New and old symmetries of the Maxwell and Dirac equations[1)]

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The symmetry properties of Maxwell's equations for the electromagnetic field and also of the Dirac and Kemmer-Duffin-Petiau equations are analyzed. In the framework of a "non-Lie" approach it is shown that, besides the well-known invariance with respect to the conformal group and the Heaviside-Larmor-Rainich transformations, Maxwell's equations have an additional symmetry with respect to the group $U(2) \otimes U(2)$ and with respect to the 23-dimensional Lie algebra A_{23}. The transformations of the additional symmetry are given by nonlocal (integro-differential) operators. The symmetry of the Dirac equation in the class of differential and integro-differential transformations is investigated. It is shown that this equation is invariant with respect to an 18-parameter group, which includes the Poincaré group as a subgroup. A 28-parameter invariance group of the Kemmer-Duffin-Petiau equation is found. Finite transformations of the conformal group for a massless field with arbitrary spin are obtained. The explicit form of conformal transformations for the electromagnetic field and also for the Dirac and Weyl fields is given.

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The physical nature of a quantity is subject to its mathematical form. . . . Mathematics loves symmetry above all.

J. C. Maxwell

Using pure thought, Maxwell robbed nature of secrets that only an entire generation later could be proved in ingenious and laborious experiments.

M. Planck

INTRODUCTION

The investigation of the symmetry of Maxwell's equations has a long and glorious history. It was by study of the symmetry of these equations that Lorentz, Poincaré, and Einstein obtained the fundamental formulas of relativistic mechanics and electrodynamics, and the subsequent generalization of the principle of relativistic invariance to all the laws of physics played a truly revolutionary role in modern natural science.

It was found that the classical results of Lorentz, Poincaré, Einstein, Bateman, and Cunningham do not exhaust all the symmetry properties of Maxwell's equations. In Refs. 15–27, new invariance groups of these equations and also of the Dirac, Kemmer-Duffin-Petiau, and other equations of relativistic physics were found. A feature of the new symmetry groups of the relativistic equations is that they include nonlocal (nonpoint) transformations of the dependent and independent variables and therefore in principle cannot be obtained in the classical approach of Lie. The book of Ref. 64 by the present authors is devoted to a detailed analysis of nonlocal symmetries.

In the present paper, which is essentially a review of our Refs. 15–35 and uses a unified algebraic approach, we obtain the classical results on the symmetry of Maxwell's equations and also establish some symmetry properties of these equations [the additional invariance with respect to the group $U(2) \otimes U(2)$ and the 23-dimensional Lie algebra A_{23}] that were not known until recently [1974 (Ref. 16), 1978 (Refs. 28, 31, and 32)]. We shall also analyze the symmetry properties of the Dirac and the Kemmer-Duffin-Petiau equations and find explicitly the conformal transformations for massless fields with arbitrary spin.

1. Before we turn to a brief exposition of the results and the basic ideas of our approach, we shall briefly review the history of the study of the symmetry of Maxwell's equations.

The modern form of Maxwell's equations was given by Hertz and Heaviside. In 1893, Heaviside,[1] having expressed them in symmetric form, noted that they are invariant under the substitution

$$\mathbf{E} \to \mathbf{H}; \quad \mathbf{H} \to -\mathbf{E}, \tag{1}$$

where E and H are the intensity vectors of the electric and magnetic fields. Larmor[2] and Rainich[3] generalized this symmetry to the family of one-parameter transformations of the form

$$\mathbf{E} \to \mathbf{E} \cos\theta + \mathbf{H} \sin\theta; \quad \mathbf{H} \to \mathbf{H} \cos\theta - \mathbf{E} \sin\theta. \tag{2}$$

At the end of the last century, the eminent Norwegian mathematician Lie created the mathematical foundations of the science of the symmetry of differential equations. Without attempting the general problem of investigating the group properties of differential equations, Lorentz,[4] Poincaré,[5,6] and Einstein[7] obtained one of the most fundamental results in this field, which was destined to play a revolutionary role in physics. Namely, Lorentz, who was not acquainted with Lie's theory, found linear transformations of the coordinates and the

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[1)]This paper commemorates the 150th anniversary of the birth of James Clark Maxwell (1831–1879).

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time (and corresponding transformations for E and H) that leave Maxwell's equations for the electromagnetic field in the absence of charges invariant.

Poincaré and, independently of him, Einstein showed that in the presence of charges and currents Maxwell's equations are invariant with respect to the same transformations if the current and charge densities are transformed appropriately. Poincaré established for the first time and studied in detail one of the most important properties of such transformations—their group structure. We emphasize that it was on the basis of the analysis of the symmetry properties of Maxwell's equations by Lorentz, Poincaré, and Einstein that the foundations of relativistic theory were laid.

In 1909, Bateman[8] and Cunningham[9] showed that Maxwell's equations are invariant with respect to nonlinear conformal transformations which can be represented as the product of the inversion transformation

$$x_\mu \to x'_\mu = x_\mu/(x_\lambda x^\lambda), \quad \mu, \lambda = 0, 1, 2, 3, \tag{3}$$

the shift transformation $x'_\mu \to x''_\mu = x_\mu - d_\mu$, and the secondary inversion $x''_\mu \to x'''_\mu = x''_\mu/(x''_\lambda x''^\lambda)$. Cunningham[9] found in explicit form linear transformations of the vectors E and H that together with (3) leave Maxwell's equations invariant.

The conformal transformations together with the Lorentz transformations and dilatations form the 15-parameter conformal group $C(1,3)$. As was shown by Bateman,[8] this group is the maximal point symmetry group of Maxwell's equations with currents and charges. The group of conformal transformations in 4-space R_4 was also studied by Lie.[10]

During the last two decades, the basic ideas and methods of classical group-theoretical analysis of differential equations have been significantly developed by Ovsyannikov and his students.[11-14] Comparatively recently, the Lie-Ovsyannikov algorithm was used in a group analysis of Maxwell's equations for the electromagnetic field in vacuum.[13] This showed that the maximal local invariance group of such equations is the 16-parameter group $C(1,3)\otimes H$, where H is the one-parameter subgroup of the Heaviside-Larmor-Rainich transformations (2).

2. The above history could create the impression that the symmetry properties of Maxwell's equations have been completely studied and that there is no hope of obtaining any new result in this field. In fact, this is not the case, since Maxwell's equations have a hidden (nongeometrical) symmetry that is not related to local coordinate transformations.[16,28,31]

As was noted in Refs. 15–35, Lie's infinitesimal method by no means enables one to find all the symmetries that a particular system of differential equations possesses. A well-known example of a "non-Lie" symmetry is the invariance of the Schrödinger equation for the hydrogen atom with respect to the group $O(4)$, which was first discovered by Fock.[36]

To see what invariance groups of differential equations can and cannot be found in the classical Lie-Ovsyannikov approach, we consider an arbitrary linear differential equation

$$\hat{L}(x, d/dx)\Psi(x) = 0, \tag{4}$$

where $\hat{L}$ is some linear operator, and Ψ is a vector function with components $\{\Psi_1, \Psi_2, \ldots, \Psi_n\}$, $x \in R_n$. In the Lie-Ovsyannikov approach, the infinitesimal operators of the invariance group of Eq. (9) are sought in the form of first-order differential operators

$$\hat{Q}_A = \xi^\mu_A(x, \Psi)\frac{\partial}{\partial x_\mu} + \eta^k_A(x, \Psi)\frac{\partial}{\partial \Psi_k}, \tag{5}$$

where $\xi^\mu_A(x, \Psi)$ and $\eta^k_A(x, \Psi)$ are unknown functions found by requiring the operators (5) to satisfy the condition for invariance of Eq. (4):

$$\hat{L}\hat{Q}_A\Psi(x) = 0. \tag{6}$$

If we require that the operators (5) form a basis of a finite-dimensional Lie algebra, the functions $\xi^\mu_A(x, \Psi)$ and $\eta^k_A(x, \Psi)$ must satisfy certain additional conditions, which follow from the relations

$$[\hat{Q}_A, \hat{Q}_B] = i f_{ABC}\hat{Q}_C, \tag{7}$$

where f_{ABC} are structure constants. We shall call a set of operators satisfying the conditions (6) and (7) an invariance algebra of Eq. (4).

It is obvious that the Lie-Ovsyannikov approach does not enable one to find all possible invariance algebras of a given differential equation, since it imposes *a priori* on the basis elements of the algebra the requirement that they belong to the class of first-order differential operators.[2)] It is clear from what we have said that the formulation of the problem of investigating the algebraic properties of differential equations can be significantly generalized by extending the class of operators $\hat{Q}_A$ satisfying (6) and (7). For example, one can seek an invariance algebra of a differential equation in the class of second-order differential operators or even integro-differential operators. It was in this way that many new invariance algebras of the Dirac,[15-19] Maxwell,[16,28,31,32] and many other equations of quantum mechanics[15-35] were found. Such invariance algebras, which are called *nongeometrical* below, correspond to nonlocal transformations of the dependent and independent variables and therefore do not generate local Lie groups.

We shall not here be concerned with the large group of problems associated with dynamical symmetry of physical systems, which have been fairly fully studied in the literature (see, for example, Refs. 38–40). Malkin and Man'ko's book[41] is devoted to the physical aspects of dynamical symmetry.

3. The main and most difficult question that arises in connection with the nongeometrical approach to the investigation of the symmetry properties of differential equations is that of finding a method of calculating constructively the operators $\hat{Q}_A$ which form an invar-

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[2)] It should be noted that the Lie-Ovsyannikov approach was significantly developed by Anderson and Ibragimov[37] by means of Lie-Bäcklund transformations.

iance algebra of a given differential equation. Generalizing the results of specific calculations of invariance algebras of the equations of quantum mechanics, one can formulate the following algorithm for finding the explicit form of such operators[24,30,33]: 1) the system of differential equations is reduced by means of a nondegenerate transformation to a canonical diagonal form, i.e., the system of differential equations is decoupled to the maximum possible extent into independent subsystems; 2) the invariance algebra of the transformed equation is found; 3) if the operators $\hat{Q}_A$ satisfy the relations (7), one can find what representation of the Lie algebra these operators realize on the solution set of the investigated equation; 4) by means of the inverse transformation, one can find the explicit form of the basis elements of the invariance algebra of the original equation; 5) the finite transformations

$$\Psi(x) \to \Psi'(x) = \exp(iQ_A\theta_A)\Psi(x), \tag{8}$$

where θ_A are the transformation parameters, can be calculated from the obtained representation of the invariance algebra.

The algorithm is based on one of the most fruitful and effective ideas in the theory of differential equations—transformation of the independent and dependent variables. We give a more detailed description of the first step of the algorithm.

An important role in the realization of the algorithm will be played by the concept of the symbol of the operator $\hat{L}(x, \partial/\partial x)$, which can be defined by means of Fourier transformation (for more detail about symbols, see, for example, Ref. 42):

$$\hat{L}(x, \partial/\partial x)\Psi(x) = (2\pi)^{-n/2}\int\limits_{D(p)} L(x, p)\exp(i\,\mathbf{x}\cdot\mathbf{p})\tilde{\Psi}(p)\,d^np, \tag{9}$$

where $\tilde{\Psi} \in C_0^\infty(R^n)$, $\tilde{\Psi} = F\Psi(x)$ is the Fourier transform of $\Psi(x)$, F is the unitary Fourier operator that maps a vector in the Hilbert space H to $\tilde{H}$; $\tilde{\Psi}(p) \in \tilde{H}$; $D(p)$ is the domain of integration; $\mathbf{x}\cdot\mathbf{p} = g^{\mu\nu}x_\mu p_\nu = g^{00}x_0p_0 + g^{11}x_1p_1 + \ldots + g^{n-1\,n-1}x_{n-1}p_{n-1}$; and $g^{\mu\nu}$ is the metric tensor of the Riemannian space R_n.

The connection between the operator $\hat{L}(x, \partial/\partial x)$ and its symbol $L(x, p)$ is given by

$$\hat{L}(x, \partial/\partial x) = F^{-1}L(x, p)F; \tag{10}$$

$$L(x, p) = F\hat{L}(x, \partial/\partial x)F^{-1}. \tag{11}$$

Equations (10) and (11) indicate a way of realizing the first step of the algorithm. Indeed, if Eq. (4) is such that the symbol of the operator $\hat{L}(x, \partial/\partial x)$ is a matrix with variable coefficients, and this is the case for an absolute majority of the equations of mathematical physics, then the problem of decoupling the system (4) into the largest possible number of decoupled equations reduces to transforming the matrix $L(x, p)$ given by (11) to diagonal or Jordan form. In the general case, such diagonalization is a fairly complicated problem, but if the vector function $\Psi(x)$ does not have too many components, the difficulties are purely technical in nature.

It should be said that as a rule the complete realization of the above algorithm for specific equations of physics and mechanics is not a simple mathematical problem. What we have said also applies fully to the Lie-Ovsyannikov algorithm.

1. DIFFERENT FORMULATIONS OF MAXWELL'S EQUATIONS

We give here the basic formulations of Maxwell's equations for the electromagnetic field in vacuum and in the presence of currents and charges. All these formulations are mathematically equivalent, but each of them may be the most convenient for solving a particular physical problem. In addition, the different forms of Maxwell's equations open up a path to entirely different generalizations of these equations.

Maxwell's equation in vector notation. Maxwell's equations for the electromagnetic field in vacuum have the form

$$\mathbf{p}\times\mathbf{E} = i\,\partial\mathbf{H}/\partial t; \quad \mathbf{p}\times\mathbf{H} = -i\,\partial\mathbf{E}/\partial t; \tag{12a}$$

$$\mathbf{p}\cdot\mathbf{E} = 0; \quad \mathbf{p}\cdot\mathbf{H} = 0, \tag{12b}$$

where $p_a = -i\partial/\partial x_a$, $a = 1, 2, 3$; $\mathbf{E} = \mathbf{E}(t, \mathbf{x})$ and $\mathbf{H} = \mathbf{H}(t, \mathbf{x})$ are the vectors of the electric and magnetic field intensities. We use the Heaviside system of units, in which $h = c = 1$.

In the presence of currents and charges, the system of Maxwell's equations takes the form

$$i\,\partial\mathbf{E}/\partial t = -\mathbf{p}\times\mathbf{H} - i\mathbf{j}; \quad \mathbf{p}\cdot\mathbf{E} = -ij_0; \tag{13a}$$

$$i\,\partial\mathbf{H}/\partial t = \mathbf{p}\times\mathbf{E}; \quad \mathbf{p}\cdot\mathbf{H} = 0, \tag{13b}$$

where $j = (j_0, \mathbf{j})$ is the 4-vector of the electric current, and the electromagnetic coupling constant is taken equal to unity.

The expression of Maxwell's equations in the form (12) or (13) was already proposed by Hertz and Heaviside. The use of vector notation makes Eqs. (12) and (13) fairly compact and elegant. However, the formulations (12) and (13) are not manifestly relativistically invariant. Examining Eqs. (12) and (13), it is not at all easy to guess that these equations could be interpreted as the equations for massless particles with helicity $\lambda = \pm 1$. Finally, equations in the form (12) or (13) cannot be directly generalized to the case of particles with arbitrary (different from ± 1) values of the helicity. The form of Eqs. (12) and (13) is also not particularly convenient for investigating their symmetry properties. Therefore, we give below other formulations of Maxwell's equations free of these shortcomings.

Maxwell's equations in operator form. To investigate the symmetry properties of Maxwell's equations, it is convenient to represent the system (12) as the result of the application of certain linear operators to the vector function

$$\varphi(t, \mathbf{x}) = \text{column of}(E_1, E_2, E_3, H_1, H_2, H_3), \tag{14}$$

where E_a and H_a are the components of the vectors of the electric and magnetic field intensities.

By the symbols $S_a (a=1, 2, 3)$ and $\sigma_\mu (\mu=0, 1, 2, 3)$ we denote the matrices

$$\left.\begin{array}{l} S_a=\begin{pmatrix}\hat{S}_a & 0\\ 0 & S_a\end{pmatrix};\quad \sigma_0=\begin{pmatrix}1 & \hat{0}\\ \hat{0} & 1\end{pmatrix};\\ \sigma_1=\begin{pmatrix}\hat{0} & 1\\ 1 & \hat{0}\end{pmatrix};\quad \sigma_2=\mathrm{i}\begin{pmatrix}\hat{0} & -1\\ 1 & \hat{0}\end{pmatrix};\quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix};\\ \hat{S}_1=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -\mathrm{i}\\ 0 & \mathrm{i} & 0\end{pmatrix};\ \hat{S}_2=\begin{pmatrix}0 & 0 & \mathrm{i}\\ 0 & 0 & 0\\ -\mathrm{i} & 0 & 0\end{pmatrix};\ \hat{S}_3=\begin{pmatrix}0 & -\mathrm{i} & 0\\ \mathrm{i} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \end{array}\right\} \tag{15}$$

where $\hat{0}$ and 1 are the three-row square null and unit matrices. Using the notation (15), we can write Eqs. (12a) in the Schrödinger form

$$\hat{L}_1\varphi(t, \mathbf{x}) = 0;\quad \hat{L}_1 = \mathrm{i}\,\partial/\partial t + \sigma_2 \mathbf{S}\cdot\mathbf{p}. \tag{16}$$

With regard to Eqs. (12b), they can also be written in the operator form

$$\hat{L}_2^a \varphi(t, \mathbf{x}) = 0, \tag{17}$$

where $\hat{L}_2^a$ is any of the three operators

$$\hat{L}_2^a = (\delta_{ab} - S_b S_a)\, p_b,\quad ab = 1, 2, 3. \tag{18}$$

Here and below, δ_{ab} is the Kronecker delta symbol, and summation from 1 to 3 is understood over repeated Latin indices.

Thus, Eqs. (12) can be represented in the form of the Schrödinger equation (16) for the six-component real function (14) and the subsidiary condition (17), which, as will be shown below, reduces the number of independent components of the function (14) to four. It is the formulation (16)–(17) that we shall mainly use below to investigate the symmetry properties of Maxwell's equations.

We consider here one further form of Eqs. (12), in which we use a three-component complex vector function:

$$\Psi=\begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\end{pmatrix}=\begin{pmatrix}H_1-\mathrm{i}E_1\\ H_2-\mathrm{i}E_2\\ H_3-\mathrm{i}E_3\end{pmatrix}. \tag{19}$$

In the notation of (15) and (19), Eqs. (12) can be rewritten as

$$\mathrm{i}\frac{\partial}{\partial t}\Psi = \hat{H}\Psi;\quad \hat{H} = \hat{\mathbf{S}}\cdot\mathbf{p}; \tag{20a}$$

$$(p_a - \hat{\mathbf{S}}\cdot\mathbf{p} S_a)\,\Psi = 0. \tag{20b}$$

Equation (20b), written down component by component, reduces to the following condition for the function (19):

$$\mathbf{p}\cdot\boldsymbol{\Psi} = 0, \tag{21}$$

where $\boldsymbol{\Psi} = (\Psi_1, \Psi_2, \Psi_3)$.

The formulation of Maxwell's equations in the form of (20a) and (21) was first proposed by Majorana (see Ref. 43).[3)] This formulation is very convenient for the corpuscular interpretation of Eqs. (12) (see Sec. 5 below).

Maxwell's equations in Dirac form. We now consider a different formulation of Eqs. (12), which was the first obtained by Borgardt,[44] and subsequently by Lomont[45] and Moses.[46] We denote by α_1, α_2, and α_3 the four-row matrices

$$\alpha_1=\mathrm{i}\begin{pmatrix}0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{pmatrix};\quad \alpha_2=\mathrm{i}\begin{pmatrix}0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix};$$
$$\alpha_3=\mathrm{i}\begin{pmatrix}0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}, \tag{22}$$

and by $\chi(t, \mathbf{x})$ the following four-component function with first component identically equal to zero:

$$\chi(t, \mathbf{x})=\begin{pmatrix}0\\ E_1-\mathrm{i}H_1\\ E_2-\mathrm{i}H_2\\ E_3-\mathrm{i}H_3\end{pmatrix}. \tag{23}$$

Using the notation (22), (23), we can write Eqs. (12) in the form

$$\left(\mathrm{i}\frac{\partial}{\partial t} - \boldsymbol{\alpha}\cdot\mathbf{p}\right)\chi = 0. \tag{24}$$

Indeed, writing out (12) and (22)–(24) in components and remembering that **E** and **H** are real, we arrive at identical systems of equations.

The matrices α_a (22) satisfy the Clifford-Dirac algebra

$$\alpha_a\alpha_b + \alpha_b\alpha_a = 2\delta_{ab}. \tag{25}$$

It should be emphasized, however, that even in the case when $\chi(t, \mathbf{x})$ is an arbitrary function Eq. (24) is not equivalent to the Dirac equation with $m=0$, and therefore the description *Maxwell's equations in Dirac form* is very conditional.

