New theory of space-time and gravitation

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Fiz. Elem. Chastits At. Yadra **13**, 757–934 (July–August 1982)

It is shown that the general theory of relativity is not satisfactory physical theory, since in it there are no laws of conservation for the matter and gravitational field taken together and it does not satisfy the principle of correspondence with Newton's theory. In the present paper, we construct a new theory of gravitation which possesses conservation laws, can describe all the existing gravitational experiments, satisfies the correspondence principle, and predicts a number of fundamental consequences.

PACS numbers: 04.20.Cv, 04.20.Me, 04.50. + h

INTRODUCTION

Einstein's general theory of relativity is one of the basic physical theories of modern times. It gives expression to the very deep idea of a connection between matter and space. This theory has explained and predicted a number of gravitational effects, and this has been a veritable triumph.

However, a number of problems in general relativity have still not found their solution. One of the most fundamental of these problems is that of the energy and momentum of the gravitational field in general relativity. Study of this problem from all sides[1-9] has led us to the conclusion that it is in principle insoluble, since the gravitational field in Einstein's theory is not a field in the spirit of Faraday and Maxwell, i.e., it is not characterized by an energy–momentum tensor density. This can be seen by comparing the physical characteristics of the gravitational field and other matter fields.

In all physical theories describing different forms of matter, one of the most important characteristics of the field is the energy–momentum tensor density, which is usually obtained by varying the Lagrangian density L of the field with respect to the components of the metric tensor g_{ni} of space–time[1)]:

$$T^{ni} = -2\delta L/\delta g_{ni} = \sqrt{-g}\,\mathrm{T}^{ni}, \tag{1}$$

where T^{ni} is the energy–momentum tensor of the field.

This characteristic reflects the existence of the field, and the nonvanishing of the energy–momentum tensor density in a region of space–time is a necessary and sufficient condition for the presence in this region of the physical field. Moreover, the energy and momentum of any physical field contribute to the total energy and momentum of the system and do not vanish identically outside the source of the field. This makes it possible to treat the transport of energy by waves in the spirit of Faraday and Maxwell, namely, one can study the distribution of the field intensity in space, determine the energy fluxes through a surface, calculate the change in the energy and momentum in processes of emission and absorption, and make other energy calculations. In the general theory of relativity, the gravitational field does not have the properties inherent in other physical fields, since it is devoid of such a characteristic.

In Einstein's theory the Lagrangian density consists of two parts: the Lagrangian density $L_g = L_g(g_{ni})$ of the gravitational field, which depends only on the metric tensor g_{ni}, and the Lagrangian density $L_M = L_M(g_{ni}, \varphi_A)$ of the matter, which depends on the metric tensor g_{ni} and on the remaining matter fields φ_A. Thus, in Einstein's theory the g_{ni} have a double meaning—they are field variables and also represent the metric tensor of space–time.

As a result of this physico-geometrical dualism, the expression for the total symmetric energy–momentum tensor density (the variation of the Lagrangian density with respect to the components of the metric tensor) is identical to the field equations (the variation of the Lagrangian density with respect to the components of the gravitational field). This has the consequence that the total symmetric energy–momentum tensor density of the system is strictly equal to zero:

$$T^{ni} + t^{ni} = 0, \tag{A}$$

where $T^{ni} = -2\delta L_M/\delta g_{ni}$ is the symmetric energy–momentum tensor density of the matter (by matter, we understand all matter fields except the gravitational field), and

$$t^{ni} = -2\delta L_g/\delta g_{ni} = -c^4\sqrt{-g}\,[R^{ni} - g^{ni}R/2]/8\pi G. \tag{2}$$

It also follows from the expression (2) that all components of the symmetric energy–momentum tensor density t^{ni} of the gravitational field vanish outside the matter.

Thus, it already follows from these results that the gravitational field in Einstein's theory does not have the properties inherent in other physical fields, since outside the source it lacks a fundamental physical characteristic—the energy–momentum tensor.

In Einstein's theory, the gravitational field is characterized by the curvature tensor R^i_{nlm}. We owe the clear

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1)Here and in what follows, Latin indices take the values 0, 1, 2, 3, and Greek indices the values 1, 2, 3. The signature of the metric is chosen in the form (+, −, −, −).

recognition of this fact to Synge (see Ref. 10, p. 8): "If we accept the idea that space–time is a Riemannian four-space (and if we are relativists we must), then surely our first task is to get the feel of it just as early navigators had to get the feel of a spherical ocean. And the first thing we have to get the feel of is the Riemann tensor, for it *is* the gravitational field. Yet, strangely enough, this most important element has been pushed into the background...." Further, Synge notes: "... in Einstein's theory, either there *is* a gravitational field or there is none, according as the Riemann tensor does not or does vanish. This is an absolute property; it has nothing to do with an observer's world-line...." Unfortunately, precisely this fundamental fact has not yet been recognized by some theoreticians concerned with problems of general relativity. The absence of such understanding leads to a lack of understanding of the very essence of general relativity.

However, this characteristic of the gravitational field (the curvature tensor) reflects rather the ability of the gravitational field to change the energy and momentum of the matter, i.e., it reflects the forces exerted by the gravitational field on the matter in accordance with the equation[11]

$$\delta^2 n^i/\delta s^2 + R^i_{mkl} u^m u^l n^k = 0, \tag{3}$$

where $u^i = dx^i/ds$ is the velocity 4-vector, and n^i is the infinitesimal vector of the geodesic deviation. But the description by means of curvature waves gives no information about the energy flux transported by the waves.

Thus, Einstein's general relativity links together matter and the gravitational field, the former being characterized, as in all physical theories, by the energy–momentum tensor, i.e., a tensor of second rank, while the latter is characterized by the curvature tensor, which is of fourth rank. It follows directly from this that in principle general relativity does not have conservation laws connecting the matter and the gravitational field. This fundamental fact, which was first established by us in Ref. 6, means that general relativity was created at the price of dispensing with conservation laws for the matter and the gravitational field considered together.

Lorentz and Levi-Civita suggested that the quantities (2) should be regarded as the components of the energy–momentum tensor density of the gravitational field, and the expression (A) as a somewhat unusual conservation law for the total energy–momentum tensor density. The conservation law (A) is unusual in that it is a local conservation law—from the change in the energy–momentum tensor of the matter at a particular point one can determine the change in the energy–momentum tensor of the gravitational field at the same point:

$$\frac{\partial}{\partial t} T^{0i} = -\frac{\partial}{\partial t} t^{0i}. \tag{4}$$

But in Einstein's theory, the tensor t^{ni} is only a characteristic of the geometry within the matter, so that in general relativity the change in the energy and momentum of the matter is directly related to only the change in the scalar curvature R and the second-rank tensor R^{ni} in the region occupied by the matter. Curvature waves, which are described by the fourth-rank tensor R^i_{nlm}, are not directly related in general relativity to the changes in the energy and momentum of the matter but only indirectly, through the metric tensor g_{ni}. Therefore, the conservation waves in general relativity are not associated with any conservation laws that link the change in the energy–momentum tensor of the matter (a tensor of second rank) to the change in the curvature tensor (a tensor of fourth rank). Thus, on the one hand, the Riemannian space–time in the general theory of relativity is an unusual source of energy and momentum, since the curvature tensor acts by virtue of Eq. (3) on the motion of particles, but, on the other, the energy of the Riemannian space–time is created without fulfillment of energy–momentum conservation laws for the matter and the gravitational field taken together.

The introduction of a conservation law on the basis of the expression (A) did not satisfy Einstein. He wrote (see Ref. 12, p. 645): "...of course, one cannot advance a *logical* objection against *such a designation*. However, I find that it is impossible to deduce from (A) the consequences that we are accustomed to draw from conservation laws. This is due to the circumstance that in accordance with (A) the components of the tensor of the *total energy* vanish everywhere" [our italics]. Einstein emphasizes further that in accordance with (A) a material system could dissolve completely and leave not a trace behind, since its energy (A) vanishes.

Einstein correctly notes that one cannot deduce from Eq. (A) the consequences that one is accustomed to draw from conservation laws, but we are concerned here not with a *designation* but with the essence of the general theory of relativity.

However, Einstein assumed that in general relativity the gravitational field with the matter must possess some conservation law (Ref. 12, p. 299): "...one must undoubtedly require the matter and the gravitational field together to satisfy the energy–momentum conservation laws."

He saw his task in the finding of conservation laws of the matter and the gravitational field analogous in their meaning to the conservation laws in classical mechanics or in the theory of the electromagnetic field. As is well known, this program led him to the introduction into a covariant theory of a noncovariant quantity—the energy–momentum pseudotensor; it was only at this price that he achieved a formal analogy with the conservation laws of classical mechanics and electrodynamics.

To obtain such "conservation laws" one usually[11] proceeds as follows.

If Einstein's equations are written in the form

$$-(c^4/8\pi G)\, g\, [R^{ik} - g^{ik}R/2] = -gT^{ik}, \tag{5}$$

then the left-hand side can be represented identically as the sum of two noncovariant quantities:

$$-(c^4/8\pi G)\, g\, [R^{ik} - g^{ik} R/2] = \frac{\partial}{\partial x^l} h^{ikl} + g\tau^{ik}, \tag{6}$$

where τ^{ik} is the energy–momentum pseudotensor of the gravitational field, and $h^{ikl} = -h^{ilk}$ is the spin pseudotensor.

Using the identity (6), Einstein's equations (5) can be written in a different, equivalent form:

$$-g\, [T^{ik} + \tau^{ik}] = \frac{\partial}{\partial x^l} h^{ikl}. \tag{7}$$

On the basis of the obvious equation

$$\frac{\partial^2}{\partial x^k \partial x^l} h^{ikl} = 0,$$

we deduce from Einstein's equations (7) the differential conservation law

$$\frac{\partial}{\partial x^k} [-g\, (T^{ik} + \tau^{ik})] = 0. \tag{8}$$

Integrating this relation over some volume and assuming that there are no matter fluxes through the surface bounding this volume, one usually obtains[12] from the expression (8) the integral "energy–momentum conservation laws" of the system:

$$\frac{d}{dt} \int (-g)\, [T^{0i} + \tau^{0i}]\, dV = -\oint (-g)\, \tau^{\alpha i}\, dS_\alpha. \tag{9}$$

Einstein (Ref. 12, p. 645) assumed that the right-hand side of this relation for $i = 0$ "certainly represents the loss of energy by the material system." In the absence of energy–momentum fluxes of the gravitational field through the surface bounding the volume of integration, we obtain from the expression (9) an energy–momentum conservation law for the system:

$$P^i = \frac{1}{c} \int (-g)\, [T^{0i} + \tau^{0i}]\, dV = \text{const}. \tag{10}$$

By means of Einstein's equations (7) the relation (10) can be rewritten in the form

$$P^i = \frac{1}{c} \oint h^{i0\alpha}\, dS_\alpha = \text{const}. \tag{11}$$

In Einstein's opinion (Ref. 12, p. 652) the four quantities P^i represent the energy ($i = 0$) and momentum ($i = 1, 2, 3$) of the physical system. It is usually asserted [Ref. 11, p. 362 (p. 283 in the English translation)] that: "The quantities P^i (the 4-momentum of field plus matter) have a completely definite meaning and are independent of the choice of the reference system to just the extent that is necessary on the basis of physical considerations." However, as we shall show below, this assertion is incorrect.

Similar results are obtained when Einstein's equations are written in mixed components:

$$\sqrt{-g}\, [T^i_k + \tau^i_k] = \partial_n \sigma^{ni}_k. \tag{12}$$

The choice of the energy–momentum pseudotensors of the gravitational field depended to a large degree on the inclinations of the authors and, as a rule, was made on the basis of secondary properties. For example, choosing h^{ikl} in the form

$$h^{ikl} = \frac{c^4}{16\pi G} \frac{\partial}{\partial x^m} [-g\, (g^{ik} g^{ml} - g^{il} g^{km})], \tag{13}$$

we obtain the symmetric Landau–Lifshitz pseudotensor, which contains only first derivatives of the metric tensor.

Choosing

$$\sigma^{ni}_k = \frac{g_{km} c^4}{16\pi G \sqrt{-g}} \frac{\partial}{\partial x^l} [-g\, (g^{im} g^{nl} - g^{nm} g^{il})], \tag{14}$$

we arrive at Einstein's pseudotensor, which is identical to the canonical energy–momentum (pseudo)tensor obtained from the noncovariant Lagrangian density

$$L_g = \sqrt{-g}\, [\Gamma^n_{mk} \Gamma^l_{nl} - \Gamma^n_{lm} \Gamma^l_{nk}]\, g^{mk}$$

of the gravitational field.

For

$$\sigma^{ni}_k = \frac{c^4 \sqrt{-g}}{16\pi G} g^{im} g^{nl}\, [\partial_l g_{km} - \partial_m g_{kl}] \tag{15}$$

we have the Lorentz pseudotensor, which is identical to the canonical energy–momentum (pseudo)tensor obtained on the basis of the noncovariant method of infinitesimal displacements from the covariant Lagrangian density $L_g = \sqrt{-g}R$ of the gravitational field. It is important to emphasize that for all the various properties of the energy–momentum pseudotensors they all possess the same property—any energy–momentum pseudotensor can vanish at any point of space.

This fact is usually seen as being a reflection of the equivalence principle. However, the assertions about the equivalence of a gravitational field and a field of inertial forces are incorrect. These two fields differ significantly, since the curvature tensor is always zero in the presence of a field of inertial forces, but in the case of a gravitational field it is nonzero. Therefore, a field of inertial forces and a gravitational field are not equivalent for all physical processes for which the curvature tensor plays a significant part. Therefore, the equivalence principle does not have a direct bearing on the general theory of relativity, though it did play a heuristic role in its construction by Einstein.

The vanishing of any energy–momentum pseudotensor at any given point of space is a consequence of the nontensorial law of transformation of their components on the transition from one coordinate system to another. Thus, all energy–momentum pseudotensors containing derivatives of the metric tensor of a Riemannian space-time of not higher than the first order vanish on the transition to a locally-geodesic (Galilean) coordinate system, since all the components of the connection Γ^i_{nm} in this system vanish. Thus, the energy and momentum of the gravitational field defined by means of energy–momentum pseudotensors can be made to vanish locally.

In contrast, the gravitational field, which is described by the curvature tensor, cannot be made to vanish by the transition to any admissible coordinate system,[2)] and therefore, because of the influence of

[2)] We define admissible transformations as coordinate transformations between frames of reference that can be realized by real physical bodies and processes. Mathematically, this condition is equivalent[13] to the requirement that in these frames of reference the quadratic form with coefficients $g_{\alpha\beta}$ should be negative definite, and the component g_{00} of the metric tensor positive:

$$g_{00} > 0, \quad g_{\alpha\beta} dx^\alpha dx^\beta < 0.$$

curvature waves on physical processes, one cannot say that a gravitational field is not present in any coordinate system. Therefore, the energy–momentum pseudotensors are not, as we have already pointed out,[1-9] energy–momentum characteristics of the gravitational field and do not reflect its existence either locally or globally. As a result, definitions of the energy and momentum of a system and of energy fluxes in the general theory of relativity based on the use of such tensors are physically meaningless.

This general conclusion was illustrated in Ref. 8 by the example of the definition in the general theory of relativity of the "inertial mass" and "energy" of a static, spherically symmetric system. On the basis of the definition (10) of the energy and momentum of a system consisting of matter and the gravitational field, one introduces in general relativity the concept of the inertial mass m of the system:

$$m=\frac{1}{c}P^0=\frac{1}{c^2}\int(-g)\,[T^{00}+\tau^{00}]\,dV=\frac{1}{c^2}\oint h^{00\alpha}\,dS_\alpha. \tag{16}$$

To calculate the inertial mass of the system, one generally uses the Schwarzschild solution. In isotropic Cartesian coordinates, the metric of the Riemannian space–time can be written in this case in the form

$$\begin{aligned} g_{\alpha\beta} &= -\delta_{\alpha\beta}\,[1+r_g/4r]^4;\\ g_{00} &= [1-r_g/4r]^2\,[1+r_g/4r]^{-2}, \end{aligned} \tag{17}$$

where $r=\sqrt{x^2+y^2+z^2}$, $r_g=2GM/c^2$, and M is the gravitational mass. These coordinates are asymptotically Galilean, since in the limit $r\to\infty$

$$g_{00}=1+O(1/r);\quad g_{\alpha\beta}=-\delta_{\alpha\beta}\,[1+O(1/r)]. \tag{18}$$

Using the covariant components (17) of the metric, we obtain from the expression (11)

$$P^0=c^3r_g/2G=Mc. \tag{19}$$

This equality of the inertial mass and the gravitational mass provided the basis for the assertions that they are equal in the general theory of relativity [see Ref. 11, p. 424 (p. 334 in the English translation)]: "...$P^\alpha=0$, $P^0=Mc$, a result which was naturally to be expected. It is an expression of the equality of gravitational and inertial mass (gravitational mass is the mass that determines the gravitational field produced by the body, the same mass that appears in the metric tensor in a gravitational field, or, in particular, in Newton's law; inertial mass is the mass that determines the ratio of energy and momentum of the body; in particular, the rest energy of the body is equal to this mass multiplied by c^2)."

However, this assertion of Einstein (Ref. 12, p. 660) and other authors is incorrect. As is shown in Ref. 8, the energy (10) of the system and, therefore, its inertial mass (16) do not have any physical meaning, since their value is even dependent on the choice of the three-dimensional coordinate system.

It is obvious that an elementary requirement which the definition of inertial mass must satisfy in any physical theory is that the value of the inertial mass should be independent of the choice of the three-dimensional coordinate system. However, in the general theory of relativity the definition (16) of the inertial mass does not satisfy this requirement. Indeed, let us make, for example, a transition from the three-dimensional Cartesian coordinates x_c^α to the new coordinates x_H^α, which are related to the old coordinates by

$$x_c^\alpha=x_H^\alpha\,[1+f(r_H)], \tag{20}$$

where $r_H=\sqrt{x_H^2+y_H^2+z_H^2}$; $f(r_H)$ is an arbitrary nonsingular function satisfying the conditions $f(r_H)\geqslant 0$ and

$$\lim_{r_H\to\infty} f(r_H)=0;\quad \lim_{r_H\to\infty} r_H\frac{\partial}{\partial r_H}f(r_H)=0. \tag{21}$$

[*Translation editor's note.* The subscripts "c" and "H" are derived from the Russian words for "old" and "new" and are retained throughout the article to simplify the composition.]

It is easily seen that the transformation (20) corresponds to a change in the numbers associated with the points of the three-dimensional space along the radius:

$$r_c=r_H\,[1+f(r_H)].$$

If the transformation (20) is to have an inverse and is to be one-to-one, it is necessary and sufficient for the condition

$$\partial r_c/\partial r_H=1+f+r_Hf'>0,$$

where

$$f'=\frac{\partial}{\partial r_H}f(r_H),$$

to be satisfied. Then the Jacobian of the transformation is also nonvanishing:

$$J=\det\left\|\frac{\partial x_c}{\partial x_H}\right\|=(1+f)^2\frac{\partial r_c}{\partial r_H}\neq 0.$$

In particular, all the requirements are satisfied by the function

$$f(r_H)=\alpha^2\sqrt{8GM/(c^2r_H)}\,[1-\exp(-\varepsilon^2r_H)], \tag{22}$$

where α and ε are arbitrary nonvanishing numbers.

Since in the given case

$$\frac{\partial r_c}{\partial r_H}=1+\alpha^2\sqrt{\frac{8GM}{c^2r_H}}\left[\frac{1}{2}+\left(\varepsilon^2r_H-\frac{1}{2}\right)\exp(-\varepsilon^2r_H)\right]>1,$$

it follows that $f(r_H)$ is a monotonic function of r_H. It is readily seen that $f(r_H)$ is a non-negative, nonsingular function in the whole of space. The Jacobian of the transformation is in this case strictly greater than unity:

$$J=(1+f)^2\,\partial r_c/\partial r_H>1.$$

Therefore, the transformation (20) with the function $f(r_H)$ defined by the expression (22) has an inverse transformation and is one-to-one.

We now calculate the inertial mass (16) in the new coordinates x_H^α. Using the transformation law of the metric tensor,

$$g_{ik}^H(x_H)=\frac{\partial x_c^l}{\partial x_H^i}\frac{\partial x_c^m}{\partial x_H^k}\,g_{lm}^c(x_c(x_H)), \tag{23}$$

we find the components of the Schwarzschild metric in the new coordinates. As a result, we obtain

$$\left.\begin{aligned} g_{00} &= \left[1-\frac{r_g}{4r_H(1+f)}\right]^2\left[1+\frac{r_g}{4r_H(1+f)}\right]^{-2}; \\ g_{\alpha\beta} &= \left[1+\frac{r_g}{4r_H(1+f)}\right]^4\Big\{-\delta_{\alpha\beta}(1+f)^2 \\ &\quad -x_\alpha^H x_\beta^H\left[(f')^2+\frac{2}{r_H}f'(1+f)\right]\Big\}. \end{aligned}\right\} \tag{24}$$

The determinant of the metric tensor (24) has the form

$$g = -g_{00}\left[1+\frac{r_g}{4r_H(1+f)}\right]^{12}(1+f)^4 \times [(1+f)^2+r_H^2(f')^2+2r_H f'(1+f)]. \tag{25}$$

It should be noted especially that the metric (24) is asymptotically Galilean:

$$\lim_{r_H\to\infty} g_{00}=1, \quad \lim_{r_H\to\infty} g_{\alpha\beta}=-\delta_{\alpha\beta}.$$

In the special case when the function $f(r_H)$ is given by the relation (22) and $r_H \to \infty$, the metric of the Riemannian space–time will have the asymptotic behavior

$$g_{00}=1+O\left(\frac{1}{r_H}\right); \quad g_{\alpha\beta}=-\delta_{\alpha\beta}+O\left(\frac{1}{\sqrt{r_H}}\right). \tag{26}$$

It can be seen from the expressions (18) and (26) that the rate at which the three-dimensional part of the metric tensor of the Riemannian space–time tends to the Galilean value even depends on the choice of the spatial coordinates, i.e., it depends on the manner in which numbers are associated with the points of space, and it is not determined by any physical conditions. Therefore, the requirements imposed on the asymptotic behavior of the three-dimensional part of the metric tensor are not physical but merely prescribe a particular way of associating numbers with the points of space. However, the theory must always ensure the possibility of choosing any admissible coordinate system. Therefore, any restriction of associating numbers with the points of space is a meaningless requirement.

Substituting the contravariant component of the metric (24) in the expression (16), we obtain

$$m=\frac{c^2}{2G}\lim_{r_H\to\infty}\{r_g+r_H^3(f')^2\}. \tag{27}$$

Thus, the inertial mass depends essentially on the rate at which f' tends to zero as $r_H \to \infty$. In particular, choosing the function $f(r_H)$ in the form (22), we obtain for the inertial mass from the expression (27)

$$m = M(1+\alpha^4). \tag{28}$$

It follows from this that for the inertial mass (16) of the system consisting of matter and the gravitational field it is possible to obtain in the general theory of relativity, by virtue of the arbitrariness of α, any preassigned number $m \geqslant M$, depending on the choice of the spatial coordinates, although the gravitational mass M of the system and, therefore, all the three general relativistic effects remain unchanged. We note also that for more complicated transformations of the spatial coordinates that leave the metric asymptotically Galilean the inertial mass (16) of the system can take any preassigned value, either positive or negative.

Thus, we see that in general relativity the inertial mass, which was first introduced by Einstein and was then taken over by many authors,[11,14-18] depends on the choice of the three-dimensional coordinate system and therefore has no physical meaning. Therefore, assertions to the effect that the inertial and gravitational masses are equal in Einstein's theory are also devoid of any physical meaning. There is equality only in a narrow class of three-dimensional coordinate systems, and since the inertial and gravitational masses have different transformation laws, their equality ceases to hold on the transition to different three-dimensional coordinate systems. In addition, the definition (16) of the inertial mass in general relativity does not satisfy the principle of correspondence with Newton's theory. Indeed, since the inertial mass m in Einstein's theory depends on the choice of the three-dimensional coordinate system, its expression in the general case of an arbitrary three-dimensional coordinate system does not go over into the corresponding expression of Newton's theory, in which the inertial mass does not depend on the choice of the spatial coordinates. Thus, in the general theory of relativity there is no classical Newtonian limit, and, therefore, it also does not satisfy the correspondence principle.

The situation is the same with regard to the energy fluxes of gravitational radiation. In general relativity, the "intensity of gravitational radiation" is defined by means of the components $\tau^{0\alpha}$ of the energy–momentum pseudotensor:

$$\frac{dI}{d\Omega} = -\lim_{r\to\infty} c(-g)r^2\tau^{0\alpha}n_\alpha. \tag{29}$$

Because of the essential nonlinearity of Einstein's equations, it is customary to make a restriction to the first approximation in a small wave perturbation when one is investigating wave solutions. In this case, making calculations in asymptotically Cartesian coordinates, we obtain the following expression for the gravitational radiation intensity:

$$\frac{dI}{d\Omega}=\frac{G}{36\pi c^5}\left\{\frac{1}{4}(\dddot{D}_{\alpha\beta}n^\alpha n^\beta)^2+\frac{1}{2}\dddot{D}_{\alpha\beta}\dddot{D}^{\alpha\beta}+\dddot{D}_{\alpha\beta}\dddot{D}^{\alpha\gamma}n^\beta n_\gamma\right\}, \tag{30}$$

where

$$D^{\alpha\beta}=\int dV\,(3x^\alpha x^\beta-\gamma^{\alpha\beta}x_\varepsilon x^\varepsilon)\,T^{00},$$

and the dots denote derivatives with respect to the time.

Integration of the expression (30) over a sphere gives Einstein's well-known quadrupole formula, which usually serves as a proof of the "reality" of the existence of an energy flux of gravitational waves from a radiating island system:

$$I=-dE/dt=(G/45c^5)\,\dddot{D}_{\alpha\beta}\dddot{D}^{\alpha\beta}. \tag{31}$$

Thus, the expressions (30) and (31) appear to confirm the conclusion drawn by Einstein (Ref. 12, p. 642): "...the radiation intensity cannot become negative in any direction and, *a fortiori*, the total radiation intensity cannot become negative...."

However, this conclusion is incorrect. In general relativity, as is shown in Ref. 9, the gravitational radiation intensity (30), and also the total intensity, both as defined by Einstein, depend essentially on the choice of the coordinates. Therefore, by an appropriate admissible coordinate transformation that leaves the metric of the Riemannian space–time asymptotically Galilean at infinity, these quantities can be made to vanish or to become negative in the region of space enclosed between two spheres with radii $r_1 = ct - u_1$ and $r_2 = ct - u_2$. It follows from this that the gravitational radiation intensity (30) and the total intensity (31) are not energy–momentum characteristics of the gravitational field in general relativity, since radiation, as objective physical reality, cannot be annihilated by any admissible coordinate transformation. For example, in electrodynamics, it is easy to show that the energy flux of electromagnetic radiation cannot be made to vanish by any admissible coordinate transformation: If the energy flux of electromagnetic waves through some surface is nonvanishing in one frame of reference, then after transition to any other admissible frame of reference it cannot be made to vanish or, *a fortiori*, change sign.

Thus, the formula (31) for calculating the energy loss of a source due to gravitational radiation is not in principle contained in general relativity, since Einstein's theory has no possibility for energy calculations.

The vanishing of the energy flux of gravitational radiation determined using energy–momentum pseudotensors in the lowest order of perturbation theory when the coordinates are chosen appropriately is a reflection of the general assertion to the effect that for any energy–momentum pseudotensor it is possible to choose a coordinate system in which the energy flux of gravitational radiation is always strictly equal to zero. A coordinate system with such properties can be found for any energy–momentum pseudotensor by reducing the components $g^{0\alpha}$ of the metric tensor to a form that ensures fulfillment of the condition $\partial_n \sigma_0^{n\alpha} \equiv 0$ for the energy–momentum pseudotensors with mixed components. Then in these coordinate systems it follows from the expressions (12) and (9) that there will be no energy flux of the gravitational waves:

$$\frac{d}{dt} \int dV (-g)^{1/2} [T_0^0 + \tau_0^0] \equiv 0.$$

Corresponding coordinate systems can be found similarly for another type of energy–momentum pseudotensor.

However, in the given coordinate systems there still remain a multitude of solutions of Einstein's equations for which the curvature tensor is nonzero, so that in these coordinate systems there exist curvature waves capable of transmitting energy and momentum to physical bodies. This assertion is most readily illustrated using the example of the Lorentz pseudotensor. In this case, going over to a synchronous frame ($g_{00} = 1, g_{0\alpha} = 0$), we can, as is readily seen from the expressions (15) and (12), make the total energy–momentum density of the gravitational field vanish: $\sqrt{-g}[T_0^i + \tau_0^i] \equiv 0$. It follows from this that outside matter energy and momentum of the gravitational field must also be absent. But curvature waves, which are solutions of Einstein's equations, exist in a synchronous frame of reference and, influencing physical processes, change their energy and momentum.

Recently, it has been asserted[19-21] that by using the Hamiltonian formalism in general relativity one can ostensively obtain an expression for the mass of a system consisting of matter and the gravitational field and prove its positive definiteness. On the basis of these assertions it was concluded prematurely in Ref. 126 that the problem of the energy and momentum of the gravitational field in Einstein's theory has been solved. However, such assertions merely tell us that the authors of Refs. 19–21 and 126 do not understand the essence of the problem. Indeed, it is readily seen that all the investigations of Refs. 19–21 are based essentially on the requirement of a definite law of asymptotic behavior of the metric tensor of Riemannian space-time at spatial infinity[126]:

$$g_{ik} = \delta_{ik} + O(1/r); \quad \partial_n g_{ik} = O(1/r^2). \tag{32}$$

It is this requirement that makes it possible to obtain an expression for the mass of a system in general relativity and prove its positive definiteness. However, this requirement is not a physical requirement; this can be seen fairly easily by considering the example of the Schwarzschild solution, which in the isotropic Cartesian coordinates (17) has the asymptotic behavior (18), which satisfies all the conditions (32).

If we now arithmetize (associate numbers with) the points of three-dimensional space in a different way (the arithmetization of three-dimensional space is always arbitrary, and all theories must admit an arbitrariness in the choice of the arithmetization), then, as is readily seen, we obtain in the general case a different law of asymptotic behavior of the spatial part of the metric tensor of a Riemannian space-time.

In particular, after the transformation (20) the components of the metric tensor, as we have seen, have the asymptotic behavior

$$g_{00} = 1 + O(1/r); \qquad g_{\alpha\beta} = -\delta_{\alpha\beta} + O(1/\sqrt{r}).$$

It follows from this that the asymptotic behavior of the three-dimensional part of the metric tensor of the Riemannian space-time is determined by the method of arithmetization of the points of space and is not dictated by any physical requirements. But a change in the asymptotic behavior of the three-dimensional part of the metric tensor when the arithmetization of the points of space is changed entirely vitiates the great efforts made in Refs. 19–21 in the proof of mathematical propositions. Mathematical proofs are, of course, very important elements of theoretical physics. But these proofs have meaning only when a problem is given a correct physical formulation. Otherwise, these proofs, however elegant, are of no value for physics. As Academician A. N. Krylov liked to say[125]: "Mathematics, like a millstone, grinds whatever is put into the mill, and just as you cannot obtain wheat flour by grinding goosefoot you cannot obtain truth from false premises by covering pages with formulas." All this applies in full measure to the cycle of papers we are discussing. On the one hand, the papers of Refs. 19–21 and 126 are false in the actual physical formulation of the problem. On the other hand, the expression for the mass of a system used in these papers (see, for example, Ref.

126),

$$m=\oint \partial_i q^{ik}\, dS_k=\oint \frac{\partial}{\partial x^m}\,[-g\,(g^{00}g^{im}-g^{0i}g^{0m})]\, dS_i,$$

also depends explicitly on the method of arithmetization of the points of three-dimensional space, or, in other words, it is not a scalar with respect to the choice of of the three-dimensional coordinate system, which is physically meaningless, since by an appropriate choice of the three-dimensional coordinate system the mass of the system can be made equal to any preassigned number.

It should also be noted that the approach based on the Hamiltonian technique is conceptually close to the pseudotensor formalism, which can be seen particularly easily from the formula given above for the mass of the system, and it is merely embellished by a number of mathematical artifices. We have pointed out above the error in Refs. 19–21 and 126. To this we could also add that the authors of Refs. 19–21 and 126 have not understood a fundamental fact, namely, in general relativity there are in principle no energy–momentum conservation laws of the matter and gravitational field taken together, so that it is impossible to introduce in the theory the concepts of energy and momentum of a system. Below, we shall return once more to this question and explain why general relativity in certain coordinate systems gives physically acceptable formulas that are nevertheless not contained in the theory.

Another approach to the energy–momentum problem in general relativity, which has been used primarily in approximate calculations, is based ostensibly on the derivation of integrals of the motion from the equations of motion of the matter constructed on the basis of the covariant conservation equation of the energy–momentum tensor of the matter. In such an approach, the nonconservation of the energy of the matter that is apparently obtained at a certain stage of the approximate calculations is usually explained by the emission of gravitational waves by the matter, which makes it possible to determine their energy and also the force of gravitational radiational damping.

This approach has led to contradictory results. For example, in Refs. 22–24 it was concluded that the energy of gravitational waves has a negative sign, since the energy of a system is increased after it has radiated gravitational waves. At the same time, the results of the analogous papers of Refs. 25–27 indicate a decrease in the energy of a system radiating gravitational waves, so that they must transmit positive energy.

However, strictly speaking, from the covariant conservation equation for the energy–momentum tensor of the matter one can obtain only a trivial conservation law of the form (4). Indeed, by virtue of the conservation law

$$\nabla_n T^{ni}=\partial_n T^{ni}+\Gamma^i_{nm}T^{nm}=0 \tag{33}$$

for the energy–momentum tensor of the matter, we obtain

$$\nabla_n\,[(-g)^a T^n_i]=(-g)^a g_{ik}\nabla_n T^{nk}=0.$$

It follows that

$$\partial_n\{(-g)^a T^n_i\}=(-g)^a\,[\Gamma^m_{ni}T^n_m+2a\Gamma^m_{nm}T^n_i].$$

Using the Einstein equations

$$T^n_i=-t^n_i,$$

where

$$t^n_i=(-c^4\sqrt{-g}/8\pi G)\,[R^n_i-\delta^n_i R/2],$$

we obtain

$$\partial_n\{(-g)^a T^n_i\}=(-g)^a\,[-\Gamma^m_{ni}t^n_m-2a\Gamma^m_{nm}t^n_i]. \tag{34}$$

It follows from the relation

$$\nabla_m R^m_n=\frac{1}{2}\,\frac{\partial}{\partial x^n}\,R$$

that

$$(-g)^a\,[\Gamma^m_{ni}t^n_m+2a\Gamma^m_{nm}t^n_i]=\partial_n\,[(-g)^a t^n_i].$$

Substituting this expression in the right-hand side of Eq. (34), we obtain the conservation law

$$\partial_n\,\{(-g)^a\,[T^n_i+t^n_i]\}=0. \tag{35}$$

Similarly, we can obtain a conservation law in the form

$$\partial_n\,\{(-g)^a\,[T^{ni}+t^{ni}]\}=0. \tag{36}$$

From the expression (36), by virtue of Eqs. (2) and (6), there also follow two differential relations:

$$\partial_n\partial_m h^{inm}=0;\ \partial_n\,\{(-g)^a\,[T^{ni}+\tau^{ni}]\}=0,$$

which reflect only the local fulfillment of the Einstein equations and are not any conservation laws.

Thus, the covariant conservation equation (33) and Einstein's equations lead us to the relations (35) and (36), which are trivially satisfied by virtue of the field equations. A special case of these relations is Eq. (4).

As is shown in Ref. 6, the nonunique results obtained in approximate calculations of the energy and the force of the gravitational radiational damping are a simple consequence of the arbitrary transfer of some of the terms of the tensor t^{0i} in (4) from the right to the left, after which the right-hand side of the resulting expression is declared to be the energy flux of the gravitational waves. It is obvious that such a procedure is quite meaningless, since it gives different results depending on whether what we transfer to the left has a positive or negative value.

Thus, summarizing what we have said above, we arrive at the following conclusions:

1. The general theory of relativity does not and cannot have energy–momentum conservation laws for the gravitational field and matter taken together.

2. An inertial mass defined in general relativity has no physical meaning.

3. Einstein's quadrupole formula for gravitational radiation is not a consequence of general relativity.

4. It does not follow from general relativity in principle that a binary system loses energy through gravitational radiation.

5. The general theory of relativity does not have a classical Newtonian limit, and therefore it does not satisfy one of the most fundamental principles of physics—the correspondence principle.

Thus, the gravitational field in general relativity is quite different from other physical fields and is not a field in the spirit of Faraday and Maxwell. All this indicates that general relativity is not a satisfactory physical theory. It should be noted that general relativity is only one of the possible realizations of Einstein's great idea about the Riemannian geometry of space-time, and therefore when we say that Einstein's theory is unsatisfactory we mean that this particular realization is unsatisfactory. Since the theories of other physical fields contain an energy-momentum conservation law for the different forms of matter taken together, and at the present time there are no experimental data indicating violation of such a law (and, moreover, the development of physics has always shown it to be unshakeable), we have no grounds for rejecting it. Therefore, we shall assume that a conservation law connecting the energy and momentum of the different forms of matter must be the basis of any physical theory. Only experimental data could force us to abandon this position. The conservation law must be valid for all matter fields, including the gravitational field. Therefore, the problem of constructing a classical theory of gravitation satisfying all the requirements imposed on a physical theory is an urgent problem of modern times.

What are the possible ways we can take? What can we retain from Einstein's great creation and what must we eliminate to ensure that in the new theory of gravitation the fundamental laws of physics hold, namely, the energy-momentum conservation law for the matter and the gravitational field taken together and the correspondence principle? To answer these questions, we consider what ideas lie at the basis of Einstein's general relativity.

In our view, the deepest of them is the idea of a Riemannian geometry of space-time with metric tensor g_{ni} determined by the matter. Another hypothesis, on which the entire ediface of general relativity is founded, is the unity of gravitation and the metric of space-time. This unity is achieved by describing gravitation by the metric tensor g_{ni}.

These two hypotheses, as was very clearly established by Hilbert, lead in the simplest case to Einstein's famous equations of general relativity. Since general relativity departs from the usual ideas about the gravitational field as a field in the spirit of Faraday and Maxwell, we must, in constructing a new theory of gravitation analogous to the theories of other physical fields with the usual properties of the gravitational field as a carrier of energy and momentum, retain and enrich Einstein's first idea and give up his second hypothesis. This is the approach we have chosen.

The papers of Refs. 28–33, in which we formulated a field theory of gravitation, were devoted to the solution of this problem. We should say that during these last years our views have undergone a certain evolution, so that Refs. 1–7 and 28–31 were in a certain sense stages leading to our present ideas, the exposition of which is the aim of the present paper.

1. THE GEOMETRY OF SPACE-TIME AND CONSERVATION LAWS

In any physical theory in which the field variable is a tensor quantity, the form of the differential field equations must be independent of the choice of the coordinates in which a given process is described. This can be achieved in two ways: either when the field equations contain only covariant derivatives in the space-time metric that is natural for this process or by constructing from the field functions and their partial derivatives a tensor quantity. In the latter case the field equations are essentially nonlinear.

In constructing general relativity, Einstein followed the second path, relating the metric tensor g_{ni} of Riemannian space-time to the matter by means of the nonlinear equations (A). Thus arose the idea that matter influences the space-time metric.

However, as we have noted above, such an approach does not enable us to regard the gravitational field in general relativity as a physical field possessing energy and momentum. In addition, the natural geometry of the gravitational field in general relativity became the geometry of Riemannian space-time, which did not follow from any experimental facts but was rather a hypothesis to the effect that the gravitational field acts on itself in a definite manner.

However, the action of the gravitational field need not necessarily reduce to a change in the geometry, though it may be nonlinear. In this connection, the problem arises of choosing a natural geometry for the gravitational field that enables one to regard it as a physical field possessing an energy-momentum density.

Any physical field corresponds to a certain natural geometry such that in the absence of interaction with other fields the front of a free wave of this physical field moves along the geodesics of the natural space-time.

