2}\varphi|\varphi|^2 - 3\varphi\Phi_y; \quad \partial_+^2\Phi = -3\partial_y|\varphi|^2,$$

are encountered in the study of nonlinear two-dimensional waves in a plasma[102,110] and surface waves on water of finite depth.[111,112] The second of these equations, as was shown by Anker and Freeman,[112] is integrable, and in its framework one can have the coalescence and decay of solitons if the usual decay conditions $k_1+k_2+k_3=0$, $\omega_1+\omega_2+\omega_3=0$ are satisfied. Unfortunately, the question of the stability of the soliton solutions remains open (we recall Berryman's instructive lesson).

Planar solitons of the first equations were investigated in the framework of the initial-value problem (54) in Ref. 104 by means of the code described in Ref. 34 and above. It was shown that planar solitons of the form $\varphi_s=\sqrt{2}\,\mathrm{sech}\,z$ are stable with respect to perturbations even in z, which agrees with the results of Refs. 2, 102, and 104, but are unstable with respect to perturbations odd in z of the type

$$\delta\varphi(z, y, t) = a(\partial_z\varphi_s)\cos(ky) \quad (a = 0.1).$$

The initial linear stage in the development of this instability agrees well with the theoretical predictions of Yajima[102] and has characteristic exponential growth. When the amplitude of the perturbation reaches half the unperturbed value, the instability goes over into the nonlinear regime shown in Fig. 11. It can be seen from these figures that the instability leads to the breakup of the soliton into pieces and to dissipation of the energy originally stored in the soliton, so that, in contrast to the preceding case, it can be called *anticollapse*.

Unfortunately, the conditions of the experiment (the influence of neighboring periods) did not make it possible to establish uniquely whether some nonlinear quasisteady state of soliton type is established or whether the energy is dispersed throughout the space entirely uniformly (see Ref. 2, p. 74).

The planar sech solitons of the relativistic Klein-Gordon equation are also subject to similar forms of transverse instability. We note here that so far there has been no regorous investigation of the stability of bell solitons of bion type for the Higgs (φ_-^4) and sine-Gordon equations.[28)] The Higgs and SG kinks, like the bell solitons of the Kadomtsev-Petviashvili equation, are stable.

It should be noted that, like infinite charged or current planes in electrodynamics, planar solitons have infinite energy, charge, etc. In this connection, in spaces of more than one dimension it is natural to regard as solitonlike solutions only field configurations that have finite values of the considered quantities,

[27)]Note that they also investigated the influence of Landau damping and a pump field on collapse and showed that in the two-dimensional case the birth and death of solitons is periodic in nature.

[28)]The considerations pointing to equivalence of SG bions and S3 solitons discussed above suggest that the former may also be unstable.

i.e., particlelike solutions. For such solutions, it is possible to formulate a theorem on the stability of quasisolitons (Derrick-Hobart theorem). A Hamiltonian system does not have stable steady quasisoliton solutions in a space of more than one dimension if the potential does not depend on the derivatives of the field functions.[113,114] For systems with an isotopic symmetry group, this theorem can be transformed into a Q theorem,[2] which states that the quasisoliton solution $\varphi_s = R(r)\exp(-i\omega t)$ is stable when $dQ/d\omega < 0$ (and unstable otherwise), where Q is the conserved quantity determined by the internal symmetry group. Thus, in the simplest case of $U(1)$ symmetry group.

$$Q = -\frac{i}{2}\int (\varphi_t^*\varphi - \varphi^*\varphi_t)\, d^D x$$

(D is the dimension of space).

For spinor fields, I am not aware of the existence of general propositions such as the Derrick-Hobart theorem or the Q theorem.

Below, we consider the properties of quasisolitons in two- and three-dimensional spaces, taking a number of concrete examples. We emphasize here that, as in the case of the discovery of the KdV solitons themselves, almost all the main results were obtained by means of the computer.

Collapse of Higgs bubbles. There has recently been a strong increase in interest in the study of nonlinear field theories and their applications in elementary-particle physics. It is possible that this is due to the successes of models based on the theory of gauge fields and also the significant increase in the mathematical understanding of the structure of the investigated objects. Whatever the truth, the number of publications has increased exponentially from the middle of the seventies, which indicates the youth and promise of this direction. The basis for constructing extended quantum objects is usually provided by particlelike (or quasisoliton) solutions of nonlinear classical field equations (bags, solitons, strings). This makes it possible to simulate and study the structure of elementary particles and not regard them as given, as in ordinary linear quantum field theory. In this connection, one of the main tasks of this direction is to study the structure of particlelike solutions at the classical level. As we have already seen above, in two-dimensional space-time the understanding of such structures has advanced very far as a result, on the one hand, of the development of powerful methods of analytic investigation based on spectral transformation[29)] and, on the other, of numerous and fruitful numerical studies.

At the same time, the detailed study of particlelike solutions in multidimensional spaces is in fact only just beginning, although the first investigations in this direction appeared more than two decades ago. At the present time, the main contribution in this direction is made by numerical investigations. We describe here only the results obtained in the study of particlelike solutions in the framework of the nonlinear Klein-Gordon and Schrödinger equations.

One of the best known early soliton models of a hadron, the so-called SLAC bag,[116] took the form of a Higgs wall within which quarks are "glued." In this connection, interest in the behavior of a Higgs wall in three-dimensional space has arisen. The study of this phenomenon by means of the computer led to the discovery of previously unknown objects, which have been called pulsons.

We consider the initial-value problem

$$(\Box - 1 + \chi^2)\,\chi = 0; \tag{55}$$

$$\chi(r, 0) = \mathrm{th}\,[(r - R_0)/\sqrt{2}]. \tag{56}$$

The function (56) is a fixed kink solution of Eq. (55) in the flat (x, t) case. Using the expressions

$$E = 4\pi \int \mathscr{H}\, dr;\quad \mathscr{H} = (r^2/2)\,\{\chi_t^2 + \chi_r^2 + (\chi^2 - 1)/2\},$$

we can readily calculate the field energy of a formation of the form (56), which is a spherically symmetric bubble. The entire energy $E \propto R_0^2$ of this bubble is concentrated in a shell of thickness $\Delta r \approx \sqrt{2}$.

We shall assume that at the initial time $R_0 \gg 1$. Since the bubble tends to decrease its potential energy, and at the initial time potential energy is all it has by virtue of $v = 0$, it begins to collapse, as under the influence of surface tension. In Ref. 117 (see also Ref. 2) an equation describing this collapse process for $R_0 \gg 1$ was obtained:

$$\frac{d^2}{dt^2} R_0 + \frac{2}{R_0}\left[1 - \left(\frac{dR_0}{dt}\right)^2\right] = 0$$

together with the solution $R_0 = R_{in}\,\mathrm{cn}(\sqrt{2}/R_{in}\, t, 1/2)$, where cn is the elliptic cosine with modulus $k^2 = \frac{1}{2}$.

Introduction of a problem (55)–(56) into the computer shows that in the initial stage the behavior of the bubble is indeed well described by the above formula. Appreciable deviations begin in the region $R_0 \approx 1$. During the collapse of the bubble, its potential energy is transformed into kinetic energy, so that near the center the collapse process is replaced by an expansion process (the bubble is "reflected" from the center), and some of the energy is lost on radiation. When it attains a diameter somewhat less than the initial diameter (due to the loss through radiation), the bubble again begins to collapse, etc. Its behavior is very similar to that of a pendulum in the presence of friction. The number of oscillations depends on the energy loss through radiation. Bogolyubskiĭ and the present author made a series of numerical experiments for different values of R_0.[118] We found that a few (usually 2–5) pulsations are sufficient for most of the energy of the bubble to be radiated to infinity. In the same series of experiments we found that bubbles with sufficiently large initial diameter end their lives by giving birth to certain new long-lived formations. Bubbles with small diameters disappear entirely. A second series of calculations was made with a view to studying the behavior of the bubbles

$$\varphi(r) = 4\,\mathrm{arctg}\,[\exp(r - R_0)],\quad R_0 \gg 1, \tag{57}$$

in the framework of the sine-Gordon equation. Because of the integrability of the sine-Gordon equation in two-

[29)]Following Calogero,[115] this is the name we give to the inverse scattering method.

dimensional space-time, we assumed that the lifetime of the bubbles (57) would be very long. The results of the calculations refuted this expectation. The radiation accompanying collapse of the bubble was very appreciable and the picture overall is qualitatively similar to that described above. Thus, in these experiments the integrability of the sine-Gordon equation in the (x, t) world was in no way manifested in the real four-dimensional space-time.

