Analysis of hard inclusive processes in quantum chromodynamics

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The two simplest inclusive processes, namely, e^+e^- annihilation into hadrons and deep inelastic lepton-hadron scattering, are used to illustrate one of the approaches to the investigation of hard processes in QCD based on systematic analysis of the asymptotic behavior of the Feynman diagrams in the α representation.

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INTRODUCTION

Hard processes, i.e., scattering processes in which not only the energy of the colliding particles but also the momentum transferred to one or several particles detected in the final state is large, are the main source of information about the structure of strongly interacting elementary particles (hadrons) and their interactions at distances short compared with the hadron radius. However, the production of particles with large (compared with the characteristic hadronic scale of order 1 GeV) momentum transfer is a comparatively rare event and is accompanied in the majority of cases by the production of a large number of particles with small momentum transfer due to peripheral interactions of the hadrons. In such a situation, it is very convenient to use the inclusive description,[1] when in the final state one separates a few particles (generally, not more than two) with large momentum transfer and the remaining particles (usually denoted by X) are not detected; thus, there is an automatic summation with respect to their characteristics.

Among inclusive processes, an important part is played by deep inelastic lepton-hadron processes such as $e^+e^- \to X$ (e^+e^- annihilation into hadrons), $eh \to eX$, $\mu h \to \mu X$, $\nu h \to \nu X$ (deep inelastic e, μ, and ν scattering), $h_1h_2 \to \mu^+\mu^- X$, $h_1h_2 \to e^+e^- X$ (production of a massive lepton pair), $e^+e^- \to hX$, etc. According to modern ideas, leptons are point particles, and in the processes we have just listed they can be used, as it were, to probe the internal structure of extended particles, i.e., hadrons. As early as the sixties, the possibility of pointlike behavior of the structure functions of deep inelastic scattering was pointed out.[2] The experimental discovery of such behavior (scale invariance[3] or scaling[4]) led to the creation of the parton model[5] (see also Ref. 6), in which hadrons are regarded as systems consisting of pointlike objects, or partons.

It was also established that the asymptotic behavior of the structure functions of deep inelastic scattering is uniquely related to the singularities of the commutators of the corresponding lepton currents on the light cone.[7] The detailed analysis of the singularities on the light cone made in Ref. 8 showed that scaling (or self-similar[3]) behavior of the structure functions of deep inelastic scattering does not contradict the fundamental principles of quantum field theory. However, attempts to obtain scaling behavior of the structure functions of deep inelastic scattering in the framework of concrete renormalizable field-theoretical models were not crowned with success. It was established, in particular, that in the higher orders of perturbation theory logarithmic breaking of scaling is characteristic for such models.[9] Methods were developed for summing the logarithmic contributions on the basis of various approaches to the analysis of the asymptotic behavior of the Feynman diagrams, and it was shown that, depending on the behavior of the effective coupling constant at large momentum transfers, logarithmic or power-law deviations from scaling are ultimately possible.[10,11] An approach was subsequently developed[12] that could readily produce the results obtained earlier by direct summation and could also give new results in situations when direct summation methods are ineffective. In the approach of Ref. 12, wide use is made of powerful methods such as operator expansions,[13,14] the renormalization group,[15-19] and dispersion relations.[20]

The discovery of asymptotic freedom in non-Abelian gauge theories[21] opened the way to the modern understanding of the nature of scaling in deep inelastic scattering and led to the creation of quantum chromodynamics (QCD),[22] which is regarded as the most probable candidate for the fundamental theory of strong interactions. The cornerstone of QCD is the hypothesis that hadrons are constructed from quarks[23] that have an additional degree of freedom, color[24-26] (see also the review of Ref. 27), and interact with one another through the non-Abelian gauge field of the gluons.[1)] It is assumed (but has not yet been proved) that the quarks, gluons, and other colored objects do not exist in the free state and that hadrons are colorless bound states.

The smallness of the effective coupling constant $\alpha_s(Q)$ in QCD at large Q offers hope that perturbation theory is applicable for calculating processes involving interactions at short distances. However, in any scattering process we are concerned not only with short but also large distances, i.e., besides large Q^2 there are always also small momentum invariants p^2 (for example, the masses of the hadrons, the mean "virtualities" of the quarks, gluons, etc.). In perturbation theory,

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[1)] In QCD, color symmetry is assumed to be exact. As a result of this, the quarks in QCD have a fractional electric charge that depends on their flavor (u, d, s, c, b, . . .) but not on their color. Recently, there has been much discussion of integral-charge models with spontaneously broken color symmetry.[28] The methods presented in the present review for analyzing hard processes can also be used in such models, but we restrict the group of considered problems to quantum chromodynamics.

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this is manifested in the occurrence of large (as $Q^2 \to \infty$) logarithmic contributions of the type $(g^2 \ln Q^2/p^2)^N$, whose presence indicates that direct application of perturbation theory is meaningless. However, for the form factors of deep inelastic scattering the existence of operator expansions on the light cone guarantees the possibility of separating the contributions of large (~$1/p$) and small (~$1/Q$) distances:

$$W(Q^2, p^2) = w(Q^2/\mu^2, \alpha_s(\mu)) \otimes f(\mu^2, p^2)\{1 + O(1/Q^2)\}. \quad (1)$$

The scale $1/\mu$ in Eq. (1) is the boundary between the large and small distances. Taking it in the region where the effective coupling constant $\alpha_s(\mu)$ is small, we can calculate $w(Q^2/\mu^2, \alpha_s(\mu))$ perturbatively. The functions $f(\mu^2, p^2)$ accumulate information about the dynamics at large distances, and therefore they cannot be calculated perturbatively in QCD. Therefore, the function $W(Q^2, p^2)$ itself cannot yet be calculated in QCD, but perturbation theory can be used to find the law of variation of $W(Q^2, p^2)$ with increasing Q^2.

The methods based on operator expansions do not apply, however, to more complicated hard processes such as $h_1 h_2 \to \mu^+\mu^- X$. Nevertheless, investigation of this process in the framework of a number of simple field-theoretical models[29,30] has shown that in this case too the contributions of the large and small distances factorize asymptotically. The assumption of such factorization in QCD was initially supported by the results of calculations of the lowest radiative corrections,[30,31] and was then proved[32-36] in all perturbation orders. Factorization of the contributions of the large and small distances is now the cornerstone of all perturbative calculations in QCD.

In the present review, analysis of the simplest characteristics of two processes—the total cross section of e^+e^- annihilation into hadrons and the structure functions of deep inelastic lepton-hadron scattering—is used to illustrate one of the approaches to factorization; it was developed in Ref. 33 (a more elementary exposition is given in Refs. 32 and 37). Since the approach based on operator expansions can also be used to analyse these characteristics, we have the possibility to compare the two approaches and, in particular, to give a diagram interpretation of operator expansions.

In the review, we also discuss a problem that has recently attracted much attention (see Refs. 34–45), namely, the problem of the nonuniqueness of the results of perturbative calculations in QCD and the connection between field-theoretical approaches and the parton model. A separate section is devoted to the factorization of the contributions of the small and large distances in non-Abelian gauge theories, of which QCD is one.

Concentrating our attention on the fundamental aspects of the analysis of hard processes in QCD, we have not had a possibility to pay attention to more practical aspects of the application of QCD to the analysis of specific processes (the technique for calculating Feynman diagrams, comparison of the predictions of QCD with experimental data, etc.). Readers interested in these questions are recommended to the reviews of Refs. 46–49.

As we have already emphasized, methods based on factorization of the contributions of large and small distances are particularly valuable in that they also work in the cases when operator expansions fail. The corresponding processes (such as $h_1 h_2 \to \mu^+\mu^- X$, $h_1 h_2 \to h_3 X$, etc.) are currently the subject of intense theoretical investigation (see, for example, Refs. 50–52), and many important results (large radiative corrections in the $h_1 h_2 \to h_3 X$ process,[52] noncanceling of some infrared contributions in the $h_1 h_2 \to \mu^+\mu^- X$ process,[53] for example) have recently been obtained. We therefore believe that the results of the analysis of such processes warrant a separate exposition.

1. ANALYSIS OF THE ASYMPTOTIC BEHAVIOR OF THE FEYNMAN DIAGRAMS

α Representation. Our approach to hard processes in QCD is based on analysis of the asymptotic behavior of the perturbation-theory diagrams that contribute to the characteristics of the processes in which we are interested. There are several methods of analysis of the asymptotic behavior of Feynman diagrams.[10,55-58] From our point of view, the most convenient one for the analysis of the general structure of the contributions is the parametric representation[59,60] based on the simple formula

$$\frac{1}{m_\sigma^2 - k^2 - i\varepsilon} = i\int_0^\infty d\alpha_\sigma \exp\{i\alpha_\sigma(k^2 - m_\sigma^2 + i\varepsilon)\}. \quad (2)$$

Applying it to all propagators of a diagram and calculating in the standard manner the resulting Gaussian integrals with respect to k (see, for example, Ref. 17), we obtain for the investigated contribution $T(p_1, \ldots, p_n; m_\sigma)$ the required α representation:

$$T(p_1, \ldots, p_n; m_\sigma) = \frac{P(\text{c.c.})}{(4\pi)^{2d}} \int_0^\infty \prod_\sigma d\alpha_\sigma D^{-d/2}(\alpha) G(\alpha_\sigma; p_1, \ldots, p_n; m_\sigma) \times \exp\left\{iQ(\alpha; p_1, \ldots, p_n)/D(\alpha) - i\sum_\sigma \alpha_\sigma(m_\sigma^2 - i\varepsilon)\right\}, \quad (3)$$

where d is the number of space-time dimensions, $p_1, \ldots, p_n$ are the momenta corresponding to the external lines of the diagram, m_σ is the mass of the particle corresponding to the line σ, $P(\text{c.c.})$ is the product of coupling constants (it is assumed that the amplitude T has "natural" normalization, in which the tree diagrams do not have factors of the type π or 2π), and D, Q, G are functions uniquely related to the structure of the considered diagram. In particular, $D(\alpha)$ is given by the sum

$$D(\alpha) = \sum_{X_i} \alpha_{i_1} \ldots \alpha_{i_z} \quad (4)$$

over all possible sets $X_i \equiv \{\sigma_{i_1}, \ldots, \sigma_{i_z}\}$ having the property that after removal of the lines $\sigma_{i_1}, \ldots, \sigma_{i_z}$ from the original diagram $\mathcal{G}$ it, i.e., $\mathcal{G}$, remaining connected, is transformed into some tree diagram $\mathcal{G}_i$ (we recall that a tree diagram is one that does not contain closed cycles). It is obvious that the number of lines in each such set X_i is equal to z, the number of independent cycles of $\mathcal{G}$. The function $Q(\alpha; \{p_k\})$ has a similar but more complicated structure:

$$Q(\alpha; \{p_k\}) = \sum_{\tilde{X}_j} \alpha_{j_1} \ldots \alpha_{j_{z+1}} (p_{k_1} + \ldots + p_{k_j})^2, \quad (5)$$

where $Y_j \equiv \{\sigma_{j_1}, \ldots, \sigma_{j_{s+1}}\}$ is some set of lines $\{\sigma_{j_\nu}\}$ having the property that after removal of Y_j from the original diagram $\mathcal{G}$ it is transformed into a diagram $\mathcal{G}_j$ consisting of two connected tree components, one of which contains the external momenta $p_{k_1}, \ldots, p_{k_l}$ (we recall that by virtue of the momentum conservation law $(p_{k_1} + \ldots + p_{k_l})^2 = (p_{k_{l+1}} + \cdots + p_{k_n})^2$, where $p_{k_{l+1}}, \ldots, p_{k_n}$ are the momenta in the second component, and therefore Eq. (5) defines the function $Q(\alpha; \{p\})$ uniquely). The rules (4) and (5) were established in Refs. 60–66. Similar rules hold for the pre-exponential factor $G(\alpha; \{p\}; m)$ due to the numerators of the propagators and vertex factors of the type γ_μ and k_μ (see Refs. 67 and 68).[2)] Note that after calculation of the traces $G(\alpha; \{p\}; m)$ is a polynomial in $(p_i p_j)$ and m^2 with coefficients that depend on the α parameters.

Ultraviolet Divergences. The integrals in (3) over some of the regions of the α space, in which the function $D(\alpha)$ vanishes, can in fact diverge, which corresponds to the well-known ultraviolet divergences. To eliminate them, it is necessary to make a preliminary regularization of the original expression. In the case of analysis in the α representation, it is most convenient to use dimensional regularization,[70] which is based on transition to $d = 4 - 2\varepsilon$:

$$d^4k \to d^{4-2\varepsilon}k\,(\mu_R^2)^\varepsilon, \tag{6}$$

where μ_R is a constant with the dimensions of mass. As a result of the substitution (6), the integrand for the regularized amplitude T_ε acquires with (3) the additional factor

$$(4\pi\mu_R^2)^{-\varepsilon} D^\varepsilon(\alpha). \tag{7}$$

and the logarithmic divergences are transformed into poles with respect to ε that can be removed by means of the R operation[71] (see also Refs. 17, 69, and 72), i.e., by the introduction of appropriate counterterms in the original Lagrangian.

The "Physical" Meaning of the α Parameters. In accordance with the basic formula given by Eq. (2), the parameter α_σ is Fourier dual to the virtuality of the line σ. In this sense, small α_σ correspond to a large virtuality $p^2 - m_\sigma^2 \sim 1/\alpha_\sigma$, and the limit $\alpha_\sigma \to \infty$ to zero virtuality, i.e., to the situation when the particle is on the mass shell. We note that since (by virtue of the uncertainty relation) particles can be in a virtual state for a bounded time $\Delta t \sim 1/(p^2 - m^2)$ but in the free state, when $p^2 = m^2$, for an unbounded time, the parameter α_σ can be interpreted as the time interval (or distance) between the ends of the line σ. As we shall see later, there is much truth in this analogy, but it need not be taken literally; in particular, two lines σ_1 and σ_2 with coincident ends may correspond to entirely different values of the α parameters, i.e., to different time intervals (or distances) between these ends. In this respect, it is more consistent to interpret the α parameter as the resistance of a part of an electric circuit associated with the considered Feynman diagram (see, for example, §127 in Ref. 73). Then $\alpha_\sigma = 0$ is analogous to short circuiting of the line σ, and $\alpha_\sigma = \infty$ to an infinite resistance, i.e., to removal of the line σ from the diagram. This analogy is justified, in particular, by the fact that for $\alpha_\sigma = 0$ the functions $D(\alpha)$ and $Q(\alpha; p)$ go over into the functions $\tilde{D}(\alpha)$ and $\tilde{Q}(\alpha; p)$ corresponding to the diagram $\tilde{\mathcal{G}}$ obtained from the original $\mathcal{G}$ by collapsing the line σ into a point. Similarly, proceeding from (4) and (5), we can show that $Q(\alpha; p)/D(\alpha)$ in the limit $\alpha_\sigma \to \infty$ goes over into the function $\tilde{Q}(\alpha; p)/\tilde{D}(\alpha)$ constructed for the diagram $\tilde{\mathcal{G}}$ obtained from $\mathcal{G}$ by removing the line σ.

General Analysis of the Contributions. When analyzing hard processes, we are dealing with a kinematic situation in which the scalar products $(p_i p_j)$ of the external momenta can be divided into two groups. The first contains "large" invariants $(p_i p_j) \sim Q^2$, and the second "small" $(p_k p_l) \sim p^2 \ll Q^2$. In this case, the expression (3) can be written in the form

$$T(Q^2, p^2) = R_\varepsilon \left\{ \frac{P(\text{c.c.})}{(4\pi)^{2s}} (4\pi\mu_R^2)^\varepsilon \int_0^\infty \prod_\sigma d\alpha_\sigma D^{-2+\varepsilon}(\alpha) \right.$$
$$\left. \times \exp\left\{ iQ^2 \sum_{ij} \omega_{ij} \frac{A_{ij}(\alpha)}{D(\alpha)} + p^2 I\left(\alpha, \frac{m^2}{p^2}\right) \right\} \left(\sum_{n=0} (Q^2)^n g_n(\alpha, p^2, m) \right) \right\}, \tag{8}$$

where $\omega_{ij} = 2(p_i p_j)/Q^2$, and R_ε is the R operation that removes the poles with respect to ε. Note that by construction $D(\alpha) \geq 0$ and $A_{ij}(\alpha) \geq 0$ [see (4) and (5)]. As a rule, we shall denote the coefficient of Q^2 in the exponential of the expression (8) by $F(\alpha, \omega)$.

At large Q^2, the main contribution to the integral (8) is obviously given by integration over the region of the α parameters in which $Q^2 F(\alpha, \omega) \leq O(1)$, since otherwise the exponential in (8) oscillates rapidly, which leads to strong suppression. In particular, integration over the region in which $F(\alpha, \omega) > \rho$ gives an exponentially decreasing contribution $O(\exp[-Q^2\rho])$. Thus, all contributions that have power-law ($O(Q^{-N})$) behavior as $Q^2 \to \infty$ are due to integration over the regions of the α space within which the function $F(\alpha, \omega)$ vanishes. There are two possibilities: Either all the terms $A_{ij}(\alpha)/D(\alpha)$ vanish simultaneously, or a zero result for $F(\alpha, \omega)$ is obtained only after summation over i and j. In what follows, since $A_{ij}(\alpha)$ and $D(\alpha)$ are in accordance with (4) and (5) sums of products of α parameters, the equation $A_{ij}/D = 0$ will mean that either $A_{ij}(\alpha)$ vanishes rapidly than $D(\alpha)$ when some of the α parameters tend to zero or that $D(\alpha)$ tends more rapidly than $A_{ij}(\alpha)$ to ∞ when some of the α parameters tend to ∞. Accordingly, there are three main mechanisms responsible for the vanishing of $F(\alpha, \omega)$.

1. Short-Distance Regime (SDR). In this case, $F(\alpha, \omega) = 0$ because some of the α parameters $\alpha_{\tau_1}, \ldots, \alpha_{\tau_n}$ are equal to zero (we shall sometimes say that the corresponding lines $\tau_1, \ldots, \tau_n$ are "heavy"). The simultaneous tending to zero of the set of α parameters $\{\alpha_{\tau_1}, \ldots, \alpha_{\tau_n}\}$ can be conveniently described by means of the substitution

$$\lambda(V) = \sum_{j=1}^{n} \alpha_{\tau_j}; \tag{9}$$
$$\alpha_{\tau_j} = \lambda(V)\beta_{\tau_j}, \tag{10}$$

where V is the subgraph composed of the lines $\tau_1, \ldots, \tau_n$

[2)] A detailed exposition of the technique of the α representation can be found in Refs. 17 and 69.

and having $z(V)$ cycles.

We investigate the behavior of the functions $D(\alpha)$ and $Q(\alpha;\{p\})$ in the limit $\lambda(V)\to 0$. In accordance with (4), the function $D(\alpha)$ is given by the sum over all possible sets of lines whose removal breaks all the z cycles of the considered diagram, including the $z(V)$ cycles lying entirely within the subgraph V. Therefore, each of the terms of the sum (4) contains not less than $z(V)$ of the α parameters of the lines τ belonging to the subgraph V,[3] and in the limit $\lambda(V)\to 0$ we have

$$D(\alpha) = [\lambda(V)]^{z(V)}\{O(1) + O(\lambda(V))\}. \qquad (11)$$

The terms in the sum (5) for $Q(\alpha,p)$ are given by a product of $z+1$ of the α parameters, i.e., for each term there is always a cycle that gives two (or more) α parameters to the corresponding product.

The function $A_{ij}(\alpha)$ decreases as $\lambda(V)\to 0$ faster than $D(\alpha)$ only in the case when for all the terms in $A_{ij}(\alpha)$ the cycle giving two α parameters is entirely contained in V. If this condition is satisfied, then in the limit $\lambda(V)\to 0$ the function $A_{ij}(\alpha)$ behaves as $[\lambda(V)]^{z(V)+1}$. Otherwise, the representation (11) holds for $A_{ij}(\alpha)$ and the ratio $A_{ij}(\alpha)/D(\alpha)$ tends to a constant as $\lambda(V)\to 0$.

Thus, by no means every graph V ensures fulfillment of the equation $F(\alpha,\omega)=0$ for $\lambda(V)=0$. However, there exists a rule that enables us to find readily (solely on the basis of the "appearance" of the diagram) the subgraphs V responsible for the power-law SDR contributions. We note first that for $\alpha_{\tau_1}=\ldots=\alpha_{\tau_n}=0$ the function $F(\alpha,\omega)$ goes over into the function $\tilde{F}(\alpha,\omega)$ corresponding to the diagram $\mathcal{G}$ ("reduced" diagram; cf. Refs. 56 and 57) for which the lines $\tau_1,\ldots,\tau_n$ are contracted to points. The requirement $F(\alpha,\omega)=0$ for $\lambda(V)=0$ means that $\tilde{F}(\alpha,\omega)=0$. In a scalar theory, this is equivalent to the contribution of the reduced diagram being independent of Q^2, i.e., independent of the large momentum invariants. Thus, the problem reduces to finding *subgraphs V whose contraction to a point transforms the original diagram $\mathcal{G}$ into a diagram $\tilde{\mathcal{G}}$ whose contributions (in the scalar theory) does not depend on* Q^2.[4] It is readily seen that $\tilde{\mathcal{G}}$ in this case must have the form of a rosette (Fig. 1) possessing the property that all the invariants $(p_i^{(a)}p_j^{(a)})$ that can be formed from the momenta $p_i^{(a)}$, $p_j^{(a)}$ belonging to the same component of the rosette must be small: $(p_i^{(a)}p_j^{(a)})=O(p^2)$. Otherwise, when $(p_i^{(a)}p_j^{(a)})=O(Q^2)$, the contribution of the corresponding component and, therefore, the entire diagram will depend on Q^2.

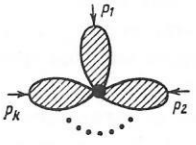

FIG. 1.

[3] The decomposition of an arbitrary diagram into cycles is not unique, but our arguments are valid for any decomposition.