The propagation of the wave front of a massless field (the equation of the characteristics),[11]

$$g^{ni}\frac{\partial\psi}{\partial x^n}\frac{\partial\psi}{\partial x^i}=0, \tag{37}$$

and also the motion of free material particles (the Hamilton-Jacobi equation),

$$g^{ni}\frac{\partial\psi}{\partial x^n}\frac{\partial\psi}{\partial x^i}=1, \tag{38}$$

are determined by the metric tensor of the geometry that is natural for these processes.

The problem of the choice of the natural geometry reduces to the question of what effective metric tensor is used to contract the highest derivatives in the Lagrangian density. It is entirely possible that one could have the situation already invisaged by Lobachevskiĭ,[34] in which different physical phenomena are described in terms of different natural geometries.

It follows from Eqs. (37) and (38) that the natural geometry of a physical theory admits experimental determination on the basis of data on the motion of test particles and fields. Study of the motion of test particles with mass and of massless fields makes it possible to determine the metric tensor of natural space–time up to a constant factor.[35] Thus, study of the motion of different forms of matter makes it possible to verify experimentally the nature of the space–time geometry of the world. Therefore, the development of our knowledge about nature has been accompanied by the development of our ideas about space and time.

Thus, Newtonian mechanics (mechanical phenomena) in conjunction with Galileo's principle of relativity (as we now know) again confirmed that space is Euclidean and time absolute, i.e., the same in all coordinate systems. However, the connection between Galileo's principle of relativity and the geometry in Newtonian mechanics was not established, and therefore it was regarded as an independent principle applicable only to inertial coordinate systems. Initially, this principle applied only to mechanical phenomena, but then in Poincaré's papers[36] it was extended to all physical phenomena and formulated as follows: "...the laws of physical phenomena will be the same for an observer at rest as for an observer in a state of uniform translational motion, so that we do not have and cannot have any means to distinguish whether we are in such motion or not." It should be noted that although this principle did appear natural, its true nature was not clear.

Subsequently, Faraday–Maxwell electrodynamics (electrodynamic phenomena) in conjunction with the principle of relativity led to the discovery of the pseudo-Euclidean space-time geometry of the world. Here, we are indebted in the first place to Minkowski. In Ref. 37, he wrote: "The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality...." Later, he noted that "...only the four-dimensional world in space and time is given by phenomena, but the projection in space and in time may still be undertaken with a certain degree of freedom...."

It was Minkowski who first discovered that the space-time in which all physical processes take place is but one, and its geometry pseudo-Euclidean. The subsequent study of the strong, electromagnetic, and weak interactions showed that for the fields associated with these interactions the natural geometry is pseudo-Euclidean geometry.

However, as a result of this discovery the principle of relativity lost its fundamental role and was transformed into a special consequence of the fact that all physical processes take place in a space–time whose geometry is pseudo-Euclidean. Thus, the space-time geometry acquired a fundamental role. The content of the principle of relativity is contained in the proposition that there exists a class of inertial frames of reference in which all physical processes take place in the same manner. In mathematical language, this means that the equations which describe the physical processes are form-invariant with respect to Lorentz transformations. In the limiting case $v/c \to 0$, the Lorentz transformations go over into Galileo transformations and ensure the form invariance of the equations of Newtonian mechanics.

But, as is shown in Refs. 38 and 39, the assertion that all physical phenomena occur in a space-time whose geometry is pseudo-Euclidean is much richer in content that the principle of relativity, since it enables one to formulate a generalized principle of relativity valid in not only inertial but also noninertial coordinate systems. In this connection, it should be noted that in the scientific literature it is commonly stated that the special theory of relativity applies only to the description of phenomena in inertial frames of reference, whereas the description of phenomena in noninertial frames is the prerogative of the general theory.

But this is incorrect. It is a trivial consequence of Minkowski's fundamental discovery that to describe physical phenomena one can use any class of admissible frames of reference—inertial and noninertial. The curvature tensor of this space-time, which determines its entire intrinsic geometry, remains equal to zero in both inertial and noninertial frames of reference. Therefore, in the framework of the special theory of relativity it is still entirely possible to describe physical phenomena in noninertial frames of reference as well. This was clearly understood by Fock.[13]

Before we turn to the formulation of the generalized principle of relativity, we shall briefly recall the difference between the concepts of general covariance and form invariance.

Because the definitions of these two concepts are similar, the concept of covariance is quite often used in the meaning of form invariance, which it does not have.

An equation is said to be *covariant* under a certain coordinate transformation if the new unknown functions which occur in it and are expressed in the new variables satisfy equations of the same form as the old functions in the old variables. Thus, the requirement of covariance of equations is not the reflection of a physical principle but a methematical requirement.

As was shown by Fock,[13] if an equation is to be covariant, it must transform under arbitrary admissible coordinate transformations in accordance with a tensor law. Let us explain this by an example. The equations of relativisic mechanics

$$Du^i(x)/Ds = \mathscr{F}^i(x) \tag{39}$$

are covariant, since by virtue of the tensorial nature under an arbitrary admissible coordinate transformation

$$x'^i = x'^i(x) \tag{40}$$

the new functions, expressed in the new variables

$u'^i(x')$, will satisfy an equation of the same form as the original equation (39):

$$D'u'^i(x')/D's' = \mathcal{F}'^i(x'),$$

i.e., on the transition from the coordinates x to the coordinates x' all quantities in (39) are replaced by the corresponding quantities with primes. It is to be emphasized especially that the functional dependence of the metric tensor g'_{ni} of space-time on the new coordinates can change under the transformations (40). This means that if in the initial coordinate system the metric tensor g_{ni} is a particular function of the coordinates x, in the primed coordinate system it can be an entirely different function of the coordinates x'. Since the metric tensor of space-time or its derivatives always occur in covariant equations, the functional form of the covariant equations changes under the transformation (40) in the general case.

This can be readily seen by noting that under the coordinate transformations (40) the metric tensor of space-time transforms in accordance with the law

$$g'_{ni}(x') = \frac{\partial x^l}{\partial x'^n}\frac{\partial x^m}{\partial x'^i} g_{lm}(x(x')).$$

Thus, the functional form of the covariant equations of physical processes is not preserved under the transformations (40), and therefore in different coordinate systems the description of the phenomena is different, i.e., in the general case the same phenomena will evolve differently in different coordinate systems.

The requirement of *form invariance* of the metric under certain coordinate transformations, i.e., the requirement that the functional dependence of the metric tensor should not change under these transformations, is more stringent than the requirement of covariance of the equations. This requirement restricts the class of coordinate systems to those between which the transformations leave the functional form of the metric tensor of space-time unchanged, i.e., the functional dependence of the tensor g_{ni} on the coordinates x in one frame is the same as the dependence of the tensor g'_{ni} on the coordinates x' in any other frame of the class. However, this requirement guarantees that for the complete group of transformations that leave the metric form-invariant the functional form of the field equations will be unchanged. Therefore, in all frames of reference connected by transformations that leave the metric form-invariant all physical phenomena will evolve in the same manner if the initial and boundary conditions correspond, so that it will be impossible to establish in which of the frames we are.

Thus, covariance and form invariance are different concepts. The transformations that ensure covariance of the field equations include in the general case transformations between different admissible but physically inequivalent (for the description of physical phenomena) coordinate systems. In contrast, the transformations that ensure form invariance of the metric tensor of space-time (and, therefore, form invariance of the equations) include only transformations between physically equivalent coordinate systems, in which all physical phenomena evolve in the same manner for corresponding initial and boundary conditions.

Since the space-time geometry on the transition between different frames of reference does not change and remains pseudo-Euclidean, for any inertial or noninertial or noninertial frame of reference there exists a 10-parameter group of coordinate transformations that leave the metric form-invariant. Thus, for pseudo-Euclidean space-time each frame of reference is associated with an infinite set of other frames of reference, the transformations between them leaving the metric form-invariant.

In pseudo-Euclidean space-time there is a generalized principle of relativity[38,39]: For whatever physical frame of reference we have chosen (inertial or noninertial), there always exists an infinite set of other frames of reference such that all physical phenomena in them take place in the same manner as in the original frame of reference. Therefore, we do not have and cannot have any means to distinguish experimentally in which frame of reference we are among this infinite set.

Thus, the Minkowski geometry has a general nature, being the natural geometry for all the known fields. Pseudo-Euclidean space-time is not *a priori*, given from the very beginning and existing independently. Its existence cannot be separated from the existence of matter, since it is the geometry in which matter evolves. The possibility of obtaining conservation laws for a closed system of interacting fields depends to a large degree on the nature of the space-time geometry.

Mathematically, the conservation laws for energy, momentum, and angular momentum are a reflection of definite properties of space-time, namely, its homogeneity and isotropy. There exist three types of spaces[40] possessing the properties of homogeneity and isotropy to such an extent that they admit the introduction of ten integrals of the motion for a closed system. These are a space of constant negative curvature (Lobachevskii space), a space of zero curvature (pseudo-Euclidean space), and a space of constant positive curvature (Riemannian space). The first two spaces are infinite and have infinite volume, while the third space is closed and has a finite volume but no boundaries.

2. FUNDAMENTALS OF THE FIELD APPROACH TO THE DESCRIPTION OF THE GRAVITATIONAL INTERACTION

Thus, if the gravitational field is to be regarded as a physical field in the spirit of Faraday and Maxwell with the usual properties of being a carrier of energy and momentum, it is sufficient to choose one of the above geometries as the natural geometry for the gravitational field. Since the experimental data obtained from the study of strong, weak, and electromagnetic interactions indicate that the natural space-time geometry for the fields associated with these interactions is pseudo-Euclidean, we can, at least at the present stage of our knowledge, assume that this geometry is the unique natural geometry for all physical processes, including gravitational processes.

This assumption is one of the basic propositions of the field approach to the theory of the gravitational interaction that we have developed. It is obvious that it leads to the fulfillment of all the conservation laws for energy, momentum, and angular momentum and to all ten integrals of the motion for the system consisting of the gravitational field and the remaining matter fields. In the field approach, the gravitational field, like all other physical fields, is characterized by its energy-momentum tensor, which makes its contribution to the total energy-momentum tensor of the system. This is the main fundamental difference between our approach and Einstein's general relativity. It should also be noted that in pseudo-Euclidean space-time the integration of tensorial quantities is not only simple but has a quite definite meaning.

Another key question that arises in the construction of the theory of the gravitational field concerns the nature of the interaction of the gravitational field with matter. When it acts on matter, the gravitational field can effectively change the geometry of the matter if the gravitational field occurs in the terms involving the highest derivatives in the equation of motion of the matter. Then the motion of material bodies and other physical fields in the pseudo-Euclidean space-time under the influence of the gravitational field will be indistinguishable from their motion in some effective Riemannian space-time. Experimental data indicate that the gravitational field acts in a universal manner on matter, so that this effective Riemannian space-time will be the same for all matter irrespective of its form.

This leads us to a proposition which we call the *identity principle* (*geometrization principle*), which we define as follows: The equations of motion of matter under the influence of the gravitational field in pseudo-Euclidean space-time with metric tensor γ_{ni} can be identically represented as the equations of motion of matter in some effective Riemannian space-time with metric tensor g_{ni} that depends on the gravitational field and on the metric tensor γ_{ni}.

This principle was introduced and formulated in Ref. 28, although essentially it was already advanced in Ref. 3. The description of the motion of matter under the influence of the gravitational field in pseudo-Euclidean space-time is physically identical to the description of the motion of matter in a corresponding effective Riemannian space-time. In such an approach, the gravitational field (as a physical field) is apparently eliminated from the description of the motion of the matter, and its energy, speaking figuratively, is expended on the forming of the effective Riemannian space-time. Thus, the effective Riemannian space-time becomes a carrier of energy and momentum; in accordance with the identity principle, it contains as much energy as is contained in the gravitational field, and therefore the propagation of curvature waves in the Riemannian space-time reflects the ordinary transport of energy by gravitational waves in the pseudo-Euclidean space-time. This means that in the field approach the curvature waves in Riemannian space-time are direct consequences of the existence of gravitational waves in the spirit of Faraday and Maxwell, and these possess an energy-momentum density.

It must be emphasized that the identity principle does not follow from any other physical principles. It is an independent principle, determining, on the one hand, the equivalence of the description of the motion of matter and, on the other, the nature of the interaction of the gravitational field with matter, corresponding, thus, to a definite choice of the Lagrangian density of the interaction between them. In particular, the identity principle reflects the physical fact that the inertial mass of a point body is equal to its gravitational mass.

We note also that in introducing the geometrization principle we retain Einstein's great idea of a Riemannian space-time geometry for matter. However, this does not mean that we necessarily return to general relativity. Einstein's theory is a particular realization of this idea, and not vice versa. Therefore, the idea of the gravitational field as a physical field that carries energy, combined with the identity principle, leads to other equations of the gravitational field, different from Einstein's equations, and changes our ideas about space-time and gravitation. The new theory of gravitation realizing this idea can describe all the existing gravitational experiments, satisfies the correspondence principle, and has a number of fundamental consequences.

It must be emphasized that the field approach to the theory of the gravitational interaction does not particularize in advance the nature of the gravitational field. We do not know what is the nature of the real gravitational field. It is possible that for its adequate description it is necessary to use spin tensors or a scalar field. At the present, the lack of experimental material in the field of gravitation leaves a large scope for theoretical constructions, and only time and new experimental facts will permit a definitive choice to be made between the theoretical variants.

3. SYMMETRIC TENSOR GRAVITATIONAL FIELD

One of the possible realizations of the field approach is to use a symmetric tensor field of second rank to describe the gravitational field. It should be noted that many authors have already attempted to formulate a theory of gravitation in flat space-time using for this purpose different fields: scalar, vector, and symmetric tensor fields. However, these attempts were occasional in nature and did not contain a clear formulation of the field-theoretical requirements on the theory of gravitation. As a result, the simplest variants proposed in Refs. 41–71 either contradicted the available experimental data or were not logically consistent and required the formulation of additional conditions to ensure that the energy of gravitational waves is positive definite.[72]

This circumstance led Thirring,[73] and subsequently other authors,[74,75] to the assertion that any route to the construction of a theory of gravitation on the basis of flat space-time that starts from the idea of the gravitational field as a physical field in the spirit of Faraday and Maxwell will necessarily lead to Einstein's general relativity.

However, the analysis of Einstein's theory that we made in Refs. 1–8, and also the search for other possibilites in the construction of the theory of gravitation[28-31] showed that such an assertion is quite unjustified. On the one hand, Einstein's theory departed from the concept of the gravitational field as a physical field possessing energy and momentum and introduced a field of a new type, namely, a field characterized by the curvature tensor, and, on the other hand, it lacks a fundamental principle, namely, conservation laws for the energy and momentum of the matter and the gravitational field taken together. This is a price too high to pay for the explanation of a small number of gravitational experiments. It is therefore necessary to compare the different classes of gravitational theories that traditionally use a symmetric tensor field of second rank and find which of them introduces the gravitational field in the physically most acceptable way.

In the construction of the theory of gravitation a key feature is the choice of the natural geometry for the gravitational field. For linear theories, the natural geometry is the geometry of flat space-time, and theories of gravitation with linear equations of the free gravitational field can be formulated in terms of flat space-time with metric tensor γ_{ni}. We shall say that theories of gravitation formulated in terms of flat space-time are *theories of class A*. Theories of class A can also be nonlinear, but it is important that this nonlinearity does not occur in the terms with the highest derivatives in the field equations and, thus, does not change the geometry of the natural space-time. Thus, in theories of class A there is a unique flat space-time, which guarantees the existence of all ten conservation laws for a closed system; in contrast, the Riemannian space-time in terms of which the motion of matter is described is an effective space-time and arises as a result of the action of the gravitational field φ_{ni} on the matter.

Among the theories of class A, we must distinguish the subclass of bimetric theories in which the gravitational field φ_{ni} in combination with the metric tensor γ_{ni} forms in the Lagrangian density L_g of the gravitational field a new field variable—the metric tensor g_{ni} of the effective Riemannian space-time in terms of which the equations of motion of the matter are formulated, the natural geometry for the new field variable being pseudo-Euclidean geometry:

$$L = L_g(\gamma_{ni}, g_{ni}(\gamma_{mk}, \varphi_{mk})) + L_M(g_{ni}, \varphi_A).$$

An example of a nonlinear theory of this subclass is Rosen's theory[65] with Lagrangian density

$$L_g = \frac{\sqrt{-\gamma}}{64\pi}\gamma^{ik}g^{nm}g^{pl}\left[D_i g_{nl} D_k g_{mp} - \frac{1}{2} D_i g_{nm} D_k g_{pl}\right],$$

where γ is the determinant of the metric tensor of flat space-time, D_i is the covariant derivative in the flat space-time, and in what follows it is assumed that $G = c = 1$.

In bimetric theories, the gravitational field φ_{ni} is actually absent, since the field variable is the metric tensor g_{ni}; therefore, there is here no sufficiently deep physical justification of the connection between the effective Riemannian space-time and the unique flat space-time.

In theories of class A, we actually have two physical space-times—the flat space-time with metric tensor γ_{ni}, in terms of which the equations of the gravitational field are formulated, and the non-Euclidean space-time with metric tensor g_{ni}, in terms of which the motion of the matter is formulated. Both of them are real observable space-times. The front of a gravitational wave moves along geodesics of the flat space-time, and therefore gravitational waves can be used to determine the geometry of pseudo-Euclidean space-time. The front of an electromagnetic wave moves along geodesics of the effective Riemannian space-time, and therefore electromagnetic waves and massive particles can be used to determine the geometry of this Riemannian space-time.

If in a nonlinear theory of the tensor field φ_{ni} the nonlinear terms are contracted with the derivatives in the Lagrangian density (in the terms with the highest derivatives in the field equations), then for such a theory a non-Euclidean space-time with an effective metric tensor $g_{ni} = g_{ni}(\gamma_{mk}, \varphi_{mk})$ becomes the natural space-time. We shall say that theories of gravitation formulated in terms of the effective Riemannian space-time are *theories of class B*. The Lagrangian density of theories of this class has the form

$$L = L_g(g_{ni}, \varphi_{ni}) + L_M(g_{ni}, \varphi_A).$$

Theories of this class warrant special study.

Since the flat space-time in theories of this class is not observed, we here obviously lack an adequate justification of the connection $g_{ni} = g_{ni}(\gamma_{mk}, \varphi_{mk})$ between the common Riemannian space-time and the gravitational field φ_{ni}. The common Riemannian space-time in theories of this class arises on the basis of the gravitational field φ_{ni} and the unobservable flat space-time. It should also be noted that the equations of the gravitational field in theories of class B are necessarily nonlinear.

A subclass of geometrized theories of class B is the set of theories with complete geometrization, in which the Lagrangian density of the gravitational field depends only on the metric tensor g_{ni}:

$$L = L_g(g_{ni}) + L_M(g_{ni}, \varphi_A).$$

Einstein's theory belongs to this subclass of theories and corresponds to the special choice of the Lagrangian density in the form $L_g = \sqrt{-g}R$. In theories with complete geometrization, the flat space-time is completely eliminated from the description of the motion of both the matter and the gravitational field. Neither the gravitational field φ_{ni} nor the metric tensor γ_{ni} appear anywhere in the theory. At the same time, the quantities g_{ni} have a double meaning: as variables of a physical field and as the metric tensor of space-time. This has the consequence that in theories of this subclass the gravitational field is not a Faraday-Maxwell field possessing energy and momentum.

It must be emphasized that theories of the classes A and B are essentially different theories of gravitation. By no transformation of the field variables or transformation of the coordinates can one transform a theory of one class into a theory of the other.

Thus, analyzing the existing possibilities, we conclude that only theories of class A introduce the gravitational field in a manner that is maximally acceptable from the physical point of view. Theories of this class enable us to regard the gravitational field as a physical field in the spirit of Faraday and Maxwell and possess all ten integrals of the motion for a closed system of interacting fields. The effective Riemannian space-time used to describe the motion of matter in theories of this class reflects in a natural manner the existence of the physical gravitational field and the unique pseudo-Euclidean space-time.

Therefore, we again arrive at the need for an immediate study of the possibilities of constructing a theory of gravitation which realizes the field approach to the description of the gravitational interaction.

4. CONSERVATION LAWS FOR THE GRAVITATIONAL FIELD AND MATTER

We now study the nature of the conservation laws for all local theories of class A without making a particular choice of the Lagrangian density. Proceeding from the basic principles of the field approach, we write the Lagrangian density of the system consisting of matter and the gravitational field in the following form for theories of this class:

$$L = L_g(\gamma_{ni}, \varphi_{ni}) + L_M(g_{ni}, \varphi_A), \tag{41}$$

where γ_{ni} is the metric tensor of the pseudo-Euclidean space-time with signature (+, –, –, –), φ_{ni} is the gravitational field, and φ_A are the remaining matter fields.

Without loss of generality, we shall assume that the metric tensor g_{ni} of the Riemannian space-time is a local function that depends on the metric tensor of the flat space-time, the gravitational field φ_{ni}, and their second derivatives to the second order inclusive:

$$g_{ml} = g_{ml}(\gamma_{ni},\ \partial_p\gamma_{ni},\ \partial_{pk}\gamma_{ni};\ \varphi_{ni},\ \partial_p\varphi_{ni},\ \partial_{pk}\varphi_{ni}), \tag{42}$$

where

$$\partial_{nm}\varphi = \partial^2\varphi/\partial x^n\partial x^m.$$

We shall assume that the Lagrangian density L_M of the matter depends only on the fields φ_A, their partial derivatives of first order, and the metric tensor g_{ni}. It is easy to see that in this case the Lagrangian density of the matter will contain the partial derivatives of the gravitational field up to the second order. We shall assume that the Lagrangian density of the gravitational field depends on the metric tensor γ_{ni}, the gravitational field φ_{ni}, and their partial derivatives to the third order inclusive. To obtain the conservation laws, we use the covariant method of infinitesimal displacements.

Since the action J is a scalar, under an arbitrary infinitesimal coordinate transformation

$$x'^i = x^i + \xi^i(x) \tag{43}$$

the variations δJ_M of the matter action and δJ_G for the gravitational field will be zero.

Since the Lagrangian density of the matter contains both covariant and contravariant components of the metric tensor of the Riemannian space-time, we shall vary the Lagrangian density with respect to them as if they were independent, and then take into account the relationships between their variations:

$$\delta g^{nm} = -g^{ni}g^{ml}\delta g_{il};$$

then the symmetric energy-momentum tensor density of the matter T^{ni} in the Riemannian space-time will have the form

$$T^{ni} = -2\frac{\Delta L_M}{\Delta g_{ni}} = -2\left(\frac{\delta L_M}{\delta g_{ni}} - g^{im}g^{nl}\frac{\delta L_M}{\delta g^{ml}}\right), \tag{44}$$

where $\delta L/\delta\varphi$ is the Euler-Lagrange variation:

$$\frac{\delta L}{\delta\varphi} = \frac{\partial L}{\partial\varphi} - \partial_n\left(\frac{\delta L}{\partial(\partial_n\varphi)}\right) + \partial_{nl}\left(\frac{\partial L}{\partial(\partial_{nl}\varphi)}\right) - \partial_{nli}\left(\frac{\partial L}{\partial(\partial_{nli}\varphi)}\right). \tag{45}$$

We shall proceed similarly when varying with respect to the components γ_{ni} and γ^{in} of the metric tensor of the flat space-time.

We write the variation of the matter action integral under the transformation (43) in the form

$$\delta J_M = \int d^4x\left\{\frac{\Delta L_M}{\Delta g_{ni}}\delta g_{ni} + \frac{\delta L_M}{\delta\varphi_A}\delta\varphi_A + \mathrm{Div}\right\} = 0, \tag{46}$$

where Div stands for divergence terms whose allowance leads to relations unimportant for the purposes of our study.

Introducing the notation

$$\left.\begin{aligned} t_M^{mn} &= -2\frac{\Delta L_M}{\Delta\gamma_{nm}} = -2\left(\frac{\delta L_M}{\delta\gamma_{nm}} - \gamma^{ns}\gamma^{mk}\frac{\delta L_M}{\delta\gamma^{sk}}\right); \\ t_{Mi}^{n} &= \gamma_{im}t_M^{nm}, \end{aligned}\right\} \tag{47}$$

for the energy-momentum tensor density of the matter in the flat space-time, we can write the variation δJ_M of the action integral under the coordinate transformations (43) in a different form equivalent to the expression (46):

$$\delta J_M = \int d^4x\left\{\frac{\delta L_M}{\delta\varphi_{nm}}\delta\varphi_{nm} + \frac{\Delta L_M}{\Delta\gamma_{mn}}\delta\gamma_{nm} + \frac{\delta L_M}{\delta\varphi_A}\delta\varphi_A + \mathrm{Div}\right\} = 0. \tag{48}$$

The variations $\delta\gamma_{nm}$, $\delta\varphi_{nm}$, $\delta\varphi_A$, and δg_{nm} under the coordinate transformation (43) have the form

$$\left.\begin{aligned} \delta\gamma_{nm} &= -\gamma_{nl}D_m\xi^l - \gamma_{ml}D_n\xi^l; \\ \delta\varphi_{nm} &= -\varphi_{nl}D_m\xi^l - \varphi_{ml}D_n\xi^l - \xi^l D_l\varphi_{nm}; \\ \delta\varphi_A &= -\xi^l D_l\varphi_A + F_{A;\,l}^{B;\,n}\varphi_B D_n\xi^l; \\ \delta g_{nm} &= -g_{nl}D_m\xi^l - g_{ml}D_n\xi^l - \xi^l D_l g_{nm}. \end{aligned}\right\} \tag{49}$$

Using these equations, we can write the variation (48) of the matter action as

$$\begin{aligned} \delta J_M = \int d^4x\Big\{\xi^l\Big[2D_n\left(\frac{\delta L_M}{\delta\varphi_{nm}}\varphi_{ml}\right) \\ - D_n t_{Ml}^{n} - \frac{\delta L_M}{\delta\varphi_{nm}}D_l\varphi_{mn} - D_n\left(\frac{\delta L_M}{\delta\varphi_A}F_{A;\,l}^{B;\,n}\varphi_B\right) \\ - \frac{\delta L_M}{\delta\varphi_A}D_l\varphi_A\Big] + \mathrm{Div}\Big\} = 0. \end{aligned} \tag{50}$$

Because the displacement vector ξ^l in the expression (50) is arbitrary, we have the identity

$$D_l t^l_{Mn} - 2D_l\left(\frac{\delta L_M}{\delta \varphi_{lm}}\varphi_{mn}\right) + \frac{\delta L_M}{\delta \varphi_{lm}} D_n \varphi_{lm} + D_l\left(\frac{\delta L_M}{\delta \varphi_A} F^{B;\,l}_{A;\,n}\varphi_B\right) + \frac{\delta L_M}{\delta \varphi_A} D_n \varphi_A \equiv 0. \qquad (51)$$

We obtain another important identity by substituting the relations (49) in the expression (46):

$$D_l(g_{nm}T^{lm}) - \frac{1}{2}T^{lm}D_n g_{ml} = -D_l\left(\frac{\delta L_M}{\delta \varphi_A} F^{B;\,l}_{A;\,n}\varphi_B\right) - \frac{\delta L_M}{\delta \varphi_A} D_n \varphi_A. \qquad (52)$$

We now express the covariant derivatives on the left-hand side of the identity (52) in terms of the partial derivatives and the connection γ^i_{nl} of the flat space-time. Bearing in mind that T^{ni} is a tensor density of weight 1, we obtain

$$\partial_i(g_{nm}T^{mi}) - \frac{1}{2}T^{im}\partial_n g_{im} = -D_i\left(\frac{\delta L_M}{\delta \varphi_A} F^{B;\,i}_{A;\,n}\varphi_B\right) - \frac{\delta L_M}{\delta \varphi_A} D_n \varphi_A.$$

But the left-hand side of this expression is the covariant divergence in the Riemannian space-time of the energy-momentum tensor density T^i_n of the matter:

$$\partial_i(g_{nm}T^{mi}) - \frac{1}{2}T^{im}\partial_n g_{im} = \partial_i T^i_n - \Gamma^m_{ni}T^i_m = \nabla_i T^i_n = g_{nm}\nabla_i T^{im},$$

where Γ^m_{ni} is the connection of the Riemannian space-time.

Therefore, the relation (52) takes the form

$$g_{ni}\nabla_m T^{im} = -D_i\left(\frac{\delta L_M}{\delta \varphi_A} F^{B;\,i}_{A;\,n}\varphi_B\right) - \frac{\delta L_M}{\delta \varphi_A} D_n \varphi_A.$$

Subtracting this equation from the expression (51), we obtain

$$D_i t^i_{Mn} - 2D_i\left(\frac{\delta L_M}{\delta \varphi_{im}}\varphi_{mn}\right) + \frac{\delta L_M}{\delta \varphi_{im}} D_n \varphi_{im} = g_{ni}\nabla_m T^{im}. \qquad (53)$$

It must be emphasized that this identity is valid irrespective of the fulfillment of the equations of motion of the matter and the gravitational field.

Similarly, from the invariance of the action of the gravitational field under the transformation (43) we obtain

$$D_i t^i_{gn} - 2D_i\left(\frac{\delta L_g}{\delta \varphi_{im}}\varphi_{mn}\right) + \frac{\delta L_g}{\delta \varphi_{im}} D_n \varphi_{im} = 0. \qquad (54)$$

For the symmetric energy-momentum tensor density t^i_{gn} of the gravitational field we have as usual

$$t^i_{gn} = -2\gamma_{nm}\Delta L_g/\Delta\gamma_{im}. \qquad (55)$$

It follows from Eqs. (53) and (54) that

$$D_i(t^i_{Mn} + t^i_{gn}) - 2D_i\left(\frac{\delta L}{\delta \varphi_{im}}\varphi_{mn}\right) + \frac{\delta L}{\delta \varphi_{im}} D_n \varphi_{im} = \nabla_i T^i_n. \qquad (56)$$

If the equations of the gravitational field are satisfied,

$$\delta L/\delta\varphi_{im} = \delta L_g/\delta\varphi_{im} + \delta L_M/\delta\varphi_{im} = 0, \qquad (57)$$

the expression (56) simplifies:

$$D_i(t^i_{Mn} + t^i_{gn}) = g_{ni}\nabla_m T^{im}. \qquad (58)$$

This equation is a manifestation of the identity principle. It follows from it that the covariant divergence in the pseudo-Euclidean space-time of the sum of the energy-momentum tensor densities of the matter and the gravitational field has been transformed into the covariant divergence in the Riemannian space-time of the energy-momentum tensor density of the matter alone. Thus, these are different forms of expression of the same fact.

If the equations of motion of the matter are satisfied,

$$\delta L_M/\delta\varphi_A = 0, \qquad (59)$$

the expression (51) simplifies:

$$D_l t^i_{Mn} - 2D_i\left(\frac{\delta L_M}{\delta \varphi_{im}}\varphi_{mn}\right) + \frac{\delta L_M}{\delta \varphi_{im}} D_n \varphi_{im} = 0, \qquad (60)$$

and from the relation (52) there automatically follows a covariant conservation equation in the Riemannian space-time:

$$\nabla_n T^{ni} = 0. \qquad (61)$$

This result is common to all theories with geometrized Lagrangian density of the matter and is not tied to any particular variant of the theory of gravitation.

We see further that if Eqs. (57) of the gravitational field are satisfied the relations (60) and (54) yield a covariant conservation law for the total symmetric energy-momentum tensor density in the pseudo-Euclidean space-time:

$$D_i(t^i_{Mn} + t^i_{gn}) = 0. \qquad (62)$$

Thus, on the basis of the Lagrangian formalism we have obtained an energy-momentum conservation law for the matter and the gravitational field in the pseudo-Euclidean space-time. This fundamental law of nature means that in the field theory of gravitation there are no processes (irrespective of the erudition of their inventors) that do not conserve energy and momentum. It also follows from the expression (62) that the gravitational field, considered in the pseudo-Euclidean space-time, behaves like all other physical fields. It possesses energy and momentum and contributes to the total energy-momentum tensor density of the system.

On the basis of Eq. (62) and the identity (58), we have

$$D_i(t^i_{Mn} + t^i_{gn}) = g_{nm}\nabla_i T^{im} = 0.$$

Therefore, the conservation law for the total energy-momentum tensor density (62) and the conservation law in the form (61) when the gravitational field equations (57) and the equations of motion of the matter (59) are satisfied are simply different forms of expression of the same conservation law. The conservation law (62) expresses the fact that in the pseudo-Euclidean space-time the total energy-momentum tensor density of the system consisting of the matter and the gravitational field is conserved. This law has the customary form of a conservation law. The conservation law (61) in the Riemannian space-time is not a conservation law in the usual sense, since the energy-momentum tensor density T^{ni} of the matter should not be conserved: $\partial_n T^{ni} \neq 0$.

Einstein already pointed out (Ref. 12, p. 492): "...the presence of the second term on the left-hand side indicates from the physical point of view that for the matter alone the energy and momentum conservation laws are not satisfied in their true sense; more precisely, they are satisfied only when the g_{ni} are constant, i.e., when the components of the gravitational field intensity are equal to zero. This second term is the expression for the momentum and, accordingly, for the energy that in unit time and in unit volume are transmitted to the matter from the gravitational field...."

In this case, the second term in (61) expresses the energy influence of the gravitational field on the matter and shows that the matter acquires energy that is, as it were, stored in the Riemannian geometry. In this case, one can say that the energy of the gravitational field has been used to create the Riemannian geometry. From the expression (61) it cannot be seen what quantity is conserved.

The absence of conservation laws in the true sense is a feature of the entire subclass of gravitational theories with complete geometrization and not only Einstein's theory. The Lagrangian density L_g of the gravitational field of theories of this subclass depends on the field φ_{ni} and on the metric tensor γ_{ni} only through the metric tensor g_{ni} of the Riemannian space-time. In theories of this subclass the symmetric energy-momentum tensor density of the matter and the gravitational field in the pseudo-Euclidean space-time is given by

$$-\frac{1}{2}t^{ni}=\frac{\Delta L}{\Delta\gamma_{ni}}=\frac{\Delta L_g}{\Delta\gamma_{ni}}+\frac{\Delta L_M}{\Delta\gamma_{ni}}=\frac{\Delta L}{\Delta g_{ml}}\frac{\partial g_{ml}}{\partial\gamma_{ni}}$$
$$-\partial_p\left[\frac{\Delta L}{\Delta g_{lm}}\frac{\partial g_{lm}}{\partial(\partial_p\gamma_{ni})}\right]+\partial_{pq}\left[\frac{\Delta L}{\Delta g_{lm}}\frac{\partial g_{lm}}{\partial(\partial_{pq}\gamma_{ni})}\right]$$
$$-\gamma^{is}\gamma^{np}\left\{\frac{\Delta L}{\Delta g_{lm}}\frac{\partial g_{lm}}{\partial\gamma^{sp}}-\partial_q\left[\frac{\Delta L}{\Delta g_{lm}}\frac{\partial g_{lm}}{\partial(\partial_q\gamma^{sp})}-\partial_h\left(\frac{\Delta L}{\Delta g_{lm}}\frac{\partial g_{lm}}{\partial(\partial_{qh}\gamma^{sp})}\right)\right]\right\}.$$

Since the equations of the gravitational field in a geometrized theory are

$$\Delta L/\Delta g_{lm}=\delta L/\delta g_{lm}-g^{li}g^{mn}\delta L/\delta g^{ni}=0,$$

the symmetric energy-momentum tensor density of the matter and the gravitational field in the pseudo-Euclidean space-time vanishes by virtue of the equations of the gravitational field:

$$\Delta L/\Delta\gamma_{ni}=-t^{ni}/2=0. \qquad (63)$$

We obtain a similar conclusion—that the symmetric energy—momentum tensor density vanishes, for the free gravitational field as well. But the equations of the free gravitational field contain solutions for which the curvature tensor R^{i}_{nlm} is nonzero. Therefore, in theories with complete geometrization the vanishing of the energy-momentum tensor density of the free gravitational field does not lead to vanishing of the field φ_{ni} and, therefore, there exists a certain fictitious field that does not possess an energy-momentum density but leads to curving of space-time (the formation of a Riemannian geometry). It follows that a theory of a physical gravitational field possessing energy and momentum and based on flat space-time cannot in principle be reduced to general relativity. Therefore, the claims made by various authors[74,75] that such a reduction is unavoidable are incorrect. The subclass of gravitational theories with complete geometrization are in principle incapable of introducing the concept of a gravitational field possessing energy and momentum.

Thus, we arrive at the following conclusions: 1. In local theories of class A, the gravitational field, described in the pseudo-Euclidean space-time, is a physical field with energy and momentum. On the basis of the identity principle, the motion of the matter can be described in an effective Riemannian space-time, the creation of which is made possible by the energy and momentum of the gravitational field. In this approach, the geometrical description arises on the basis of field-theoretical ideas about the gravitational field, and it is based on the conservation laws. 2. In the subclass of theories with complete geometrization, the gravitational field and the matter have a common geometry, but the gravitational field loses the properties of a physical field and does not possess energy or momentum. In this approach, we do not have the field-theoretical notions of a gravitational field as a field in the spirit of Faraday and Maxwell.

The general theory of relativity realizes this possibility of construction of a theory. It introduced a field of a new type, described by the curvature tensor, which is not a Faraday-Maxwell field. In this case there are no conservation laws of the matter and the gravitational field taken together, so that this theory does not satisfy the principle of correspondence with Newton's theory of gravitation.

5. GAUGE-INVARIANT TENSOR FIELD

In this section, we shall formulate all our relations and equations in Cartesian coordinates, although, of course, they could also be written covariantly and in an arbitrary curvilinear coordinate system.

We consider theories of class A with Lagrangian density in the form (41). The equations of the gravitational field and the equations of motion of the matter are

$$\delta L_g/\delta\varphi_{nm}+\delta L_M/\delta\varphi_{nm}=0; \qquad (64)$$
$$\delta L/\delta\varphi_A=0. \qquad (65)$$

Among the set of theories with Lagrangian density (41) there are theories in which the action integral is invariant under the gauge transformation

$$\varphi_{ni}\to\varphi_{ni}+\partial_i a_n+\partial_n a_i, \qquad (66)$$

where a_i is an arbitrary gauge 4-vector. From the invariance of the action integral of the free gravitational field under the gauge transformation (66), we have

$$\delta J_g=\int d^4x\,[-2a_n\partial_i(\delta L_g/\delta\varphi_{ni})+\mathrm{Div}]=0.$$

Because the gauge vector a_n is arbitrary,

$$\partial_i(\delta L_g/\delta\varphi_{ni})=0.$$

From this equation and the field equation (64) we obtain a conservation equation for the source of the gravitational field:

$$\partial_i(\delta L_M/\delta\varphi_{ni})=0.$$

It is well known[76] that in electrodynamics the invariance of the Lagrangian density $L=L_A+L_M$ under the gauge transformation $A_i\to A_i+\partial_i f$ of the vector potential leads to analogous conservation equations:

$$\partial_i\delta L_A/\delta A_i=0;\ \partial_i\delta L_M/\delta A_i=0.$$

Since the source in the field equations in a gauge theory is conserved, it is generally assumed that the source in the equations of a gauge theory of gravitation is the total energy-momentum tensor of the system consisting of the matter and the gravitational field. This has the consequence that the field equations become nonlinear, and it is usually stated that the systematic inclusion of such nonlinearities can lead to Einstein's nonlinear theory of gravitation.[74,75,77]

However, in reality this hypothesis leads in the first place to the loss of the gravitational field's property of being a carrier of energy and momentum. If we assume the possibility of identifying the source $\delta L_M/\delta\varphi_{ni} = J^{ni}/2$ with the total energy-momentum tensor density $t^{ni} = t_g^{ni} + t_M^{ni}$, it follows directly from this that the energy-momentum tensor of the free gravitational field (for $L_M = 0$) is equal to zero. Such a theory does not possess the properties characteristic of other physical systems, and therefore we regard it as unacceptable.

In accordance with Noether's theorem, invariance of the action integral with respect to a group of transformations entails the existence of definite conserved quantities. Invariance with respect to coordinate transformations leads, as is well known, to conservation of the energy-momentum tensor density t^{ni}. The invariance of the action integral with respect to the gauge transformations (66) leads to conservation of the current j^{ni}. Since coordinate and gauge transformations are quite different transformations, t^{ni} and j^{ni} are, of course, entirely different physical quantities.

The problem of constructing a gauge-invariant theory of a tensor field is in the first place the problem of constructing a conserved tensor current j^{ni}, or, in other words, the problem of constructing a Lagrangian L_M of the matter that leads to a conserved variation $\delta L_M/\delta\varphi_{ni}$. To solve this problem, it is necessary to consider the question of the spin states of a field described by a symmetric tensor of second rank.