Pulsons. Their formation, stability, and interaction. As we have seen above, in a flat (x, t) world there exist long-lived bound states of solitons in the framework of the Klein-Gordon equation, i.e., bions. It is natural to consider whether there exist analogs of bions in two- and three-dimensional spaces. In accordance with the Derrick-Hobart theorem, stable stationary solitons do not exist for the sine-Gordon and $\varphi^4_{\pm}$ theories. In the investigations made at Dubna in the framework of the φ^4_+ theory stable oscillator solutions were not found. The situation changes on the transition to the sine-Gordon and φ^4_- theories.

As we have already noted, the evolution of Higgs and sine-Gordon bubbles of sufficiently large diameter (or, which is the same thing, sufficient mass) results in the formation of a concentration of field energy at the center of the bubbles. The radial distribution of the field in the concentration has the form of a bell, and the field function oscillates about the vacuum value fixed by the boundary conditions. In the experiments we have described,[119] these values were $\chi^{\mathrm{v}}(t, \infty) = -1$ for the φ^4_- theory and $\varphi^{\mathrm{v}}_{\mathrm{SG}}(t, \infty) = 0$ for the sine-Gordon equation.

Such concentrations of energy, called pulsons, arose for fairly arbitrary initial conditions, provided $\chi(0, 0) > C_1 \gtrsim 1$, $\varphi(0, 0) > C_2 \gtrsim 2\pi$.

The radiation from the pulsons was found to be slight, so that their lifetime is farily long—of the order of thousands of oscillation periods (we recall the Higgs bions in a flat world). These results were confirmed later by a group at the Institute of Theoretical and Experimental Physics at Moscow[120] for pulsons of the φ^4_- theory and by a Danish group[16] for SG pulsons.

For a multivacuum sine-Gordon theory heavy pulsons, which have no analogs in a flat world, were found in Ref. 121. Their amplitudes lie in the range $\varphi(0, t) \in (3\pi, 4\pi)$. In the range $(2\pi, 3\pi)$, these pulsons become stable and rapidly go over into the 2π pulsons described above. Finally, we note that the pulsons described above are stable with respect to azimuthal perturbations.[122]

One could imagine that rotation of a weakly asymmetric Higgs bubble could delay its collapse and significantly extend its lifetime (so far as I know, this idea was put forward independently by Kudryatsev and Scott). Our preliminary investigations have not yet led to a positive result in the x, y, t case.

Interaction of pulsons. Above, we have considered the dynamical properties of pulsons relating to their formation and stability. However, this is only the first step. The real dynamics—in which the particular properties of solitons are fully manifested—is their interaction.

Therefore, we consider the results on the collision of two-dimensional cylindrically symmetric (in the rest frame) pulsons of the Klein-Gordon equation (φ^4_+ theory) obtained at Dubna.[122] In a series of calculations, a study was made of the head-on collision of two *unstable* Klein-Gordon pulsons of the form

$$\psi(x, y, t) = Af\left[u_0 \sqrt{\gamma_i^2 (x - v_i t)^2 + y^2}\right] \cos\left[\sqrt{1 - u_0^2}\, \gamma_i (t - v_i x)\right],$$

where v_i is the velocity of pulson i in units of the velocity of light, $v_1 = -v_2 = v = 0.2,\ 0.3,\ 0.4,\ 0.6$, $\gamma_i^2 = (1 - v_i^2) = (1 - v_i^2)^{-1}$, and the function $f(r)$ in the pulson rest frame is the solution of the boundary-value problem

$$f_{rr} + f_r/r - f + f^3 = 0;\ f_r(0) = f(\infty) = 0.$$

The collision time was taken to be shorter than the time required for the development of the instability. The behavior of the pulsons recalls the behavior of one-dimensional ls solitons or Higgs kinks. If the velocity of the pulsons exceeds a certain critical value $v_{\mathrm{cr}} \approx 0.3$, they emerge from the interaction and only then "decay" in accordance with the dissipative (spreading) or singular (collapse) mode. For $v \leqslant 0.3$, the pulsons coalesce, subsequently collapsing. A remarkable fact is that for $v > v_{\mathrm{cr}}$ the number of unstable quasiparticles is conserved in the interaction, and they live about as long as in the free state. And this despite the fact that the collision of the pulsons introduces a large, albeit self-consistent, perturbation of order unity for each of them. Here, already in the three-dimensional world we have become witnesses of the manifestation of "soliton" properties (as understood by Zabusky and Kruskal) by unstable pulsons.

Cylindrically symmetric Q solitons and their interactions. Nonrelativistic models. We now turn to the study of the dynamical properties of stable Q solitons in models with the simplest internal symmetry $U(1)$. As we have already noted, stable nontopological solitons exist only in theories with internal symmetry. Naturally, the investigations were commenced with the simplest models ($U(1)$ symmetry), and then ever higher symmetries were included, some of which could be spontaneously broken.

We note here that by gradually making the models more complicated and choosing the parameters appropriately Lee and Friedberg[3] succeeded in describing with satisfactory accuracy (relative to the experimental data) properties of nucleons such as the relative magnetic moments of the proton and neutron, the ratio of the β-decay constants (axial and vector), and the mean-square electromagnetic radius of the proton.

If the static properties of solitons are amenable to analytic investigation in the framework of even complicated models, the interaction of non-one-dimensional solitons has as yet proved possible only by means of the computer. Here too it is natural to advance in the investigation from simple to complicated models.

The simplest models, whose study was the subject of several papers in 1977–1979, were nonrelativistic models described by the Schrödinger equation with various forms of nonlinearity. We have here the very impressive films of Tappert showing the interaction of cylindrical solitons in the framework of the Schrödinger equation with exponential nonlinearity,

$$i\varphi_t + \Delta_{rr}\varphi + \varphi\,(1 - \exp[-\alpha \mid \varphi \mid^2])/\alpha = 0,$$

simulating the behavior of packets of Langmuir waves near stationary states.

In the second study,[36] an investigation was made of the collision of cylindrically symmetric gaussons (in the terminology of Ref. 36) in the framework of the equation

$$i\varphi_t + (\Delta_{rr}/2 + b \ln [a^D \mid \varphi \mid^2])\,\varphi = 0, \tag{58}$$

where a and b are constants and D is the dimension of space.

This equation has an exact one-soliton solution—a gausson of the form

$$\left.\begin{aligned} &\varphi(\mathbf{x}, t) = (A/\sqrt{\pi})^{D/2} \exp\{i[-\omega t + \mathbf{v}\mathbf{x} + \vartheta_0]\} \\ &\times \exp\{-b(\mathbf{x} - \mathbf{v}t - \mathbf{x}_0)^2\}; \\ &\omega = v^2/2 + 2Db\,[1 - \ln(Aa/\sqrt{\pi})], \end{aligned}\right\} \tag{59}$$

where $\mathbf{v}$ is the velocity of the gausson and ϑ_0 is its initial phase.