The main advantage of writing Maxwell's equations in the form (22)–(24) is that this formulation can be readily generalized to the case of a relativistic massless field of arbitrary spin. However, Eqs. (22)–(24) are not invariant with respect to spatial inversion and, in addition, the condition of vanishing of the first component of the function $\chi(t, \mathbf{x})$ (23) is not very attractive from a purely esthetic point of view.

Following Refs. 47 and 48, we give one further formulation of Eqs. (12), in which, as in (15)–(17), we use a real vector function, which this time has eight components:

$$\Psi = \text{column of}\,(H_1, H_2, H_3, \varphi_1, E_1, E_2, E_3, \varphi_2). \tag{26}$$

Equations (12) can be represented in the form of the following system for the function (26):

$$\left.\begin{array}{l} \hat{L}_1\Psi = 0;\quad \hat{L}_1 = \gamma_\mu p^\mu;\\ \hat{L}_2\Psi = 0;\quad \hat{L}_2 = \gamma_\mu p^\mu S_{\sigma\nu} S^{\sigma\nu}, \end{array}\right\} \tag{27}$$

where γ_μ and $S_{\sigma\nu}$ are 8×8 matrices:

$$\left.\begin{array}{l} \gamma_0=\begin{pmatrix}\tilde{0} & \tilde{I}\\ \tilde{I} & \tilde{0}\end{pmatrix};\quad \gamma_a=-\mathrm{i}\begin{pmatrix}\alpha_a & \tilde{0}\\ \tilde{0} & -\alpha_a\end{pmatrix};\quad S_{ab}=\begin{pmatrix}\tilde{S}_{ab} & 0\\ 0 & \tilde{S}_{ab}\end{pmatrix};\\ S_{0a}=\mathrm{i}\begin{pmatrix}\tilde{0} & -\tilde{S}_{0a}\\ \tilde{S}_{0a} & 0\end{pmatrix};\ \tilde{S}_{ab}=-\mathrm{i}\varepsilon_{abc}\tilde{S}_{0c}=\varepsilon_{abc}\begin{pmatrix} & & & 0\\ & \hat{S}_c & & 0\\ & - & & 0\\ 0 & 0 & 0 & 0\end{pmatrix}; \end{array}\right\} \tag{28}$$

[3)]Such a formulation (but in component rather than matrix form) was used much earlier by Bateman and Zimmermann.[63]

$\tilde{0}$ and $\tilde{1}$ are the null and unit 4×4 matrices, and α_a and $\hat{S}_c$ are the matrices (22) and (15).

Substituting (26) and (28) in (27) and writing out the obtained system component by component, we arrive at Eqs. (12) for **E** and **H** and the following conditions for φ_1 and φ_2:

$$p_a\varphi_1=\frac{\partial}{\partial t}\varphi_1=p_a\varphi_2=\frac{\partial}{\partial t}\varphi_2=0, \tag{29}$$

from which we conclude that φ_1 and φ_2 are constants which, without loss of generality, we can assume are zero.

Maxwell's equations in the form (26)–(28) also admit the most direct generalization to the case of fields with arbitrary spin[47,48] and, in contrast to (22) and (23), are invariant with respect to spatial inversion (see Sec. 2).

Equations in the Kemmer-Duffin-Petiau form. In all the above formulations, Maxwell's equations were expressed as the result of applying two (or four) linear operators to a certain vector function. However,

$$(\beta_\mu p^\mu+\beta\varkappa)\Psi=0, \tag{30}$$

where β_μ are the 10-row Kemmer-Duffin-Petiau matrices, which satisfy the algebra

$$\beta_\mu\beta_\nu\beta_\lambda+\beta_\lambda\beta_\nu\beta_\mu=g_{\mu\nu}\beta_\lambda+g_{\nu\lambda}\beta_\mu; \tag{31}$$

$\beta=\beta_5^2$, $\beta_5=\varepsilon_{\mu\nu\rho\sigma}\beta_\mu\beta_\nu\beta_\rho\beta_\sigma/4!$, and $g_{\mu\nu}$ is the metric tensor: $g_{\mu\nu}=\mathrm{diag}(1,-1,-1,-1)$.

We shall show that Eqs. (12) can indeed be represented in the form (30)–(31). Choosing β_μ and Ψ in the form

$$\beta_0=i\begin{pmatrix}\hat{0}&\hat{0}&-1&0\\ \hat{0}&\hat{0}&\hat{0}&0\\ 1&\hat{0}&\hat{0}&0\\ 0&0&0&0\end{pmatrix};\quad \beta_a=\begin{pmatrix}\hat{0}&\hat{0}&\hat{0}&\lambda_a\\ \hat{0}&\hat{0}&-S_a&0\\ \hat{0}&S_a&\hat{0}&0\\ -\lambda_a^+&0&0&0\end{pmatrix};$$
$$\beta_5=i\begin{pmatrix}\hat{0}&-1&\hat{0}&0\\ 1&\hat{0}&\hat{0}&0\\ \hat{0}&\hat{0}&\hat{0}&0\\ 0&0&0&0\end{pmatrix}; \tag{32}$$

where $\hat{S}_a$ are the matrices (15),

$$\lambda_1=\begin{pmatrix}i\\0\\0\end{pmatrix};\quad \lambda_2=\begin{pmatrix}0\\i\\0\end{pmatrix};\quad \lambda_3=\begin{pmatrix}0\\0\\i\end{pmatrix}, \tag{33}$$

$\hat{0}$ and 1 are the null and unit 3×3 matrices, and $\hat{0}$ are null matrices of appropriate dimension, we arrive at the system of equations

$$\begin{gathered} i\,\partial A_b/\partial t+i\,\partial A_0/\partial x_b=-i\varkappa E_b;\quad \varkappa\mathbf{H}=\mathrm{curl}\,\mathbf{A};\\ i\,\partial\mathbf{E}/\partial t=-\mathbf{p}\times\mathbf{H};\quad \mathbf{p}\cdot\mathbf{E}=0, \end{gathered} \tag{34}$$

from which Eqs. (12) for **E** and **H** follow directly.

Maxwell's equations were apparently first written in the form (30)–(31) by Fedorov (Ref. 49; see also Refs. 50 and 51).

We now turn to Maxwell's equations with currents and charges, Eqs. (13), and show that they can be expressed in the form of a system of two equations of the type (30). We denote by $\tilde{\Psi}(t,\mathbf{x})$ a ten-component function of the form

$$\tilde{\Psi}=\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\\ \mathbf{j}\\ j_0\end{pmatrix}=\text{column of}\,(E_1,\ E_2,\ E_3,\ H_1,\ H_2,\ H_3,\ j_1,\ j_2,\ j_3,\ j_0). \tag{35}$$

Then Eqs. (13) can be written in the form of the system

$$\left.\begin{aligned}\hat{L}_1\tilde{\Psi}=0;\quad &\hat{L}_1=(1-\beta_5^2)(\beta_\mu p^\mu+1);\\ \hat{L}_2\tilde{\Psi}=0;\quad &\hat{L}_2=\beta_\mu p^\mu\beta_5,\end{aligned}\right\} \tag{36}$$

where β_μ and β_5 are the matrices (32). Substituting (32) and (35) in (36), we arrive at Eqs. (13).

The formulation of Maxwell's equations in the form (35)–(36) is in many respects more convenient than the usual notation for these equations in the vector form (13). In particular, Eqs. (36) are manifestly invariant with respect to the Poincaré group (see Sec. 2).

We note also that the system of equations (13) can be written in the form of a single equation of the type (30), where β_μ are 16×16 matrices satisfying the algebra (29). Such matrices can be chosen in the form[52]

$$\beta_\mu=(\gamma_\mu^1+\gamma_\mu^2)/2, \tag{37}$$

where $\{\gamma_\mu^1\}$ and $\{\gamma_\mu^2\}$ are two mutually commuting sets of Dirac matrices:

$$\gamma_\mu^\alpha\gamma_\nu^\alpha+\gamma_\nu^\alpha\gamma_\mu^\alpha=2g_{\mu\nu};\quad [\gamma_\mu^1,\ \gamma_\nu^2]=0,\ \alpha=1,\ 2.$$

If now the matrix β in (30) is chosen in the form

$$\beta=\beta_5^2+(1+\gamma_\mu^1\gamma^{2\mu})(\gamma_5^1-\gamma_5^2)/4\varkappa, \tag{38}$$

Eqs. (30), (37), and (38) are equivalent to the system (13) of Maxwell's equations with currents and charges.

There are other formulations of Maxwell's equations (for example, using the 4-potential A_μ), on which we shall not dwell here.

2. RELATIVISTIC AND CONFORMAL INVARIANCE OF MAXWELL'S EQUATIONS

We describe here the invariance algebra of Maxwell's equations in the class of first-order differential operators and find explicitly the finite transformations. We also obtain the explicit form of conformal transformations for a massless field with arbitrary spin.

Definition of the invariance algebra. We turn to an investigation of the symmetry properties of Maxwell's equations. We proceed from the formulation of these equations given by (36). We also consider the somewhat more general system given by Eqs. (31) and (36), where $\tilde{\Psi}(t,\mathbf{x})$ is an arbitrary complex function and β_μ are arbitrary [not necessarily identical to (32)] 10-row matrices satisfying the Kemmer-Duffin-Petiau algebra.

We denote by $\{Q_A\}$ $(A=1,2,\ldots,N,N<\infty)$ a set of linear operators that are defined on a set everywhere dense in the space of the 10-component square-integrable functions $\Psi(t,\mathbf{x})$ and form a finite-dimensional Lie algebra.

Definition 1. Equations (36) are invariant with respect to the algebra $\{Q_A\}$ if the operators Q_A satisfy the conditions

$$\left.\begin{array}{l}[\hat{L}_1,\ Q_A]=f_A^1\hat{L}_1+g_A^1\hat{L}_2;\\ [\hat{L}_2,\ Q_A]=f_A^2\hat{L}_1+g_A^2\hat{L}_2,\end{array}\right\} \qquad (39)$$

where $f_A^1,\ g_A^1,\ f_A^2,\ g_A^2$ are certain operators defined on the set of solutions of Eqs. (36), and the symbol $[A,B]$ denotes the commutator: $[A,B]=AB-BA$.

For if (39) is satisfied, the transformation $\Psi\to Q_A\Psi$ carries a solution of Eq. (23) into another solution.

Thus, the problem of describing the invariance algebra of Maxwell's equations reduces to finding the largest possible class of operators Q_A satisfying the conditions (39). Note that Definition 1 does not contain any requirements concerning the general form of the operators Q_A—they could, for example, be differential operators including derivatives of order higher than the first and also integro-differential operators. This is the fundamental difference between our formulation of the problem and the classical Lie-Ovsyannikov approach, in which the infinitesimal operators of the invariance group of a differential equation, which obviously form the invariance algebra of the given equation, always belong to the class of first-order differential operators.

Invariance algebra of Maxwell's equations in the class of first-order differential operators. We consider the problem of finding the invariance algebra of Eqs. (36) in the class of first-order differential operators; this consists of finding all possible operators of the form

$$Q_A=B_A(t,\ \mathbf{x})+C_A^b(t,\ \mathbf{x})\,\partial/\partial x_b+D_A(t,\ \mathbf{x})\,\partial/\partial t \qquad (40)$$

that satisfy the conditions (39) and form a finite-dimensional Lie algebra. In Eq. (40), $C_A^b(t,\mathbf{x})$ and $D_A(t,\mathbf{x})$ are infinitely differentiable functions, and $B_A(t,\mathbf{x})$ are 10×10 matrices whose elements are also infinitely differentiable.

Theorem 1. The invariance algebra of Eqs. (36) in the class of first-order differential operators is the 15-dimensional Lie algebra whose basis elements are given by

$$P_\mu=p_\mu=ig_{\mu\nu}\partial/\partial x_\nu;\quad J_{\mu\nu}=x_\mu p_\nu-x_\nu p_\mu+S_{\mu\nu};\quad D=x_\mu p^\mu+ik;$$
$$K_\mu=2x_\mu D-x_\nu x^\nu p_\mu+2S_{\mu\nu}x^\nu, \qquad (41)$$

where

$$S_{\mu\nu}=i(\beta_\mu\beta_\nu-\beta_\nu\beta_\mu);\quad k=\beta_\mu\beta^\mu=3-\beta_5^2. \qquad (42)$$

Proof. Using the relations

$$\beta_5^3=\beta_5,\quad (1-\beta_5^2)\beta_\mu=\beta_\mu\beta_5^2, \qquad (43)$$

which follow from the algebra (31), we can verify directly that the operators (36) and (41) satisfy the conditions

$$[P_\mu,\ \hat{L}_\alpha]=[J_{\mu\nu},\ \hat{L}_\alpha]=[D,\ \hat{L}_\alpha]=[K_\mu,\ \hat{L}_\alpha]=0,\ \alpha=1,\ 2, \qquad (44)$$

which are identical to (39) when $g_A^\alpha=f_A^\alpha\equiv0$.

Using (31), we can readily show that the operators (41) form a 15-dimensional Lie algebra, since they satisfy the commutation relations

$$\left.\begin{array}{l}[P_\mu,\ P_\nu]=0;\quad [J_{\mu\nu},\ P_\lambda]=i(g_{\nu\lambda}P_\mu-g_{\mu\lambda}P_\nu);\\ [J_{\mu\nu},\ J_{\lambda\sigma}]=i(g_{\mu\lambda}J_{\nu\sigma}+g_{\nu\sigma}J_{\mu\lambda}-g_{\nu\lambda}J_{\mu\sigma}-g_{\mu\sigma}J_{\nu\lambda});\\ [J_{\mu\nu},\ K_\lambda]=i(g_{\nu\lambda}L_\mu-g_{\mu\lambda}K_\nu);\\ [K_\mu,\ P_\nu]=-2i(g_{\mu\nu}D+J_{\mu\nu});\quad [K_\mu,\ K_\nu]=0;\\ [D,\ P_\mu]=-iP_\mu;\quad [D,\ K_\mu]=iK_\mu;\quad [J_{\mu\nu},\ D]=0.\end{array}\right\} \qquad (45)$$

Thus, the operators (41) do indeed form the invariance algebra of Maxwell's equations. This proves the theorem.

The relations (45) define the Lie algebra of the conformal group $C(1,3)$. This algebra contains the Poincaré subalgebra generated by the operators P_μ and $J_{\mu\nu}$ and given by the relation (45a).

Corollary 1. Each equation of (36) is separately invariant with respect to the algebra $C(1,3)$.

This follows directly from the fact that in accordance with (44) the operator $\hat{L}_1$ from the first equation of (36) and the operator $\hat{L}_2$ from the second commute with all basis elements of the algebra $C(1,3)$ given by (41).

Corollary 2. Maxwell's equations for the electromagnetic field in vacuum are invariant with respect to the algebra $C(1,3)$.

Indeed, Maxwell's equations without currents and charges, given by Eqs. (12), can be represented in the form of the system (36) with subsidiary condition

$$\hat{L}_3\tilde{\Psi}\equiv(1-\beta_5^2)\,\tilde{\Psi}=0 \qquad (46)$$

imposed on its solution set [here, the matrices β_μ must have the form (32)]. But the matrix $\hat{L}_3$ commutes with the generators (41) and, therefore, Eq. (46), like (36), is invariant with respect to the algebra $C(1,3)$.

It follows from the symmetry of Eqs. (36) with respect to the algebra (41) that these equations are also invariant with respect to the set of transformations

$$\Psi\to\exp(iQ_A\theta_A)\,\Psi,\quad A=1,\ 2,\ \ldots,\ 15, \qquad (47)$$

where Q_A is an arbitrary operator of the set (41), and θ_A are real parameters. We obtain below in explicit form all transformations of the type (47), which form a representation of the conformal group. As was already shown by Bateman,[8] the conformal group is the maximal local group of transformations of the variables $\mathbf{x}$ and t that leave invariant Maxwell's equations with currents and charges.

Invariance of the equations for the electromagnetic field in vacuum with respect to the algebra $C(1,3)\oplus H$. It was shown above that Eqs. (36) and (46) are invariant with respect to the 15-dimensional algebra $C(1,3)$. It can be shown that the invariance algebra of these equations in the class of first-order differential operators can be extended to a 16-dimensional Lie group, as is established in the following theorem.

Theorem 2. The system of equations (36) and (46) is invariant with respect to the 16-dimensional Lie algebra whose basis elements are given by Eqs. (41) and (48):

$$F = \beta_5. \tag{48}$$

Proof. Using Eqs. (43), we obtain for $\hat{L}_1$ and $\hat{L}_2$ from (36) and for $\hat{L}_3$ from (46) the relations $[\hat{L}_1, \beta_5] = -L_2$, $[\hat{L}_2, \beta_5] = \hat{L}_1 - \hat{L}_3 - \beta_5 \hat{L}_2$, $[\hat{L}_3, \beta_5] = 0$, from which it follows directly that the operator (41) satisfies the invariance condition of Eqs. (36) and (46). The operator F given by (48) commutes with all the operators (41). This means that the operators (41) and (48) form the algebra $C(1,3) \oplus H$, where H consists of the single element (48). By virtue of Corollary 2 to Theorem 1, this algebra is an invariance algebra of Eqs. (36) and (46). The theorem is proved.

As we shall see below, the operator (48) generates the Heaviside-Larmor-Rainich transformations (2). In Ref. 13, it is shown that the algebra $C(1,3) \oplus H$ is the maximal invariance algebra of Maxwell's equations for the electromagnetic field in vacuum. In Sec. 3, we shall find new invariance algebras of Maxwell's equations whose basis elements are nonlocal (intergro-differential) operators.

Remark. All the formulations of Maxwell's equations for the electromagnetic field in vacuum considered in Sec. 1 are also invariant with respect to the algebra $C(1,3)$. The basis elements of this invariance algebra belong in all cases to the class of first-order differential operators and are given by Eqs. (41), in which

$$\begin{cases} S_{ab} = \varepsilon_{abc} S_c \\ S_{0a} = i\sigma_2 S_a \end{cases} \quad \text{for} \quad (16), (17); \tag{49a}$$

$$\begin{cases} S_{ab} = \varepsilon_{abc} \hat{S}_c \\ S_{0a} = i\hat{S}_a \end{cases} \quad \text{for} \quad (20); \tag{49b}$$

$$S_{\mu\nu} = \tilde{S}_{\mu\nu} \quad \text{for} \quad (23), (24), \tag{49c}$$

and, finally, $S_{\mu\nu}$ has the form (28) for (27). Here, S_a, $\hat{S}_a$, and $\tilde{S}_{\mu\nu}$ are the matrices (15) and (28). At the same time, Eqs. (16), (17), and (27) are invariant with respect to the larger algebra $C(1,3) \oplus H$, where H consists of a single element F, equal to σ_2 for Eqs. (16) and (17) and $\gamma_0\gamma_1\gamma_2\gamma_3$ for Eqs. (27), whereas for Eqs. (20) and (23), (24) the maximal invariance algebra in the class of linear first-order differential operators is $C(1,3)$.

Transformations of discrete symmetries. The Lie algebras considered above are the maximal invariance algebras of Maxwell's equations in the class of first-order differential operators, but, as we shall see below, they do not exhaust all the symmetry properties of these equations. A well-known example of a symmetry not included by the invariance algebras considered above is the invariance of Maxwell's equations with respect to the discrete transformations

$$\left.\begin{array}{l} \mathbf{x} \to \mathbf{x}; \quad t \to t; \\ \mathbf{E}(t, \mathbf{x}) \to -\mathbf{E}(t, -\mathbf{x}); \ \mathbf{H}(t, \mathbf{x}) \to \mathbf{H}(t, -\mathbf{x}); \\ \mathbf{j}(t, \mathbf{x}) \to -\mathbf{j}(t, -\mathbf{x}); \quad j_0(t, \mathbf{x}) \to j_0(t, -\mathbf{x}); \end{array}\right\} \tag{50a}$$

$$\left.\begin{array}{l} \mathbf{x} \to \mathbf{x}, \quad t \to -t; \\ \mathbf{E}(t, \mathbf{x}) \to \mathbf{E}(-t, \mathbf{x}); \quad H(t, \mathbf{x}) \to -\mathbf{H}(-t, \mathbf{x}); \\ \mathbf{j}(t, \mathbf{x}) \to -\mathbf{j}(-t, \mathbf{x}); \quad j_0(t, \mathbf{x}) \to j_0(-t, \mathbf{x}); \end{array}\right\} \tag{50b}$$

$$\left.\begin{array}{l} \mathbf{x} \to \mathbf{x}, \quad t \to t; \\ \mathbf{E}(t, \mathbf{x}) \to \mathbf{E}^*(t, \mathbf{x}); \quad \mathbf{H}(t, \mathbf{x}) \to \mathbf{H}^*(t, \mathbf{x}); \\ \mathbf{j}(t, \mathbf{x}) \to \mathbf{j}^*(t, \mathbf{x}); \quad j_0(t, \mathbf{x}) \to j_0^*(t, \mathbf{x}). \end{array}\right\} \tag{50c}$$

The transformations (50) are called spatial inversion P, time reversal T, and charge conjugation C.