As is shown in Refs. 78 and 79, a symmetric tensor φ_{nl} of second rank can be represented as a sum of irreducible representations: one representation with spin 2, one with spin 1, and two representations with spin 0:

$$\varphi_{ni} = (P_2 + P_1 + P_0 + P_{0'})_{ni}^{lm}\varphi_{lm}.$$

The quantities P_S can be conveniently expressed in the momentum representation. We introduce the auxiliary operators

$$X_{ni} = (1/\sqrt{3})(\gamma_{ni} - q_n q_i/q^2);\quad Y_{ni} = q_n q_i/q^2,$$

by means of which the operators P_S can be represented in the form

$$\left.\begin{aligned} &P_0 = X_{ni}X^{lm};\quad P_{0'} = Y_{ni}Y^{lm};\\ &P_1 = (\sqrt{3}/2)(X_i^l Y_n^m + X_n^l Y_i^m + X_i^m Y_n^l + X_n^m Y_i^l);\\ &P_2 = (3/2)(X_i^l X_n^m + X_n^l X_i^m) - X_{ni}X^{lm}.\end{aligned}\right\} \tag{67}$$

In the X representation, the projection operators P_S are nonlocal integro-differential operators:

$$P_{ni}^{lm}\varphi_{lm} = \int d^4y P_{ni}^{lm}(x, y)\,\varphi_{lm}(y).$$

Using the expressions (67), we readily see that only the operators P_2 and P_0 are conserved:

$$q_l P_{2ni}^{lm} = q_m P_{2ni}^{lm} = q_l P_{0ni}^{lm} = q_m P_{0ni}^{lm} = 0.$$

Therefore, if the Lagrangian density L_g of the gravitational field contains the field φ_{ni} only in the combination

$$f_{ni} = [(P_2 + \alpha P_0)\,\varphi]_{ni}, \tag{68}$$

then the Lagrangian density and, therefore, the equations of the free gravitational field will be invariant under the gauge transformation (66). However, the use of the expression (68) is not entirely convenient, since it is an integro-differential expression and therefore leads to nonlocal field equations.

If the field equations are to be local, we need a differential connection between the fields f_{ni} and φ_{ni}. This can be achieved by taking, for example, the combination

$$f_{ni} = \Box^2[(P_2 + \alpha P_0)\,\varphi]_{ni}.$$

In this case, the tensor f_{ni} will contain the fourth derivatives of the field functions φ_{ni}. But among all values of α the value $\alpha = -2$ is distinguished in the sense that it enables us to write f_{ni} in the form of a combination of, not fourth derivatives of the field φ_{ni}, but only the second derivatives:

$$f_{ni} = \Box\,[(P_2 - 2P_0)\,\varphi]_{ni}. \tag{69}$$

It is readily seen that the operator $\Box(P_2 - 2P_0)$ is a gauge-invariant and local operator of the lowest order, namely, in a theory that uses a symmetric tensor field of the second rank there is no other local operator that uses lower derivatives and leads to gauge invariance. Thus, we have

$$\begin{aligned} f_{ni} = \Box\,\theta_{ni} - \partial_i\partial^m\theta_{mn} - \partial_n\partial^m\theta_{mi}\\ + \gamma_{ni}\partial^l\partial^m\theta_{lm};\ \partial^i f_{ni} = 0,\end{aligned} \tag{70}$$

where we have introduced the notation

$$\theta_{lm} = \varphi_{lm} - \gamma_{lm}\varphi_n^n/2. \tag{71}$$

In this case, the vector field and the spin-0′ field, which are not invariant under the gauge transformation (66), are eliminated from the theory.

Since the entire theory must be gauge-invariant, we assume that the fields φ_{lm} occur in the connection equations $g_{ni} = g_{ni}(\varphi_{lm})$ only through the field f_{lm}. Moreover, we assume that the metric tensor g_{ni} of the Riemannian space-time is a local function of the fields f_{lm} alone and of the metric tensor of the flat space-time. With regard to the form of this function, we shall not here make any assumptions apart from the requirement that the quadratic form with coefficients $g_{\alpha\beta}$ be negative definite, and the component g_{00} be a positive quantity. Then the parameter x^0 will have the nature of time, and the parameters x^α the nature of spatial coordinates in the Riemannian space-time.

6. EQUATIONS OF THE GRAVITATIONAL FIELD IN THE FIELD THEORY OF GRAVITATION

The gauge-invariant theory leading to linear equations of the free gravitational field is the simplest variant among all theories of class A. In what follows, we shall call this theory of gravitation the field theory of gravitation. Using derivatives of the fields f_{nm} of order not higher than the first, we can write the Lagrangian density of the gravitational field of the field theory in the most general form as

$$L_g = \frac{1}{64\pi}\{\partial_i f_{nm}\partial^i f^{nm} - b\partial_i f\partial^i f - m_g^2[\alpha f_{nm}f^{nm} + \beta f^2]\},$$

where $f = f_n^n$.

For $\alpha \neq 0$ or $\beta \neq 0$, the obtained equations will describe a gravitational field whose quantum (graviton) has a nonvanishing rest mass. Since we expect the front of a gravitational wave to propagate with the fundamental velocity $v = c$, the graviton rest mass must be equal to zero. For this, it is necessary to set $\alpha = \beta = 0$.

Choosing different values of b, we can realize different physical situations. It can be shown that the energy of the free gravitational field has positive sign if $b \leq \frac{1}{2}$. In addition, if $b < \frac{1}{2}$ a scalar component of gravitational waves is emitted, the magnitude of this component and its energy depending essentially on b, since for $b < \frac{1}{2}$ the scalar component carries positive energy. However, in what follows we do not require this degree of generality, since we assume that gravitational waves (gravitons) are characterized by the spin value $S = 2$ and a positive-definite energy. Therefore, in what follows we set $b = \frac{1}{2}$ in order to eliminate emission of the scalar component.

Thus, we arrive at a Lagrangian density of the free gravitational field in the form

$$L_g = (1/64\pi)\{\partial_i f_{nm}\partial^i f^{nm} - \partial_i f\partial^i f/2\}. \tag{72}$$

This Lagrangian density of the gravitational field is the simplest Lagrangian density invariant under the gauge transformations (66) of the field φ_{ni}. The field f_{nm} can also be subjected to the gauge transformation

$$f_{nm} \to f_{nm} + \partial_n a_m + \partial_m a_n - \gamma_{nm}\partial_i a^i, \tag{73}$$

which does not violate the conditions $\partial^n f_{nm} = 0$, if the gauge vectors a^n satisfy the homogeneous equations $\Box a^n = 0$.

We must emphasize the fact that the symmetric fields f_{nm} are not independent by virtue of the four conditions $\partial^n f_{nm} = 0$ which they satisfy, so that in the derivation of the field equations the Euler-Lagrange variation must be taken with respect to the field φ_{ni}, since only this field has all ten components independent.

Note also that in the considered case the variation of the matter Lagrangian density can be obtained in two ways: either directly, using the Euler-Lagrange variation (45) with respect to the field φ_{ni}, or using the fact that the matter Lagrangian density contains the gravitational field only through the field f_{nm} (70), and the field f_{nm}, in its turn, occurs in the matter Lagrangian density through the metric tensor of the Riemannian space-time. In both cases, we obtain the same result.

Introducing the notation

$$h^{lm} = \frac{1}{2}T^{np}\frac{\partial g_{np}}{\partial f_{ik}}(\delta_i^l\delta_k^m + \delta_k^l\delta_i^m - \gamma_{ik}\gamma^{lm}) \tag{74}$$

and using the relation (70), we obtain the gravitational field equations (64) in the form

$$\Box^3\theta^{lm} - \partial^l\partial_n\Box^2\theta^{nm} - \partial^m\partial_n\Box^2\theta^{nl} + \gamma^{lm}\partial_n\partial_p\Box^2\theta^{np} = -16\pi J^{lm}, \tag{75}$$

where

$$J^{lm} = \Box h^{lm} - \partial^l\partial_n h^{nm} - \partial^m\partial_n h^{nl} + \gamma^{lm}\partial_n\partial_i h^{ni}.$$

Using the definition (70), we can write the field equations (75) as

$$\Box^2 f^{lm} = -16\pi J^{lm}. \tag{76}$$

It is easy to see that the gravitational field equations in either the form (75) or the form (76) are invariant under the gauge transformations (66) with arbitrary gauge vector a^n. If we take the total divergence with respect to one of the indices in the field equations (75) and (76), we obtain $0 = 0$ identically. Therefore, although the field φ_{ni} does have ten independent components, the structure of the equations is such that the four components with spins 1 and 0′ are automatically eliminated from the equations, with the consequence that the field equations contain only six independent components with spin 2 and 0. For them, we have six independent field equations, since the four conditions (70) hold by virtue of the gauge invariance.

The gravitational field equations (76) can be simplified by using the gauge transformation (66) and imposing subsidiary conditions on the field functions. The arbitrariness in the choice of the gauge means that when concrete problems are solved it is necessary to determine the gauge conditions explicitly in some manner, for example, by imposing subsidiary conditions. The fact that the Euler-Lagrange variation in a gauge theory satisfies the four identities $\partial_i \delta L/\delta\varphi_{im} = 0$ also means that when the field equations are solved in a concrete problem it is necessary to impose at least four subsidiary conditions on the field. Under the gauge transformations (66), it follows by virtue of the relation (71) that the fields θ_{nm} are subjected to the gauge transformation

$$\theta_{nm} \to \theta_{nm} + \partial_n a_m + \partial_m a_n - \gamma_{nm}\partial_k a^k. \tag{77}$$

The most general subsidiary conditions linear in the field $\Box^2\theta^{nm}$ are the conditions

$$\partial_n\Box^2\theta^{nm} = A\partial^m\Box^2\theta_n^n. \tag{78}$$

When (78) are satisfied, the gravitational field equations can be written in the form

$$\Box^3\theta^{lm} - 2A\partial^l\partial^m\,\Box^2\theta_n^n + A\gamma^{ml}\Box^3\theta_n^n = -16\pi J^{lm}.$$

It is easy to see that the left-hand sides of the equations are also conserved when the subsidiary conditions (78) are taken into account. For $A = 0$, we obtain the gravitational field equations in the simplest form

$$\Box^3\theta^{lm} = -16\pi J^{lm} \tag{79}$$

with the subsidiary conditions

$$\partial_n\Box^2\theta^{nm} = 0. \tag{80}$$

Thus, the gravitational field equations in our case are equations with higher derivatives, and (79) are also invariant under the gauge transformations (77) that do not violate the subsidiary conditions (80).

We introduce the field H^{nm} in accordance with the equation

$$\Box H^{nm} = h^{nm}, \qquad (81)$$

and then Eqs. (79) become

$$\Box^3\theta^{lm} = -16\pi\Box\{h^{lm} - \partial^l\partial_n H^{nm} - \partial^m\partial_n H^{nl} + \gamma^{lm}\partial_n\partial_p H^{np}\}.$$

Since in what follows we shall be interested only in causally determined solutions, we can in accordance with Ref. 80 "divide" these equations by the d'Alembert operator. Introducing the notation

$$\psi_{nm} = \Box\theta_{nm}, \qquad (82)$$

we obtain the gravitational field equations for causally determined solutions in the form

$$\Box\psi^{lm} = -16\pi\{h^{lm} - \partial^l\partial_n H^{nm} - \partial^m\partial_n H^{nl} + \gamma^{lm}\partial_n\partial_p H^{np}\}.$$

The tensor current on the right-hand side of this equation satisfies outside the source the condition

$$\Box\{h^{lm} - \partial^l\partial_n H^{nm} - \partial^m\partial_n H^{nl} + \gamma^{lm}\partial_n\partial_p H^{np}\} = 0.$$

Therefore, outside the matter this tensor current can be eliminated by a gauge transformation. Indeed, since the subsidiary conditions (80) admit the transformations (77) with gauge 4-vector satisfying

$$\Box^3 a^n = 0, \qquad (83)$$

we have the possibility of making the gauge transformation

$$\psi^{nm} \to \psi^{nm} + \partial^n\Box a^m + \partial^m\Box a^n - \gamma^{nm}\partial_l\Box a^l. \qquad (84)$$

Outside the source, we choose as gauge 4-vector a vector satisfying the condition $\Box^2 a^n = 16\pi\partial_m H^{mn}$. Since the equations $\Box H^{nm} = 0$ hold outside the source, and the gauge 4-vector satisfies Eq. (83) in this region, the subsidiary conditions (80) are satisfied automatically as a result. After the gauge transformation (84), we obtain outside the source the gravitational field equations in the form

$$\Box\psi^{nm} = 0.$$

This means that the tensor current

$$I^{nm} = h^{nm} - \partial^n\partial_l H^{lm} - \partial^l\partial^m H^n_l + \gamma^{nm}\partial_l\partial_p H^{pl} \qquad (85)$$

is nonzero only within the matter. Therefore, in the given gauge the gravitational field equations become

$$\Box\psi^{nm} = -16\pi I^{nm}. \qquad (86)$$

These equations admit the gauge transformations (77) on the class of vectors that satisfy the condition $\Box^2 a^n = 0$. Therefore, we shall solve Eqs. (86) with the subsidiary conditions $\partial_l\psi^{ln} = 0$, which leave the possibility of making gauge transformations only on this class. This choice of the subsidiary conditions is in accordance with Fock's theorem,[13] according to which a solution of the homogeneous wave equation $\Box\partial_l\psi^{ln} = 0$ that is bounded in the whole of space and satisfies the Sommerfeld radiation condition vanishes identically: $\partial_l\psi^{ln} = 0$.

Thus, we obtain the gravitational field equations

$$\Box\psi^{nm} = -16\pi I^{nm} \qquad (87)$$

with the subsidiary conditions

$$\partial_n\psi^{nm} = 0. \qquad (88)$$

We note further that the expression $\Box f_{nm}$ can, using the notation (82), be written in the form

$$\Box f_{nm} = \Box\psi_{nm} - \partial_n\partial^l\psi_{lm} - \partial_m\partial^l\psi_{ln} + \gamma_{nm}\partial^l\partial^p\psi_{lp}.$$

This expression is also invariant under the transformations (84) with any gauge vector a^n, but the operator $\Box f_{nm}$ in this case has the original form. We can simplify this operator if we bear in mind that in the gauge we have chosen the subsidiary conditions (88) are satisfied. In this case,

$$\Box f_{nm} = \Box\psi_{nm}. \qquad (89)$$

Note that the operator $\Box f_{nm}$ (89) is also invariant under the gauge transformations (84) that do not violate the subsidiary conditions (88). The relations (89) make it possible to rewrite the gravitational field equations in the form

$$\Box f^{nm} = -16\pi I^{nm} \qquad (90)$$

with the subsidiary conditions

$$\partial_n f^{nm} = 0. \qquad (91)$$

It should be emphasized especially that the tensor current I^{nm} on the right-hand side of (90) is concentrated exclusively in the matter. We note also that the equations (90) of the field theory of gravitation can be formulated not only for inertial but also for noninertial coordinate systems, and that on the transition from one noninertial coordinate system to another the field equations are form-invariant for each infinite set of noninertial coordinate systems. In inertial coordinate systems, the field equations are Lorentz-invariant on the transition from one inertial system to another. This leads us to the necessity of extending[30] the principle of relativity, which we formulate in the following form: There are no physical phenomena, not even gravitational, by means of which it is possible to establish whether we are at rest or in a state of uniform translational motion.

We emphasize that the principle of relativity does not require the propagation velocity of an electromagnetic wave front—the velocity of light—to be constant. It is natural that in the presence of an interaction with external gravitational fields the velocity of light, like the velocity of motion of any bodies, will not be constant.

7. MINIMAL-COUPLING EQUATION

To close the theoretical scheme, we must now specify the equation connecting the metric tensor g_{ni} of the effective Riemannian space-time to the gravitational field f_{ni}.

Since the choice of the connection equation in the field theory of gravitation is equivalent to the choice of the Lagrangian density of the interaction between the gravitational field and the other matter fields, we shall construct the connection equation in the same way as the interaction Lagrangian density is constructed in theories of other physical fields. For example, in electrodynamics the "minimal Lagrangian" is chosen as the interaction Lagrangian density.

Therefore, in the field theory of gravitation too it is appropriate to choose as the connection equation the minimal-coupling equation, which is the minimum necessary for describing the existing experiments for a weak gravitational field. In the usually considered linear approximation, the tensor current I^{nm} in Eq. (85) must be taken in the absence of a gravitational field. Since in this approximation the only physical symmetric tensor of second rank satisfying a conservation law is the energy-momentum tensor of the matter, we require the following correspondence to hold: In the zeroth approximation in the gravitational field, the tensor current I^{nm} must automatically go over into the energy-momentum tensor of the matter:

$$I^{nm}(f_{lp}=0)=T^{nm}. \qquad (92)$$

This requirement makes it possible to establish uniquely the structure of the connection equation $g_{ni}=g_{ni}(f_{lm})$ in the linear approximation. Indeed, using the expressions (74), (81), and (85), we find that the correspondence requirement (92) leads to the following equation in the linear approximation:

$$g_{nm}=\gamma_{nm}+f_{nm}-\gamma_{nm}f/2. \qquad (93)$$

One could imagine that the relation (93) is the minimal-coupling equation and is satisfied always and not only in the linear approximation in the weak field f_{nm}. But then a theory with such a connection equation would belong to the class of so-called quasilinear theories of gravitation (in Will's terminology). However, as is shown in Ref. 81, any quasilinear, asymptotically Lorentz-invariant theory of gravitation contradicts the results of experiments. Therefore, the relation (93) must be only an expansion of the minimal-coupling equation to terms linear in the weak field f_{nm}. Thus, the minimal-coupling equation must be a quadratic equation in the field f_{nm}:

$$g_{nm}=\gamma_{nm}+f_{nm}-\gamma_{nm}f/2 + [b_1 f_{nl} f^l_m + b_2 f_{nm} f + b_3 \gamma_{nm} f_{il} f^{il} + b_4 \gamma_{nm} f^2]/4, \qquad (94)$$

with as yet undetermined minimal-coupling parameters b_1, b_2, b_3, b_4.

As we shall see in what follows, the condition for the post-Newtonian expressions for the inertial and gravitational masses of a static spherically symmetric body to be equal leads to the relation $2(b_1+b_2+b_3+b_4)=1$ between the minimal-coupling parameters.

One could consider other, more complicated connection equations that only in the weak-field approximation go over into the minimal-coupling equation (94). However, at the present time there are no grounds for such a complication, since Eq. (94) describes all gravitational experiments.

Therefore, all the following treatment will be based on Eq. (94), and as the main physical requirement, which imposes definite restrictions on the values of the minimal-coupling parameters, we shall take the condition that there should be no singularities in the metric of the effective Riemannian space-time for finite values of the matter density in the source of the gravitational field. This assumption precludes the appearance in the field theory of gravitation of objects like black holes. In addition, we require that the theory should not throw up a paradox of Olbers type in a description of a model of the Universe.

It should be noted that, by virtue of the minimal-coupling equation (94), the nondiagonal components of the metric tensor g_{nm} of the Riemannian space-time can be nonzero even when the nondiagonal components of the gravitational field f_{nm} are equal to zero.

If the nondiagonal components of the tensor g_{nm} are to vanish when the corresponding nondiagonal components of the gravitational field vanish, it is necessary and sufficient to set $b_1=0$. In this case, we arrive at the simplest minimal-coupling equation:

$$g_{nm}=\gamma_{nm}+f_{nm}-\gamma_{nm}f/2 + [b_2 f_{nm} f + b_3 \gamma_{nm} f_{il} f^{il} + b_4 \gamma_{nm} f^2]/4. \qquad (95)$$

The condition of equality of the post-Newtonian expressions for the gravitational and inertial masses of a static spherically symmetric gody requires the simplest minimal-coupling parameters to satisfy $2(b_2+b_3+b_4)=1$.

8. CONSERVATION LAWS IN THE FIELD THEORY OF GRAVITATION

In Sec. 4, we obtained conservation laws that hold for all theories of gravitation of class A. The existence in theories of this class of the differential conservation law (62) for the total symmetric energy-momentum tensor density of the system in the flat space-time makes it possible to obtain a corresponding integral conservation law.

In Cartesian coordinates,

$$\partial_n [t_g^{ni}+t_M^{ni}]=0. \qquad (96)$$

Integrating this expression over a certain volume V for $i=0$ and assuming that there are no matter fluxes through the surface bounding this volume, we obtain

$$-\frac{\partial}{\partial t}\int dV\,[t_g^{00}+t_M^{00}]=\int dS_\alpha t_g^{0\alpha}. \qquad (97)$$

Thus, when gravitational waves are emitted the source energy must change. Further, if gravitational waves carry positive energy, the source energy must decrease.

All these conclusions and relations are also valid for the field theory of gravitation, which is a concrete representative of theories of class A. Since the symmetric and canonical energy-momentum tensors differ by the divergence of an antisymmetric tensor of third rank, the conservation laws (62) and (96) also hold for the canonical energy-momentum tensor.

The canonical energy-momentum tensor of the free gravitational field can be obtained as follows. We write down the equation

$$\frac{\partial L_g}{\partial x^p}=\partial_n\left[\frac{\partial L_g}{\partial(\partial_n f_{lm})}\partial_p f_{lm}\right]-\partial_p f_{lm}\partial_n\left[\frac{\partial L_g}{\partial(\partial_n f_{lm})}\right]. \qquad (98)$$

In accordance with (90), the free gravitational field satisfies the equation

$$\partial_n\,[\partial L_g/\partial(\partial_n f_{lm})] = \Box f_{lm} = 0,$$

and therefore the expression (98) means that the divergence of the canonical energy-momentum tensor of the free gravitational field is equal to zero. Hence, we obtain

$$\tilde{t}^n_{gp} = -L_g\delta^n_p + \frac{\partial L_g}{\partial(\partial_n f_{lm})}\partial_p f_{lm}. \tag{99}$$

Using the expression (72) for the Lagrangian density of the free gravitational field, we obtain

$$\tilde{t}^n_{gp} = \frac{1}{64\pi}\Big\{-\delta^n_p\Big[\partial_i f_{lm}\partial^i f^{lm} - \frac{1}{2}\partial_i f\partial^i f\Big] + 2\partial_p f_{lm}\partial^n f^{lm} - \partial_p f\partial^n f\Big\}. \tag{100}$$

To obtain the symmetric energy-momentum tensor t^{ni}_g of the gravitational field, we must write the Lagrangian density L_g of the gravitational field and the expression for f_{ni} in a manifestly covariant form. Going over in the expression (72) from the Cartesian coordinate system to an arbitrary curvilinear system, we obtain

$$L_g = \frac{\sqrt{-\gamma}}{64\pi}\gamma^{ik}\Big[\gamma^{ln}\gamma^{mp} - \frac{1}{2}\gamma^{lm}\gamma^{np}\Big]D_i f_{lm} D_k f_{np}. \tag{101}$$

Similarly, from the expression (69) we obtain

$$f_{ik} = \gamma^{lm}[D_l D_m \varphi_{ik} - D_i D_l \varphi_{mk} - D_k D_l \varphi_{mi} + D_i D_k \varphi_{lm} + \gamma_{ik}\gamma^{pn}(D_l D_n \varphi_{mp} - D_n D_p \varphi_{lm})]. \tag{102}$$

To simplify the following expressions, we also introduce the notation

$$\begin{aligned}\Lambda^{ik} = {} & -A^{lm}[\partial_l\partial_m\varphi^{ik} - \partial^i\partial_l\varphi^k_m - \partial^k\partial_l\varphi^i_m + \partial^i\partial^k\varphi_{lm}] + fA^{ik}/2\\ & + A^n_n[f^{ik} - \gamma^{ik}f/2] + \partial_s\{\varphi^i_n[-\partial^s A^{kn} + 2\partial^n A^{sk} + 2\gamma^{sk}\partial_l A^{ln}\\ & - \gamma^{kn}\partial_l A^{ls} - \partial^k A^{sn} + \gamma^{kn}\partial^s A^l_l - 2\gamma^{sk}\partial^n A^l_l]\\ & + \varphi^s_n[\partial^i A^{kn} - \partial^n A^{ik} - \gamma^{ik}\partial_l A^{ln} + \gamma^{ik}\partial^n A^l_l] + 2\gamma^{ks}A^{np}\partial^i\varphi_{np}\\ & - A^{sn}\partial^i\varphi^k_n - 3A^{kn}\partial^i\varphi^s_n + 2A^{ks}\partial^i\varphi^n_n - \gamma^{ik}A^{nm}\partial^s\varphi_{nm} + 3A^{kn}\partial^s\varphi^i_n\\ & - A^{ik}\partial^s\varphi^n_n - 2\gamma^{sk}A^{ln}\partial_l\varphi^i_n - 2A^{sk}\partial_l\varphi^{li} + A^{ns}\partial_n\varphi^{ik} + \gamma^{ik}A^{ln}\partial_l\varphi^s_n\\ & + A^{ik}\partial_n\varphi^{ns} + A^l_l[2\partial^i\varphi^{ks} - 2\gamma^{ks}\partial^i\varphi^n_n - 2\partial^s\varphi^{ik} + \gamma^{ik}(\partial^s\varphi^n_n - \partial_n\varphi^{ns})\\ & + 2\gamma^{ks}\partial_n\varphi^{ni}]\}/2.\end{aligned} \tag{103}$$

The symmetric energy-momentum tensor of the gravitational field can be obtained by substituting the expressions (101) and (102) in the relation (55). In a Cartesian coordinate system, we have

$$\begin{aligned}t^{ik}_g = {} & \frac{1}{64\pi}\Big\{-\gamma^{ik}\Big[\partial_l f_{nm}\partial^l f^{nm} - \frac{1}{2}\partial_n f\partial^n f\Big] + 2\partial^i f_{nm}\partial^k f^{nm} - \partial^i f\partial^k f\Big\}\\ & + \frac{1}{16\pi}\Big\{\partial_l f^{ni}\partial^l f^k_n - \frac{1}{2}\partial_l f^{ik}\partial^l f\Big\} - \frac{1}{32\pi}\partial_l\{f^i_p[\partial^l f^{kp} + \partial^k f^{lp}] - f^{ik}\partial^l f\\ & + f^k_n[\partial^l f^{ni} + \partial^i f^{nl}] - f^{nl}[\partial^i f^k_n + \partial^k f^i_n]\} - 2\Lambda^{(ik)},\end{aligned} \tag{104}$$

where, as usual, symmetrization is carried out with respect to indices within the round brackets:

$$\Lambda^{(ik)} = (\Lambda^{ik} + \Lambda^{ki})/2.$$

The tensor A^{nm} in the expression (103) has in this case the form

$$A^{nm} = -(1/32\pi)\,\Box\,[f^{nm} - \gamma^{nm}f/2].$$

Outside the matter $\Box f_{nm} = 0$, and therefore the expression for t^{ik}_g simplifies appreciably:

$$t^{ik}_g = \tilde{t}^{ik}_g + \frac{1}{32\pi}\partial_l\{f^l_n[\partial^i f^{kn} + \partial^k f^{ni}] - f^i_n\partial^k f^{nl} - f^k_n\partial^i f^{nl}\}, \tag{105}$$

where $\tilde{t}^{ik}_g$ is the canonical energy-momentum tensor (100) of the free gravitational field.

We show that in the wave zone the symmetric energy-momentum tensor t^{ik}_g of the gravitational field differs from the canonical tensor $\tilde{t}^{ik}_g$ only by nonwave terms which decrease faster than $1/r^2$. In the wave zone, we have the expansion

$$f_{nm} = \frac{a_{nm}(t-r,\theta,\varphi)}{r} + O\Big(\frac{1}{r^2}\Big),$$

and therefore for an arbitrary function $F(f_{nm})$

$$\partial_\alpha F = n_\alpha\frac{\partial}{\partial t}F + O\Big(\frac{1}{r}F\Big),$$

where $n_\alpha = x_\alpha/r$. Therefore, the expression (105) can be written in the form

$$\begin{aligned}t^{ik}_g = {} & \tilde{t}^{ik}_g + \frac{1}{32\pi}\frac{\partial}{\partial t}\{[f^{0l} + n_\alpha f^{\alpha l}][\partial^i f^k_l + \partial^k f^i_l] - f^i_n\partial^k[f^{0n} + n_\alpha f^{\alpha n}]\\ & - f^k_l\partial^i[f^{0l} + n_\alpha f^{\alpha l}]\} + O\Big(\frac{1}{r^3}\Big).\end{aligned}$$

Denoting differentiation with respect to the time by a dot, we obtain from the subsidiary conditions (91)

$$\dot{f}^{0m} + n_\alpha\dot{f}^{\alpha m} = O(1/r^2). \tag{106}$$

Integrating this expression with respect to the time and setting the constants of integration equal to zero, since waves should not have a time-dependent part, we obtain

$$f^{0m} + n_\alpha f^{\alpha m} = O(1/r^2). \tag{107}$$

It follows from this that in the wave zone the symmetric energy-momentum tensor of the gravitational field differs from the canonical one by a nonwave quantity that decreases faster than $1/r^2$ with increasing r:

$$t^{ik}_g = \tilde{t}^{ik}_g + O(1/r^3). \tag{108}$$

Therefore, in the wave zone calculations made using the symmetric and the canonical energy-momentum tensors of the gravitational field give the same result. These tensors are also equivalent for the calculation of the integrated characteristics of gravitational radiation. Indeed, from the expression (105) we obtain

$$t^{00}_g = \tilde{t}^{00}_g + \frac{1}{16\pi}\partial_\alpha\{f^{\alpha l}\dot{f}^0_l - f^0_l\dot{f}^{\alpha l}\},$$

and therefore

$$\int t^{00}_g\,dV = \int\tilde{t}^{00}_g\,dV + \frac{1}{16\pi}\int dS_\alpha[f^{\alpha l}\dot{f}^0_l - f^0_l\dot{f}^{\alpha l}].$$

If the boundary of the region of integration is in the wave zone, then on the basis of Eqs. (106) and (107) we have

$$f^{\alpha l}\dot{f}^0_l - f^0_l\dot{f}^{\alpha l} = n_\beta[f^{\alpha l}\dot{f}^\beta_l - \dot{f}^{\alpha l}f^\beta_l] + O(1/r^3).$$

Taking as surface of integration a sphere of radius r ($dS_\alpha = -r^2 n_\alpha d\Omega$), we obtain

$$\int dV t^{00}_g = \int dV\tilde{t}^{00}_g + O\Big(\frac{1}{r}\Big). \tag{109}$$

In addition, it follows from (108) that

$$\int t^{0\alpha}_g dS_\alpha = \int\tilde{t}^{0\alpha}_g dS_\alpha + O\Big(\frac{1}{r}\Big). \tag{110}$$

Thus, from the expressions (109) and (110) it is obvious that the two tensors are equivalent for calculating the integrated characteristics of gravitational radiation. As will be shown in Sec. 9, the components $\tilde{t}^{00}_g$ and $\tilde{t}^{0\alpha}_g$ are quantities of positive sign, and, moreover, only the transverse component of a gravitational wave contribute to the energy and the momentum. Therefore, on the basis of the expression (97) the source energy decreases when waves are radiated.

To obtain the symmetric energy-momentum tensor density t_M^{in} of the matter in flat space-time, we note that the metric tensor γ_{ni} occurs in the matter Lagrangian density only through the metric tensor of the Riemannian space-time. Therefore, the tensor density t_M^{ni} can be written in the form

$$t_M^{ni} = T^{ni}[1 - f/2 + (b_3/4) f_{lm}f^{lm} + (b_4/2) f^2] + f^{ni}T^{lm}\gamma_{lm} - [b_1 T^{lm} f_l^n f_m^i + b_2 T^{lm} f_{lm} f^{ni} + 2b_3 f_p^i f^{np} T^{lm}\gamma_{lm} + 2b_4 f^{ni} f T^{lm}\gamma_{lm}]/4 - 2\Lambda^{(ni)}. \qquad (111)$$

We obtain the expression for Λ^{ni} from (103) by setting

$$A^{lm} = -T^{lm}/2 + \gamma^{lm}T^{ni}\gamma_{ni}/4 - [b_1 T^{ln} f_n^m + b_1 T^{nm} f_n^l + b_2 \gamma^{lm} T^{ni} f_{ni} + b_2 T^{lm} f + 2b_3 f^{lm} T^{ni}\gamma_{ni} + 2b_4 \gamma^{lm} f T^{ni}\gamma_{ni}]/8. \qquad (112)$$

9. EMISSION OF GRAVITATIONAL WAVES IN THE FIELD THEORY OF GRAVITATION

One of the most important problems in the theory of gravitation and in the whole of modern physics is the problem of the emission and detection of gravitational waves. The recent interest in this problem has led to a large number of experimental and theoretical studies aimed both at improving the experimental methods and techniques as well as at the design of possible emitters and detectors of gravitational waves. The great interest in these questions is explained by the fact that the problem of gravitational waves is important both theoretically and for applications. Several theories of gravitation have by now been proposed that give a fairly satisfactory description of the existing post-Newtonian experiments but differ strongly in their description of gravitational waves. The experimental proof of the existence of gravitational waves and the study of their properties will permit not only a choice of the theory that describes reality adequately but also a further improvement of it. In addition, the existence of gravitational waves and the possibility of their detection will open up the field of gravitational-wave astronomy and new communication channels.

The problem of the emission and detection of gravitational waves in the field theory of gravitation contains a number of aspects. In the present paper, we concentrate our main attention on only some of them, namely, we shall investigate wave solutions of the field theory of gravitation in the weak-field approximation, we shall study the interaction of weak gravitational waves with the field of a magnetic dipole, and we shall also mention a number of gravitational-wave experiments that make it possible to verify the predictions of the field theory of gravitation and Einstein's general relativity with regard to the properties of weak gravitational waves in the presence of external gravitational fields.

The gravitational field equations of the field theory of gravitation in the gauge we have chosen are

$$\Box f^{lm} = -16\pi I^{lm}, \qquad (113)$$

and the tensor current I^{lm} (85) is defined only in the matter.

Since the metric tensor g_{nm}, and also the energy-momentum tensor of the free gravitational field, i.e., the field outside matter, depend only on the fields f_{nm}, we shall solve the field equations (113) for f_{nm}. We write the tensors f^{lm} and I^{lm} as Fourier integrals with respect to the time. In the spectrum $I^{ln}(\omega, \mathbf{r})$, we separate the static part $J_0(\mathbf{r})$. It is obvious that the static part of the tensor current $J_0(\mathbf{r})$ will give only static solutions, and we therefore omit it; then for the Fourier amplitudes we obtain the gravitational field equations

$$\Delta\tilde{f}^{nm} + \omega^2\tilde{f}^{nm} = 16\pi\tilde{I}^{nm}. \qquad (114)$$

We place the origin of a Cartesian coordinate system at some point of the source, and then in this system we can write the solution of the field equations in the form

$$R = |\mathbf{r} - \mathbf{r}'|;$$

$$\tilde{f}^{nm} = -4\int \frac{\exp(i\omega R)}{R}\tilde{I}^{nm}(\mathbf{r}', \omega)\, d^3\mathbf{r}'. \qquad (115)$$

Using the Lorentz conditions (91), $i\omega\tilde{f}^{0m} = \partial_\alpha\tilde{f}^{\alpha m}$, we express the components $\tilde{f}^{0n}$ in terms of the spatial components:

$$\tilde{f}^{00} = -\frac{1}{\omega^2}\partial_\alpha\partial_\beta\tilde{f}^{\alpha\beta}; \quad \tilde{f}^{0\alpha} = -\frac{i}{\omega}\partial_\beta\tilde{f}^{\alpha\beta}.$$

Outside the source of the gravitational waves, choosing the gauge

$$f^{nm} \to f^{nm} + \partial^n a^m + \partial^m a^n - \gamma^{nm}\partial_l a^l, \qquad (116)$$

which is compatible with the Lorentz condition (91) for $\Box a^n = 0$, we can impose on the wave components f'^{nm} a further four conditions in accordance with the number of independent gauge vectors. As such conditions, we can choose $\tilde{f}' = 0, \tilde{f}'^{0\alpha} = 0$ (the TT gauge).

As a result of this gauge, we obtain

$$\tilde{f}'^{\alpha\beta} = \tilde{f}^{\alpha\beta} - \gamma^{\alpha\beta}\tilde{f}/2 - \left(\frac{i}{\omega}\right)[\partial^\beta\tilde{f}^{0\alpha} + \partial^\alpha\tilde{f}^{0\beta}] - \frac{1}{\omega^2}\partial^\alpha\partial^\beta[\tilde{f}^{00} - \tilde{f}/2]. \qquad (117)$$

Using the Lorentz conditions (91), we can write these expressions in the form

$$\tilde{f}'^{\alpha\beta} = \tilde{P}^{\alpha\beta} - \frac{1}{\omega^2}[\partial^\beta\partial_\eta\tilde{P}^{\alpha\eta} + \partial^\alpha\partial_\eta\tilde{P}^{\beta\eta}] + \frac{1}{2\omega^2}\gamma^{\alpha\beta}\partial_\eta\partial_\gamma\tilde{P}^{\eta\gamma} + \frac{1}{2\omega^4}\partial^\alpha\partial^\beta\partial_\gamma\partial_\eta\tilde{P}^{\gamma\eta}, \qquad (118)$$

where we have introduced the notation

$$\tilde{P}^{\alpha\beta} = \tilde{f}^{\alpha\beta} - \gamma^{\alpha\beta}\tilde{f}_\eta^\eta/3. \qquad (119)$$

Thus, a wave solution to the field equations contains in the general case six nonvanishing spatial components $\tilde{f}'^{\alpha\beta}$, but among them only two components are independent on account of the three Lorentz conditions (91) (the fourth Lorentz condition is trivial by virtue of the TT gauge) and the vanishing $\tilde{f}' = 0$ of the trace. These subsidiary conditions are the well-known subsidiary conditions for an irreducible representation with spin 2 in the TT gauge, and therefore a free gravitational wave has spin 2, and the scalar component corresponding to the irreducible representation with spin 0 is not radiated in the form of gravitational waves.

It is customary to write the wave solutions to the equations of the gravitational field in a somewhat different form making it possible to demonstrate clearly the quadrupole nature of the emitted gravitational waves.