In the cylindrically symmetric geometry, we obtain

$$\varphi(\mathbf{r}, t) = (A/\sqrt{\pi}) \exp\{i[-\omega t + \mathbf{v}\mathbf{r} + \vartheta_0]\} \exp\{-b(\mathbf{r} - \mathbf{v}t - \mathbf{r}_0)^2\} \tag{60}$$

and $\omega = v^2/2 + 4b[1 - \ln(Aa/\sqrt{\pi})]$. For $A = e\sqrt{\pi}/a$, we have the particular solution $\omega = v^2/2$, which was used in the investigations of Ref. 36. The procedure of the numerical calculation consisted of the formal solution of (58) with neglect of the time dependence of $\varphi(r, t)$ in the argument of the logarithm in connection with the smallness of the time step δt:

$$\begin{aligned} \varphi(x, y, t + \delta t) &= \exp\{-i\, b\delta t \ln \mid \varphi(x, y, t) \mid^2\} \\ &\times \exp(-i\,\delta t\Delta_{rr}/2)\,\varphi(x, y, t). \end{aligned}$$

Further, in each step a Fourier transformation was used, so that

$$\exp(-i\delta t\Delta_{rr}/2) \to \exp[i(k_x^2 + k_y^2)\,\delta t/2].$$

The inverse transformation yielded the function $F(x, y, t + \delta t) = \exp(-i\,\delta t\Delta_{rr}/2)\varphi(x, y, t)$, which it is sufficient to multiply by $\exp\{-i\,\delta t b \ln|\varphi(x, y, t)|^2$ to obtain the required solution on the neighboring time layer. A (128×128)-point spatial grid was used and the numerical values $a = e$ and $b = 400$ of the constants were taken, so that $|\varphi|^2 \approx 1$ and $\Delta r \approx 0.035$. The time step was $\Delta t \approx 0.002$. At the initial time two identical gaussons moving toward each other with impact parameter (relative angular momentum $l = 2vp$) equal or not equal to zero were specified. The numerical experiments showed that there exists a certain resonance region with respect to the energy (or velocity) of the colliding solitons, ΔE_{res}, in which a third gausson can be produced as a result of the collision if the difference between the initial phases of the gaussons differs from π (Fig. 12). If $\delta\vartheta_{in} = \pi$, the gaussons repel one another in the resonance region as well if $l = 0$ (Fig. 12e). However, for $l \neq 0$ and $\delta\vartheta_{in} = \pi$ we see that there is overlapping of the gaussons, and their scattering takes place as if there

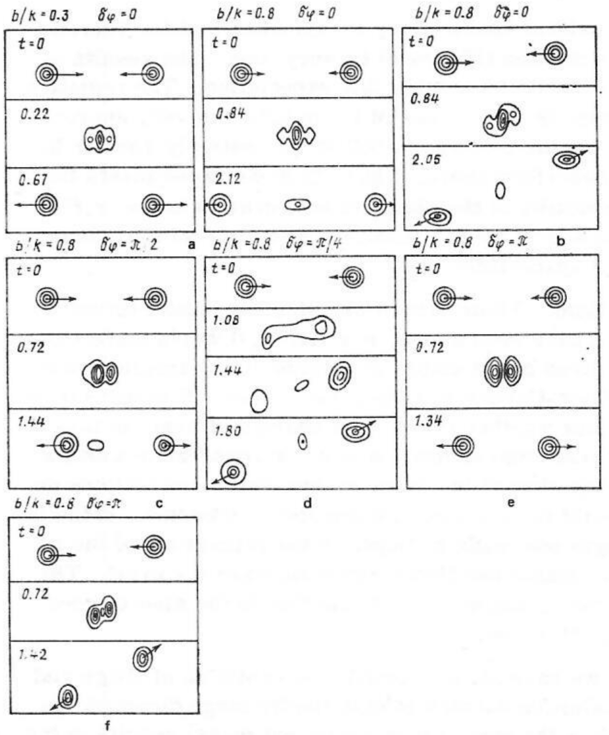


FIG. 12.

were an attractive force between them (there is even a time lag as the gaussons move apart; cf. Figs. 12e and 12f). In the figures, we have plotted the function $|\varphi|^2$, and the lines correspond to geodesics with values 0.1, 0.4, 0.7, and 1.1 of the initial height of the gausson. These results show, first, that the interactions of two-dimensional gaussons with $l = 0$ and one-dimensional gaussons are qualitatively the same. In both cases, there is a resonance region with respect to the energy in which a third gausson can be produced. Outside this region, the interaction is quasielastic and depends strongly on the phase difference of the colliding gaussons, becoming quasielastic as $\vartheta \to \pi$ even in the resonance region.

The presence of angular momentum is effectively equivalent to the appearance of an additional phase difference—as if $\delta\vartheta \neq \pi$ at the collision time. We emphasize that only the phase difference acquired by the solitons up to the collision time is important. This difference may depend on the differences between the velocities and amplitudes of the solitons. In the considered case of Schrödinger gaussons $\delta\vartheta = \delta\vartheta_{in}$; for S3 solitons $\delta\vartheta = L/2/v_1 + v_2(A_2^2 - A_1^2) + \delta\vartheta_{in}$, etc.

The question naturally arises of whether bound states of solitons are possible in the investigated model (58). Although they were not found in Ref. 36, the matter cannot be regarded as settled, since, as we have already seen [see Eq. (59)], the interaction of only "uncharged" gaussons, for which $\omega = 0$ when $v = 0$, was investigated.

In Ref. 123 there was a discussion of the more general solutions of gausson type that also exist in the framework of the Klein-Gordon equation with nonlinearity of the form (58). In accordance with Ref. 123a, besides real (stationary in the rest frame) and complex (charged) gaussons of the type (59) there can exist *stable* nonradiating gaussons of pulsating type (G pulsons). Hitherto, all the pulsons known to us, except the SG and S3 bions in the (x, t) world, were radiating, albeit weakly, quasisolitons. Although the stability of real and complex G pulsons was not rigorously proved, it was confirmed in the numerical experiments. Moreover, it was found that the radiation from an excited G pulson is very little.[123b] In this sense, the model with the nonlinearity $\varphi \ln|\varphi|^2$ is as yet unique and has features similar to the integrable [sine-Gordon in the (x, t) world] models.

In Ref. 124 the scattering of α particles by one another is investigated in the framework of time-dependent Hartree-Fock theory. Rather convincing arguments are given for reducing the three-dimensional model to a two-dimensional model described by the nonrelativistic (Schrödinger) $\varphi^4 - \alpha\varphi^6$ model:

$$i\varphi_t + \Delta_{rr}\varphi + a\varphi + b\varphi|\varphi|^2 - c\varphi|\varphi|^4 = 0, \qquad (61)$$

where a, b, and c are constants, and $\varphi(x, y, t)$ is the single-particle wave function.

The reader can find details about the physical formulation of the problem and the limits of applicability of the developed theory in Ref. 124 and the literature quoted there. Head-on collisions of two identical cylindrically symmetric quasisolitons were studied on the basis of (61) at different energies ($v_1 = -v_2 = v$) and different impact parameters p or, which is the same thing, angular momenta l. The constants a, b, and c were chosen experimentally on the basis of the binding energy and the mean-square radius of the α particle.

In the numerical experiments, it was found that there is a fairly narrow resonance region of impact parameters in which the inelasticity of the soliton interaction increases strongly, i.e., there are increases in the excitation energy $E_{exc} = K_{in} - K_{fin}$ (K_{in} and K_{fin} are the kinetic energies of the solitons before and after the collision, respectively), the delay time τ_d (the difference between the kinematic times of the interacting and noninteracting solitons), and the deflection angle ϑ.

With decreasing energy of the colliding solitons, $K_{in} \propto v^2$, the resonance region expands, and for a certain value of K_{in} bound states appear in the center of the resonance region ($\tau_d \to \infty$).

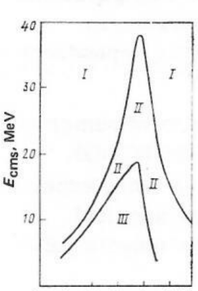


FIG. 13. I is the region of quasielastic scattering of the solitons (α particles); II is the region of strongly inelastic scattering; III is the region of formation of bound states.