Using the notation (32) and (35), we can rewrite the transformations (50) in the form

$$\left.\begin{array}{l} \tilde{\Psi}(t, \mathbf{x}) \to P\tilde{\Psi}(t, \mathbf{x}) = (1 - 2\beta_0^2)\tilde{\Psi}(t, -\mathbf{x}); \\ \tilde{\Psi}(t, \mathbf{x}) \to T\tilde{\Psi}(t, \mathbf{x}) = (1 + 2\beta_1^2)(1 + 2\beta_2^2)(1 + 2\beta_3^2)\tilde{\Psi}(-t, \mathbf{x}); \\ \tilde{\Psi}(t, \mathbf{x}) \to C\tilde{\Psi}(t, \mathbf{x}) = \tilde{\Psi}^*(t, \mathbf{x}). \end{array}\right\} \tag{51}$$

Using the relations (31), we can readily show that the transformations (51) leave Eqs. (46) invariant, since

$$[P, \hat{L}_1] = [P, \hat{L}_2]_+ = [T, \hat{L}_1]_+ = [T, \hat{L}_2] = [C, \hat{L}_1] = [C, \hat{L}_2] = 0,$$

where $\hat{L}_1$ and $\hat{L}_2$ are the operators (36); the symbol $[A, B]_+$ denotes the anticommutator, $[A, B]_+ = AB + BA$. The operators (51) satisfy the following commutation relations together with the generators (41) of the group $C(1,3)$:

$$\left.\begin{array}{l} [P, P_0] = [P, P_a]_+ = [P, J_{ab}] = [P, J_{0a}]_+ = 0; \\ [T, P_0]_+ = [T, P_a] = [T, J_{ab}] = [T, J_{0a}]_+ = 0; \\ [C, P_\mu]_+ = [C, J_{\mu\nu}]_+ = 0; \\ [P, D] = [P, K_0] = [P, K_a]_+ = 0; \\ [T, D] = [T, K_0]_+ = [T, K_a] = 0; \ [C, D]_+ = [C, K_\mu]_+ = 0; \\ [P, T] = [P, C] = [T, C] = 0, \quad T^2 = P^2 = C^2 = 1. \end{array}\right\} \tag{52}$$

The commutation and anticommutation relations (52) can serve as an abstract definition of the operators P, T, and C. We see that Maxwell's equations are invariant with respect to the set of operators $\{P_\mu, J_{\mu\nu}, D, K_\mu, P, T, C\}$ forming the algebra (52), which is not a Lie algebra.

Explicit form of the transformations of the conformal group for E, H, j, and $\mathbf{j_0}$. We find now in explicit form a representation of the conformal group realized on the solution set of Maxwell's equations with currents and charges, i.e., we calculate finite transformations of the coordinates and time, the vectors **E** and **H**, and the current 4-vector generated by the generators (41).

A representation of the conformal group on the solution set of the system (36) is realized by operators of the form

$$U = \exp(iQ_A\theta_A), \quad A = 1, 2, \ldots, 15, \tag{53}$$

where Q_A are the generators of (36), θ_A are arbitrary real parameters, and summation from 1 to 15 over the repeated index A is understood. Since the generators (38) form a finite-dimensional Lie algebra, the operator (51) can always be represented in the form

$$U = U_5U_4U_3U_2U_1,$$

where

$$\left.\begin{array}{l} U_1 = \exp(iP_\mu a^\mu) = \exp(ip_\mu a^\mu), \quad \mu = 0, 1, 2, 3; \\ U_2 = \exp(iJ_a\theta_a); \quad J_a = \varepsilon_{abc}J_{bc}/2; \\ U_3 = \exp(iJ_{0a}\lambda_a); \quad U_4 = \exp(iD\lambda_0); \\ U_5 = \exp(iK_\mu b^\mu), \end{array}\right\} \tag{54}$$

and a_μ, θ_a, λ_μ, b_μ are real parameters. Therefore, to determine the explicit form of the finite transformations of the conformal group it is sufficient to specify the action of the operator (54).

The transformations of the vectors **E** and **H** and of the 4-vector $j=(\mathbf{j}, j_0)$ generated by the operators (54) are well known. The transformations of the Poincaré group generated by U_1, U_2, and U_3 were already found by Lorentz, Poincaré, and Einstein; the dilation transformations realized by the operator D were described for an arbitrary field by Weyl, and, finally, the strictly conformal transformations generated by the operators K_μ were described by Cunningham.[9] However, so far as we know the explicit form of conformal transformations for **E** and **H** has been nowhere given, although in Ref. 53 there is a very complicated formula for transforming the electromagnetic field tensor.

Here, we write out explicitly the transformations of the conformal group for **E**, **H**, and j, and we also give below the proof of these formulas and establish the transformation law for a conformally invariant field of arbitrary spin.

The conformal transformations for the independent variables **x** and t are given by[4)]

$$\left.\begin{aligned} &\mathbf{x}\to\mathbf{x}'=\mathbf{x}-\mathbf{a};\\ &t\to t'=t-a_0;\end{aligned}\right\} \tag{55a}$$

$$\left.\begin{aligned} &\mathbf{x}\to\mathbf{x}''=\mathbf{x}\cos\theta-\boldsymbol{\theta}\times\mathbf{x}\sin\theta/\theta+\boldsymbol{\theta}\boldsymbol{\theta}\cdot\mathbf{x}\,(1-\cos\theta)/\theta^2;\\ &t\to t''=t;\end{aligned}\right\} \tag{55b}$$

$$\left.\begin{aligned} &\mathbf{x}\to\mathbf{x}'''=\mathbf{x}\operatorname{ch}\lambda-\boldsymbol{\lambda}t\operatorname{sh}\lambda/\lambda+\boldsymbol{\lambda}(\boldsymbol{\lambda}\cdot\mathbf{x})(\operatorname{ch}\lambda-1)/\lambda^2;\\ &t\to t'''=t\operatorname{ch}\lambda-\mathbf{x}\cdot\boldsymbol{\lambda}\operatorname{sh}\lambda/\lambda;\end{aligned}\right\} \tag{55c}$$

$$x_\mu\to x_\mu^{\mathrm{IV}}=\exp(-\lambda_0)\,x_\mu; \tag{55d}$$

$$x_\mu\to x_\mu^{\mathrm{V}}=(x_\mu+b_\mu x_\nu x^\nu)/(1+2b_\nu x^\nu+b_\nu b^\nu x_\lambda x^\lambda), \tag{55e}$$

where $\theta=(\theta_1^2+\theta_2^2+\theta_3^2)^{1/2}$, $\lambda=(\lambda_1^2+\lambda_2^2+\lambda_3^2)^{1/2}$.

The transformations (55a)–(55c) conserve the quadratic form $x_0^2-\mathbf{x}^2$ and form the group $P(1,3)$, which is called the *Poincaré group*. Equations (55d) and (55e) define scale and strictly conformal transformations, which together with (55a)–(55c) form the conformal group $C(1,3)$. The operators (54) realize a representation of this group on the solution set of Eqs. (36) and generate the following transformations of the function $\tilde{\Psi}(t,\mathbf{x})$:

$$\tilde{\Psi}(t,\mathbf{x})\to\tilde{\Psi}'(t,\mathbf{x})=U_1\tilde{\Psi}(t,\mathbf{x})=\tilde{\Psi}(t',\mathbf{x}'); \tag{56a}$$

$$\begin{aligned}\tilde{\Psi}(t,\mathbf{x})\to\tilde{\Psi}''(t,\mathbf{x})&=U_2\tilde{\Psi}(t,\mathbf{x})=\exp(\mathrm{i}\mathbf{S}\cdot\boldsymbol{\theta})\,\tilde{\Psi}(t'',\mathbf{x}'')\\ &=[1+\mathrm{i}\mathbf{S}\cdot\boldsymbol{\theta}\sin\theta/\theta+(\mathbf{S}\cdot\boldsymbol{\theta})^2(\cos\theta-1)/\theta^2]\tilde{\Psi}(t'',\mathbf{x}'');\end{aligned} \tag{56b}$$

$$\begin{aligned}\tilde{\Psi}(t,\mathbf{x})\to\tilde{\Psi}'''(t,\mathbf{x})&=U_3\tilde{\Psi}(t,\mathbf{x})=\exp(\mathrm{i}S_{0a}\lambda_a)\,\tilde{\Psi}(t''',\mathbf{x}''')\\ &=[1+\mathrm{i}S_{0a}\lambda_a\operatorname{sh}\lambda/\lambda+(S_{0a}\lambda_a)^2(1-\operatorname{ch}\lambda)/\lambda^2]\,\tilde{\Psi}(t''',\mathbf{x}''');\end{aligned} \tag{56c}$$

$$\tilde{\Psi}(t,\mathbf{x})\to\tilde{\Psi}^{\mathrm{IV}}(t,\mathbf{x})=U_4\tilde{\Psi}(t,\mathbf{x})=\exp(-k\lambda_0)\,\tilde{\Psi}(t^{\mathrm{IV}},\mathbf{x}^{\mathrm{IV}}); \tag{56d}$$

$$\begin{aligned}\tilde{\Psi}(t,\mathbf{x})\to\tilde{\Psi}^{\mathrm{V}}(t,\mathbf{x})&=[\varphi\beta_5^2+\varphi^2(1-\beta_5^2)]\\ &\times[\varphi+2\mathrm{i}S_{\mu\nu}b^\mu x^{\mathrm{V}\nu}(b_\nu x^{\mathrm{V}\nu}-1)-2(S_{\mu\nu}b^\mu x^{\mathrm{V}\nu})^2]\,\tilde{\Psi}(t^{\mathrm{V}},\mathbf{x}^{\mathrm{V}}),\end{aligned} \tag{56e}$$

where $S_a=\varepsilon_{abc}S_{bc}/2$, $\varphi=1-2b^\nu x_\nu^V+b_\nu b^\nu x_\lambda^V x^{V\lambda}$, and $S_{\mu\nu}$ and k are the matrices (42). Substituting in (56) the expressions (35) for the function $\tilde{\Psi}(t,\mathbf{x})$ and (32) for the matrices β_μ, we obtain conformal transformations for the intensity vectors of the electric and magnetic fields and for the 4-vector of the electric current in the form

$$\mathbf{E}\to\mathbf{E}'=\mathbf{E};\quad \mathbf{H}\to\mathbf{H}'=\mathbf{H};\quad j_\mu\to j'_\mu=j_\mu; \tag{57a}$$

$$\left.\begin{aligned} &\mathbf{E}\to\mathbf{E}''=\mathbf{E}\cos\theta-\boldsymbol{\theta}\times\mathbf{E}\sin\theta/\theta+\boldsymbol{\theta}(\boldsymbol{\theta}\cdot\mathbf{E})(1-\cos\theta)/\theta^2;\\ &\mathbf{H}\to\mathbf{H}''=\mathbf{H}\cos\theta-\boldsymbol{\theta}\times\mathbf{H}\sin\theta/\theta+\boldsymbol{\theta}(\boldsymbol{\theta}\cdot\mathbf{H})(1-\cos\theta)/\theta^2;\\ &\mathbf{j}\to\mathbf{j}''=\mathbf{j}\cos\theta-\boldsymbol{\theta}\times\mathbf{j}\sin\theta/\theta+\boldsymbol{\theta}(\boldsymbol{\theta}\cdot\mathbf{j})(1-\cos\theta)/\theta^2;\\ &j_0\to j_0''=j_0;\end{aligned}\right\} \tag{57b}$$

$$\left.\begin{aligned} &\mathbf{E}\to\mathbf{E}'''=\mathbf{E}\operatorname{ch}\lambda-\boldsymbol{\lambda}\times\mathbf{H}\operatorname{sh}\lambda/\lambda+\boldsymbol{\lambda}(\boldsymbol{\lambda}\cdot\mathbf{E})(1-\operatorname{ch}\lambda)/\lambda^2;\\ &\mathbf{H}\to\mathbf{H}'''=\mathbf{H}\operatorname{ch}\lambda+\boldsymbol{\lambda}\times\mathbf{E}\operatorname{sh}\lambda/\lambda+\boldsymbol{\lambda}(\boldsymbol{\lambda}\cdot\mathbf{H})(1-\operatorname{ch}\lambda)/\lambda^2;\\ &\mathbf{j}\to\mathbf{j}'''=\mathbf{j}-\boldsymbol{\lambda}j_0\operatorname{sh}\lambda/\lambda-\boldsymbol{\lambda}(\boldsymbol{\lambda}\cdot\mathbf{j})(1-\operatorname{ch}\lambda)/\lambda^2;\\ &j_0\to j_0''=j_0\operatorname{ch}\lambda-\boldsymbol{\lambda}\cdot\mathbf{j}\operatorname{sh}\lambda/\lambda;\end{aligned}\right\} \tag{57c}$$

$$\begin{aligned}&\mathbf{E}\to\mathbf{E}^{\mathrm{IV}}=\exp(-2\lambda_0)\,\mathbf{E};\quad \mathbf{H}\to\mathbf{H}^{\mathrm{IV}}=\exp(-2\lambda_0)\,\mathbf{H};\\ &j_\mu\to j_\mu^{\mathrm{IV}}=\exp(-3\lambda_0)\,j_\mu;\end{aligned} \tag{57d}$$

$$\left.\begin{aligned} &\mathbf{E}\to\mathbf{E}^{\mathrm{V}}=\varphi[(b_\mu x^{\mathrm{V}\mu}-1)^2\mathbf{E}\\ &+(b^\mu x_\mu^{\mathrm{V}}-1)(b_0\mathbf{x}\times\mathbf{H}-x_0^{\mathrm{V}}\mathbf{b}\times\mathbf{H}+\mathbf{b}\mathbf{x}^{\mathrm{V}}\cdot\mathbf{E}-\mathbf{x}^{\mathrm{V}}\mathbf{b}\cdot\mathbf{E})\\ &+\mathbf{b}\times\mathbf{x}^{\mathrm{V}}(x_0^{\mathrm{V}}\mathbf{b}\cdot\mathbf{H}-b_0\mathbf{x}^{\mathrm{V}}\cdot\mathbf{H}+\mathbf{b}\cdot\mathbf{x}^{\mathrm{V}}\times\mathbf{E})\\ &+(\mathbf{b}x_0^{\mathrm{V}}-b_0\mathbf{x}^{\mathrm{V}})(\mathbf{b}\cdot\mathbf{x}^{\mathrm{V}}\times\mathbf{H}-x_0^{\mathrm{V}}\mathbf{b}\cdot\mathbf{E}+b_0\mathbf{x}^{\mathrm{V}}\cdot\mathbf{E})];\\ &\mathbf{H}\to\mathbf{H}^{\mathrm{V}}=\varphi[(b^\mu x_\mu^{\mathrm{V}}-1)^2\mathbf{H}\\ &+(b^\mu x_\mu^{\mathrm{V}}-1)(x_0^{\mathrm{V}}\mathbf{b}\times\mathbf{E}-b_0\mathbf{x}^{\mathrm{V}}\times\mathbf{E}+\mathbf{b}\mathbf{x}^{\mathrm{V}}\cdot\mathbf{H}-\mathbf{x}^{\mathrm{V}}\mathbf{b}\cdot\mathbf{H})\\ &+\mathbf{b}\times\mathbf{x}^{\mathrm{V}}(\mathbf{b}\cdot\mathbf{x}^{\mathrm{V}}\times\mathbf{H}+b_0\mathbf{x}^{\mathrm{V}}\cdot\mathbf{E}-x_0\mathbf{b}\cdot\mathbf{E})\\ &+(\mathbf{b}x_0^{\mathrm{V}}-\mathbf{x}^{\mathrm{V}}b_0)(b_0\mathbf{x}^{\mathrm{V}}\cdot\mathbf{H}-x_0^{\mathrm{V}}\mathbf{b}\cdot\mathbf{H}-\mathbf{b}\cdot\mathbf{x}^{\mathrm{V}}\times\mathbf{E})];\\ &j_\lambda\to j_\lambda^{\mathrm{V}}=\varphi^2\{\varphi j_\lambda-2[b_\lambda(1-2x_\nu^{\mathrm{V}}b^\nu)+x_\lambda^{\mathrm{V}}b_\nu b^\nu]x_\mu^{\mathrm{V}}j^\mu\\ &+2(x_\lambda^{\mathrm{V}}-b_\lambda x_\nu^{\mathrm{V}}x^{\mathrm{V}\nu})b_\mu j^\mu\}.\end{aligned}\right\} \tag{57e}$$

In Eqs. (57), we have for brevity omitted the arguments of the functions **E**, **H**, and j_μ.

The relations (55) and (57) give the explicit form of transformations of the conformal group for the electric and magnetic field intensity vectors and the current 4-vector. These relations simplify appreciably if a restriction is made to single-parameter transformations, when only one of the parameters in (57) is nonzero. Thus, setting $\mathbf{b}=0$ and $b_0=b$ in (57e), we obtain

$$\left.\begin{aligned} &\mathbf{E}^{\mathrm{V}}(\tilde{t},\tilde{\mathbf{x}})=\tilde{\varphi}[(bt-1)^2\mathbf{E}-b^2\mathbf{x}\mathbf{x}\cdot\mathbf{E}+2b(bt-1)\mathbf{x}\times\mathbf{H}];\\ &\mathbf{H}^{\mathrm{V}}(\tilde{t},\tilde{\mathbf{x}})=\tilde{\varphi}[(bt-1)^2\mathbf{H}-b^2\mathbf{x}\mathbf{x}\cdot\mathbf{H}-2b(bt-1)\mathbf{x}\times\mathbf{E}];\\ &\mathbf{j}^{\mathrm{V}}(\tilde{t},\tilde{\mathbf{x}})=\tilde{\varphi}^2[\tilde{\varphi}\mathbf{j}+2\mathbf{x}\mathbf{x}\cdot\mathbf{j}b^2-2b(1-bt)\mathbf{x}j_0];\\ &j_0^{\mathrm{V}}(\tilde{t},\tilde{\mathbf{x}})=\tilde{\varphi}^2[\tilde{\varphi}j_0-2b(1-bt)\mathbf{x}\cdot\mathbf{j}],\end{aligned}\right\} \tag{58}$$

where

$$\tilde{\varphi}=1-2bt+b^2x_\nu x^\nu;\quad \tilde{x}_\mu=(x_\mu-bx_\nu x^\nu\delta_{\mu0})/\tilde{\varphi}.$$

Integration of representations of the conformal algebra corresponding to arbitrary spin. We shall prove Eqs. (57), which give the finite transformations of the conformal group. Simultaneously, we shall solve the more general problem of obtaining in explicit form the group of transformations generated by the generators (46), where $S_{\mu\nu}$ are arbitrary matrices satisfying the algebra $O(1,3)$:

$$[S_{\mu\nu},S_{\rho\lambda}]=\mathrm{i}(g_{\mu\lambda}S_{\nu\rho}+g_{\nu\rho}S_{\mu\lambda}-g_{\mu\rho}S_{\nu\lambda}-g_{\nu\lambda}S_{\mu\rho}). \tag{59}$$

The generators (41) have the form

$$Q_A=\eta_A^\mu(x)\,\partial/\partial x_\mu+C_A(x),\quad A=1,2,\ldots,15, \tag{60}$$

where $\eta_A^\mu(x)$ are functions of $x=(x_0,x_1,x_2,x_3)$, and $C_A(x)$ are matrices whose elements may also depend on x. The operators (59) generate finite transformations of the conformal group of the form

$$\Psi(x)\to\Psi'(x'),$$

where $\Psi(x)$ are vector functions forming the linear

[4)] *Translator's Note.* The Russian notation for the trigonometric, inverse trigonometric, hyperbolic trigonometric functions, etc., is retained here and throughout the article in the displayed equations.

representation space of the group $C(1,3)$. The explicit form of these transformations can be obtained by integrating the Lie equations[54]

$$\partial x'_\mu/\partial\theta_A = \eta^\mu_A(x')\, x'_\mu;\quad x'_\mu\,|_{\theta_A=0} = x_\mu; \tag{61}$$
$$\partial\Psi'/\partial\theta_A = C_A(x')\,\Psi';\quad \Psi'\,|_{\theta_A=0} = \Psi, \tag{62}$$

where θ_A are the transformation parameters.

Each of Eqs. (61)–(62) determines a system of ordinary differential equations with given initial condition, i.e., a Cauchy problem with a unique solution. We shall find this solution for the case when the operators Q_A (60) are the generators of the strictly conformal transformations K_μ (41), since the transformations generated by the remaining generators of the conformal group, i.e., P_μ, $J_{\mu\nu}$, and D (41), are well known.