In our case, it is also possible to express the obtained solution in terms of generalized quadrupole moments of the tensor current I^{nm}. For this we note that the spatial components $f^{\alpha\beta}$ (115) can be written on the basis of the conservation $\partial_n I^{nm}=0$ of the tensor current in the form

$$\tilde{f}^{\alpha\beta}=2\omega^2\left\{\int \frac{\exp(i\omega R)}{R}\,\tilde{I}^{00}x^\alpha x^\beta\,dV+\frac{2i}{\omega}\partial_\eta\int\frac{\exp(i\omega R)}{R}\,\tilde{I}^{0\eta}x^\alpha x^\beta\,dV -\frac{1}{\omega^2}\partial_\eta\partial_\gamma\int\frac{\exp(i\omega R)}{R}\,\tilde{I}^{\eta\gamma}x^\alpha x^\beta\,dV\right\}. \quad (120)$$

This relation is exact. It simplifies significantly if the linear dimensions of the source are appreciably smaller than the distance from its center to the point of of observation. Omitting the nonwave terms, which decrease faster than $1/r$, we obtain

$$\tilde{f}^{\alpha\beta}=\frac{2\omega^2}{r}\int dV x^\alpha x^\beta \exp(i\omega R)\,[\tilde{I}^{00}+2n_\varepsilon\tilde{I}^{0\varepsilon}+n_\varepsilon n_\gamma\tilde{I}^{\varepsilon\gamma}],$$

where $n_\alpha=x_\alpha/r$ and $n_\alpha n^\alpha=-1$. Then (119) can be written in the form

$$\tilde{P}^{\alpha\beta}=\frac{2\omega^2}{r}\int dV\left[x^\alpha x^\beta-\frac{1}{3}\gamma^{\alpha\beta}x_\varepsilon x^\varepsilon\right] \times\exp(i\omega R)\,[\tilde{I}^{00}+2n_\varepsilon\tilde{I}^{0\varepsilon}+n_\varepsilon n_\gamma\tilde{I}^{\varepsilon\gamma}]. \quad (121)$$

Introducing the projection operators

$$Z^{\alpha\beta}=\gamma^{\alpha\beta}+n^\alpha n^\beta, \quad (122)$$

which satisfy the conditions

$$Z^{\alpha\beta}\gamma_{\alpha\beta}=2;\quad Z^{\alpha\beta}Z_{\beta\varepsilon}=Z^\alpha_\varepsilon,$$

we rewrite (118) as

$$\tilde{f}'^{\alpha\beta}=[Z^\alpha_\varepsilon Z^\beta_\gamma-Z^{\alpha\beta}Z_{\varepsilon\gamma}/2]\,\tilde{P}^{\varepsilon\gamma}. \quad (123)$$

Substituting the expression (121) in the Fourier integral, we obtain

$$P^{\alpha\beta}=-\frac{2}{r}\frac{d^2}{dt^2}\int dV\left(x^\alpha x^\beta-\frac{1}{3}\gamma^{\alpha\beta}x_\varepsilon x^\varepsilon\right)[I^{00}+2n_\varepsilon I^{0\varepsilon}+n_\varepsilon n_\gamma I^{\varepsilon\gamma}]_{\text{ret}}. \quad (124)$$

Here $[\ldots]_{\text{ret}}$ means that the expression in the square brackets is taken at the retarded time $t'=t-R$. If we introduce the traceless generalized quadrupole-moment tensor

$$\mathscr{D}^{\alpha\beta}=D^{\alpha\beta}+2n_\varepsilon D^{\alpha\beta\varepsilon}+n_\varepsilon n_\gamma D^{\alpha\beta\varepsilon\gamma}, \quad (125)$$

where

$$\left.\begin{aligned} D^{\alpha\beta}&=\int dV\,(3x^\alpha x^\beta-\gamma^{\alpha\beta}x_\varepsilon x^\varepsilon)\,[I^{00}]_{\text{ret}};\\ D^{\alpha\beta\varepsilon}&=\int dV\,(3x^\alpha x^\beta-\gamma^{\alpha\beta}x_\delta x^\delta)\,[I^{0\varepsilon}]_{\text{ret}};\\ D^{\alpha\beta\varepsilon\gamma}&=\int dV\,(3x^\alpha x^\beta-\gamma^{\alpha\beta}x_\delta x^\delta)\,[I^{\varepsilon\gamma}]_{\text{ret}},\end{aligned}\right\} \quad (126)$$

then the components (123) of the gravitational wave can be written in the form

$$f'^{\alpha\beta}=-\frac{2}{3r}\left(Z^\alpha_\varepsilon Z^\beta_\gamma-\frac{1}{2}Z^{\alpha\beta}Z_{\varepsilon\gamma}\right)\ddot{\mathscr{D}}^{\varepsilon\gamma}. \quad (127)$$

Here and in what follows, the dot denotes the derivative with respect to the time.

Bearing in mind that $\partial_\varepsilon f_{\alpha\beta}=n_\varepsilon\dot{f}_{\alpha\beta}$, for the components $\tilde{t}^\alpha_{g0}$ and $\tilde{t}^0_{g0}$ of the energy-momentum tensor of the gravitational wave we obtain the expression

$$\tilde{t}^\alpha_{g0}=n^\alpha\tilde{t}^0_{g0}=\frac{1}{32\pi}n^\alpha\dot{f}_{\beta\varepsilon}\dot{f}^{\beta\varepsilon}.$$

Then for the intensity of the radiation of energy of the gravitational waves into the element of solid angle $d\Omega$ we have

$$dI/d\Omega=(1/32\pi)\,r^2\dot{f}_{\alpha\beta}\dot{f}^{\alpha\beta}\geqslant 0. \quad (128)$$

It can be seen from this expression that the intensity is positive for all values of the components of the tensor $\dot{f}_{\alpha\beta}$ provided all components are not equal to zero. If they all vanish, $\dot{f}_{\alpha\beta}=0$, and then $dI/D\Omega=0$ as well.

Using the relations (122) and (127), we can write the expression (128) in the form

$$\frac{dI}{d\Omega}=\frac{1}{36\pi}\left\{\frac{1}{4}(\dddot{\mathscr{D}}^{\alpha\beta}n_\alpha n_\beta)^2+\frac{1}{2}\dddot{\mathscr{D}}_{\alpha\beta}\dddot{\mathscr{D}}^{\alpha\beta}+\dddot{\mathscr{D}}_{\alpha\beta}\dddot{\mathscr{D}}^{\beta\gamma}n_\gamma n^\alpha\right\}. \quad (129)$$

We consider first the case of emission of weak gravitational waves, which is the most common in practice. In the usually considered linear approximation, the tensor I^{nm} (85) must be taken subject to the condition that a gravitational field is absent. It follows from the expressions (74) and (94) that in this case

$$I^{nm}=T^{nm}. \quad (130)$$

When studying gravitational waves whose wavelength is appreciably greater than the source diameter, the retardation in the system can be ignored, and the expressions in the square brackets in (126) can be taken at the time $t'=t-R$. Then for the loss of energy in all directions per unit time we obtain the expression

$$I=-dE/dt=(G/45c^5)\,\dddot{D}_{\alpha\beta}\dddot{D}^{\alpha\beta}, \quad (131)$$

where

$$D^{\alpha\beta}=\int dV\,(3x^\alpha x^\beta-\gamma^{\alpha\beta}x_\varepsilon x^\varepsilon)\,T^{00}(t-R,\mathbf{r}')$$

and we have explicitly introduced the gravitational constant G and the velocity of light c.

This expression agrees with the results[82,83] of indirect measurements of the energy loss by the binary pulsar system PSR 1913+16 through the presumed radiation of gravitational waves. Since the calculation of the "energy loss" usually made in general relativity using energy-momentum pseudotensors in the weak-field approximation leads to the expression (131), it was concluded in Ref. 83 that the results of the observations agree with the prediction of Einstein's theory.

However, as is shown in Refs. 5 and 9, Eq. (131) is not a consequence of Einstein's general relativity. In Einstein's theory, one can speak only of curvature waves, with which transfer of energy to matter is associated, although conservation laws in their usual sense do not hold; as a result, calculation of the energy loss by a source and determination of the energy fluxes of gravitational waves are impossible in general relativity.

We shall now show why the calculation of the energy fluxes of gravitational waves in general relativity leads to physically acceptable expressions in a small class of coordinate systems. The main reason for this is that in pseudo-Euclidean space-time one can construct a theory of gravitation which possesses energy-momentum conservation laws for the matter and the gravitational field taken together:

$$D_i\,[t^{ik}_g+t^{ik}_M]=0. \quad (132)$$

Indeed, in this theory, from the conservation law (132) written down in a Cartesian coordinate system,

$$\partial_i\,[t^{ik}_g+t^{ik}_M]=0, \quad (133)$$

one can obtain the expression (128) for the intensity of the gravitational radiation.

For energy calculations in general relativity, one generally uses the relation

$$\partial_i \left[-g\left(T^{ik}+\tau^{ik}\right)\right]=0, \tag{134}$$

where τ^{ik} is the Landau-Lifshitz energy-momentum pseudotensor:

$$\begin{aligned}\tau^{ik}=\frac{c^4}{16\pi G}\{&(2\Gamma^n_{ml}\Gamma^p_{np}-\Gamma^n_{lp}\Gamma^p_{mn}-\Gamma^n_{nl}\Gamma^p_{mp})(g^{il}g^{km}-g^{ik}g^{lm})\\&+g^{il}g^{mn}(\Gamma^k_{pl}\Gamma^p_{mn}+\Gamma^k_{mn}\Gamma^p_{pl}-\Gamma^k_{np}\Gamma^p_{ml}-\Gamma^k_{ml}\Gamma^p_{np})\\&+g^{kl}g^{mn}(\Gamma^i_{pl}\Gamma^p_{mn}+\Gamma^i_{mn}\Gamma^p_{pl}-\Gamma^i_{np}\Gamma^p_{ml}-\Gamma^i_{ml}\Gamma^p_{np})\\&+g^{lm}g^{np}(\Gamma^i_{nl}\Gamma^k_{mp}-\Gamma^i_{ml}\Gamma^k_{np})\};\end{aligned} \tag{135}$$

the relation (134), naturally, differs from the covariant conservation law (132) of the field theory of gravitation. But in Cartesian coordinates, the expressions (133) and (134) have the same form. Moreover, in the lowest nonvanishing approximation the expression (135) for the components $\tau^{0\alpha}$ of the energy-momentum pseudotensor in Cartesian coordinates is equal to the expression for the components of the energy-momentum tensor of the field theory of gravitation:

$$(-g)\,\tau^{0\alpha}=t_g^{0\alpha}=\frac{G\dot{h}_{\beta\varepsilon}\dot{h}^{\beta\varepsilon}}{32\pi c^6 r^2}\,n^\alpha+O\left(\frac{1}{r^3}\right).$$

Therefore, in the Cartesian coordinate system general relativity appears to match the field theory of gravitation. It is for this reason that in the small class of coordinate systems close to a Cartesian system we obtain from the relation (134) in general relativity the expressions (30) for the "intensity of the gravitational radiation" and (31) for the "total intensity."

This circumstance created the illusion that Eqs. (30) and (31) for calculating the energy loss are a consequence of general relativity. However, because of the different transformation laws of the energy-momentum tensor of the field theory of gravitation and the energy-momentum pseudotensors of general relativity their expressions will be different in different coordinate systems even in the lowest nonvanishing approximation. Moreover, as is shown in Ref. 9, calculation of the intensity of gravitational radiation by means of the expression (134) leads to physically absurd results. In contrast, calculation of the energy fluxes of gravitational radiation in the field theory of gravitation has a well-defined physical meaning in any admissible coordinate system by virtue of the tensor nature of the conservation laws (132).

To see this, we consider the following simple example. Suppose a source of island type emits weak gravitational waves during an infinite interval of time, so that the radiation process can be regarded as steady and the influence of the initial conditions no longer significant. In this case, the metric tensor of the Riemannian space-time after transition to the TT gauge in the wave zone takes the form

$$g_{ni}=\gamma_{ni}+\frac{G}{c^4}\frac{h_{ni}}{r}, \tag{136}$$

where $h_{0i}=0$, and

$$h^{\alpha\beta}=-\tfrac{2}{3}[Z^\alpha_\varepsilon Z^\beta_\nu-\tfrac{1}{2}Z^{\alpha\beta}Z_{\varepsilon\nu}]\,\dddot{D}^{\varepsilon\nu}.$$

To simplify the following calculations, we introduce the notation $u=ct-r$ and besides the Cartesian coordinates also use the spherical coordinates $z=r\cos\theta$, $x=r\sin\theta\cos\varphi$, $y=r\sin\theta\sin\varphi$.

We also introduce the quantities

$$\mu_i=\partial u/\partial x^i;\quad n_i=-\partial r/\partial x^i;$$
$$m_i=r\partial\theta/\partial x^i;\quad l_i=r\sin\theta\,\partial\varphi/\partial x^i,$$

with the components

$$\begin{aligned}\mu_i&=\{1;\ -\sin\theta\cos\varphi;\ -\sin\theta\sin\varphi;\ -\cos\theta\};\\n_i&=\{0;\ -\sin\theta\cos\varphi;\ -\sin\theta\sin\varphi;\ -\cos\theta\};\\m_i&=\{0;\ \cos\theta\cos\varphi;\ \cos\theta\sin\varphi;\ -\sin\theta\};\\l_i&=\{0;\ -\sin\varphi;\ \cos\varphi;\ 0\}.\end{aligned}$$

If we use the metric tensor γ_{ni} to raise the indices of these quantities, then

$$\begin{aligned}n_in^i=m_im^i=l_il^i=n_i\mu_i=-1;\ \mu_i\mu^i&=n_im^i=n_il^i=\\=m_il^i=0;\quad \mu_im^i&=\mu_il^i=0.\end{aligned}$$

It is obvious that in our chosen Cartesian coordinates the intensity of the gravitational radiation and the total intensity will, on the basis of the expression (136), be determined by Eqs. (30) and (31), respectively.

We now make a transformation from the Cartesian coordinates x_c^α to the new coordinates x_H^α, which are related to the old coordinates by the relation $t_c=t_H$:

$$x_c^\alpha=x_H^\alpha\left\{1+\frac{GF(ct_H-r_H,\ \theta_H,\ \varphi_H)}{c^5 r_H^{3/2}}\left[1-\exp(-\varepsilon^2 r_H^2)\right]\right\}, \tag{137}$$

where F is some arbitrary function that is bounded for all values of $u=ct_H-r_H$ and angles θ_H and φ_H: $\max|F|=A<\infty$.

It is easy to see that the transformation (137) corresponds to a change in the arithmetization of the points of three-dimensional space along the radius:

$$r_c=r_H\left\{1+\frac{GF(u_H,\ \theta_H,\ \varphi_H)}{c^5 r_H^{3/2}}\left[1-\exp(-\varepsilon^2 r_H^2)\right]\right\}.$$

For this, we write down equations that follow from the expression (137):

$$\begin{aligned}\cos\theta_H&=z_H/r_H=z_c/r_c=\cos\theta_c;\\\sin\theta_H\cos\varphi_H&=x_H/r_H=x_c/r_c=\sin\theta_c\cos\varphi_c;\\\sin\theta_H\sin\varphi_H&=y_H/r_H=y_c/r_c=\sin\theta_c\sin\varphi_c.\end{aligned}$$

From this we obtain $\theta_H=\theta_c$ and $\varphi_H=\varphi_c$. Therefore, the values of the angles θ and φ of any point of space in the new coordinate system are equal to the values of the angles of this point in the old system. It must be emphasized especially that the Cartesian coordinates are fundamental, and we shall regard the spherical coordinates r, θ, and φ as nothing more than convenient notation for certain combinations of the Cartesian coordinates.

If the function F depends on the time, then the transformation (137) in the general case describes transition to a different coordinate system that executes radial motions with respect to the old coordinate system.

If the transformation is to have an inverse and to be one-to-one, it is necessary and sufficient for the condition $\partial r_c/\partial r_H>0$ to hold; then the Jacobian of the transformation (137) will also be nonzero:

$$J=\det\left\|\frac{\partial x_c}{\partial x_H}\right\|=\left[1+\frac{GF}{c^5 r_H^{3/2}}\left(1-\exp\left(-\varepsilon^2 r_H^2\right)\right)\right]^2\frac{\partial r_c}{\partial r_H}>0.$$

Since in our case

$$\frac{\partial r_c}{\partial r_H}=1-\frac{G\left(F+2r_H\,\partial F/\partial u\right)}{2c^5 r_H^{3/2}}\left[1-\exp\left(-\varepsilon^2 r_H^2\right)\right]+\frac{2GF\varepsilon^2}{c^5}\sqrt{r_H}\exp\left(-\varepsilon^2 r_H^2\right),$$

the condition $\partial r_c/\partial r_H>0$ will certainly be satisfied when

$$\frac{G\varepsilon^{1/2}}{c^5}\left\{2\varepsilon A\sqrt{q}\exp\left(-q^2\right)+\frac{A\varepsilon+2qB}{2q^{3/2}}\left[1-\exp\left(-q^2\right)\right]\right\}<1, \tag{138}$$

where $q=\varepsilon r_H$ and $B=\max\left|\partial F/\partial u\right|<\infty$. In this case, the transformation (137) will be nonsingular and one-to-one in the whole of space.

It is easy to see that by the choice of an appropriate value of ε this condition can always be satisfied. Indeed, for $q\geqslant 0$ the functions

$$f_1(q)=2\sqrt{q}\exp\left(-q^2\right)+\frac{1}{2q^{3/2}}\left[1-\exp\left(-q^2\right)\right];$$

$$f_2(q)=\frac{1}{\sqrt{q}}\left[1-\exp\left(-q^2\right)\right]$$

are non-negative and vanish for $q=0$ and $q\to\infty$, so that they must have maxima in the interval $0<q<\infty$.

It is obvious that the absolute maxima of these functions in the given interval are finite:

$$\operatorname{Max} f_1(q)=H<\infty;\quad \operatorname{Max} f_2(q)=L<\infty.$$

Therefore if the condition (138) is to hold, it is sufficient for ε to satisfy the condition $\varepsilon<\alpha^2$, where α is the unique real root of the equation

$$AH\alpha^3+BL\alpha-c^5/G=0.$$

We now calculate the energy flux and the total intensity of gravitational radiation in the new coordinates.

Using the transformation of the metric tensor

$$g_{ik}^H(x_H)=\frac{\partial x_c^l}{\partial x_H^i}\frac{\partial x_c^m}{\partial x_H^k}g_{lm}^c(x_c)$$

and retaining only the terms linear in the coupling constant G/c^4, we find that the asymptotic expression for the metric as $r_H\to\infty$ in the new coordinates has the form

$$g_{ik}=\gamma_{ik}+G/c^4\left[a_{ik}/\sqrt{r_H}+h_{ik}/r_H+b_{ik}/r_H^{3/2}\right]+O\left(\left[Ga_{ik}/c^4\sqrt{r_H}\right]^2\right), \tag{139}$$

where the components of the tensors a_{ik} and b_{ik} are

$$a_{ik}=\frac{1}{c}\frac{\partial F}{\partial u}\left(\mu_i n_k+\mu_k n_i\right);$$

$$b_{ik}=\frac{1}{c}F\left[2\gamma_{ik}-2\mu_i\mu_k+n_i n_k+2\left(\mu_i n_k+\mu_k n_i\right)\right]+\frac{1}{c}\frac{\partial F}{\partial\theta}\left[n_i m_k+n_k m_i\right]+\frac{1}{c\sin\theta}\frac{\partial F}{\partial\varphi}\left[n_i l_k+n_k l_i\right].$$

For the tensor h_{ik} in the new coordinates, we have $h_{0i}=0$,

$$h^{\alpha\beta}=-\frac{2}{3}\left[Z_\varepsilon^\alpha Z_\gamma^\beta-\frac{1}{2}Z^{\alpha\beta}Z_{\varepsilon\gamma}\right]\ddot{D}^{\varepsilon\gamma}(u_H)$$

It is easy to see that in the new coordinate system the metric tensor (139) is asymptotically Galilean:

$$g_{00}=1+O\left(\left[\frac{Ga_{ik}}{c^4 r_H^{1/2}}\right]^2\right);\quad g_{\alpha\beta}=\gamma_{\alpha\beta}+O\left(\frac{Ga_{ik}}{c^4\sqrt{r_H}}\right).$$

To be required accuracy, the determinant of the metric tensor (139) can be written in the form

$$-g=1-\frac{G}{c^5}\left[\frac{2}{\sqrt{r_H}}\frac{\partial F}{\partial u_H}-\frac{3F}{r_H^{3/2}}\right].$$

For the components of the connection of the Riemannian space-time, we obtain from the expression (139)

$$\begin{aligned}\Gamma_{ks}^i=\frac{G}{c^5}\Big\{&\frac{\partial^2 F}{\partial u_H^2}\frac{n^i\mu_k\mu_s}{\sqrt{r_H}}+\frac{1}{2r_H}\left[\dot{h}_k^i\mu_s+\dot{h}_s^i\mu_k-\dot{h}_{ks}\mu^i\right]\\&+\frac{1}{2r_H^{3/2}}\frac{\partial F}{\partial u_H}\left[-2\mu^i\mu_k\mu_s+n^i\left(\mu_k n_s+\mu_s n_k+4\mu_k\mu_s\right)+\right.\\&\left.+2\mu_k\delta_s^i+2\mu_s\delta_k^i-2\mu^i\gamma_{ks}-2\left(\mu^i+n^i\right)\left(m_k m_s+l_k l_s\right)\right]\\&+\frac{1}{2r_H^{3/2}}\frac{\partial^2 F}{\partial u_H\,\partial\theta}\left[n^i\left(\mu_k m_s+\mu_s m_k\right)+\mu^i\left(n_k m_s+n_s m_k\right)\right.\\&\left.-m^i\left(\mu_k n_s+\mu_s n_k\right)\right]+\frac{1}{2r_H^{3/2}\sin\theta}\frac{\partial^2 F}{\partial u_H\,\partial\varphi}\left[n^i\left(\mu_k l_s+\mu_s l_k\right)\right.\\&\left.+\mu^i\left(n_k l_s+n_s l_k\right)-l^i\left(\mu_s n_k+\mu_k n_s\right)\right]+O\left(\left[\frac{Ga_{ik}}{c^4\sqrt{r_H}}\right]^2\right).\end{aligned}$$

Substituting these expressions in the relation (135), for the components $\tau^{0\alpha}$ of the energy-momentum pseudotensor we have

$$-g\tau^{0\alpha}=\frac{G}{32\pi c^6 r^2}n^\alpha\left[\dot{h}_{\beta\varepsilon}\dot{h}^{\beta\varepsilon}-8\frac{\partial F}{\partial u}\frac{\partial^2 F}{\partial u^2}\right]+O\left(\frac{1}{r^{5/2}}\right).$$

Therefore, the intensity of the gravitational radiation emitted into the element of solid angle in the new coordinates will have the form

$$\frac{dI}{d\Omega}=\frac{G}{32\pi c^5}\left[\dot{h}_{\beta\varepsilon}\dot{h}^{\beta\varepsilon}-8\frac{\partial F}{\partial u}\frac{\partial^2 F}{\partial u^2}\right].$$

Using the relation

$$h^{\alpha\beta}=-\frac{2}{3}\left[Z_\varepsilon^\alpha Z_\gamma^\beta-\frac{1}{2}Z^{\alpha\beta}Z_{\varepsilon\gamma}\right]\ddot{D}^{\varepsilon\gamma},$$

we obtain

$$\frac{dI}{d\Omega}=\frac{G}{36\pi c^5}\left\{\frac{1}{4}\left(\dddot{D}_{\alpha\beta}n^\alpha n^\beta\right)^2+\frac{1}{2}\dddot{D}^{\alpha\beta}\dddot{D}_{\alpha\beta}+\dddot{D}_{\alpha\beta}\dddot{D}^{\beta\gamma}n^\alpha n_\gamma-9\frac{\partial F}{\partial u}\frac{\partial^2 F}{\partial u^2}\right\}. \tag{140}$$

Integrating this expression over all directions, we obtain the total radiation in the new coordinates:

$$I=\frac{G}{45c^5}\left[\dddot{D}_{\alpha\beta}\dddot{D}^{\alpha\beta}-\frac{45}{4\pi}\int d\Omega\frac{\partial F}{\partial u}\frac{\partial^2 F}{\partial u^2}\right]. \tag{141}$$

Thus, both the intensity of the gravitational radiation and the total intensity as calculated in general relativity using an energy-momentum pseudotensor depend on the choice of the coordinates even under the condition of an asymptotically Galilean metric and can be reduced to a physically absurd form by an appropriate choice of the coordinates. To see this, we consider the following two special cases of the function F.

1. Suppose F has the form

$$F=\pm\frac{a}{3}\int_{-\infty}^{u}du\left\{\sqrt{Q(u)}-c_1\int_{-\infty}^{u}du\left[\theta\left(u-u_3\right)-\theta\left(u-u_4\right)\right]\right\}, \tag{142}$$

where

$$Q(u)=2\int_{-\infty}^{u}du\left[\theta\left(u-u_1\right)-\theta\left(u-u_2\right)\right]\left[\frac{1}{4}\left(\dddot{D}_{\alpha\beta}n^\alpha n^\beta\right)^2+\frac{1}{2}\dddot{D}_{\alpha\beta}\dddot{D}^{\alpha\beta}+\dddot{D}_{\alpha\beta}\dddot{D}^{\alpha\gamma}n^\beta n_\gamma\right];$$

$$c_1=\frac{1}{u_4-u_3}\sqrt{Q(\infty)};$$

$$\theta(x)=\{0\ \text{for}\ x<0,\ 1\ \text{for}\ x>0\};$$

and $-\infty<u_1<u_2<u_3<u_4<\infty$ and a are arbitrary constants.

For the transformation (137) to be admissible, it is necessary and sufficient for the function F and its derivative $\partial F/\partial u$ to be bounded for all values of u, θ, φ and, in addition,

$$g_{00} > 0; \; g_{\alpha\beta}\, dx^\alpha dx^\beta < 0.$$

It is easy to see that the transformation (137) with the function F defined by the expression (142) is admissible. Indeed, it follows from the expression (142) that for all values of the variable $u = ct - r$ the function F is finite: for $u < u_1$, it is identically equal to zero, $F \equiv 0$, in the interval $u_1 < u < u_4$ the function F is bounded, and for $u > u_4$, $F = \text{const}$. The derivative of F is also bounded:

$$\frac{\partial F}{\partial u} = \pm \frac{a}{3} \left\{ \sqrt{Q(u)} - c_1 \int_{-\infty}^{u} du\, [\theta(u-u_3) - \theta(u-u_4)] \right\}.$$

Analysis shows that the remaining conditions can be readily ensured by choice of an appropriate $\varepsilon > 0$ in the expression (137).

Substituting the expression (142) in (140), for the intensity of the gravitational radiation we obtain

$$\frac{dI}{d\Omega} = \frac{G}{36\pi c^5} \Big[\frac{1}{4} (\dddot{D}_{\alpha\beta} n^\alpha n^\beta)^2 + \frac{1}{2} \dddot{D}_{\alpha\beta} \dddot{D}^{\alpha\beta} + \dddot{D}_{\alpha\beta} \dddot{D}^{\alpha\gamma} n^\beta n_\gamma \Big] \Big\{ 1 - a^2 [\theta(u-u_1) - \theta(u-u_2)] + \frac{a^2 c_1}{\sqrt{Q(u)}} \int_{-\infty}^{u} du\, [\theta(u-u_3) - \theta(u-u_4)] \Big\} + \frac{G a^2 c_1}{36\pi c^5} [\theta(u-u_3) - \theta(u-u_4)] \Big\{ \sqrt{Q(u)} - c_1 \int_{-\infty}^{u} du\, [\theta(u-u_3) - \theta(u-u_4)] \Big\}.$$

It follows from this expression that for values of the variable u in the interval (u_1, u_2) the intensity of the gravitational radiation emitted into the element of solid angle is

$$\frac{dI}{d\Omega} = \frac{G(1-a^2)}{36\pi c^5} \Big\{ \frac{1}{4} (\dddot{D}_{\alpha\beta} n^\alpha n^\beta)^2 + \frac{1}{2} \dddot{D}_{\alpha\beta} \dddot{D}^{\alpha\beta} + \dddot{D}_{\alpha\beta} \dddot{D}^{\alpha\gamma} n^\beta n_\gamma \Big\}. \tag{143}$$

The total intensity of the gravitational radiation integrated over all directions is then

$$I = \frac{G(1-a^2)}{45 c^5} \dddot{D}_{\alpha\beta} \dddot{D}^{\alpha\beta}. \tag{144}$$

It can be seen from the expressions (143) and (144) that in the interval $u_1 < u = ct - r < u_2$ the intensity of the gravitational radiation per element of solid angle and the total intensity depend on the choice of the arbitrary constant a, and as a result of a suitable choice of a can be made equal to zero (for $a^2 = 1$) and also negative (for $a^2 > 1$).

This result can be interpreted in two ways. On the one hand, under the condition $u_1 < u < u_2$ the expressions (143) and (144) determine the energy flux density through any element of a spherical surface of a certain radius r and the total intensity of the gravitational radiation through this sphere during the time interval $t_1 = (r + u_1)/c < t < (r + u_2)/c = t_2$. Therefore, depending on the choice of a, the intensity of the gravitational radiation, and also the total intensity through this sphere can be made either equal to zero or to negative values in any preassigned time interval $t_1 < t < t_2$.

On the other hand, the expressions (143) and (144) determine at each given time t the energy flux density, and also the total intensity of the gravitational radiation in the region of space between the two spheres with radii $r_1 = ct - u_1$ and $r_2 = ct - u_2$. Therefore, depending on the coice of a, the energy flux density and the total intensity of the gravitational radiation can be made to vanish or become negative in the entire region of space between the two spheres with radii $r_1 = ct - u_1$ and $r_2 = ct - u_2$.

2. Suppose the function F has the form

$$F = \pm \frac{a}{3} \int_{-\infty}^{u} du \Big\{ \sqrt{\frac{2a^2}{45} \int_{-\infty}^{u} du\, \dddot{D}_{\alpha\beta} \dddot{D}^{\alpha\beta} [\theta(u-u_1) - \theta(u-u_2)]} - c_2 \int_{-\infty}^{u} du\, [\theta(u-u_3) - \theta(u-u_4)] \Big\}, \tag{145}$$

where

$$c_2 = \frac{1}{u_4 - u_3} \sqrt{\frac{2a^2}{45} \int_{-\infty}^{\infty} du\, \dddot{D}_{\alpha\beta} \dddot{D}^{\alpha\beta} [\theta(u-u_1) - \theta(u-u_2)]}.$$

One can show similarly that the transformation (137) with function F determined by the expression (145) is admissible. In this case, for $u_1 < u < u_2$ the intensity of the gravitational radiatin,

$$\frac{dI}{d\Omega} = \frac{G}{36\pi c^5} \Big\{ \frac{1}{4} (\dddot{D}_{\alpha\beta} n^\alpha n^\beta)^2 + \frac{5 - 2a^2}{10} \dddot{D}_{\alpha\beta} \dddot{D}^{\alpha\beta} + \dddot{D}_{\alpha\beta} \dddot{D}^{\beta\gamma} n_\gamma n^\alpha \Big\}$$

and the total intensity

$$I = \frac{G(1-a^2)}{45 c^5} \dddot{D}_{\alpha\beta} \dddot{D}^{\alpha\beta}$$

will also depend on the choice of a and can be either positive or negative. For $a^2 = 1$, the intensity of the gravitational radiation will have different signs, positive in some directions and negative in others. In this case, the total intensity will be equal to zero.

In contrast, the components $\tilde{t}_g^{0\alpha}$ of the energy-momentum tensor of the field theory of gravitation after transformation to the new coordinates (137) have the form

$$\tilde{t}_g^{0\alpha} = \frac{G n^\alpha}{32\pi c^6 r_H^2} \dot{h}_{\beta\varepsilon} \dot{h}^{\beta\varepsilon} \Big[1 + O\Big(\frac{1}{\sqrt{r_H}} \Big) \Big]. \tag{146}$$

Therefore, the expressions for the intensity of the gravitational radiation and the total intensity calculated in the field theory of gravitation in the new coordinate system will be equal to the corresponding expressions (129) and (131) calculated in the old coordinate system:

$$I = \frac{G}{45 c^5} \dddot{D}_{\alpha\beta} \dddot{D}^{\alpha\beta} \Big[1 + O\Big(\frac{1}{\sqrt{r}} \Big) \Big]. \tag{147}$$

It should also be noted that in both the old and the new coordinate systems the asymptotic expression as $r \to \infty$ for the components of the curvature tensor are the same:

$$\left. \begin{aligned} R_{0\alpha 0\beta} &= -\frac{G}{2c^6 r} \ddot{h}_{\alpha\beta}; \\ R_{\alpha\beta 0\varepsilon} &= \frac{G}{2c^6 r} (\ddot{h}_{\alpha\varepsilon} n_\beta - \ddot{h}_{\beta\varepsilon} n_\alpha); \\ R_{\alpha\beta\gamma\varepsilon} &= \frac{G}{2c^6 r} (\ddot{h}_{\alpha\varepsilon} n_\beta n_\gamma + \ddot{h}_{\beta\gamma} n_\alpha n_\varepsilon - \ddot{h}_{\alpha\gamma} n_\beta n_\varepsilon - \ddot{h}_{\beta\varepsilon} n_\alpha n_\gamma). \end{aligned} \right\} \tag{148}$$

Thus, Einstein's quadrupole formula for the intensity of gravitational radiation is not a consequence of general relativity, since, as we have seen, the energy loss due to gravitational radiation can be made equal to zero or even negative, depending on the choice of the coordinate system. Therefore, this formula cannot be used (since it does not exist) for any energy calculations in general relativity. Thus, Einstein's theory is incapable of explaining the reason for the observed[83] energy loss by the binary pulsar system PSR 1913+16.

In the field theory of gravitation, the gravitational field, like all other physical fields, possesses energy and momentum, and when a slowly moving source emits weak gravitational waves its energy decreases in accordance with (131). Therefore, the experimental proof of the existence of gravitational waves as a physical field carrying energy and thus decreasing the source energy would be a confirmation of the ideas developed here, since general relativity cannot explain the energy loss by matter through the radiation of gravitational waves.

To conclude this section, we discuss briefly the question of the calculation of the Riemann tensor in the field theory of gravitation. In Einstein's theory, it is possible to have a situation[2,4] in which the energy-momentum pseudotensor of the gravitational waves is equal to zero but the components of the Riemann tensor are not. This fact beautifully illustrated the invalidity of interpreting the energy-momentum pseudotensors as energy characteristics of the gravitational field.

If the components of the energy-momentum tensor of gravitational waves in the field theory of gravitation are equal to zero, then the Riemann tensor is also identically equal to zero, i.e., the formation of a Riemannian space-time always requires energy and momentum of the gravitational field. It should be noted that the Riemannian space-time metric is defined only within the matter. The components of the metric tensor g_{ni}, and also the curvature tensor R^i_{nlm} can be calculated at any point, outside the matter as well. But then one must always bear in mind the need to gauge appropriately the fields f_{nm} outside the matter, since the physical quantities do not depend on the components of the field f_{nm} which change under gauge transformations. These components do not occur in the expression for the energy-momentum tensor of the gravitational field. By an appropriate gauge transformation they can always be made equal to zero. Therefore, in a calculation outside the matter of geometrical characteristics of space-time such as, for example, the metric tensor g_{ni} or the Riemann tensor R^i_{nlm} we must substitute in the connection equation (94) only those components f_{nm} that occur in the energy-momentum tensor of the gravitational field; we shall set all the other field components equal to zero, because they can be made to vanish by an appropriate gauge transformation. Thus, our theory will be internally consistent.

Suppose all components of the canonical energy-momentum tensor of free gravitational waves are equal to zero; then from the expression (100) for $n=p=0$ we obtain

$$\dot{f}_{lm}\dot{f}^{lm} - \dot{f}^2/2 = 0. \tag{149}$$

We show that in the TT gauge for a free gravitational wave all the components are identically equal to zero under the condition (149), and then in this gauge all the components of the metric tensor of the Riemannian space-time are equal to the components of the metric tensor of the flat space-time: $g_{ni}=\gamma_{ni}$. Therefore, the curvature tensor in the case of a vanishing energy-momentum tensor of the gravitational field is also equal to zero.

We consider some point. We orient the x axis of a Cartesian coordinate system to make it pass through the point of observation. Around this point, we take a sufficiently small region for us to regard the gravitational wave as plane within it. Then all its components will depend only on the difference $t-x$. In this case, the conditions $\partial_n f^{nm}=0$ take the form

$$\dot{f}^{00} = \dot{f}^{01} = \dot{f}^{11}; \quad \dot{f}^{02} = \dot{f}^{12}; \quad \dot{f}^{03} = \dot{f}^{13}.$$

Integrating these equations and setting the constants of integration equal to zero, since the gravitational waves do not have a time-dependent part, we obtain

$$f^{00} = f^{01} = f^{11}; \quad f^{02} = f^{12}; \quad f^{03} = f^{13}.$$

In the TT gauge, all these components are equal to zero. In addition, from the vanishing $f^n_n=0$ of the trace we have $f^{22}=-f^{33}$. From the condition (149) of vanishing of the energy-momentum tensor we obtain

$$2(\dot{f}_{23})^2 + (\dot{f}_{22} - \dot{f}_{33})^2/2 = 0,$$

and then the transverse components of the gravitational wave also vanish, $f_{23}=f_{22}=f_{33}=0$.

Thus, in the TT gauge we deduce from the condition of vanishing of the energy-momentum tensor of a free gravitational wave that all components of this wave vanish. Then all the components of the metric tensor of the Riemannian space-time are equal to the corresponding components of the metric tensor of the pseudo-Euclidean space-time, $g_{ni}=\gamma_{ni}$, which leads to the vanishing of all components of the Riemann tensor: $R^i_{nlm}=0$.

10. POST-NEWTONIAN APPROXIMATION OF THE FIELD THEORY OF GRAVITATION

To facilitate the comparison of the results of experiments made within the solar system with the predictions of metric theories of gravitation, Nordtvedt and Will[84] developed a formalism which has become known as the parametrized post-Newtonian (PPN) formalism.

In this formalism, the Riemannian space-time metric produced by a body consisting of an ideal fluid is expressed as the sum of all possible generalized gravitational potentials with arbitrary coefficients, which are called post-Newtonian parameters. Using the revised Will-Nordtvedt parameters, the Riemannian space-time metric can be written in the form

$$\left.\begin{aligned} g_{00} &= 1 - 2U + 2\beta U^2 - (2\gamma + 2 + \alpha_3 + \xi_1)\Phi_1 + \xi_1 A \\ &\quad + 2\xi_w \Phi_w - 2[(3\gamma + 1 - 2\beta + \xi_2)\Phi_2 + (1+\xi_3)\Phi_3 \\ &\quad + 3(\gamma + \xi_4)\Phi_4] - (\alpha_1 - \alpha_2 - \alpha_3) w_\alpha w^\alpha U \\ &\quad + \alpha_2 w^\alpha w^\beta U_{\alpha\beta} - (2\alpha_3 - \alpha_1) w^\alpha V_\alpha; \\ g_{0\alpha} &= \frac{1}{2}(4\gamma + 3 + \alpha_1 - \alpha_2 + \xi_1) V_\alpha + \frac{1}{2}(1 + \alpha_2 - \xi_1) W_\alpha \\ &\quad - \frac{1}{2}(\alpha_1 - 2\alpha_2) w_\alpha U + \alpha_2 w^\beta U_{\alpha\beta}; \\ g_{\alpha\beta} &= (1 + 2\gamma U)\gamma_{\alpha\beta}, \end{aligned}\right\} \tag{150}$$

where w^α are the spatial components of the velocity of the frame of reference with respect to a hypothetical universal rest frame. For some theories of gravitation, this is the velocity of the center of mass of the solar system with respect to the rest frame of the Universe.

The generalized gravitational potentials have the form

$$\left.\begin{aligned} &U(\mathbf{r}, t) = \int \frac{\rho_0(\mathbf{r}', t)}{R}\, dV;\quad R = |\mathbf{r}-\mathbf{r}'|;\\ &\Phi_1 = -\int \frac{\rho_0 v_\alpha v^\alpha}{R}\, dV;\quad \Phi_2 = \int \frac{\rho_0 U}{R}\, dV;\\ &\Phi_3 = \int \frac{\rho_0 \Pi}{R}\, dV;\quad \Phi_4 = \int \frac{p}{R}\, dV;\\ &A = \int \frac{\rho_0 v_\alpha v_\beta R^\alpha R^\beta}{R^3}\, dV;\quad V_\alpha = -\int \frac{\rho_0 v_\alpha}{R}\, dV;\\ &W_\alpha = \int \frac{\rho_0 v_\beta R^\beta R_\alpha}{R^3}\, dV;\quad U_{\alpha\beta} = \int \frac{\rho_0 R_\alpha R_\beta}{R^3}\, dV;\\ &\Phi_w = \int \frac{\rho_0(\mathbf{r}', t)\,\rho_0(\mathbf{r}'', t)}{|\mathbf{r}-\mathbf{r}'|^3}\,(x^\alpha - x'^\alpha)\\ &\quad\times\left[\frac{x_\alpha - x''_\alpha}{|\mathbf{r}'-\mathbf{r}''|} - \frac{x'_\alpha - x''_\alpha}{|\mathbf{r}-\mathbf{r}''|}\right] d^3\mathbf{r}'\, d^3\mathbf{r}'';\\ &R^\alpha = x^\alpha - x'^\alpha, \end{aligned}\right\} \tag{151}$$

where ρ_0 is the invariant mass density of the body, v^α are the velocity components of the elements of the ideal fluid, p is the isotropic pressure, and $\rho_0\Pi$ is the internal energy density of the ideal fluid.

Each metric theory of gravitation will correspond to a certain set of values of the ten parameters $\beta, \gamma, \alpha_1, \alpha_2, \alpha_3, \xi_1, \xi_2, \xi_3, \xi_4, \xi_\omega$. Therefore, from the point of view of experiments made in the solar system, one theory of gravitation differs from another only by the values of these parameters. To find theories of gravitation that in the post-Newtonian limit can describe the Experiments made in the solar system, it is sufficient to determine from these experiments the values of the ten post-Newtonian parameters and to select only those theories of gravitation whose post-Newtonian approximation leads to values of the parameters agreeing with those obtained from the experiments. Then all such theories of gravitation will be indistinguishable from the point of view of all experiments made with post-Newtonian accuracy.