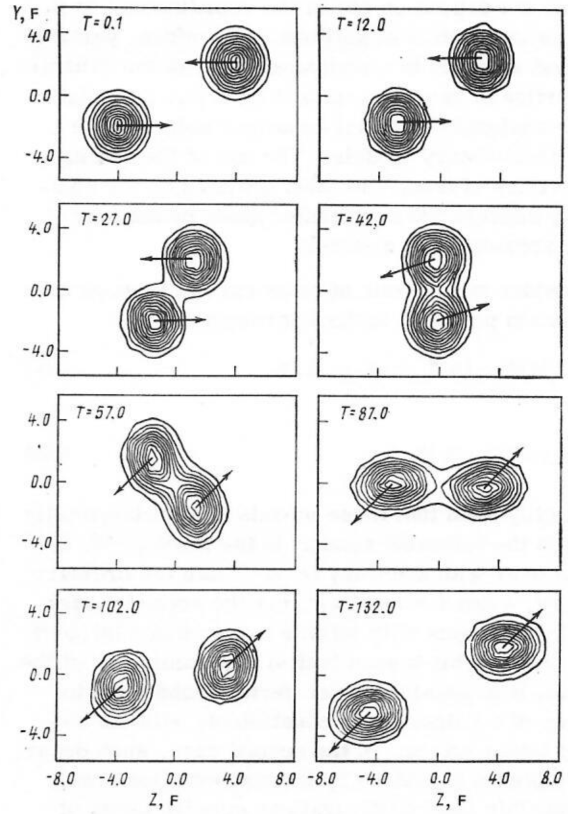


FIG. 14.

Figure 13 shows schematically the regions in which there is elastic scattering, strongly inelastic scattering, and the formation of bound states in the K_{in}, l plane. An example of weakly inelastic interaction of solitons is given in Fig. 14.

Multidimensional solitons and their interactions. Relativistic theory. Below, we discuss some properties of solitonlike solutions that from our point of view are nontrivial. Particular attention will be paid to their dynamic characteristics. We shall see that even in the framework of fairly simple models of nonlinear classical field theory such solutions can have not only nontrivial but, at the first glance, striking properties very reminiscent of those of complex real objects.

As we have already noted, one of the possibilities of obtaining stable solitonlike solutions is to introduce a certain isotopic symmetry group of the Lagrangian and the conservation laws associated with it. We discuss the properties of models with the simplest $U(1)$ group, which leads to a conservation law for the "isocharge" Q. The conservation of Q means that there are restrictions on the possible forms of perturbations, namely, $\delta Q[\varphi] = 0$, which leads to the stability condition $\omega/Q\, dQ/d\omega < 0$ for the solitonlike solutions. It is obvious that real stationary field configurations cannot satisfy this condition and will be unstable. These conclusions were confirmed earlier in numerical experiments of various groups.

At the present time, only one nonlinear relativistically invariant model is known that is integrable in three-dimensional space-time. It is described by a self-duality equation in a 4-space with metric (2, 2) when the potentials do not depend on one of the coordinates. In it, there is no interaction of solitons. Therefore, particular interest attaches to a computer study of the dynamical properties of two-dimensional (2, 1) and then also three-dimensional (3, 1) well-localized solutions for *different* field-theory models. The use of the qualitative properties of these solutions obtained by the computer may suggest ways of further study by analytic (possibly approximate) methods.

We consider two models of classical field theory with an interaction potential in the Lagrangian[30]:

$$U = -\ln(1 + |\varphi|^2) - (m^2 - 1)|\varphi|^2 \tag{62}$$

and

$$U = -|\varphi|^2 \ln(|\varphi|^2). \tag{63}$$

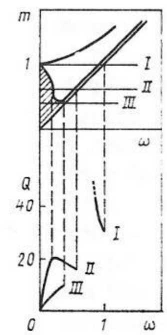
It is readily seen that these models are fundamentally different in the following sense: In the limit $\varphi \to 0$, the first goes over with accuracy $O(|\varphi|^2)$ into the ordinary free theory, since $U \approx -m^2|\varphi|^2$, but the second model contains constituents with infinite mass, since $\ln(|\varphi|^2) \to -\infty$ as $\varphi \to 0$. This means that in the framework of the first model it is possible under certain conditions to have decay of a solution into constituents with the radiation of linear waves. In the second case, such decay is impossible (is forbidden by the conservation laws), and all possible field configurations consist solely of nonlinear solutions; models of the second type are sometimes called *confining models*. As a result, in the framework of the first model the solitonlike solutions can decay, whereas in the framework of the second instability of the solitonlike solutions is manifested in the form of their collapse. The second model is also interesting in that in it solitonlike solutions can be found in explicit form for any dimension D. Moreover, it follows from the Q theorem that irrespective of D stable $U(1)$-symmetric solitonlike solutions of the form $\varphi = \psi(r)\exp(-i\omega t)$ exist for $\omega > \omega_{cr} = 2^{-1/2}$. In this sense, the model $\ln(|\varphi|^2)$ is dimensionally invariant and differs qualitatively from the model (62), for which ω_{cr} depends strongly on D and m. Figure 15 shows the regions of existence and stability of the solitonlike solutions and the corresponding dependences $Q(\omega)$ for the model (62).

Hypothesis: The nature of the interaction of solitons in collisions is determined by the dispersion dependence $Q(\omega)$ and not by the type of the model (irrespective of the nature of the instability—decay or collapse).

This hypothesis was verified in a series of numerical experiments.[125] In the calculations two parameters were varied—the velocity v of the relative motion of the quasisolitons and their charge Q. In both cases, four types of interaction were found:

1) elastic and quasielastic interaction of quasisolitons;

2) decay (collapse) of quasisolitons after interaction;



3) decay (collapse) through a short-lived bound state (resonance)[31];

4) a long-lived bound state of two quasisolitons, a bion, which indicates that in reality there is a model-independent nature of soliton interaction, at least in the framework of the considered models.

The two central types of interaction are possibly only in the region $dQ/d\omega \approx 0$. This gives grounds for assuming that such forms of interaction will also be present in models whose Lagrangians admit higher symmetry groups, provided the dependence of the corresponding "isocharge" ("isospin," etc.) has extremal points like the considered models.

A more detailed investigation of the interaction of quasisolitons showed[125d] that it also depends on the impact parameter p or, which is the same thing, the angular momentum l and the initial phase difference $\Delta\vartheta$. The numerical experiments showed that:

a) there exists a certain resonance region with respect to the angular momentum l in which the inelasticity of the interaction of the quasisolitons increases sharply (see also Ref. 124);

b) a purely antisymmetric initial field configuration, $\Delta\vartheta = \pi \pm 2\pi n$, leads to elastic repulsion of the solitons.

As we have already noted, stationary configurations of real fields cannot be stable, i.e., real quasisolitons do not exist. Moreover, it is not in all systems with internal isosymmetry and not in all cases that stable quasisolitons will exist. Such solutions can arise in systems in which the surfaces of constant energy in the functional space can have conditional or local minima. The following question arises naturally: In such systems, do there exist nonstationary stable configurations of real fields? In such a case, the nonstationarity plays the part of a stabilizing factor like the dependence $\exp(-i\omega t)$ in the case of the $U(1)$ group.[32]

This conjecture was verified in the series of numerical experiments in the framework of the model (62) made at Dubna.[125c] The results obtained in these experiments seemed at the first glance to be paradoxical. Placing, for example, unstable solitonlike objects suf-

30) For the φ_D^4 theory when $D > 1$ stable solitonlike solutions do not exist even in systems with an isogroup.

31) We recall the strange resonances in the (1, 1)-dimensional Higgs model (φ_-^4).

32) We recall here Kapitsa's problem of a pendulum with a rocking point of suspension.