Theorem 3. The finite transformations generated by K_μ (41), where $S_{\mu\nu}$ are matrices that realize an arbitrary representation of the algebra $O(1,3)$ (59), and k is an arbitrary number, are given by

$$\Psi(x)\to\Psi'(x') = \hat{\varphi}^k \exp\{2iS_{\mu_0\nu}x^\nu \operatorname{arctg}[a_{\mu_0}/(b_{\mu_0}x^{\mu_0}-1)]/a_{\mu_0}\}\,\Psi(x); \tag{63}$$
$$x_\mu\to x'_\mu = (x_\mu - \delta_{\mu\mu_0}b_{\mu_0}x_\lambda x^\lambda)/(1-2x_\mu b^{\mu_0} + b_{\mu_0}b^{\mu_0}x_\lambda x^\lambda), \tag{64}$$

where

$$\hat{\varphi} = 1 - 2b_{\mu_0}x^{\mu_0} + g_{\mu_0\mu_0}b^2_{\mu_0}x_\lambda x^\lambda;\quad a_{\mu_0} = b_{\mu_0}\sqrt{x_\nu x^\nu - x_{\mu_0}x^{\mu_0}}; \tag{65}$$

the index μ_0 takes one fixed value; b_{μ_0} is the transformation parameter.

Proof. By direct verification we can readily show that the transformations (63) and (64) satisfy the Lie equations (61) and (62). Indeed, comparing the generators K_μ (41) with (60), we obtain

$$C_\mu(x) = -2kx_\mu + 2iS_{\mu\nu}x^\nu;\quad \eta^\nu_\mu(x) = 2x_\mu x^\nu - \delta_{\mu\nu}x_\lambda x^\lambda. \tag{66}$$

Using (66), we rewrite the Lie equations (61) and (62) for the case of strictly conformal transformations in the form

$$dx'_\mu/db_{\mu_0} = 2x'_\mu x'^{\mu_0} - x'_\lambda x'^\lambda\delta_{\mu\mu_0};\quad x'_\mu\,|_{b_{\mu_0}=0} = x_\mu; \tag{67}$$
$$\partial\Psi'/\partial b_{\mu_0} = 2(iS_{\mu_0\nu}x'^\nu - kx'_{\mu_0}); \tag{68a}$$
$$\Psi'\,|_{b_{\mu_0}=0} = \Psi. \tag{68b}$$

It is readily seen that the transformation (64) satisfies Eqs. (67). We see that a solution of Eqs. (68) is given by Eq. (63). Differentiating (63) with respect to b_{μ_0} and taking into account the readily verified identities

$$\frac{d}{db_{\mu_0}}\{2iS_{\mu_0\nu}b^{\mu_0}x^\nu \operatorname{arctg}[a_{\mu_0}/(b_{\mu_0}x^{\mu_0}-1)]/a_{\mu_0}\} = 2iS_{\mu_0\nu}x^\nu;$$
$$\frac{d}{db_{\mu_0}}\hat{\varphi}^k = -2k\hat{\varphi}^k x'_{\mu_0},$$

we obtain Eq. (68a). Setting $b_{\mu_0}=0$ in (63), we arrive at the initial condition (68b).

Thus, the transformations (63) and (64) do indeed satisfy the equations and initial conditions (67), (68) and, by virtue of the uniqueness of the solution of the Cauchy problem, give the unique solution of the Lie equations for the conformal transformations generated by K_μ (41). The theorem is proved.

Using the fact that the generators K_μ form an Abelian subalgebra, we can readily find from (63) and (64) the general form of the strictly conformal transformations:

$$\Psi(x)\to\Psi'(x') = \hat{\varphi}^k \exp\{2iS_{\mu\nu}b^\mu x^\nu \operatorname{arctg}[a/(b_\mu x^\mu - 1)]/a\}\,\Psi(x), \tag{69}$$

where

$$a = [b_\mu b^\mu x_\nu x^\nu - (b_\nu x^\nu)^2]^{1/2};\quad \varphi = 1 - 2b_\mu x^\mu + b_\nu b^\nu x_\mu x^\mu; \tag{70}$$

x'_μ is given by Eq. (55e).

Equations (55e) and (70) give the explicit form of the conformal transformations for an arbitrary representation of the group $C(1,3)$ generated by generators of the form (41). Given some definite finite-dimensional representation of the algebra (59), the exponential in (69) can readily be represented in the form of a polynomial in powers of the matrix $S_{\mu\nu}b^\mu x^\nu$. For the case when the matrices $S_{\mu\nu}$ have the form (42), Eq. (69) reduces to (56e). But if the matrices $S_{\mu\nu}$ form the representation $D(0,1/2)$ of the algebra $O(1,3)$,

$$S_{ab} = \varepsilon_{abc}\sigma_c/2;\quad S_{0a} = i\sigma_a/2,$$

where σ_a are the Pauli matrices and $k=3/2$, which corresponds to a representation of the conformal algebra realized on the solution set of the Weyl equation, then Eq. (69) takes the form

$$\Psi(x)\to\Psi'(x')$$
$$= (1 - 2b_\mu x^\mu + b_\nu b^\nu x_\mu x^\mu)\,[b_\mu x^\mu - 1 + \boldsymbol{\sigma}\cdot(x_0\mathbf{b} - b_0\mathbf{x} + i\mathbf{x}\times\mathbf{b})]\,\Psi(x), \tag{71}$$

where $\Psi(x)$ is a two-component Weyl spinor. For the solution set of the massless Dirac equation, to which there corresponds the matrices $S_{\mu\nu} = i[\gamma_\mu,\gamma_\nu]/4$, where γ_μ are the Dirac matrices, we obtain from (69) the conformal transformations in the form

$$\Psi(x)\to\Psi'(x')$$
$$= (1 - 2b_\mu x^\mu + b_\nu b^\nu x_\mu x^\mu)\,(b_\mu x^\mu - 1 + i\gamma_\mu\gamma_\nu b^\mu x^\nu)\,\Psi(x), \tag{72}$$

where $\Psi(x)$ is a four-component Dirac bispinor.

Note that Eq. (69) is also valid if k is not a number but an arbitrary matrix that commutes with $S_{\mu\nu}$.

3. NONGEOMETRICAL SYMMETRY OF MAXWELL'S EQUATIONS

We consider here the hidden (nongeometrical[24,30]) symmetry of Maxwell's equations, which cannot be found by the classical Lie-Ovsyannikov approach. By means of a non-Lie method of investigation of the symmetry properties of differential equations (see Refs. 24, 28, 30, and 32) it has been shown that these equations have not only conformal invariance but also an additional symmetry with respect to an eight-dimensional Lie algebra isomorphic to the algebra $U(2)\otimes U(2)$, and also with respect to the 23-dimensional algebra A_{23}, which includes the coordinates, $P(1,3)$ and $U(2)\otimes U(2)$.

Invariance with respect to the algebra A_8. We consider the problem of finding invariance algebras of Maxwell's equations for the electromagnetic field in vacuum in the class of integro-differential operators. We proceed from the formulation of these equations given by (14)–(18). Following the first step of the algorithm outlined briefly in the Introduction, we go over from Eqs. (16) and (17) to equations in the momentum space:

$$L_1\varphi(t, \mathbf{p}) = 0; \quad L_1 = i\partial/\partial t + \sigma_2 \mathbf{S}\cdot\mathbf{p}; \tag{73a}$$

$$L_2\varphi(t, \mathbf{p}) = 0; \quad L_2 = p_1 - \mathbf{S}\cdot\mathbf{p}S_1, \tag{73b}$$

where $\varphi(t,\mathbf{p})$ is the Fourier transform of the vector function (14):

$$\varphi(t, \mathbf{p}) = (2\pi)^{-3/2} \int d^3x\varphi(t, \mathbf{x}) \exp(-i\mathbf{p}\cdot\mathbf{x}); \tag{74}$$

and $\mathbf{p}=(p_1, p_2, p_3)$, where p_1, p_2, and p_3 are independent variables.

From the condition that the function $\varphi(t,\mathbf{x})$ be real, we find that $\varphi(t,\mathbf{p})$ must satisfy

$$\varphi^*(t, \mathbf{p}) = \varphi(t, -\mathbf{p}). \tag{75}$$

The invariance condition (39) in terms of the operators (73) takes the form

$$[L_1, Q_A] = f_A^1 L_1 + g_A^1 L_2; \quad [L_2, Q_A] = f_A^2 L_1 + g_A^2 L_2, \tag{76}$$

where L_1 and L_2 are the operators (73), Q_A are the symbols of the basis elements of the invariance algebra of the original system (16)–(17), and f_A^1, g_A^1, f_A^2, g_A^2 are 6×6 matrices, in the general case dependent on p_μ and x_μ, $\mu=0, 1, 2, 3$.

We consider the problem of describing all possible (up to equivalence) operators $Q_A=Q_A(\mathbf{p})$ satisfying the conditions (76). We require that these operators should not carry (74) out of the class of functions satisfying the condition (75). This requirement can be written in the form

$$Q_A^*(\mathbf{p}) = Q_A(-\mathbf{p}). \tag{77}$$

Theorem 4.[28,31,32] Maxwell's equations (16)–(17) are invariant with respect to the 8-dimensional Lie algebra A_8, whose basis elements belong to the class of integro-differential operators. The symbols of these basis elements have the form

$$\left.\begin{aligned} &Q_1 = \sigma_3\mathbf{S}\cdot\mathbf{p}D/p; \quad Q_2 = i\sigma_2; \quad Q_3 = -\sigma_1\mathbf{S}\cdot\mathbf{p}D/p; \quad Q_4 = -\sigma_1 D;\\ &Q_5 = \mathbf{S}\cdot\mathbf{p}/p; \quad Q_6 = -\sigma_3 D; \quad Q_7 = I; \quad Q_8 = i\sigma_2\mathbf{S}\cdot\mathbf{p}/p, \end{aligned}\right\} \tag{78}$$

where

$$\left.\begin{aligned} D = \Big\{ &\sum_{a\neq b\neq c} [(p_a^2p_b^2 + p_a^2p_c^2 - p_b^2p_c^2)(1-S_a^2) + p_1p_2p_3S_aS_bp_c]\\ &- pp_1p_2p_3[1-(\mathbf{S}\cdot\mathbf{p})^2/p^2]\Big\}\delta^{-1};\\ \delta = [&p_1^4(p_2^2-p_3^2)^2 + p_2^4(p_3^2-p_1^2)^2 + p_3^4(p_1^2-p_2^2)^2]^{1/2}/\sqrt{2}; \end{aligned}\right\} \tag{79}$$

σ_a and S_a are the matrices (15). The operators (78) satisfy the algebra

$$\left.\begin{aligned} &[Q_a, Q_b] = -[Q_{3+a}, Q_{3+b}] = -\varepsilon_{abc}Q_c;\\ &[Q_{3+a}, Q_b] = \varepsilon_{abc}Q_{3+c}, \quad a, b, c = 1, 2, 3;\\ &[Q_7, Q_A] = [Q_8, Q_A] = 0, \quad A = 1, 2, \ldots, 8, \end{aligned}\right\} \tag{80}$$

which is isomorphic to the Lie algebra of the group $U(2)\otimes U(2)$.

Proof. The validity of the theorem is most readily seen by direct verification. For this, it is sufficient to use the identities

$$\left.\begin{aligned} &D\sigma_a = \sigma_a D; \quad D\mathbf{S}\cdot\mathbf{p} = -\mathbf{S}\cdot\mathbf{p}D; \quad D(\mathbf{S}\cdot\mathbf{p})^2 = Dp^2;\\ &D^2\mathbf{S}\cdot\mathbf{p} = \mathbf{S}\cdot\mathbf{p}; \quad L_2\mathbf{S}\cdot\mathbf{p} = 0; \quad [D, L_2] = (D + pp_1p_2p_3\delta^{-1})L_2, \end{aligned}\right\} \tag{81}$$

from which the relations (76) and (80) follow. The same identities (81) can be readily verified by writing the matrices D and $\mathbf{S}\cdot\mathbf{p}$ in the explicit form [see (15) and (79)]

$$\left.\begin{aligned} &D = D_0 + D_1; \quad D_0 = [(\mathbf{S}\cdot\mathbf{p})^2/p^2 - 1]\,pp_1p_2p_3\delta^{-1};\\ &D_1 = \begin{pmatrix} \hat{D}_1 & \hat{0}\\ \hat{0} & \hat{D}_1\end{pmatrix}; \quad \mathbf{S}\cdot\mathbf{p} = \begin{pmatrix} \hat{\mathbf{S}}\cdot\mathbf{p} & \hat{0}\\ \hat{0} & \hat{\mathbf{S}}\cdot\mathbf{p}\end{pmatrix}, \end{aligned}\right\} \tag{82}$$

where $\hat{0}$ are square three-row null matrices, and

$$\left.\begin{aligned} &\hat{D}_1 = \begin{pmatrix} f-2p_2^2p_3^2 & p_1p_2p_3^2 & p_1p_2^2p_3\\ p_1p_2p_3^2 & f-2p_1^2p_3^2 & p_1^2p_2p_3\\ p_1p_2^2p_3 & p_1^2p_2p_3 & f-2p_1^2p_2^2\end{pmatrix}\delta^{-1};\\ &f = p_1^2p_2^2 + p_1^2p_3^2 + p_2^2p_3^2; \quad \hat{\mathbf{S}}\cdot\mathbf{p} = i\begin{pmatrix} 0 & -p_3 & p_2\\ p_3 & 0 & -p_1\\ -p_2 & p_1 & 0\end{pmatrix}. \end{aligned}\right\} \tag{83}$$

Since $\mathbf{S}\cdot\mathbf{p}D_0 = D_0\mathbf{S}\cdot\mathbf{p}\equiv 0$ in accordance with (82) and (83), verification of the relations (81) reduces to multiplication of the matrices $\hat{D}_1$ and $\hat{\mathbf{S}}\cdot\mathbf{p}$, D, $\mathbf{S}\cdot\mathbf{p}$, and L_2. It can also be seen from (78) and (82) that the operators Q_A satisfy (77). The theorem is proved.

Thus, we have found a new invariance algebra of Maxwell's equations, its basis elements being the operators (78). These operators are defined on the set of vectors $\varphi(t,\mathbf{p})$, which are the Fourier transforms of the solutions of Maxwell's equations (16) and (17). Each matrix Q_A (78) can be associated with an integral operator $\hat{Q}_A$ defined on the space of the functions $\varphi(t,\mathbf{x})$ (14):

$$\begin{aligned} \hat{Q}_A\varphi(t, \mathbf{x}) &= (2\pi)^{-3/2}\int d^3pQ_A\varphi(t, \mathbf{p})\exp(i\mathbf{p}\cdot\mathbf{x})\\ &= (2\pi)^{-3}\int d^3pd^3x'Q_A\varphi(t, \mathbf{x}')\exp[i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')]. \end{aligned} \tag{84}$$

The integral operators (84) satisfy the invariance condition of Eqs. (16) and (17) and form a Lie algebra isomorphic to $U(2)\oplus U(2)$. In contrast to the basis elements of the conformal algebra, these operators generate nonlocal transformations of the function $\varphi(t,\mathbf{x})$ (14) and, hence, of the electric and magnetic field vectors. Therefore, the invariance algebra of Maxwell's equations found here cannot in principle be obtained in the classical Lie-Ovsyannikov approach.

We emphasize that symmetry with respect to the algebra A_8 is not a specific property of Maxwell's equations in the form (16)–(17), but it can be established for any formulation of these equations. The validity of such an assertion is demonstrated below, where we find the symmetry transformations generated by the algebra A_8 directly in terms of the electric and magnetic field intensities.

Finite transformations of the vectors E and H generated by the nongeometrical invariance algebra. We shall give a different proof of Theorem 4, from which it follows that the invariance algebra of Maxwell's equations found above is in a certain sense maximally large. The basic idea of the proof is to transform Eq. (73) to an equivalent diagonal form for which the theorem is obvious.

The operators L_1 and L_2 in (73) do not commute with one another and therefore cannot be diagonalized simultaneously. To avoid this difficulty, we consider instead of L_2 the operator L_3:

$$L_3 = L_4L_2 \equiv p^2 - (\mathbf{S}\cdot\mathbf{p})^2; \quad L_4 = p_1 + iS_2p_3 - iS_3p_2, \tag{85}$$

which commutes with L_1. It follows from (73b) and (85) that $L_2 \equiv p^{-2}L_2L_3$. It follows that if L_3 satisfies the conditions

$$[L_3, Q_A] = f_A^3 L_1 + g_A^3 L_3, \tag{86}$$

then for L_2 the relations (76) hold with

$$f_A^2 = L_2 f_A^3/p; \quad g_A^2 = [Q_A, L_2/p^2] L_4 + L_2 g_A^3 L_4/p^2. \tag{87}$$

If the matrix L_2 satisfies (76), then for L_3 (85) Eq. (86) is satisfied, and the first equation of (76) can be rewritten in the form

$$[L_1, Q_A] = f_A^1 L_1 + \tilde{g}_A^1 L_3, \tag{88}$$

where $\tilde{g}_A' = g_A' L_2/p$. Therefore, the invariance condition (76) is equivalent to the conditions (86) and (88) imposed on L_1 (73a) and L_3 (85).

To diagonalize L_1 and L_3, we use the operator

$$W = U_4 U_3 U_2 U_1, \tag{89}$$

where

$$\left.\begin{aligned}
&U_1 = \exp(-P_+ DS\cdot\hat{\mathbf{p}}\pi/2) = P_- - P_+ DS\cdot\hat{\mathbf{p}};\\
&U_2 = \exp\{-i\varepsilon_{abc}S_a(p_b - p_c)\operatorname{arctg}[\tilde{p}/(p_1+p_2+p_3)]/2p\};\\
&U_3 = \exp[i(S_2 - S_1)\pi/4\sqrt{2}];\\
&U_4 = [1 - i(S_1S_2 + S_2S_1 + 1 - S_3^2)]/\sqrt{2};
\end{aligned}\right\} \tag{90}$$

$$\hat{\mathbf{p}} = \mathbf{p}/p; \quad \tilde{p} = [(p_1-p_2)^2 + (p_2-p_3)^2 + (p_3-p_1)^2]^{1/2}; \quad P_\pm = (1 \mp \sigma_2)/2.$$

As a result of simple calculations, we obtain

$$WL_1W^+ = L_1' = i\,\partial/\partial t + \Gamma_0 p; \quad WL_3W^+ = (1 - \Gamma_0^2)\,p^2, \tag{91}$$

where Γ_0 is the diagonal matrix

$$\Gamma_0 = -i(S_1S_2 + S_2S_1)S_3 = \operatorname{diag}(1, -1, 0, 1, -1, 0). \tag{92}$$

After the transformation (91), the invariance conditions (86) and (88) take the form

$$[L_1', Q_A'] = f_A'^1 L_1' + \tilde{g}_A^1 L_3'; \quad [L_3', Q_A'] = f_A'^3 L_1' + g_A'^3 L_3'. \tag{93}$$

Using (91) and (92), we can readily find the general form of the matrix $Q_A(\mathbf{p})$ satisfying the conditions (93):

$$Q_A(p) = \begin{pmatrix} a & 0 & 0 & c & 0 & 0\\ 0 & b & 0 & 0 & d & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ e & 0 & 0 & f & 0 & 0\\ 0 & g & 0 & 0 & h & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \hat{F}(p) L_3', \tag{94}$$

where $\hat{F}(p)$ is an arbitrary 6×6 matrix whose elements depend on p; $a, b, \ldots, h$ are arbitrary functions of p. At the same time, by virtue of (73b), (85), and (91) we can, without loss of generality, set $\hat{F}(p) = 0$.

Thus, there are altogether eight linearly independent matrices that satisfy the conditions (93). We choose these matrices in the form

$$Q_a' = i\sigma_a; \quad Q_{3+a}' = i\Gamma_0 Q_a'; \quad Q_7' = -1; \quad Q_8' = -i\Gamma_0, \tag{95}$$

where the matrices σ_2 and Γ_0 are given by Eqs. (15) and (92), and 1 is the unit matrix. By means of the operator (89), we obtain the explicit form of these matrices on the solution set of the original system (73): $Q_A = W^\dagger Q_A' W$, where Q_A are the operators (78). Theorem 4 is proved.

Our proof admits a simple generalization to the case of the equations for massless fields of arbitrary spin when, for example, these equations are formulated as in Refs. 47 and 48.