The further selection of a theory of gravitation adequate for the description of reality entails a raising in the accuracy of the measurements to the post-Newtonian level and the search for possibilities of studying the properties of gravitational waves and of phenomena in strong gravitational fields.

We now find the set of values of the post-Newtonian parameters that corresponds to the field theory of gravitation. We write the gravitational field equations of this theory for the calculation of the post-Newtonian approximation in the form

$$\Box^2 f^{nm} = -16\pi J^{nm};\quad \Box = \partial_i \partial^i. \tag{152}$$

If we use the notation (66), then for the tensor current we obtain the expression

$$J^{nm} = \Box h^{nm} - \partial^n \partial_l h^{lm} - \partial^m \partial_l h^{ln} + \gamma^{nm} \partial_l \partial_i h^{li}. \tag{153}$$

Following Fock,[13] to construct a post-Newtonian approximation valid in the solar system we consider a problem of astronomical type. We shall assume that the components of the energy-momentum tensor of the matter are equal to zero in the whole of space except for certain regions. Within each such region the enerby-momentum tensor must correspond to our adopted ideal-fluid model and satisfy the covariant conservation equation in the Riemannian space-time. The energy-momentum tensor of the matter will depend on the physical properties of the model of the celestial bodies and also on the metric of the Riemannian space-time. Therefore, the construction of the energy-momentum tensor of the matter and the determination of the metric tensor of the Riemannian space-time must be done simultaneously.

We use the circumstance that within the solar system the maximal values of the gravitational potential, the square v^2 of the characteristic velocity (the velocity of the celestial bodies with respect to the center of mass of the solar system), the specific pressure p/ρ_0, and the specific internal energy Π have approximately the same small order of magnitude ε^2, where $\varepsilon \approx 10^{-3}$ is a dimensionless parameter. Therefore, in the solar system we have the estimates

$$\left.\begin{aligned} &U = O(\varepsilon^2);\quad v^\alpha = O(\varepsilon);\\ &\Pi = O(\varepsilon^2);\quad p/\rho_0 = O(\varepsilon^2). \end{aligned}\right\} \tag{154}$$

In addition, we consider the field in the near zone, i.e., at distances from the Sun appreciably less than the wavelength of a gravitational wave radiated by object in the solar system moving with characteristic velocity $v \sim \varepsilon$: $R/\lambda \sim R\partial/\partial t \sim \varepsilon$. In this case, the variations in time of all quantities are due in the first place to the motion of the matter. Therefore, the partial derivatives with respect to the time are small compared with the partial derivatives with respect to the coordinates:

$$\partial/\partial t = O(\varepsilon)\, \partial/\partial x^\alpha. \tag{155}$$

We shall solve the problem of simultaneous determination of the energy-momentum tensor of the matter and the metric tensor of the Riemannian space-time in successive stages, each of which corresponds to expansion of the exact solutions of the problem in powers of the dimensionless parameter ε.

We have the following exact relations: the energy-momentum tensor density of an ideal fluid,

$$T^{nm} = \sqrt{-g}\,[(p + \mathscr{E})\, u^n u^m - p g^{nm}], \tag{156}$$

the covariant continuity equation

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} [\sqrt{-g}\, \rho_0 u^i] = 0, \tag{157}$$

and the conservation equation for the energy-momentum tensor density of the matter in the Riemannian space-time:

$$\nabla_n T^{nm} = \partial_n T^{nm} + \Gamma^m_{nl} T^{nl} = 0, \tag{158}$$

where $\mathscr{E}$ is the total energy density of the ideal fluid, and u^i is the velocity 4-vector.

For our purposes, it is more convenient to write the gravitational field equations (152) and the minimal-coupling equation (94) in the form

$$\Box^2 \chi^{nm} = -16\pi A^{nm}; \tag{159}$$

$$g_{nm}=\gamma_{nm}+\chi_{nm}+\frac{1}{4}[b_1\chi_{nl}\chi_m^l+b_3\gamma_{nm}\chi^{li}\chi_{li} - (b_1+b_2)\chi_{nm}\chi+(b_4+b_2/2+b_1/4)\chi^2\gamma_{nm}], \quad (160)$$

where we have introduced the notation

$$\chi_{nm}=f_{nm}-\gamma_{nm}f/2;\quad \chi=\chi_n^n; \quad (161)$$

$$A^{nm}=\Box[h^{nm}-\gamma^{nm}h_l^l/2]-\partial^n\partial_l h^{lm}-\partial^m\partial_l h^{ln}. \quad (162)$$

We expand all quantities in (158)–(161) in series in the small parameter ε. If we ignore the energy losses due to radiation of gravitational waves, these expansions must also be valid when the sign of the time is reversed, i.e., under the coordinate transformation $x^{0\prime}=-x^0$. In this case, the sign of the components $v^\alpha, \chi^{0\alpha}, T^{0\alpha}, g_{0\alpha}, A^{0\alpha}, \partial/\partial x^0$ is reversed. Since $v\sim\varepsilon$ and $\partial/\partial x^0 \sim \varepsilon\partial/\partial x^\alpha$, the dimensionless parameter ε also changes sign when the sign of the time is reversed. It follows that if the energy loss due to radiation of gravitational waves can be ignored the expansions of the components $v^\alpha, \chi^{0\alpha}, T^{0\alpha}, g_{0\alpha}, A^{0\alpha}$ contain only odd powers of the parameter ε.

We write the expansions of the tensor current A^{nm} and the field χ^{nm} in the form

$$\chi^{nm}=\overset{(1)}{\chi}{}^{nm}+\overset{(2)}{\chi}{}^{nm}+\ldots; \quad (163)$$

$$A^{nm}=\overset{(0)}{A}{}^{nm}+\overset{(1)}{A}{}^{nm}+\ldots, \quad (164)$$

where the components of the zeroth, $\overset{(0)}{A}{}^{nm}$, first, $\overset{(1)}{A}{}^{nm}$, and second, $\overset{(2)}{A}{}^{nm}$, approximations have the orders of magnitude

$$\left.\begin{array}{l}\overset{(0)}{A}{}^{0\alpha}=O(\varepsilon);\ \overset{(0)}{A}{}^{00}=O(1);\ \overset{(0)}{A}{}^{\alpha\beta}=O(1);\\ \overset{(1)}{A}{}^{0\alpha}=O(\varepsilon^3);\ \overset{(1)}{A}{}^{\alpha\beta}=O(\varepsilon^2);\ \overset{(1)}{A}{}^{00}=O(\varepsilon^2);\\ \overset{(2)}{A}{}^{0\alpha}=O(\varepsilon^5);\ \overset{(2)}{A}{}^{00}=O(\varepsilon^4);\ \overset{(2)}{A}{}^{\alpha\beta}=O(\varepsilon^4).\end{array}\right\} \quad (165)$$

Using the expansions (163) and (164) and the estimate (155), we rewrite the gravitational field equations (159) in the form of a series of successive approximations:

$$\Delta^2\overset{(1)}{\chi}{}^{nm}=-16\pi\overset{(0)}{A}{}^{nm}; \quad (166)$$

$$\Delta^2\overset{(2)}{\chi}{}^{nm}=-16\pi\overset{(1)}{A}{}^{nm}+2\frac{\partial^2}{\partial t^2}\Delta\overset{(1)}{\chi}{}^{nm}. \quad (167)$$

From the expressions (74) and (94), we obtain

$$h^{nm}=T^{nm}+\frac{b_1}{4}[T^{nl}\chi_l^m+T^{ml}\chi_l^n]-\frac{b_1+b_2}{4}\chi T^{nm}+\frac{b_3}{2}\chi^{nm}T^{li}\gamma_{li} - \frac{b_1+b_2}{4}\gamma^{nm}T^{li}\chi_{li}+\left(\frac{b_2}{4}+\frac{b_1}{8}+\frac{b_4}{2}\right)\gamma^{nm}\chi T^{li}\gamma_{li}. \quad (168)$$

Then for the tensor current A^{nm} we have

$$\left.\begin{array}{l}\overset{(0)}{A}{}^{nm}=-\Delta\left[\overset{(0)}{T}{}^{nm}-\frac{1}{2}\gamma^{nm}\overset{(0)}{T}{}^{li}\gamma_{li}\right];\\ \overset{(1)}{A}{}^{nm}=\frac{\partial^2}{\partial t^2}\left[\overset{(0)}{T}{}^{nm}-\frac{1}{2}\gamma^{nm}\overset{(0)}{T}{}^{li}\gamma_{li}\right]\\ +\partial^n(\overset{(1)}{\Gamma}{}^m_{li}\overset{(0)}{T}{}^{li})+\partial^m(\overset{(1)}{\Gamma}{}^n_{li}\overset{(0)}{T}{}^{li})-\Delta\left[\overset{(1)}{T}{}^{nm}-\frac{1}{2}\gamma^{nm}\overset{(1)}{T}{}^{li}\gamma_{li}\right.\\ +\frac{b_1}{4}(\overset{(0)}{T}{}^{nl}\overset{(1)}{\chi}{}_l^m+\overset{(0)}{T}{}^{ml}\overset{(1)}{\chi}{}_l^n)-\frac{b_1+b_2}{4}\overset{(0)}{T}{}^{nm}\overset{(1)}{\chi}+\frac{b_2}{4}\gamma^{nm}\overset{(0)}{T}{}^{li}\overset{(1)}{\chi}{}_{li}\\ \left.+\frac{b_3}{2}\overset{(1)}{\chi}{}^{nm}\overset{(0)}{T}{}^{li}\gamma_{li}-\left(\frac{b_2}{8}+\frac{b_3}{4}+\frac{b_4}{2}\right)\gamma^{nm}\overset{(0)}{T}{}^{li}\gamma_{li}\chi\right],\end{array}\right\} \quad (169)$$

where $\Delta=-\partial^\alpha\partial_\alpha$.

To determine the post-Newtonian parameters, it is sufficient to determine the components $g_{\alpha\beta}$ to accuracy ε^2, the components $g_{0\alpha}$ to accuracy ε^3, and the component g_{00} to accuracy ε^4. It follows from Eqs. (160) that for this it is necessary to determine the components of the field $\chi^{\alpha\beta}$ to accuracy ε^2, $\chi^{0\alpha}$ to accuracy ε^3, and χ^{00} to accuracy ε^4.

In the initial approximation, we assume that the metric tensor of the Riemannian space-time is equal to the metric tensor of the pseudo-Euclidean space-time, i.e., we ignore the gravitational forces. Then Eqs. (157) and (158) take the form

$$\left.\begin{array}{l}\frac{\partial}{\partial x^i}(\rho u^i)=O(\varepsilon^2);\\ \partial_n T^{n0}=O(\varepsilon^3);\ \partial_n T^{n\alpha}=O(\varepsilon^2).\end{array}\right\} \quad (170)$$

Using the estimates (154), we obtain from these equations

$$T^{00}=\rho_0[1+O(\varepsilon^2)];\quad T^{\alpha\beta}=\rho_0 O(\varepsilon^2);\quad T^{0\alpha}=\rho_0 v^\alpha[1+O(\varepsilon^2)].$$

Therefore, the components of the tensor current A^{nm} in the zeroth approximation can be written as

$$\overset{(0)}{A}{}^{00}=-\Delta\rho_0/2;\quad \overset{(0)}{A}{}^{0\alpha}=-\Delta(\rho_0 v^\alpha);\quad \overset{(0)}{A}{}^{\alpha\beta}=\gamma^{\alpha\beta}\Delta\rho_0/2. \quad (171)$$

Then from Eq. (166) we obtain

$$\overset{(1)}{\chi}{}^{00}=-2U;\quad \overset{(1)}{\chi}{}^{\alpha\beta}=2U\gamma^{\alpha\beta};\quad \overset{(1)}{\chi}{}^{0\alpha}=4V^\alpha. \quad (172)$$

As a result, the components of the metric tensor (160) of the Riemannian space-time in the first approximation can be written as

$$\left.\begin{array}{l}g_{00}=1-2U+O(\varepsilon^4);\ g_{0\alpha}=4V_\alpha[1+O(\varepsilon)^2];\\ g_{\alpha\beta}=\gamma_{\alpha\beta}[1+2U]+O(\varepsilon^4).\end{array}\right\} \quad (173)$$

Knowledge of the metric in this approximation makes it possible to determine the components of the energy-momentum tensor of the matter in the following approximation. Using the expressions (173), we find

$$\left.\begin{array}{l}\sqrt{-g}=1+2U+O(\varepsilon^4);\ u_0=1+U-v_\alpha v^\alpha/2;\\ \Gamma^0_{00}=-\partial U/\partial t+O(\varepsilon^5);\ \Gamma^\alpha_{00}=\partial^\alpha U+O(\varepsilon^4);\\ \Gamma^0_{0\alpha}=-\partial_\alpha U+O(\varepsilon^4);\ \Gamma^\alpha_{\beta\gamma}=O(\varepsilon^2);\\ \Gamma^\alpha_{0\beta}=O(\varepsilon^3);\ \Gamma^0_{\alpha\beta}=O(\varepsilon^2).\end{array}\right\} \quad (174)$$

We also introduce the conserved mass density ρ in accordance with the equation $\rho=\sqrt{-g}\rho_0 u^0$. To obtain the metric in the following approximation, we must construct the energy-momentum tensor density of the matter, which must satisfy the conservation equations (158) by virtue of the covariant continuity equation

$$\frac{1}{\sqrt{-g}}\left[\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x^\alpha}(\rho v^\alpha)\right]=0 \quad (175)$$

and the equations of motion of the ideal fluid in the Newtonian approximation[13]:

$$\rho\frac{dv^\alpha}{dt}=\gamma^{\alpha\beta}\left[-\rho\frac{\partial U}{\partial x^\beta}+\frac{\partial p}{\partial x^\beta}\right]+\rho O(\varepsilon^4);$$
$$\rho d\Pi/dt=-p\partial_\alpha v^\alpha;\ d/dt=\partial/\partial t+v^\beta\partial/\partial x^\beta.$$

It is readily seen that these conditions are satisfied by the following components of the energy-momentum tensor density of the matter:

$$T^{00}=\rho[1-v_\alpha v^\alpha/2+\Pi+U]+\rho O(\varepsilon^4);$$
$$T^{0\alpha}=\rho v^\alpha[1-v_\beta v^\beta/2+\Pi+U]+pv^\alpha+\rho O(\varepsilon^4);$$
$$T^{\alpha\beta}=\rho v^\alpha v^\beta-p\gamma^{\alpha\beta}+\rho O(\varepsilon^4).$$

As a result, we obtain the following relation between the conserved, ρ, and invariant, ρ_0, mass densities:

$$\rho=\rho_0[1+3U-v_\alpha v^\alpha/2]+\rho O(\varepsilon^4). \quad (176)$$

To determine the post-Newtonian approximation, it remains to determine $\overset{(2)}{\chi}{}^{00}$. Using the expression (169), (172), and (174), we can write Eq. (167) for the component $\overset{(2)}{\chi}{}^{00}$ in the form

$$\Delta^2\overset{(2)}{\chi}{}^{00} = 8\pi \frac{\partial^2}{\partial t^2} \rho_0$$
$$16\pi\Delta \left\{ \frac{3}{2} p + \rho_0 \left[\frac{\Pi}{2} - v_\alpha v^\alpha - 2(b_1 + b_2 + b_3 + b_4) U \right] \right\}. \tag{177}$$

Solving this equation, we obtain

$$\overset{(2)}{\chi}{}^{00} = -\frac{\partial^2}{\partial t^2} \int \rho_0 R dV - 4\Phi_1 - 2\Phi_3 - 6\Phi_4 + 8(b_1 + b_2 + b_3 + b_4)\Phi_2. \tag{178}$$

Using the expressions (160), (172), and (178), we write the metric of the Riemannian space-time in the post-Newtonian approximation in the form

$$\left.\begin{aligned} g_{00} &= 1 - 2U + 2\beta U^2 - 4\Phi_1 + 4(\beta - 2)\Phi_2 - 2\Phi_3 - 6\Phi_4 \\ &\quad - \frac{\partial^2}{\partial t^2} \int \rho_0 R dV + O(\varepsilon^6); \quad g_{0\alpha} = 4V_\alpha + O(\varepsilon^5), \\ g_{\alpha\beta} &= \gamma_{\alpha\beta}(1 + 2U) + O(\varepsilon^4), \end{aligned}\right\} \tag{179}$$

where $\beta = 2(b_1 + b_2 + b_3 + b_4)$.

To determine the post-Newtonian parameters of our theory, we must go over to the coordinate system in which we have written down the post-Newtonian expansion (150) of the metric. If we introduce the coordinate transformation

$$x'^n = x^n + \xi^n(x) \tag{180}$$

and assume that $\xi^\alpha(x) \sim O(\varepsilon^2)$ and $\xi^0(x) \sim O(\varepsilon^3)$, then the metric (179) in the new coordinate system will have the form

$$\left.\begin{aligned} g'_{00} &= g_{00} - 2\partial_0\xi_0 + O(\varepsilon^6); \\ g'_{0\alpha} &= g_{0\alpha} - \partial_0\xi_\alpha - \partial_\alpha\xi_0 + O(\varepsilon^5); \\ g'_{\alpha\beta} &= g_{\alpha\beta} - \partial_\alpha\xi_\beta - \partial_\beta\xi_\alpha + O(\varepsilon^4). \end{aligned}\right\} \tag{181}$$

As "canonical" coordinate system, it is usual to choose a coordinate system in which the nondiagonal components of the spatial part of the metric tensor g_{ni} are equal to zero,

$$g_{12} = g_{13} = g_{23} = 0,$$

and, in addition, the component g_{00} does not contain terms of the form

$$\frac{\partial^2}{\partial t^2} \int \rho_0 R\, dV.$$

These requirements make it possible to determine the 4-vector uniquely with the required accuracy. In our case, for the transition to the canonical coordinate system it is necessary to choose the 4-vector ξ^n as follows:

$$\xi^\alpha(x) = 0; \quad \xi^0(x) = -\frac{1}{2} \frac{\partial}{\partial t} \int \rho_0 R dV.$$

Using the continuity equations (175), we obtain

$$\partial_\alpha \xi_0 = [V_\alpha - W_\alpha]/2.$$

As a result, we have the following expression for the metric tensor of the effective Riemannian space-time:

$$\left.\begin{aligned} g_{00} &= 1 - 2U + 2\beta U^2 - 4\Phi_1 + 4(\beta - 2)\Phi_2 - 2\Phi_3 - 6\Phi_4 + O(\varepsilon^6); \\ g_{0\alpha} &= (7/2) V_\alpha + W_\alpha/2 + O(\varepsilon^5); \quad g_{\alpha\beta} = \gamma_{\alpha\beta}[1 + 2U] + O(\varepsilon^4). \end{aligned}\right\} \tag{182}$$

Thus, the post-Newtonian approximation of the field theory of gravitation leads to the Riemannian space-time metric (182), which contains only the single unknown constant β.

For the case when the source of the gravitational field is a static, spherically symmetric body of radius a, this metric becomes

$$\left.\begin{aligned} g_{00} &= 1 - 2M/r + 2\beta M^2/r^2 + O(M^3/r^3); \\ g_{\alpha\beta} &= \gamma_{\alpha\beta}(1 + 2M/r) + O(M^2/r^2); \quad g_{0\alpha} = 0, \end{aligned}\right\} \tag{183}$$

where M is the gravitational mass of the source of the field:

$$M = 4\pi \int_0^a \rho_0 \left[1 + \Pi + \frac{3p}{\rho_0} + 2(2 - \beta) U \right] r^2 dr. \tag{184}$$

Using the Newtonian virial theorem for static bodies,

$$3 \int p\, dV = \frac{1}{2} \int \rho_0 U\, dV,$$

and also the relation (176) between the conserved and the invariant mass densities, we reduce the expression (184) to the form

$$M = 4\pi \int_0^a \rho \left[1 + \Pi + \left(\frac{3}{2} - 2\beta \right) U \right] r^2 dr. \tag{185}$$

As we shall see below, if the post-Newtonian expressions for the gravitational (185) and the inertial [see Eq. (200)] masses of a static, spherically symmetric body are to be equal it is necessary to set $\beta = 1$, and then the post-Newtonian parameters of the field theory of gravitation will have the values

$$\left.\begin{aligned} &\gamma = \beta = 1; \quad \alpha_1 = \alpha_2 = \alpha_3 = 0; \\ &\xi_1 = \xi_2 = \xi_3 = \xi_4 = \xi_w = 0. \end{aligned}\right\} \tag{186}$$

For comparison, we note that in general relativity the post-Newtonian parameters have the same values (see Ref. 84). It should be noted that in Einstein's theory all the parameters α and ξ are equal to zero. For a long time this was regarded as a property of Einstein's theory alone and was taken to be one of its achievements. However, as we see, in the field theory of gravitation these parameters are also equal to zero. The remaining parameters in general relativity and in the field theory of gravitation are equal to unity.

Since the post-Newtonian parameters of Einstein's theory and of the field theory of gravitation are equal, these two theories will be indistinguishable from the point of view of all experiments made with post-Newtonian accuracy of the measurements in the gravitational field of the solar system.

As is shown in Ref. 85, the vanishing of the three parameters α has a definite physical meaning: Any theory of gravitation in which $\alpha_1 = \alpha_2 = \alpha_3 = 0$ has no preferred universal rest frame in the post-Newtonian limit. In this case, on transition from a universal rest frame to a moving system the Riemannian space-time metric in the post-Newtonian limit is form-invariant and the velocity w^α of the new coordinate system with respect to the rest frame does not occur explicitly in the metric.

It follows from the expression (186) that in the field theory of gravitation there is no universal preferred rest frame.

A linear dependence of the parameters ξ and α also has a definite physical meaning. As is shown in Ref. 86, if

$$\left.\begin{array}{l}\alpha_1=\xi_3=0;\quad \alpha_2-\xi_1-2\xi_w=0;\\ \xi_2=\xi_w;\quad \alpha_3+\xi_1+2\xi_w=0;\\ 3\xi_4+2\xi_w=0;\quad \xi_1+2\xi_w=0\end{array}\right\} \tag{187}$$

then from the post-Newtonian equations of motion one can deduce quantities that do not depend on the time in the post-Newtonian approximation. However, they can be interpreted as the energy, momentum, and angular momentum of the system, i.e., as integrals of the motion, only in those theories of gravitation that possess conservation laws for the energy-momentum tensor of the matter and the gravitational fields.

Thus, in Einstein's theory the relations (187) are satisfied, but the time-independent (in the post-Newtonian approximation) quantities are not, as is shown by a detailed analysis (see also Sec. 11), integrals of the motion of the system consisting of the matter and the gravitational field.

In the field theory of gravitation, an isolated system in pseudo-Euclidean space-time has all ten conservation laws in their usual sense, which in the post-Newtonian approximation lead to ten integrals of the motion of the system, and therefore in the post-Newtonian approximation the field theory of gravitation has ten time-independent quantities. The fulfillment of the relations (187) in the field theory of gravitation confirms this conclusion.

11. CONSERVATION LAWS IN THE POST-NEWTONIAN APPROXIMATION OF THE FIELD THEORY OF GRAVITATION

In the field theory of gravitation, the gravitational field, considered in pseudo-Euclidean space-time, behaves like all other physical fields. It possesses energy and momentum and contributes to the total energy-momentum tensor density of the system. The covariant conservation law for the total energy-momentum tensor density in the pseudo-Euclidean space-time, written down in a Cartesian coordinate system, has its usual meaning:

$$\partial_i\,[t_g^{ni}+t_M^{ni}]=0, \tag{188}$$

where t_g^{ni} is the symmetric energy-momentum tensor density (104) of the gravitational field, and t_M^{ni} is the symmetric energy-momentum tensor density (111) of the matter.

Using the differential conservation law (188), we can obtain a corresponding integral conservation law:

$$-\frac{\partial}{\partial t}\int dV\,[t_g^{0n}+t_M^{0n}]=\int dS_\alpha\,[t_g^{n\alpha}+t_M^{n\alpha}].$$

If there is no energy flux of the matter and the gravitational field through a surface bounding the considered volume,

$$\int dS_\alpha\,[t_g^{\alpha n}+t_M^{\alpha n}]=0, \tag{189}$$

then we arrive at a conservation law for the total 4-momentum of the isolated system:

$$(d/dt)\,P^n=0,$$

where

$$P^n=\int dV\,[t_g^{0n}+t_M^{0n}]. \tag{190}$$

In this case, because of the symmetry of the total energy-momentum tensor density, the angular-momentum tensor of the system is also conserved:

$$(d/dt)\,M^{ni}=0,$$

where

$$M^{ni}=\int dV\,\{x^n\,[t_g^{0i}+t_M^{0i}]-x^i\,[t_g^{0n}+t_M^{0n}]\}. \tag{191}$$

By virtue of the conservation of the components

$$M^{0\alpha}=x^0\int dV\,[t_g^{0\alpha}+t_M^{0\alpha}]-\int dV x^\alpha\,[t_g^{00}+t_M^{00}]$$

the center of mass of the isolated system, which is determined by the expression

$$X^\alpha=\int x^\alpha\,[t_g^{00}+t_M^{00}]\,dV\Big/\int dV\,[t_g^{00}+t_M^{00}]=(P^\alpha t-M^{0\alpha})/P^0, \tag{192}$$

executes uniform rectilinear motion with velocity

$$(d/dt)\,X^\alpha=P^\alpha/P^0.$$

Thus, to describe the motion of an isolated system consisting of matter and the gravitational field it is sufficient to determine the 4-momentum P^n (190). It should be noted that in any real system, because of the motion of its constituent parts, the thermal motion of the matter, and other reasons, gravitational waves may be emitted; any real system exchanges matter with other systems in the form of electromagnetic radiation and in the form of particles, atoms, etc. Therefore, in the most general case the energy fluxes of the matter and the gravitational field cannot be ignored; indeed, there exist many astrophysical processes in which these energy fluxes play a leading part, and it is only when they are taken into account that one can understand and predict many astrophysical processes. But for systems for which the energy fluxes of the matter and the gravitational field are small, the condition (189) for the system to be isolated is satisfied with a certain degree of accuracy. Then to the same degree of accuracy it can be asserted that the 4-momentum of this system is conserved. It is just such a situation that obtains for systems to which the post-Newtonian formalism applies. In this case, the condition (189) is satisfied in the post-Newtonian approximation, and one can determine a conserved 4-momentum of the system.

We find the post-Newtonian expression for the 4-momentum of an isolated system in the field theory of gravitation. The total symmetric energy-momentum tensor density in the flat space-time has the form

$$\begin{aligned}t^{ni}=t_g^{ni}+t_M^{ni}=\frac{1}{64\pi}\Big\{-\gamma^{ni}\Big[\partial_l f_{ms}\partial^l f^{ms}-\frac{1}{2}\partial_l f\partial^l f\Big]\\ -\partial^n f\partial^i f+2\partial^n f_{lm}\partial^i f^{lm}-2f^{im}\,\Box\, f_m^n-2f^{nm}\,\Box\, f_m^i+2f^{ni}\,\Box\, f\Big\}\\ -\frac{1}{32\pi}\partial_l\{f_m^i\partial^n f^{lm}+f_m^n\partial^i f^{lm}-f^{lm}(\partial^i f_m^n+\partial^n f_m^i)\}-2\Lambda^{(ni)}\\ +T^{ni}\Big[1-\frac{1}{2}f+\frac{b_3}{4}f_{lm}f^{lm}+\frac{b_4}{2}f^2\Big]+\frac{1}{2}f^{ni}\gamma_{lm}T^{lm}\\ -\frac{1}{4}[b_1T^{lm}f_l^i f_m^n+b_2f^{ni}T^{lm}f_{lm}+2b_3f_s^n f^{is}T^{lm}\gamma_{lm}+2b_4f^{ni}fT^{lm}\gamma_{lm}],\end{aligned} \tag{193}$$

where Λ^{ni} is determined by the expression (103), and the tensor A^{lm} in this case has the form

$$A^{lm} = -\frac{1}{32\pi}\left\{\Box\left(f^{lm} - \frac{1}{2}\gamma^{lm}f\right) + 16\pi\left(h^{lm} - \frac{1}{2}\gamma^{lm}h_n^n\right)\right\}. \qquad (194)$$

It follows from the expression (193) that the components t^{00} and $t^{0\alpha}$ of the total symmetric energy-momentum tensor of the system can be determined up to terms $t^{00} \sim \rho O(\varepsilon^2), t^{0\alpha} \sim \rho O(\varepsilon^3)$ inclusive. Therefore, we shall omit all terms of higher order, for example, Λ^{00} and $\Lambda^{0\alpha}$, since $\Lambda^{00} \sim \rho O(\varepsilon^4), \Lambda^{0\alpha} \sim \rho O(\varepsilon^5)$. Bearing in mind that

$$\partial_\alpha \partial^\alpha U = 4\pi\rho_0; \quad \partial V^\beta/\partial x^\beta = \partial U/\partial t,$$

we find from the expressions (193) and (172)

$$\left.\begin{aligned} t^{00} &= \rho\left[1 + v^2/2 + \Pi - U/2\right] - \frac{1}{8\pi}\partial_\alpha\left[U\partial^\alpha U\right] + \rho O(\varepsilon^4); \\ t^{0\alpha} &= \rho v^\alpha\left[1 + \frac{v^2}{2} + \Pi + U\right] + pv^\alpha + 2\rho V^\alpha \\ &+ \frac{1}{4\pi}\left\{\frac{\partial U}{\partial t}\partial^\alpha U + 2\partial_\beta\left[U\partial^\alpha V^\beta - V^\beta\partial^\alpha U\right]\right\} + \rho O(\varepsilon^5). \end{aligned}\right\} \qquad (195)$$

To find the 4-momentum of the system in the post-Newtonian approximation, we integrate the expressions (195) over the whole of space. Using the equations

$$\int \frac{\partial U}{\partial t}\partial^\alpha U\, dV = 2\pi\int\rho\left[Uv^\alpha + W^\alpha\right]dV;$$

$$\int \rho V^\alpha\, dV = -\int \rho U v^\alpha\, dV;$$

$$\int \partial_\alpha\left(U\partial^\alpha U\right)dV = \int dS_\alpha U\partial^\alpha U = 0,$$

we finally obtain

$$\left.\begin{aligned} P^0 &= \int dV\rho\left[1 + \Pi + \frac{v^2}{2} - \frac{1}{2}U\right]; \\ P^\alpha &= \int dV\left\{\rho v^\alpha\left[1 + \Pi + \frac{v^2}{2} - \frac{1}{2}U\right] + pv^\alpha + \frac{1}{2}\rho W^\alpha\right\}. \end{aligned}\right\} \qquad (196)$$

Using the expressions (196) and (191), we can readily determine in the post-Newtonian approximation the conserved angular-momentum tensor of the system,

$$\left.\begin{aligned} M^{0\alpha} &= -\int dV\rho x^\alpha\left[1 + \Pi + \frac{v^2}{2} - \frac{1}{2}U\right] + P^\alpha t; \\ M^{\alpha\beta} &= \int dV\rho\left\{x^\alpha\left[v^\beta\left(1 + \Pi + \frac{v^2}{2} - \frac{1}{2}U + \frac{p}{\rho}\right) + \frac{1}{2}W^\beta\right]\right. \\ &\left. - x^\beta\left[v^\alpha\left(1 + \Pi + \frac{v^2}{2} - \frac{1}{2}U + \frac{p}{\rho}\right) + \frac{1}{2}W^\alpha\right]\right\}, \end{aligned}\right\} \qquad (197)$$

and also the coordinates of the center of mass of the system:

$$X^\alpha = \int dV\rho x^\alpha\left[1 + \Pi + \frac{v^2}{2} - \frac{1}{2}U\right]\Big/P^0. \qquad (198)$$

It should be noted that in the system of units we have adopted the expression (190) for the component P^0 of the 4-momentum of the isolated system is equal to the expression for the inertial mass of this system. Therefore, in the post-Newtonian approximation the inertial mass is

$$m = \int dV\rho\left[1 + \Pi + \frac{v^2}{2} - \frac{1}{2}U\right]. \qquad (199)$$

For a static, spherically symmetric body the post-Newtonian expression for the inertial mass is

$$m = 4\pi\int_0^a r^2 dr\rho\left[1 + \Pi - \frac{1}{2}U\right]. \qquad (200)$$

The expressions obtained for the inertial and gravitational masses, (200) and (185), respectively, make it possible to determine the numerical value of the parameter β in the field theory of gravitation. Indeed, as follows from the expressions (185) and (200), the condition of equality of these masses lead uniquely to $\beta=1$.

In contrast to the conservation law (188), the covariant conservation law for the energy-momentum tensor density of the matter in the Riemannian space-time,

$$\nabla_n T^{ni} = \partial_n T^{ni} + \Gamma^i_{nm}T^{nm} = 0, \qquad (201)$$

does not express in explicit form the conservation of any quantity but simply reflects the fact that the energy-momentum tensor density of the matter is not conserved: $\partial_n T^{ni} \neq 0$. However, as is shown in Sec. 3 of the present paper, the conservation law (188) and the conservation law (201) are simply different forms of expression of the same conservation law in the field theory of gravitation. This general result can be confirmed in any stage of approximate calculations. Therefore, in the field theory of gravitation the integrals of the motion (196) and (197) can also be obtained from the conservation equation (201) in the post-Newtonian approximation.

We show, for example, that the post-Newtonian integrals of the motion determined in the field theory of gravitation from the covariant equation (201) in the Riemannian space-time are equal to the integrals of the motion (196) obtained in the pseudo-Euclidean space-time from the conservation law (188). It must be emphasized that to compare the integrals of the motion the calculations in the two cases must be made in the same coordinate system, since different expressions for the integrals of the motion correspond to different coordinate systems. Therefore, we shall calculate the integrals of the motion in a "noncanonical" coordinate system of the Riemannian space-time in which the metric tensor has the form (179) and corresponds to the coordinate system of the pseudo-Euclidean space-time in which we found the integrals of the motion (196) and (197). We note also that in the given case transition to the canonical coordinate system, in which the metric tensor g_{ni} has the form (182), does not change the post-Newtonian expressions for the integrals of the motion in the field theory of gravitation. But in the general case different expressions for the integrals of the motion correspond to different coordinate systems.

Using the post-Newtonian expansion (179) of the metric tensor and the definition (156), we find the components of the energy-momentum tensor density of the matter to post-Newtonian accuracy:

$$\left.\begin{aligned} T^{00} &= \rho\left[1 + \Pi + v^2/2 + U\right] + \rho O(\varepsilon^4); \\ T^{0\alpha} &= \rho v^\alpha\left[1 + \Pi + v^2/2 + U\right] + pv^\alpha + \rho O(\varepsilon^5); \\ T^{\alpha\beta} &= \rho v^\alpha v^\beta\left[1 + \Pi + v^2/2 + U\right] + \\ &+ pv^\alpha v^\beta - p(1 - U)\gamma^{\alpha\beta} + \rho O(\varepsilon^6). \end{aligned}\right\} \qquad (202)$$

We write down Eqs. (201) component-by-component:

$$\left.\begin{aligned} &\partial_0 T^{00} + \partial_\alpha T^{0\alpha} + \Gamma^0_{00}T^{00} + 2\Gamma^0_{0\alpha}T^{0\alpha} + \Gamma^0_{\alpha\beta}T^{\alpha\beta} = 0; \\ &\partial_0 T^{0\alpha} + \partial_\beta T^{\alpha\beta} + \Gamma^\alpha_{00}T^{00} + 2\Gamma^\alpha_{0\beta}T^{0\beta} + \Gamma^\alpha_{\beta\varepsilon}T^{\beta\varepsilon} = 0. \end{aligned}\right\} \qquad (203)$$

Since the components of the energy-momentum tensor density of the matter are known to accuracy

$$T^{00} \sim \rho O(\varepsilon^4); \; T^{0\alpha} \sim \rho O(\varepsilon^5); \; T^{\alpha\beta} \sim \rho O(\varepsilon^6),$$

the connection of the Riemannian space-time must be obtained to accuracy

$$\Gamma^0_{00} \sim O(\varepsilon^5); \quad \Gamma^0_{0\alpha} \sim O(\varepsilon^4); \quad \Gamma^0_{\alpha\beta} \sim O(\varepsilon^3);$$
$$\Gamma^\alpha_{00} \sim O(\varepsilon^6); \quad \Gamma^\alpha_{0\beta} \sim O(\varepsilon^5); \quad \Gamma^\alpha_{\beta\varepsilon} \sim O(\varepsilon^4).$$

Using the post-Newtonian expansion (179) of the metric, we determine the connection of the Riemannian space-time to the required accuracy:

$$\left.\begin{aligned}
&\Gamma^0_{00} = -\partial U/\partial t + O(\varepsilon^5); \quad \Gamma^0_{0\alpha} = -\partial_\alpha U + O(\varepsilon^4);\\
&\Gamma^\alpha_{\beta\varepsilon} = \delta^\alpha_\varepsilon \partial_\beta U + \delta^\alpha_\beta \partial_\varepsilon U - \gamma_{\beta\varepsilon}\partial^\alpha U + O(\varepsilon^4);\\
&\Gamma^\alpha_{0\beta} = \delta^\alpha_\beta\, \partial U/\partial t + 2\gamma^{\alpha\varepsilon}(\partial V_\varepsilon/\partial x^\beta - \partial V_\beta/\partial x^\varepsilon) + O(\varepsilon^5);\\
&\Gamma^0_{\alpha\beta} = O(\varepsilon^3);\\
&\Gamma^\alpha_{00} = 4\partial V^\alpha/\partial t + \gamma^{\alpha\beta}(1-2U)\,\partial U/\partial x^\beta\\
&\quad -\frac{\gamma^{\alpha\beta}}{2}\frac{\partial}{\partial x^\beta}[2\beta U^2 - 4\Phi_1 + 4(\beta-2)\Phi_2\\
&\qquad -2\Phi_3 - 6\Phi_4 - w] + O(\varepsilon^6),
\end{aligned}\right\} \quad (204)$$

where we have introduced the notation

$$w = \frac{\partial^2}{\partial t^2} Q; \quad Q = \int dV \rho R.$$

Substituting the expressions (202) and (204) in the first equation of (203), we obtain

$$\frac{\partial}{\partial t}\left[\rho\left(1+\frac{v^2}{2}+\Pi+U\right)\right] + \partial_\alpha\left[\rho v^\alpha\left(1+\Pi+U+\frac{v^2}{2}\right)+pv^\alpha\right]$$
$$-\rho\frac{\partial U}{\partial t} - 2\rho v^\alpha \partial_\alpha U = \rho O(\varepsilon^5). \quad (205)$$

Using the expressions (204), we write the second equation of (203) in the form

$$\frac{\partial}{\partial t}\left[\rho v^\alpha\left(1+\Pi+\frac{v^2}{2}+U\right)+pv^\alpha\right]$$
$$+\partial_\beta\left[\rho v^\alpha v^\beta\left(1+\Pi+U+\frac{v^2}{2}\right)+pv^\alpha v^\beta - p(1-2U)\gamma^{\alpha\beta}\right]$$
$$+\rho\left(1+\Pi+U+\frac{v^2}{2}\right)\partial^\alpha U + 4\rho\frac{\partial V^\alpha}{\partial t}$$
$$-(2+2\beta)\rho U\,\partial^\alpha U + \rho\partial^\alpha[2\Phi_1 - 2(\beta-2)\Phi_2 + \Phi_3 + 3\Phi_4 + w/2]$$
$$+2\rho v^\alpha\frac{\partial U}{\partial t} + 4\rho v^\beta(\partial_\beta V^\alpha - \partial^\alpha V_\beta) + 2\rho v^\alpha v^\beta \partial_\beta U$$
$$+\rho v^2\,\partial^\alpha U + p\,\partial^\alpha U = \rho O(\varepsilon^6). \quad (206)$$

To simplify these expressions, we use the continuity equation of the ideal fluid,

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x^\alpha}(\rho v^\alpha) = 0; \quad \rho = \rho_0[1+3U+v^2/2+O(\varepsilon^4)],$$

the Newtonian equations of motion of an elastic body,

$$\rho\, dv^\alpha/dt = \gamma^{\alpha\beta}(-\rho\,\partial U/\partial x^\beta + \partial p/\partial x^\beta); \quad \rho\, d\Pi/dt = -p\,\partial_\alpha v^\alpha,$$

and also the relations

$$\partial V_\alpha/\partial x^\beta - \partial V_\beta/\partial x^\alpha = \partial W_\alpha/\partial x^\beta - \partial W_\beta/\partial x^\alpha; \quad \partial_\alpha\partial^\alpha U = 4\pi\rho_0;$$
$$\rho\frac{\partial U}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(\rho U) + \frac{1}{8\pi}\frac{\partial}{\partial x^\alpha}\left[\frac{\partial U}{\partial t}\partial^\alpha U - \gamma^{\alpha\beta}U\frac{\partial^2 U}{\partial t\,\partial x^\beta}\right].$$

As a result of these transformations, we obtain from the expression (205)

$$\frac{\partial}{\partial t}\left[\rho\left(1+\Pi-\frac{1}{2}U+\frac{v^2}{2}\right)\right] + \partial_\alpha\left[\rho v^\alpha\left(1+\Pi+U+\frac{v^2}{2}\right)\right.$$
$$\left.+pv^\alpha - \frac{1}{8\pi}\frac{\partial U}{\partial t}\partial^\alpha U + \frac{1}{8\pi}\gamma^{\alpha\beta}U\frac{\partial^2 U}{\partial t\,\partial x^\beta}\right] = \rho O(\varepsilon^4). \quad (207)$$

We reduce the expression (206) to the form

$$\frac{\partial}{\partial t}[\rho v^\alpha(1+\Pi+v^2/2+U)+pv^\alpha]$$
$$+\partial_\beta[\rho v^\alpha v^\beta(1+\Pi+v^2/2+U)+pv^\alpha v^\beta - p\gamma^{\alpha\beta}]$$
$$+4\rho\, dV^\alpha/dt + 2\rho\frac{d}{dt}(Uv^\alpha) + \rho_0\gamma^{\alpha\beta}\frac{\partial U}{\partial x^\beta} + (4-2\beta)\rho U\,\partial^\alpha U$$
$$+\rho(\Pi+2v^2)\,\partial^\alpha U + 3p\,\partial^\alpha U + 3\rho\gamma^{\alpha\beta}\,\partial\Phi_4/\partial x^\beta - 4\rho v^\beta\,\partial^\alpha V_\beta$$
$$+\rho\partial^\alpha[2\Phi_1+\Phi_3-2(\beta-2)\Phi_2+w/2] = \rho O(\varepsilon^6). \quad (208)$$

We integrate these expressions over the whole of space. We note first that

$$\left.\begin{aligned}
&\int dV\rho_0\left[\Pi\frac{\partial U}{\partial x^\beta}+\frac{\partial\Phi_3}{\partial x^\beta}\right]=0;\\
&\int dV\rho_0\left[U\frac{\partial U}{\partial x^\beta}+\frac{\partial\Phi_2}{\partial x^\beta}\right]=0;\\
&\int dV\left[p\frac{\partial U}{\partial x^\beta}+\rho_0\frac{\partial\Phi_4}{\partial x^\beta}\right]=0;\\
&\int dV\rho_0\left[v^2\frac{\partial U}{\partial x^\beta}+\frac{\partial\Phi_1}{\partial x^\beta}\right]=0.
\end{aligned}\right\} \quad (209)$$

Indeed, let us consider, for example, the first relation. Using the expressions (151), we obtain

$$\int dV\,dV'\rho_0\rho_0'\left[\frac{\Pi(x_\beta-x'_\beta)}{|x-x'|^3}+\frac{\Pi'(x_\beta-x'_\beta)}{|x-x'|^3}\right].$$

Because of the antisymmetry of the integrand with respect to the substitution $\rho_0 \leftrightarrow \rho_0'$, $x_\beta \leftrightarrow x'_\beta$, this integral vanishes. The remaining relations in (209) can be proved similarly.