[4] Such subgraphs are sometimes called essential[69] or Q^2 subgraphs.[10]

2. Infrared Regime (IR). In this case $F(\alpha,\omega)=0$ because some of the α parameters $\alpha_{\sigma_1},\alpha_{\sigma_2},\ldots,\alpha_{\sigma_n}$ become infinite (we shall say that the corresponding lines are "soft"). It is convenient to describe the simultaneous tending of the parameters $\alpha_{\sigma_1},\ldots,\alpha_{\sigma_n}$ to ∞ (or, which is the same thing, the tending of $1/\alpha_{\sigma_1},\ldots,1/\alpha_{\sigma_n}$ to zero) by means of the substitution

$$1/\varkappa(S)=\sum_{i=1}^{n} 1/\alpha_{\sigma_i}, \qquad (12)$$

where $S\equiv\{\sigma_1,\ldots,\sigma_n\}$ is the subgraph composed of the lines $\sigma_1,\ldots,\sigma_n$. The search for the subgraphs S responsible for the infrared regime is facilitated by a rule analogous to the Q^2-subgraph rule: It is necessary to find a set of lines $\{\sigma_1,\ldots,\sigma_n\}$ whose removal from the original diagram $\mathcal{G}$ gives a diagram $\tilde{\mathcal{G}}$ whose contribution (in the scalar theory) does not depend on large invariants. One can show that for such subgraphs $F(\alpha,\omega)\sim 1/\varkappa(S)$ as $\varkappa(S)\to\infty$.

Remark. As a rule, the infrared regime makes a contribution only if the soft lines correspond to massless particles, since otherwise there is a cutoff factor $\exp(-i\alpha_\sigma m_\sigma^2)$. However, this factor can be compensated by a term from $Q(\alpha;p)/D(\alpha)$ [see (3)] if some of the particles corresponding to the external lines of the diagram are on the mass shell $p_i^2=m_\sigma^2$. This possibility makes the analysis more complicated. To simplify the analysis, we shall consider only off-shell amplitudes. In QCD this is also justified by the fact that quarks and gluons probably do not have a mass shell.

3. Pinch Regime (PR). In pseudo-Euclidean space-time, the invariants (p_ip_j) can be positive or negative. Therefore, when there are invariants of both signs the function $F(\alpha,\omega)$ can vanish for nonvanishing finite α because of compensation of the terms in $F(\alpha,\omega)$. An important part in the analysis of the contributions due to the pinch regime is played by Tiktopoulos's theorem,[56] which states that the pinch regime works only in the case when $F(\alpha,\omega)$ can be represented as a product of not less than two functions $F_1(\alpha)$ and $F_2(\alpha)$, each of which vanishes for nonzero α. In many important cases, it is fairly easy to show that there is no such factorization and the pinch regime can be discounted, despite the presence of invariants of both signs.

Besides the three main regimes listed above, it is also possible to have various combined regimes. For example, if $F(\alpha,\omega)\sim O(\lambda(V))+O(1/\varkappa(S))$ as $\lambda(V)\to 0$, $\varkappa(S)\to\infty$ for certain subgraphs V and S, we have an SDR–IR regime.

Analysis of the Behavior of the Contributions Associated with the Short-Distance and Infrared Regimes. When the pinch regime does not work, the analysis of the contributions is greatly simplified. Using the rules formulated above, it is easy in each concrete case to find the subgraphs responsible for the short-distance and infrared regimes. Integration over the corresponding region of the α parameters gives a certain power-law contribution $O(Q^{-N})$. Asymptotically, of course, only the contributions with the smallest possible N are really important. The value of the exponent N can be estimated on the basis of the fact that the main contri-

bution to the integral (3) is associated with the region of the space of the α parameters in which the phase of the exponential is of order 1.

Contributions Due to the Short-Distance Regime. Since $F(\alpha, \omega) \sim \lambda(V)$ for a Q^2-subgraph V, the main contribution is made by integration over the region in which $Q^2\lambda(V) \lesssim 1$. This means that in the momentum representation the region in which $|k_i^2| \gtrsim Q^2$ for all internal lines of the subgraph V is the most important. Large virtual momenta correspond to short times (or distances), i.e., to contraction of the subgraph V to a point. As we have established above, the weakly connected components of the reduced diagram that result from such contraction must not depend on the large momentum invariants, and we can therefore assume that momenta with small ($O(p^2)$) virtuality pass through them. Thus, the corresponding contribution to $T(Q^2, p^2)$ has the structure

$$T^V_{(\mathrm{SDR})}(Q^2,\ p^2) \propto E_V^{\{\alpha\}\{\mu\}}(Q)\,[f_1(p_1)\ \dots\ f_n(p_n)]_{\{\alpha\}\{\mu\}}, \tag{13}$$

where $\{\alpha\}$ and $\{\mu\}$ are, respectively, the Dirac and vector indices associated with the external quark lines and the gluon lines of the subgraph V. Since $|k_i^2| \gtrsim Q^2$ for all internal lines of the subgraph V, it follows that $E_V(Q) \lesssim Q^{d(V)}$, where

$$d(V) = 4 - \sum_{j \in V} d_j, \tag{14}$$

and d_j is the dimension (in mass units) of the field corresponding to the j-th external line of the subgraph V (we recall that expressions of the type (14) are widely used to calculate degrees of divergence[17,72,73]). It is also necessary to take into account the additional factors associated with the external quark and gluon lines of the subgraph V. Each quark index α of the function $f_{\{\alpha\mu\}}(p)$ gives a Dirac spinor [for example, $u_\alpha(p)$], each gluon index μ can give p_μ, and the product of momenta $p_i^{(a)}$ and $p_j^{(b)}$ belonging to different components of a rosette (see Fig. 1) behaves as Q^2. Thus, each external gluon line of V must add to $T_V(Q^2, p^2)$ an additional factor Q, and each quark line a factor $Q^{1/2}$. It is easy to note the correlation between the additional factor Q^{N_j} and the spin of the corresponding particle, $N_j = s_j$, and obtain as a result the estimate

$$T^V_{(\mathrm{SDR})}(Q) \lesssim Q^{4-\Sigma t_j}, \tag{15}$$

where $t_j \equiv d_j - s_j$ is the twist[14,74] of the j-th external line of the subgraph V. In QCD, $t = 1$ for fields associated with quarks and fictitious particles (Faddeev-Popov ghosts[75]), and $t = 0$ for the vector potential A_μ. Thus, the leading contribution associated with the short-distance regime is given by integration with respect to $\lambda(V) \approx 0$ for subgraphs V having the minimal possible number of external lines with twist 1 and an arbitrary number of external gluon lines.

Contributions Due to the Infrared Regime. The contributions due to the infrared regime can be estimated similarly. It must be borne in mind though that, in contrast to $Q^2\lambda(V)$, the quantity $Q^2/\varkappa(S)$ has a nonvanishing dimension. It is therefore helpful to go over to dimensionless $\tilde{\alpha}$ parameters: $\alpha = \tilde{\alpha}/p^2$. As a result, the exponential factor (8) is transformed into

$$\exp\left\{i\,\frac{Q^2}{p^2}\,F(\tilde{\alpha},\ \omega) + iI(\tilde{\alpha})\right\} \tag{16}$$

and the main contribution to the infrared regime will be made by integration over the region $Q^2/(p^2\varkappa(S)) \approx 1$ or, using the ordinary α parameters, over the region $\alpha_0 \gtrsim Q^2/p^4$. As is shown by a detailed analysis, in the momentum representation this corresponds to integration over $|k_\sigma| < p^2/Q$ for all soft lines.[5)] Finally, for the contribution of the infrared regime we obtain the estimate (Fig. 2)

$$T^S_{(\mathrm{IR})}(Q) \leqslant \Big[\int\limits_{k_j \leqslant p^2/Q} \Big\{\prod_{j=1}^{n} d^4k_j k_j^{-4+2d_j}\Big\}\,\delta^4(\Sigma k_j)\,k^{4-\Sigma d_j}\Big]\,Q^{\Sigma s_j}. \tag{17}$$

We explain the structure of the expression (17): $k_j^{-4+2d_j}$ is the estimate for the propagator of the j-th external line of the subgraph S; $k^{4-\Sigma d_j}$ gives the estimate of the contribution of the internal loops of the subgraph S [cf. (14)]; $\delta^4(\Sigma k_j)$ reflects the momentum conservation law; and the factor $Q^{\Sigma s_j}$, as for the short-distance regime, is due to the spin structure of the external line of the subgraph S. Transforming (17), we arrive at the simple estimate

$$T^S_{(\mathrm{IR})}(Q) \leqslant Q^{-\Sigma t_j}. \tag{18}$$

Thus, the leading infrared-regime contributions are also due to configurations with the smallest possible number of external lines of the subgraph S. An estimate of the contribution of the combined regime (Fig. 3) is obviously given by the product of (15) and (18):

$$T^{V\ S}_{(\mathrm{SDR})\,(\mathrm{IR})} \leqslant Q^{4-\sum\limits_{i\in V} t_i - \sum\limits_{j\in S} t_j}, \tag{19}$$

where the subscripts i and j label the external lines of the subgraphs V and S. The lines connecting V to S must be taken into account only once; we shall always include them in S.

Physically, the integration over the small α thus corresponds to a certain subprocess at short distances, and the integration over the region $\varkappa(S) \to \infty$ to the exchange of soft particles. We note that in QCD exchange of soft gluons ($t = 0$) does not, in contrast to the exchange of soft massless quarks ($t = 1$), lead in accordance with (18) and (19) to a power-law suppression of the corresponding contribution and, therefore, in QCD (in the cases when the infrared regime contributes) it is necessary to sum over all possible soft gluon exchanges.

Remark. At the first glance, it appears that the estimates (15), (18), and (19) have not been obtained with sufficient rigor. However, a detailed analysis of the various possibilities in the α representation gives the

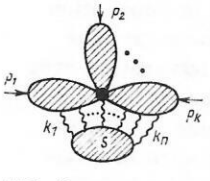

FIG. 2.

[5)]Thus, the limit $\alpha_0 \to \infty$ corresponds to the limit $k \to 0$, and not simply $k^2 \to 0$.

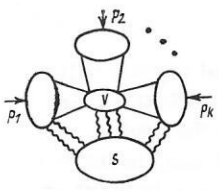

FIG. 3.

same estimates (see, for example, the appendix to the first of the papers quoted in Ref. 33).

Mellin Representation. The contributions of the Feynman diagrams in the limit $Q^2 \to \infty$ have a generalized power-law asymptotic behavior: $T(Q^2, p^2) \sim Q^{-N}(\ln Q^2/p^2)^n$. However, the presence of the dimensionless factors of the type $\ln Q^2/p^2$ is not reflected in Eqs. (15), (18), and (19). When analyzing the logarithmic corrections, it is very convenient to express the investigated amplitude in the form of a Mellin integral:

$$T(Q^2, p^2) = \frac{1}{2\pi i} \int_{-\delta-i\infty}^{-\delta+i\infty} \Gamma(-J)(Q^2/p^2)^J \Phi(J)\, dJ. \tag{20}$$

In such an expression, the asymptotic behavior of the amplitude $T(Q^2, p^2)$ is determined by the extreme right-hand singularity of its modified Mellin transform $\Phi(J)$ in the complex J plane. In particular, if $\Phi(J) \sim (J - J_0)^{-n}$, then $T(Q) \sim (Q^2)^{J_0}(\ln Q^2/p^2)^{n-1}$. The power-law corrections to the leading contribution are due to the singularities of the function $\Phi(J)$ at $J = J_0 - 1$, $J = J_0 - 2$, etc.

One frequently also uses the α representation for the Mellin transform $\Phi(J)$ of the contribution of some diagram G:

$$\Phi_G(J) = R_\varepsilon \left\{ \frac{P(\text{c.c.})}{(4\pi)^{2z}} (4\pi\mu_R^2)^{z\varepsilon} \times \int_0^\infty \prod_\sigma d\alpha_\sigma D^{-2+\varepsilon}(\alpha)\, (iF(\alpha, \omega))^J \exp\{ip^2 I(\alpha)\} \times \sum_{n=0}^{n_{\max}} \frac{\Gamma(n-J)}{\Gamma(-J)} g_n(\alpha, \omega, p^2). \right. \tag{21}$$

In the language of the Mellin representation, the expression (15) means, for example, that the integration in (21) over the region $\lambda(V) \approx 0$ gives the pole of the function $\Phi_G(J)$ at the point

$$J = J_0 = 2 - \sum_{i \in V} t_i/2. \tag{22}$$

Multipole poles $(J - J_0)^{-n}$, leading to logarithmic contributions, arise in the cases when the pole $(J - J_0)^{-1}$ can be obtained as a result of several independent integrations.

Region of Applicability of Perturbation Theory in QCD. As we have established above, the generalized power-law contributions can arise as a result of integration over different regions of the virtual momenta: The short-distance regime corresponds to integration over the region $k \gtrsim Q$, the pinch regime to the region $k \sim p$, and the infrared region to $k \lesssim p^2/Q$. Since in QCD the coupling constant is small only in the region of sufficiently large momenta (for example, for $k \gtrsim 1$ GeV), one can have hopes for perturbation theory in QCD only in the case when the pinch and infrared regimes, and also the combined regimes, do not contribute, or when the corresponding contributions have a power-law ($O(1/Q^2)$) suppression compared with the contributions due solely to the short-distance regime.

2. ANALYSIS OF THE TOTAL CROSS SECTION OF e^+e^- ANNIHILATION INTO HADRONS

The Parton Model. According to the ideas of the parton model, the transition of an e^+e^- pair into hadrons (in the lowest order in the electromagnetic coupling constant $\alpha \approx 1/137$) is realized in two stages. First, as a result of the subprocess $e^+e^- \to \gamma^* \to q\bar{q}$, a $q\bar{q}$ pair is formed, and then, as they move apart, the quarks fragment into the observed hadrons. It is also assumed that the $q\bar{q}$ pair is transformed into hadrons with 100% probability. It follows that the total cross section of the process e^+e^- → "any hadrons" is equal in the parton model to the sum of the cross section for production of the possible $q\bar{q}$ pairs. Further, each such cross section differs from the cross section of the process $e^+e^- \to \mu^+\mu^-$ only by the square of the electric charge of the corresponding quark (measured in units of the electron charge). Thus, in the parton model[76]

$$R(s) \equiv \frac{\sigma_{\text{tot}}(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \sum_{j=1}^{N_c} \sum_{i=1}^{N_f(s)} Q_{ij}^2, \tag{23}$$

where s is the square of the total energy of the e^+e^- pair in the center-of-mass system, i is the quark flavor (u, $d, s, c, \ldots$), and j is the quark color. The quantity $N_f(s)$ is determined by the number of quarks species whose production thresholds lie below s. The expression (23) is too crude. In particular, accurate allowance for threshold effects for heavy quarks[77] leads to the replacement of (23) by the expression

$$R(s) = 3\left\{ \sum_{i=u,d,s} Q_i^2 + \sum_{i=c,b,\ldots} Q_i^2 \theta(s - 4m_i^2) \sqrt{1 - 4m_i^2/s}\,(1 - m_i^2/2s) \right\}, \tag{24}$$

in which the new quark species come into play smoothly and not abruptly. The opening of the channels associated with the production of heavy quarks is also accompanied by the appearance of narrow resonances. In particular, the experimental curve for the contribution of particles containing the c quark to $R(s)$ begins with two narrow peaks at $\sqrt{s} = m_{J/\psi}$ and $\sqrt{s} = m_{\psi'}$. Thus, in contrast to the naive parton ideas, hadrons that contain c quarks are not produced in the region $m_{J/\psi} < \sqrt{s} < m_{\psi'}$ or even up to $\sqrt{s} = 2m_D$, i.e., at energies clearly sufficient for the production of a $c\bar{c}$ pair. When $\sqrt{s} > 2m_D$, the cross section $\sigma_{\text{tot}}(e^+e^- \to \text{charm})$ has a number of broader peaks, and it is only when $\sqrt{s} \gtrsim 6$ GeV that the charm contribution to $R(s)$ becomes more or less constant. Thus, in the near-threshold (resonance) region the parton notions (especially the assumption of a 100% probability of transition of a produced $c\bar{c}$ pair into hadrons) are too crude and one can only hope that the expression (24) corresponds to the real picture "on the average" and not literally. This proposition can be given mathematical meaning.

Dispersion Relations and Sum Rules. In accordance with the optical theorem, $\sigma_{\text{tot}}(e^+e^- \to \text{hadrons})$ is proportional to the imaginary part of the function $T_{\text{had}}(q^2)$ corresponding to the hadron contribution to the vacuum po-

larization:

$$T_{\text{had}}(q^2) = \int d^4x \exp(iqx) \langle 0|S^+T(J^\mu(x) J_\mu(0) S)|0\rangle, \quad (25)$$

where S is the scattering matrix and J^μ is the hadronic electromagnetic current (see Refs. 3, 5, and 46). In the timelike region, the function $T(q^2)$ has a cut and σ_{tot}, and hence $R(s)$, is proportional to the discontinuity across this cut at $q^2 = s$. For our purposes, it is convenient to separate from T_{had} all trivial factors and consider the function $t(Q^2)$ ($Q^2 = -q^2$), which is related to $R(s)$ by

$$R(s) = (1/2\pi i)\,[t(-s+i\varepsilon) - t(-s-i\varepsilon)]. \quad (26)$$

Calculating $t(Q^2)$ perturbatively, we can take into account the corrections for the final-state interaction of the quarks and thus make the parton expression (24) more accurate. However, the narrow peaks in the near-threshold region ($s \sim 4m_q^2$) mentioned above manifest themselves in the higher perturbation orders by large contributions of the type $(\alpha_s/\sqrt{s-4m_q^2})^n$, which render straightforward use of perturbation theory meaningless. A more correct procedure is to use perturbation theory to calculate $t(Q^2)$ in the spacelike region $\mathrm{Re}\,Q^2 > 0$, where the dangerous resonance denominators degenerate into power-law corrections $\sim(4m_q^2 + Q^2)^{-N}$, which can be ignored for sufficiently large Q^2. Then, using Cauchy's formula, we can write down the dispersion relation

$$t(Q^2) = t(0) - Q^2 \int_{4m_\pi^2}^{\infty} R(s)\, ds/[s(s+Q^2)], \quad (27)$$

which connects the experimentally observed function $R(s)$ to the perturbatively calculated function $t(Q^2)$.[78] Note that since $R(s) \to \text{const}$ as $s \to \infty$, Cauchy's formula was applied to $t(Q^2)/Q^2$ in order to obtain (27).

To compare (27) with experimental data, we must formally know $R(S)$ at all energies, which is impossible. Since the weight function in (27) decreases fairly rapidly (as $1/s$) with increasing s, in the region of large s we can assume with good accuracy that $R(s)$ is constant (see, for example, Ref. 79). A different approach[80-82] based on the use of finite-energy sum rules[82] also leads to relations of the type

$$\int_{4m_\pi^2}^{s_0} R^{\text{exp}}(s)\, s^n\, ds = \int_0^{s_0} R^{QCD}(s)\, s^n\, ds, \quad (28)$$

where $R^{\text{QCD}}(s)$ is given by the analytic continuation of the function $t(Q^2)$ calculated perturbatively with allowance for only the leading power-law contributions.[6)] Thus, in both approaches the main problem is to calculate the function $t(Q^2)$ in QCD.

General Analysis of the Contributions. The diagrams that contribute to $t(Q^2)$ have the structure of Fig. 4a. To eliminate the dependence of such a diagram on Q^2, we must at least ensure that a momentum q whose square is large enters and leaves the diagram at the same point (see Fig. 4b). This can be achieved only by contracting to a point a certain subgraph V (see Fig. 4a) that contains both photon vertices. Further, since the diagram that results (see Fig. 4b) does not depend on Q^2, such contraction is not only necessary but also sufficient. This means that all the power-law contributions to $t(Q^2)$ are due to the short-distance regime, and (15) can be used to estimate them. It must be borne in mind that the function $t(Q^2)$ in which we are interested differs from the function $T(Q^2)$, to which Eq. (15) applies directly, by a factor $O(Q^2)$: $t(Q^2) \sim T(Q^2)/Q^2$. In addition, the external photon lines, which formally correspond to spin 1, do not introduce additional Q factors in $T(Q^2)$ in the considered problem, and therefore

$$t^V_{(\text{SDR})}(Q^2) \propto Q^{-\sum_{i=1}^{L} d_i}, \quad (29)$$

where the summation is over the external lines of the subgraph V, excluding the photon lines. Note that additional Q^{s_i} factors [see the discussion preceding Eq. (15)] arise only when the considered amplitude depends on momenta P whose squares are small ($P^2 \ll Q^2$) while the scalar products with the other momenta are large ($Pq \sim Q^2$). Naturally, such factors are absent for amplitudes that depend on only one (even a large one) momentum q.

Further, since $d^i \geq 1$ for all fields, the leading asymptotic contribution $t^{as}(Q^2)$ is given by the subgraphs with the smallest number of external lines, this being two in the given case: $t^{as}(Q^2) \sim Q^0$. Since any subgraph V not identical to the complete diagram $\mathcal{G}$ has more than two external lines, its SDR contribution $t^V_{\text{SDR}}(Q^2)$ is in accordance with (29) suppressed in a power-law manner compared with $t^{\mathcal{G}}_{\text{SDR}}(Q^2)$. Thus, to obtain $t^{as}(Q^2)$, it is necessary to integrate over the region $\lambda(\mathcal{G}) < 1/\mu^2$, where $\lambda(\mathcal{G}) = \alpha_{\sigma_1} + \ldots + \alpha_\sigma$ is the sum of the α_σ parameters of all the lines σ of the considered diagram, and $\mu \sim Q$ is some scale with the dimensions of mass.

Divergences and Logarithmic Contributions. In the Mellin representation (20), a pole at $J = 0$ corresponds to the behavior $t(Q^2) \sim Q^0$. In addition, it is also necessary to take into account the effects due to the *ultraviolet divergences* of some of the subgraphs within $\mathcal{G}$. With allowance for the substitution $D^2(\alpha) \to D^{2-2\varepsilon}(\alpha)(\mu^2)^\varepsilon$, which is associated with dimensional regularization [see (7)], integration over the region $\lambda(\mathcal{G}) < 1/\mu^2$ gives

$$\Phi(J, \varepsilon) \propto \frac{1}{J + z\varepsilon} \left(\frac{1}{\mu^2}\right)^{J-1+z\varepsilon} (4\pi\mu^2)^{z\varepsilon}\, \varphi(J, \varepsilon, g). \quad (30)$$

Note that since there is only one subgraph that gives a pole at $J \sim 0$ in the short-distance regime, and this pole has already been separated in (30), the function $\varphi(J, \varepsilon, g)$ can have singularities with respect to J only at $J \approx -1$ or further to the left. Going over from $\Phi(J)$ to $t(Q^2)$ by taking the residue at the point $J = -z\varepsilon$, we ob-

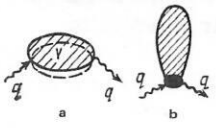

FIG. 4.