It follows from our results that Maxwell's equations (73) are invariant with respect to the eight-parameter transformations

$$\varphi(t, \mathbf{p}) \to \varphi'(t, \mathbf{p}) = \exp(Q_A\theta_A)\,\varphi(t, \mathbf{p}), \tag{96}$$

where θ_A are arbitrary real parameters. Using the relations (73), we can rewrite Eq. (96) in the form

$$\varphi'(t, \mathbf{p}) = \begin{cases} (\cos\theta_A + Q_A\sin\theta_A)\,\varphi(t, \mathbf{p}), & A = 1, 2, 3, 8;\\ (\operatorname{ch}\theta_A + Q_A\operatorname{sh}\theta_A)\,\varphi(t, \mathbf{p}), & A = 4, 5, 6, 7. \end{cases} \tag{97}$$

Substituting in (97) the explicit form of the operators Q_A (78) and using for the components of the function $\varphi(t, \mathbf{p})$ the notation

$$\varphi(t, \mathbf{p}) = \text{column of}\,(\tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{H}_1, \tilde{H}_2, \tilde{H}_3), \tag{98}$$

where $\tilde{E}_a$ and $\tilde{H}_a$ are the Fourier transforms of the components of the electric and magnetic field vectors, we obtain the transformation law for $\tilde{E}_a$ and $\tilde{H}_a$ in the form

$$\left.\begin{aligned}
&\tilde{E}_a \to \tilde{E}_a\cos\theta_1 + i\varepsilon_{abc}\hat{p}_b D_{cd}\tilde{E}_d\sin\theta_1;\\
&\tilde{H}_a \to \tilde{H}_a\cos\theta_1 - i\varepsilon_{abc}\hat{p}_b D_{cd}\tilde{H}_d\sin\theta_1;
\end{aligned}\right\} \tag{99a}$$

$$\left.\begin{aligned}
&\tilde{E}_a \to \tilde{E}_a\cos\theta_2 + \tilde{H}_a\sin\theta_2;\\
&\tilde{H}_a \to \tilde{H}_a\cos\theta_2 - \tilde{E}_a\sin\theta_2;
\end{aligned}\right\} \tag{99b}$$

$$\left.\begin{aligned}
&\tilde{E}_a \to \tilde{E}_a\cos\theta_3 - i\varepsilon_{abc}\hat{p}_b D_{cd}\tilde{H}_d\sin\theta_3;\\
&\tilde{H}_a \to \tilde{H}_a\cos\theta_3 - i\varepsilon_{abc}\hat{p}_b D_{cd}\tilde{E}_d\sin\theta_3;
\end{aligned}\right\} \tag{99c}$$

$$\left.\begin{aligned}
&\tilde{E}_a \to \tilde{E}_a\operatorname{ch}\theta_4 - D_{ab}\tilde{H}_b\operatorname{sh}\theta_4;\\
&\tilde{H}_a \to \tilde{H}_a\operatorname{ch}\theta_4 - D_{ab}\tilde{E}_b\operatorname{sh}\theta_4;
\end{aligned}\right\} \tag{99d}$$

$$\left.\begin{aligned}
&\tilde{E}_a \to \tilde{E}_a\operatorname{ch}\theta_5 + i\varepsilon_{abc}\hat{p}_b\tilde{E}_c\operatorname{sh}\theta_5;\\
&\tilde{H}_a \to \tilde{H}_a\operatorname{ch}\theta_5 + i\varepsilon_{abc}\hat{p}_b\tilde{H}_c\operatorname{sh}\theta_5;
\end{aligned}\right\} \tag{99e}$$

$$\left.\begin{aligned}
&\tilde{E}_a \to \tilde{E}_a\operatorname{ch}\theta_6 - D_{ab}\tilde{H}_b\operatorname{sh}\theta_6;\\
&\tilde{H}_a \to \tilde{H}_a\operatorname{ch}\theta_6 + D_{ab}\tilde{E}_b\operatorname{sh}\theta_6;
\end{aligned}\right\} \tag{99f}$$

$$\tilde{E}_a \to \tilde{E}_a\exp\theta_7; \quad \tilde{H}_a \to \tilde{H}_a\exp\theta_7; \tag{99g}$$

$$\left.\begin{aligned}
&\tilde{E}_a \to \tilde{E}_a\cos\theta_8 + i\varepsilon_{abc}\hat{p}_b\tilde{H}_c\sin\theta_8;\\
&\tilde{H}_a \to \tilde{H}_a\cos\theta_8 - i\varepsilon_{abc}\hat{p}_b\tilde{E}_c\sin\theta_8.
\end{aligned}\right\} \tag{99h}$$

The equations of (99b) give the Heaviside-Larmor-Rainich transformations.[1-3] The remaining relations (99), which extend the transformations (99b) to the eight-parameter group A_8, which is locally isomorphic to $U(2)\otimes U(2)$, correspond to nonlocal (integral) transformations of the vectors $\mathbf{E}(t, \mathbf{x})$ and $\mathbf{H}(t, \mathbf{x})$, which have the form

$$\left.\begin{aligned}
&\mathbf{E}(t, \mathbf{x}) = (2\pi)^{-3/2}\int d^3p\,\tilde{\mathbf{E}}\exp(i\mathbf{p}\cdot\mathbf{x});\\
&\mathbf{H}(t, \mathbf{x}) = (2\pi)^{-3/2}\int d^3p\,\tilde{\mathbf{H}}\exp(i\mathbf{p}\cdot\mathbf{x}).
\end{aligned}\right\} \tag{100}$$

The transformations (99a)–(99c) and (99h) conserve the bilinear form

$$\mathscr{E} = \int d^3x\,(\mathbf{E}^2 + \mathbf{H}^2), \tag{101}$$

which is associated with the energy of the electromagnetic field. The remaining transformations (99d)–(99g) do not conserve (101). However, there exists a positive-indefinite bilinear form that is invariant with respect to all the transformations (99):

$$\begin{aligned}
(\varphi_1, \varphi_2) &= \int d^3p\,\varphi_1^\dagger(t, \mathbf{p})\,\sigma_2 D\varphi_2(t, \mathbf{p})\\
&= (2\pi)^{-3}\int d^3p\,d^3x\,d^3x'\,\varphi^+(t, \mathbf{x})\,\sigma_2 D\varphi(t, \mathbf{x}')\exp[i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')].
\end{aligned} \tag{102}$$

Thus, Maxwell's equations for the electromagnetic field in vacuum are not only invariant with respect to the well-known conformal group but are also invariant with respect to the eight-parameter group of integral transformations (99). Apart from the substitution $\theta_b \to i\theta_b$, $b=4, 5, 6, 7$, the transformations (99) are identical to the ones obtained in Refs. 28, 31, and 32. Note that the symmetry of Eqs. (73) with respect to the transformations (99) can be readily verified directly.

Invariance with respect to the 23-dimensional Lie algebra. We have shown above that there exist two sets of operators, $\{P_\mu, J_{\mu\nu}, D, K_\mu\}$ (41), (49a) and $\{Q_A\}$ (78), which form invariance algebras of Maxwell's equations. However, as is readily seen, the operators (41), (49a), and (78) do not together form a closed algebra. We prove here a theorem that establishes symmetry of Maxwell's equations with respect to a 23-dimensional Lie algebra that includes the subalgebras $C(1,3)$ and A_8. This unification of the algebras $C(1,3)$ and A_8 can be realized if the basis elements of the conformal algebra are specified in the class of integral operators.

Theorem 5. Equations (73) are invariant with respect to the 23-dimensional Lie algebra whose basis elements are given by Eqs. (78) and (103):

$$\left.\begin{array}{l} P_\mu = ip_\mu; \quad J_{\mu\nu} = i(x'_\mu p_\nu - x'_\nu p_\mu); \\ D = i(x'_\mu p^\mu - i); \quad K_\mu = i(2x'_\mu D - x'_\nu x'^\nu p_\mu), \end{array}\right\} \tag{103}$$

where $x'_a = W^\dagger i\partial/\partial p_a W$, $x'_0 = t$, and W is the operator (89).

Proof. The propositions of the theorem become almost obvious in the representation in which the operators L_1 (73a) and L_3 (85) have the diagonal form (91), (92). In such a representation, the operators (78) take the form (95), and for the operators (103) we obtain the expressions

$$\left.\begin{array}{l} P'_\mu = ip_\mu; \quad J'_{\mu\nu} = i(x_\mu p_\nu - x_\nu p_\mu); \quad D' = i(x_\mu p^\mu + i); \\ K'_\mu = i(2x_\mu D' - x_\nu x^\nu p_\mu), \end{array}\right\} \tag{104}$$

where $x_\mu = i\partial/\partial p\mu$; $B'_a = WB_aW^\dagger$, B_a is an arbitrary operator from (103). By direct verification we see that the operators L'_1 and L'_3 [(91), (92)] and the generators (78) and (104) satisfy the invariance conditions (86) and (88):

$$\left.\begin{array}{l} [L'_1, P'_\mu] = [L'_1, J'_{ab}] = [L'_1, Q'_A] = 0; \\ \{L'_1, J'_{0a}] = -ip_a p^{-2} L'_3 + ip_a p^{-1} Q'_7 L'_1; \quad [L'_1, D'] = iL'_1; \\ \{L'_1 K_0] = -2\{[x_0 + (\mathbf{x}\cdot\mathbf{p} - i) Q'_7 p^{-1}] L'_1 - (\mathbf{x}\cdot\mathbf{p} - i) p^{-1} L'_3\}; \\ [L'_1, K_a] = -2\{(x_a + Q'_7 x_0 p_a p^{-1}) L'_1 - x_0 p_a p^{-2} L'_3\}; \\ [L'_3, P'_\mu] = [L'_3, L'_{ab}] = [L'_3, Q'_A] = 0; \\ [L'_3, J'_{0a}] = -2p_a p_0 p^{-2} L'_3; \quad [L'_3, D'] = -2L'_3; \\ [L'_3, K'_0] = -2[2x_0 - (\mathbf{x}\cdot\mathbf{p} + \mathbf{p}\cdot\mathbf{x}) p_0 p^{-2}] L'_3; \quad [L'_3, K'_a] \\ = -2[2x_a + i2p_a D' p^{-1} - (\mathbf{x}\cdot\mathbf{p} + \mathbf{p}\cdot\mathbf{x}) p_a p^{-2}] L'_3. \end{array}\right\} \tag{105}$$

The operators (104) belong to the algebra $C(1,3)$, since they satisfy the commutation relations (45). In addition, these operators commute with the matrices Q'_A (95), which, in their turn, form a representation of the algebra A_8 (80). We therefore conclude that the operators (95) and (104) form a basis of the algebra $C(1,3)\oplus A_8$. The theorem is proved.

With each operator (78) and (103) defined in the space of the functions $\varphi(t, \mathbf{p})$ (74) we can associate an integral transformation in the space H of functions $\varphi(t, \mathbf{x})$ (14):

$$\begin{array}{l} \varphi(t, \mathbf{x}) \to \varphi'(t, \mathbf{x}) = (2\pi)^{-3} \int d^3p\, d^3x' \exp(iG_\alpha\theta_\alpha) \\ \qquad \times \varphi(t, \mathbf{x}) \exp[i\mathbf{p}\cdot(\mathbf{x} - \mathbf{x}')], \end{array} \tag{106}$$

where G_α is one of the operators (78) and (103), θ_α is a transformation parameter, $\alpha = 1, 2, \ldots, 23$. The transformations (106) leave Maxwell's equations (16) and (17) invariant and form a representation of the group $C(1,3)\otimes A_8$.

Thus, we have obtained a 23-parameter symmetry group of Maxwell's equations for the electromagnetic field in vacuum, this including $C(1,3)$ and A_8 as subgroups. Since the transformations (106) are nonlocal the corresponding invariance group cannot in principle be found in the Lie-Ovsyannikov approach,[11-14] in which the infinitesimal operators belong to the class of first-order differential operators and generate point transformations.

Symmetry with respect to transformations that do not change the time. Since the creation of the special theory of relativity, it has been assumed that the Lorentz transformations (55a)–(56c) are the only symmetry transformations of Maxwell's equations that can be associated with transition to a new inertial frame of reference. Therefore, the very posing of the question of the existence of transformations of solutions of Maxwell's equations that form a representation of the Poincaré group but do not change the time might appear rather unexpected. However, the posing of such a question is entirely justified, and such transformations do indeed exist.[16,28,30] Namely, there exist transformations of the form

$$t \to t' = t; \quad \mathbf{x} \to \mathbf{x}' = \mathbf{f}(\mathbf{x}, t, \lambda_1, \ldots, \lambda_{10}); \tag{107a}$$

$$\left.\begin{array}{l} \mathbf{E} \to \mathbf{E}' = \mathbf{g}\left(\mathbf{E}, \mathbf{H}, \frac{\partial \mathbf{E}}{\partial x_a}, \frac{\partial^2 \mathbf{E}}{\partial x_a \partial x_b}, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_{10}\right); \\ \mathbf{H} \to \mathbf{H}' = \mathbf{h}\left(\mathbf{E}, \mathbf{H}, \frac{\partial \mathbf{E}}{\partial x_a}, \frac{\partial \mathbf{H}}{\partial x_a}, \frac{\partial^2 \mathbf{E}}{\partial x_a \partial x_b}, \ldots \right. \\ \qquad \left. \ldots, \lambda_1, \lambda_2, \ldots, \lambda_{10}\right), \end{array}\right\} \tag{107b}$$

where λ_α are real parameters that realize a representation of the group $P(1,3)$ and leave Maxwell's equations invariant.

It is shown in Refs. 15–17 and 29 that the Schrödinger-Klein-Gordon-Fock equation and other relativistic equations are also invariant with respect to transformations of the form (107a) as well as with respect to the transformations of the independent variables of the Lorentz group. In other words, all relativistic equations have dual symmetry.[29]

Here, we consider the formulation of Maxwell's equations given by (23) and (24). The symmetry of Eqs. (23) and (24) also has a dual nature, since the generators (41), (49b), and (28) of the Poincaré group on the solution set of these equations can also be represented in the form

$$\left.\begin{array}{l} P_0 = H = -\boldsymbol{\alpha}\cdot\mathbf{p}; \quad P_a = p_a = -i\,\partial/\partial x_a; \\ J_{ab} = x_a p_b - x_b p_a + \tilde{S}_{ab}; \quad J_{0a} = tp_a - x_a H + \tilde{S}_{0a}, \end{array}\right\} \tag{108}$$

where α_a and $\tilde{S}_{\mu\nu}$ are the matrices (22) and (28). The operators (108), like (41), (49c), and (28), satisfy the conditions of invariance of Eqs. (22)–(24) and the commutation relations (45a), i.e., they form an invariance algebra of Maxwell's equations. However, in contrast to the generators (41) and (49c), the operators (108) commute with t, i.e., they generate transformations of the Poincaré group that do not change the time.

We shall find in explicit form the transformation group generated by the operators (108). Formally, they can be written as

$$\Psi \to \Psi' = W\Psi; \quad W = \exp(i\theta_A Q_A), \tag{109}$$

where Q_A are the generators (108), θ_A are real parameters, and Ψ is the vector function (23).

We restrict ourselves to the case when Q_A are identical to the generators J_{0a} (108) [the calculation of the explicit form of the transformations generated by the remaining generators P_μ and J_{ab} does not present difficulties; cf. (56a) and (56b)]. The generators J_{0a} given by Eqs. (108) cannot be represented in the form of sums of commuting quantities with one a numerical matrix and the other expressible in terms of first-order differential operators. Although the operator

$$W = \exp(iJ_{0a}\theta_a), \quad a = 1, 2, 3, \tag{110}$$

can always be represented in the form of a finite series in powers of spin matrices, the explicit calculation of this series is a rather difficult problem.

We transform the operator (110) to a form that does not contain matrices in the argument of the exponential. We use the identity

$$iJ_{0a}\theta_a = A_+P_+ + A_-P_-, \tag{111}$$

where

$$P_\pm = (1 \pm H/p)/2; \quad A_\pm = i(tp_a \mp x_a p - S_{0a})\theta_a. \tag{112}$$

The operators (112) satisfy the relations

$$\left.\begin{array}{l} P_\pm P_\pm = P_\pm; \quad P_\pm P_\mp = P_\mp P_\pm = 0; \quad P_\pm A_\mp P_\mp = 0; \\ P_\mp A_\pm P_\pm = 0; \quad A_\pm P_\pm A_\pm P_\pm = A_\pm^2 P_\pm, \end{array}\right\} \tag{113}$$

and, using them, we can readily obtain the following expression for the operator (110):

$$\begin{aligned} \exp(iJ_{0a}\theta_a) &\equiv \exp(A_+P_+ + A_-P_-) = \exp(A_+P_+) \\ &\times \exp(A_-P_-) = \left(1 + A_+P_+ + \frac{1}{2!}A_+^2P_+ + \dots\right) \\ &\times \left(1 + A_-P_- + \frac{1}{2!}A_-^2P_- + \dots\right) = [\exp(A_+)P_+ + P_-] \\ &\times [\exp(A_-)P_- + P_+] = \exp(A_+)P_+ + \exp(A_-)P_-. \end{aligned} \tag{114}$$

But the calculation of the exponentials of the operators A_+ and A_- (112) is not difficult, since $A_\pm$ consist of two commuting terms, one of which can be expressed in terms of spin matrices. Indeed,

$$\begin{aligned} \exp(A_\pm) &\equiv \exp[i(tp_a \mp x_a p + S_{0a})\theta_a] \\ &= \exp[i(tp_a \mp x_a p)]\exp(iS_{0a}\theta_a), \end{aligned} \tag{115}$$

where the last exponential can be written in the form of a finite sum in powers of the matrices $S_{0a}\theta_a$:

$$\exp(iS_{0a}\theta_a) \equiv N(\theta) = 1 + iS_{0a}\theta_a \operatorname{sh}\theta/\theta + (S_{0a}\theta_a)^2(1 - \operatorname{ch}\theta)/\theta^2, \tag{116}$$

where $\theta = (\theta_1^2 + \theta_2^2 + \theta_3^2)^{1/2}$.

Substituting (115) and (116) in (114), we obtain

$$W = \exp(iJ_{0a}\theta_a) = N(\theta)[\exp(iB_+)P_+ + \exp(iB_-)P_-], \tag{117}$$

where we have introduced the notation

$$B_\pm \equiv (tp_a \mp x_a p)\theta_a.$$

Similarly, we can calculate the inverse operator

$$W^{-1} = N(-\theta)[\exp(-iB_+)P_+ + \exp(-iB_-)P_-]. \tag{118}$$

Equations (116)–(118) give the required representation of the operators W; it does not contain spin matrices in the argument of the exponential. Using (116)–(118), we can readily find explicitly the transformation law of the function $\Psi(t, \mathbf{p})$ in the momentum space:

$$\Psi(t, \mathbf{p}) \to \Psi'(t, \mathbf{p}) = W\Psi(t, \mathbf{p}) = N(\theta)[\Psi_+(t, \mathbf{p}') + \Psi_-(t, \mathbf{p}'')], \tag{119}$$

where

$$\begin{aligned} p_a' &= p_a \operatorname{ch}\theta - \theta_a p \operatorname{sh}\theta/\theta + \theta_a\theta_b p_b(\operatorname{ch}\theta - 1)/\theta^2; \\ p_a'' &= p_a \operatorname{ch}\theta + \theta_a p \operatorname{sh}\theta/\theta + \theta_a\theta_b p_b(\operatorname{ch}\theta - 1)/\theta^2; \quad \Psi_\pm = P_\pm\Psi. \end{aligned} \tag{120}$$

To the transformations (119) there correspond nonlocal (integral) transformations of the function (23):

$$\begin{aligned} \Psi(t, \mathbf{x}) \to \Psi'(t, \mathbf{x}) &= N(\theta)(2\pi)^{-3/2}\int d^3p \exp(i\mathbf{p}\cdot\mathbf{x}) \\ &\times [\Psi_+(t, \mathbf{p}') + \Psi_-(t, \mathbf{p}'')], \end{aligned} \tag{121}$$

where $N(\theta)$, $\Psi_\pm(t, \mathbf{p})$, $\mathbf{p}'$, and $\mathbf{p}''$ are given by Eqs. (116) and (120). The transformations (121) together with (56a) and (56b) form a representation of the group $P(1, 3)$ but they leave the time invariant, $t' = WtW^+ = t$.

Non-Lie symmetry of Maxwell's equations in a conducting medium. We investigate here the nongeometrical symmetry of Maxwell's equations in a conducting medium:

$$\left.\begin{array}{c} i\,\partial\mathbf{E}/\partial t = -\mathbf{p}\times\mathbf{H} + i\sigma\mathbf{E}; \quad i\,\partial\mathbf{H}/\partial t = \mathbf{p}\times\mathbf{E}; \\ \mathbf{p}\cdot\mathbf{E} = \mathbf{p}\cdot\mathbf{H} = 0, \end{array}\right\} \tag{122}$$

where σ is the conductivity. We shall show that Eqs. (122), like Eqs. (12) for the electromagnetic field in vacuum, are invariant with respect to the algebra A_8.

Using the notation (14) and (15), we write the system (122) in the form

$$\hat{L}_1\varphi(t, \mathbf{x}) = 0; \quad \hat{L}_1 = i\partial/\partial t + \sigma_2\mathbf{S}\cdot\mathbf{p} + (i/2)(1 + \sigma_3)\sigma; \tag{123}$$

$$\hat{L}_2\varphi(t, \mathbf{x}) = 0; \quad \hat{L}_2 = p_1 - \mathbf{S}\cdot\mathbf{p}S_1. \tag{124}$$

Finding an invariance algebra of Eqs. (123) and (124) means determining a set of operators $\{Q_A\}$ that form a finite-dimensional Lie algebra and satisfy the invariance conditions (39). Since here we shall seek an invariance algebra in the class of integro-differential operators, we go over from (123) and (124) to the equivalent system of equations in the momentum representation:

$$\begin{aligned} L_1\varphi(t, \mathbf{p}) = 0; \quad L_1 &= i\partial/\partial t + \sigma_2\mathbf{S}\cdot\mathbf{p} + (i/2)(1 + \sigma_3)\sigma; \\ L_2\varphi(t, \mathbf{p}) = 0; \quad L_2 &= p_1 - \mathbf{S}\cdot\mathbf{p}S_1, \end{aligned} \tag{125}$$

where $\varphi(t, \mathbf{p})$ is the Fourier transform of $\varphi(t, \mathbf{x})$, and p_a are the independent variables, $-\infty < p_a < \infty$.