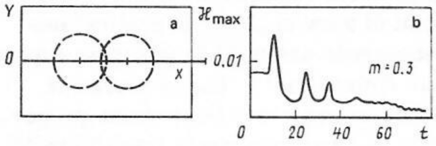


FIG. 16.

ficiently close to each other—so that the kinematic time of their interaction was shorter than the decay time of each of them (Fig. 16a)—we found that if the quasisolitons approached each other at sufficiently low velocities they formed a bound state, i.e., a two-dimensional bion. The amplitude at the center of the bion oscillated regularly, decreasing only very slightly during the calculation (of several oscillation periods: see Fig. 16b). Further study showed that similar objects can arise from a sufficiently heavy single-soliton initial state. The behavior in time and the form of the discovered bions agree qualitatively with those of the pulsons described above (Fig. 17). This showed that the existence of pulsons is not a privilege of systems with a degenerate vacuum as in the case of the Higgs and sine-Gordon field equations, in which the field function oscillates between two adjacent vacua. We note that similar pulsons also arise naturally in the framework of the system (63).

This confirmed the hypothesis that in models with a sufficiently complicated constant-energy surface that admit the existence of stable charged solitons there exist long-lived (stable) pulsons, i.e., quasiperiodic solutions.[33)]

This hypothesis enables us to consider all the results so far obtained on the interaction of quasisolitons from a common point of view. In the considered models there exist, besides the stable (and unstable) charged ($Q \neq 0$) quasisolitons, stable (and unstable) charged ($Q \neq 0$) and uncharged ($Q = 0$) pulsons. This means that for definite parameters of the quasisolitons (both charged and uncharged) the evolution of their interaction either ends with the formation of a corresponding pulson (bound state) or passes through a pulson phase (resonance) before the development of instability (decay).

From this point of view, the appearance of a stable pulson as a result of the collision of two unstable quasisolitons ceases to be paradoxical. Pulsons like those described above were found in the numerical experiments of Ref. 127 in a space of dimension (1, 1) in collisions of charged quasisolitons with different total "isocharges" ($Q = 2Q_1$, $Q_1 = Q_2$ and $Q = Q_1 + Q_2 = 0$, $Q_1 = -Q_2$). In the first case, we evidently have the formation of a charged pulson, since $Q_p \propto d\vartheta/dt \neq 0$ (see Fig. 2 in Ref. 127); in the second case, the formation of an uncharged pulson: $Q_p \to 0$ (see p. 383 of Ref. 127). The region of initial conditions for the formation of an uncharged pulson is much narrower.

Finally, pulsons were also detected in Ref. 124, but in the framework of the Schrödinger equation with nonlinearity of the form $\varphi^3 - \alpha\varphi^5$. Do non-one-dimensional

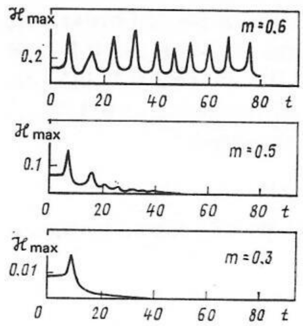


FIG. 17.

pulsons exist in the framework of the Schrödinger equation with other forms of nonlinearity? In the numerical experiments made by the Polish group[36] non-one-dimensional pulsons were not found, at least in the case of the gausson model. A possible explanation of the stability of the discovered pulsons by means of a certain adiabatic invariant can be found in Ref. 126. We note also that stable bound states formed by unstable constituents have already been long known in nuclear physics (the deuteron). As in our case, this state bears little resemblance to a bound state of two classical objects of the Earth-Moon, binary-star, etc., type. On the formation of the bound state, the constituents that make up the system lose their individuality.

A few words about the method used in the calculations. In Ref. 124, a 25×25 grid was used in the two-dimensional calculations, a difference scheme of fourth order (step equal to 0.8) was used for the integration with respect to x and y, and the integration with respect to t was made by means of a prediction-correction method of fifth order of accuracy with step $\Delta t = 0.15$. The calculations were made with an IBM 370/165. The computing time for one variant with given K_{in} and l was about 15 min. A three-dimensional calculation with 15 points in the perpendicular direction took 15 times longer, i.e., about 4 h. However, the authors of Ref. 124, referring to Flocard, assert that they can improve the technique of integration with respect to the time and reduce the computing time by almost an order of magnitude. It then becomes realistic to investigate three-dimensional interactions with the same degree of symmetry, i.e., head-on collisions of identical solitons with initial phases equal in modulus.

In Ref. 125, a single-soliton solution $\tilde{\psi}(r)$ was first found with given accuracy and then approximated by a set of Gaussian exponentials (mixed method; see above):

$$\tilde{\psi}(r) \approx \sum_i^n \alpha_i \exp\{\beta_i (r - \delta_i)^2\}.$$

Choosing $\alpha_i, \beta_i, \delta_i$, it is possible to approximate $\tilde{\psi}(r)$ for $n = 3$ in such a way that for all ω

$$\max |\tilde{\psi}(r) - \sum_i \alpha_i \exp\{\beta_i (r - \delta_i)^2\}| \leqslant 0.005 \tilde{\psi}(0).$$

By a Lorentz transformation, a moving soliton is obtained:

$$\varphi(x, y, t) = \sum_{i=1}^{3} \alpha_i \exp\{\beta_i (\sqrt{\gamma^2 (x - vt)^2 + y^2} - \delta_i)^2\} \times \exp\{-i\omega\gamma(t - vx)\}.$$

[33)]This conjecture was made earlier in Ref. 126.

The collision of these solitons was studied by means of a symmetric difference scheme of second order with respect to x, y, and t. The time step was $\Delta t=0.1$, and the step $\Delta x=\Delta y$ was chosen in the interval 0.1–0.4, depending on the parameter ω. At the points $t=0, 1, 2, \ldots$ the field energy density was calculated:

$$\mathcal{H} = |\varphi_t|^2 + |\varphi_x|^2 + |\varphi_y|^2 + \ln(1+|\varphi|^2).$$

The accuracy of the calculations was tested by means of the relation

$$dE/dt=0, \quad E=\int \mathcal{H}\, dx\, dy,$$

and the maximum of

$$\varepsilon_T = (E_0 - E_T)/E_0$$

did not exceed 0.01; here $E_0=E(t=0)$ and $E_T=E(t=T)$. The calculations were made with a BÉSM-6 computer (for more details, see Ref. 125).

CONCLUSIONS

In our exposition, the main attention has been concentrated on the mathematical rather than the physical aspects of soliton theory, and we have emphasized the priority of the computer in the discovery of the various unusual properties of solitons. This was done deliberately, since the length of the paper does not permit a sufficiently full review of the applications of this theory in the different fields of physics (and I have not set myself such a task). It is sufficient to say that a number of review papers and even books have already been published in which these applications are treated. We mention here the reviews of Rajaraman,[128] Coleman,[129] Lee,[130] Rebbi,[131] Faddeev and Korepin,[132] and Wadati, Matsumoto, and Umezawa[134] on solitons in particle physics and the reviews of Bishop,[11] Ichikawa,[94] Davydov,[60] Scott and Luzader,[60] Maki,[133] and Wadati[137] on solitons in the theory of the condensed state, plasmas, and biology. Finally, there has recently been a growth in the number of publications on the experimental confirmation of the existence of solitons, especially in the physics of the condensed state, where solitons have made it possible to predict phenomena subsequently discovered (see Ref. 135 and the popular article of Bullough[136]). We can say with confidence that at the present time this direction continues to develop successfully in both its mathematical and physical aspects.

In two-dimensional space-time new powerful methods of investigation of integrable systems based on spectral transformation and the group-theoretical, algebraic, and geometrical approaches have already been developed and are still being developed. The obtained results are very impressive.[15,25b] For nonintegrable systems, the methods of numerical investigations are constantly being improved, and an adequate soliton perturbation theory is also being steadily developed.[25c,25d]

Studies on the dynamics of non-one-dimensional solitons have only just begun, but the already existing data offer hope that in the near future we shall witness new and possibly unexpected discoveries in this field, in which numerical investigations will not yield their leading role.

We note that the obtained results are of undoubted interest from the point of view of physical applications. The discovered resonances are used to interpret experiments with nuclear molecules.[138] The existence of multidimensional pulsons may cast light on the properties of the soft mode in structural phase transitions,[139] and also a long Josephson transmission line.[140] This list could be continued.