[6)] Allowance for the first power-law corrections to $t(Q^2)$ makes it possible to obtain a number of predictions[79] about the properties of resonances in excellent agreement with the experimental data. However, in the present paper we shall restrict ourselves to analyzing the leading power-law contributions.

tain

$$t(Q^2) = R_\varepsilon \left\{ \left(\frac{4\pi\mu^2}{Q^2} \right)^{z\varepsilon} \Gamma(1+z\varepsilon)\, \varphi(J=-z\varepsilon,\ \varepsilon,\ g) \right\} \{1 + O(1/Q^2)\}. \quad (31)$$

The ultraviolet divergences mentioned above correspond to poles of the function $\varphi(J, \varepsilon, \mu)$ with respect to ε, and each pole $1/\varepsilon^k$ leads in accordance with the expansion

$$\frac{1}{\varepsilon^k} (4\pi\mu^2 \exp(-\gamma_E)/Q^2)^{z\varepsilon} = \frac{1}{\varepsilon^k} \sum_{n=0}^{\infty} \frac{[z\varepsilon \ln(4\pi \exp(-\gamma_E)\mu^2/Q^2)]^n}{n!} \quad (32)$$

to the appearance of a logarithmic contribution $O(\ln^k(\mu^2/Q^2))$ that is finite in the limit $\varepsilon \to 0$. The factor $\exp(-\gamma_E)$ appears because

$$\Gamma(1+x) = \exp\{-\gamma_E x + O(x^2)\}, \quad (33)$$

where $\gamma_E = 0.577\,215\,664\,9$ is Euler's constant (see, for example, Ref. 83).

Remark. There is a divergence in t already in the zeroth order in the strong interaction. This is due to the fact that with respect to the electromagnetic interaction the corresponding simplest diagram (Fig. 5a) is a constituent part of the self-energy correction to the photon propagator. Its contribution behaves as $Q_i^2 \ln(Q^2/\mu^2)$, where Q_i is the charge of the quark, which gives in accordance with (26) a constant contribution to $R(s)$ equal to Q_i^2.

To eliminate the divergence in the original diagram in Fig. 5a, it is customary to consider, not the function $t(Q^2)$ itself, but its derivative[78]:

$$D(Q^2) = Q^2 \frac{d}{dQ^2} t(Q^2). \quad (34)$$

In the lowest approximation (and for vanishing quark masses) $D(Q^2)$ is given by the same expression as $R(s)$:

$$D^{(0)}(Q^2) = N_c \sum_{i=1}^{Nf} Q_i^2 \quad (35)$$

[see (23)]. Generally speaking, the diagrams of the following order (see Figs. 5b and 5c) contain ultraviolet divergences. However, their total contribution to $D(Q^2)$ is free of divergences, which is due to conservation of the electromagnetic current. Because of this, the contribution of the diagrams in Figs. 5b and 5c to $D(Q^2)$ does not contain a logarithmic $[O(\ln Q^2/\mu^2)]$ contribution. Such contributions appear only in the following order in g^2.

Logarithmic Contributions and the Renormalization Group. Thus, after removal of the ultraviolet divergences, i.e., the poles with respect to ε, the amplitude $D(Q^2)$ is (up to power-law corrections that we do not consider) a function of $\ln(Q^2/\mu^2)$ and the coupling constant g:

$$D(Q^2) = D^{\text{lead}}(\ln(\mu^2/Q^2), g)\{1 + O(1/Q^2)\}. \quad (36)$$

However, since $D(Q^2)$ is directly related to an observable quantity (the cross section of e^+e^- annihilation into hadrons), it must not depend on our freedom in choosing the parameter μ^2. This can be ensured only if different g correspond to different μ^2, i.e., if g in reality is a function of μ^2: $g = \bar{g}(\mu^2)$. The set of transformations $\{\mu \to \mu', g \to g'\}$ that leave the physical quantities [such as $T(Q^2)$] unchanged form the renormalization group.[15-17] Using the renormalization-group method, one can show (see, for example, Refs. 17, 84, and 85) that in renormalizable theories (in particular, in QCD) the requirement that some (arbitrary) physical quantity $T_i(Q)$ be independent of the choice of the parameter μ gives the same (for all T_i) form of the dependence of the *effective coupling constant* $\bar{g}(\mu^2)$ (see Ref. 86) on its argument μ^2. The function $\bar{g}(\mu^2)$ plays an exceptionally important part in all perturbative calculations in QCD, and it is therefore worth beginning with a detailed discussion of the dependence of $\bar{g}(\mu^2)$ on μ^2 and only then returning to an investigation of the perturbation series for $D(Q^2)$.

FIG. 5.

The Effective Coupling Constant. The explicit form of the function $\bar{g}(\mu^2)$ can, in principle, be calculated as follows. Let $f(\bar{g}(\mu^2), \ln\mu^2/Q^2)$ be some quantity that does not change when μ is varied:

$$f(g, l) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} g^n l^k a_{nk}. \quad (37)$$

Since $df/dl = 0$ for all l and, in particular, for $l = 0$, by calculating successively the coefficients a_{n0} and a_{n1} we can find (in the form of a series in g) the function

$$\beta(g) = \frac{dg}{dl}\Big|_{l=0} = \sum_k \beta_k g^k, \quad (38)$$

which gives the required dependence of $\bar{g}(\mu^2)$ on μ^2. Indeed, integrating (38), we obtain

$$\ln(\mu^2/\tilde{\mu}^2) = \int_{\bar{g}(\tilde{\mu})}^{g(\mu)} \frac{dx}{\beta(x)} \equiv \Phi(\bar{g}(\mu^2)) - \Phi(\bar{g}(\tilde{\mu}^2)). \quad (39)$$

the so-called *Gell-Mann–Low equation*,[16] from which it follows that

$$\bar{g}(\mu^2) = \Phi^{-1}[\ln(\mu^2/\tilde{\mu}^2) + \Phi(\bar{g}(\tilde{\mu}^2))]. \quad (40)$$

Thus, knowing $\bar{g}$ at a certain point $\tilde{\mu}^2$, we can in the framework of perturbation theory find $\bar{g}(\mu^2)$ as a function of μ^2. It is readily noted that the combination $\ln(\mu^2/\tilde{\mu}^2) + \Phi(\bar{g}(\tilde{\mu}^2))$ in (39) is in fact independent of $\tilde{\mu}$. This is possible if

$$\Phi(\bar{g}(\tilde{\mu}^2)) = \ln(\tilde{\mu}^2/\Lambda^2), \quad (41)$$

where Λ is a parameter with the dimensions of mass. As a result, $\bar{g}(\mu) = \Phi^{-1}[\ln(\mu^2/\Lambda^2)]$.

It must be borne in mind that Φ is determined by the expression (39) only up to an arbitrary constant. However, making the substitution

$$\Phi \to \Phi_\varphi \equiv \Phi + 2\varphi, \quad (42)$$

we must in accordance with the definition (41) of Λ also make the substitution $\Lambda \to \Lambda_\varphi \equiv \Lambda \exp(-\varphi)$. Then $\bar{g}(\mu^2)$ is unchanged, and we are actually dealing with a one-parameter family of expressions for $\bar{g}(\mu^2)$ in terms of $\ln\mu^2$:

$$\bar{g}(\mu^2) = \Phi_\varphi^{-1}[\ln(\mu^2/[\Lambda \exp(-\varphi)]^2)]. \quad (43)$$

Behavior of the Effective Coupling Constant in Massless Chromodynamics. At the present time, the function

$B_{QCD}(g)$ has been calculated up to terms $O(g^7)$ (see Refs. 21 and 87–89):

$$\frac{dG}{dL}\Big|_{MS} = -\sum_{k=0}^{\infty} b_k G^{k+2} = -\Big[\Big(11-\frac{2N_f}{3}\Big)G^2 + \Big(102-\frac{38}{3}N_f\Big)G^3 + \Big(\frac{2857}{2}-\frac{5033}{18}N_f+\frac{325}{24}N_f^2\Big)G^4\Big] + O(G^5), \quad (44)$$

where $G = (\bar{g}/4\pi)^2$, and N_f is the number of quark flavors $(u, d, s, \ldots)$.

In the lowest approximation, (44) gives

$$\alpha_s(Q) \equiv \frac{\bar{g}^2(Q^2)}{4\pi} = \frac{4\pi}{(11-2N_f/3)\ln(Q^2/\Lambda^2)}, \quad (45)$$

i.e., $\alpha_s(\mu)$ decreases with increasing μ for $N_f \leq 16$. This property of the theory ("asymptotic freedom"[21,46]) is the cornerstone of all approaches that use perturbation theory in quantum chromodynamics.

Under the substitution $\Lambda \to \Lambda \exp(-\varphi)$, Eq. (45) keeps its structure up to terms $O(1/\ln^2 Q^2)$, and therefore, to fix the arbitrariness in the determination of the parameter Λ, it is necessary to take into account the G^3 term in the expansion (44). The Gell-Mann–Low equation (39) reduces with allowance for (41) in this case to

$$L \equiv \ln\Big(\frac{\mu^2}{\Lambda^2}\Big) = \frac{1}{b_0 G} + \Big(\frac{b_1}{b_0^2}\ln G + \Delta\Big) + \Big(\frac{b_2 b_0 - b_1^2}{b_0^3}G\Big) + O(G^2), \quad (46)$$

where Δ is the parameter that reflects the discussed arbitrariness, and b_i are the coefficients of the β function [see (44)]. At small G, this equation can be solved iteratively:

$$\left.\begin{aligned} &G^{(0)} = 1/(b_0 L): \quad G^{(1)} = (1/b_0 L)\{1 - L_1/L\}^{-1}; \\ &G^{(2)} = (1/b_0)\Big\{L - L_1 + \frac{b_1}{b_0^2}\ln(1-L_1/L) - \frac{b_2 b_0 - b_1^2}{b_0^3 L}(1-L_1/L)^{-1}\Big\}^{-1} \end{aligned}\right\} \quad (47)$$

etc., where

$$L_1 = (b_1/b_0^2)\ln(b_0 L) - \Delta. \quad (48)$$

Frequently, an expansion of $\alpha_s(\mu, \Delta)$ in a series in $1/L$ is used:

$$\alpha_s(\mu, \Delta) = \frac{4\pi}{b_0 L}\Big\{1 - \frac{L_1}{L} + \frac{1}{L^2}\Big(L_1^2 - \frac{b_1}{b_0^2}L_1 + \frac{b_2 b_0 - b_1^2}{b_0^4}\Big) + O(1/L^3)\Big\}. \quad (49)$$

It can be seen from (47) and (48) that the choice $\Delta = 0$ is not particularly felicitous, since for the physically most interesting values of the logarithm $3 < L < 10$, i.e., in the region $0.5 > \alpha_s > 0.1$, the ratio L_1/L, which characterizes the two-loop correction (45), is between 80 and 30%. Usually, one chooses

$$\Delta_c = (b_1/b_0^2)\ln b_0. \quad (50)$$

Then the expression (49) does not contain terms of the form const$/L^2$:

$$\alpha_s(\mu^2, \Delta_c) = \frac{4\pi}{b_0}\Big\{\frac{1}{L} - \frac{b_1}{b_0^2}\frac{\ln L}{L^2} + O(1/L^3)\Big\}. \quad (51)$$

The value of the first correction in (51) is 30–15%. It is also easy to find a value of the parameter Δ that in a certain sense is optimal:

$$\Delta_{opt} = (b_1/b_0^2)\ln(4b_0), \quad (52)$$

for which L_1/L is not more than 7% in the complete region $\infty > L > 3$. We note further that since the choice $\Delta = \Delta_{opt}$ leads in the region in which we are interested ($G \lesssim 0.05$) to minimization with respect to the contribution of the two-loop correction not only in the expression (49) but also in the original expression (46), a significant deviation of Δ from Δ_{opt} leads to an appreciable growth of the coefficients of the leading powers in the expansion of $\alpha_s(\mu, \Delta)$ with respect to $1/L$, and the choice (52) is equivalent to summation of the large corrections to the factor $\exp(\Delta - \Delta_{opt})$ and leads to redefinition of the parameter Λ:

$$\Lambda(\Delta_{opt}) = \Lambda(\Delta)\exp[(\Delta - \Delta_{opt})/2]. \quad (53)$$

In particular, $\Lambda(\Delta_{opt}) \approx 0.6\Lambda(\Delta_c)$ for $N_f = 3, 4$.

Thus, if we take $\Delta = \Delta_{opt}$, the simplest expression (45) gives a very good approximation for $\alpha_s(Q)$ in the region $L > 3$. If desired, the following two corrections in (49) can be taken into account.

Remark. The coefficients $b_2, b_3, \ldots$ depend, in contrast to b_0, b_1, on the prescription used to subtract the ultraviolet divergences (see, for example, Ref. 90). In particular, the value of the coefficient b_2 (Ref. 89) given in (44) is calculated in the minimal subtraction (MS) scheme.[91] The form of b_2 in other schemes is as yet unknown. Further, it follows from (44) and (49) that the relative contribution of the $O(G^4)$ terms to the right-hand side of the Gell-Mann–Low equation for $G \lesssim 0.05$ does not exceed 3%, and therefore the curve for $\alpha_s^{(2)}(Q, \Delta_{opt})$ obtained with allowance for the three-loop corrections differs but little from the curve $\alpha_s^{(0)}(Q, \Delta_{opt})$ up to $L \approx 1$ (see Ref. 92).

Analysis of the Structure of the Perturbation Series for $D(Q)$. Having obtained explicit expressions for $\alpha_s(\mu)$ in QCD, we turn to the analysis of the function $D(Q)$:

$$D(Q) = N_c\sum_{i=1}^{N_f} Q_i^2 \sum_{n=0}^{\infty} d_n(Q^2/\mu^2)\Big(\frac{\alpha_s(\mu)}{\pi}\Big)^n. \quad (54)$$

Choice of the Argument of the Effective Coupling Constant. As we have already noted, the sum of the series (54) does not depend on the parameter μ. Using the renormalization-group formulas, we can go over from expansion with respect to $\alpha_s(\mu)$ to expansion with respect to $\alpha_s(\tilde{\mu})$, and from the point of view of the renormalization group all such expansions are equally valid: *within the renormalization group there is no criterion on the basis of which preference could be given to one expansion rather than another.* In reality, however, we are dealing solely with the sum of the first few terms of the perturbation series; this sum has a nontrivial dependence on μ and, if μ is not chosen felicitously, may differ strongly from the true value of $D(Q)$. The deviation will be compensated only by the contributions of higher orders in the expansion with respect to $\alpha_s(\mu)$. Thus, large corrections in the expansion (54) must be regarded as an indication that a poor choice has been made of the parameter μ. It is therefore meaningful to choose μ to obtain the smallest corrections. Suppose that we use the standard subtraction scheme[17,72,73] or, as it is called, the momentum subtraction scheme (MOM scheme) and $\bar{g}(\mu)$ corresponds to a quark-gluon vertex with ingoing momenta k_i with virtuality μ^2: $k_i^2 = -\mu^2$. Since the main contribution to $D(Q)$ is due to integration over virtual momenta $k_i^2 \sim -Q^2$, it is sensible in the MOM scheme to take $\mu^2 = Q^2$,

i.e., to expand $D(Q)$ with respect to $\alpha_s(Q)$. For such a choice, the contributions containing $\ln(Q^2/\mu^2)$ are absorbed by the effective coupling constant, and there is every reason to expect the remaining contributions to give an admittedly nonzero but fairly small correction. Let us see to what extent such expectations are confirmed by the results of direct calculations of the coefficients d_n.

In accordance with (35), $d_0 = 1$. The contributions of the two-loop diagrams (see Figs. 5b and 5c) differ only by the color factor $C_F = (N_c^2 - 1)/2N_c$ (in QCD, $C_F = 4/3$) from their quantum-electrodynamic analogs, for which the corresponding calculations were already made in Ref. 93. Using Ref. 93, we can set $d_1 = 3C_F/4$,[94] i.e., in QCD $d_1 = 1$. At the present time, the coefficients d_2 is also known,[95-97] and, moreover, in different renormalization schemes. In the MOM scheme[97]

$$d_2(\mu^2/Q^2)|_{\text{MOM}} = \frac{85}{36\sqrt{3}} Cl_2\left(\frac{\pi}{3}\right) - \frac{43}{48} + \left(\frac{23}{24} - \zeta(3)\right) b_0 + \frac{b_0}{4} \ln\left(\frac{\mu^2}{Q^2}\right) \approx -2.193 + 0.162 N_f + \frac{b_0}{4} \ln\left(\frac{\mu^2}{Q^2}\right), \quad (55)$$

where $Cl_2(\theta)$ is the Clausen function[98]

$$Cl_2(\theta) = \sum_{n=1}^{\infty} \sin(n\theta)/n^2, \quad (56)$$

$Cl_2(\pi/3) = 1.014\,942\ldots$, $\zeta(n)$ is the zeta function (see Ref. 83), $\zeta(3) = 1.2021\ldots$.

It follows from (55) that in the MOM scheme for $N_f = 3$ and $\mu^2 = Q^2$ the contribution of the $O(\alpha_s^2)$ correction is (for $\alpha_s/\pi \approx 0.1$) approximately 17% of the $O(\alpha_s)$ contribution. This clearly indicates that the physical arguments advanced above with regard to the optimal choice of the parameter μ are not far from the truth.

Schemes Based on Dimensional Renormalization. It must be said that the MOM scheme is very inconvenient in direct calculations and in this respect is very inferior to 't Hooft's MS scheme,[91] which we have already mentioned. The coefficient d_2 was in fact calculated for the first time in this scheme in Ref. 95:

$$d_2\left(\frac{\mu^2}{Q^2}\right)\Big|_{\text{MS}} = \left\{\frac{1}{12} + \left[\frac{11}{2} - 4\zeta(3) + (\ln(4\pi) - \gamma_E) + \ln\frac{\mu^2}{Q^2}\right]\frac{b_0}{4}\right\} \approx 7.359 - 0.441 N_f + (b_0/4)\ln(\mu^2/Q^2). \quad (57)$$

The result (57) was reproduced by other authors.[96,97]

It follows from (57) that in the minimal subtraction scheme the choice $\mu^2 = Q^2$ is far from optimal, since the $O(\alpha_s^2)$ contribution is (for $\alpha_s/\pi \approx 0.1$) about 60% of the $O(\alpha_s)$ contribution. However, 40% of them are due to the contribution $\ln(4\pi) - \gamma_E$, which arises because in accordance with (32) the logarithms $\ln(\mu^2/Q^2)$ always occur in the MS scheme in the combination $\ln(4\pi\mu^2 \exp(-\gamma_E)/Q^2)$. But if we take $\mu^2 = \exp(\gamma_E)Q^2/4\pi$, the meaningless terms of the type $(\ln(4\pi) - \gamma_E)^k$ do not arise in any order in α_s. Usually, the "natural" choice $\mu^2 = Q^2$ is maintained, and to eliminate the factor $4\pi\exp(-\gamma_E)$ the actual prescription of the dimensional regularization is changed, (6) being replaced by the substitution

$$d^4k/(2\pi)^4 \to [d^{4-2\varepsilon}k/(2\pi)^{4-2\varepsilon}](\mu^2 \exp(\gamma_E)/4\pi)^{\varepsilon}, \quad (58)$$

and the corresponding renormalization prescription being called the $\overline{\text{MS}}$ scheme.[38] The expansion with respect to $\bar{g}(Q^2)$ in the $\overline{\text{MS}}$ scheme gives the same series as the expansion with respect to $\bar{g}(Q^2\exp(\gamma_E)/4\pi)$ in the MS scheme. In terms of the Λ parameters, this means that

$$\Lambda^2_{\overline{\text{MS}}} = 4\pi \exp(-\gamma_E)\Lambda^2_{\text{MS}} \approx (2.656\Lambda_{\text{MS}})^2, \quad (59)$$

if the parameter Δ in (46) is the same for both schemes. In what follows, we shall always assume that in all schemes $\Delta = \Delta_{\text{opt}}$ [see (52)].

In Ref. 99, one further variant of minimal subtraction schemes was proposed; it is the G scheme,

$$\Lambda_G = e\Lambda_{\overline{\text{MS}}}, \quad (60)$$

and it has a number of technical advantages in the explicit calculations of the coefficient d_2.

The OMS Scheme and Expansion with Respect to $1/L$. If instead of (59) we take

$$\Lambda_{\text{OMS}} = 2.104\Lambda_{\overline{\text{MS}}}, \quad (61)$$

then the coefficient $d_2^{\text{OMS}}(\mu^2/Q^2)$ in the corresponding scheme, which we shall call the optimized minimal subtraction scheme (OMS scheme) is for $N_f = 3$ equal to $d_2^{\text{MOM}}(\mu^2/Q^2)$. This does not mean that the OMS and MOM schemes are identical. In particular, the coefficients d_2^{MOM} and d_2^{OMS} differ for $N_f \neq 3$:

$$d_2^{\text{OMS}}(\mu^2/Q^2) = -2.105 + 0.133 N_f + (b_0/4)\ln(\mu^2/Q^2). \quad (62)$$

In addition, since $d(\ln(k\mu)) = d(\ln\mu)$, the MS, $\overline{\text{MS}}$, and OMS schemes have in accordance with (38) the same β function, which, beginning with the third coefficient, is not necessarily equal to the β function in the MOM scheme.[90] Similarly, the coefficients d_n^{OMS} and d_n^{MOM} for $n \geq 3$ may also differ even when $N_f = 3$.

If we set $\mu = Q$ and express $\alpha_s(Q)$ by means of Eq. (49), then $D(Q)$ is given by the series

$$D(Q) = \sum_{n=0}^{\infty} \sum_{k=\max[0,\, n-1]}^{\infty} \frac{(\ln\ln(Q^2/\Lambda_S^2))^k}{(\ln(Q^2/\Lambda_S^2))^n} a_{nk}^S, \quad (63)$$

in which the coefficients a_{nk}^{OMS} and a_{nk}^{MOM} for $N_f = 3$ are equal to each other for all n and k. Indeed, the change in the coefficients $a_{nk}^S \to a_{nk}^{S'}$ associated with the transition $S \to S'$ from the one scheme to the other can be compensated in (63) solely by transition from the parameter Λ_S associated with the scheme S to the Λ parameter of the scheme S': $\Lambda_S \to \Lambda_{S'} = \varkappa_{SS'}\Lambda_S$. It follows from this, first, that the first three coefficients (a_{00}, a_{10}, and a_{11}) in (63) do not depend on the chosen scheme and, second, that, fixing the coefficient $a_{20} \equiv 4d_2(1)/b_0$, we uniquely fix all the remaining coefficients a_{nk}.[41,42]

Minimal-Sensitivity Principle. Thus, if the series (63) was obtained in the scheme S from the series (54) for $\mu = Q$, to obtain the series (63) with the same coefficients a_{nk} in some different scheme S' it is necessary in the original series (54) to replace $\mu = Q$ by $\mu = \Lambda_{S'}Q/\Lambda_S$. In other words, if the final result for $D(Q)$ is represented in the form of the series (63), then the arbitrariness in the choice of the scheme is equivalent to the arbitrariness in the choice of the argument for $\bar{g}(\mu^2)$. Therefore, investigating the dependence of the approximants $D_n(\mu^2/Q^2, \alpha_s(\mu))$ corresponding to the sum of the first $n+1$ terms of the series (54) on the choice of the parameter μ, one can get a fairly complete pic-

ture of the manner in which the results of calculating $D(Q)$ perturbatively depend on the chosen renormalization prescription.