In addition, we use the obvious equations

$$\int \rho_0 v^\varepsilon\frac{\partial V_\varepsilon}{\partial x^\beta}dV = \int\rho_0 v^\varepsilon\frac{\partial W_\varepsilon}{\partial x^\beta}dV = 0;$$
$$\int\rho_0\frac{\partial U}{\partial x^\beta}dV = 0; \quad \int\rho v^\beta\frac{\partial^3 Q}{\partial t\,\partial x^\alpha\,\partial x^\beta}dV = 0;$$
$$\frac{d}{dt}\int\rho f\,dV = \int dV\rho\left[\frac{d}{dt}f + fO(\varepsilon^2)\right].$$

We also use the circumstance that integrals over a volume of a spatial divergence vanish after transformation into surface integrals.

As a result, from the expression (207) in the post-Newtonian approximation of the field theory of gravitation we obtain the energy integral $dP^0/dt=0$, where

$$P^0 = \int d V\rho\left[1+\Pi+\frac{v^2}{2}-\frac{U}{2}\right] = \text{const}. \quad (210)$$

Bearing in mind that

$$\rho\partial^\alpha\frac{\partial^2}{\partial t^2}Q = \frac{\partial}{\partial t}\left(\rho\partial^\alpha\frac{\partial}{\partial t}Q\right) + \partial_\beta\left[\rho v^\beta\partial^\alpha\frac{\partial}{\partial t}Q\right] - \rho v^\beta\partial^\alpha\partial_\beta\frac{\partial}{\partial t}Q,$$

we obtain $dP^\alpha/dt=0$ from the expression (208). It follows that

$$P^\alpha = \int dV\left[\rho v^\alpha(1+\Pi+v^2/2+3U)\right.$$
$$\left.+pv^\alpha + 4\rho V^\alpha + \frac{1}{2}\partial^\alpha\frac{\partial}{\partial t}Q\right] = \text{const}. \quad (211)$$

Since

$$\partial^\alpha\frac{\partial}{\partial t}Q = W^\alpha - V^\alpha; \quad \int dV\rho V^\alpha = -\int dV\rho U v^\alpha,$$

the expression (211) can be reduced to the form

$$P^\alpha = \int dV\rho\left[v^\alpha\left(1+\Pi+\frac{v^2}{2}-\frac{U}{2}+\frac{p}{\rho}\right)+\frac{W^\alpha}{2}\right]. \quad (212)$$

Thus, in the field theory of gravitation the post-Newtonian integrals of the motion determined from the conservation law (188) in the pseudo-Euclidean space-time and from the covariant equation (201) in the Riemannian space-time are identical. This is a direct consequence of the fact that in the field theory of gravitation the conservation law (188) and Eq. (201) are different forms of expression of the same conservation law. In the field theory, the gravitational field is a physical field possessing energy and momentum densities, and it contributes to the total energy-momentum tensor of the system. It is the presence in the field theory of gravitation of ordinary conservation laws that enables us to carry out energy calculations, including the finding of the post-Newtonian expressions for the integrals of the motion.

In general relativity, the field is not a field in the spirit of Faraday and Maxwell, as a result of which there is no possibility in Einstein's theory of making calculations of the gravitational field energy. However, in general relativity post-Newtonian integrals of the motion of an isolated system are usually obtained from the covariant equation (201) and lead to the expressions (210) and (212).

It is easy to understand the reason for this agreement. The point is that in general relativity one can radily deduce from the covariant conservation equation (201) either the conservation law (36), which is trivially satisfied on the basis of the Einstein equations, or a relation containing an energy-momentum pseudotensor:

$$\partial_i[-g(T^{ik}+\tau^{ik})]=0. \quad (213)$$

In the field theory of gravitation, the conservation equation (201) is equivalent to a conservation law for the energy-momentum tensor of the matter and gravitational field taken together:

$$D_i[t_g^{ik}+t_M^{ik}]=0. \quad (214)$$

From the relation (213) in general relativity one can obtain integral quantities that are independent of the time in the lowest approximation:

$$J^k=\int dV\,(-g)[T^{0k}+\tau^{0k}]. \quad (215)$$

However, they cannot be interpreted as integrals of the motion, since the quantities (215) reflect the choice of the coordinates rather than physical characteristics of the system consisting of the matter and the gravitational field. In particular, by an appropriate choice of the coordinates that leaves the metric tensor of the Riemannian space-time asymptotically Galilean, these quantities can be made equal to any preassigned values, both positive and negative. Therefore, the quantities (215) are not integrals of the motion in general relativity. However, in the lowest approximation in the Cartesian coordinates the expressions (213) and (214) are equal, which leads to equality in this coordinate system of the expressions for the quantities (215) and the expressions (190) for the integrals of the motion of the field theory of gravitation.

It is because one can construct a theory of gravitation in pseudo-Euclidean space-time possessing a tensor conservation law for the energy and momentum of the matter and the gravitational field taken together that the lowest approximations for (215) in a small class of coordinate systems give results that agree with the expressions for the integrals of the motion (196) of the field theory of gravitation. However, because of the different transformation laws in any other coordinate system these expressions differ appreciably, though the quantities (190) in the field theory of gravitation retain the meaning of integrals of the motion. Since the quantities (215) in the general theory of relativity do not have the physical meaning of integrals of the motion, it is easy to obtain physically absurd results from the expression (215) in other coordinate systems.

Therefore, the interpretation that is usually made in Einstein's theory of these quantities as the energy and momentum of an isolated system is incorrect.

To conclude this section, we note that in the Newtonian approximation the energy of a static field in the field theory of gravitation, calculated using the canonical energy-momentum tensor (100), is positive:

$$\int \tilde{t}_g^{00}\,dV=-\frac{1}{8\pi}\int dV\,\partial_\alpha U\,\partial^\alpha U>0,$$

but if the symmetric energy-momentum tensor (104) is used, it is negative:

$$\int t_g^{00}\,dV=\frac{3}{8\pi}\int dV\,\partial_\alpha U\,\partial^\alpha U<0.$$

It is well known that in electrodynamics the opposite situation obtains, namely, the energy of an electrostatic field calculated using the canonical energy-momentum tensor is negative but according to the symmetric tensor it is positive. From this analogy it can be concluded that a static gravitational field is a field of attractive forces, whereas in electrodynamics charges of the same sign produce a field of repulsive forces.

Calculation of the total energy of the matter and the static gravitational field in the Newtonian approximation give the same result when either the canonical or the symmetric energy-momentum tensor is used:

$$P^0=\int dV\,[t_g^{00}+t_M^{00}]=\int dV\,[\tilde{t}_g^{00}+\tilde{t}_M^{00}] =\int dV\rho\left[1+\Pi+\frac{v^2}{2}-\frac{U}{2}\right].$$

It follows from this expression that the energy of two particles at rest increases with increasing distance between them, which also indicates that forces of attraction act between them.

12. GRAVITATIONAL EXPERIMENTS IN THE SOLAR SYSTEM

We consider what restrictions are imposed on the values of the post-Newtonian parameters by experiments made in the solar system.

We shall analyze these experiments in the following order. First, we consider the standard effects—the deflection of light and radio waves in the field of the Sun, the advance of the perihelion of Mercury, and measurement of the time delay of a radio signal in the gravitational field of the Sun; we then consider the Nordtvedt effect, and also the effects associated with nonvanishing of the parameters α_1, α_2, α_3, ξ_w. We shall not consider the red shift in the gravitational field of the Sun, since it is completely described in the Newtonian approximation.[87]

Deflection of light and radio waves in the gravitational field of the sun. According to the calculations in Ref. 88, light rays and radio waves, which can be regarded as massless particles with impact parameter b, are deflected in the gravitational field of the Sun through the angle

$$\delta\varphi=2\,(1+\gamma)\,M/b.$$

Analysis of the experimental results obtained by observing the bending in the gravitational field of the Sun of the rays of light from distant stars, and also the radio waves emitted by quasars, indicates the value $\gamma = 1 \pm 0.2$ of the post-Newtonian parameter γ.[72]

Time delay of radio signals in the field of the sun. Another independent method of determining the post-Newtonian parameter γ is to measure the time delay of radio signals in the field of the Sun.[89] The time of propagation of radio signals (measured by clocks on the Earth) sent from the Earth to a reflector somewhere else in the solar system and back differs from the time of this process in the absence of a gravitational field.

The experiments made to measure the time delay use the surfaces of planets, and also transponders on satellites as reflectors. These experiments[90] give the value $\gamma = 1 \pm 0.002$.

In the field theory of gravitation, as in Einstein's general theory of relativity, the parameter γ has the value $\gamma = 1$, which is in good agreement with the results of these experiments.

Precession of a gyroscope in orbit. If $\alpha_1 = \alpha_2 = \alpha_3 = 0$, measurement of the precession of a gyroscope moving in orbit around the Earth will be a third independent method for measuring the parameter γ. According to the calculations in Ref. 91, the angular velocity of precession of a gyroscope in a circular orbit in the Earth's field is

$$\Omega = \frac{2\gamma + 1}{2} m \frac{[\mathbf{r}\mathbf{v}]}{r^3} + \frac{\gamma + 1}{2r^3}\left(-\mathbf{J} + \frac{3\mathbf{r}(\mathbf{r}\mathbf{J})}{r^2}\right),$$

where m is the mass of the Earth, $\mathbf{v}$ is the linear velocity of the gyroscope with respect to the center of the Earth, $\mathbf{J}$ is the angular momentum of the Earth, and $\mathbf{r}$ is the radius vector of the point at which the gyroscope is situated.

The present level of development of the relevant technology[92-94] offers hope that this experiment will be performed in the near future.

Advance of mercury's perihelion. The advance of Mercury's perihelion is influenced by various factors in addition to the post-Newtonian corrections in the equation of motion. These include the attraction exerted by the planets in the solar system and a quadrupole moment of the Sun. The only undetermined factor among them is the quadrupole moment of the Sun; the influence of all other factors can be calculated with sufficient accuracy.

The total advance of Mercury's perihelion due to the presence of a quadrupole moment J_2 of the Sun and the post-Newtonian corrections in the equation of motion is[95]

$$\delta\varphi = 42.98[(2 + 2\gamma - \beta)/3] + 1.3 \cdot 10^5 J_2$$

(in seconds of arc per century). It follows from the results of the observations[72] that $\delta\varphi = 41.4 \pm 0.9$ seconds of arc per century.

Measurements of the apparent shape of the Sun made by Dicke and Goldberg[96] gave for J_2 the value $J_2 = (2.5 \pm 0.2) \times 10^{-5}$, while the later measurements of Hill *et al.*[95] showed that $J_2 < 0.5 \times 10^{-5}$. Comparison of the observed displacements of the perihelia of Mercury and Mars[97] gave a bound for J_2: $J_2 \leq 3 \times 10^{-5}$.

Thus, because of the absence of direct measurements of the quadrupole moment of the Sun, there remains a large uncertainty in the value of β determined from the advance of Mercury's perihelion: $\beta = 1^{+0.4}_{-0.2}$. Note that in the field theory of gravitation the parameter β has the value $\beta = 1$ and permits description of this effect to within the errors.

The nordtvedt effect and lunar laser ranging. Until recently, the requirements imposed on possible theories of gravitation reduced to the need to obtain Newton's law of gravitation in the weak-field limit and also to describe the three effects accessible to observation: the gravitational red shift in the field of the Sun, the bending of light rays passing near the Sun, and the advance of Mercury's perihelion. Recently, great attention has been devoted to the establishment in the various theories of gravitation of relations between the inertial and gravitational masses of a body and the search for possibilities of verifying these relations in an experiment.

In any theory of gravitation, as was noted by Bondi,[127] we can distinguish three kinds of mass, depending on the measurements by means of which they are determined: the inertial mass m_i, the passive gravitational mass m_p, and the active gravitational mass m_a.

The *inertial mass* is the mass which occurs in (and is defined by) Newton's second law:

$$m_i a^\alpha = F^\alpha.$$

The *passive gravitational mass* is the mass on which the gravitational field acts, i.e., the mass determined by the expression

$$F^\alpha = -m_p \nabla^\alpha V.$$

The *active gravitatinal mass* is the mass that is the source of the gravitational field.

In Newtonian mechanics, Newton's third law requires equality of the active and passive masses, $m_a = m_p$, irrespective of the size and composition of the bodies; the equality of the inertial mass to the two remaining masses is regarded as an empirical fact. In Einstein's theory for point bodies, the inertial and passive gravitational masses are equal. Equality of the active and passive gravitational masses is not postulated. In some theories of gravitation, all three masses may be different for a given body. It is therefore necessary to establish by means of experiments the correspondence between these three masses.

One of the first attempts to measure the ratio of the passive gravitational mass m_p to the inertial mass m_i was made in the last century by Bessel and Eötvös. These measurements established that for bodies of laboratory size the ratio of the gravitational mass to the inertial mass can differ from unity by not more than 10^{-9} irrespective of the matter of which the body is composed. This result prompted Einstein to formulate the equivalence principle.

However, although this result was interpreted as proving equality of the gravitational and inertial masses to a very high accuracy, this does not mean that bodies of large dimensions will also have equal gravitational and inertial masses to the same accuracy. For bodies of laboratory size, the gravitational self-energy, the energy of elastic deformations of the body, etc., are very small quantities compared with the total energy of the body. In particular, for a body of mass M having characteristic dimension a the ratio of the gravitational self-energy of a body to its total energy is

$$\frac{GM^2/a}{Mc^2}=\frac{GM}{c^2a}\sim\frac{G\rho a^2}{c^2},$$

where ρ is the density of the body.

This ratio is 10^{-25} in order of magnitude for bodies of laboratory size. Therefore, if the measurements have an accuracy of 10^{-9}, nothing can be said about how the gravitational self-energy is distributed between the inertial and gravitational masses of the body. And even the gravimetric experiments made with higher accuracy (10^{-11} in the experiments of Dicke's group[128] and 10^{-12} in the experiments of Braginskiĭ's group[104]) do not provide an answer to this question.

It can therefore be asserted that the gravimetric measurements tell us that there is equality of the gravitational and inertial masses of a point body, i.e., a body having a negligibly small size, so that the gravitational self-energy, the energy of elastic deformations, etc., are also negligibly small. To establish whether the gravitational and inertial masses of an extended body are equal, it is necessary to increase significantly the accuracy of gravimetric experiments with bodies of laboratory size, which at the present level of development of technology is still impossible, or to make measurements with larger bodies, for example, planets, for which the ratio of the gravitational self-energy to the total energy is appreciably higher than in bodies of laboratory size.

However, since gravimetric measurement of the ratio of the passive gravitational mass of an extended body (a planet) to its inertial mass is impossible, it was necessary to make a theoretical study of the motion of extended bodies in the gravitational field of other bodies with a view to establishing what features in the motion of an extended body were derived from a possible inequality between its inertial and gravitational masses.

One such effect is a possible deviation at the post-Newtonian level of the motion of the center of mass of an extended body from motion along a geodesic of Riemannian space-time. The possibility of such an effect was pointed out by Dicke,[129] who suggested that the ratio of the gravitational and inertial masses for astronomical bodies could differ slightly from unity if the gravitational self-energy of these bodies changes with their position in the gravitational field of other bodies. Subsequently, this effect was investigated by Nordtvedt,[130] Will,[131] and Dicke.[132]

Using a model of coherent particles, Nordtvedt[130] made a very detailed study of this effect, which subsequently became known as the Nordtvedt effect, and showed that it can occur in several metric theories of gravitation.

Will,[131] who had in mind subsequent application of the results of his calculations to a system consisting of the Sun and one of its planets, showed on the basis of the parametrized post-Newtonian expansion that the equations of motion of the center of mass of an extended body (a planet) in the gravitational field of a point source at rest (the Sun) will have the form

$$Ma^{\alpha}=-(M_0/R^2)\,f^{\alpha}, \tag{216}$$

where M is the mass of the extended body, M_0 is the active gravitational mass of the point source at rest, a^α are the components of the acceleration of the center of mass of the extended body, and R is the distance between the point body and the center of mass of the extended body.

For the vector f^α in this case the following expression was obtained:

$$f^{\alpha}/M=n^{\alpha}[1-(4\beta-\alpha_1-\gamma-3-\xi_1+\alpha_2)\,\Omega^{\varepsilon}_{\varepsilon}] + (\xi_2+\alpha_2-\xi_1)\,\Omega^{\alpha\beta}n_{\beta}, \tag{217}$$

where $n^\alpha=R^\alpha/R$, and $\Omega^\varepsilon_\varepsilon$ and $\Omega^{\alpha\beta}$ are the post-Newtonian corrections:

$$\Omega^{\alpha\beta}=-\frac{1}{2M}\int\rho(x,t)\,\rho(x',t)\,\frac{(x^{\alpha}-x'^{\alpha})(x^{\beta}-x'^{\beta})}{|x-x'|^3}\,dx\,dx';$$

$$\Omega^{\varepsilon}_{\varepsilon}=\gamma_{\alpha\varepsilon}\Omega^{\alpha\varepsilon}=\frac{1}{2M}\int\frac{\rho(x,t)\,\rho(x',t)}{|x-x'|}\,dx\,dx'.$$

Assuming that the vector f^α is related to the tensor $m_p^{\alpha\beta}$ of the passive gravitational mass by

$$f^{\alpha}=-m_p^{\alpha\beta}n_{\beta}, \tag{218}$$

Will[103] obtained for this tensor the expression

$$m_p^{\alpha\beta}/M=-\gamma^{\alpha\beta}[1-(4\beta-\alpha_1-\gamma-3-\xi_1+\alpha_2)\,\Omega^{\varepsilon}_{\varepsilon}] - (\xi_2+\alpha_2-\xi_1)\,\Omega^{\alpha\beta}. \tag{219}$$

In such an approach, the presence of the post-Newtonian corrections in the expression (217) for the vector f^α is interpreted as the result of violation in some theories of gravitation of the equality of the passive gravitational mass and the inertial mass of an extended body at the post-Newtonian level. At the same time, it is asserted that equality of the inertial and passive gravitational masses in the post-Newtonian approximation will mean that the center of mass of an extended body moves along a geodesic of Riemannian space-time.

However, under the conditions of a real experiment it is also rather difficult to determine whether the center of mass of an extended body does or does not move along a geodesic of Riemannian space-time. Therefore, it was suggested that the values of all the necessary post-Newtonian parameters should be determined from experiments and, using Will's expression (219), one should find an answer to the already academic questions of the relationship between the tensor of the passive gravitational mass of an extended body and its inertial mass and also establish the nature of the motion of the center of mass of such a body with respect to a geodesic of Riemannian space-time. As a result of a calculation of the motion of the Earth-Moon system in the gravitational field of the Sun, Nordtvedt[98,100] pointed out a number of possible anomalies in the motion of the Moon, the observation of which would permit measurement of various combinations of the post-Newtonian parameters. One such anomaly is a polarization of the Moon's orbit in the direction of the Sun with amplitude $\delta r = \eta L$, where L is a constant of order 10 m:

$$\eta = -(1/3)\,(\xi_2 + \alpha_2 - \xi_1) + (4\beta - \xi_1 - \gamma - \alpha_1 + \alpha_2 - 3) - (10/3)\,\xi_w. \quad (220)$$

To detect this effect, the data obtained by lunar laser ranging were analyzed. The analysis of one of the groups[101] led to the conclusion that $\eta = 0 \pm 0.03$. The other group[102] obtained a similar result: $\eta = -0.001 \pm 0.015$. Using these estimates and Will's theoretical expression (219) for the tensor of the passive gravitational mass, the authors of Refs. 101 and 102 concluded that the ratio of the passive gravitational mass of an extended body to its inertial mass is equal to unity within the errors of the measurement:

$$|\, m_p^{\alpha\beta}/M - \delta^{\alpha\beta} \,| < 1.5 \cdot 10^{-11}.$$

Thus, the data obtained from the lunar laser ranging appeared to confirm (and such a conclusion was drawn in Refs. 72, 101, and 102) that the passive gravitational mass of an extended body is equal to its inertial mass and that the center of mass of an extended body moves along a geodesic of Riemannian space-time.

However, these conclusions are incorrect, since they are based on the use of false premises. Below, we shall show especially that the center of mass of an extended body in the post-Newtonian approximation does not move along a geodesic of Riemannian space-time at all in any metric theory of gravitation that possesses conservation laws for the energy and momentum of the matter and the gravitational field taken together. In addition, as will be seen from what follows, the lunar laser ranging experiments in conjunction with other experiments indicate that the ratio of the passive gravitational mass of the Earth to its inertial mass is not equal to unity but differs from it by

$$|\, m_p^{\alpha\beta}/M - \delta^{\alpha\beta} \,| \approx 10^{-8}.$$

It should be noted that the equality or nonequality of the passive gravitational mass of an extended body to its inertial mass cannot serve as an indicator of whether the center of mass of the extended body moves along a geodesic of Riemannian space-time or not. As we shall see below, equality of these masses guarantees only that the post-Newtonian equations of motion of the center of mass of an extended body are the same as the corresponding equations of Newton's theory, which is in no way a condition for them to be the same as the equations of geodesics. In addition, the introduction of the concept of a tensor of the passive gravitational mass, as done by Will for the solar system, is not possible in the general case of an arbitrary post-Newtonian system (see Sec. 17). The solar system is a very specific (though, perhaps, common) type of post-Newtonian system characterized by the fact that one of the bodies of the system (the Sun) has a mass greatly exceeding the mass of the remaining bodies of the system taken together. This has the consequence that in the frame of reference associated with the center of mass of such a post-Newtonian system the massive body has a negligibly small velocity, whereas the other bodies of the system move around it with appreciable velocities $v \approx 10^{-4}$ sec. In this case, on the right-hand side of the equations of motion (216) of the extended body one can separate a factor (as Will did) that includes all the characteristics of the extended body and call it the tensor of the passive gravitational mass of the body.

However, in an arbitrary post-Newtonian system all bodies can have comparable masses; the velocities of their motion relative to the frame of reference associated with the center of mass of the system may be quantities of the same order. A separation of the characteristics of the bodies on the right-hand side of the equations of motion (216) in the post-Newtonian approximation no longer occurs, and therefore the introduction of a tensor of the passive gravitational mass is not possible in the general case.

These conclusions apply fully to the field theory of gravitation, which is a definite representative of metric theories of gravitation and possesses conservation laws for the energy and momentum of the matter and the gravitational field taken together.

It should also be noted that in the field theory of gravitation $\eta = 0$ and, thus, it makes it possible to describe the lunar laser ranging experiments to within the errors of measurement.

Effects associated with the existence of a preferred frame of reference. Theories of gravitation in which at least one of the parameters $\alpha_1, \alpha_2, \alpha_3$ is nonzero have a preferred frame of reference. The predictions of such theories of gravitation for the standard effects can agree with the results of observations only if the solar system is a preferred frame of reference. It is more sensible to assume that the solar system, which moves with respect to other stellar systems, is not distinguished compared with them, and therefore cannot be a preferred universal rest frame for such theories. Since a preferred rest frame must be distinguished in some way from other systems, it is more sensible to associate the rest frame with the center of mass of the Galaxy or even the Universe. Then the solar system will be in motion relative to the perferred rest frame with a velocity of about $10^{-3}c$, i.e., of the same order as the orbital velocity of the solar system with respect to the center of the Galaxy. In such a case, it will be possible to observe a number of effects associated with the motion relative to the preferred rest frame,[72] which will make it possible to estimate the parameters $\alpha_1, \alpha_2, \alpha_3$.

In theories of gravitation with a preferred rest frame, the gravitational constant G measured in gravimetric experiments will depend on the motion of the Earth with respect to such a frame.

For the relative value $\Delta G/G$, we have[72]

$$\Delta G/G = \left(\frac{\alpha_2}{2} + \alpha_3 - \alpha_1\right) \mathbf{wv} + (1/4)\,\alpha_2\,[(\mathbf{ve}_r)^2 + 2\,(\mathbf{we}_r)\,(\mathbf{ve}_r) + (\mathbf{we}_r)^2],$$

where $\mathbf{v}$ is the orbital velocity of the Earth around the

Sun, w is the velocity of the Sun with respect to the preferred rest frame, and e_r is a unit vector directed from the gravimeter to the center of the Earth.

Because of the rotation of the Earth about its axis, the vector e_r changes its orientation with respect to the vectors v and w, which leads to a periodic change of the scalar products $(v \cdot e_r)$ and $(w \cdot e_r)$ with period approximately equal to 12 h. This leads to corresponding periodic changes in the values of the acceleration of free fall, and for a point of observation at latitude θ we have

$$\Delta g/g \approx 3\alpha_2 \cdot 10^{-8} \cos^2\theta.$$

Will,[103] analyzing the results of gravimetric experiments, found that the relative variations of g do not exceed $|\Delta g/g| < 10^{-9}$; hence, we obtain $|\alpha_2| < 3\times 10^{-2}$.

The motion of the Earth around the Sun also leads to a periodic variation of $(w \cdot v)$ with a period of the order of a year. This variation gives rise to a contraction and expansion of the Earth, which, in its turn, leads to periodic variations in the angular velocity of the Earth's rotation as a result of the change of its moment of inertia:

$$\Delta\omega/\omega \approx 3\cdot 10^{-9}\,(\alpha_3 + (2/3)\,\alpha_2 - \alpha_1).$$

It follows from the results of the observations that

$$|\alpha_3 + (2/3)\,\alpha_2 - \alpha_1| < 0.2.$$

Motion of the solar system with respect to the center of the Universe may lead to an anomalous displacement $\delta\varphi_0$ of the perihelia of the planets. For Mercury,[72] the additional contribution to the displacement of the perihelion (in seconds of arc per century) has the form

$$\delta\varphi_0 = 35\alpha_1 + 8\alpha_2 - 4\cdot 10^4\alpha_3.$$

Comparison with the observations and combining of all these estimates of the parameters α give

$$|\alpha_1| < 0.2;\ |\alpha_2| < 3\cdot 10^{-2};\ |\alpha_3| < 2\cdot 10^{-5}.$$

In the field theory of gravitation, as in Einstein's theory, $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and, therefore, all these effects are absent.

Effect of anisotropy with respect to the center of the galaxy. In theories of gravitation for which $\xi_w \neq 0$, it is possible to have anisotropy effects due to the influence of the gravitational field of the Galaxy.[81] If it is assumed that the mass M of the Galaxy is concentrated at the center of the Galaxy at distance R from the solar system, then the gravitational field of the Galaxy leads to periodic changes in the readings of a gravimeter with period 12 h:

$$\Delta G/G = \xi_w\,[1 - 3K/(mr^2)]\,(M/R)\,(e_r e_R),$$

where K is the moment of inertia, m is the mass, r is the radius of the Earth, e_r is the unit vector directed from the gravimeter to the center of the Earth, and $e_R = R/R$.

Another effect is the anomalous displacement of the perihelia of the planets due to the anisotropy produced by the Galaxy:

$$\delta\varphi_0 = \frac{\pi\xi_w}{2}\,\frac{M}{R}\cos^2\beta\cos^2(\omega - \lambda),$$

where λ and β are the angular coordinates of the center of the Galaxy, and ω is the angle of the perihelion of the planet in geocentric coordinates. Comparison with the observations gives $|\xi_w| < 10^{-2}$ as an upper limit.

In the field theory of gravitation, as in Einstein's theory, $\xi_w = 0$, and all anisotropy effects associated with the gravitational field of the Galaxy are absent.

Concluding our review of the gravitational experiments, we conclude that the field theory of gravitation makes it possible to describe all the experimental facts. It should be noted that in the post-Newtonian limit the quadratic terms in the connection equation (94) are indistinguishable, and no experiment in the gravitational field of the solar system at the post-Newtonian level is capable of determining the minimal-coupling parameters separately.

As will be shown below in Sec. 16, measurement of the deceleration parameter of an expanding homogeneous Universe in the neighborhood of the present epoch makes it possible to determine the value of a different linear combination of these minimal-coupling parameters. Experiments in the gravitational field of the solar system can be used to determine the minimal-coupling parameters only after an increase in the accuracy of the measurements to the post-Newtonian level.

In conclusion, we note that in the field theory of gravitation the equivalence principle is valid only for point bodies. For extended bodies moving in a weak gravitational field it is satisfied approximately to the accuracy with which the gravitational field can be assumed to be homogeneous in the region occupied by the body. In this case, we can "eliminate" the gravitational field by going over to a coordinate system in which $g_{ni} = \gamma_{ni}$ in the region occupied by the matter. As follows from the experiment of Braginskii's group[104] with bodies of laboratory size in a sufficiently uniform gravitational field, the equivalence principle holds for the strong, electromagnetic, and weak interactions to the accuracy achieved in these experiments. However, for extended bodies, when allowance is made for the gravitational field this principle is valid strictly in neither Einstein's general theory of relativity nor the field theory of gravitation.

13. STATIC, SPHERICALLY SYMMETRIC GRAVITATIONAL FIELD

For a static source of radius a with spherically symmetric distribution of the matter the gravitational field equations (90) and the expression (85) for the tensor current simplify appreciably.

On the basis of the symmetry of the problem, let us determine what components of the tensors I_{lm} and h^{lm} will be nonvanishing in this case. We place the origin of a spherical coordinate system at the center of the source. When it is rotated through an arbitrary angle, the physical situation must not change by virtue of the spherical symmetry of the matter distribution. Therefore, after the rotation transformation the components of the tensors I_{lm} and h^{lm} must be the same functions of the transformed argument as the original functions of their original arguments, i.e., these tensors must be form-invariant under a rotation transformation of the coordinate system. It follows that in a spherical coordinate system the only components of the tensors I_{lm} and h^{lm} which can be nonvanishing are the components $(00), (0r), (rr), (\theta\theta), (\varphi\varphi)$, since it is only in this case that the tensors I_{lm} and h^{lm} are form-invariant under the rotation transformation.

It follows from the expressions (85) and (74) that $I_{0r}=0$. Therefore, for a static, spherically symmetric matter distribution the tensor I_{lm} has the components

$$I_{lm}=\{I_{00}, I_{rr}, I_{\theta\theta}, I_{\varphi\varphi}=I_{\theta\theta}\sin^2\theta\}.$$

In this case, the gravitational field equations (90) can be written in the form of a system of ordinary differential equations of second order:

$$\left.\begin{aligned}&f''_{00}+\frac{2}{r}f'_{00}=16\pi I_{00}(r);\\&f''_{rr}+\frac{2}{r}f'_{rr}-\frac{4}{r^2}f_{rr}+\frac{4}{r^2}\left(\frac{f_{\theta\theta}}{r^2}\right)=16\pi I_{rr}(r);\\&\left(\frac{f_{\theta\theta}}{r^2}\right)''+\frac{2}{r}\left(\frac{f_{\theta\theta}}{r^2}\right)'-\frac{2}{r^2}\left(\frac{f_{\theta\theta}}{r^2}\right)+\frac{2}{r^2}f_{rr}=16\pi\frac{I_{\theta\theta}(r)}{r^2}.\end{aligned}\right\}\quad(221)$$

(Here and in what follows, the prime denotes the derivative with respect to r.)

As boundary conditions for these equations, we require the functions f_{00}, f_{rr}, and $f_{\theta\theta}/r^2$ to be bounded at $r=0$ and to vanish as $r\to\infty$. Then the solution of the gravitational field equations (221) will be unique.

However, the components of the tensor current in the expression (221) are not independent by virtue of the conditions $D^l I_{lm}=0$. In our case, these conditions take the form

$$I'_{rr}+\frac{2}{r}\left(I_{rr}-\frac{1}{r^2}I_{\theta\theta}\right)=0.\quad(222)$$

We use this to express the component $I_{\theta\theta}$ and substitute it in Eqs. (221). Integrating the equations and bearing in mind that outside the source $I_{rr}=0$, we write the components of the gravitational field in the form

$$\left.\begin{aligned}&f_{00}=-16\pi\left\{\frac{1}{r}\int_0^r r_0^2\,dr_0\,I_{00}+\int_r^a r_0\,dr_0\,I_{00}\right\};\\&f_{rr}=-\frac{16\pi}{3}\left\{\frac{1}{r^3}\int_0^r r_0^4\,dr_0\,I_{rr}+\int_r^a\frac{dr_0}{r_0}I_{rr}\right\};\\&\frac{f_{\theta\theta}}{r^2}=-\frac{16\pi}{3}\left\{-\frac{1}{2r^3}\int_0^r r_0^4\,dr_0\,I_{rr}+\int_r^a\frac{dr_0}{r_0}I_{rr}\right\}.\end{aligned}\right\}\quad(223)$$

We consider the exterior $(r>a)$ solution. Introducing the quantities

$$M=4\pi\int_0^a r_0^2\,dr_0\,I_{00};\quad \mu=\frac{4\pi}{3}\int_0^a r_0^4\,dr_0\,I_{rr},\quad(224)$$

we obtain for the exterior solution the expressions

$$\begin{aligned}f_{00}&=4M/r;\ f_{rr}=-4\mu/r^3;\\f_{\theta\theta}&=2\mu/r;\ f_{\varphi\varphi}=f_{\theta\theta}\sin^2\theta.\end{aligned}\quad(225)$$

As was already noted in Sec. 6, the fields f_{lm} can be subjected to the gauge transformation

$$f_{lm}\to f_{lm}+D_l a_m+D_m a_l-\gamma_{lm}D_n a^n.\quad(226)$$

with gauge vector a^n satisfying the equation $D_l D^l a^n=0$. Under the transformation, the Lagrangian density of the gravitational field changes only by a four-dimensional divergence, which is unimportant for the theory, and the change in the metric tensor g_{ni} of the Riemannian space-time under the transformation (226) corresponds to coordinate transformations of the Riemannian space-time and can always be eliminated by a suitable choice of the coordinates.

We use the gauge transformation (226) to simplify the exterior solution (225). By virtue of the symmetry of the problem, we choose the gauge vector a_n, which satisfies the condition $D_l D^l a^n=0$, in the form $a_r=-\mu/r^2$, $a_0=a_\theta=a_\varphi=0$. Then, as a result of this gauge transformation, we obtain for the exterior solution

$$f_{00}=-4M/r;\ f_{rr}=f_{\theta\theta}=f_{\varphi\varphi}=0.\quad(227)$$

To obtain the metric tensor for a static, spherically symmetric source, we must substitute the gravitational field components (227) in the minimal-coupling equation (94). We then obtain $(r>a)$

$$g_{00}=1-2M/r+2M^2/r^2;\ g_{\alpha\beta}=\gamma_{\alpha\beta}[1+2M/r+4\lambda M^2/r^2],\quad(228)$$

where $\lambda=b_3+b_4$.

It is easy to see that the component g_{00} of the metric tensor (228) of the effective Riemannian space-time does not have physical singularities outside the source:

$$g_{00}\neq 0;\ |g_{00}|<\infty$$

for $r>a$.

If the spatial components $g_{\alpha\beta}$ of the metric tensor (228) are to have no physical singularities outside the source of the gravitational field, we require fulfillment of the condition

$$\lambda=b_3+b_4\geqslant 0.\quad(229)$$

From the relations (224) and (85),

$$M=8\pi\int_0^a r_0^2\,dr_0\left[I_{00}-\frac{1}{2}I_n^n\right]=8\pi\int_0^a r_0^2\,dr_0\left[h_{00}-\frac{1}{2}h_n^n\right].$$

To obtain the post-Newtonian expansion of M, it is necessary, as usual, to make the calculations in successive stages. We first obtain the expression for M in the Newtonian approximation, in which w completely ignore the influence of gravitation on the energy-momentum tensor of the matter, and then, using the Newtonian approximation, we find the post-Newtonian expression. As a result

$$M=4\pi\int_0^a r^2\,dr\left\{\rho\left[1+\Pi-\frac{U}{2}+O(\varepsilon^4)\right]\right\}.$$

As one would expect, for the static, spherically symmetric body the post-Newtonian expansion of the total mass is equal to the expression (200).

14. NEW MECHANISM OF RELEASE OF ENERGY IN ASTROPHYSICAL OBJECTS

Since the metric tensor of the effective Riemannian space-time in the field theory of gravitation for a static, spherically symmetric source differs appreciably from the Schwarzschild solution in general relativity, the description of the phenomena that occur in strong gravitational fields will be different in these theories.

This makes it possible to study some new effects of the field theory of gravitation which differ strongly from the general relativistic effects in strong gravitational fields.

One of them[105] is a new mechanism of release of energy by astrophysical objects. It can be readily understood on the basis of the following simple arguments. Using the expression (228) for the metric tensor of the effective Riemannian space-time, we can find an expression for the force exerted by a static, spherically symmetric source of the field on a test body of mass M_0 at rest outside the source ($r \geqslant a$). The radial component of this force is

$$F^r = -\frac{M_0 m (1-2m/r)}{r^2 [1+2m/r+4\lambda m^2/r^2]}, \qquad (230)$$

where m is the inertial mass of the source.

It follows from this expression that for $m/r < \frac{1}{2}$ the force acting on the test body is a force of attraction, whereas for $m/r > \frac{1}{2}$ it is a force of repulsion. Thus, in the field theory of gravitation with minimal coupling the forces of gravitational attraction go over into forces of gravitational repulsion with increasing potential. This property of the gravitational interaction is fundamentally new and quite different from the properties of the gravitational interaction in general relativity. In particular, it follows that collapse is impossible in the field theory of gravitation.