Theorem 6. Equations (125) are invariant with respect to the algebra A_8. Its basis elements on the solution set of Eqs. (125) are given by

$$\left.\begin{array}{l} Q_1 = \sigma_3\mathbf{S}\cdot\hat{\mathbf{p}}D; \quad Q_2 = ih\mathbf{S}\cdot\hat{\mathbf{p}}/|h|; \quad Q_3 = ih\sigma_3 D/|h|; \\ Q_{3+a} = ihQ_a/|h|; \quad Q_7 = h/|h|; \quad Q_8 = I, \end{array}\right\} \tag{126}$$

where σ_a, S_a, and D are the matrices (6), (103), and (104), and

$$\left.\begin{aligned}&\hat{\mathbf{p}}=\mathbf{p}/p;\quad h=\sigma_2\mathbf{S}\cdot\mathbf{p}+(i/2)\sigma_3\sigma;\\&h=\sqrt{h^2}=[(\mathbf{S}\cdot\mathbf{p})^2-\sigma_2/4]^{1/2}\\&=(p^2-\sigma^2/4)^{1/2}(\mathbf{S}\cdot\hat{\mathbf{p}})^2+(i/2)\,\sigma\,[1-(\mathbf{S}\cdot\hat{\mathbf{p}})^2].\end{aligned}\right\}\tag{127}$$

Proof. As we did above [see (85)–(88)], we consider instead of (125) the system of equations

$$\left.\begin{aligned}&L_1\varphi(t,\ \mathbf{p})=0;\\&L_3\varphi(t,\ \mathbf{p})=0;\\&L_3=1-(\mathbf{S}\cdot\hat{\mathbf{p}})^2,\end{aligned}\right\}\tag{128}$$

where L_1 is the operator from (125).

The investigation of the symmetry properties of the system (128) is a simpler problem because the operators L_1 and L_3 commute. At the same time, as can be shown in complete analogy with (85)–(88), the invariance algebras of Eqs. (125) and (128) are identical.

Using the identities (81) and taking into account the anticommutativity of the Pauli matrices, we can readily show that the operators (126) satisfy the invariance conditions (86) and (88) of Eqs. (128) and the commutation relations (80), i.e., they form the invariance algebra A_8 of the system (128) and, hence, the system (125). The theorem is proved.

Thus, Maxwell's equations for the electromagnetic field in a conducting medium have the same nongeometrical invariance algebra as the corresponding equations in the absence of currents and charges. It also follows from Theorem 6 that the system (125) is invariant with respect to a group of transformations of the form (96), where Q_A are the operators (126).

To conclude the section, we note that the nongeometrical approach to the investigation of the symmetry of Maxwell's equations was used not only in our papers of Refs. 15–25 but also in Refs. 55 and 56. The approach can also be effective in an investigation of the group properties of the new formulation of electrodynamics proposed in Refs. 57 and 58.

4. SYMMETRY OF THE DIRAC AND THE KEMMER-DUFFIN-PETIAU EQUATIONS

This section is basically devoted to investigation of the symmetry of the Dirac equation

$$L\Psi(t,\ \mathbf{x})\equiv(\gamma_\mu p^\mu-m)\,\Psi(t,\ \mathbf{x})=0,\tag{129}$$

where $\Psi(t,\mathbf{x})$ is a four-component wave function, and γ_μ are 4×4 matrices satisfying the algebra

$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2g_{\mu\nu}.\tag{130}$$

Following Refs. 15–18, 19, 27, and 31, we establish invariance algebras of Eq. (129) in the class of differential and integro-differential operators. We consider separately the case $m=0$ and show that the symmetry of the Dirac equation for massless particles is determined by the same 23-dimensional Lie algebra as the symmetry of Maxwell's equations.

We shall also investigate the symmetry of the Kemmer-Duffin-Petiau equation.

Invariance algebra of the Dirac equation in the class of differential operators. It is well known that Eq. (129) is invariant with respect to the Poincaré group. The generators of this group on the solution set of Eq. (129) have the form

$$P_\mu=p_\mu;\quad J_{\mu\nu}=x_\nu p_\mu-x_\nu p_\mu+S_{\mu\nu},\tag{131}$$

where $S_{\mu\nu}=(i/4)(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)$. The operators (131) commute with L in (129) and satisfy the commutation relations (45a), i.e., they form an invariance algebra of Eq. (129). It is shown in Refs. 14 and 59 that the Lie algebra spanned by the basis elements (131) is the maximal invariance algebra of the Dirac equation in the class of first-order differential operators.

However, the invariance with respect to the algebra (131) does not exhaust all the symmetry properties of the Dirac equation. If we extend the class of operators belonging to the invariance algebra, we can prove the following theorem.[18,19]

Theorem 7. The Dirac equation (129) is invariant with respect to the algebra A_8 over the field of real numbers. The basis elements of this algebra on the solution set of Eqs. (129) are given by

$$\left.\begin{aligned}&\Sigma_{\mu\nu}=(i/2)\,[\gamma_\mu,\ \gamma_\nu]+(i/m)\,(1-i\gamma_5)\,(\gamma_\mu p_\nu-\gamma_\nu p_\mu);\\&\Sigma_0=1;\quad\Sigma_1=\gamma_5-(i/m)\,(1-i\gamma_5)\,\gamma_\mu p^\mu.\end{aligned}\right\}\tag{132}$$

where $\gamma_5=\gamma_0\gamma_1\gamma_2\gamma_3$, and 1 is the unit matrix.

Proof. The theorem is most readily proved by direct verification. Using the relations (130), we obtain

$$\left.\begin{aligned}&[\Sigma_{\mu\nu},\ L]=(i/m)\,(\gamma_\mu p_\nu-\gamma_\nu p_\mu)\,L;\quad[\Sigma_0,\ L]=0;\\&[\Sigma_1,\ L]=-(2i/m)\,\gamma_5\gamma_\mu p^\mu L;\quad[\Sigma_1,\ \Sigma_2]=[\Sigma_\alpha,\ \Sigma_{\mu\nu}]=0;\\&[\Sigma_{\mu\nu},\ \Sigma_{\lambda\sigma}]=2i\,(g_{\mu\sigma}\Sigma_{\nu\lambda}+g_{\nu\lambda}\Sigma_{\mu\sigma}-g_{\mu\lambda}\Sigma_{\nu\sigma}-g_{\nu\sigma}\Sigma_{\mu\lambda}),\end{aligned}\right\}\tag{133}$$

from which it can be seen that the operators (132) satisfy the invariance condition of Eq. (129) and form an 8-dimensional Lie algebra isomorphic to the algebra A_8 (80). This isomorphism is established by the relations

$$\Sigma_{ab}\leftrightarrow\varepsilon_{abc}Q_c;\quad\Sigma_{0a}\leftrightarrow Q_{3+a};\quad\Sigma_0\leftrightarrow Q_7;\quad\Sigma_1\leftrightarrow Q_8.$$

Over the field of the real numbers all the elements of the algebra A_8 given by (132) are linearly independent. To see this, it is sufficient to subject the operators (132) to the transformation

$$\left.\begin{aligned}&\Sigma_{\mu\nu}\to\Sigma'_{\mu\nu}=V\Sigma_{\mu\nu}V^{-1}=(i/2)\,[\gamma_\mu,\ \gamma_\nu];\\&\Sigma_0\to\Sigma'_0=V\Sigma_0V^{-1}=1;\quad\Sigma_1\to\Sigma'_1=V\Sigma_1V^{-1}=\gamma_5,\end{aligned}\right\}\tag{134}$$

where

$$V=\exp\left[-(1-i\gamma_5)\,\gamma_\mu p^\mu/2m\right]=1-(1-i\gamma_5)\,\gamma_\mu p^\mu/2m.\tag{135}$$

The theorem is proved.

We note that (132) do not form a basis of the Lie algebra over the field of complex numbers, since

$$(\Sigma_{ab}-i\varepsilon_{abc}\Sigma_{0c})\,\Psi=(\Sigma_1-i\Sigma_0)\,\Psi=0,\tag{136}$$

where Ψ is an arbitrary solution of Eq. (129).

Thus, the Dirac equation has the same nongeometrical invariance algebra as the system (12) of Maxwell's equations. Although the operators (132) include derivatives with respect to the independent variables of order not higher than the first, they do not generate local transformations of the wave function $\Psi(t,\mathbf{x})$. If we use the notation of Ovsyannikov,[11,12] the generators (132) can be classified as second-order differential operators, including derivatives of the form $\partial^2/\partial x_\mu \partial \Psi_\alpha$.

It follows from the foregoing theorem that the Dirac equation is invariant with respect to the eight-parameter group of transformations

$$\left.\begin{aligned}&\Psi \to (\cos\theta_{ab} - \gamma_a\gamma_b \sin\theta_{ab})\,\psi\\&\quad + (1/m)\sin\theta_{ab}\,(1 - i\gamma_5)\,(\gamma_a\,\partial\Psi/\partial x_b - \gamma_b\,\partial\Psi/\partial x_a);\\&\Psi \to (\mathrm{ch}\,\theta_{0a} - \gamma_0\gamma_a\,\mathrm{sh}\,\theta_{0a})\,\Psi\\&\quad + (1/m)\,\mathrm{sh}\,\theta_{0a}\,(1 - i\gamma_5)\,(\gamma_0\,\partial\Psi/\partial x_a - \gamma_a\,\partial\Psi/\partial x_0);\\&\Psi \to (\mathrm{ch}\,\theta_1 + i\gamma_5\,\mathrm{sh}\,\theta_1)\,\Psi + (1/m)\,\mathrm{sh}\,\theta_1\,(1 - i\gamma_5)\\&\quad \times \gamma_\mu\,\partial\Psi/\partial x_\mu;\quad \Psi \to \exp(i\theta_0)\,\Psi,\end{aligned}\right\} \tag{137}$$

where θ_0, θ_1, and $\theta_{\mu\nu}$ are arbitrary real parameters. The transformations (137) are unitary in the indefinite metric

$$(\Psi_1, \Psi_2) = \int d^3x \Psi^+ \gamma_0 \Psi. \tag{138}$$

Thus, the Dirac equation is not only invariant with respect to the Poincaré group but also has an additional symmetry with respect to the transformations (137), which form a representation of the group $U(2)\otimes U(2)$. The main difference between the transformations (137) and the Lorentz transformations for the bispinor Ψ is that in the first case the transformed function depends not only on Ψ (and on the transformation parameters) but also on the derivatives $\partial\Psi/\partial x_\mu$.

The question arises of whether it is possible to combine the symmetry group of the Dirac equation given by the transformations (137) with the Poincaré group. This is in fact possible, since the operators (131) and (132) form an 18-dimensional Lie algebra.

Theorem 8. The Dirac equation (129) is invariant with respect to an 18-dimensional Lie algebra whose basis elements are given by Eqs. (131) and (132). These operators satisfy the commutation relations (45a), (133), and the commutation relations

$$\left.\begin{aligned}&[P_\mu, \Sigma_{\lambda\sigma}] = [P_\mu, \Sigma_\alpha] = [J_{\mu\nu}, \Sigma_\alpha] = 0;\quad [J_{\mu\nu}, \Sigma_{\lambda\sigma}]\\&\quad = i\,(g_{\mu\sigma}\Sigma_{\nu\lambda} + g_{\nu\lambda}\Sigma_{\mu\sigma} - g_{\mu\lambda}\Sigma_{\nu\sigma} - g_{\nu\sigma}\Sigma_{\mu\lambda}).\end{aligned}\right\} \tag{139}$$

Proof. The proof reduces to verifying the validity of the relations (139), which can be readily done by using the relations (130).

From what we have proved above, we conclude that the Dirac equation is invariant with respect to the 18-parameter transformation group of the form (11). This group is isomorphic to the group $P(1,3)\otimes U(2)\otimes U(2)$ and includes the inhomogeneous Lorentz transformations for the bispinor $\Psi(x)$, and also the transformations whose explicit form is given in (137).

Invariance algebra of the Dirac equation in the class of integro-differential operators. We now show that in the class of nonlocal (integro-differential) transformations the symmetry of the Dirac equation is even higher. Namely, we have the following theorem.[16,18,19]

Theorem 9. The Dirac equation (129) is invariant with respect to the algebra A_8 over the field of complex numbers. The basis elements of the invariance algebra belong to the class of integro-differential operators and are given by

$$\left.\begin{aligned}&\tilde{\Sigma}_{\mu\nu} = (i/2)\,[\gamma_\mu, \gamma_\nu] + (\gamma_\mu p_\nu - \gamma_\nu p_\mu)\,(1 - i\gamma_5 H/E)/2m;\\&\tilde{\Sigma}_0 = 1;\quad \tilde{\Sigma}_1 = H/E,\end{aligned}\right\} \tag{140}$$

where

$$H = \gamma_0\gamma_a p_a + \gamma_0 m;\quad E = \sqrt{H^2} = \sqrt{p^2 + m^2}. \tag{141}$$

Instead of the proof, we give the explicit form of the operator that diagonalizes Eq. (129) and simultaneously transforms the operators (140) to a representation in which they do not depend on p_a:

$$V = \exp\,[iS_{0a}p_a\,\mathrm{arth}\,(p/E)/p] \times \exp\,[\gamma_a p_a\,\mathrm{arctg}\,(p/m)/p]. \tag{142}$$

Using the operator (142), we obtain

$$\left.\begin{aligned}&\tilde{\Sigma}'_{ab} = V\tilde{\Sigma}_{ab}V^{-1} = (i/2)\,\gamma_a\gamma_b;\quad \tilde{\Sigma}'_1 = V\tilde{\Sigma}_1 V^{-1} = \gamma_0;\\&\tilde{\Sigma}'_{0a} = V\tilde{\Sigma}_{0a}V^{-1} = (i/2)\,\gamma_4\gamma_a;\\&\tilde{L}' = V\gamma_0 L V^{-1} = i\,(\partial/\partial t) - \gamma_0 E,\end{aligned}\right\} \tag{143}$$

where L is the Dirac operator (129). In the representation (143), the propositions of the theorem become obvious.

The operators (143), in contrast to (132), define a basis of the Lie algebra over the field of complex numbers. It therefore follows from Theorem 9 that Eq. (129) is invariant with respect to a 16-parameter Lie group including transformations of the form

$$\Psi' = \exp\,(i\tilde{\Sigma}_{\mu\nu}\tilde{\theta}_{\mu\nu})\,\Psi;\quad \Psi'' = \exp\,(i\tilde{\Sigma}_\alpha\tilde{\theta}_\alpha)\,\Psi, \tag{144}$$

where $\tilde{\theta}_{\mu\nu} = \theta^1_{\mu\nu} + i\theta^2_{\mu\nu}$, $\tilde{\theta}_\alpha = \theta^1_\alpha + i\theta^2_\alpha$, $\alpha = 1, 2$; $\theta^1_{\mu\nu}$, $\theta^2_{\mu\nu}$, θ^1_α, θ^2_α are real parameters. Substituting (140) in (144) and taking into account the fact that the square of any of the operators (140) is equal to the identity operator, we obtain the transformations (144) in the form

$$\left.\begin{aligned}&\Psi \to \Psi' = (\cos\tilde{\theta}_{ab} - \gamma_a\gamma_b \sin\tilde{\theta}_{ab})\,\Psi\\&\quad + \sum_\varepsilon (1 - i\varepsilon\gamma_5)\sin\tilde{\theta}_{ab}\,(\gamma_a\,\partial\Psi^\varepsilon/\partial x_b - \gamma_b\,\partial\Psi^\varepsilon/\partial x_a)/m;\\&\Psi \to \Psi'' = (\mathrm{ch}\,\tilde{\theta}_{0b} - \gamma_0\gamma_b\,\mathrm{sh}\,\tilde{\theta}_{0b})\,\Psi\\&\quad + \sum_\varepsilon (1 - i\varepsilon\gamma_5)\,\mathrm{sh}\,\tilde{\theta}_{0b}\,(\gamma_0\,\partial\Psi^\varepsilon/\partial x_b - \gamma_b\,\partial\Psi^\varepsilon/\partial x_0)/m;\\&\Psi \to \Psi''' = \mathrm{ch}\,\tilde{\theta}_1\Psi + \sum_\varepsilon \varepsilon\,\mathrm{sh}\,\tilde{\theta}_1\Psi^\varepsilon;\quad \Psi \to \Psi''' = \exp\,(i\tilde{\theta}_0)\,\Psi,\end{aligned}\right\} \tag{145}$$

where $\Psi^\varepsilon = (1/2)\,(1 - \varepsilon H/E)\,\Psi$, $\varepsilon = \pm 1$; $\tilde{\theta}_{\mu\nu}$ and $\tilde{\theta}_\alpha$ are complex parameters. The transformations (145) form a group that is locally isomorphic to the group $GL(2, C)\otimes GL(2, C)$.

The operators (140), in contrast to (132), do not form a closed algebra together with the generators (131) of the Poincaré group. However, on the solution set of Eq. (129) the generators (131) can also be represented in the form

$$\left.\begin{aligned}&P_0 = H;\quad P_a = p_a = -i\partial/\partial x_a;\\&J_{ab} = x_a p_b - x_b p_a + S_{ab};\quad J_{0a} = tp_a - [x_a, H]_+/2.\end{aligned}\right\} \tag{146}$$

The operators (140) and (146) satisfy the commutation relations (45a), (133), and (139) and, therefore, define a basis of the algebra $P(1,3)\oplus U(2)\oplus U(2)$.

Symmetry of the eight-component Dirac equation. We consider the eight-component Dirac equation

$$\tilde{L}\tilde{\Psi}(t, x) = 0;\quad \tilde{L} = \Gamma_\mu p^\mu - m, \tag{147}$$

where $\Gamma_\mu (\mu = 0, 1, 2, 3)$ are 8×8 matrices that together with $\Gamma_4, \Gamma_5, \Gamma_6$ satisfy the Clifford algebra.

Choosing Γ_μ and Ψ in the form

$$\Gamma_\mu = \begin{pmatrix} \gamma_\mu & 0 \\ 0 & \gamma_\mu \end{pmatrix}; \quad \tilde{\Psi} = \begin{pmatrix} \Psi \\ \Psi^c \end{pmatrix}; \quad \Psi^c = \gamma_2 \Psi^*, \tag{148}$$

where γ_μ are the four-row Dirac matrices, and Ψ is a four-component wave function, we obtain from (147) a system of equations identical to the Dirac equation (129) and the equation that is the adjoint of (129). In the general case, when Γ_μ are arbitrary 8×8 matrices satisfying the Clifford algebra Eq. (147) admits very different interpretations, including that of an equation of motion for particles with spin 1 (see Refs. 45, 47, and 48) and 3/2 (Ref. 60).

As a result of the increase in the number of components of the wave function, Eq. (147) has a higher symmetry than the Dirac equation (129). Apart from the almost obvious invariance with respect to the Poincaré group, whose generators are given by Eqs. (131), in which $S_{\mu\nu} = (i/4)[\Gamma_\mu, \Gamma_\nu]$, Eq. (147) has a hidden nongeometrical symmetry described by the following theorem.

Theorem 10. The eight-component Dirac equation (147) is invariant with respect to a 16-dimensional Lie algebra defined over the field of complex numbers. The basis elements of this algebra belong to the class of differential operators and are given by

$$\left.\begin{array}{l} \Sigma_{nl} = i[\Gamma_n, \Gamma_l]/2 + (1 + i\Gamma_6)(\Gamma_n p_l - \Gamma_l p_n)/m; \\ \Sigma_0 = 1, \quad l, n = 0, 1, \ldots, 5. \end{array}\right\} \tag{149}$$

The proof repeats almost literally the proof of Theorem 7. We emphasize that by definition

$$p_{3+a}\tilde{\Psi}(t, \mathbf{x}) \equiv -i\frac{\partial}{\partial x_{3+a}}\tilde{\Psi}(t, \mathbf{x}) \equiv 0,$$

and therefore those of the generators (149) for which $l, n > 3$ reduce on the set of functions $\Psi(t, \mathbf{x})$ to numerical matrices.

The generators Σ_{mn} and Σ_0 satisfy the commutation relations

$$[\Sigma_{mn}, \Sigma_{m'n'}] = 2i(g_{mn'}\Sigma_{nm'} + g_{nm'}\Sigma_{mn'} - g_{mm'}\Sigma_{nn'} - g_{nn'}\Sigma_{mm'}); \quad [\Sigma_0, \Sigma_{mn}] = 0. \tag{150}$$

The algebra (150) is isomorphic to the algebra $O(1, 5) \oplus T_1 \sim GL(4)$, where T_1 is the one-dimensional subalgebra realized by the unit matrix.