In conclusion, we emphasize that the idea of the importance or even the necessity of investigations in this field begins to penetrate into ever wider groups of the scientific community, as can be seen from the following predictions: "...we must take into account all possible ways by which the list of elementary particles that arise from the equations of quantum field theory can be extended. Here, of course, we have ordinary bound states and Goldstone bosons and fermions—there may also be solitons (such as monopoles) and other particlelike solutions and bound states including them." And further: "in any quantum field theory we must understand the extent to which all these objects could masquerade at present energies as elementary particles,"—Gell-Mann in Ref. 141. And even terser: "...if supergravity or some similar future theory is correct, then there must be only an indirect relation between the elementary fields of the theory and the particles that appear to us today to be elementary. If the known fermions behave, for a given handedness, like a complex spinor representation of SO_{4n+2} or E_6, then the relation is not even that of a composite model. All or most of the familiar particles would have to correspond to particle-like solutions of the fundamental equations, with a different algebraic behavior from that of the fundamental fields." (Gell-Mann, Ramond, and Slansky in Ref. 142.) Similar ideas were also often put forward earlier by Faddeev.

In this connection, it is appropriate to recall the aphorism: "...nature is complicated and nonlinear equations are complicated, and therefore nature should be simulated by means of nonlinear equations...," which can be found in Ref. 143 by Wheeler. Indeed, as we have seen above, the dynamics of even simple nonlinear systems is very rich, varied, and sometimes unexpected.

1A. Scott, F. Chu, and D. McLaughlin, Proc. IEEE 61, 1443 (1973).

2V. G. Makhankov, Phys. Rep. 35, 1 (1978).

3R. Friedberg, T. D. Lee, and A. Sirlin, Phys. Rev. D 13, 2739 (1976); Nucl. Phys. B115, 1, 32 (1976); R. Friedberg and T. D. Lee, Phys. Rev. D 15, 1694 (1977); 16, 1096 (1977); Preprint CU-TP-118, Columbia Univ., New York (1978); T. D. Lee, Preprint CU-TP-127, Columbia Univ., New York (1978).

4R. Muira, SIAM Rev. 18, 412 (1976).

5V. E. Zakharov, Zh. Eksp. Teor. Fiz. 65, 219 (1973) [Sov. Phys. JETP 38, 108 (1974)].

[6]V. E. Zakharov and A. B. Shabat, Zh. Eksp. Teor. Fiz. **61**, 118 (1971) [Sov. Phys. JETP **34**, 62 (1972)].
[7]C. Cercignani, Riv. Nuovo Cimento **7**, 429 (1977).
[8]K. Spatschek, Fortschr. Phys. **24**, 687 (1976); Y. Lee and G. Morales, "Soliton-like structures in plasmas," Invited paper at the soliton Conf., Tuscon, Arizona (1976).
[9]D. ter Haar, "Non-linear propagation behavior in optical and other physical systems," Preprint 54/77, Univ. of Oxford (1978).
[10]A. Barone *et al.*, Riv. Nuovo Cimento **1**, 227 (1971).
[11]A. Bishop, Phys. Scr. **20**, 409 (1979).
[12]V. E. Zakharov, Zh. Eksp. Teor. Fiz. **62**, 1745 (1972) [Sov. Phys. JETP **35**, 908 (1972)].
[13]K. Nishikawa, H. Hojo, K. Mima, and H. Ikegi, Phys. Rev. Lett. **33**, 148 (1974).
[14]V. Makhankov, Preprint E5-8389 [in English], JINR, Dubna (1974); Phys. Lett. **A50**, 42 (1974).
[15]S. P. Novikova (ed.), Teoriya solitonov (Theory of Solitons), Nauka, Moscow (1980).
[16]P. Christiansen and O. Olsen, Phys. Lett. **A68**, 185 (1978); Phys. Scr. **20**, 531 (1979); P. Christiansen and P. Lomdahl, Preprint DCAMM No. 198, Lyngby, Denmark (1980); Physica **D2**, 482 (1981).
[17]T. Bock and M. Kruskal, Phys. Lett. **A74**, 173 (1979).
[18]P. Dodd, R. Bullough, and S. Duckworth, J. Phys. A **8**, L64 (1975).
[19]S. Duckworth *et al.*, Phys. Lett. **A57**, 19 (1976).
[20]M. Ablowitz, M. Kruskal, and J. Ladik, SIAP Reports No. 870.
[21]D. Peregrine, J. Fluid Mech. **25**, 321 (1966).
[22]T. Benjamin, J. Fluid Mech. **25**, 241 (1966); **29**, 559 (1967).
[23]H. One, J. Phys. Soc. Jpn. **39**, 1082 (1975).
[24]M. Ablowitz, Y. Kodama, and J. Satsuma, Phys. Lett. **A73**, 283 (1979); Phys. Rev. Lett. **46**, 687 (1981).
[25]Various papers in: a) R. Muira (ed.), Bäcklund Transformations. The Inverse Scattering Methods, Solitons and their Application (Lecture Notes in Mathematics), Springer, Berlin-New York (1976); b) F. Calogero (ed.), Non-Linear Evolution Equations Solvable by the Spectral Transform, Pitman, London (1978); c) A. Bishop and T. Schneider (eds.), Solitons and Condensed Matter Physics (Springer Series in Solid-State Sciences, Vol. 8) (1978); d) H. Wilhelmsson *et al.*, Solitons in Theoretical Physics. A Topical Issue of Phys. Scr., Vol. 19, (1979).
[26]See Ref. 25b, pp. 97–126.
[27]V. E. Zakharov and A. V. Mikhaĭlov, Pis'ma Zh. Eksp. Teor. Fiz. **27**, 47 (1978) [JETP Lett. **27**, 42 (1978)].