Since in the considered problem the lowest approximation D_0 does not depend on μ, it is worth analyzing, not D_n, but the difference $\Delta D_n \equiv D_n - D_0$ or its ratio to D_0. The first approximant is

$$\Delta D_1 (\mu^2/Q^2, \alpha_s(\mu))/D_0 = 4/b_0 \ln [(\mu^2/Q^2)(Q^2/\Lambda^2)]. \quad (64)$$

In Fig. 6, we have plotted the corresponding curves for $\ln(Q^2/\Lambda^2) = 4$ (curve A_1) and for $\ln(Q^2/\Lambda^2) = 7$ (curve B_1).

It can be seen that $\Delta D_1/D_0$ has a pronounced dependence on the parameter μ in its entire range of variation, and in this sense the first approximant is very far from the limiting form $\Delta D_\infty(\mu^2/Q^2, \alpha_s(\mu)) = D(Q) - D_0(Q)$, which represents a certain horizontal straight line. Allowance for the following contribution somewhat improves the situation, and curves A_2 and B_2 in Fig. 6, which correspond to the second approximant (for $\ln Q^2/\Lambda^2 = 4$ and 7; $N_f = 3$),

$$\frac{\Delta D_2^{\overline{\text{MS}}}(\mu^2/Q^2, \alpha_s(\mu))}{D_0} = \frac{4}{b_0 L_\mu}\left\{1 - \frac{b_1}{b_0^2}\frac{\ln(L_\mu 4)}{L_\mu} + \frac{4}{b_0}\frac{0.133 N_f - 2.105}{L_\mu} + \frac{\ln(\mu^2/Q^2)}{L_\mu}\right\}, \quad (65)$$

where $L_\mu \equiv \ln(\mu^2/\Lambda^2)$, have a section within which $\Delta D_2(\mu^2/Q^2, \alpha_s(\mu))$ depends rather weakly on μ. Taking into account the following contributions, we must obtain ever flatter curves, approaching in the limit $n \to \infty$ the straight line $D(Q) - D_0(Q)$. On this basis, we must expect to obtain the best approximation for $D(Q)$ by choosing μ in the region where ΔD_2 has a weak dependence on μ. These arguments are the basis of the *minimal sensitivity principle*, which was formulated in Ref. 45 as follows: "If an approximant depends on unphysical parameters, then their values should be chosen so as to minimize the sensitivity of the approximant to small variations in those parameters." Here, "unphysical" means parameters on the particular value of which the true value of the investigated quantity D must not depend. In our example, μ is such a parameter.

Stevenson's criterion[45] is not a mathematical theorem. Rather, it is a rule of optimal behavior under conditions of acute lack of information. However, a number of arguments can be advanced for it. First, the values of the parameter μ_{opt} found from the condition

$$\frac{d}{d\mu}\Delta D_2^{\overline{\text{MS}}}(\mu^2/Q^2, \alpha_s(\mu))|_{\mu=\mu_{opt}} = 0 \quad (66)$$

are equal to 1.25 and 1.52 for curves A_2 and B_2 (see Fig. 6), which is in complete agreement with the assumption that in the MOM and OMS schemes the optimal choice for μ must be near Q. Second, since the curves corresponding to ΔD_1 and ΔD_2 intersect at a point near μ_{opt} (at $\mu = 1.65Q$ in case A and $\mu = 1.95Q$ in case B), the difference between ΔD_1 and ΔD_2, i.e., the value of the three-loop correction, is small for $\mu = \mu_{opt}$. This agrees with the requirement that for the optimal choice of μ the series (54) converges to the exact value of $D(Q)$ fairly rapidly. The effectiveness of Stevenson's criterion was demonstrated for a number of exactly solvable problems.[45,100] However, one can also invent a nondenumerable set of "counterexamples" for which the flat sections of the curves for $\Delta D_n(\mu)$ are fairly close to the exact value ΔD only for very large n. We note that in such a situation perturbation theory fails altogether, and not only the minimal-sensitivity principle. The very use of perturbation theory means that we implicitly assume a fairly rapid convergence of the curves for ΔD_n to the straight line corresponding to the exact result, and in such a situation there is every reason to have confidence in Stevenson's criterion.

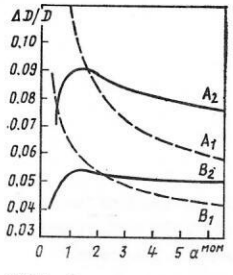

FIG. 6.

Minimization Principles as a Method of Solving the Problem of the Renormalization-Group Nonuniqueness. The transition to a different renormalization scheme corresponds in Fig. 6 to only a change of scale along the horizontal axis; the vertical scale and, therefore, the value of ΔD_2 at the maximum of the curve [i.e., $\Delta D_2(\mu = \mu_{opt})$] does not change. Thus, the use of Stevenson's criterion gives the same result for $\Delta D_2(\mu_{opt})$ in all the renormalization schemes.

The renormalization-group nonuniqueness can be eliminated in other ways. In particular, it was suggested in Refs. 43 and 44 that one should regard as optimal the value of the parameter μ for which the difference $\Delta D_2 - \Delta D_1$, i.e., the value of the three-loop correction, is equal to zero (*correction-minimization principle*). The value of ΔD_2 at the point of intersection of the curves ΔD_1 and ΔD_2 (like the value of ΔD_2 at the maximum of the corresponding curve) does not change under renormalization-group transformations, and, therefore, the final result for ΔD must be independent of the choice μ and (or) the renormalization scheme.

3. FUNDAMENTALS OF THE ANALYSIS OF DEEP INELASTIC LEPTON-HADRON SCATTERING

Introduction. In experiments on deep inelastic scattering one studies the hadron structure functions W_1, W_2, W_3 (see Refs. 3–14, 46–49, 51, and 101), which can be expressed formally in terms of the matrix element of the product of the corresponding hadronic currents[7)]:

$$W_{\mu\nu}(P, q) = \frac{1}{4\pi}\int d^4x \exp(i qx)\langle P|J_\mu^+(x) J_\nu(0)|P\rangle = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) W_1 + \frac{1}{m_h^2}\left(P_\mu - \frac{Pq}{q^2} q_\mu\right)\left(P_\nu - \frac{Pq}{q^2} q_\nu\right) W_2 - i\frac{\varepsilon_{\mu\nu\alpha\beta} P^\alpha q^\beta}{2m_h^2} W_3, \quad (67)$$

where m_h is the hadron mass; the currents J^+ and J are

[7)] To simplify the exposition, we shall consider only deep inelastic scattering by an unpolarized target. This means that in (67) averaging over the hadron spin is understood. Deep inelastic scattering by a polarized target was considered in Refs. 102–104.

taken in the Heisenberg representation. In theoretical analysis, it is more convenient to work with contractions, $W^{\mu}_{\mu}, P_{\mu}P_{\nu}W^{\mu\nu}$, and $\varepsilon_{\mu\nu\alpha\beta}W^{\mu\nu}q^{\alpha}P^{\beta}$, linear combinations of which give W_1, W_2, W_3. In what follows, we shall restrict ourselves to analyzing the form factor $W \equiv -W^{\mu}_{\mu}$. Other contractions can be analyzed by similar methods.

The form factor $W(P, q)$ depends on the momenta P_2 and q only through their scalar products $P^2 = m_h^2$, $q^2 \equiv -Q^2$, $2(Pq) \equiv \omega Q^2$. The kinematic region of deep inelastic scattering is determined by the conditions

$$q^2 < 0,\ Q^2 \gtrsim 1\ \text{GeV}^2,\ (q+P)^2 \gg m_h^2. \tag{68}$$

It follows from the last condition that $\omega > 1$. As in the analysis of $\sigma_{\text{tot}}(e^+e^- \to \text{hadrons})$, in the present case too it is sensible to introduce the amplitude

$$T(\omega, Q^2) = \frac{1}{4\pi}\int d^4x \exp(i\,qx)\langle P|S^+T(j_\mu(x)\,j^\mu(0)\,S)|P\rangle, \tag{69}$$

which is defined in the entire complex ω plane (j_μ are the currents in the interaction representation). The amplitude $T(\omega, Q^2)$ corresponding to virtual Compton scattering through zero angle has on the real axis cuts for $|\omega| > 1$. The function $W(\omega, Q^2)$ is given by the discontinuity across the right-hand cut; using the charge symmetry of the currents j_μ, we can also express $W(\omega, Q^2)$ in terms of the discontinuity across the left-hand cut (see, for example, Ref. 12). In the region $(\omega - 1)Q^2 \sim m_h^2$, the form factor $W(\omega, Q^2)$ has resonance peaks, for the calculation of which the simplest variant of perturbation theory in QCD (without allowance for power-law corrections) does not apply. Therefore, in the resonance region the predictions of QCD correspond only "on the average" to the real behavior of the form factors. The connection between $W(\sigma, Q^2)$ for $\sigma > 1$ and the amplitude $T(\omega, Q^2)$ calculated in the framework of perturbation theory for $\omega < 1$ is given by the dispersion relation [cf. (27)]

$$T(\omega, Q^2) = \int_1^{\infty}\frac{d\sigma}{\sigma-\omega}W(\sigma, Q^2) + \int_{-\infty}^{-1}\frac{d\sigma}{\sigma-\omega}W(\sigma, Q^2). \tag{70}$$

Note that in the considered case $W(\sigma, Q^2) = -W(-\sigma, Q^2)$.

Analysis of Contributions. The general structure of the diagrams that contribute to $T(\omega, Q^2)$ is shown in Fig. 7. As in the analysis of $\sigma_{\text{tot}}(e^+e^- \to \text{hadrons})$, we have a situation in which the square of one of the external momenta q is large. However, in contrast to $\sigma_{\text{tot}}(e^+e^- \to \text{hadrons})$, the form factor W also depends on one momentum P whose square is small, though the scalar product Pq is large.

In this case, the function $F(\alpha, \omega)$ [see (8)] can be written in the general case in the form[10,33]

$$F(\alpha, \omega) = \{A(\alpha) - \omega[A_s(\alpha) - A_u(\alpha)]\}/D(\alpha), \tag{71}$$

where $A, A_s, A_u, D \geq 0$. It can be shown[33] that

$$|[A_s(\alpha) - A_u(\alpha)]/A(\alpha)| < 1. \tag{72}$$

Therefore, in the region $|\omega| < 1$ the function $F(\alpha, \omega)$ cannot vanish because of the compensation of nonvanishing terms. Thus, the pinch regime does not operate in this region.[8)] We can restrict ourselves to analyzing the short-distance and infrared regimes, i.e., for a start we can find ways of eliminating the dependence of

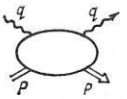

FIG. 7.

the diagrams in Fig. 7 on the variables Q^2 and Pq. We must ensure that the momentum q whose square is large enters and leaves the diagram at the same point, which can be achieved by contracting to a point a certain subgraph V containing both photon vertices. Then, as is readily seen, the diagram (in the scalar theory) also loses its dependence on Pq. Therefore, as in the problem considered in the previous section, such contraction is not only necessary but also sufficient, and all the power-law contributions to $T(\omega, Q^2)$ are determined solely by the short-distance regime.

Besides the two necessary photon lines, the subgraph V must also have L other external lines ($L \geq 2$). Bearing in mind that the external photon lines, as in the problem of $\sigma_{\text{tot}}(e^+e^- \to \text{hadrons})$, do not introduce additional Q factors in the amplitude, we obtain the estimate

$$T^V_{(\text{SDR})}(\omega, Q^2) \leqslant Q^{2-\sum_{i=1}^{L} t_i}, \tag{73}$$

which holds (in four dimensions) for all theories with a dimensionless coupling constant [see (15) and the discussion preceding it]. For methodological reasons, we shall restrict ourselves in the present section to considering deep inelastic scattering in a theory with Yukawa interaction in which the gluons have spin 0 (and twist 1) and there are no fields with zero twist. The analysis of deep inelastic scattering in QCD is given in Sec. 5.

If all the fields have twist equal to unity, the leading power-law contribution $O(Q^0)$ to $T(\omega, Q^2)$ is given in accordance with (73) by integration with respect to the small $\lambda(V)$ of the subgraphs V that have two external lines (besides the two photon lines). For any diagram, such subgraphs can be ordered (Fig. 8):

$$V_n \supset V_{n-1} \supset \ldots \supset V_2 \supset V_1. \tag{74}$$

Remark. If deep inelastic scattering by a composite particle is considered, the maximal subgraph V_n is not identical to the complete diagram.

Analysis of the General Structure of the Leading Power-Law Contributions. The number of subgraphs

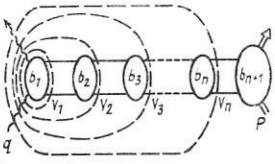

FIG. 8.

[8)] Since $F(\alpha, \omega)$ cannot be represented as the product of two functions $F_1(\alpha, \omega)$ and $F_2(\alpha, \omega)$ that are each the difference of two functions, it follows from Tiktopoulos's theorem[56] that the pinch regime also makes no contribution when $|\omega| > 1$.

$V_1, \ldots, V_n$ responsible for the leading contribution $O(Q^0)$ is determined by the number of two-particle divisions in the (P,P) channel, and it is worth representing each of the investigated diagrams in the form of a certain generalized ladder composed of *two-particle-irreducible* [with respect to the (P,P) channel] subgraphs $b_1, \ldots, b_{n+1}$ (see Fig. 8).

In the language of the Mellin representation (20), the total leading contribution $T^{\text{lead}}(Q)$ can be determined as the result of integration around a contour of small radius $\delta \ll 1$ surrounding the point $J = 0$. The contributions $r(Q)$ due to the singularities of the function $\Phi(J)$ at $J = -1$ and to the left are power-law corrections to $T^{\text{lead}}(Q)$. In accordance with the general analysis, the leading contribution (or, which is the same thing, the pole singularity at $J = 0$) arises only as the result of integration over a region of the space of the α parameters in which at least one of the sums $\lambda(V_i)$ [see (9)] can vanish. Since it follows from $\lambda(V_i) = 0$ that $\lambda(V_j) = 0$ for all smaller subgraphs $V_j \subset V_i$, each leading contribution T_i can be characterized by the maximal subgraph V_i for which integration over the region $\lambda(V_i) \sim 0$ gives the pole $1/J$. In other words,

$$T(Q) = T^{\text{lead}}(Q) + r(Q) = \sum_{i=1}^{n} T_i(Q) + r(Q). \tag{75}$$

The structure of the expression (75) can be reproduced explicitly by the following construction. We integrate first over the region in which the sum of the α parameters of the lines forming the largest subgraph V_n is small: $\lambda(V_n) < 1/\mu^2$, where μ is some scale. We denote the corresponding contribution by $T_n(Q, \mu)$, and then in the remaining region of integration we separate the subregion in which $\lambda(V_{n-1}) < 1/\mu^2$. We note the result of integration by $T_{n-1}(Q, \mu)$, etc., until all possibilities have been exhausted. In such a construction, the contribution T_i to (75) is due to integration over the region in which $\lambda(V_i) < 1/\mu^2$ but $\lambda(V_j) > 1/\mu^2$ for all subgraphs V_j containing V_i.

Frequently, it is more convenient to have a formal definition of the procedure of separating the contributions that leads to (75). Let M_i be the operation that separates the leading SDR contribution of the subgraph V_i. For example, in the above construction M_i corresponds to integration over the region in which $\lambda(V_i) < 1/\mu^2$. Then the procedure is none other than application of the identity

$$\begin{aligned} 1 \equiv M_n &+ (1-M_n)M_{n-1} + (1-M_n)(1-M_{n-1})M_{n-2} + \ldots \\ &+ (1-M_n)(1-M_{n-1})\ldots(1-M_2)M_1 \\ &+ (1-M_n)(1-M_{n-1})\ldots(1-M_2)(1-M_1) \end{aligned} \tag{76}$$

to the amplitude T, written in the α representation. In what follows, we shall denote the operation $(1 - M_i)$ corresponding to the condition $\lambda(V_i) > 1/\mu^2$ by R_i. In this notation,

$$T_i = R_n \ldots R_{i+1} M_i T; \tag{77}$$

$$r = R_n R_{n-1} \ldots R_1 T. \tag{78}$$

Remark. Definition of the operation M_i in terms of a simple restriction on the region of integration is too crude. Such a definition does not take into account the fact that integration over the region $\lambda(V_i) \sim 0$ can lead not only to the pole $1/J$ but also to contributions regular at $J = 0$, for example, poles $1/(J+n)$ $(n = 1, 2, 3, \ldots)$ situated at the point $J = -1$ and to its left. To formulate a more accurate definition of the operation M_i, we analyze in more detail the structure of the integral with respect to $\lambda(V_i)$.

Separation of the Leading Contributions. The integral with respect to $\lambda(V_i)$ appears as a result of the substitution

$$\begin{aligned} &\lambda_i \equiv \lambda(V_i) = \sum_{\sigma \in V_i} \alpha_\sigma;\ \alpha_\sigma = \lambda_i \beta_\sigma; \\ &\prod d\alpha_\sigma = \lambda_i^{l-1} \prod d\beta_\sigma \delta(1 - \Sigma\beta_\sigma), \end{aligned} \tag{79}$$

where l is the number of internal lines of the subgraph V_i. Using (79), we have

$$\begin{aligned} \Phi(J, \omega, p^2) = &\int_0^\infty \lambda_i^{J-1} d\lambda_i \int_0^1 \prod_{\sigma \in V_i} d\beta_\sigma \delta(1 - \Sigma\beta_\sigma) \\ \times &\int \prod_{\bar{\sigma} \in \bar{V}_i} d\alpha_{\bar{\sigma}} \varphi(J, \omega, \beta_\sigma, \alpha_{\bar{\sigma}}, \lambda_i; p^2) \exp[i p^2 I(\alpha_\sigma, \alpha_{\bar{\sigma}})]. \end{aligned} \tag{80}$$

where the lines $\bar{\sigma}$ belong to the subgraph $\bar{V}_i = \mathcal{G} \setminus V_i$. Note that the p^2 dependence of the function φ in (80) is related to the pre-exponential factor $G(\alpha, p)$ in (3).

Expanding the integrand in a series in λ_i,

$$\begin{aligned} &\varphi(J, \omega, \beta_\sigma, \alpha_{\bar{\sigma}}, \lambda_i; p^2) \exp[i p^2 I(\alpha_\sigma, \alpha_{\bar{\sigma}})] \\ &= \varphi(J, \omega, \beta_\sigma, \lambda_i = 0, p^2) \exp[i p^2 I_0(\alpha_{\bar{\sigma}})] + \sum_{k=1}^{\infty} \lambda_i^k \varphi_k(J, \beta_\sigma, \alpha_{\bar{\sigma}}, p^2) \end{aligned} \tag{81}$$

(where $I_0(\alpha_{\bar{\sigma}}) \equiv I(\alpha_\sigma = 0, \alpha_{\bar{\sigma}})$ is the function $I(\alpha)$ constructed for the diagram for which the subgraph V_i is contracted to a point), we see that integration over the region $\lambda_i \approx 0$ gives the pole $1/J$ only for the first term in (81). Therefore, only this term makes a contribution to T_i, and the contributions of the terms with $k \geq 1$ must be included in $T_{i-1}, \ldots, T_1$ or r. Thus, the operation M_i must separate from the integrand in (81) the part of it that after integration over the region $\lambda(V_i) < 1/\mu^2$ does indeed give the pole $1/J$. We denote this operation by $M_i^{(1)}$:

$$\begin{aligned} M_i^{(1)}\Phi(J, \omega, p^2) = &\left(\frac{1}{\mu^2}\right)^J \frac{1}{J} \int_0^1 \prod_\sigma d\beta_\sigma \delta(1 - \Sigma\beta_\sigma) \\ &\times \int_0^\infty \prod_{\bar{\sigma}} d\alpha_{\bar{\sigma}} \varphi(J, \omega, \beta_\sigma, \alpha_{\bar{\sigma}}, \lambda_i = 0, p^2). \end{aligned} \tag{82}$$

We bear in mind further that the leading contribution $T^{\text{lead}}(Q)$ is given by an integral around a small contour surrounding the point $J = 0$. This means that if we expand the expression $\Phi_i^{(1)}(J, \omega, p^2) \equiv (\mu^2)^J M_i^{(1)} \Phi(J, \omega, p^2)$ in a Laurent series at $J = 0$, only the pole terms [we denote their sum by $\Phi_i^{(1)\,\text{pole}}(J, \omega, p^2)$] contribute to $T^{\text{lead}}(Q)$, and the terms regular at $J = 0$ must be added to r. We denote the corresponding operation by $M_i^{(2)}$:

$$M_i^{(2)} \Phi_i^{(1)}(J, \omega, p^2) = \Phi_i^{(1)\,\text{pole}}(J, \omega, p^2). \tag{83}$$

Finally, $M_i = M_i^{(2)} M_i^{(1)}$, and the expressions (82) and (83) give the complete definition of the operation M_i.

Remark. The contribution of the region of large λ_i can be limited by introducing in the integral over λ_i any function that decreases sufficiently rapidly when $\lambda_i \gtrsim 1/\mu^2$, for example, $\exp(-\lambda_i \mu^2)$. In the last case, $1/J$ in

(82) must be replaced by $\Gamma(J) = \Gamma(J+1)/J$. In other words, it is possible to have different definitions of the operation $M_i^{(1)}$, these differing from (82) by some factor regular at $J = 0$.