Thus, however great the gravitational forces that compress an astrophysical object, the compression must necessarily be halted when the size of the object approaches its Schwarzschild radius, after which there must necessarily be an expansion of the matter, which can be accompanied by ejection of some of the mass of the object. In addition, static astrophysical objects with $m/a \gtrsim \frac{1}{2}$ will be in a state of unstable equilibrium, from which sooner or later they will go over into a stable static state $m/a \ll \frac{1}{2}$ by the ejection of some of their mass, which will be accompanied by the release of some of the internal energy of this object in the form of radiation.

As a result, the following questions arise. At what mean value of the gravitational potential are the various astrophysical objects (giant stars, supermassive star clusters, etc.) in a state of unstable equilibrium? How can these objects be in this state?

Strictly speaking, to answer these questions we should choose a model of an astrophysical object, and then, by simultaneous solution of the gravitational field equations and the equations of motion of the matter with allowance for the equation for the change in the entropy and the equation of state of the matter, construct a model of the internal structure of such an astrophysical object. Then an investigation of the stability of the model with respect to different perturbations (random perturbation of the radius of the object, a small change in its mass due to the capture of surrounding matter, combustion of matter within it, etc.) would answer the questions posed above. However, this problem is not amenable to analytic solution and requires the use of numerical calculations on a computer, which makes the analysis of the problem much harder.

It is much easier to obtain qualitative estimates with order-of-magnitude accuracy, since we can use the well-known scheme of analytic investigation of the stability of astrophysical objects.[106] For such an analysis, we need the averaged equation of the perturbations of an astrophysical object in the neighborhood of the static state. For simplicity, we consider the spherically symmetric case. As a model of the matter of this object, we shall consider an ideal fluid, whose energy-momentum tensor in the effective Riemannian space-time has the form

$$T^{ni} = (\varepsilon + p) u^n u^i - p g^{ni} \qquad (231)$$

where $u^i = dx^i/ds$ is the velocity 4-vector, ε is the mass density, and p is the isotropic pressure.

In the field theory of gravitation, as in any other metric theory, the energy-momentum tensor of the matter satisfies the covariant conservation equation

$$\nabla_n T^{ni} = 0. \qquad (232)$$

We project this equation onto the direction of the velocity 4-vector u^i and onto the direction orthogonal to it. As a result, we obtain the covariant continuity equation of an ideal fluid:

$$\nabla_n[(\varepsilon + p) u^n] = u^n \nabla_n p \qquad (233)$$

and the equations of motion

$$(\varepsilon + p) u^n \nabla_n u^i = (g^{ni} - u^n u^i) \nabla_n p. \qquad (234)$$

For spherically symmetric motion and distribution of the matter, the continuity equation (233) has the form

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial t}\left[(\varepsilon + p)\sqrt{-g}\, u^0\right] + \frac{1}{\sqrt{-g}} \frac{\partial}{\partial r}\left[(\varepsilon + p)\sqrt{-g}\, u^0 v^r\right] = u^0\left[\frac{\partial p}{\partial t} + v^r \frac{\partial p}{\partial r}\right], \qquad (235)$$

and the equation of motion (234) for the radial component of the velocity 4-vector can be written as follows:

$$(\varepsilon - p)\left[\frac{\partial u^r}{\partial t} + v^r \frac{\partial u^r}{\partial r}\right] = (g^{nr} - u^n u^r)\, \partial_n p - (\varepsilon + p)\,[\Gamma^r_{00} u^0 u^0 + 2\Gamma^r_{0r} u^0 u^r + \Gamma^r_{rr} u^r u^r]. \qquad (236)$$

We consider a spherically symmetric astrophysical object in a static state. We investigate the stability of this object with respect to small perturbations of its parameters (the mass, radius, matter density, etc.). We shall make such an investigation in the framework of perturbation theory.

We expand all quantities in Eqs. (235) and (236) in series in small perturbations, restricting ourselves to linear terms in the perturbation. In the original unperturbed state, we shall assume that all the quantities in Eqs. (235) and (236) are independent of the time. We also use the fact that the component g_{0r} of the effective Riemannian space-time metric tensor is equal to zero for a static, spherically symmetric source.

Then in the zeroth approximation, we obtain from Eq. (236)

$$\overset{(0)}{g^{rr}}\, \partial_r \overset{(0)}{p} - (\overset{(0)}{\varepsilon} + \overset{(0)}{p})\, \overset{(0)}{\Gamma^r_{00}}\, \overset{(0)}{g^{00}} = 0. \qquad (237)$$

From the continuity equation (235) in the same approximation, we have

$$\frac{d}{dt}\left[\sqrt{-\overset{(0)}{g}}\,(\overset{(0)}{\varepsilon}+\overset{(0)}{p})\,\overset{(0)}{u^0}\right]=0. \tag{238}$$

We reduce the equation of motion (236) in the approximation linear in the perturbation to the form

$$(\overset{(0)}{\varepsilon}+\overset{(0)}{p})\,\overset{(0)}{g^{00}}\,\frac{d^2}{dt^2}\,\delta r=\partial_r\overset{(0)}{p}\,\delta g^{rr}+\overset{(0)}{g^{rr}}\delta\,\partial_r p-(\delta\varepsilon+\delta p)\,\overset{(0)}{\Gamma^r_{00}}\overset{(0)}{g^{00}}$$
$$-(\overset{(0)}{\varepsilon}+\overset{(0)}{p})\,\overset{(0)}{g^{00}}\delta\Gamma^r_{00}-(\overset{(0)}{\varepsilon}+\overset{(0)}{p})\,\overset{(0)}{\Gamma^r_{00}}\delta g^{00}. \tag{239}$$

We average Eqs. (237) and (239) over the volume of the astrophysical object. Using the qualitative estimates usually made in this approach,[105] we obtain for the averaged components of the metric

$$\left.\begin{aligned}\overline{g}_{00}&=1-2m/a+2m^2/a^2;\\ \overline{g}_{\alpha\beta}&=\gamma_{\alpha\beta}\,[1+2m/a+4\lambda m^2/a^2].\end{aligned}\right\} \tag{240}$$

For the purposes of our investigation, however, it is more convenient to write these expressions, not in terms of the inertial mass m of the source of the gravitational field, but in terms of the total rest mass M of the particles which constitute the matter of the astrophysical source:

$$M=4\pi\int_0^a r^2\,dr\,\sqrt{-g}\;\varepsilon u^0. \tag{241}$$

It follows from the expressions (200) and (241) that

$$m=M\,[1-M/2a+O\,(M^2/a^2)].$$

Therefore, the relations (240) become

$$\left.\begin{aligned}\overline{g}_{00}&=1-2M/a+3M^2/a^2+O\,(M^3/a^3);\\ \overline{g}_{\alpha\beta}&=\gamma_{\alpha\beta}\,[1+2M/a+(4\lambda-1)\,M^2/a^2+O\,(M^3/a^3)].\end{aligned}\right\} \tag{242}$$

Then averaging of Eq. (237) leads to the equation

$$\frac{\overline{p}}{a}=(\overline{\varepsilon}+\overline{p})\,\frac{M}{a^2}\,\frac{[1-3M/a+O\,(M^2/a^2)]}{[1-2M/a+3M^2/a^2]}. \tag{243}$$

Proceeding similarly, and using (238) and (243), we obtain from (239) the averaged equation of perturbations of the spherically symmetric astrophysical object:

$$(\overline{\varepsilon}+\overline{p})\,\overline{g}^{00}\delta\ddot{a}=\overline{g}^{rr}[-\delta\overline{p}/a+(\overline{p}/a^2)\,\delta a$$
$$-(\delta\overline{\varepsilon}+\delta\overline{p})\,\overline{\Gamma}_{r,00}\overline{g}^{00}-(\overline{\varepsilon}+\overline{p})\,\overline{\Gamma}_{r,00}\delta\overline{g}^{00}-(\overline{\varepsilon}+\overline{p})\overline{g}^{00}\delta\Gamma_{r,00}]. \tag{244}$$

We consider a spherically symmetric astrophysical object of radius a in a static state. We investigate the stability of this state with respect to small perturbations of the radius. We shall assume that the entire process takes place adiabatically and that the number of particles and, therefore, the total rest mass of the particles of this object are conserved. Since averaging of the expression (241) gives

$$M=\frac{4\pi}{3}\,a^3\varepsilon\left[1+\frac{2M}{a}+(4\lambda-1)\,\frac{M^2}{a^2}\right], \tag{245}$$

the condition for the total rest mass of the particles to remain unchanged leads to the following relation between the mean mass density $\delta\overline{\varepsilon}$ and the perturbation δa of the object's radius:

$$\frac{\delta\overline{\varepsilon}}{\overline{\varepsilon}}=-\frac{3\delta a}{a}\,\frac{[1+M/a]}{[1+2M/a+(4\lambda-1)\,M^2/a^2]}. \tag{246}$$

Introducing the mean adiabatic exponent

$$\overline{\Gamma}_1=(\overline{\varepsilon}/\overline{p})\,(\partial\overline{p}/\partial\overline{\varepsilon})_{\text{ad}}, \tag{247}$$

we obtain for the perturbation $\delta\overline{p}$ of the mean pressure

$$\delta\overline{p}=\overline{\Gamma}_1\overline{p}\,\delta\overline{\varepsilon}/\overline{\varepsilon}. \tag{248}$$

Substituting Eqs. (248) in the averaged equation (244) of the perturbations and using Eqs. (242) and (243), we obtain

$$\delta\ddot{a}+\omega^2\delta a=0, \tag{249}$$

where we have introduced the notation

$$\omega^2=\frac{M}{a^3}\left\{3\,(\overline{\Gamma}_1-1)\left(1+\frac{M}{a}\right)\left(1-3\,\frac{M}{a}\right)\left(1-3\,\frac{M}{a}+6\,\frac{M^2}{a^2}\right)\right.$$
$$\left.-\left(1-6\,\frac{M}{a}+3\,\frac{M^2}{a^2}\right)\left[1+\frac{2M}{a}+O\left(\frac{M^2}{a^2}\right)\right]\right\}\left[1+\frac{2M}{a}\right.$$
$$\left.+O\left(\frac{M^2}{a^2}\right)\right]^{-2}\left[1-\frac{2M}{a}+\frac{3M^2}{a^2}\right]^{-1}. \tag{250}$$

For $\omega^2>0$, this equation describes harmonic oscillations of the radius of the astrophysical object about the position of equilibrium, so that for $\omega^2>0$ it will be stable with respect to small perturbations of its radius that do not change the total rest mass of the particles which constitute the object. For $\omega^2<0$, Eq. (249) describes exponential growth of the initial perturbation $\delta a(0)$ with the passage of time. Thus, for $\omega^2\leqslant 0$ the static, spherically symmetric object will be unstable with respect to small perturbations of the radius.

We determine the values of the mean adiabatic exponent $\overline{\Gamma}_1$ of the matter for which the static astrophysical object will be unstable. By virtue of the equation

$$[1-2M/a+3M^2/a^2]>0$$

it follows from the condition $\omega^2\leqslant 0$ and the expression (250) that the static, spherically symmetric object will be unstable with respect to small perturbations of the radius if the mean adiabatic exponent satisfies the inequality

$$\overline{\Gamma}_1\leqslant\frac{[4-19M/a+O\,(M^2/a^2)]}{3\,[1-3M/a+6M^2/a^2]\,[1-3M/a]\,[1+M/a]}=F\left(\frac{M}{a}\right). \tag{251}$$

Strictly speaking, this applies only to the region of values $M/a<1$ of the averaged gravitational potential, since we have used the approximate expansion (242). However, this expression can be extrapolated qualitatively to the region $M/a\sim 1$ as well. At $x=0$, the function $F(x)$ has the value 4/3, and as x tends to 1/3 it decreases to $-\infty$. Thus, if the mean adiabatic exponent of the matter of the astrophysical object in some range of values of M/a is higher than the value determined by the expression (251), then in this region the object will be stable with respect to small perturbations of the radius.

As follows from the results of observations,[106] stars of the main sequence have a relatively low temperature; therefore, for such stars the gas pressure dominates over the radiation pressure, so that for them $\overline{\Gamma}_1=5/3$. For high-temperature stars, the radiation pressure can be higher than the gas pressure and the adiabatic exponent is just higher than $\overline{\Gamma}_1\approx 4/3$. However, such stars have not yet been observed,[106] since their lifetime is fairly short.

It must, however, be noted that the above values of the adiabatic exponent can be used only under the condition that the averaged pressure of the star is positive: $\overline{p}>0$. From Eq. (243) we can readily obtain an expression for the averaged pressure:

$$\bar{p} = \frac{\bar{\varepsilon}(M/a)(1-3M/a)}{1-3M/a+6M^2/a^2}. \qquad (252)$$

It follows from this that for $M/a<1/3$ the averaged pressure is positive, and for $M>a/3$ it is negative.

Thus, our qualitative analysis shows that for values $M/a<1/3$ of the averaged potential of astrophysical objects they will be stable with respect to small perturbations of their radius, but for $M/a\geqslant 1/3$ they will be unstable. Moreover, the transition from the stability region to the instability region will take place abruptly—from infinitely great stability to infinitely great instability. This means that the instability can be explosive in nature, as a result of which any small perturbation of the radius of an astrophysical object will destroy its equilibrium and may lead to ejection of matter.

We consider a spherically symmetric static astrophysical object. We study the consequences of capture by this object of matter from the surrounding space. As follows from the expression (245), a change in the mass of the object will necessarily be accompanied by a change in the mean matter density of the object and its radius:

$$\frac{\delta M}{M} = \frac{\delta\bar{\varepsilon}}{\bar{\varepsilon}}\left[1+3\frac{M}{a}+O\left(\frac{M^2}{a^2}\right)\right]+3\frac{\delta a}{a}\left[1+2\frac{M}{a}+O\left(\frac{M^2}{a^2}\right)\right]. \qquad (253)$$

But the change in the mean matter density of the object and its radius naturally lead to a change in the mean gravitational potential of the object:

$$\delta\left(\frac{M}{a}\right) = \frac{\delta\bar{\varepsilon}}{\bar{\varepsilon}}\frac{M}{a}\left[1+3\frac{M}{a}+O\left(\frac{M^2}{a^2}\right)\right] + 2\frac{\delta a}{a}\frac{M}{a}\left[1+3\frac{M}{a}+O\left(\frac{M^2}{a^2}\right)\right]. \qquad (254)$$

We determine the change in the mean gravitational potential when the object captures mass in the following two limiting cases corresponding to a change in only one of the two parameters of the object: either the mean mean density of its matter or its radius.

Suppose the averaged matter density of the object does not change when it captures mass: $\delta\bar{\varepsilon}=0$. Then in accordance with the expression (253), capture of mass δM will lead to a change in the radius of the object equal to

$$\delta a/a = (\delta M/3M)\,[1-2M/a+O(M^2/a^2)]. \qquad (255)$$

Substituting this relation in Eq. (254), we obtain

$$\delta(M/a) = (2/3)(\delta M/a)\,[1+M/a+O(M^2/a^2)]. \qquad (256)$$

Thus, for unchanged mean matter density, capture of mass by the astrophysical object leads to an increase in the mean gravitational potential of this object.

Suppose the radius of the object does not change when it captures mass: $\delta a=0$. Then capture of the mass δM will lead to a change in the mean matter density of the object:

$$(\delta\bar{\varepsilon}/\bar{\varepsilon}) = (\delta M/M)\,[1-3M/a+O(M^2/a^2)]. \qquad (257)$$

For the change in the mean gravitational potential of the object, we obtain from the expressions (254) and (257)

$$\delta(M/a) = (\delta M/a)\,[1+O(M^2/a^2)]. \qquad (258)$$

Therefore, in this case too the capture of mass by the astrophysical object will lead to an increase in the mean gravitational potential of the object.

Thus, any astrophysical object that captures matter gravitationally will have a tendency to increase its mean gravitational potential. Our above qualitative analysis of the evolution of astrophysical objects shows that in the field theory of gravitation with minimal coupling objects in the region of values $M/a<1/3$ of the mean gravitational potential are stable with respect to small perturbations of their radius for unchanged rest mass.

However, the mean gravitational potential of these objects increases when they capture surrounding matter. Once the average potential reaches the value $M/a=1/3$, the object goes over abuptly from the infinitely stable state to the infinitely unstable state with respect to small perturbations of its radius. Therefore, once the critical value of the mean gravitational potential has been reached, even small perturbations of the radius of the object lead unavoidably to an expansion of the matter, and this may be accompanied by the ejection of some of the mass of the object and the release of energy.

Therefore, instead of the gravitational collapse that occurs because of the instability of astrophysical objects in general relativity, the new theory has a new mechanism of release of energy.

15. GRAVITATIONAL FIELD OF A NONSTATIC, SPHERICALLY SYMMETRIC SOURCE

In Einstein's theory, the gravitational field of a nonstatic, spherically symmetric source outside the matter must, by virtue of Birkhoff's theorem, be a static field with metric corresponding to the Schwarzschild solution.

We show that in the field theory of gravitation the gravitational field outside the matter in the case of a nonstatic, spherically symmetric source is a static field with components given by Eqs. (227) and (228). We consider the case when the matter is distributed in a certain sphere of radius a spherically symmetrically and its motion is also spherically symmetric in radial directions.

By virtue of the symmetry of the problem, the nonvanishing components of the tensors $T^{ni}, h^{ni}, I_{ni}, f_{ni}$ will be the diagonal components, and also the components T^{0r}, I_{0r}, h^{0r}, and f_{0r}. All the components of these tensors, apart from the $(\varphi\varphi)$ components, will depend on r and t. For the $(\varphi\varphi)$ components, we have

$$T^{\varphi\varphi} = T^{\theta\theta}/\sin^2\theta;\ h^{\varphi\varphi} = h^{\theta\theta}/\sin^2\theta;$$
$$I_{\varphi\varphi} = I_{\theta\theta}\sin^2\theta;\ f_{\varphi\varphi} = f_{\theta\theta}\sin^2\theta.$$

The velocity 4-vector of the matter is $u^i=\{u^0(r,t), u^r(r,t),0,0\}$.

We expand the components of the tensor current I_{lm} and the gravitational field f_{lm} in Fourier integrals with respect to the time:

$$f_{lm} = \int d\omega \exp(-\mathrm{i}\omega t) f_{lm}(\omega, r);$$
$$I_{lm} = \int d\omega \exp(-\mathrm{i}\omega t) I_{lm}(\omega, r).$$

In the spectrum $I_{lm}(\omega, r)$, we separate the static part $I_{lm}(r)$. It is obvious that the static part will give the static solutions considered in the previous section. Therefore, in what follows we shall understand by $I_{ml}(\omega, r)$ the nonstatic part.

The field equations (90) for the considered case will have the form of ordinary differential equations:

$$\left.\begin{aligned} &f''_{00} + \frac{2}{r} f'_{00} + \omega^2 f_{00} = 16\pi I_{00};\\ &f''_{0r} + \frac{2}{r} f'_{0r} + \left(\omega^2 - \frac{2}{r^2}\right) f_{0r} = 16\pi I_{0r};\\ &f''_{rr} + \frac{2}{r} f'_{rr} + \frac{4}{r^2} f^{\theta}_{\theta} + \left(\omega^2 - \frac{4}{r^2}\right) f_{rr} = 16\pi I_{rr};\\ &f^{\theta\,''}_{\theta} + \frac{2}{r} f^{\theta\,'}_{\theta} - \frac{2}{r^2} f_{rr} + \left(\omega^2 - \frac{2}{r^2}\right) f^{\theta}_{\theta} = 16\pi I^{\theta}_{\theta}. \end{aligned}\right\} \qquad (259)$$

As boundary conditions for these equations, it is natural to require the functions f_{00}, f_{0r}, f^{θ}_{θ}, and f_{rr} to be bounded as $r \to 0$ and also to require fulfillment of the radiation conditions as $r \to \infty$. From the conditions of conservation of the tensor current, $D_l I^l_n = 0$, we have

$$\left.\begin{aligned} &\mathrm{i}\omega I_{00} + I'_{0r} + (2/r) I_{0r} = 0;\\ &\mathrm{i}\omega I_{0r} + I'_{rr} + (2/r) I_{rr} - (2/r^3) I_{\theta\theta} = 0. \end{aligned}\right\} \qquad (260)$$

Solving Eqs. (259) using the relations (260), we obtain

$$f_{rr} = (A_1 + 2A_2)/3;$$
$$f_{\theta\theta} = (r^2/3)(A_1 - A_2);$$

$$f_{00} = -\frac{8\pi^2}{\sqrt{r}} \left\{ H^{(1)}_{1/2}(\omega r) \int_0^r x^{3/2}\, dx\, I_{0r} J_{3/2}(\omega x) + J_{1/2}(\omega r) \int_r^a x^{3/2}\, dx\, I_{0r} H^{(1)}_{3/2}(\omega x) \right\};$$

$$f_{0r} = -\frac{8\pi^2 \mathrm{i}}{\sqrt{r}} \left\{ H^{(1)}_{3/2}(\omega r) \int_0^r x^{3/2}\, dx\, J_{3/2}(\omega x) I_{0r} + J_{3/2}(\omega r) \int_r^a x^{3/2}\, dx\, H^{(1)}_{3/2}(\omega x) I_{0r} \right\},$$

where

$$A_1 = -\frac{8\pi^2 \mathrm{i}\omega}{\sqrt{r}} \left\{ H^{(1)}_{1/2}(\omega r) \int_0^r x^{5/2} [\mathrm{i} I_{0r} J_{1/2}(\omega x) + I_{rr} J_{3/2}(\omega x)]\, dx + J_{1/2}(\omega r) \int_r^a x^{5/2} [\mathrm{i} I_{0r} H^{(1)}_{1/2}(\omega x) + I_{rr} H^{(1)}_{3/2}(\omega x)]\, dx \right\};$$

$$A_2 = \frac{4\pi^2 \mathrm{i}\omega}{\sqrt{r}} \left\{ H^{(1)}_{5/2}(\omega r) \int_0^r x^{5/2}\, dx\, [\mathrm{i} I_{0r} J_{5/2}(\omega x) - I_{rr} J_{3/2}(\omega x)] + J_{5/2}(\omega r) \int_r^a x^{5/2}\, dx\, [\mathrm{i} I_{0r} H^{(1)}_{5/2}(\omega x) - I_{rr} H^{(1)}_{3/2}(\omega x)] \right\}.$$

Using outside the matter the gauge transformation

$$f_{ni} \to f_{ni} + D_n a_i + D_i a_n - \gamma_{ni} D_l a^l,$$

we impose on the gravitational field components two conditions: $f = 0$, $f^{00} = 0$. If the condition $D_l f^{lm} = 0$ is not to be violated by the gauge transformation, the gauge 4-vector must satisfy outside the matter the equation $D_l D^l a_i = 0$. Choosing gauge vectors in the form

$$a_0 = \frac{2\pi^2 \mathrm{i}}{\omega\sqrt{r}} H^{(1)}_{1/2}(\omega r) \int_0^a x^{3/2}\, dx\, \{ I_{0r} [J_{3/2}(\omega x) + \omega x J_{1/2}(\omega x)] - \mathrm{i}\omega x I_{rr} J_{3/2}(\omega x) \};$$

$$a_r = \frac{2\pi^2}{\sqrt{r}} H^{(1)}_{3/2}(\omega r) \int_0^a x^{5/2}\, dx\, [I_{0r} J_{5/2}(\omega x) + \mathrm{i} I_{rr} J_{3/2}(\omega x)];$$

$$a_\theta = a_\varphi = 0,$$

we readily see that all components of the nonstatic gravitational field vanish outside the matter: $f_{nm} = 0$.

Thus, for a nonstatic source with spherically symmetric distribution and motion of the matter the gravitational field outside the matter will be a static field with components given by Eqs. (227) and (228).

16. NONSTATIONARY MODEL OF A HOMOGENEOUS UNIVERSE

The field theory of gravitation makes it possible to construct nonstationary models of the Universe capable of describing the cosmological red shift and free of divergences of Newtonian type. These models correspond to a flat Universe.

It should be noted that in the field theory of gravitation the model of the Universe characterizes only a part of it with linear dimension $r \sim cT$, where T is the age of the Universe. From this point of view, the "creation" of the Universe means that in the past the matter density in a given, fairly large section of the Universe was high. The further evolution of this region of the Universe can be described by the considered model. All other regions of the Universe can develop independently of the development of the given region and even in accordance with quite different laws. But in the field theory of gravitation observation of them is possible.

Astronomical observations show[107,108] that the matter is distributed very nonuniformly in the Universe. Most of the matter is in planets and stars, and only a small part of the total mass is in the form of interstellar gas and radiation. However, if one averages over regions of space whose linear dimensions are appreciably greater than the distances between clusters of galaxies, the matter density of the part of the Universe accessible to observation is constant and does not depend on the position of the center of the region of averaging. Therefore, from the physical point of view it is natural to consider a model of a homogeneous isotropic Universe as a first step.

In such an approach, the inhomogeneity of the matter distribution which appears on averaging over smaller regions of space (clusters of galaxies, galaxies, etc.) can be taken into account by introducing small inhomogeneous perturbations on the background cosmological field of the homogeneous Universe. The homogeneous isotropic Universe is described by the interval

$$dS^2 = U(t)\, dt^2 - V(t)\, [dx^2 + dy^2 + dz^2]. \qquad (261)$$

We shall regard the matter in the Universe as an ideal fluid with energy-momentum tensor density

$$T^{ni} = \sqrt{-g}\, [(\varepsilon + p) u^i u^n - p g^{ni}].$$

By virtue of the homogeneity and isotropy of the Universe,

$$\varepsilon = \varepsilon(t);\ p = p(t);$$
$$u^\alpha = 0;\ u^0 \neq 0;\ u^0 u^0 g_{00} = 1.$$

Then the components of the energy-momentum tensor density of the matter take the form

$$T^{00} = \varepsilon \sqrt{V^3/U};\ T^{\alpha\beta} = -p\sqrt{UV}\, \gamma^{\alpha\beta}. \qquad (262)$$

Using the expression (261) for the interval, we determine the connection of the Riemannian space-time:

$$\left.\begin{array}{l}\Gamma^0_{00}=\dot U/2U;\ \Gamma^0_{0\alpha}=0;\ \Gamma^\alpha_{00}=0;\\ \Gamma^0_{\alpha\beta}=\dot V\gamma_{\alpha\beta}/2U;\ \Gamma^\alpha_{0\beta}=\dot V\delta^\alpha_\beta/2V;\ \Gamma^\alpha_{\beta\varepsilon}=0,\end{array}\right\} \tag{263}$$

where the dot denotes differentiation with respect to t.

Substituting the expressions (262) and (263) in the covariant conservation equation (201) for the energy-momentum tensor density of the matter, we obtain

$$\frac{d}{dt}(\varepsilon\sqrt{V^3})+p\frac{d}{dt}\sqrt{V^3}=0. \tag{264}$$

The solution of Eq. (264) has the form

$$\ln V=-\frac{2}{3}\int\limits_{\varepsilon_0}^{\varepsilon}\frac{d\varepsilon'}{\varepsilon'+p(\varepsilon')}. \tag{265}$$

The connection equation can be written in most general form as

$$g_{ni}=\gamma_{ni}f_1+f_{ni}f_2+f_{im}f_{nl}A^{lm}, \tag{266}$$

where f_1 and f_2 are certain scalar functions of the invariants $I_1=f$, $I_2=f_{nm}f^{nm}$, etc., and the tensor A^{lm} is constructed from the tensors $\gamma^{nm}, f^{nm}, f^{ni}f^m_i$ ... and the invariants.

For a homogeneous Universe, the gravitational field equations (90) take the form

$$\left.\begin{array}{l}\ddot f_{00}=\ddot f_{0\alpha}=0;\\ \ddot f_{\alpha\beta}=-16\pi\{h_{\alpha\beta}+\gamma_{\alpha\beta}h_{00}\}\end{array}\right\} \tag{267}$$

Using the definition (70), we obtain

$$f_{00}=0;\ f_{0\alpha}=0. \tag{268}$$

By virtue of the isotropy of the Universe, the spatial components of the gravitational field must be

$$f_{\alpha\beta}=\gamma_{\alpha\beta}F(t). \tag{269}$$

Then

$$g_{00}=U(F);\ g_{\alpha\beta}=\gamma_{\alpha\beta}V(F). \tag{270}$$

One can show that

$$\frac{\partial g_{00}}{\partial f_{\alpha\beta}}=\frac{1}{3}\gamma^{\alpha\beta}\frac{dU}{dF};\ \gamma^{\varepsilon\eta}\frac{\partial g_{\varepsilon\eta}}{\partial f_{\alpha\beta}}=\gamma^{\alpha\beta}\frac{dV}{dF}.$$

Therefore, the field equations (267) take the form

$$\ddot F=\frac{64\pi}{3}\left\{\varepsilon\sqrt{V^3}\frac{d}{dF}\sqrt{U}-p\sqrt{U}\frac{d}{dF}\sqrt{V^3}\right\}. \tag{271}$$

As initial conditions for Eq. (271), we take the conditions at the contemporary epoch $t=0$:

$$\varepsilon=\varepsilon_0;\ U=V=1;\ dV/dt=2H, \tag{272}$$

where H is Hubble's constant. It must be noted especially that the initial conditions are chosen on the basis of the assumption that the energy density of the matter is nonzero, $\varepsilon_0\neq 0$. Therefore, the following calculation will apply only to the given case. It follows from experiments[17] that 20×10^9 years $>1/H>7.5\times 10^9$ years. For such a choice of the initial conditions, the cosmological field at the present time will be the pseudo-Euclidean background on which we consider all the other physical processes. It follows from the conditions (272) that

$$F(0)=0;\ dF/dt\,|_{t=0}=-4H.$$

We reduce Eq. (271) to the form

$$\frac{d}{dt}[\dot F^2+c_1]=\frac{128\pi}{3}\left\{\varepsilon\sqrt{V^3}\frac{d}{dt}\sqrt{U}-p\sqrt{U}\frac{d}{dt}\sqrt{V^3}\right\}.$$

Using the conservation equation (264), we obtain

$$\dot F^2+c_1=(128\pi/3)\,\varepsilon\sqrt{V^3U}. \tag{273}$$

It is interesting to note that Eq. (273) is a modified form of expression of the conservation law for the energy density of the matter and the gravitational field in flat space-time. Indeed, if we use the definitions (104) and (111), the connection equation (266), the components (262) of the energy-momentum tensor density of the matter in the Riemannian space-time, and also Eqs. (268) and (269), then we obtain

$$t^{00}_M=\varepsilon\sqrt{UV^3};\ t^{00}_g=-\frac{3}{128\pi}\dot F^2;\ t^{0\alpha}_M=t^{0\alpha}_g=0. \tag{274}$$

Therefore, the conservation law for the energy-momentum tensor density (96) in the flat space-time is

$$\frac{\partial}{\partial t}[t^{00}_M+t^{00}_g]=0.$$

It follows from this that $t^{00}_M+t^{00}_g=\text{const}$.

Using the initial conditions (272) and the expressions (274), we find

$$\varepsilon\sqrt{V^3U}-\frac{3}{128\pi}\dot F^2=\frac{3}{128\pi}c_1,$$

where

$$c_1=16H^2(\alpha-1);\ \alpha=8\pi\varepsilon_0/3H^2.$$

Thus, the total energy density of the matter and the gravitational field of the Universe in the flat space-time is constant at all stages of its evolution. This means that the energy of the Universe does not change during the evolution but is merely redistributed between the matter and the gravitational field.

Using the initial conditions, we write the solution of Eq. (273) in the form

$$t=-\frac{1}{4H}\int\limits_0^F\frac{dF'}{\sqrt{1-\alpha+(\alpha\varepsilon/\varepsilon_0)\sqrt{V^3U}}}. \tag{275}$$

The expressions (266), (270), and (275) determine parametrically the entire evolution of the homogeneous isotropic Universe, including the singular state (or hot Universe) for arbitrary equation of state $p=p(\varepsilon)$ of the matter and connection equation (266) given in the most general form.

In the expressions (270) and (261), we go over to the proper time. In the time interval in which $U(t)$ is nonzero, we can go over to a proper time $\tau(t)$ such that $\sqrt{U(t)}dt=d\tau$. Then the interval becomes

$$dS^2=d\tau^2-V(\tau)[dx^2+dy^2+dz^2]. \tag{276}$$

Assuming that the present time is $\tau(0)=0$, we obtain expressions that determine the evolution of the Universe given parametrically:

$$\tau=-\frac{1}{4H}\int\limits_0^F\frac{\sqrt{U}\,dF'}{\sqrt{1-\alpha+(\alpha\varepsilon/\varepsilon_0)\sqrt{UV^3}}}; \tag{277}$$

$$\ln V(F)=-\frac{2}{3}\int\limits_{\varepsilon_0}^{\varepsilon}\frac{d\varepsilon'}{\varepsilon'+p(\varepsilon')}. \tag{278}$$

For the functions U and V, we obtain from the minimal-coupling equation (94)

$$U=1-(3/2)F+(1/4)(9b_4+3b_3)F^2;$$
$$V=1-(1/2)F+(1/4)(b_1+3b_2+3b_3+9b_4)F^2. \quad (279)$$

We investigate the obtained solutions in the neighborhood of the present epoch ($|\tau|\ll 1/4H$) of proper time. We shall assume that in the neighborhood of the present epoch of proper time the pressure is negligibly small compared with the energy density: $p\ll\varepsilon$. Therefore, from Eq. (278),

$$\varepsilon=\varepsilon_0/\sqrt{V^3(F)}. \quad (280)$$

Substituting the expressions for U, V, ε in the integral (277) and integrating, we obtain

$$\tau=-\frac{1}{4H}\left[F-\frac{3}{8}\left(1-\frac{\alpha}{2}\right)F^2+O(F^3)\right].$$

Expressing F by means of this relation and substituting it in the expression for $V(F)$, we obtain

$$V(\tau)=1+2H\tau+H^2\tau^2\left[\frac{3}{2}\alpha\right.$$
$$\left.-3+4(b_1+3b_2+3b_3+9b_4)\right]+O(H^3\tau^3).$$

The metric (276) with cosmological scale factor $V(\tau)$ leads to experimentally observable effects. One of them is the cosmological red shift discovered in 1929 by Hubble.[109] This effect consists of a red shift of the spectral lines emitted by distant galaxies, the magnitude of the shift being directly proportional to the distance from the considered galaxy to the Earth. In general relativity, this effect was predicted by the Soviet scientist Friedmann[110] in 1922.

In the field theory of gravitation, the model of a homogeneous Universe in the neighborhood of the contemporary epoch (for $H\tau\ll 1$ or $\tau\ll 10^{10}$ years) also describes a cosmological red shift $\Delta\omega=-HL\omega$ of the frequency.

The deceleration parameter $q=1-2V\ddot{V}/\dot{V}^2$ of the "expanding" Universe in the neighborhood of the contemporary epoch $\tau=0$ is

$$q_0=4-(3/2)\alpha-4(b_1+3b_2+3b_3+9b_4). \quad (281)$$

For comparison, we point out that in Einstein's theory the deceleration parameter of a homogeneous Universe is $q_0=\alpha/2$. In Einstein's theory of gravitation the deceleration parameter is one of the most important quantities that characterize the homogeneous Universe as a whole: For deceleration parameter $q_0<\frac{1}{2}$ ($\alpha<1$) the Universe is open, and for $q_0>\frac{1}{2}$ ($\alpha>1$) it is closed (it has finite volume but no boundaries). In the field theory of gravitation, there is no such relationship—the Universe has infinite volume for all values of α and q_0.

It follows from estimates of the mass of matter in galaxies[107] that $\varepsilon_0=3\times 10^{-31}$ g/cm^3. In this case,

$$\alpha=0.06. \quad (282)$$

Then in Einstein's theory the deceleration parameter is $q_0=0.03$ and the Universe is open and expand forever. However, measurements of the deceleration parameter gave a different result.

Thus, in Ref. 111 it is concluded that the value of q_0 is in the range between 2 and 32, and that the most probable value is $q_0=5$. Thus, in Einstein's theory the deceleration parameter obtained from the observations contradicts the observed matter density in the galaxies, which is much higher than is required for correspondence. To eliminate this discrepancy between the characteristics of the cosmological solution of Einstein's theory and their observational values, attempts are currently being made to increase ε_0 (by searching for missing matter in galaxies) and to decrease the value of q_0 obtained expreimentally (by assuming the existence of a strong evolution of the luminosity function with the red shift). These attempts have not yet led to resolution of this problem.

In the field theory of gravitation, in contrast to general relativity, the deceleration parameter is determined not only by the mean matter density ε_0 (the parameter $\alpha=8\pi\varepsilon_0/3H^2$) but also by the minimal-coupling parameters, and therefore measurement of the deceleration parameter q_0 makes it possible, without using post-Newtonian experiments in the solar system, to measure

$$b_1+3b_2+3b_3+9b_4=-\left[q_0+\frac{3}{2}\alpha-4\right]/4. \quad (283)$$

The behavior of the model of the homogeneous Universe in the distant past depends strongly on the form of the connection equation in strong gravitational fields.

If the equation $V(F_1)=0$ has real roots, then for $F=F_1$ the determinant of the metric tensor, and also its spatial components vanish. Therefore, it is natural to assume that at $F=F_1$ a singular state of the Universe is realized. Near the singular state, the Universe is dominated by ultrarelativistic particles, for which the equation of state is $p=\varepsilon/3$. Substituting this equation in the expression (278), we obtain

$$\varepsilon=\varepsilon_0/V^2. \quad (284)$$

It follows from this that when the function $V(F)$ vanishes the density of the total energy of the Universe becomes infinite, and at $F=F_1$ a singular state of the Universe is indeed realized.

A certain epoch $\tau=\tau_m$ in the past corresponds to the smallest positive root F^* of the equation $V(F)=0$. It is natural to call $T=-\tau_m$ the age of the Universe, and

$$T=\frac{1}{4H}\int_0^{F^*}\frac{\sqrt{U}\,dF}{\sqrt{1-\alpha+\alpha(\varepsilon/\varepsilon_0)\sqrt{UV^3}}}. \quad (285)$$

We introduce the time $\tau_0=T+\tau$ measured from the singular state:

$$\tau_0=\frac{1}{4H}\int_F^{F^*}\frac{\sqrt{U}\,dE}{\sqrt{1-\alpha+\alpha(\varepsilon/\varepsilon_0)\sqrt{UV^3}}}.$$

In the neighborhood of the singular state (for $F\sim F^*$), the relation (284) holds, and therefore

$$\tau_0=\frac{1}{4H}\int_F^{F^*}\frac{\sqrt{U}\,dF}{\sqrt{1-\alpha+\alpha\sqrt{U/V}}}. \quad (286)$$

The expression (286) determines the dependence of the proper time in the neighborhood of the singular state on the gravitational field F and, thus, makes it possible to find the behavior of the function $V(\tau)$ in the given neighborhood.

It should be noted that if the equation $V(F_1)=0$ does not have real roots, the model of the Universe does not have a singular state. In this case, it is possible to have an Olbers paradox, i.e., a divergence of the integral of the luminosity of all stars. Indeed, the total energy ρ of starlight at the present time $\tau=0$ is[17]

$$\rho = \int_{-\infty}^{0} Z(\tau)\, V^2(\tau)\, d\tau, \qquad (287)$$

where $Z(\tau)$ is the luminosity density of the stars: $Z(\tau) = \int n(\tau, L) dL$, in which $n(\tau, L)$ is the density of stars with absolute luminosity L at the time τ. If the integral (287) is to converge, we require either the existence of a singular state of the Universe $[V(F_1)=0]$ at a finite F_1, when the integral (287) is effectively truncated at the lower limit at a certain $\tau=\tau(F_1)$, or sufficiently rapid decrease to zero of $V(\tau)$ with increasing $|\tau|$:

$$\tau V(\tau)\, Z(\tau) \to 0; \quad |\tau| \to \infty. \qquad (288)$$

We introduce the notation

$$b_1 + 3b_2 + 3b_3 + 9b_4 = w; \; 3b_3 + 9b_4 = k. \qquad (289)$$

Using this notation, we rewrite the expression (279) in the form

$$U = 1 - (3/2)\, F + (k/4)\, F^2; \; V = 1 - F/2 + wF^2/4. \qquad (290)$$

We now study the influence of the coefficients k and w on the behavior of the model of the Universe. We require that the theory should contain no Olbers-type paradox or physical singularities of the metric of the Universe for finite values of the matter energy density. As follows from the expression (290), the first of these requirements can be satisfied only subject to the condition $w \leq \frac{1}{4}$. By virtue of the relations (281) and (289), this condition leads to a restriction on the value of the deceleration parameter in the neighborhood of the contemporary epoch: $q_0 \geq 3-(3/2)\,\alpha$. The second requirement imposes restrictions on the values of the real roots of the equation $U(F)=0$ and, thus, on the value of the coefficient k.