[28]A. V. Mikhaĭlov, Pis'ma Zh. Eksp. Teor. Fiz. **23**, 356 (1976) [JETP Lett. **23**, 320 (1976)].
[29]K. Polmayer, Commun. Math. Phys. **46**, 207 (1976); F. Lund and T. Regge, Phys. Rev. D **14**, 1524 (1976).
[30]F. Lund, Phys. Rev. Lett. **38**, 1175 (1977); P. P. Kulish, Teor. Mat. Fiz. **33**, 272 (1977); **26**, 198 (1976).
[31]J. Eilbeck, see Ref. 25c, pp. 28–43.
[32]R. Richtmyer and K. Morton, Difference Methods for Initial Value Problems, Interscience, London (1967); A. Mitchell, Computational Methods in Partial Differential Equations, Wiley, London (1969); W. Ames, Numerical Methods for Partial Differential Equations, Nelson, London (1977); D. Jacobs *et al.*, The State of the Art in Numerical Analysis, Academic Press, London (1977); A. A. Samarskiĭ, Vvedenie v teoriyu raznostnykh skhem (Introduction to the Theory of Difference Schemes), Nauka, Moscow (1971); A. A. Samarskiĭ and A. V. Gulin, Ustoĭchivost' raznostnykh skhem (Stability of Difference Schemes), Nauka, Moscow (1973).
[33]J. Cooley, P. Lewis, and P. Welch, IEE Trans. Educ. E-12, 27 (1969); J. Cooley and J. Tukey, Math. Comp. **19**, 297 (1965).
[34]N. Pereira, R. Sudan and J. Denavit, Phys. Fluids **20**, 271, 936 (1977).
[35]B. Fornberg and G. Whitham, Philos. Trans. R. Soc. London **289**, 373 (1978).
[36]J. Oficjalski and I. Bialynicki-Birula, Acta Phys. Pol. **B9**, 759 (1978).
[37]N. Zabusky and M. Kruskal, Phys. Rev. Lett. **15**, 240 (1965).
[38]Kh. Abdulloev, I. Bogolubsky, and V. Makhankov, Phys. Lett. **A56**, 427 (1976).
[39]T. Benjamin, J. Bona, and J. Mahony, Philos. Trans. R. Soc. London **272**, 47 (1972).
[40]J. Eilbeck and G. McGuire, J. Comp. Phys. **19**, 43 (1975); **23**, 63 (1977).
[41]J. Hammack, J. Fluid Mech. **60**, 769 (1973).
[42]R. Hirota, J. Math. Phys. **14**, 810 (1973).
[43]J. Boussinesq, Mem. Acad. Sci. (Paris) **23**, 1 (1877).
[44]I. Bogolubsky, Comp. Phys. Commun. **13**, 149 (1977).
[45]T. Ogino and S. Takeda, J. Phys. Soc. Jpn. **39**, 1365 (1975).
[46]L. M. Degtyarev, V. G. Makhan'kov, and L. I. Rudakov, Zh. Eksp. Teor. Fiz. **67**, 533 (1974) [Sov. Phys. JETP **40**, 264 (1974)].
[47]J. Bona, W. Pritchard, and L. Scott, Preprint No. 94, Fluid Mechanics Research Institute, Univ. of Essex (1978).
[48]A. Santarelli, Nuovo Cimento **B46**, 179 (1978).
[49]C. Karney, A. Sen, and F. Chu, Preprint PPPL-1455, Plasma Physics Laboratory, Princeton Univ. (1978).
[50]C. Karney, A. Sen, and F. Chu, Preprint PPPL-1452, Plasma Physics Laboratory, Princeton Univ. (1978).
[51]R. Hirota, J. Phys. Soc. Jpn. **33**, 1456 (1972).
[52]J. Meiss and N. Pereira, Phys. Fluids **21**, 700 (1978).
[53]R. Joseph, J. Math. Phys. **18**, 2251 (1977).
[54]N. Yajima and A. Outi, Prog. Theor. Phys. **45**, 1997 (1971).
[55]V. E. Zakharov and A. B. Shabat, Zh. Eksp. Teor. Fiz. **64**, 1627 (1973) [Sov. Phys. JETP **37**, 823 (1973)].
[56]Kh. Abdulloev, I. Bogolubsky, and V. Makhankov, Phys. Lett. **A48**, 161 (1974).
[57]J. Satsuma and N. Yajima, Suppl. Prog. Theor. Phys. **55**, 284 (1974).
[58]H. Chen, Y. Lee, and N. Pereira, Phys. Fluids **22**, 187 (1979).
[59]V. Fedyanin and V. Makhankov, Phys. Scr. **20**, 552 (1979).
[60]A. Davydov, Phys. Scr. **20**, 387 (1979); A. Scott and S. Luzader, Phys. Scr. **20**, 395 (1979).
[61]Kh. Abdulloev, I. Bogolubsky, and V. Makhankov, Nucl. Fusion **15**, 21 (1975).
[62]G. L. Lamb, Rev. Mod. Phys. **43**, 99 (1971).
[63]N. Yajima and M. Oikawa, Prog. Theor. Phys. **56**, 1719 (1976).
[64]S. Thornhill and D. ter Haar, Phys. Rep. **C43**, 43 (1978).
[65]N. Yajima *et al.*, Res. Inst. Appl. Phys. Reports 22, **70**, 89 (1975); Preprint IPPJ-319, Nagoya Univ. (1978).
[66]J. Denavit and W. Kruer, Phys. Fluids **14**, 1782 (1971); A. Langdon, Preprint UCRL-80446 (1977); UCRL-80666 (1978); Phys. Fluids **22**, 163 (1979).
[67]E. Valeo and W. Kruer, Phys. Rev. Lett. **33**, 750 (1974); Y. Lee and G. Morales, Invited paper at the Soliton Conf., Tucson, Arizona (1976) and references therein.
[68]I. Bogolubsky and V. Makhankov, Preprint E4-9426 [in English], JINR, Dubna (1975); Fiz. Plazmy **2**, 974 (1976).
[69]S. V. Manakov, Zh. Eksp. Teor. Fiz. **67**, 543 (1976) [Sov. Phys. JETP **40**, 269 (1976)].
[70]A. V. Mikhaĭlov, Pis'ma Zh. Eksp. Teor. Fiz. **30**, 443 (1979) [JETP Lett. **30**, 414 (1979)].
[71]R. Sagdeev and A. Galeev, Non-Linear Plasma Theory, Benjamin, New York (1969), p. 92.
[72]C. Montes, Plasma Physics, Non-Linear Theory and Experiments, Plenum, New York (1977), p. 222; Astrophys. J. **216**, 329 (1977).
[73]V. Makhankov and B. Shchinov, Comp. Phys. Commun. **4**, 327 (1972); Communication R9-6432, JINR, Dubna (1972).
[74]C. Montes, J. Peyraund, and M. Hénon, Preprint LPMC, Nice (1978); Phys. Fluids **22**, 176 (1979).
[75]J. Fernandes and G. Reinisch, Physica **A91**, 393 (1978).
[76]J. Fernandes *et al.*, Phys. Lett. **A66**, 175 (1978).