Logarithmic Contributions. The Mellin transform $\Phi_i(J)$ of the contribution T_i has a fairly complicated structure at $J = 0$. This is due to the fact that V_i (for $i > 1$) contains smaller subgraphs $V_{i-1}, \ldots, V_1$ for which integration over the region $\lambda_k \approx 0$ also gives a leading contribution, i.e., a pole at $J = 0$. This means that if we apply the substitution

$$\sum_{\sigma \in V_{i-1}} \beta_\sigma = \rho_{i,\,i-1}; \quad \beta_\sigma = \rho_{i,\,i-1}\beta_\sigma^{(1)} \tag{84}$$

to the β_σ parameters of the lines σ belonging to the subgraph V_{i-1}, the integral over λ_i and β_σ has the form

$$\int_0^\infty \lambda_i^{J-1}\, d\lambda_i \int_0^1 (\rho_{i,\,i-1})^{J-1}\, d\rho_{i,\,i-1} \times \int_0^1 \prod_{\sigma \in V_i \setminus V_{i-1}} d\beta_\sigma \delta\Big(1 - \rho_{i,\,i-1} - \sum_{\sigma \in V_i \setminus V_{i-1}} \beta_\sigma\Big) \times \int_0^1 \prod_{\sigma \in V_{i-1}} d\beta_\sigma^{(1)} \delta\Big(1 - \sum_{\sigma \in V_{i-1}} \beta_\sigma^{(1)}\Big). \tag{85}$$

Thus, integration over the region $\lambda_i < 1/\mu^2$, $\rho_{i,\,i-1} \approx 0$ gives a factor $(1/\mu^2)^J/J^2$, which corresponds to a logarithmic contribution. Note that the logarithm arises as the result of integration over the region $\lambda_{i-1} \equiv \lambda_i \rho_{i,\,i-1} \ll \lambda_i$, i.e., in a situation in which the "distances" within V_{i-1} are much less than within V_i as a whole. If the subgraph V_{i-1} also contains a smaller subgraph V_{i-2}, then integration over the region $\rho_{i,\,i-1,\,i-2} \approx 0$, where

$$\rho_{i,\,i-1,\,i-2} = \sum_{\sigma \in V_{i-2}} \beta_\sigma^{(1)}; \quad \beta_\sigma^{(1)} = \rho_{i,\,i-1,\,i-2}\beta_\sigma^{(2)}, \tag{86}$$

gives one further pole J^{-1}. Therefore, the region $\lambda_{i-2} \ll \lambda_{i-1} \ll \lambda_i$ gives $\ln^2(Q^2/\mu^2)$, and for a diagram G consisting of n blocks the function $\Phi_G(J)$ has at $J = 0$ poles up to n-th order, i.e.,

$$T_G(Q) = \sum_{i=0}^{n-1} \Big(\ln \frac{Q^2}{\mu^2}\Big)^k a_k(\mu^2, \rho^2) + r_G(Q, p) \equiv T_G^{\text{lead}}(Q, p) + r_G(Q, p). \tag{87}$$

To go over in (87) to a different μ, it is necessary to use the identity

$$\ln(Q^2/\mu^2) \equiv \ln(Q^2/\tilde{\mu}^2) + \ln(\tilde{\mu}^2/\mu^2) \tag{88}$$

and to group the terms with the same power of $\ln(Q^2/\tilde{\mu}^2)$. The corresponding coefficient is $a_k(\tilde{\mu}^2)$.

Remark. Since the substitution (88) does not result in terms of the form $O(1/Q^2)$, i.e., power-law corrections, the sum of the logarithmic terms in (87) and the remainder $r(Q,p)$ do not depend on μ. This is related to the fact that the division into the leading contribution, given by the integral around the small contour surrounding the point $J = 0$, and the power-law corrections corresponding to the poles at $J = -1, -2, \ldots$ is unique.

Subtractional Procedure. The action of the operation $R_i \equiv (1 - M_i)$ can be illustrated by the example of the expression (85). We consider the contribution

$$T_{i-1} = R_n \ldots R_i M_{i-1} T. \tag{89}$$

For the pole part of the integral over λ_i, i.e., for

$$\int (d\lambda_i/\lambda_i)\, \lambda_i^J, \tag{90}$$

the operation $R_i = (1 - M_i)$ corresponds, roughly speaking, to integration over the region $\lambda_i > 1/\mu^2$. We now note that the operation M_{i-1}, which in (89) precedes the operation R_i, includes in accordance with (82) the procedure $\lambda_{i-1} \to 0$, and, therefore, for the contribution T_{i-1} the operation R_i corresponds to the restriction $\lambda(V_i \setminus V_{i-1}) > 1/\mu^2$ on the α parameters of the subgraph $V_i \setminus V_{i-1}$, which is the difference between the subgraphs V_i and V_{i-1} and is outside V_{i-1}. Similarly, the operations $R_{i+1}, \ldots, R_n$ reduce with allowance for the procedure $\lambda_{i-1} \to 0$ to the restriction

$$\lambda(V_k \setminus V_{i-1}) > 1/\mu^2; \quad k \geqslant i \tag{91}$$

for the subgraphs $V_{i+1} \setminus V_{i-1}, \ldots, V_n \setminus V_{i-1}$, which, like $V_i \setminus V_{i-1}$, all lie outside the subgraph V_{i-1}. In what follows, we shall denote the analog of the condition (91) by $\text{Reg}_{\mu^2}^{\text{UV}}(\bar{V}_{i-1})$, and the analog of the condition $\lambda(V_{i-1}) < 1/\mu^2$, i.e., the operation M_{i-1}, by $\text{Reg}_{\mu^2}^{\text{IR}}(\bar{V}_{i-1})$. By virtue of (91), the integral (85) and the analogous integrals over $\lambda_{i+1}, \ldots, \lambda_n$ for the contribution T_{i-1} are all finite at $J = 0$.

Thus, after application of the procedure $\lambda_{i-1} \to 0$ the integrals over $\lambda(V_k \setminus V_{i-1})$ at $J = 0$ diverge formally at the lower limit, i.e., in the ultraviolet region, but the operation $\text{Reg}_{\mu^2}^{\text{UV}}(\bar{V}_{i-1})$ gives a procedure for subtracting the corresponding singularities. In the case when the contributions $O(\lambda_i)$ are taken in the expansion with respect to λ_i [see (91)], the integrals over λ_i are not singular at $J = 0$, and for such contributions we obviously have $M_i = 0$ and $R_i = 1$, i.e., R_i does not give any restrictions on the integral over λ_i, but they are "not necessary," since these integrals converge at $J = 0$.

Remark 1. In the considered problem $V_k \supset V_i$, and the fulfillment of the condition $\lambda(V_i \setminus V_{i-1}) > 1/\mu^2$ automatically entails fulfillment of the condition (91) for all $k > i$. Further, the conditions (91) will certainly be satisfied if

$$\alpha_1^{(i)} + \alpha_2^{(i)} > 1/\mu^2 \tag{92}$$

for the lines $\sigma_1^{(i)}$ and $\sigma_2^{(i)}$ connecting the blocks b_i and b_{i-1}. The condition (92) together with the condition $\lambda(V_i) < 1/\mu^2$ can be interpreted as follows: the subgraph V_{i-1} corresponds to "short distances," the block b_i is separated from it by "large distances," and the distances between the blocks $b_i, \ldots, b_n$ (and within these blocks) can be arbitrary.

Remark 2. If we use the definition of the operation M_i, which differs from (82) by a certain factor $k(J)$, then, naturally, the operation R_i is also changed. In particular, if M_i signifies introduction of the weight $\exp(-\lambda_i\mu^2)$ in the integral over λ_i, then R_i will correspond to integration over λ_i with weight $(1 - \exp(-\lambda_i\mu^2))$.

Factorization of the Contributions of the Large and Short Distances. Since $\text{Reg}_{\mu^2}^{\text{IR}}(V_i)$ acts only on the α parameters of the lines of the subgraph V_i, and $\text{Reg}_{\mu^2}^{\text{UV}}$ on the α parameters of the lines of the subgraph $\bar{V}_i = G \setminus V_i$, the procedure $R_n \ldots R_{i+1} M_i$ *factorizes*:

$$R_n \ldots R_{i+1} M_i = \text{Reg}_{\mu^2}^{\text{UV}}(\bar{V}_i)\, \text{Reg}_{\mu^2}^{\text{IR}}(V_i). \tag{93}$$

It is more important that the dependence of the func-

tion $\varphi(J, \omega, \beta_\sigma, \alpha_{\bar\sigma}, \lambda_i = 0; p^2)$ in (82) on the parameters $\alpha_{\bar\sigma}$ and β_σ, which correspond to the subgraphs $\overline{V}_i$ and V_i, respectively, also factorizes:

$$\varphi(J, \omega, \beta_\sigma, \alpha_{\bar\sigma}, \lambda_i = 0; \quad p^2) = C(J, \beta_\sigma, \omega) \otimes f(\alpha_{\bar\sigma}, p^2). \tag{94}$$

In particular, it follows from (94) that the coefficient of the pole $1/J$ in (82) is the product of the contributions of the subgraphs V_i and $\overline{V}_i$, namely,

$$\Big[\int_0^1 \prod_{\sigma\in V_i} \delta\Big(1 - \sum \beta_\sigma\Big) C(J, \omega, \beta_\sigma)\Big]$$
$$\otimes \Big[\int_0^\infty \prod_{\bar\sigma\in\overline{V}_i} d\alpha_{\bar\sigma} f(\alpha_{\bar\sigma}, p^2) \exp(ip^2 I_0(\alpha_{\bar\sigma}))\Big]. \tag{95}$$

For the contribution $T_i(Q)$, this means that

$$T_i(Q) \sim \tilde{C}(Q^2, \omega, V_i) \otimes f(p^2, V_i). \tag{96}$$

The absence of a Q^2 dependence in $f(p^2, \overline{V}_i)$ reflects the fact that the contraction of the subgraph V_i to a point, which is the analog of the procedure $\lambda_i \to 0$ in $\varphi(J, \omega, \beta_\sigma, \alpha_{\bar\sigma}, \lambda_i = 0; p^2)$, eliminates the dependence on Q^2. The transition $\lambda_i \to 0$ also has the consequence that in the factor C_i associated with the subgraph V_i there is no p^2 dependence due to the exponential factor $\exp[ip^2 I(\alpha)]$. The function $C(Q^2, \omega, V_i)$ could have a dependence on p^2 due to the pre-exponential factor $G(\alpha, p)$. But in the given case, as follows from dimensional considerations, p^2 can appear only in the form of the power-law correction p^2/Q^2, which, by definition, must not occur in T_i. Thus, in the construction of $C(V_i)$ the small variables p^2 must be equated to zero.

In the simplest cases (for example, in the scalar theories φ^3 and φ^4), the expression (94), which expresses the factorization of the contributions associated with the large and small distances, can be proved directly in the α representation without great difficulty (see Ref. 33). The analysis of theories in which there are particles with nonzero spin is significantly complicated by the presence of the pre-exponential factor $G(\alpha, p)$ due to the numerators of the spinor propagators, vertices with derivatives, etc. The factorization properties of this factor can be obtained only after very laborious analysis. For 4-leg diagrams in a theory with Yukawa interaction, such an analysis was made in Ref. 68. However, in vector theories the subgraph V_i can have an arbitrary number of external vector lines, and this leads to such complications that even in the simplest (Abelian) vector theories the analysis of the factorization by means of the α representation appears to be impossible. In non-Abelian theories (in particular, in QCD) there are further complications associated with the presence of derivatives at the three-gluon vertices and the presence in all vertices of color matrices.

In Ref. 33 it was proposed, on the basis of the topological structure of the subgraphs responsible for the leading contribution, established by means of the (fairly simple) analysis in the α representation, to make the further analysis in the coordinate representation, in which the factorization properties are almost trivial. In this way, one can also establish the connection between the standard approach to the analysis of the asymptotic behavior of deep inelastic scattering based on operator expansions[12,46] and the direct summation methods.

Analysis in the Coordinate Representation. Using (93), we can write the expression (75) in the form

$$T_G(q, P) = \sum_{i=1}^{n} \int \exp(iqx)\, d^4x$$
$$\times R_\varepsilon\Big[\int d\xi\, d\eta\, \mathrm{Reg}^{IR}_{\mu^2} C_{\{\alpha\}}(V_i, q, \xi, \eta, \mu_R, g)$$
$$\times \mathrm{Reg}^{UV}_{\mu^2}(\overline{V}_i, \xi, \eta, P; \varepsilon, \mu_R, g)\Big] + r_G(q, P), \tag{97}$$

where ξ and η are the coordinates of the vertices of the subgraph V_i from which the two lines joining the blocks b_i and b_{i+1} emanate (Fig. 9a).

Note that since G and V_i do not diverge as a whole, the R operation in (97) factorizes,

$$R_\varepsilon\{C(V_i, \varepsilon) f(\overline{V}_i, \varepsilon)\} = \{R_\varepsilon C(V_i, \varepsilon)\}\{R_\varepsilon f(\overline{V}_i, \varepsilon)\}, \tag{98}$$

and its use does not change the structure of the expression (97).

Further, if the external lines of the subgraph V_i are spinor lines, it is worthwhile, using the Fierz identity

$$\delta^\alpha_{\alpha'} \cdot \delta^{\beta'}_\beta = \sum_{a=S, V, T, A, P} (\Gamma_a)^\alpha_\beta (\Gamma^a)^{\beta'}_{\alpha'}, \tag{99}$$

to eliminate the Dirac indices $\{\alpha\}$. For the spin-averaged amplitudes $T(q, P)$, only the S and V projections make a nonvanishing contribution, the V projection being predominant, since in contrast to the S expression it gives an extra P_μ factor and, thus, realizes the upper bound in (73).

The representation (97) is valid for any diagram, and therefore, summing over all diagrams, we obtain

$$T(q, P) = \sum_k \int \exp(iqx)\, dx \int d\xi\, d\eta\, [\mathrm{Reg}^{IR}_{\mu^2} C_k(x, \xi, \eta, \mu_R, \bar g(\mu_R))]$$
$$\times [\mathrm{Reg}^{UV}_{\mu^2} f_k(\xi, \eta, P, \mu_R, \bar g(\mu_R))] + R(q, P), \tag{100}$$

where k labels the different types of two-particle intermediate states (quark, gluon, etc.).

The functions C and f, which are the sums of all possible Feynman diagrams with fixed external lines, can be written compactly in the form of the total Green's functions

$$C_k(x, \xi, \eta; \mu_R, \bar g(\mu_R); \mu^2) \tag{101}$$
$$= \mathrm{Reg}^{IR}_{\mu^2} \langle 0 \mid R_\varepsilon S^+ T(j^\mu(x)\, j_\mu(0)\, \chi_k(\xi)\, \chi_k(\eta)\, S \mid 0\rangle;$$
$$f_k(\xi, \eta; P; \mu_R, \bar g(\mu_R); \mu^2) = \langle P \mid R_\varepsilon S^+ T(\mathcal{O}_k(\varepsilon, \eta; \mu^2) S) \mid P\rangle, \tag{102}$$

where S is the S matrix, χ_k are the currents conjugate to the fields φ_k, and $\mathcal{O}_k$ is a bilocal operator. In particular, for scalar fields

$$\mathcal{O}^{(\varphi)}(\xi, \eta; \mu^2) = \mathrm{Reg}^{UV}_{\mu^2}(:\varphi(\xi)\varphi(\eta):), \tag{103}$$

and for spinor fields

$$\mathcal{O}^{(\Psi)}_\nu(\xi, \eta; \mu^2) = \mathrm{Reg}^{UV}_{\mu^2}(:\bar\psi(\xi)\gamma_\nu\psi(\eta):). \tag{104}$$

The normal product symbol (: :) corresponds to the

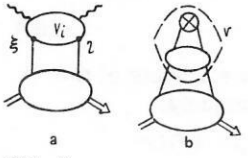

FIG. 9.

fact which is obvious from Fig. 9a that the fields $\varphi(\xi)$ and $\varphi(\eta)$ [or $\overline{\psi}(\xi)$ and $\psi(\eta)$] are paired with operators in the S matrix and not with one another. Further, the lines connecting b_i and b_{i+1} must be included in f_i and not C_i, and therefore f_i will be expressed in terms of an ordinary Green's function and C_i in terms of an "amputated" Green's function containing instead of the fields φ the corresponding currents χ.

Expansion with Respect to Local Operators. In the coordinate representation, the analog of expansion in powers of $\lambda(V)$ is expansion of the bilocal operator $\mathcal{O}_k(\xi, \eta)$ in a series with respect to $\Delta = \xi - \eta$:

$$\mathcal{O}_k(\xi,\ \eta;\ \mu^2) = \sum_{n=0}^{\infty} \frac{\Delta^{\nu_1} \dots \Delta^{\nu_n}}{n!} O_{k\nu_1\dots\nu_n}\left(\frac{\xi+\eta}{2};\ \mu^2\right). \tag{105}$$

The coefficients of this expansion are the local operators $O_{k\nu_1\dots\nu_n}$.

Using the standard methods (see, for example, Ref. 17), we can readily establish that the matrix elements of the local operators $O_{k\nu_1\dots\nu_n}$ have additional divergences that arise in the subgraphs v containing a new vertex (with n derivatives) and having two external lines (see Fig. 9b), i.e., in subgraphs that have the form of the difference $v = V_j \setminus V_i$ of two subgraphs V_j and V_i responsible for the leading contribution. Such divergences, which correspond in the α representation to singularities of the integrals (85) at $J = 0$, are not eliminated by the ordinary R operation. Their appearance reflects the fact that the product of two quantum-field operators at one point is not defined. However, as we have seen above, the operation $\mathrm{Reg}^{UV}_{\mu^2}$ [for example, the condition (91)] ensures that the integrals over λ in which we are interested are finite. Therefore, with allowance for the operation $\mathrm{Reg}^{UV}_{\mu^2}$, which, thus, can be interpreted as the R operation for the new vertices, the expansion (105) is well defined.

Re-Expansion with Respect to Irreducible Operators. The matrix element $\langle P|O_{\nu_1\dots\nu_n}|P\rangle$ is, in general, a sum of very varied structures of the type $(P_{\nu_1}P_{\nu_2}\dots P_{\nu_n})$, $(g_{\nu_1\nu_2}P_{\nu_3}\dots P_{\nu_n})$, etc., each such structure occurring in the sum with a corresponding coefficient that depends on the dynamics at large distances. To introduce order in this chaos, it is worthwhile re-expanding the operators $O_{\nu_1\dots\nu_n}$ with respect to irreducible operators, for which the form of the tensor structure is uniquely determined. For scalar operators $O^{(\varphi)}_{\nu_1\dots\nu_n}$, which are automatically symmetric with respect to $\nu_1, \dots, \nu_n$, this reduces to expansion with respect to traceless tensors:

$$\overset{\nu_1}{\Delta} \dots \overset{\nu_n}{\Delta} (\varphi_k \partial_{\nu_1} \dots \partial_{\nu_n}\varphi)$$
$$= \sum_{l=0}^{[n/2]} \frac{n!\,(n-2l+1)}{l!\,(n-l+1)!}\left(\frac{\Delta^2}{4}\right)^l \{\Delta^{\nu_1} \dots \Delta^{\nu_{n-2l}}\}(\varphi\,(\partial^2)^l\{\partial_{\nu_1}\dots\partial_{\nu_{n-2l}}\}\varphi), \tag{106}$$

where $\{\ \}$ denotes the taking of the traceless part of a tensor:

$$g^{\mu\nu}\{O_{\mu\nu}\dots\} = 0. \tag{107}$$

The tensor structure of the matrix elements of the traceless operators is determined uniquely:

$$i^n\langle P \mid R_\varepsilon S^+ T\,([\mathrm{Reg}^{UV}_{\mu^2} : \varphi\,(\partial^2)^l\{\partial_{\nu_1}\dots\partial_{\nu_{n-2l}}\}\varphi :]\,S) \mid P\rangle$$
$$= 2\{P_{\nu_1}\dots P_{\nu_{n-2l}}\}(M^2)^l A^{(\varphi)}_{n,\,l}(\mu^2,\ \mu_R^2,\ \overline{g}(\mu_R)). \tag{108}$$

The factor $(M^2)^l$, where M is a parameter with the dimensions of mass, is separated in order to make the coefficient $A_{n,\,l}$ dimensionless.

For spinor fields, it is necessary to separate first from $O^{(\psi)}_{\nu\nu_1\dots\nu_n}$ an operator symmetric with respect to all indices:

$$\overline{\psi}\gamma_\nu\,\partial_{\nu_1}\dots\partial_{\nu_n}\psi = (\overline{\psi}\gamma_\nu\,\partial_{\nu_1}\dots\partial_{\nu_n}\overline{\psi})_{\mathrm{sym}}$$
$$+\frac{1}{n+1}\sum_{i=1}^{n}\overline{\psi}\,(\gamma_\nu\partial_{\nu_i} - \gamma_{\nu_i}\partial_\nu)\,\partial_{\nu_1}\dots\partial_{\nu_{i-1}}\partial_{\nu_{i+1}}\dots\partial_{\nu_n}\psi, \tag{109}$$

and then, using expressions analogous to (106), re-expand the operators in (109) with respect to traceless operators. The tensor structure of the matrix elements in this case is also determined uniquely, for example,

$$i^{n-1}\langle P \mid \overline{\psi}_a\{\gamma_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_n}\}\psi_a \mid P\rangle$$
$$= 2\{P_{\mu_1}\dots P_{\mu_n}\}[\tilde{f}_a(n) + (-1)^n\tilde{f}_a(n)]. \tag{110}$$

The re-expansion (106) is particularly helpful in that it also automatically leads to an expansion of the coordinate factor with respect to the traceless combinations $\{\Delta^{\mu_1}\dots\Delta^{\mu_{n-2l}}\}(\Delta^2)^l$. Since the matrix element $\langle P|O((\xi+\eta)/2)|P\rangle$ does not depend on ξ, η (by virtue of translational invariance), the tensor structure of the result of integration over ξ, η for each traceless combination is fixed uniquely:

$$\int d\xi\,d\eta\,\{\Delta^{\mu_1}\dots\Delta^{\mu_{n-2l}}\}(\Delta^2)^l C^{(\varphi)}(x,\ \xi,\ \eta;\ \mu^2)$$
$$= \{x^{\mu_1}\dots x^{\mu_{n-2l}}\}(x^2)^{l-d_\varphi}E^{(\varphi)}_{n,\,l}(x^2\mu^2). \tag{111}$$

Substituting (105), (106), and (111) in (97) and integrating over x, we obtain

$$T^{(\varphi)}(q,\ P) = \sum_{n,\,l=0}^{\infty} C^{(\varphi)}_{n,\,l}(Q^2/\mu^2)\,A^{(\varphi)}_{n,\,l}(\mu^2)\,(M^2)^l\{2qP\}^{n-2l}/(Q^2)^n. \tag{112}$$

In (112), we have introduced notation that will be used in what follows:

$$\{AB\}^n = \{A_{\mu_1}\dots A_{\mu n}\}\{B^{\mu_1}\dots B^{\mu n}\}. \tag{113}$$

In accordance with (112), the leading contribution to $T(q,P)$ corresponds to the operators $\varphi\{\partial_{\mu_1}\dots\partial_{\mu_n}\}\varphi$ with minimal twist, which in the given case is equal to two. The operators $O_l \equiv \varphi(\partial^2)^l\{\partial\dots\partial\}\varphi$, which have twist $2+2l$, give when $l \geq 1$ power-law corrections $O(Q^{-2l})$ to $T^\varphi(q,P)$ [cf. (73)]. Similarly, the leading contribution to $T^{(\psi)}(q,P)$ is due to the operators $\overline{\psi}\{\gamma_{\mu 1}\partial_{\mu_2}\dots\partial_{\mu_n}\}\psi$, and the operators containing ∂^2 or $\gamma_\mu\partial^\mu$, like all operators antisymmetric with respect to any pair of indices, lead to power corrections. Thus, we arrive at the standard representation[12,14,46]

$$T(q,\ P) = \int \exp(iqx)\,dx\Big[\sum_{i=\psi,\ \varphi}\{x^{\nu_1}\dots x^{\nu_n}\}(x^2)^{d^i_0}$$
$$\times E^i_n(x^2\mu^2)\langle P \mid O^i_{\nu_1\dots\nu_n}(0,\ \mu^2) \mid P\rangle + \text{"higher twists"}\Big], \tag{114}$$

i.e., at the operator expansion for $T(q,P)$.