The operators (149) form a closed algebra together with the generators of the Poincaré group, since

$$\left.\begin{array}{l} [P_\mu, \Sigma_{mn}] = 0; \quad [J_{\mu\nu}, \Sigma_{m'n'}] = 0; \\ [J_{\mu\nu}, \Sigma_{m'\lambda}] = i(g_{\mu\gamma}\Sigma_{m'\nu} - g_{\nu\lambda}\Sigma_{m'\mu}); \\ [J_{\mu\nu}, \Sigma_{\lambda\rho}] = i(g_{\mu\rho}\Sigma_{\nu\lambda} + g_{\nu\lambda}\Sigma_{\mu\rho} - g_{\mu\lambda}\Sigma_{\nu\rho} - {}_{\nu\rho}\Sigma_{\mu\lambda}), \end{array}\right\} \tag{151}$$

where $m, n = 0, 1, \ldots, 6$; $3 < m', n' \leqslant 6$; $\mu, \nu, \rho, \lambda \geqslant 3$. It follows from this in particular that Eq. (147) is invariant with respect to a 26-dimensional group of transformations that includes the inhomogeneous Lorentz group and transformations of the form

$$\left.\begin{array}{l} \tilde{\Psi} \to \tilde{\Psi}' = (\cos\theta_{kl} - \Gamma_k\Gamma_l \sin\theta_{kl})\Psi + i(1 + i\Gamma_6) \\ \times(\Gamma_k \partial\Psi/\partial x_l - \Gamma_l \partial\Psi/\partial x_k)\sin\theta_{kl}/m, \quad k, \; l \neq 0; \\ \tilde{\Psi} \to \tilde{\Psi}'' = (\mathrm{ch}\,\theta_{0k} - \Gamma_0\Gamma_k\,\mathrm{sh}\,\theta_{0k})\Psi + i(1 + i\Gamma_6) \\ \times(\Gamma_0\,\partial\Psi/\partial x_k - \Gamma_k\,\partial\Psi/\partial x_0)\,\mathrm{sh}\,\theta_{0k}/m; \\ \tilde{\Psi} \to \tilde{\Psi}''' = \exp(i\theta)_0\Psi, \end{array}\right\} \tag{152}$$

where θ_{kl}, θ_0, θ_{0k} are arbitrary parameters. For k, $l > 3$ Eqs. (152) define matrix transformations of the function $\Psi(t, \mathbf{x})$.

In complete analogy with the results of the previous subsection, we can show that Eq. (147) is invariant with respect to a 42-dimensional Lie algebra isomorphic to $P(1, 3) \otimes GL(4, C) \otimes GL(4, C)$. The basis elements of this algebra belong to the class of integro-differential operators and are given by Eqs. (131), in which $S_{\mu\nu} = i[\Gamma_\mu, \Gamma_\nu]/4$, and by

$$\left.\begin{array}{l} \Sigma_{kl} = i[\Gamma_k, \Gamma_l]/2 + (\Gamma_k p_l - \Gamma_l p_k)(1 + i\Gamma_6 H/E)/m; \\ \Sigma_{5+k\,5+l} = \Sigma_{kl}\Sigma_1; \Sigma_0 = 1; \quad \Sigma = H/E, \end{array}\right\} \tag{153}$$

where $H = \Gamma_0\Gamma_a p_a + \Gamma_0 m$, $k, l = 0, 1, 2, 3, 4, 5$.

Symmetry of the Dirac equation for a massless particle. We now consider the Dirac equation in the special case when the parameter m, the mass of the particle, is equal to zero.

It is well known that the Dirac equation for $m = 0$ is invariant with respect to the 16-dimensional conformal algebra. Here, we show that the symmetry of this equation is higher. If we extend the class of operators to which the basis elements of the invariance algebra belong, including in it integro-differential operators, we can prove the following proposition.

Theorem 11.[31] The Dirac equation (129) for $m = 0$ is invariant with respect to a 23-dimensional Lie algebra isomorphic to the algebra $C(1, 3) \oplus A_8$. The basis elements of this invariance algebra belong to the class of integro-differential operators and are given by

$$\left.\begin{array}{l} P_0 = p_0 = i\partial/\partial t; \; P_a = p_a = -i\partial/\partial x_a; \; J_{ab} = x_a p_b - x_b p_a + S_{ab}; \\ J_{0a} = x_0 p_a - x_a p_0 + iH(1 - i\gamma_5)\gamma_a\gamma_b p_b/2p^2 + (1/2)\hat{\Sigma}_{0a}, \\ D = x_\mu p^\mu + i; \\ K_\mu = (-x_\nu x\nu + J_{ab}S_{ab}p^{-2} + p^{-2})p_\mu \\ + 2[x_\mu + (1 - \delta_{\mu 0})(1 - \gamma_0)S_{\mu b}\,p_b p^{-1}]D; \\ \hat{\Sigma}_{0a} = \gamma_4(p_a + \gamma_0 S_{ab}p_b)/p; \; \Sigma_0 = 1; \\ \hat{\Sigma}_1 = iH/p; \; \hat{\Sigma}_{ab} = i\Sigma_1\varepsilon_{abc}\Sigma_{0c}, \end{array}\right\} \tag{154}$$

where

$$H = \gamma_0\gamma_a p_a; \; p = \sqrt{H^2} = (p_1^2 + p_2^2 + p_3^2)^{1/2}.$$

We outline the proof. By means of the transformation

$$Q_A \to Q'_A = WQ_A W^{-1}; \quad \hat{L} \to \hat{L}' = WLW^{-1}, \quad \Psi \to \Psi' = W\Psi,$$

where Q_A is an arbitrary generator in the set defined by Eqs. (154), and

$$W = W^{-1} = [1 + \gamma_0 + (1 - \gamma_0)\,\varepsilon_{abc}S_{ab}\hat{p}_c]; \quad \hat{L} = \gamma_0 L = i\partial/\partial t - H; \quad \hat{p}_c = p_c/p, \tag{155}$$

we reduce Eq. (129) (for $m = 0$) and the operators (154) to the form

$$\left.\begin{array}{l} L'\Psi' = 0; \; L' = i\partial/\partial t - i\gamma_5 p; \; P'_\mu = P_\mu; \\ J'_{ab} = J_{ab}; \; J'_{0a} = x_0 p_a - x_a p_0 + S_{0a}; \end{array}\right\} \tag{156}$$

$$\left.\begin{array}{l} D' = D; \; K'_\mu = 2x_\mu D - x_\nu x^\nu p_\mu; \\ \hat{\Sigma}'_{\mu\nu} = 2S_{\mu\nu} = i[\gamma_\mu, \gamma_\nu]/2; \; \hat{\Sigma}'_0 = 1; \; \hat{\Sigma}'_1 = \gamma_5. \end{array}\right\} \tag{157}$$

The operators (157) satisfy the commutation relations (45a), (133), and (151) and the conditions of invariance of Eq. (156):

$$[L', P'_\mu]=[L', J'_{\mu\nu}]=[L', \Sigma'_{\mu\nu}]=[L', \Sigma'_a]=0;$$
$$[L', K'_0]=2i[x_0+(x_a p_a-i)i\gamma_5 p^{-1}]L';$$
$$[L', K'_a]=2i(x_a+ix_0\gamma_5 p_a/p)L';$$
$$[L', D']=iL';, [L', J'_{0a}]=\gamma_5\hat{p}_a L'. \quad (158)$$

From this we conclude that the operators (154) form an invariance algebra of Eq. (129) with $m=0$ that is isomorphic to $C(1,3)\oplus A_8$.

Thus, the Dirac equation for a massless particle is invariant with respect to a 23-dimensional Lie algebra including the subalgebras $C(1,3)$ and A_8. As we have shown above, Maxwell's equations (12) have the same symmetry. It can be shown (using, for example, the method proposed in Refs. 47 and 48) that all relativistic equations for massless fields invariant with respect to the transformations P, T, and C have this symmetry.

Symmetry of the Kemmer-Duffin-Petiau equation. We consider now the Kemmer-Duffin-Petiau equation

$$(\beta_\mu p^\mu - m)\Psi(t, \mathbf{x})=0, \quad (159)$$

where Ψ is a ten-component wave function, and β_μ are 10×10 matrices satisfying the algebra

$$\beta_\mu\beta_\nu\beta_\lambda+\beta_\lambda\beta_\nu\beta_\mu=i(g_{\mu\nu}\beta_\lambda+g_{\nu\lambda}\beta_\mu). \quad (160)$$

It is well known that Eq. (159) is invariant with respect to the Poincaré group. The generators of this group on the solution set of Eq. (159) have the form given by Eqs. (131), in which $S_{\mu\nu}=i(\beta_\mu\beta_\nu-\beta_\nu\beta_\mu)$. It can be shown that Eq. (159) also has a hidden (nongeometrical) symmetry, this being larger than in the case of the Dirac equation.

Theorem 12. Equation (159) is invariant with respect to an 18-dimensional Lie algebra defined over the field of real numbers. The basis elements of this algebra belong to the class of differential operators and are given by

$$A_b^a=C_{ad}C_{db};\ \tilde{A}_b^a=\varepsilon_{adc}C_{0b}C_{dc}/2,\ a\neq b;$$
$$A_a^a=2/3-C_{bc}^2;\ \tilde{A}_a^a=\varepsilon_{adc}C_{0a}C_{dc}/6-C_{0a}C_{bc};\ (a, b, c)$$
is a cyclic permutation of $(1, 2, 3)$;
$$A_0=1,\ \tilde{A}_0=\varepsilon_{abc}C_{0a}C_{bc}/4;\ a,\ b,\ c,\ d=1,\ 2,\ 3, \quad (161)$$

where

$$C_{\mu\nu}=S_{\mu\nu}+(a_\mu p_\nu-a_\nu p_\mu)/m;\ S_{\mu\nu}=i[\beta_\mu, \beta_\nu];\ a_\mu=S_{4\mu}+iS_{5\mu},\ S_{4\mu}=i\beta_\mu;$$
$$S_{5\mu}=i[\beta_5, \beta_\mu];\ \beta_5=\varepsilon_{\mu\nu\rho\sigma}\beta_\mu\beta_\nu\beta_\rho\beta_\sigma/4!. \quad (162)$$

Proof. Using the fact that the matrices β_μ and β_5 satisfy the Kemmer-Duffin-Petiau algebra (160), and the matrices $S_{kl}(k, l=0, 1, \ldots, 5)$ satisfy the algebra $O(1, 5)$, we obtain

$$[C_{\mu\nu}, L_1]=f_{\mu\nu}^1 L;\ f_{\mu\nu}^1=i(L+2m)(\beta_\mu p_\nu-\beta_\nu p_\mu)/m^2. \quad (163)$$

From (163) we conclude that the operators $C_{\mu\nu}$ [and, therefore, also A_a^b, $\tilde{A}_a^b$, A_0, $\tilde{A}_0$ (161)] satisfy the condition of invariance of Eq. (159).

The operators (161) satisfy the commutation relations

$$[A_0, A_b^a]=[A_0, \tilde{A}_b^a]=[\tilde{A}_0, A_b^a]=[\tilde{A}_0, \tilde{A}_b^a]=[A_0, \tilde{A}_0]=0;$$
$$[A_a^b, A_c^d]=-[\tilde{A}_a^b, \tilde{A}_c^d]=if_{abcd}^{kl}A_k^l;\ [A_a^b, \tilde{A}_c^d]=if_{abcd}^{kl}\tilde{A}_{kl}, \quad (164)$$

where $a, b, c, d, k, l=1, 2, 3$; f_{abcd}^{kl} are the structure constants of the group $SU(3)$ in the Okubo basis.

Equations (164) can be directly verified. This is most readily done by means of a preliminary transformation $\lambda_n\to V\lambda_n V^{-1}$, where $V=\exp[ia_\mu p^\mu/m]$ and $C_{\mu\nu}\to C'_{\mu\nu}=VC_{\mu\nu}V^{-1}=i[\beta_\mu, \beta_\nu]$. The theorem is proved.

Thus, besides the symmetry with respect to the algebra $P(1,3)$, the Kemmer-Duffin-Petiau equation (159) is also invariant with respect to the 18-dimensional Lie algebra whose basis elements are given by Eqs. (161). It follows that the Kemmer-Duffin-Petiau equation is invariant with respect to transformations of the form

$$\Psi\to\Psi'=\exp(iA_a^b\theta_a^b)\Psi;\ \Psi\to\Psi''=\exp(iA_0\theta_0)\Psi;$$
$$\Psi\to\Psi'''=\exp(i\tilde{A}_a^b\tilde{\theta}_a^b)\Psi;\ \Psi\to\Psi^{IV}=\exp(i\tilde{A}_0\tilde{\theta}_0)\Psi, \quad (165)$$

where θ_a^b, $\tilde{\theta}_a^b$, θ_0, $\tilde{\theta}_0$ are real parameters. In accordance with (164), the transformations (165) form an 18-parameter Lie group that is locally isomorphic to $U(3)\otimes U(3)$.

The operators (164) together with the generators of the Poincaré group form a closed 28-dimensional Lie algebra. This follows from the relations

$$[P_\mu, C_{\lambda\sigma}]=0;\ [J_{\mu\nu}, C_{\lambda\sigma}]=i(g_{\mu\sigma}C_{\nu\lambda}+g_{\nu\lambda}C_{\mu\sigma}-g_{\mu\lambda}C_{\nu\sigma}-g_{\nu\sigma}C_{\mu\lambda}). \quad (166)$$

Thus, the Kemmer-Duffin-Petiau equation is invariant with respect to a 28-dimensional Lie group that includes the inhomogeneous Lorentz transformations and the transformations (165). The transformations (165) can be readily found in explicit form, since the corresponding exponentials reduce to polynomials of $(A_a^b\theta_a^b)^n$, $(\tilde{A}_a^b\tilde{\theta}_a^b)^n$, $(A_0\theta_0)^n$, $(\tilde{A}_0\tilde{\theta}_0)^n$, $n=0, 1, 2$:

$$\exp(iA_a^b\theta_a^b)=1+(\cos\theta_a^b-1)(A_a^b)^2+i\sin(\theta_a^b)A_a^b;$$
$$\exp(i\tilde{A}_a^b\tilde{\theta}_a^b)=1+(\operatorname{ch}\tilde{\theta}_a^b-1)(\tilde{A}_a^b)^2+\operatorname{sh}(\tilde{\theta}_a^b)A_a^b;$$
$$\exp(iA_0\theta_0)=\exp(i\theta_0);\ \exp(i\tilde{A}_0\tilde{\theta}_0)=1+(\operatorname{ch}\tilde{\theta}_0-1)\tilde{A}_0^2+\operatorname{sh}(\tilde{\theta}_0)\tilde{A}_0. \quad (167)$$

It is easy to show that the general transformation of the function $\Psi(x)$, including the Lorentz transformations and (165), has the form

$$\Psi(x)\to\Psi'(x')=A\psi(x)+B\Psi(x')+C_\mu\frac{\partial\Psi(x)}{\partial x_\mu}+D_{\mu\nu}\frac{\partial^2\Psi(x)}{\partial x_\mu\partial x_\nu}+E_{\mu\nu\lambda}\frac{\partial^3\Psi(x)}{\partial x_\mu\partial x_\nu\partial x_\lambda}+F_{\mu\nu\lambda\sigma}\frac{\partial^4\Psi(x)}{\partial x_\mu\partial x_\nu\partial x_\lambda\partial x_\sigma}, \quad (168)$$

where A, B, C_μ, $D_{\mu\nu}$, $E_{\mu\nu\lambda}$, $F_{\mu\nu\lambda\delta}$ are numerical matrices.

Nongeometrical symmetry of the Dirac and Kemmer-Duffin-Petiau equations for particles interacting with an external field. Hitherto, in this section we have investigated the nongeometrical symmetry of the Dirac and Kemmer-Duffin-Petiau equations for noninteracting particles. It can be shown that the introduction of minimal coupling leads in the general case to a restriction of the nongeometrical symmetry of the equations of motion. However, for some classes of external fields the symmetry remains. In addition, the equations describing anomalous couplings of Pauli type have nongeometrical symmetry.

Here, we give without proof some of the results obtained in Refs. 25 and 27 relating to the symmetry of the equations of motion for interacting particles.

Theorem 13. The Dirac equation with Pauli-type interaction

$$L\Psi=0;\ L=\gamma_\mu\pi^\mu+(i/4m)\times(1-i\gamma_5)\gamma_\mu\gamma_\nu F_{\mu\nu}+m, \tag{169}$$

where $\pi_\mu=p_\mu-eA_\mu$, $F_{\mu\nu}=-i[\pi_\mu,\pi_\nu]$, and A_μ is the vector potential of the electromagnetic field, is invariant with respect to the algebra A_8. The explicit form of the basis elements of this algebra can be obtained from (132) by means of the substitution $p_\mu\to\pi_\mu$. Thus, Eq. (169) has the same nongeometrical symmetry as the Dirac equation (129) for a noninteracting particle.

Theorem 14. The Dirac equation for a particle in a homogeneous magnetic field.

$$\pi_0\Psi=\hat{H}\Psi;\quad \hat{H}=\gamma_0\gamma_a\pi_a+\gamma_0 m, \tag{170}$$

where

$$\pi_0=p_0;\quad \pi_3=p_3;\quad \pi_1=p_1-eA_1(x_1,\ x_2);$$
$$\pi_2=p_2-eA_2(x_1,\ x_2),$$

is invariant with respect to the algebra A_8. Its basis elements are given by the integro-differential operators

$$\left.\begin{array}{l}\Sigma_{12}=i\gamma_3\gamma_0\gamma_a\pi_a/|\gamma_0\gamma_a\pi_a|;\ \Sigma_{31}=i\gamma_5(\gamma_3 m+p_3)/(p_3^2+m^2)^{1/2};\\ \Sigma_{32}=i\Sigma_{12}\Sigma_{31};\ \Sigma_{0a}=i\varepsilon_{abc}\Sigma_{bc}H/2|H|;\ \Sigma_0=1;\ \Sigma_1=iH/|H|.\end{array}\right\} \tag{171}$$

Note that an analogous result is valid for the Dirac equation describing the motion of a particle in a constant electric field.

We now consider the generalized Kemmer-Duffin-Petiau equation describing the motion of a particle with spin 1, charge e, and anomalous moment q in the homogeneous magnetic field $\mathbf{H}=(0,0,H)$:

$$[\beta_\mu\pi^\mu+m+(eq/4m)S_{\mu\nu}F^{\mu\nu}]\Psi=0, \tag{172}$$

where

$$\pi_0=p_0;\ \pi_1=p_1-eHx_2;\ \pi_2=p_2;\ \pi_3=p_3;\ S_{\mu\nu}F^{\mu\nu}=2S_{12}H.$$

Theorem 15. Equation (172) has six linearly independent integrals of the motion associated with the nongeometrical symmetry. For $q=1$, Eq. (172) is invariant with respect to a 10-dimensional Lie algebra including the subalgebra $O(4)$.

Because they are cumbersome, we shall not give here the explicit form of the basis elements listed in the invariance-algebra theorem (see Refs. 25 and 27).

5. CONSERVATION LAWS

It is well known that the symmetry of Maxwell's equations with respect to the Poincaré group results in the existence of various integral combinations of the electric and magnetic field intensity vectors that are conserved in time. In this section, we consider the new quantities that arise in addition to the classical conservation laws (for the energy, momentum, and angular momentum of the electromagnetic field) as a consequence of the symmetry of Maxwell's equations with respect to the algebra A_8. We also consider the new integrals of the motion for the Dirac field associated with the nongeometrical symmetry of the Dirac equation.

Classical integrals of the motion of the electromagnetic field.
the electromagnetic field in vacuum that the following quantities are conserved in time:

$$\mathscr{E}=\int d^3x\,(\mathbf{E}^2+\mathbf{H}^2)/2; \tag{173a}$$

$$\mathscr{P}=\int d^3x\mathbf{E}\times\mathbf{H}; \tag{173b}$$

$$\mathscr{L}=\int d^3x\mathbf{x}\times(\mathbf{E}\times\mathbf{H}); \tag{173c}$$

$$\mathscr{N}=\int d^3x\,[t\mathbf{E}\times\mathbf{H}-\mathbf{x}(\mathbf{E}^2+\mathbf{H}^2)/2], \tag{173d}$$

and these determine the energy, momentum, angular momentum, and center of energy of the field.

The existence of the classical integrals of the motion listed in (173) follows from the symmetry of Maxwell's equations with respect to the Poincaré group. Proceeding from the Lagrangian formulation of these equations and using Noether's theorem, one can show that the conservation of the energy (173a) and the momentum (173b) follows directly from the invariance of the Lagrangian with respect to displacements with respect to the time and spatial coordinates, that the conservation of the angular momentum is a consequence of the invariance of the Lagrangian with respect to spatial rotations, and, finally, that the conservation of the center of energy follows from the invariance with respect to Lorentz transformations.