[77]A. Seeger, H. Donth, and A. Kochendörfer, Z. Phys. **B134**, 173 (1953).
[78]J. K. Perring and T. H. R. Skyrme, Nucl. Phys. **31**, 550 (1962).
[79]A. E. Kudryavtsev, Pis'ma Zh. Eksp. Teor. Fiz. **22**, 178 (1975) [JETP Lett. **22**, 82 (1975)].
[80]B. Getmanov, Preprint R2-10208 [in Russian], JINR, Dubna (1976); Phys. Lett. **B66**, 39 (1977).
[81]D. McLaughlin and A. Scott, J. Math. Phys. **14**, 1817 (1973).
[82]P. Caudrey, J. Eilbeck, and J. Gibbon, Nuovo Cimento **B25**, 497 (1975).
[83]M. Ablowitz *et al.*, Phys. Rev. Lett. **30**, 1262 (1973); L. D. Faddeev and L. A. Takhtadzhan, Teor. Mat. Fiz. **21**, 160 (1974).
[84]P. Dodd and R. Bullough, Proc. R. Soc. London Ser. A **351**, 499; **352**, 481 (1976).
[85]a) M. Fogel *et al.*, Phys. Rev. B **15**, 1578 (1977); b) M. Rice *et al.*, Phys. Rev. Lett. **36**, 423 (1976).
[86]a) D. Kraup, SIAM J. Appl. Math. **31**, 121 (1976); b) D. Kaup and A. Newell, Proc. R. Soc. London Ser. A **361**, 413 (1978); c) D. Kaup and A. Newell, "Prediction of a non-linear oscillating dipolar excitation in one-dimensional condensates," Preprint, Clarkson College of Technology (1978).
[87]a) T. Schneider and E. Stoll, Phys. Rev. Lett. **31**, 1254 (1973); Phys. Rev. B **17**, 1302 (1978); b) E. Stoll, T. Schneider, and A. Bishop, "Evidence for breather excitation in the sine-Gordon chain," Preprint, IBM Research Laboratory, PACS No. 0340 (1978).
[88]J. Krumhansl and J. Schrieffer, Phys. Rev. B **11**, 3535 (1975).
[89]E. Laedke and K. H. Spatschek, Phys. Rev. Lett. **41**, 1798 (1978).
[90]E. Ott and R. Sudan, Phys. Fluids **12**, 2388 (1969); **13**, 1432 (1970).
[91]J. Keener and D. McLaughlin, Phys. Rev. A **16**, 777 (1977).
[92]M. Fogel *et al.*, Phys. Rev. Lett. **36**, 1411 (1976).
[93]A. Bishop, "Solitons and physical perturbation," Preprint, Queen Mary College (1977); in: Solitons in Action (Proc. of a Workshop, Redstone Arsenal, 1977, eds. K. Lonngren and A. Scott), Academic Press, New York (1978) [Russian translation published by Mir, Moscow (1981), p. 72].
[94]Y. Ichikawa, Phys. Scr. **20**, 296 (1979).
[95]V. I. Karpman and E. Maslov, Zh. Eksp. Teor. Fiz. **73**, 537 (1977); **75**, 504 (1978) [Sov. Phys. JETP **46**, 281 (1977); **48**, 252 (1978)]; Phys. Lett. **A60**, 307 (1977); **61**, 355 (1977).
[96]A. Bondeson, M. Lisak, and D. Anderson, Phys. Scr. **20**, 343 (1979).
[97]B. Cohen and K. Watson, Phys. Fluids **19**, 345 (1976).
[98]V. Arnol'd, Sov. Math. Surveys **18**, 9 (1963); J. Moser, "Nearly integrable and integrable systems," Preprint of a lecture at the Eastern Theoretical Physics Conf., Rochester, New York (1976); Lectures on Celestial Mechanics, Springer, New York (1971) (with C. Seigel).
[99]J. Berryman, Phys. Fluids **19**, 771 (1976).
[100]L. Zastavenko, J. Appl. Math. Mech. **29**, 497 (1965).
[101]V. Katyshev, N. Makhaldiani, and V. Makhankov, Phys. Lett. **A66**, 456 (1978).
[102]N. Yajima, Prog. Theor. Phys. **52**, 1066 (1974).
[103]V. E. Zakharov and A. M. Rubenchik, Zh. Eksp. Teor. Fiz. **65**, 997 (1973) [Sov. Phys. JETP **38**, 494 (1974)]; L. M. Degtyarev, V. E. Zakharov, and L. I. Rudakov, Zh. Eksp. Teor. Fiz. **68**, 115 (1975) [Sov. Phys. JETP **41**, 57 (1975)]; G. Schmidt, Phys. Rev. Lett. **34**, 724 (1975).
[104]N. Pereira, A. Sen, and A. Bers, Phys. Fluids **21**, 117 (1978).
[105]A. Hasegawa and F. Tappert, Appl. Phys. Lett. **23**, 142 (1973).
[106]N. Pereira, Phys. Fluids **20**, 1735 (1977).
[107]S. Watanabe, J. Phys. Soc. Jpn. **45**, 276 (1978).
[108]J. Curie *et al.*, Phys. Rev. B **15**, 5567 (1977).
[109]K. Nakajima *et al.*, J. Appl. Phys. **45**, 4095 (1974); K. Nakajima, Y. Sawada, and Y. Onodera, J. Appl. Phys. **46**, 5272 (1975).
[110]A. Sen *et al.*, Report PLE-PRR 77-16, MIT (1977).
[111]D. Benney and G. Roskes, Stud. Appl. Math. **48**, 377 (1969); N. Freeman and A. Darey, Proc. R. Soc. London Ser. A **344**, 427 (1975).
[112]D. Anker and N. Freeman, Proc. R. Soc. London Ser. A **360**, 529 (1978).
[113]R. Hobart, Proc. Phys. Soc. **82**, 201 (1963); **85**, 610 (1965).
[114]G. Derrick, J. Math. Phys. **5**, 1252 (1964).
[115]F. Calogero, See the Introduction in Ref. 25b.
[116]W. Bardeen *et al.*, Phys. Rev. D **11**, 109 (1975).
[117]N. Voronov, I. Kobzarev, and N. Konyukhova, Pis'ma Zh. Eksp. Teor. Fiz. **22**, 590 (1975) [JETP Lett. **22**, 290 (1975)].
[118]I. L. Bogolyubskiĭ and V. G. Makhan'kov, Pis'ma Zh. Eksp. Teor. Fiz. **24**, 15 (1976) [JETP Lett. **24**, 12 (1976)].
[119]I. L. Bogolyubskiĭ and V. G. Makhan'kov, Pis'ma Zh. Eksp. Teor. Fiz. **25**, 120 (1977) [JETP Lett. **25**, 107 (1977)].
[120]T. Belova *et al.*, Zh. Eksp. Teor. Fiz. **73**, 1611 (1977) [Sov. Phys. JETP **46**, 846 (1977)].
[121]I. L. Bogolubsky, Phys. Lett. **A61**, 205 (1977).
[122]I. L. Bogolubsky, V. G. Makhankov, and A. B. Shvachka, Phys. Lett. **A63**, 225 (1977).
[123]a) G. Marques and I. Ventura, Preprint IFUSP P-83 (1976); b) I. L. Bogolyubskiĭ, Zh. Eksp. Teor. Fiz. **76**, 422 (1979) [Sov. Phys. JETP **49**, 213 (1979)].
[124]Sandhya Devi, M. Strayer, and J. Irvine, "TDHF calculation for nuclear collisions: model studies of alpha-alpha scattering," Preprint DL/NSF/D83, Daresbury Laboratory, (1978).
[125]a) V. Makhankov, G. Kummer, and A. Shvachka, "Interaction of non-one-dimensional classical Q solitons," Preprints E-2-11579 [in English], R2-11913 [in Russian], JINR, Dubna; Phys. Scr. **20**, 454 (1979); Phys. Lett. **A70**, 171 (1979); b) V. G. Makhan'kov and A. B. Shvachka, Preprint R2-13041 [in Russian], JINR, Dubna (1980); Physica **D3**, 396 (1981); c) V. G. Makhan'kov, G. Kummer, and A. B. Shvachka, Preprint R2-13042 [in Russian], JINR, Dubna (1980); Physica **D3**, 344 (1981); d) V. G. Makhan'kov, I. L. Bogolyubskiĭ, G. Kummer, and A. B. Shvachka, Preprint R2-80-367 [in Russian], JINR, Dubna (1980); Phys. Scr. **23**, 767 (1981).
[126]S. V. Manakov, Pis'ma Zh. Eksp. Teor. Fiz. **25**, 589 (1977) [JETP Lett. **25**, 553 (1977)]; V. Makhankov, Phys. Scr. **20**, 558 (1979).
[127]Yu. A. Simonov and J. A. Tjon, Phys. Lett. **B85**, 380 (1979).
[128]R. Rajaraman, Phys. Rep. **C21**, 227 (1975).
[129]S. Coleman, "Classical lumps and their quantum descendants," in: New Phenomena in Subnuclear Physics (ed. A. Zichichi), Plenum, New York (1977).
[130]T. D. Lee, Phys. Scr. **20**, 440 (1979); Phys. Rep. **C23**, 254 (1970).
[131]C. Rebbi, "Solitons in particle physics," Preprint BNL-24920, Upton, New York (1978); Sci. Am. **240**, 92 (1979).
[132]L. D. Faddeev and V. E. Korepin, Phys. Rep. **C42**, 1 (1978).
[133]K. Maki, Physica **B90**, 84 (1977).
[134]M. Wadati, H. Matsumoto, and H. Umezawa, Phys. Rev. B **18**, 4077 (1978); "Extended objects created by Goldstone bosons," Phys. Rev. D **18**, 520 (1978); Phys. Lett. **B73**, 448 (1978).
[135]H. Mikeska, J. Phys. C **11**, L. 29 (1978); J. Kjems and M. Steiner, Phys. Rev. Lett. **41**, 1137 (1978).
[136]R. Bullough, Phys. Bull. 78 (1978).
[137]M. Wadati, Phys. Rep. **C50**, 87 (1979).
[138]D. A. Bromely, "Nuclear molecules," Sci. Am. **239**, No. 6, 58 (1978).
[139]T. Schneider and E. Stoll, Phys. Rev. B **13**, 1216 (1976); Ferroelectrics **24**, 67 (1980).
[140]P. Christiansen, P. Lomdahl, and N. Zabusky, Preprint DCAMM; Lyngby, Denmark (1981); Appl. Phys. Lett. **39**, 170 (1981).
[141]M. Gell-Mann, Preprint CERN TH. 2855, Geneva (1980).
[142]M. Gell-Mann, P. Ramond, and R. Slansky, in: Supergravity

(Eds. P. Van Neiuwenhuizen and D. Freedman), North-Holland, Amsterdam (1979).
[143]J. A. Wheeler, Geometrodynamics, Academic Press, New York (1962) [Russian translation published by IL, Moscow (1963)].

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