Integrating over x, we obtain [cf. (112)]

$$T(\omega, Q^2) = \sum_{a=\psi,\ \varphi}\sum_{n}^{\infty}\frac{2^n\{Pq\}^n}{(Q^2)^n}C^a_n(Q^2/\mu^2,\ g)\,A^a_n(\mu^2) + Q(1/Q^2). \tag{115}$$

Operator Expansions and Asymptotic Behavior of $T(Q^2, p^2)$ as $Q^2 \to \infty$. Usually, the operator expansion for deep inelastic scattering is written in the form

$$T(J(x)\,J(0)) = (x^2)^{d_\varphi} \sum_{\varphi,\psi} \sum_{n=0}^{\infty} \{x^{\mu_1} \dots x^{\mu_n}\} \times E_n^i(x^2\mu^2)\, O_{\nu_1 \dots \nu_n}(0;\ \mu^2) + \quad \text{"higher twists"}\ , \tag{116}$$

i.e., entirely in the coordinate representation, without any reference to the momentum variables q and P. Such an expression of the representation (115) is, however, justified only to the extent that the asymptotic behavior of the function $T(q,P)$, the Fourier transform of the matrix element $\langle P|TJ(x)J(0)|P\rangle$, stands in the Bjorken limit ($Q^2 \to \infty$, $(Pq) \to \infty$, $\omega \equiv 2(Pq)/Q^2$ fixed) in a one-to-one correspondence with the singularities of the product of the currents $J(x)$ and $J(0)$ on the light cone.[8,105,106] The behavior of the product of the currents at $x^2 = 0$ has been investigated in many studies (Refs. 7, 8, 14, 74, 101, and 105–111). In particular, in Refs. 109 and 110 a mathematically rigorous proof was given of the validity of the expansions on the light cone in the framework of perturbation theory (see also Ref. 117). However, there exist processes whose cross sections can also be expressed in terms of the matrix element of a product of currents, and there is no connection between the asymptotic behavior as $Q^2 \to \infty$ and the singularities of the product $J(x)J(0)$ at $x^2 = 0$. As an example, we can give the process of inclusive production of a massive lepton pair in a collision of two hadrons A and B:

$$\frac{d\sigma}{dQ^2}\Big|_{AB\to\mu^+\mu^-X} \propto \int dx \exp(ikx)\, dk\, \langle AB|J(x)J(0)|AB\rangle\, \delta^+(k^2 - Q^2), \tag{117}$$

where Q is the invariant mass of the $\mu^+\mu^-$ pair. In addition, there are many processes (including, in particular, all purely hadronic hard processes) whose cross sections cannot be related in any simple manner to the matrix elements of a product of currents. Thus, the approach based on operator expansions (in the standard understanding of this expression as an expansion of a product of operators) has a very restricted region of applicability. It was this fact that stimulated the development of the approaches of Refs. 33–36, in which the investigated object is the amplitude $T(q,P)$ irrespective of whether or not it can be expressed in terms of a product of currents.

4. FACTORIZATION AND MODIFIED PARTON MODEL

We now turn to a discussion of the practically most important consequences of the representation (115).

Xi Scaling. The contribution of operators that have twist equal to two contains power-law corrections $O(m_p^2/Q^2)$ due to the nonvanishing mass of the target (for example, a proton). These corrections arise only from the factor $\{P_{\mu_1} \dots P_{\mu_n}\}$, and therefore they can be calculated exactly by using some fairly simple properties of traceless combinations. For example, in 4-dimensional space-time

$$\{AB\}^n = (\sqrt{A^2B^2}/2)^n\, C_n^1((AB)/\sqrt{A^2B^2}), \tag{118}$$

where C_n^1 is the Gegenbauer polynomial (see, for example, Ref. 83)

$$C_n^1(\cos\theta) = \sin[(n+1)\theta]/\sin\theta. \tag{119}$$

Combining the expressions (118) and (119), we readily obtain

$$2^r \{P_{\mu_1} \dots P_{\mu_r}\}\, q^{\mu_1} \dots q^{\mu_r}/(Q^2)^r = [Q^2/2(Pq)]\,[(1/\xi_-)^{r+1} - (1/\xi_+)^{r+1}]\,(\sqrt{1 + m_h^2Q^2/(Pq)^2})^{-1}, \tag{120}$$

where $Q^2 = -q^2$, $m_h^2 = P^2$, and ξ_+ and ξ_- are given by

$$\xi_\pm = Q^2/[(Pq) \mp \sqrt{(Pq)^2 + m_h^2Q^2}]. \tag{121}$$

Using (120), we can rewrite the representation (115) in the form

$$T(q,P) = \frac{x}{\sqrt{1+4m_h^2x^2/Q^2}} \sum_a \sum_{n=2}^{\infty} \left[\left(\frac{1}{\xi_-}\right)^{n+1} - \left(\frac{1}{\xi_+}\right)^{n+1}\right] \times [1 + (-1)^n]\, C_n^a(Q^2/\mu^2, g)\, A_n^a(\mu^2) \equiv T_- - T_+, \tag{122}$$

where $x \equiv 1/\omega = Q^2/(2Pq)$ is the ordinary Bjorken variable. Since $|\xi_+| > 1$, the series for T_+ always converges and, therefore, only the amplitude T_- has a cut for $|x| < 1$, the discontinuity across it giving the structure function $W(x,Q^2)$. To obtain an explicit expression for $W(x,Q^2)$, we write (122) in the form of the Mellin integral

$$T_-(x,Q^2) = \frac{x}{\sqrt{1+4m_h^2x^2/Q^2}} \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{dj}{2\pi i} \frac{\pi}{\sin\pi(j+1)} \times (1/\xi_-)^{j+1}[1 + \exp(i\pi j)]\,[M(j,Q^2) + r(j,Q^2)], \tag{123}$$

where $M(j,Q^2)$ and $r(j,Q^2)$ are the analytic continuations of the coefficients $M_n \equiv E_nA_n$ and $r_n = O(1/Q^2)$ from even n to the complex j plane. Further, we use the fact that

$$(-\omega - i\varepsilon)^{j+1} - (-\omega + i\varepsilon)^{j+1} = 2i\omega^{j+1}\sin\pi(j+1). \tag{124}$$

As a result, we obtain a representation for $\tilde{W}(\omega,Q^2) \equiv W(x,Q^2)$:

$$\tilde{W}(\omega,Q^2) = \frac{T(\omega+i\varepsilon,\ Q^2) - T(\omega-i\varepsilon,\ Q^2)}{2\pi i} = \frac{x}{\sqrt{1+4m_h^2x^2/Q^2}} \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{dj}{2\pi i} \left(\frac{1}{\xi}\right)^{j+1} [M(j,Q^2) + r(j,Q^2)], \tag{125}$$

from which it follows that if the power-law corrections and the logarithmic dependence on Q^2 in $M(j,Q^2)$ are ignored

$$W(x,Q^2,m_h^2)\sqrt{1+4m_h^2x^2/Q^2}/x \tag{126}$$

will be a function of only a single modified scaled variable ξ,[112,113]

$$\xi = \frac{2x}{1+\sqrt{1+4m_h^2x^2/Q^2}} = x - x^3\frac{m_h^2}{Q^2} + \dots, \tag{127}$$

which goes over in the limit $Q^2 \to \infty$ into the ordinary variable x.

Moments of the Structure Functions. Applying to (125) the inverse transformation, we arrive at Nachtmann's formula[112]

$$\int_0^1 \frac{dx}{x} W(x,Q^2)\,\xi^{n+1} = C_n \otimes A_n + r_n(M^2/Q^2). \tag{128}$$

In reality, (128) is a rigorous consequence of the original expression (115), and (125) is valid for all x only when $C_n \otimes A_n + r_n$ are uniformly bounded for different n.

The correct derivation of (128) is based on the use of the dispersion relation (70):

$$T(\omega, Q^2)=\int_1^\infty \left\{\frac{d\sigma}{\sigma-\omega}\tilde{W}(\sigma, Q^2)+(\omega\to-\omega)\right\}, \quad (129)$$

which relates $\tilde{W}(\sigma, Q^2)$, which is observed experimentally, to the amplitude $T(\omega, Q^2)$ in the unphysical region $|\omega|<1$ (this region is also sometimes called the Euclidean region, since it is always possible to make a Wick rotation $k_0\to ik_0$ because of the absence of cuts for $|\omega|<1$). Expanding the integrand in (129) with respect to ω/σ, we obtain

$$T(\omega, Q^2)=2\sum_{n=0, 2, 4, \ldots}\int_1^\infty \frac{d\sigma}{\sigma^{n+1}}\omega^{n+1}W(\omega, Q^2). \quad (130)$$

If we ignore the target mass, comparison of the coefficients of equal powers of ω in (130) and (115) gives the *Polyakov-Cornwall-Norton sum rule*[114,115]:

$$W_n(Q^2)=\int_1^\infty \frac{W(\omega, Q^2)}{\omega^{n+1}}d\omega=\sum_{a=\varphi, \psi} C_n^a(Q^2/\mu^2, g)A_n^a(\mu^2)+r_n(M^2/Q^2), \quad (131)$$

which *for even n* relates $C_n\otimes A_n$ and r_n to the moments of the structure function $\tilde{W}(\omega, Q^2)$.

Similarly, using a number of formulas for the Gegenbauer polynomials C_n^1, we can obtain from (129) the Nachtmann moments (128) (see Ref. 112).

Connection with the Parton Model. Since $r_n(M^2/Q^2)$ contains power-law corrections, for every n there exists a Q_0^2 that in general depends on n and is such that when $Q^2>Q_0^2$ it is possible to ignore r_n. In this case, the use of Eq. (128) makes it possible to express the moments of the structure function $W(x, Q^2)$ in terms of $C(n, Q^2/\mu^2)$ and $A_n^a(\mu^2)$. Of these two quantities, the first describes the interaction at short distances, and therefore in asymptotically free theories the function $C(n, Q^2/\mu^2)$ can be calculated perturbatively. The coefficients A_n^a accumulate information about the interaction at large distances, and for them perturbation theory in realistic field-theoretical models of the strong interactions is inapplicable. These coefficients are to be regarded as phenomenological parameters. It is well known that an analogous situation obtains in the parton model[5]: $W(x, Q^2)$ can be expressed in terms of the parton distribution functions $f_a(x)$, which cannot be calculated in the framework of the model itself:

$$W(x, Q^2)=\sum_a e_a^2[f_a(x)+f_{\bar{a}}(x)]. \quad (132)$$

For the moments,

$$W_n(Q^2)=\sum_a e_a^2\int_0^1[f_a(x)+f_{\bar{a}}(x)]x^n dx/x. \quad (133)$$

Thus, in the parton model the coefficients W_n are proportional to the moments of the parton distribution functions.

Modified Parton Picture. We now attempt, without invoking any ideas from the parton model, to analyze the properties of the functions whose moments are A_n.

In accordance with Carlson's well-known theorem (see, for example, Ref. 116), the sequence of numbers $\{A_n\}$ defines two analytic functions $A^+(n)$ and $A^-(n)$ which are identical to A_n, respectively, for even and odd n:

$$A_a(n)=\frac{1+(-1)^n}{2}A_a^+(n)+\frac{1-(-1)^n}{2}A_a^-(n)\equiv\tilde{f}_a(n)+(-1)^n\tilde{f}_{\bar{a}}(n). \quad (134)$$

This is due to the fact that in the general case the integration over β in the moment formula

$$A_a(n, \mu^2)=\int_{-\infty}^\infty \beta^n f_a(\beta, \mu^2)d\beta/\beta \quad (135)$$

is along the entire real axis. Defining $f_a(\beta)=f_{\bar{a}}(-\beta)$ for $\beta<0$, we can go over to an integral from 0 to ∞, the factor $(-1)^n$ then appearing in front of $f_{\bar{a}}$.

Then, substituting (135) in

$$T_{\text{quark}}(q, P)=\int\exp(iqx)dx\int d\xi d\eta\sum_a C_{\mu_1}^a(x, \xi, \eta; \mu^2)\times\sum_n\frac{(-i)^n}{n!}(\xi-\eta)_{\mu_2}\ldots(\xi-\eta)_{\mu_n}\{P^{\mu_1}\ldots P^{\mu_n}\}(1+(-1)^n)A(n, \mu^2) \quad (136)$$

[see (100), (105), and (110)], we arrive at the parton representation for $T(q, P)$:

$$T_{\text{quark}}(q, P)=\sum_a\int_0^\infty\frac{d\beta}{\beta}t_a(q, \beta P; \mu^2)|_{P^2=0}[f_a(\beta, \mu^2)+f_{\bar{a}}(\beta, \mu^2)], \quad (137)$$

where

$$t_a(q, \beta P; \mu^2)=\int\exp(iqx)\int\exp[-i(P\xi)\beta+i(P\eta)\beta]C_a(x, \xi, \eta; \mu^2)d\xi d\eta \quad (138)$$

is none other than the amplitude T for the subprocess $\gamma_a^*\to\gamma_a^*$, in which parton a has momentum βP (Fig. 10).

Remark. Since $T(\omega, Q^2)$ is an even function of the parameter ω, only $A^+(n)$ contributes to (136), and therefore the sum $f_a+f_{\bar{a}}$ occurs in (137).

Structure of the Amplitude in the Complex ω Plane. The amplitude $T(q, P)$ is regular for $|\mathrm{Re}\,\omega|<1$ and has two cuts beginning at the points $\omega=\pm1$. Similarly, the function $t_a(q, \beta P)$ has cuts beginning at the points $\omega=\pm1/\beta$. It follows, in particular, that the function $f(\beta)$ is equal to zero for $\beta>1$, since otherwise the amplitude $T(q, P)$ would have singularities for $|\mathrm{Re}\,\omega|<1$.

We note further that only the functions $t_a(q, \beta P; \mu^2)$ are responsible for the cuts with respect to ω on the right-hand side of the expression (137). This means that for the discontinuity across the given cut, i.e., for $\tilde{W}(\omega, Q^2)$,

$$\tilde{W}_{\text{quark}}(\omega, Q^2)=\sum_a\int_{1/\omega}^1 d\beta\tilde{w}_a(\beta\omega, Q^2/\mu^2)[f_a(\beta, \mu^2)+f_{\bar{a}}(\beta, \mu^2)], \quad (139)$$

which is analogous to (137), in which $w_a(\omega\beta, Q^2/\mu^2)$ is the discontinuity of the function $t(\beta P, q; \mu^2)$ across the corresponding cut. Calculating the moments

$$\int_1^\infty\frac{d\omega}{\omega^{n+1}}\tilde{W}_{\text{quark}}(\omega, Q^2)=\sum_a\left[\int_1^\infty w_a(\zeta, Q^2/\mu^2)\frac{d\zeta}{\zeta^{n+1}}\right][\tilde{f}_a(n, \mu^2)+\tilde{f}_{\bar{a}}(n, \mu^2)], \quad (140)$$

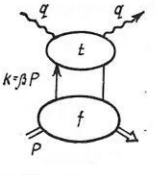

FIG. 10.

we conclude that $C_n^q(Q^2/\mu^2)$ in (115) are the moments of the "structure functions" $w_a(\zeta, Q^2/\mu^2)$ for the quark subprocess.

Gluon Contribution. Because the gluon field is real, the gluon and antigluon are identical and, therefore,

$$\mathrm{Reg}^{LY}_{\mu^2} \langle P \mid S^+ T(:\varphi\{\partial_{\mu_1} \dots \partial_{\mu_n}\}\varphi : S) \mid P\rangle = \frac{1+(-1)^n}{2}\{P_{\mu_1} \dots P_{\mu_n}\}\int_0^1 \frac{d\beta}{\beta} f_g(\beta, \mu^2). \quad (141)$$

Thus, in the general case $W = W_{\text{quark}} + W_{\text{gluon}}$, where

$$\tilde{W}_{\text{gluon}}(\omega, Q^2) = \int_{1/\omega}^1 d\beta \tilde{w}_g(\omega\beta, Q^2/\mu^2) f_g(\beta, \mu^2). \quad (142)$$

However, in contrast to the quark contribution, the gluon contribution is nonvanishing only if allowance is made for the quark-gluon interactions, since neutral gluons cannot directly interact with the electromagnetic field.

Gluons also manifest themselves in the moment formula

$$W_n(Q^2) = \sum_{a=1}^{N_f} C_n^a(Q^2/\mu^2)(f_a(n, \mu^2) + f_{\bar{a}}(n, \mu^2)) + C_n^g(Q^2/\mu^2) f_g(n, \mu^2) + r_n(M^2/Q^2). \quad (143)$$

Remark. The relations (134) and (141) are valid for all n. However, it is only for even n that the quantities $[f_a(n) + f_{\bar{a}}(n)]$ are proportional to the matrix elements of the local operators (108) and (110). For odd n, these matrix elements for the quark operators are proportional to $f_a - f_{\bar{a}}$, and for the gluon operator are equal to zero. In other words, Eq. (143) for arbitrary n is given by analytic continuation from even n.

Singlet and Nonsinglet Contributions. Equation (143) contains contributions of two types, quark and gluon, and, in addition, the quarks themselves can be of different species. Since the quark masses are negligibly small compared with Q^2, the differences between the functions $C^q(V, \xi, \eta, \dots)$ for the different quark species are due solely to the differences between the electric charges e_a. It must also be borne in mind that the subgraphs V_i corresponding to the quark contribution can be of two types (Fig. 11). In the first, we have the subgraphs that do not contain gluon divisions in the channel (P, P) (see Fig. 11a). The corresponding contribution C_I^q is proportional to e_a^2:

$$C_I^q = e_a^2 C^{NS}. \quad (144)$$

In the second, we have the subgraphs containing at least one gluon division in the channel (P, P) (see Fig. 11b). The corresponding contribution C_{II} is proportional to $\langle e^2\rangle$, the mean square of the quark charge,

$$\langle e^2\rangle = \frac{1}{N_f}\sum_{a=1}^{N_f} e_a^2, \quad (145)$$

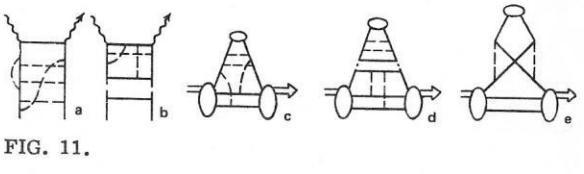

FIG. 11.

since in the quark cycle between the gluon division and the photon vertices it is necessary to sum over all quark species. Thus,

$$W_n = C^{NS}\sum_{a=1}^{N_f} e_a^2 A_n^+(n) + \langle e^2\rangle C_{II}\sum_{a=1}^{N_f} A_a^+(n) + \langle e^2\rangle C_g f_g\langle n\rangle, \quad (146)$$

and the functions C^{NS} and C_{II} do not depend on the quark species. We introduce the function $C^S = C^{NS} + C_{II}$, which corresponds to the sum of the contributions of all the subgraphs with quark lines. Then

$$W_n = C_n^{NS}\sum_{a=1}^{N_f}(e_a^2 - \langle e^2\rangle) A_a^+(n) + \langle e^2\rangle\{C_n^S\sum_{a=1}^{N_f} A_a^+(n) + C_n^g\tilde{f}_g(n)\}. \quad (147)$$

From the group point of view [we are referring to the quark flavor group $SU(N_f)$], this corresponds to decomposition of the matrix $\mathrm{diag}(e_a^2)$ into a singlet component proportional to the unit matrix and a nonsinglet (traceless) component: $\mathrm{diag}(e_a^2) = \langle e^2\rangle 1 + \mathrm{diag}(e_a^2 - \langle e^2\rangle)$, which is achieved by means of the Fierz identity for the matrices of the group $SU(N_f)$:

$$C^{ab} f_{ab} = \frac{1}{N_f}\mathrm{Sp}(C1)\,\mathrm{Tr}(f1) + \frac{1}{2}\sum_{k=1}^{N_f^2-1}\mathrm{Tr}(\lambda^k C)\,\mathrm{Tr}(\lambda_k f). \quad (148)$$

The matrices λ_k are normalized in (148) in accordance with $\mathrm{Tr}(\lambda_k\lambda_i) = 2\delta_{ik}$. To go over from (148) to (147) it is necessary to take into account the relation

$$\frac{1}{2}\sum_k \mathrm{Tr}(\lambda^k C_q)\lambda^k = \mathrm{diag}(e_a^2 - \langle e^2\rangle) C^{NS}. \quad (149)$$

The subgraphs $\bar{V}$ that contribute to f_a can also be decomposed into singlet and nonsinglet components. One frequently also uses a division of the quark distribution functions into the "valence" and "sea" parts. *Valence quarks* are the *quarks of the minimal quark set* having the quantum numbers of the investigated hadron, and the *sea quarks* are the *quarks which arise as a result of the production of* $q\bar{q}$ *pairs*. In this connection, diagrams having gluon divisions (see Fig. 11c) contribute only to the "sea" component. It is a fairly wide misconception that diagrams without gluon divisions contribute only to the distribution functions of the valence quarks. (This corresponds to identification of the concepts of *valence* and *nonsinglet* quarks.) Such an identification is valid for the diagram shown in Fig. 11d, but there are diagrams (see Fig. 11e) that contribute to the antiquark distribution function but do not have purely gluon divisions in the (P, P) channel.[117]

Factorization and the Renormalization Group. As we have already pointed out, the parameter μ can be interpreted as a renormalization parameter of the vertices specific for the composite operators. In other words, the μ dependence of $\tilde{f}_n(\mu, \dots)$ is due to the renormalization-group logarithms in the matrix elements of the composite operators. Since $M_n(Q)$ does not depend on the choice of μ, this means that the dependence of $M_n(Q)$ on Q due to the *mass logarithms* $[\ln(Q^2/P^2)]^N$ is uniquely related to the dependence of $\tilde{f}_n(\mu^2, \dots)$ on μ due to the *renormalization-group logarithms* $[\ln(\mu^2/p^2)]^N$. Since the vertex functions corresponding to the composite operators are renormalized multiplicatively,

$$\mu^2 \frac{d}{d\mu^2} \tilde{f}_n(\mu^2, \mu_R^2 = \mu^2, g) \equiv \left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(g) \frac{\partial}{\partial g}\right) \tilde{f}_n(\mu^2, g) = \gamma_n(g) \tilde{f}_n(\mu^2, g), \quad (150)$$

to calculate their dependence on μ it is sufficient to know the *anomalous dimensions* $\gamma_n(g)$. Here and in what follows, to avoid complications with the $\ln(\mu^2/\mu_R^2)$ contributions, we shall assume that the parameter μ and the renormalization parameter μ_R of the ordinary R operation are equal: $\mu_R = \mu$. This standard device (of decreasing the number of independent renormalization parameters) is used, as a rule, in all renormalization-group calculations (see, for example, Ref. 17).