It is natural to ask what conserved quantities are associated with the symmetry of Maxwell's equations with respect to the nongeometrical transformations. To answer this question, it is impossible to use Noether's theorem, since the nongeometrical transformations (99) and (100) have a nonlocal nature. Therefore, it is necessary to use a different method to find the conserved quantities, and this consists of calculating the mean values of the basis elements of the invariance algebra in a corresponding scalar product. We shall show how in such an approach it is possible to obtain the classical integrals of the motion (173).

We proceed from the formulation (73) of Maxwell's equations in the momentum space. The generators of the Poincaré group on the solution set of Eqs. (73) are given by

$$\left.\begin{array}{l}P_0=-\sigma_2 S\cdot\mathbf{p}\equiv\hat{H};\ P_a=p_a;\\ J_a=-i\left(\mathbf{p}\times\frac{\partial}{\partial\mathbf{p}}\right)_a+S_a;\ J_{0a}=tp_a-i\hat{H}\frac{\partial}{\partial p_a},\end{array}\right\} \tag{174}$$

where p_a are independent variables, $-\infty<p_a<\infty$, and σ_2 and S_a are the matrices (15). The operators (174) are Hermitian with respect to the indefinite scalar product

$$(\varphi_1,\ \varphi_2)=\int d^3p\varphi_1^+(t,\ \mathbf{p})\,(\hat{H}/2\mathbf{p}^2)\,\varphi_2(t,\ \mathbf{p}), \tag{175}$$

where $\varphi_\alpha(t,\mathbf{p})$ are arbitrary square-integrable solutions of Eqs. (73).

From the invariance of Eqs. (73) with respect to the algebra (174) there follows the conservation in time of the mean values of the operators P_μ and $J_{\mu\nu}$ in the metric (175):

$$\frac{\partial}{\partial t}\langle Q_A\rangle=0,$$

where

$$\langle Q_A\rangle=(\varphi,\ Q_A\varphi)\equiv\int d^3p\varphi^+(t,\ \mathbf{p})\,(\hat{H}/2p^2)\,Q_A\varphi(t,\ \mathbf{p}), \qquad (176)$$

in which Q_A is any of the generators (174). Substituting in (176) the expressions (174) for the generators of the group $P(1,3)$ and using for the vector function $\varphi(t,\mathbf{p})$ the notation (98), we obtain after simple calculations

$$\langle P_0\rangle=\mathscr{E};\ \langle P_a\rangle=\mathscr{P}_a;\ \langle J_a\rangle=\mathscr{L}_a;\ \langle J_{0a}\rangle=\mathscr{N}_a, \qquad (177)$$

where $\mathscr{T}$, $\mathscr{P}_a$, $\mathscr{L}_a$, and $\mathscr{N}_a$ are integrals of the motion (173).

We give the proof of the first equation of (177). Substituting in (176) the expression (174) for the generator $Q_A=P_0=\hat{H}$ and bearing in mind that $\hat{H}^2\varphi=p^2\varphi$, we find

$$\langle P_0\rangle=(\varphi,\ H\varphi)\equiv\int d^3p\varphi^+(t,\ \mathbf{p})\,\varphi(t,\ \mathbf{p})/2. \qquad (178)$$

Substituting in (178) the expression (98) for the vector function $\varphi(t,\mathbf{p})$ and using (75), we obtain

$$\langle P_0\rangle=\int d^3p\,[\mathbf{E}(t,\ -\mathbf{p})\cdot\mathbf{E}(t,\ \mathbf{p})+\mathbf{H}(t,\ -\mathbf{p})\cdot\mathbf{H}(t,\ \mathbf{p})]/2. \qquad (179)$$

Expressing $\mathbf{E}(t,\mathbf{p})$ and $\mathbf{H}(t,\mathbf{p})$ by means of Fourier transformation in terms of $\mathbf{E}(t,\mathbf{x})$ and $\mathbf{H}(t,\mathbf{x})$ and integrating over p by means of the relation

$$\int d^3p\exp(i\mathbf{p}\cdot\mathbf{x})=(2\pi)^3\,\delta(\mathbf{x}).$$

we arrive at the expression

$$\langle P_0\rangle=\int d^3x\,[\mathbf{E}^2(t,\ \mathbf{x})-\mathbf{H}^2(t,\ \mathbf{x})]/2. \qquad (180)$$

All the remaining relations (173) and (177) can be proved similarly.

Integrals of the motion that follow from the nongeometrical symmetry of Maxwell's equations. The classical integrals of the motion for the electromagnetic field can be represented as the mean values (in the quantum-mechanical sense) of the basis elements of the algebra $P(1,3)$. One can calculate similarly the new integrals of the motion associated with the nongeometrical symmetry of Maxwell's equations considered in Sec. 3.

We shall find the conserved quantities corresponding to the basis elements of the algebra A_8. The operators (78) commute with L_1 from Eq. (73a), and the following integrals are conserved in time:

$$\langle Q_A\rangle=\int d^3p\varphi^+(t,\ \mathbf{p})\,MQ_A\varphi(t,\ \mathbf{p})/2p, \qquad (181)$$

where M is an arbitrary operator defined on the solution set of Eqs. (73), and Q_A are the operators (78).

We restrict ourselves to the choice $M=\hat{H}/p$. In this case, (181) determines the mean values of the operators (78) in the same metric in which the generators of the Poincaré group are given [see (173), (176), and (177)].

We shall calculate successively the mean values of all the operators (78). To obtain real quantities, we multiply Q_A (78) by i and denote $iQ_A=\tilde{Q}_A$. Choosing $A=3$, $\tilde{Q}_A=iQ_3=-\sigma_2$, we obtain

$$\begin{aligned}\langle\tilde{Q}_3\rangle&=-\int d^3p\varphi^+(t,\ \mathbf{p})\,(\hat{H}/2p^2)\,\sigma_2\varphi(t,\ \mathbf{p})\\&=-\int d^3p\varphi^+(t,\ \mathbf{p})\,\mathbf{S}\cdot\mathbf{p}\varphi(t,\ \mathbf{p})/2p^2.\end{aligned} \qquad (182)$$

Substituting in (182) the explicit form (15) of the matrices S_a and the expression (98) for the function $\varphi(t,\mathbf{p})$, we arrive at the formula

$$\langle\tilde{Q}_3\rangle=i\int d^3p\mathbf{p}\cdot[\mathbf{E}^*(t,\ \mathbf{p})\times\mathbf{E}(t,\ \mathbf{p})+\mathbf{H}^*(t,\ \mathbf{p})\times\mathbf{H}(t,\ \mathbf{p})]/2p^2. \qquad (183)$$

If we require that E and H satisfy the condition of reality, then in accordance with (183) and (75)

$$\langle\tilde{Q}_3\rangle=i\int d^3p\mathbf{p}\cdot[\mathbf{E}(t,\ -\mathbf{p})\times\mathbf{E}(t,\ \mathbf{p})-\mathbf{H}(t,\ -\mathbf{p})\times\mathbf{H}(t,\ \mathbf{p})]. \qquad (184)$$

Thus, from the invariance of Maxwell's equations with respect to the Heaviside-Larmor-Rainich transformations the conservation in time of the integral (184) follows.

We now find the mean value of the operator Q_2 (73):

$$\langle\tilde{Q}_2\rangle=\int d^3p\varphi^+(t,\ \mathbf{p})\,\sigma_3D\varphi(t,\ \mathbf{p})/2p. \qquad (185)$$

Using the explicit form (82)–(83) of the matrix D and also (73) and (98), we obtain

$$\varphi'(t,\ \mathbf{p})=D\varphi=D_1\varphi=(i/\delta)\begin{pmatrix}p_2p_3(p_3\dot{H}_3-p_2\dot{H}_2)\\p_1p_3(p_1\dot{H}_1-p_3\dot{H}_3)\\p_1p_2(p_2\dot{H}_2-p_1\dot{H}_1)\\-p_2p_3(p_3\dot{E}_3-p_2\dot{E}_2)\\-p_1p_3(p_1\dot{E}_1-p_3\dot{E}_3)\\-p_1p_2(p_2\dot{E}_2-p_1\dot{E}_1)\end{pmatrix}+(f/\delta)\begin{pmatrix}E_1\\E_2\\E_3\\H_1\\H_2\\H_3\end{pmatrix}; \qquad (186)$$

where δ and f are the functions (79) and (83), and $\dot{H}_a=\partial H_a/\partial t$, $\dot{E}_a=\partial E_a/\partial t$. Substituting in (185) the expression (98) for $\varphi^+(t,\mathbf{p})$, and also the explicit form (15) of the matrix σ_3 and (186) of the function $\varphi'(t,\mathbf{p})$, we obtain

$$\begin{aligned}\langle\tilde{Q}_2\rangle=&\int d^3p\,\Big\{f\big[\mathbf{H}^*(t,\ \mathbf{p})\cdot\mathbf{H}(t,\ \mathbf{p})-\mathbf{E}^*(t,\ \mathbf{p})\cdot\mathbf{E}(t,\ \mathbf{p})\big]\\&+\sum_a p_a^2\,[\dot{H}_a^*(t,\ \mathbf{p})\cdot\dot{H}_a(t,\ \mathbf{p})-\dot{E}_a^*(t,\ \mathbf{p})\cdot\dot{E}_a(t,\mathbf{p})]\Big]\Big\}\Big/2\delta p.\end{aligned} \qquad (187)$$

If we now impose on $\mathbf{E}(t,\mathbf{p})$ and $\mathbf{H}(t,\mathbf{p})$ the reality conditions (75), the integral (187) takes the form

$$\begin{aligned}\langle\tilde{Q}_2\rangle=&\int d^3p\,\Big\{f\big[\mathbf{E}(t,\ \mathbf{p})\cdot\mathbf{E}(t,\ -\mathbf{p})-\mathbf{H}(t,\ \mathbf{p})\cdot\mathbf{H}(t,\ -\mathbf{p})\big]\\&+\sum_a p_a^2\,[\dot{E}_a(t,\ \mathbf{p})\,\dot{E}_a(t,\ -\mathbf{p})-\dot{H}_a(t,\ \mathbf{p})\,\dot{H}_a(t,\ -\mathbf{p})\big]\Big\}\Big/2\delta p.\end{aligned} \qquad (188)$$

Similarly, we find that

$$\left.\begin{aligned}&\langle\tilde{Q}_1\rangle=\int d^3p\,\Big[f\mathbf{E}(t,\ -\mathbf{p})\cdot\mathbf{H}(t,\ \mathbf{p})\\&\quad+\sum_a p_a^2\dot{E}_a(t,-\mathbf{p})\cdot\dot{H}_a(t,\ \mathbf{p})\Big]\Big\}\Big/\delta p;\\&\langle\tilde{Q}_8\rangle=\int d^3p\,[\mathbf{E}(t,\ -\mathbf{p})\cdot\mathbf{E}(t,\ \mathbf{p})+\mathbf{H}(t,\ -\mathbf{p})\cdot\mathbf{H}(t,\ \mathbf{p})]/2p;\\&\langle\tilde{Q}_4\rangle=\langle\tilde{Q}_5\rangle=\langle\tilde{Q}_6\rangle=\langle\tilde{Q}_7\rangle=0.\end{aligned}\right\} \qquad (189)$$

Equations (189) are valid only for the vectors $\mathbf{E}(t,\mathbf{p})$ and $\mathbf{H}(t,\mathbf{p})$ that satisfy the condition (75). For complex E and H, the integrals $\langle Q_B\rangle (B=4, 5, 6, 7)$ are not equal to zero but define certain conserved quantities, whose explicit form is not given here.

Equations (184), (188), and (189) determine integral combinations of the Fourier transforms of the components of the electric and magnetic field vectors that are conserved in time by virtue of Maxwell's equations. By means of a Fourier transformation, these conserved quantities can be expressed in terms of $\mathbf{E}(t, \mathbf{x})$ and $\mathbf{H}(t, \mathbf{x})$. Thus, for $\langle \hat{Q}_1\rangle$ we obtain

$$\langle \widetilde{Q}_1\rangle = (2\pi)^{-3} \int d^3x\, d^3x'\, d^3p\, (\delta p)^{-1} \times \Big\{ f\mathbf{E}(t, \mathbf{x})\cdot \mathbf{H}(t, \mathbf{x}') - \sum_a p_a^2 \dot{E}_a(t, \mathbf{x})\, \dot{H}_a(t, \mathbf{x}') \exp[i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')]\Big\}. \quad (190)$$

In contrast to the classical integrals of the motion (173), the conserved quantity (190) depends not only on the electromagnetic field but also, roughly speaking, on an infinite number of derivatives $\partial E/\partial x_a$, $\partial H/\partial x_a$, $\partial^2 E/\partial x_a \partial x_b$, $\partial^2 H/\partial x_a \partial x_b$.

Proceeding from the invariance algebra of Maxwell's equations given by the relations (78), we can also construct integrals of the motion that depend on a finite number of derivatives. We use the nonuniqueness in the definition of the operator M in (181) and choose

$$M = \begin{cases} \hat{H}\delta, & A=1, 2; \\ \hat{H}p, & A=3; \\ \hat{H}, & A=8, \end{cases} \quad (191)$$

where $\hat{H}$ and δ are given by (174) and (79). Proceeding further by analogy with (181)–(190), we obtain the conserved quantities

$$\langle \widetilde{Q}_1\rangle' = \int d^3p \varphi^\dagger(t, \mathbf{p})\, \hat{H}\delta \hat{Q}_1 \varphi(t, \mathbf{p})/2p = \sum_{\substack{a,b,c\\ b, c\neq a}} \int d^3x \left(\frac{\partial^2 E_a}{\partial x_b^2}\frac{\partial^2 H_a}{\partial x_c^2} - \frac{\partial \dot{E}_a}{\partial x_a}\frac{\partial \dot{H}_a}{\partial x_a}\right); \quad (192a)$$

$$\langle \widetilde{Q}_2\rangle' = \int d^3p \varphi^\dagger(t, \mathbf{p})\, \hat{H}\delta\, \widetilde{Q}_2 \varphi(t, \mathbf{p})/2p = \sum_{\substack{a,b,c\\ b, c\neq a}} \int d^3x \left(\frac{\partial^2 E_a}{\partial x_b^2}\frac{\partial^2 E_a}{\partial x_c^2} - \frac{\partial^2 H_a}{\partial x_b^2}\frac{\partial^2 H_a}{\partial x_c^2} + \frac{\partial \dot{H}_a}{\partial x_a}\frac{\partial \dot{H}_a}{\partial x_a} - \frac{\partial \dot{E}_a}{\partial x_a}\frac{\partial \dot{E}_a}{\partial x_a}\right)\Big/ 2; \quad (192b)$$

$$\langle \widetilde{Q}_3\rangle' = \int d^3p \varphi^\dagger(t, \mathbf{p})\, \hat{H} p Q_3 \varphi(t, \mathbf{p})/2p = \int d^3x\, [\dot{\mathbf{E}}(t, \mathbf{x})\cdot \mathbf{H}(t, \mathbf{x}) - \mathbf{E}(t, \mathbf{x})\cdot \dot{\mathbf{H}}(t, \mathbf{x})]/2; \quad (192c)$$

$$\langle \widetilde{Q}_8\rangle' = \int d^3p \varphi^\dagger(t, \mathbf{p})\, \hat{H} Q_8 \varphi(t, \mathbf{p})/2p = \int d^3x\, (\mathbf{E}^2 + \mathbf{H}^2)/2. \quad (192d)$$

Thus, apart from the classical integrals of the motion (192), it follows from Maxwell's equations that the three further independent integral quantities given by (192a)–(192c) are conserved in time [with regard to (192d), this integral is identical to (173a)]. The question of the physical interpretation of the quantities (192a)–(192c) as yet remains open.

We note also that the conservation laws for the quantities (192a)–(192c) can be formulated by means of the continuity equation

$$p_\mu j_a^\mu = 0, \quad (193)$$

where

$$j_a^0 = \varphi^\dagger B_a \varphi; \quad j_a^b = \varphi^\dagger \sigma_2 S_b B_a \varphi;$$

$B_a (a=1, 2, 3)$ are differential operators commuting with $L_1 = i\partial/\partial t - \hat{H} = i\partial/\partial t + \sigma_2 \mathbf{S}\cdot\mathbf{p}$ (12a),

$$B_1 = -\widetilde{D}\sigma_1; \quad B_2 = -\widetilde{D}\sigma_3, \quad B_3 = \mathbf{S}\cdot\mathbf{p}, \quad (194)$$

where

$$\widetilde{D} = \sum_{a\neq b\neq c} [(p_a^2 p_b^2 + p_a^2 p_c^2 - p_b^2 p_c^2)(1 - S_a^2) + p_1 p_2 p_3 S_a S_b p_c].$$

Because the operators (194) commute with L_1 (12a), Eqs. (193) follow directly from (12).

Conservation laws for the Dirac field. We now find conserved quantities associated with the nongeometrical symmetry of the Dirac equation.

With any Hermitian operator Q defined on the solution set of the Dirac equation (129) we can associate the current 4-vector

$$j_\mu^\theta = \overline{\Psi}\gamma_\mu Q \Psi, \quad (195)$$

where $\overline{\psi} = \psi^\dagger \gamma_0$. It is easy to show that if the operator Q satisfies the invariance condition of the Dirac equation (129), then the continuity equation (193) holds for the current (195), so that by virtue of Gauss's theorem the following integral is conserved in time:

$$I^\theta = \int d^3x \overline{\Psi}\gamma_0 Q \Psi = \int d^3x\, \Psi^\dagger Q \Psi. \quad (196)$$

Choosing as $\{Q\}$ the generators of the Poincaré group, we obtain from here conservation laws for the energy, momentum, angular momentum, and center of energy of the electron-positron field.

It is now appropriate to ask what new conserved quantities can be associated with the basis elements of the nongeometrical invariance algebra of the Dirac equation given by (132). Since (136) is satisfied on the solution set of the Dirac equation, it is sufficient to consider only four of the eight operators (132), for example, Σ_{ab} and Σ_0 $(a, b = 1, 2, 3)$.

Since the operators Σ_{ab}(132) are non-Hermitian, we consider the conserved quantities corresponding to the invariant operators

$$\widetilde{\Sigma}_{ab} = M\Sigma_{ab}; \quad \widetilde{\Sigma}_0 = M\Sigma_0, \quad (197)$$

where

$$M = (H + 2\mathbf{S}\cdot\mathbf{p})/m; \quad H = \gamma_0\gamma_a p_a + \gamma_0 m; \quad S_a = i\varepsilon_{abc}\gamma_b\gamma_c/2.$$

The operator M is uniquely determined (up to a factor proportional to the unit matrix) by the requirements that the operators (197) be Hermitian and satisfy the condition of invariance of the Dirac equation.

Substituting (197) in (196), we obtain

$$\left.\begin{aligned} C_a &= (1/4)\varepsilon_{abc}\widetilde{\Sigma}_{bc} = \gamma_0 S_a + (1 - i\gamma_5)p_a/2m; \\ C_0 &= \widetilde{\Sigma}_0/2 = (\mathbf{S}\cdot\mathbf{p} + H/2)/m, \end{aligned}\right\} \quad (198)$$

and the current corresponding to (196):

$$\left.\begin{aligned} j_\mu^a &= \overline{\Psi}\gamma_\mu C_a \Psi = \overline{\Psi}\gamma_\mu\gamma_0 S_a \Psi - i\,[\overline{\Psi}\gamma_\mu(1 - i\gamma_5)\,\partial\Psi/\partial x_a \\ &\quad - (\partial\overline{\Psi}/\partial x_a)\,\gamma_\mu(1 - i\gamma_5)\,\Psi]/4m; \\ j_\mu^0 &= \overline{\Psi}\gamma_\mu C_0 \Psi = -i\,(\overline{\Psi}\gamma_\mu S_a \partial\Psi/\partial x_a \\ &\quad - (\partial\overline{\Psi}/\partial x_a)\,\gamma_\mu S_a \Psi + (\partial\overline{\Psi}/\partial t)\,\gamma_\mu\Psi/2 - \overline{\Psi}\gamma_\mu(\partial\Psi/\partial t)/2)/2m. \end{aligned}\right\} \quad (199)$$

By virtue of the Dirac equation (129) and the fact that

C_μ(198) commutes with the operator $i\partial/\partial t - H$, the currents j^ν_μ(199) satisfy the continuity equation (193), from which, in particular, there follows the conservation in time of the integral quantities

$$I^a = \int d^3x\, j_0^a = \int d^3x \overline{\Psi} S_a \Psi - i \int d^3x\, [\Psi^\dagger (1 - i\gamma_5)\, \partial\Psi/\partial x_a - (\partial\Psi^\dagger/\partial x_a)(1 - i\gamma_5)\, \Psi]/4m. \quad (200)$$

In contrast to the spin vector of the Dirac field, which is obtained using the Lagrangian formalism and Noether's theorem, the integral combinations (200) include derivatives of the bispinor and are conserved in time.

Note that the operators (198) can be represented in the form

$$C_\mu = \tilde{S}_\mu + p_\mu/2,$$

where the operators S_μ are equal to the covariant Fradkin-Good spin operators.[62]

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