Note that the presence of gluon divisions in the diagrams for $\tilde{f}_n$ (Figs. 12a–12d) leads to a mixing of the gluon and quark operators and, thus, for singlet operators γ_n is a matrix:

$$\gamma^S = \begin{pmatrix} \gamma_{qq}^S, & \gamma_{qg} \\ \gamma_{gq}, & \gamma_{gg} \end{pmatrix}. \quad (151)$$

In the nonsinglet channel, there is no such mixing, and γ^{NS} is simply a number: $\gamma^{NS} = \gamma_{qq}^{NS}$. It must be borne in mind that $\gamma_{qq}^S \neq \gamma_{qq}^{NS}$, since in the higher orders there are diagrams (see, for example, Fig. 12e) that contribute to γ_{qq}^S but not to γ_{qq}^{NS}.

The solution of Eq. (150) is well known[17,46]:

$$\tilde{f}(n, \mu^2, \bar{g}(\mu^2)) = T \exp\left(\int_{\mu_0^2}^{\mu^2} \gamma_n(\bar{g}(t)) \frac{dt}{t}\right) \tilde{f}(n, \mu_0^2, \bar{g}(\mu_0^2)), \quad (152)$$

where T denotes t ordering of the matrices γ_n (for f^{NS}, the symbol T can be omitted).

Therefore, if the coefficients $\tilde{f}(n, \ldots)$ are known for some $\mu^2 = \mu_0^2$, and $\bar{g}(\mu_0)$ is sufficiently small, we can use (152) to find $\tilde{f}(n, \mu^2, \bar{g}(\mu^2))$ as a function of μ^2 in the complete range of μ^2 values for which perturbation theory is valid. In particular, if we set $\mu = aQ$ in (147), then

$$W_n(Q) = \sum_i C_n^i(a^2, \bar{g}(a^2Q^2)) \tilde{f}_n^i(a^2Q^2). \quad (153)$$

The dependence of the matrix element $\tilde{f}_n(a^2Q^2)$ on Q^2 is given by (147), and the *coefficient function* $C_n(a^2, \bar{g}(a^2Q^2))$ is a series in $\bar{g}(a^2Q^2)$:

$$C_n(a^2, \bar{g}(a^2Q^2)) = \sum_{m=0}^{\infty} C_n(a, m) (\bar{g}^2(a^2Q^2))^m. \quad (154)$$

Thus, knowing $\gamma_n(g)$ and $C_n(a, \bar{g})$, we can find the dependence of $M_n(Q^2)$ on Q^2.

Remark 1. The applicability of renormalization-group methods to calculate the asymptotic behavior of quantities such as $T(P, q)$, which depend on both large, Q^2, and small, p^2, momentum variables, is by no means obvious because of the fact that $T(Q^2, p^2)$ contains mass logarithms $\ln(Q^2/p^2)$. The mass logarithms arise in convergent integrals, and the ordinary renormalization-group transformations associated with variation of the parameter μ_R do not affect them. Therefore, at this stage it is meaningless to use the renormalization group for their summation. If the renormalization-group method is to make it possible to obtain nontrivial information about $T(Q^2, p^2)$, it is necessary to find a procedure which possesses the necessary properties of the R operation and, in addition (which is the important thing), affects the mass logarithms. Such a procedure need not exist. But in the considered problem, the procedure for splitting the contributions of the large and small distances satisfies the formulated requirements, and this ensures that renormalization-group methods can be used to analyze the asymptotic behavior of $T(q, P)$.

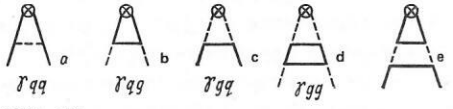

FIG. 12.

Remark 2. As in the analysis of $\sigma_{tot}(e^+e^- \to \text{hadrons})$, to fix the parameter a in (153) it is necessary to augment the renormalization-group analysis by some minimization criterion.

Evolution of the Distribution Functions. The variation of the coefficients $\tilde{f}_n(\mu^2)$ with increasing μ^2 means that the distribution functions $f(x, \mu^2)$ also evolve. Using the connection between $f(x, \mu^2)$ and $\tilde{f}(n, \mu^2)$,

$$\tilde{f}(n, \mu^2) = \int_0^1 x^{n-1} f(x, \mu^2)\, dx, \quad (155)$$

we arrive at the integro-differential equation

$$\left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(g) \frac{\partial}{\partial g}\right) f_a(x, \mu^2, g) = \sum_b \int_x^1 \frac{dy}{y} P_{ab}\left(\frac{x}{y}, g\right) f_b(y, \mu^2, g), \quad (156)$$

whose solution gives the law of evolution of the distribution functions $f(x, \mu^2)$. By definition, $P_{ab}(z, g)$ is a function whose moments are equal to $\gamma_{ab}(n)$:

$$\int_0^1 P_{ab}(z)\, z^n \frac{dz}{z} = \gamma_{ab}(n). \quad (157)$$

In the leading logarithmic approximation, i.e., when allowance is made for only the $O(g^2)$ terms in P_{ab}, Eq. (156) has the following simple parton interpretation. As we have already pointed out, the functions $f_a(x, \mu^2)$ describe the parton momentum distribution in a situation in which the hadron structure is probed at distances of order $1/\mu$. If μ, i.e., the resolution of our "partonmeter," is less than the characteristic hadron scale $1/R_{had}$, we shall not see at all that within the hadron (for example, a proton) there are smaller constituents. However, beginning with a certain $\mu = \mu_0$ the three valence quarks begin to be seen in the proton, and at a certain $\mu = \mu_1 = N\mu_0$ so do the virtual gluons emitted by the valence quarks. If μ is increased by a further N times, the partonmeter will see the virtual $q\bar{q}$ pairs surrounding the valence quarks; at $\mu = \mu_0 N^3$, the gluons emitted by the virtual quarks will become visible, and so forth (Fig. 13). Thus, at $\mu = N^k\mu_0$ the k-th level of hadron structure is seen. In such a picture, $P_{ab}(x/y)$ for $x \neq y$ characterizes the probability that parton b,

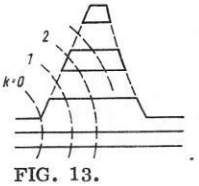

FIG. 13.

having longitudinal momentum yP, goes over (by emitting a gluon or producing a $q\bar{q}$ pair) into parton a with longitudinal momentum xP. Probing the hadron with a partonmeter with resolution $\mu = \mu_0 N^{k+1}$, we obtain the distribution function $f_a(x, \mu_0 N^{k+1})$, which differs from the functions $f_a(x, \mu_0 N^k)$ measured for $\mu = \mu_0 N^k$. This difference is due to the fact that at $\mu = \mu_0 N^k$ a certain parton b with momentum yP is observed as a point, but at $\mu = \mu_0 N^{k+1}$ one can with probability $P_{ab}(x/y)$ observe in it parton a with momentum xP. It should be borne in mind that if p_a is the total probability that parton a is observed after an increase in resolution by N times as consisting of "smaller" partons, then with probability $1 - p_a$ we shall observe at $\mu = \mu_0 N^{k+1}$ the same parton as at $\mu = \mu_0 N^k$. We write down the balance equation

$$f_a(x, \mu_0 N^{k+1}) = f_a(x, \mu_0 N^k)(1-p_a) + \int_{x+0}^{1} \frac{dy}{y} P_{ab}(x/y, g) f_b(x, \mu_0 N^k). \quad (158)$$

Replacing the difference

$$f_a(x, \mu_0 N^{k+1}) - f_a(x, \mu_0 N^k) \quad (159)$$

by the derivative $(d/dk) f_a(x, \mu_0 N^k)$ and noting that $k = \ln(\mu/\mu_0)$, we obtain from (158) the evolution equation (156), in which $P_{ab}(x/y)$ for $x = y$ must be taken equal to $[-p_a \delta_{ab} \delta(1 - x/y)]$. In accordance with their meaning, the coefficients p_a are not quantities independent of $P_{ab}(x/y)$. Explicit expressions for p_a in terms of $P_{ab}(x/y)$ (for $x \neq y$) can be obtained by using the conservation laws for the total momentum and electric charge of the system of partons in gluon-emission and pair-production events. In particular, it follows from the equation

$$\frac{d}{d(\ln \mu)} \left(\sum_a e_a \int_0^1 dx f_a(x, \mu^2) \right) = 0 \quad (160)$$

that for quarks and antiquarks $p_a = \rho_{ab}(1)$, where

$$\rho_{ab}(n) = \int_0^{1-0} z^n \frac{dz}{z} P_{ab}(z). \quad (161)$$

Similarly, from the equation

$$\frac{d}{d(\ln \mu)} \left(\sum_a \int_0^1 x\, dx f_a(x, \mu^2) \right) = 0, \quad (162)$$

which expresses the energy-momentum conservation law, we obtain the equation

$$\mathrm{Det}\,(\rho_{ab}(2) - p_a \delta_{ab}) = 0. \quad (163)$$

Solving (163), we find an explicit expression for p_g in terms of $\rho_{ab}(2)$ and p_q:

$$p_g = \rho_{gg}(2) - 2N_f \rho_{qg}(2)\, \rho_{gq}(2)/[\rho_{qq}(2) - p_q]. \quad (164)$$

Remark 1. Evolution equations were proposed by Lipatov[118] in the framework of the approach previously developed in Ref. 11 to analyze the asymptotic behavior of the structure functions in the leading logarithmic approximation. The transition from (156) to (150) by means of the inverse Mellin transformation is due to Parisi and Gross.[119,120] The parton interpretation of the evolution equations, in the form expounded above, was given in Ref. 121. However, evolution equations came to be widely used only after the appearance of Ref. 110, in which Altarelli and Parisi gave a simple derivation of the evolution equations in quantum chromodynamics. In this connection, the evolution equations in QCD are sometimes called the Lipatov-Altarelli-Parisi equations. Anticipating somewhat, we also note that the evolution equations in the different field-theoretical models differ only in the form of the kernel $P_{ab}(x/y)$. In QCD, this kernel was calculated by Dokshitser,[122] who used the Gribov-Lipatov approach[11] (see also Ref. 51), at the same time as it was calculated by Altarelli and Parisi.

Remark 2. In the analysis of the structure functions in the region of large Q^2, a very important problem is that of finding model expressions for the distribution functions satisfying the following requirements: First, they are fairly compact, second, they give a good description of the experimental data at some fixed $Q^2 = Q_0^2$, and, third, they satisfy the evolution equations with high accuracy. The simplest of the parametrizations published in the literature was proposed by Buras and Gaemers.[123] A somewhat more complicated parametrization for $f_a(x, Q^2)$ that, however, satisfies all three requirements much better was developed by Isaev and Kovalenko.[124]

5. ANALYSIS OF DEEP INELASTIC SCATTERING IN QUANTUM CHROMODYNAMICS

The efforts expended in the previous sections studying models with scalar gluons are justified by the fact that the entire program of factorization of the contributions of large and small distances can, as will be demonstrated below, be carried through in quantum chromodynamics too, and the structure of the final result for $T(\omega, Q^2)$ in QCD is given by the same expression (115) as in the scalar gluon model considered above.

Structure of the Contributions. In QCD, as in any vector theory, there are fields A_μ^a with vanishing twist. This means that the contributions of the configurations of Fig. 14a do not have additional $1/Q^2$ factors compared with the contribution of the simplest configuration (see Fig. 14b), irrespective of the number of external gluon lines of the subgraph V corresponding to the integration over the small α. It is readily seen that any of the configurations in Fig. 14a can be obtained from the corresponding simplest configuration by adding gluon lines joining the lines of the original subgraph v_0 to the remaining diagram. This means that the parton subprocess takes place not in vacuum but in the gluon field of the hadron. It is therefore appropriate to consider at once the sum of the contributions of all configurations obtained from the same simplest configuration, i.e., to sum over the external gluon lines of the subgraphs V corresponding to small α.

Quark Contribution. For the summation over the gluon lines, the coordinate representation is the most con-

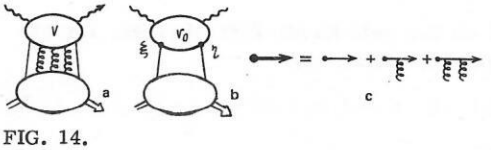

FIG. 14.

venient. Let

$$\int \prod dx_i \int d\xi\, d\eta C_v(v_0, q; \xi, \eta, \{x_i\}; \mu^2) \times \langle P \mid S^+ T(\mathrm{Reg}^{UV}_{\mu^2} : \bar\psi(\xi)\gamma^\nu\psi(\eta) : S) \mid P\rangle \quad (165)$$

be the contribution of the simplest configuration (see Fig. 14b), and x_i be the coordinate of the i-th internal vertex of subgraph v_0. The transition to the configuration in which the subgraph v has one additional external gluon line corresponds to the addition of the field $A^a_\mu(z)$ to the matrix element $\langle P \mid \bar\psi \ldots \psi \mid P\rangle$ describing the contribution of the large distances and the modification of one of the propagators [for example, $S^c(x_1 - x_2)$] corresponding to some internal line of the subgraph v_0:

$$S^c(x_1 - x_2) \to g\int d^4z S^c(x_1 - z)\gamma^\mu\tau_a S^c(z - x_2), \quad (166)$$

where $(\tau_a)_{AB}$ is a matrix of the $SU(3)_c$ gauge group in the quark (fundamental) representation $\tau_a = \lambda_a/2$, where λ_a is is a Gell-Mann matrix (see, for example, Ref. 46). It is readily seen that the sum over the gluon lines connected to the line (x_1, x_2), i.e.,

$$\mathscr{S}^c(x_1, x_2; A) = S^c(x_1 - x_2) + g\int d^4z S^c(x_1 - z)\gamma^\mu \hat A_\mu(z) S^c(z - x_2) + \ldots \quad (167)$$

(see Fig. 14c), gives formally the propagator of the quark in the external gluon field. In other words, (167) is a perturbation-theory expansion of the solution of the equation

$$(i\hat D_\mu\gamma^\mu - m_q)\mathscr{S}^c(x_1, x_2) = -\delta^4(x_1 - x_2), \quad (168)$$

where $\hat D_\mu = \partial/\partial x_1^\mu - ig\hat A_\mu$ is the covariant derivative acting on the quark field, and $A_\mu \equiv A^a_\mu\tau_a$. Thus, each propagator S^c in $C(v_0, \ldots)$ must be replaced by $\mathscr{S}^c$. The solution of Eq. (168) with allowance for the fact that $\hat A$ in the given case is a matrix can be written in the form

$$\mathscr{S}^c(x_1, x_2; A) = \hat E(x_1, x_2; A)\{S^c(x_1 - x_2) + \hat R(x_1, x_2; G)\}. \quad (169)$$

In (169), we have introduced the abbreviated notation

$$\hat E_{AB}(x, y; A) \equiv \left(P \exp ig\int_y^x \hat A_\mu(z)\, dz^\mu\right)_{AB}. \quad (170)$$

The integration in (170) is along the straight line connecting the points x and y, and P in (170) denotes the operation of ordering the matrices $\hat A$ along this path. The function $R(x, y, G)$ satisfies the equation

$$i\gamma^\mu \frac{\partial}{\partial x^\mu}\hat R(x, y; G) - g\gamma^\mu(x^\nu - y^\nu)\left\{\tau^b\int_0^1 t\, dt\, G^a_{\mu\nu}(y + t(x - y))\right. \times \tilde E_{ba}(y + t(x - y), y; A)[S^c(x - y) + \hat R(x, y; G)]\Big\} = 0, \quad (171)$$

where $\tilde E_{ba}(x, y, A)$ is given by Eq. (170), in which it is necessary to take $A_\mu \equiv A^a_\mu\sigma_a$ instead of $\hat A_\mu$ [σ_a is a matrix of the gauge group in the gluon (adjoint) representation: $(\sigma_a)_{bc} = -if_{abc}$, where f_{abc} are the structure constants of the gauge group]. In the derivation of (171), we have used the commutation rule

$$(\tau^a)_{AB}\hat E_{BC}(z, y) = \hat E_{AB}(z, y)(\tau^b)_{BC}\tilde E_{ab}(z, y), \quad (172)$$

which is based on the well-known formula (see, for example, the appendix of Ref. 72)

$$\exp(A)B\exp(-A) = B + [A, B] + (1/2!)[A, [A, B]] + \ldots \quad (173)$$

and the relation

$$[\tau_b, \tau_a] = -(\sigma_b)_{ac}\tau_c. \quad (174)$$

Bearing in mind that

$$G(z)\tilde E(z, y) = \sum_{n=0}^{\infty} G(y)\overleftarrow{\tilde D}_{\mu_1}\ldots\overleftarrow{\tilde D}_{\mu_n}\frac{(z-y)^{\mu_1}\ldots(z-y)^{\mu_n}}{n!}, \quad (175)$$

where $\tilde D_\mu = \partial/\partial y^\mu - ig\tilde A^\mu$ is the covariant derivative acting on the gluon field, we conclude that R depends on the gluon field only through $\hat G_{\mu\nu} = (i/g)[\hat D_\mu, \hat D_\nu]$ and its covariant derivatives. Equation (75) is a special case of the well-known Baker-Hausdorff theorem (see, for example, Ref. 125). We note further that, in contrast to the combination $\langle P \mid \bar\psi \ldots A_\mu \ldots A_\nu \ldots \psi \mid P\rangle$, which gives the double factor $P_\mu P_\nu$, the matrix element $\langle P \mid \bar\psi \ldots G_{\mu\nu} \ldots \psi \mid P\rangle$ cannot add $P_\mu P_\nu$ to the corresponding contribution $C^{\mu\nu}(v)$, since the tensor $\hat G_{\mu\nu}$ is antisymmetric with respect to the indices $\mu\nu$. In the best case, $\hat G_{\mu\nu}$ can add a simple factor P_μ (or P_ν). Therefore, bearing in mind that the dimension of $G_{\mu\nu}$ is two, the field $G_{\mu\nu}$ must be associated with twist 1, like fields with spin 0 and $\frac{1}{2}$. It follows from this in particular that operators of the type $\bar\psi \ldots G \ldots \psi$ have twist not less than three and, thus, the function $R(x, y; G)$ in (169) is responsible in our case only for the power-law corrections to the main contribution. Since we are interested in only the leading power-law asymptotic behavior of the amplitude, we shall in what follows ignore the contribution $\hat R$ when analyzing the configurations in Fig. 14a. Note that by solving Eq. (171) iteratively we can calculate the contributions to $\hat R$ containing G to the first, second, etc., power, i.e., we can take into account the first power corrections.

Thus, if we ignore the power-law corrections, the summation over the gluon lines joined to the spinor line (x_α, x_β) reduces to the substitution

$$S^c(x_\alpha - x_\beta) \to S^c(x_\alpha - x_\beta)\hat E(x_\alpha, x_\beta; A). \quad (176)$$

In Abelian theories the exponential factors associated with the neighboring lines are readily combined:

$$E(x, y)E(y, z) = E(x, z)[1 + O(G)]. \quad (177)$$

The $O(G)$ contributions to (117) due to replacement of the integration along the broken line (xyz) by an integration along the straight line (xz) can be ignored in the approximation of the lowest twists, and as a result, after combination of all the exponential factors, the operator $\bar\psi(\xi)\gamma_\nu\psi(\eta)$ in (165) is replaced by a gauge-invariant bilocal operator,

$$\mathcal{O}_\nu(\xi, \eta; \mu^2) = \mathrm{Reg}^{UV}_{\mu^2}\left\{\bar\psi(\xi)\gamma_\nu \exp\left(ig\int_\eta^\xi A_\mu(z)\, dz^\mu\right)\psi(\eta)\right\}, \quad (178)$$

and the coefficient function $C^\nu(v_0, \xi, \eta)$ remains unchanged. The operator (178) is then expanded in a Taylor series with respect to the local gauge-invariant operators,

$$\mathrm{Reg}^{UV}_{\mu^2}\bar\psi(\xi)\gamma_\nu E(\xi, \eta; A)\psi(\eta) = \sum_{n=0}^{\infty}\frac{1}{n!}(\xi - \eta)^{\nu_1}\ldots(\xi - \eta)^{\nu_n}\,\mathrm{Reg}^{UV}_{\mu^2}\bar\psi(\xi)\gamma_\nu D_{\nu_1}\ldots D_{\nu_n}\psi(\xi). \quad (179)$$

[cf. (175)]. Further, to separate the leading power-law contribution, it is necessary, as for nongauge theories, to symmetrize the operators with respect to the indices

$\nu, \nu_1, \ldots, \nu_n$, then expand them with respect to traceless tensors, and retain only the operators with the lowest twist [cf. (106) and (109)].

Subtractional Procedure. The operation $\mathrm{Reg}_{\mu^2}^{\mathrm{UV}}$ means, as usual, that $\lambda(v) \gtrsim 1/\mu^2$ for subgraphs v represented in the form of the difference of two subgraphs V_1 and V_2, each of which makes a leading contribution in the short-distance regime. In gauge theories, such subgraphs v can have a fairly complicated form. For example, in Fig. 15 the three subgraphs formed by the lines (1, 2, 3), (1, 4, 5), and (1, 2, 3, 4, 5) can be objects of the operation $\mathrm{Reg}_{\mu^2}^{\mathrm{UV}}$.

In non-Abelian theories, in particular in QCD, the gluon lines can be joined not only to quark but also to gluon lines, and also to the lines of fictitious particles (Faddeev-Popov "ghosts"). As a result,

$$\delta_{ab} g_{\mu\nu} D^c(x_\alpha - x_\beta) \to \mathscr{D}^c_{ab,\,\mu\nu}(x_\alpha, x_\beta)$$
$$= [\hat{E}(x_\alpha - x_\beta, A)\{g_{\mu\nu} D^c(x_\alpha - x_\beta) + O(G)\}]_{ab} \qquad (180)$$

for the gluon propagator and

$$\delta_{ab} D^c(x_\alpha - x_\beta) \to [\tilde{E}(x_\alpha, x_\beta; A)\{D^c(x_\alpha - x_\beta) + O(G)\}]_{ab} \qquad (181)$$

for the propagator of the fictitious particles. Note that the factor $E_{ab}(x, y; A)$ has the property

$$\tilde{E}_{ab}(x, y; A) = \tilde{E}_{ba}(y, x; A), \qquad (182)$$

which follows from the fact that $(\sigma_a)_{bc} = -(\sigma_a)_{cb}$.

Since there are both 3-gluon and 4-gluon vertices in QCD, we must also take into account the more exotic possibility when an external gluon line is joined, not to a propagator, but directly to a 3-gluon vertex (Fig. 16). We note further that in the original 3-gluon vertex there is a derivative that acts, for example, on the propagator $D^c(x - y)$. In accordance with (180), the summation over the gluon insertions into this line leads to the modification

$$\delta_{ab} \frac{\partial}{\partial x^\mu} D^c(x-y) \to \tilde{E}(x, y; A)$$
$$\times \left\{ \frac{\partial}{\partial x^\mu} D^c(x-y) + i g \tilde{A}_\mu(x) D^c(x-y) + O(G) \right\}, \qquad (183)$$

the second term on the right-hand side of the expression (183) corresponding to the structure of the 4-gluon vertex. Direct calculations show that the sum of all such contributions is numerically equal to the contribution of the configuration of Fig. 16, but has the opposite sign. Thus, if we ignore the terms $O(G)$, which are unimportant for the leading contribution, the summation over the gluons participating in the subprocess reduces to the substitution $\delta_{ab} \to \tilde{E}_{ab}$, $\delta_{AB} \to \hat{E}_{AB}$ for all propagators corresponding to internal lines of the subgraph v_0.

However, in contrast to Abelian theories, to combine the exponentials corresponding to the neighboring quark lines, it is necessary, using Eq. (172), to commute first one of the exponentials with a τ matrix at a corresponding quark-gluon vertex. As a result, an additional

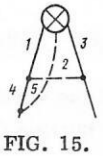

FIG. 15.

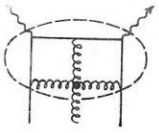
FIG. 16.

factor $\tilde{E}$ appears in accordance with (172). The same factor appears after commutation of the exponentials occurring in the modified propagators of the gluons and the ghosts with the σ matrices at the corresponding triple vertices:

$$\tilde{E}_{ab}(x, y)(\sigma_c)_{bd} = (\sigma_f)_{ab} \tilde{E}_{bd}(x, y) \tilde{E}_{fc}(x, y). \qquad (184)$$

We note that the 4-gluon vertex can be represented as a sum of three terms, each of which has the color structure of two 3-gluon vertices at one point (for example, x) and connected by the factor $\delta_{ab} \equiv E_{ab}(x, x; A)$ (Fig. 17). Bearing in mind all these remarks, we readily see that the exponential factors $\tilde{E}$ arising as the result of the commutations cancel against the analogous factors that occur in the modified propagators of the gluons and the ghosts, and for the propagator v_0 with two external quark lines there remains the factor $\tilde{E}(\xi, \eta; A)$. For the operator $\bar{\psi}(\xi)\gamma_\nu \tilde{E}(\xi, \eta; A)\psi(\eta)$ which then arises Eq. (179) holds, it being necessary to take the covariant derivatives in the quark representation. Thus, we have seen in the special example the validity of the well-known (but usually not proved) general assertion that for gauge theories it is necessary to use gauge-invariant local operators in an operator expansion.

Gluon Contribution. Configurations in which the subgraph V corresponding to short distances has only gluon external lines (Fig. 18a) give in accordance with (73) a contribution $O(Q^2)$ to $T(\omega, Q^2)$. In this case, the original subgraph V_0 has the form of a vacuum photon loop (Fig. 18b), and, since photons are colorless, all the $\hat{E}$ and $\tilde{E}$ factors due to the gluon insertions in V_0 cancel. This means that the summation over the gluons gives the factor $[1 + O(G)]$. The term with unity corresponds to the original disconnected diagram (see Fig. 18c) and therefore gives a vanishing contribution to $T(\omega, Q^2)$. In addition, since $\langle P|G|P\rangle = 0$ (by virtue of color conservation), only the $O(G)$ terms containing not less than two operators $G_{\mu\nu}$ make a nonvanishing contribution. Since the twist of $G_{\mu\nu}$ is 1, we must replace the rough estimate $T_{(g)} \sim Q^2$, which is valid for individual subgraphs, by the more accurate estimate $T_g \sim Q^{2-n_G}$, which is valid for the sum of all such subgraphs. Here, n_G is the number of fields $G_{\mu\nu}$ that occur in the corresponding composite operator.

It follows from Eqs. (169) and (171) with allowance for the $\tilde{E}$ factors in (171) that the leading gluon contribution can be written in a form analogous to (165):

$$T_{(g)}^{\mathrm{lead}}(q, P) \sim \int \exp(i qx)\, dx \int d\xi\, d\eta\, C_{(g)}^{\mu\nu}(x, \xi, \eta; \mu^2)$$
$$\times \langle P|S^+ T(\mathrm{Reg}_{\mu^2}^{UV}(\mathrm{Tr}\, \tilde{G}_{\mu\kappa}(\xi) \tilde{E}(\xi, \eta; A) \tilde{G}^{\kappa}_{\nu}(\eta))\, S)|P\rangle, \qquad (185)$$

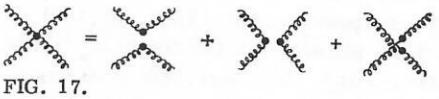
FIG. 17.

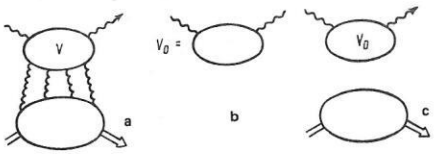

FIG. 18.

where Tr denotes the trace with respect to the matrices σ.

In QCD, it is also necessary to take into account the contribution of the subgraphs that have external lines associated with Faddeev-Popov ghosts.[73] The twist of these particles is 1, and therefore only subgraphs V with two external ghost lines (and an arbitrary number of external gluon lines) contribute to the leading asymptotic behavior. It is necessary to introduce the ghosts because in the diagrams containing gluon cycles (we note that such cycles are possible only in non-Abelian theories) we do not, in contrast to Abelian theories, have the complete compensation of the contributions of the unphysical (longitudinal and time) components of the vector field, and the role of the fictitious particles reduces to subtracting the uncompensated remainder. Therefore, allowance for the subgraphs with external lines corresponding to the fictitious particles is equivalent to the substitution

$$\widetilde{G}_{\mu\varkappa}(\xi)\,\widetilde{E}(\xi,\eta)\,\widetilde{G}^{\varkappa}_{\nu}(\eta)\to\widetilde{G}_{\mu\varkappa}(\xi)\,\widetilde{E}(\xi,\eta)\,\widetilde{G}_{\nu\varkappa'}(\eta)\,\rho^{\varkappa\varkappa'} \tag{186}$$

in the expression (185), where $\rho^{\varkappa\varkappa'}$ is the projection operator onto the states with physical polarization of the gluons:

$$\rho^{\varkappa\varkappa'}=\sum_{i=1}^{2}\varepsilon^{\varkappa}(\lambda_i)\,\varepsilon^{\varkappa'}(\lambda_i) \tag{187}$$

(see, for example, Ref. 126). The operator (186) must be expanded in a Taylor series, and the further analysis of the gluon contribution is made in the standard manner as described in Sec. 3.

The final result for $T(\omega,Q^2)$ has the same form (115), but the coefficient functions C_n^i in these two cases are obviously different. Also different are the local operators whose matrix elements give f_a, $f_{\bar a}$, and f_g:

$$\frac{i^{n-1}}{2}\langle P|S^{+}T\,(\mathrm{Reg}^{UV}_{\mu^2}:\bar\psi_a\{\gamma_{\mu_1}\hat D_{\mu_2}\dots\hat D_{\mu_n}\}\psi_a:S)|P\rangle$$
$$=\{P_{\mu_1}\dots P_{\mu_n}\}[\widetilde f_a(n,\mu^2)+(-1)^n\widetilde f_{\bar a}(n,\mu^2)]; \tag{188}$$

$$\frac{i^{n}}{2}\langle P|S^{+}T\,(\mathrm{Reg}^{UV}_{\mu^2}\mathrm{Tr}(:\widetilde G_{\alpha\mu_1}\widetilde D_{\mu_2}\dots\widetilde D_{\mu_{n-1}}\widetilde G_{\mu_n\alpha'}:\rho^{\alpha\alpha'}\}S|P\rangle$$
$$=\{P_{\mu_1}\dots P_{\mu_n}\}\frac{1+(-1)^n}{2}\widetilde f_g(n,\mu^2). \tag{189}$$

Note that in (188) and (189), as usual, averaging with respect to the spin of the target is assumed, and the presence of the subscript a of ψ_a and $\bar\psi_a$ simultaneously in (188) does not mean that there is summation over this subscript.

Parton Interpretation and the Axial Gauge. Although Eqs. (188) and (189) are very similar to Eq. (110), which, as we have seen, admits a natural parton interpretation, such an interpretation for (188) and (189) is difficult because of the presence of the fields A_μ in the covariant derivatives—Eqs. (188) and (189) give, as it were, the distribution functions of the quarks and gluons accompanied by their gluon fields. We recall that the need to take into account the fields A_μ in the matrix elements $\langle P|\bar\psi\dots A_\mu\dots\psi|P\rangle$ is due to the appearance of the additional factor $\mathscr{P}_\mu$ [see (15)]. With respect to the coefficient function, this factor plays the part of the gluon polarization. Further, since the considered gluon arises as a result of interactions at large distances, its transverse momentum is small, and it can be assumed that in a system in which the hadron moves rapidly the momentum of the gluon is also proportional to P (formally, this can be related to the fact that in the matrix element $\langle P|\dots|P\rangle$ there are no other momenta). Thus, the gluons in the covariant derivatives are longitudinally polarized. In particular, they can be eliminated by a suitable choice of the gauge. Indeed, multiplying both sides of the expressions (188) and (189) by the tensor $l^{\mu_1}\dots l^{\mu_n}$, where l^μ is a null vector, $l^2=0$, and choosing the *axial gauge* $(lA^\mu)=0$, we obtain instead of (188) the representation

$$f_a(n)+(-1)f_{\bar a}(n)$$
$$=\frac{i^{n-1}}{2}\frac{1}{(lP)^n}\langle P|S^{+}T\,(:\bar\psi_a(l_\mu\gamma^\mu)(l_\mu\partial^\mu)^{n-1}\psi_a:S)|P\rangle, \tag{190}$$

which contains an operator with ordinary derivatives.

Sometimes, other axial gauges are used. In particular, if we write the quark contribution to $T(\omega,Q^2)$ in the form

$$T^{\mathrm{quark}}(\omega,Q^2)\sim\sum_a C_n^a(Q^2/\mu^2)\,q^{\mu_1}\dots q^{\mu_n}\ \langle P|\bar\psi_a\gamma_{\mu_1}\hat D_{\mu_2}\dots\hat D_{\mu_n}\psi_a|P\rangle \tag{191}$$

and take the axial gauge $(qA)=0$,[118] then the fields A_μ in $\hat D_\mu$, i.e., the contributions of the configurations of Fig. 14a, can be ignored up to power-law corrections. This means that the weakly virtual gluons produced by the interactions at large distances have in this gauge only transverse polarization (for more detail, see Ref. 51).

Similarly, considering the expression (111) for $T(\omega,Q^2)$ in the coordinate representation, we can readily see that in the Schwinger gauge $(xA)=0$ (Ref. 127) the contributions of the configurations of Fig. 14a to $T^{\mathrm{lead}}(\omega,Q^2)$ can also be ignored. This fact can be viewed from a different side by noting that the Schwinger gauge has a remarkable property (see, for example, Ref. 128)—in it, the field A_μ can be expressed in terms of $G_{\mu\nu}$:

$$A_\mu(x)=x^\nu\int_0^1 t\,dt\,G_{\nu\mu}(tx) \tag{192}$$

and, therefore, in this gauge the field A_μ must be associated with a twist equal to unity.

The Nature of the Breaking of Scaling in QCD. The moments of the nonsinglet component of the structure function W have the simplest dependence on Q^2. In this case, the anomalous dimension $\gamma^{NS}(n,g)$ is simply a number. Because

$$f^{NS}(n,\mu^2,\bar g(\mu^2))=\exp\Big[\int_{\bar g(\mu_0^2)}^{\bar g(\mu^2)}\frac{\gamma^{NS}(n,g)}{\beta(g)}\,dg\Big]\widetilde f^{NS}(n,\mu_0^2,\bar g(\mu_0^2)) \tag{193}$$

[as is readily seen, this form of expression of the solution of the renormalization-group equation is equivalent to (152)], representing $\gamma(g)/\beta(g)$ in the form

$$\gamma(g)/\beta(g) = \Big(-\sum_{k=1}^{\infty} \gamma_k (g^2/16\pi^2)^k\Big)\Big(-\frac{1}{2}\frac{g^3}{16\pi^2}\sum_{k=1}^{\infty} b_k (g^2/16\pi^2)^k\Big)^{-1}$$
$$\equiv \frac{2}{g^2}\Big(\frac{\gamma_1}{b_0}\Big)\Big[1+\sum_{k=1}^{\infty} a_k (g^2/16\pi^2)^k\Big] \qquad (194)$$

[cf. (44)], we find that

$$W_n^{NS}(Q) = C_n^{NS}(Q^2/\mu^2, \bar{g}(\mu)) \exp\Big\{\frac{\gamma_1^{NS}(n)}{b_0}\Big[\ln\frac{\bar{g}^2(\mu^2)}{\bar{g}^2(\mu_0^2)} + \sum_{k=1}^{\infty}\frac{a_k(n)}{k}[(\bar{g}^2(\mu^2))^k - (\bar{g}^2(\mu_0^2))^k]\Big\}\tilde{f}^{NS}(n, \mu_0^2). \qquad (195)$$

Further, setting $\mu = aQ$ and using (154), we obtain

$$W_n^{NS}(Q) = A^{NS}(n)\Big(\frac{4\pi}{b_0\bar{g}^2(a^2Q^2)}\Big)^{-(\gamma_1^{NS}/b_0)} \times\Big\{1+\sum_{l=1}^{n} c_l(a, n)\Big(\frac{\bar{g}^2(aQ)}{16\pi^2}\Big)^l\Big\} \times\exp\Big\{\frac{\gamma_1^{NS}(n)}{b_0}\sum_{k=1}^{\infty}\frac{a_k(n)}{k}\Big(\frac{\bar{g}^2(a^2Q^2)}{16\pi^2}\Big)^k\Big\}. \qquad (196)$$

The entire information about the dynamics at large distances is accumulated in the coefficient $A^{NS}(n)$:

$$A^{NS}(n) = \tilde{f}^{NS}(n, \mu_0^2)\Big(\frac{b_0\bar{g}^2(\mu_0^2)}{16\pi^2}\Big)^{-\gamma_1^{NS}(n)/b_0} \exp\Big\{\frac{\gamma_1^{NS}(n)}{b_0}\sum_{k=1}^{\infty}\frac{a_k(n)}{k}\Big(\frac{\bar{g}^2(\mu_0^2)}{16\pi^2}\Big)^k\Big\}. \qquad (197)$$

In the leading order (LO), Eq. (193) with allowance for (45) gives the well-known prediction[46,129] of logarithmic breaking of scaling in QCD:

$$W_n^{NS(LO)} = \Big(\ln\frac{a^2Q^2}{\Lambda^2}\Big)^{-(\gamma_1^{NS}/b_0)} A^{NS}(n). \qquad (198)$$

The anomalous dimension γ_1^{NS} is given by the expression[199]

$$\gamma_1^{NS} = C_F\Big[1-\frac{2}{n(n+1)}+4\sum_{j=2}^{n}(1/j)\Big]. \qquad (199)$$

Note that $\gamma_1^{NS}(1) = 0$; this is due to the conservation of the vector current [cf. (160) and (161)].

Minimization. Usually, $a = 1$ is taken in (195), but in order to see to what extent a particular choice of a is justified it is necessary to calculate the next-to-leading order (NLO) correction:

$$W_n^{NS(NLO)}(Q) = W_n^{NS(LO)}(Q)\,[\alpha_s^{(NLO)}(aQ)/\alpha_s^{LO}(aQ)]^{(\gamma_1^{NS}(n)/b_0)} \times\{1+[\alpha_s^{(LO)}(aQ)/4\pi]\,[C_1^{NS}(n) - \gamma_1(n)\ln a^2 + (\gamma_2^{NS}(n)/b_0 - \gamma_1^{NS}(n)\,b_1/b_0^2)]\}, \qquad (200)$$

where $C_1^{NS}(n) \equiv C_1^{NS}(a=1, n)$. The coefficient γ_2^{NS} was calculated in Ref. 130 and was reduced to a more compact form in Ref. 131. However, even in Ref. 131 the expression for $\gamma_2^{NS}(n)$ occupies about a page of text. Therefore, we shall not give it here. The coefficient $C_1^{NS}(n)$ has the form[38]

$$C_1^{NS}(n)|_{\overline{MS}} = C_F\Big[3\sum_{j=1}^{n}(1/j) - 4\sum_{j=1}^{n}(1/j^2) - \frac{2}{n(n+1)}\sum_{j=1}^{n}(1/j) + 4\sum_{s=1}^{n}\frac{1}{s}\sum_{j=1}^{s}\frac{1}{j} + \frac{3}{n} + \frac{4}{n+1} + \frac{2}{n^2} - 9\Big]. \qquad (201)$$

In Table I, we give the numerical values of the coefficient $K^{NS}(n)$,

$$K^{NS}(n) = C_1^{NS}(n) + (\gamma_2^{NS}(n)/b_0 - \gamma_1^{NS}(n)\,b_1/b_0^2), \qquad (202)$$

TABLE I.

n	OMS (MOM)	$\overline{\text{MS}}$	MS	$a_{\text{opt}}^{\text{OMS}}$
1	0	0	0	1
2	−3.19	2.9	3.11	1.14
3	−3.06	4.16	16.02	0.97
4	−2.20	8.12	21.75	0.86
5	−1.16	10.84	26.64	0.80
6	−0.03	13.34	30.93	0.75
7	+1.14	15.65	34.76	0.71
8	2.27	17.79	38.22	0.67
9	3.37	19.77	41.37	0.65
10	4.44	21.64	44.30	0.63
11	5.46	23.39	47.01	0.62
12	6.43	25.04	49.53	0.59

which characterizes the corrections in the chosen scheme for $a = 1$, $N_f = 4$. It can be seen that the choice $a = 1$ is best justified in the OMS, while in the $\overline{\text{MS}}$ and especially the MS scheme this choice leads to large values of the coefficient $k^{NS}(a)$.

For optimal choice of the parameter a, we can use Stevenson's criterion.[45] It is only necessary to bear in mind that in the considered problem perturbation theory does not predict the absolute value of $W_n^{NS}(Q^2)$ but gives only the dependence of $W_n^{NS}(Q^2)$ on Q^2, and therefore the criterion must be applied, not to W_n^{NS}, but to the logarithmic derivative $d(\ln W_n)/d(\ln Q^2)$. In the final column of Table I, we give the values of a_{opt} in the OMS scheme for the corresponding n. It can be seen that with increasing n the value of a_{opt} decreases. This fact can be interpreted as follows. In the diagram (Fig. 19) giving the Born contribution to $C_n(Q^2)$ the virtuality of the quark line (for $\omega < 1$) is $Q^2(1-x\omega)$. Since large values of n correspond to ω near 1, $(1-\omega) \sim 1/n$, the mean virtuality of the quark decreases with increasing n, and, accordingly, the optimal value of the argument of the coupling constant $\alpha_s(\mu^2)$ must decrease.

CONCLUSIONS

The methods of analyzing the asymptotic behavior of Feynman diagrams presented in the present review can also be used to investigate more complicated processes differing, for example, from those considered above by the presence of two hadrons in the initial state (in this connection, we can speak of the inclusive production of a massive lepton pair in hadron collisions: $h_1h_2 \to \mu^+\mu^-X$) or the presence of a detected hadron in the final state (the simplest example is the process $e^+e^- \to hX$). In the latter case, we must also introduce functions for the decay of partons into hadrons, which, in contrast to the distribution functions of partons in a hadron, cannot be directly related to the matrix elements of any local operators and are expressed in terms of the so-called *cut vertices*.[36] The analysis of processes in which two or more hadrons are detected in the initial and/or final state is greatly complicated by the need to take into account the contributions associat-

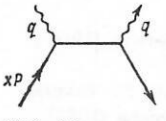

FIG. 19.

ed with the infrared regime. For the leading power-law asymptotic behavior these contributions cancel as a rule (a complete proof of this fact has not yet been given), and for the remaining contributions there is factorization of the large and small distances. The problem of infrared sensitivity in processes of this kind is at the present time the subject of intense study by specialists, and it is to be expected that the remaining problems will be solved in the very near future.

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