Depending on the coefficients k and w, it is possible to have different models of the Universe. We consider them successively:

$$\text{I. } 0 \leqslant w \leqslant 1/4 \text{ or } 4-(3/2)\,\alpha \geqslant q_0 \geqslant 3-(3/2)\,\alpha. \qquad (291)$$

In this case, both roots of the function V are positive. The smaller of them, corresponding to a singular state of the Universe, is $F^* = (1-\sqrt{1-4w})/w$. In the range of w values (291), the root F^* is in the interval $2 \leq F^* \leq 4$. Since the region of negative values of F corresponds to the future in the evolution of the Universe, the Universe will "expand" infinitely long in the case (291). At the same time, the metric (290) will not have singularities outside the singular state of the Universe if the function U does not vanish in the interval $-\infty < F < F^*$. It is readily seen that this is possible only when

$$k > 9/4. \qquad (292)$$

The time factor when the Universe is in the singular state is

$$U(F^*) = -F^* + (k-w)\, F^{*2}/4. \qquad (293)$$

It follows that in the region (291) of variation of the parameter w the time factor U for $F=F^*$ satisfies the inequalities $-5+4k \geq U \geq -2+k > \frac{1}{4}$.

The evolution of the Universe in the neighborhood of the singular state is essentially determined by the parameter w. Thus, for $w=\frac{1}{4}$ we obtain from the expressions (286), (284), and (290)

$$V \sim \tau_0^{4/3}; \quad \varepsilon \sim \tau_0^{-8/3}; \quad U \approx -5 + 4k.$$

For $w=0$,

$$V \sim \tau_0^{4/5}; \quad \varepsilon \sim \tau_0^{-8/5}; \quad U \approx -2 + k.$$

Thus, for large values of the parameter w there is a more rapid growth of the scale factor V in the neighborhood of the singular state of the Universe as time passes. For comparison we point out that in general relativity[11] the estimates $V \sim \tau_0$ and $\varepsilon \sim \tau_0^{-2}$ hold in the neighborhood of the singular state for any type of model of the Universe.

The behavior of the functions U and V in the neighborhood of the singular state essentially determines the flux densities and spectral characteristics of the fossil (background) electromagnetic, neutrino, and gravitational radiation. The frequencies of the electromagnetic and neutrino radiations and, therefore, their temperature change as a result of the influence of both the cosmological gravitational field and the Doppler effect. In contrast, the frequency and temperature of the gravitational radiation can change only as a result of the Doppler effect. Therefore, measurement of the flux density and spectral characteristics of these background radiations will make it possible to determine the behavior of the coupling equation in strong gravitational fields.

To determine the age of the Universe, we need the equation of state of the matter. Since we do not know the exact equation of state of the matter, we estimate the age of the Universe approximately. We note first that for any equation of state of the matter $0 < p \geq \varepsilon/3$. In accordance with the expressions (278) and (290), this means that for $0 \leq F \leq F^*$

$$\varepsilon_0/\sqrt{V^3} \leqslant \varepsilon \leqslant \varepsilon_0/V^2.$$

Therefore, for the age (285) of the Universe we have $T_1 \geq T \geq T_2$, where

$$T_1 = \frac{1}{4H} \int_0^{F^*} \frac{\sqrt{U}\, dF}{\sqrt{1-\alpha+\alpha\sqrt{U}}};$$

$$T_2 = \frac{1}{4H} \int_0^{F^*} \frac{\sqrt{U}\, dF}{\sqrt{1-\alpha+\alpha\sqrt{U/V}}}.$$

Analysis shows that the numerical values of T_1 and T_2 depend essentially on the values of the parameters w and k, varying in wide ranges with variation of these parameters. The minimal age of the Universe for the ranges of variation (291) and (292) is attained for $w=0$, $k \approx 9/4$: $3/4H \geq T \geq 2/9H$.

When the parameters w and k increase in the ranges (291) and (292), the age of the Universe increases monotonically. Thus, for $w=\frac{1}{4}, k=12$ we have $7/H \geq T \geq 1/H$.

It follows from the inequalities (229), (291), and (292) in the considered case that the minimal-coupling parameters b_1 and b_2, and also b_3 and b_4 do not vanish pairwise. Moreover, if one of the parameters b_1, b_2, b_3, b_4 is zero, all the remaining parameters are necessarily nonzero.

II. $w<0$ or $q_0>4-(3/2)\alpha$.

In this case, the roots of the function V have opposite signs, and, therefore, the "expansion" of the Universe will be replaced in the future by "contraction," and the Universe will return to the singular state. This occurs for scale factor $V=1-1/4w>1$. The value of the root F^* corresponding to the initial state of the Universe is in the range $0<F^*<2$. The return of the Universe to the singular state occurs at $F_2=(1+\sqrt{1-4w})/w$. It is easy to see that the value of F_2 is in the interval $0>F_2>-\infty$. It should be noted that the metric of the Universe will not have singularities between these singular states only when the function U does not have roots in the region $F_2<F<F^*$.

The evolution of the Universe in the neighborhood of the singular state, and also the age of the Universe are essentially determined in this case by the values of the parameters w and k, and, as analysis shows, the age of the Universe may be either greater or less than $1/H$.

Thus, in the field theory of gravitation nonstationary homogeneous models of the Universe describe the cosmological red shift and allow both monotonic and nonmonotonic behavior. The behavior of the model and the lifetime of the Universe depend on the deceleration parameter q_0: For values of q_0 in the range $4-(3/2)\alpha \geqslant 3-(3/2)\alpha$, the Universe will expand infinitely long, but for $q_0>4-3\alpha/2$ the expansion will give way after a certain time to contraction, and the Universe will return to the singular state.

17. MOTION OF EXTENDED BODIES IN AN ARBITRARY METRIC THEORY OF GRAVITATION

In various papers,[129-132] the possible deviation of the motion of the center of mass of an extended body from motion along a geodesic of Riemannian space-time was considered on the basis of the parametrized post-Newtonian approximation of an arbitrary metric theory of gravitation for the example of a system containing the Sun and one of its planets.

The restriction in Refs. 129–132 to the case of the solar system was evidently due to the fact that the only post-Newtonian system accessible at that time to observation was the solar system. But with the discovery[82,83] of the binary pulsar system PSR 1913+16 it has become possible to measure the orbital parameters of a post-Newtonian system consisting of two extended bodies of comparable mass. It is to be hoped that in future other such systems, and also systems consisting of a large number of bodies of comparable mass will be discovered and studied.

The motion of the center of mass of an extended body in an arbitrary binary system was investigated in Refs. 133 and 134. Besides the achievement of greater generality, the need for such a study was dictated by two further considerations. First, the solution given in the literature to the question of the relationship between the inertial and passive gravitational masses of an extended body and of the nature of the motion of its center of mass with respect to a geodesic of Riemannian space-time cannot be regarded as correct because of the use of some incorrect assumptions. Second, such study is important for any metric theory of gravitation and not only the field theory of gravitation, since measurement of the deviation of the center of mass of an extended body from motion along a geodesic of Riemannian space-time makes it possible to determine the numerical values of the post-Newtonian parameters more accurately.

It should be noted that the expression originally adopted in Refs. 133 and 134 for the Riemannian space-time metric differs somewhat from the corresponding expressions employed by other authors. Therefore, to facilitate comparison of our results with those of others authors, we choose the original Riemannian space-time metric in the form (150). In addition, for greater generality of the treatment, we shall expand the equations of motion of an extended body in the small parameter $L/R\ll 1$ (L is the characteristic linear dimension of the body and R is the distance between the bodies) to higher orders than in Refs. 133 and 134. However, as will be shown below, these changes do not affect the main conclusions about the nature of the motion of the center of mass of an extended body. We consider a problem of astronomical type, namely, we shall assume that the investigated system consists of two extended bodies moving in the gravitational field they produce and separated by a distance appreciably greater than their linear dimensions. We shall call them nominally the first and second body. We shall assume that they consist of an ideal fluid whose energy-momentum tensor density (of weight 1) has theform (156).

We shall also assume that the post-Newtonian formalism applies to the system. For this, it is necessary that the maximal values of the gravitational potential U, the square v^2 of the characteristic velocity, the specific pressure p/ρ_0, and the specific internal energy Π should have approximately the same order of magnitude ε^2, where $\varepsilon\ll 1$ is a small dimensionless parameter. In this case, the bodies will be situated in the near zone of the gravitational radiation produced by their motion. Therefore, in the region occupied by the bodies the variations of all quantities in time will be due in the first place to the motion of the matter, and, therefore, the partial derivatives of all quantities with respect to the time will be small compared with the partial derivatives with respect to the coordinates. As is well known,[84] any theory of gravitation for which the natural geometry for motion of the matter is a Riemannian geometry generates the metric (150) in the post-Newtonian approximation.

In the general case, this metric contains the ten arbitrary parameters $\gamma, \beta, \alpha_1, \alpha_2, \alpha_3, \xi_1, \xi_2, \xi_3, \xi_4, \xi_w$ and the three components w^α of the velocity of the frame of reference with respect to a hypothetical universal rest frame. Note that the model we have adopted for the extended bodies describes bodies in which the pressure is isotropic. Therefore, our calculation applies only to physical situations in which the shear stress in the extended bodies can be ignored compared with the isotropic pressure. If they cannot be ignored, it is necessary to take into account the contribution of the shear stresses to the matter energy-momentum tensor (156) and to the metric (150).

It should also be emphasized that our calculation applies only to the metric theories of gravitation that possess energy-momentum conservation laws for the matter and the gravitational field taken together. For theories of gravitation that do not possess these conservation laws, the calculation must be made specially in the framework of each of these theories, and the conclusions of the present paper do not apply to them.

Post-Newtonian equations of motion of an ideal fluid. To determine the force exerted on the first body by the second, we must construct the equations of hydrodynamics (the equations of motion of an element of ideal fluid) in the Riemannian space-time with metric (150). Following Fock,[13] to construct these equations we shall proceed from the covariant equation for the energy-momentum tensor density of the matter in the Riemannian space-time,

$$\nabla_i T^{ni} = \partial_i T^{ni} + \Gamma^n_{mi} T^{mi} = 0, \tag{294}$$

and also from the covariant continuity equation of the ideal fluid,

$$\frac{1}{\sqrt{-g}}\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^\alpha}(\rho v^\alpha)\right] = 0, \tag{295}$$

where $\rho = \rho_0 \sqrt{-g} u^0$ is the conserved mass density. It is obvious that in the Newtonian approximation the relation (294) leads to Euler's well-known equations for an ideal fluid:

$$\rho\, dv^\alpha/dt = -\rho\, \partial^\alpha U + \partial^\alpha p; \quad \rho\, d\Pi/dt = -p\, \partial_\alpha v^\alpha, \tag{296}$$

where $\Phi/dt = \partial/\partial t + v^\beta \partial/\partial x^\beta$ is the substantial derivative with respect to the time.

To obtain from the expression (294) the post-Newtonian equations of motion of an ideal fluid, we first construct the components of the energy-momentum tensor density of the matter with the necessary accuracy. Using the definition (156) of the energy-momentum tensor density of an ideal fluid, the metric (150), the expression for the velocity 4-vector,

$$u^i = \frac{dx^i/dt}{\sqrt{g_{00} + 2g_{0\alpha}v^\alpha + g_{\alpha\beta}v^\alpha v^\beta}}, \tag{297}$$

and the contravariant components of the metric tensor

$$g^{00} = 1 + 2U + O(\varepsilon^4); \; g^{0\alpha} = O^\alpha(\varepsilon^3); \; \sqrt{-g} = 1 + (3\gamma - 1)U + O(\varepsilon^4);$$
$$g^{\alpha\beta} = \gamma^{\alpha\beta}(1 - 2\gamma U) + O(\varepsilon^4),$$

we obtain the following expressions for the components of the energy-momentum tensor density (of weight 1) of the matter (156):

$$\left.\begin{aligned}
&T^{00} = \rho(1 - v_\alpha v^\alpha/2 + \Pi + U) + \rho O(\varepsilon^4);\\
&T^{0\alpha} = \rho v^\alpha(1 - v_\beta v^\beta/2 + \Pi + U) + p v^\alpha + \rho O^\alpha(\varepsilon^5);\\
&T^{\alpha\beta} = \rho v^\alpha v^\beta - p\gamma^{\alpha\beta} + \rho O^{\alpha\beta}(\varepsilon^4);\\
&\rho = \rho_0[1 + 3\gamma U - v_\alpha v^\alpha/2 + O(\varepsilon^4)].
\end{aligned}\right\} \tag{298}$$

Noting that

$$\partial_\beta U_{\delta\eta} - \partial_\delta U_{\beta\eta} = \gamma_{\beta\eta}\partial_\delta U - \gamma_{\eta\delta}\partial_\beta U;$$
$$\partial_\alpha V_\beta - \partial_\beta V_\alpha = \partial_\alpha W_\beta - \partial_\beta W_\alpha,$$

we write the components of the connection of the Riemannian space-time, which are needed for further calculations, in the form

$$\left.\begin{aligned}
&\Gamma^\alpha_{00} = \frac{1}{2}(4\gamma + 3 + \alpha_1 - \alpha_2 + \xi_1)\frac{\partial V^\alpha}{\partial t} + \frac{1}{2}(1 + \alpha_2 - \xi_1)\frac{\partial W^\alpha}{\partial t}\\
&\quad - \frac{1}{2}(\alpha_1 - 2\alpha_2) w^\alpha \frac{\partial U}{\partial t} + \alpha_2 w_\beta \frac{\partial U^{\alpha\beta}}{\partial t}\\
&\quad + \partial^\alpha U + \partial^\alpha \Phi - \xi_w \partial^\alpha \Phi_w - 2(\beta + \gamma) U \partial^\alpha U + O^\alpha(\varepsilon^6);\\
&\Gamma^\alpha_{0\beta} = \gamma\delta^\alpha_\beta \partial U/\partial t + (4\gamma + 4 + \alpha_1)(\partial_\beta V^\alpha - \partial^\alpha V_\beta)/4\\
&\quad - \alpha_1(w^\alpha \partial_\beta U - w_\beta \partial^\alpha U)/4 + O(\varepsilon^5);\\
&\Gamma^\alpha_{\beta\delta} = \gamma[\delta^\alpha_\delta \partial_\beta U + \delta^\alpha_\beta \partial_\delta U - \gamma_{\beta\delta}\partial^\alpha U] + O(\varepsilon^4),
\end{aligned}\right\} \tag{299}$$

where to shorten the expressions we have introduced the notation

$$\begin{aligned}
\Phi = {} & (2\gamma + 2 + \alpha_3 + \xi_1)\Phi_1/2 - \xi_1 A/2\\
& + (3\gamma + 1 - 2\beta + \xi_2)\Phi_2 + (1 + \xi_3)\Phi_3\\
& + 3(\gamma + \xi_4)\Phi_4 + (\alpha_1 - \alpha_2 - \alpha_3) w_\beta w^\beta U/2\\
& - \alpha_2 w^\beta w^\alpha U_{\alpha\beta}/2 + (2\alpha_3 - \alpha_1) w^\beta V_\beta/2.
\end{aligned}$$

The post-Newtonian equations of motion of the ideal fluid canbe obtained by writing down (294) for $n = \alpha$ to terms $\rho O(\varepsilon^4)$ inclusive:

$$\partial_0 T^{0\alpha} + \partial_\beta T^{\alpha\beta} + \Gamma^\alpha_{00} T^{00} + 2\Gamma^\alpha_{0\beta} T^{0\beta} + \Gamma^\alpha_{\beta\eta} T^{\beta\eta} = \rho O(\varepsilon^6). \tag{300}$$

Using the expressions (298) and (299), it is readily seen that all terms of Eq. (300), apart from the second, $\partial_\beta T^{\beta\alpha}$, can be written down to the accuracy adopted in this equation. Substituting the expressions (298) and (299) in all terms of Eq. (300) except the second, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\rho v^\alpha\left(1 + \Pi + U - \frac{v_\beta v^\beta}{2}\right) + p v^\alpha\right]\\
&+ \partial_\beta T^{\alpha\beta} + \rho\partial^\alpha U + \rho\partial^\alpha\Phi + 2\gamma\rho v^\alpha v^\beta \partial_\beta U - \xi_w \rho \partial^\alpha \Phi_w\\
&+ \rho\partial^\alpha U\left[\Pi - \frac{2\gamma + 1}{2} v_\varepsilon v^\varepsilon + \gamma\frac{p}{\rho} - (2\beta + 2\gamma - 1)U\right]\\
&+ \frac{1}{2}(4\gamma + 3 + \alpha_1 - \alpha_2 + \xi_1)\rho\frac{\partial V^\alpha}{\partial t} + \frac{1}{2}(1 + \alpha_2 - \xi_1)\rho\frac{\partial W^\alpha}{\partial t}\\
&+ \frac{1}{2}(4\gamma + 4 + \alpha_1)\rho v^\beta(\partial_\beta V^\alpha - \partial^\alpha V_\beta)\\
&+ \rho\frac{\partial U}{\partial t}\left[-\frac{1}{2}(\alpha_1 - 2\alpha_2) w^\alpha + 2\gamma v^\alpha\right] - \frac{1}{2}\alpha_1 \rho v^\beta(w^\alpha \partial_\beta U - w_\beta \partial^\alpha U)\\
&+ \alpha_2 w_\beta \rho \frac{\partial U^{\alpha\beta}}{\partial t} = \rho O(\varepsilon^6).
\end{aligned}$$

We reduce the expression to a form convenient for further investigation. After identical transformations, we obtain the post-Newtonian equations of motion of the ideal fluid:

$$\begin{aligned}
&\rho\frac{dv^\alpha}{dt} + \frac{\partial}{\partial x^\beta}\left[T^{\alpha\beta} - \rho v^\alpha v^\beta\left(1 + \Pi + U - \frac{v_\varepsilon v^\varepsilon}{2}\right) - p v^\alpha v^\beta + \gamma^{\alpha\beta} p U\right]\\
&+ \rho\partial^\alpha U - 2(\beta + \gamma)\rho U \partial^\alpha U + (\gamma - 2) p \partial^\alpha U + v^\alpha \frac{\partial p}{\partial t}\\
&+ \partial^\alpha p\left(\Pi - \frac{v_\varepsilon v^\varepsilon}{2} + \frac{p}{\rho}\right) - \gamma v_\varepsilon v^\varepsilon \rho \partial^\alpha U + (2\gamma + 1)\rho v^\alpha \frac{\partial U}{\partial t}\\
&+ 2(\gamma + 1)\rho v^\alpha v^\beta \partial_\beta U + \frac{1}{2}(4\gamma + 3 + \alpha_1 - \alpha_2 + \xi_1)\rho\frac{\partial V^\alpha}{\partial t}\\
&+ \frac{1}{2}(1 + \alpha_2 - \xi_1)\rho\frac{\partial W^\alpha}{\partial t} + \frac{1}{2}(4\gamma + 4 + \alpha_1)\rho v^\beta(\partial_\beta V^\alpha - \partial^\alpha V_\beta) + \rho\partial^\alpha\Phi\\
&- \frac{1}{2}(\alpha_1 - 2\alpha_2) w^\alpha \rho\frac{\partial U}{\partial t} - \xi_w \rho \partial^\alpha \Phi_w - \frac{1}{2}\alpha_1 \rho v^\beta(w^\alpha \partial_\beta U - w_\beta \partial^\alpha U)\\
&+ \alpha_2 w_\beta \rho \frac{\partial U^{\alpha\beta}}{\partial t} = \rho O(\varepsilon^6).
\end{aligned} \tag{301}$$

Equations of motion of the center of mass of an extended body. We consider two extended bodies occupying the volumes V_1 and V_2 and separated by a distance R appreciably greater than their linear dimensions L: $L/R \ll 1$. In this case, the conserved mass density of the ideal fluid is

$$\rho(x, t) = \rho_1(x, t) + \rho_2(x, t),$$

where the density $\rho_1(x,t)$ is nonzero in the volume V_1, and the density $\rho_2(x,t)$ is nonzero in the volume V_2. For simplicity, we shall assume that the matter distribution in the bodies is nearly spherically symmetric, as a result of which the reduced multipole moments of each of the bodies are small. Analysis shows that this condition makes it possible to avoid cumbersome calculations but in no way affects the final conclusions. Using the last of the equations in (298), we rewrite the generalized potentials (151) in the form

$$\left.\begin{aligned}
U &= \tilde{U} - 3\gamma\tilde{\Phi}_2 - \frac{1}{2}\tilde{\Phi}_1 + \int \frac{\rho_2}{|X-y|}dy + \frac{1}{2}\int \frac{\rho_2 v_\alpha v^\alpha}{|X-y|}dy \\
&\quad - 3\gamma \int \frac{[\rho_1'\rho_2'' + \rho_2'\rho_1'' + \rho_2'\rho_2'']}{|X-X'||X'-X''|}dX'\,dX'' + O(\varepsilon^6);\\
\Phi_1 &= \tilde{\Phi}_1 - \int \frac{\rho_2 v_\alpha v^\alpha}{|X-y|}dy + O(\varepsilon^6);\\
\Phi_2 &= \tilde{\Phi}_2 + \int \frac{[\rho_1'\rho_2'' + \rho_2'\rho_1'' + \rho_2'\rho_2'']}{|X-X'||X'-X''|}dX'\,dX'' + O(\varepsilon^6);\\
\Phi_3 &= \tilde{\Phi}_3 + \int \frac{\rho_2 \Pi}{|X-y|}dy + O(\varepsilon^6);\\
\Phi_4 &= \tilde{\Phi}_4 + \int \frac{p_2}{|X-y|}dy + O(\varepsilon^6);\\
V^\alpha &= \tilde{V}^\alpha - \int \frac{\rho_2 v^\alpha}{|X-y|}dy + O(\varepsilon^5);\\
A &= \tilde{A} + \int \frac{\rho_2[v_\alpha(X^\alpha - y^\alpha)]^2}{|X-y|^3}dy + O(\varepsilon^6);\\
W^\alpha &= \tilde{W}^\alpha + \int \frac{\rho_2 v_\beta (X^\beta - y^\beta)(X^\alpha - y^\alpha)}{|X-y|^3}dy + O(\varepsilon^5);\\
U^{\alpha\beta} &= \tilde{U}^{\alpha\beta} + \int \frac{\rho_2 (X^\alpha - y^\alpha)(X^\beta - y^\beta)}{|X-y|^3}dy + O(\varepsilon^4);\\
\Phi_w &= \tilde{\Phi}_w + \int \frac{[\rho_2'\rho_1'' + \rho_1'\rho_2'' + \rho_2'\rho_2'']}{|X-X'|^3}(X^\beta - X'^\beta)\\
&\quad \times \left[\frac{X_\beta - X_\beta''}{|X'-X''|} - \frac{X_\beta' - X_\beta''}{|X-X''|}\right]dX'\,dX'' + O(\varepsilon^6),
\end{aligned}\right\} \quad (302)$$

where for the gravitational self-potentials of the extended bodies we have introduced the notation

$$\left.\begin{aligned}
&\tilde{U} = \int \frac{\rho_1}{|X-y|}dy; \quad \tilde{\Phi}_1 = -\int \frac{\rho_1 v_\alpha v^\alpha}{|X-y|}dy;\\
&\tilde{\Phi}_2 = \int \frac{\rho_1 \tilde{U}}{|X-y|}dy;\\
&\tilde{\Phi}_3 = \int \frac{\rho_1 \Pi}{|X-y|}dy; \quad \tilde{\Phi}_4 = \int \frac{p_1}{|X-y|}dy;\\
&\tilde{V}^\alpha = -\int \frac{\rho_1 v^\alpha}{|X-y|}dy;\\
&\tilde{A} = \int \frac{\rho_1[v_\beta(X^\beta - y^\beta)]^2}{|X-y|^3}dy; \quad \tilde{U}^{\alpha\beta} = \int \frac{\rho_1 (X^\alpha - y^\alpha)(X^\beta - y^\beta)}{|X-y|^3}dy;\\
&\tilde{W}^\alpha = \int \frac{\rho_1 v_\beta (X^\beta - y^\beta)(X^\alpha - y^\alpha)}{|X-y|^3}dy;\\
&\tilde{\Phi}_w = \int \frac{\rho_1'\rho_1''}{|X-X'|^3}(X^\beta - X'^\beta)\\
&\quad \times \left[\frac{X_\beta - X_\beta''}{|X'-X''|} - \frac{X_\beta' - X_\beta''}{|X-X''|}\right]dX'\,dX''.
\end{aligned}\right\} \quad (303)$$

For the following, it is necessary to define the inertial mass and center of mass of an extended body. In the case of a theory of gravitation that possesses energy-momentum conservation laws of the matter and gravitational field taken together, we introduce rigorous concepts of the inertial mass and the center of mass:

$$m = \int [t_M^{00} + t_g^{00}]\,dX; \quad mX^\alpha = \int [t_M^{00} + t_g^{00}]\,x^\alpha\,dX, \quad (304)$$

where t_M^{ni} and t_g^{ni} are the energy-momentum tensors of the matter and the gravitational field, respectively, m is the inertial mass of the body, and X^α is the radius vector of its center of mass.

However, it is not possible to use this definition in the considered general case, since different metric theories of gravitation with energy-momentum conservation laws for the matter and the gravitational field can have different expressions for the tensors t_M^{ni} and t_g^{ni}. Since we wish to encompass in our investigation the largest possible class of metric theories of gravitation possessing energy-momentum conservation laws of the matter and the gravitational field, we shall, following Will,[131] use the following definitions of the mass of the bodies and coordinates of the centers of mass in our calculations:

$$M = \int \rho\,dx; \quad MX^\alpha = \int \rho x^\alpha\,dx. \quad (305)$$

The mass M is the total rest mass of the particles of the body, and in accordance with Eq. (295) it does not depend on the time. This mass is not equal to the inertial mass (304) of the body, differing from it by the post-Newtonian corrections $m = M(1+\delta)$, where $\delta \sim O(\varepsilon^2)$. However, this difference does not affect the calculation of the acceleration of the center of mass of the extended body.

Indeed, if we use the definition (304), the expression for the force exerted by the second body on the first is found in the form $F_1^\alpha = km[n^\alpha + \Delta^\alpha + O(\varepsilon^4)]$, where k is a constant, and $\Delta^\alpha \sim O(\varepsilon^2)$ are the post-Newtonian corrections.

When the definition (305) is used, we shall calculate the expression on the right-hand side already in terms of the density ρ, and therefore $F_2^\alpha = kM[n^\alpha + \Delta^\alpha + O(\varepsilon^4)]$. It is readily seen that in the two cases we arrive at the same acceleration:

$$a^\alpha = F_1^\alpha/m = F_2^\alpha/M = k[n^\alpha + \Delta^\alpha + O(\varepsilon^4)].$$

A further advantage of the definition (305) is the fairly simple connection between the coordinates of the center of mass of the body, its velocity, and acceleration. Thus, for the first body, using the covariant continuity equation (295) of the ideal fluid, we obtain

$$M_1 V_{(1)}^\alpha = \frac{d}{dt} M_1 X_{(1)}^\alpha = \int \rho_1(x, t)\,v^\alpha\,dx;$$

$$M_1 a_{(1)}^\alpha = \frac{d}{dt} M_1 V_{(1)}^\alpha = \int \rho_1(x, t)\,\frac{dv^\alpha}{dt}\,dx.$$

Therefore, we shall make the further calculation on the basis of the definition (305).

Since the metric (150) and, therefore, the equations of motion (301) are given in an arbitrary frame of reference, the velocity of the center of mass of the complete post-Newtonian system is not equal to zero in it but has the form

$$V_\Sigma^\alpha = (M_1 V_{(1)}^\alpha + M_2 V_{(2)}^\alpha)/(M_1 + M_2) \neq 0.$$

However, in accordance with Refs. 131 and 135, one can make "post-Galilean" transformations of the frame of reference which leave the metric (150) and the equations of motion (301) form-invariant and such that in the new frame the center-of-mass velocity of the complete post-Newtonian system is equal to zero.

We denote the radius vector of the center of mass of the first body in this frame by $Y^{\alpha}_{(1)}$, that of the second by $Y^{\alpha}_{(2)}$, and their difference by $Y^{\alpha}_{(1)}-Y^{\alpha}_{(2)}=R^{\alpha}$. We now integrate the post-Newtonian equations of motion (301) of the ideal fluid over the volume occupied by the first body. We expand each term in the obtained expression in powers of L/R to accuracy $O(1/R^4)$. For this, we use the expansions

$$\left.\begin{aligned}
&\frac{X_1^{\alpha}-X_2^{\alpha}}{|X_1-X_2|}=\frac{R^{\alpha}}{R}+\frac{R^{\alpha}R_{\varepsilon}(x_1^{\varepsilon}-x_2^{\varepsilon})}{R^3}+\frac{1}{2}\frac{R^{\alpha}(x_{1\varepsilon}-x_{2\varepsilon})(x_1^{\varepsilon}-x_2^{\varepsilon})}{R^3}\\
&\quad+\frac{3}{2}\frac{R^{\alpha}[R_{\varepsilon}(x_1^{\varepsilon}-x_2^{\varepsilon})]^2}{R^5}+\frac{x_1^{\alpha}-x_2^{\alpha}}{R}+\frac{R_{\varepsilon}(x_1^{\varepsilon}-x_2^{\varepsilon})(x_1^{\alpha}-x_2^{\alpha})}{R^3}\\
&\quad+\frac{3}{2}\frac{R^{\alpha}R_{\varepsilon}(x_1^{\varepsilon}-x_2^{\varepsilon})(x_1^{\beta}-x_2^{\beta})(x_{1\beta}-x_{2\beta})}{R^5}+\frac{5}{2}\frac{[R_{\varepsilon}(x_1^{\varepsilon}-x_2^{\varepsilon})]^3}{R^6}\\
&\quad+\frac{(x_1^{\alpha}-x_2^{\alpha})(x_1^{\varepsilon}-x_2^{\varepsilon})(x_{1\varepsilon}-x_{2\varepsilon})}{2R^3}+\frac{3(x_1^{\alpha}-x_2^{\alpha})[R_{\varepsilon}(x_1^{\varepsilon}-x_2^{\varepsilon})]^2}{2R^5}\\
&\quad+O\left(\frac{1}{R^4}\right);\\
&\frac{1}{|X_1-X_2|}=\frac{1}{R}+\frac{R_{\varepsilon}(x_1^{\varepsilon}-x_2^{\varepsilon})}{R^3}\\
&\quad+\frac{(x_1^{\varepsilon}-x_2^{\varepsilon})(x_{1\varepsilon}-x_{2\varepsilon})}{2R^3}+\frac{3[R_{\varepsilon}(x_1^{\varepsilon}-x_2^{\varepsilon})]^2}{2R^5}+O\left(\frac{1}{R^4}\right);\\
&\frac{X_1^{\alpha}-X_2^{\alpha}}{|X_1-X_2|^2}=\frac{R^{\alpha}}{R^2}+\frac{2R^{\alpha}R_{\varepsilon}(x_1^{\varepsilon}-x_2^{\varepsilon})}{R^4}\\
&\quad+\frac{R^{\alpha}(x_1^{\varepsilon}-x_2^{\varepsilon})(x_{1\varepsilon}-x_{2\varepsilon})}{R^4}+\frac{4R^{\alpha}[R_{\varepsilon}(x_1^{\varepsilon}-x_2^{\varepsilon})]^2}{R^6}+\frac{x_1^{\alpha}-x_2^{\alpha}}{R^2}\\
&\quad+\frac{2(x_1^{\alpha}-x_2^{\alpha})R_{\varepsilon}(x_1^{\varepsilon}-x_2^{\varepsilon})}{R^4}+O\left(\frac{1}{R^4}\right);\\
&\frac{(X_1^{\alpha}-X_2^{\alpha})(X_1^{\beta}-X_2^{\beta})}{|X_1-X_2|^3}=\frac{R^{\alpha}R^{\beta}}{R^3}+\frac{3R^{\alpha}R^{\beta}R_{\varepsilon}(x_1^{\varepsilon}-x_2^{\varepsilon})}{R^5}\\
&\quad+\frac{3R^{\alpha}R^{\beta}(x_1^{\varepsilon}-x_2^{\varepsilon})(x_{1\varepsilon}-x_{2\varepsilon})}{2R^5}+\frac{3R^{\alpha}(x_1^{\beta}-x_2^{\beta})R_{\varepsilon}(x_1^{\varepsilon}-x_2^{\varepsilon})}{2R^5}\\
&\quad+\frac{15R^{\alpha}R^{\beta}[R_{\varepsilon}(x_1^{\varepsilon}-x_2^{\varepsilon})]^2}{2R^7}+\frac{R^{\alpha}(x_1^{\beta}-x_2^{\beta})+R^{\beta}(x_1^{\alpha}-x_2^{\alpha})}{R^3}\\
&\quad+\frac{(x_1^{\alpha}-x_2^{\alpha})(x_1^{\beta}-x_2^{\beta})}{R^3}+\frac{3R^{\beta}(x_1^{\alpha}-x_2^{\alpha})R_{\varepsilon}(x_1^{\varepsilon}-x_2^{\varepsilon})}{2R^5}+O\left(\frac{1}{R^4}\right);\\
&\frac{(X_1^{\alpha}-X_2^{\alpha})(Z_1^{\beta}-X_2^{\beta})}{|X_1-X_2|^3}=\frac{R^{\alpha}R^{\beta}}{R^3}+\frac{3R^{\alpha}R^{\beta}R_{\varepsilon}(x_1^{\varepsilon}-x_2^{\varepsilon})}{R^5}\\
&\quad+\frac{3R^{\alpha}R^{\beta}(x_1^{\varepsilon}-x_2^{\varepsilon})(x_{1\varepsilon}-x_{2\varepsilon})}{2R^5}+\frac{15R^{\alpha}R^{\beta}[R_{\varepsilon}(x_1^{\varepsilon}-x_2^{\varepsilon})]^2}{2R^7}\\
&\quad+\frac{R^{\alpha}(z_1^{\beta}-x_2^{\beta})+R^{\beta}(x_1^{\alpha}-x_2^{\alpha})}{R^3}+\frac{3R^{\alpha}(z_1^{\beta}-x_2^{\beta})R_{\varepsilon}(x_1^{\varepsilon}-x_2^{\varepsilon})}{R^5}\\
&\quad+\frac{3R^{\beta}(x_1^{\alpha}-x_2^{\alpha})R_{\varepsilon}(x_1^{\varepsilon}-x_2^{\varepsilon})}{R^5}+\frac{(x_1^{\alpha}-x_2^{\alpha})(z_1^{\beta}-x_2^{\beta})}{R^3}+O\left(\frac{1}{R^4}\right);\\
&\frac{X_1^{\alpha}-X_2^{\alpha}}{|X_1-X_2|^3}=\frac{R^{\alpha}}{R^3}+\frac{3R^{\alpha}R_{\varepsilon}(x_1^{\varepsilon}-x_2^{\varepsilon})}{R^5}+\frac{x_1^{\alpha}-x_2^{\alpha}}{R^3}+O\left(\frac{1}{R^4}\right).
\end{aligned}\right\}\quad(306)$$

Here and in what follows $X_1^{\alpha}, Z_1^{\alpha}, (X_2^{\alpha})$ denote the radius vectors of the first (respectively, the second) body in a coordinate system whose origin is placed at the center of mass of the post-Newtonian system, and x_1^{α}, $z_1^{\alpha}, (x_2^{\alpha})$ denote the radius vectors of the same points but in a coordinate system whose origin is placed at the center of mass of the first (respectively, the second) body.

Introducing the notation

$$\left.\begin{aligned}
&\Omega^{\alpha\beta}_{(1)}=-\frac{1}{2M_1}\int\rho_1\rho_1'\frac{(x^{\alpha}-x'^{\alpha})(x^{\beta}-x'^{\beta})}{|x-x'|^3}\,dx\,dx';\\
&\Omega^{\alpha\beta}_{(2)}=-\frac{1}{2M_2}\int\rho_2\rho_2'\frac{(x^{\alpha}-x'^{\alpha})(x^{\beta}-x'^{\beta})}{|x-x'|^3}\,dx\,dx';\\
&Q^{\alpha\beta}_{(1)}=\frac{1}{2M_1}\int\rho_1 v^{\alpha}v^{\beta}\,dx;\quad Q=Q^{\varepsilon}_{\varepsilon};\\
&Q^{\alpha\beta}_{(2)}=\frac{1}{2M_2}\int\rho_2 v^{\alpha}v^{\beta}\,dx;\quad \Omega=\Omega^{\varepsilon}_{\varepsilon};\\
&P_{(1)}=\frac{1}{M_1}\int p_1\,dx;\quad P_{(2)}=\frac{1}{M_2}\int p_2\,dx;\quad \Pi_{(2)}=\frac{1}{M_2}\int\rho_2\Pi\,dx
\end{aligned}\right\}\quad(307)$$

and using the trivial relations

$$\int\frac{\rho\rho'(x^{\alpha}-x'^{\alpha})}{|x-x'|^3}\,dx\,dx'=0;\quad \frac{\partial U}{\partial t}=\frac{\partial V^{\beta}}{\partial x^{\beta}}+O(\varepsilon^5);$$

$$\frac{1}{M}\int\frac{\rho\rho'(x^{\alpha}-x'^{\alpha})x^{\beta}}{|x-x'|^3}\,dx\,dx'=-\Omega^{\alpha\beta},$$

we obtain a number of equations needed for the following calculations:

$$\left.\begin{aligned}
&\int\rho_1\partial^{\alpha}U\,dx=\frac{M_1M_2}{R^2}\left\{n^{\alpha}\left[1-6\gamma\Omega_{(2)}+Q_{(2)}-3\gamma\frac{M_2}{R}\right]\right.\\
&\quad\left.-3\gamma n_{\beta}\Omega^{\alpha\beta}_{(1)}\right\}-3\gamma\int\rho_1\partial^{\alpha}\widetilde{\Phi}_2\,dx-\frac{1}{2}\int\rho_1\partial^{\alpha}\widetilde{\Phi}_1\,dx;\\
&\int\rho_1 U\partial^{\alpha}U\,dx=\frac{M_1M_2}{R^2}\left[2n^{\alpha}\Omega_{(1)}-n_{\varepsilon}\Omega^{\alpha\varepsilon}_{(1)}+n^{\alpha}\frac{M_2}{R}\right]\\
&\quad+\int\rho_1\widetilde{U}\partial^{\alpha}\widetilde{U}\,dx;\\
&\int p_1\partial^{\alpha}U\,dx=\int p_1\partial^{\alpha}\widetilde{U}\,dx+\frac{M_1M_2}{R^2}n^{\alpha}P_{(1)};\\
&\int\rho_1 v_{\varepsilon}v^{\varepsilon}\partial^{\alpha}U\,dx=\int\rho_1 v_{\varepsilon}v^{\varepsilon}\partial^{\alpha}\widetilde{U}\,dx+\frac{2M_1M_2}{R^2}n^{\alpha}Q_{(1)};\\
&\int\rho_1 v^{\alpha}\frac{\partial U}{\partial t}\,dx=\int\rho_1 v^{\alpha}\frac{\partial\widetilde{V}^{\beta}}{\partial x^{\beta}}\,dx-\frac{M_1M_2}{R^2}n_{\beta}V^{\alpha}_{(1)}V^{\beta}_{(2)};\\
&\int\rho_1 v^{\alpha}v^{\beta}\partial_{\beta}U\,dx=\int\rho_1 v^{\alpha}v^{\beta}\partial_{\beta}\widetilde{U}\,dx+\frac{2M_1M_2}{R^2}n_{\beta}Q^{\alpha\beta}_{(1)};\\
&\int\rho_1\frac{\partial V^{\alpha}}{\partial t}\,dx=-\int\rho_1[\partial^{\alpha}\widetilde{\Phi}_4+v^{\alpha}v_{\beta}\partial^{\beta}\widetilde{U}-\widetilde{U}\partial^{\alpha}\widetilde{U}]\,dx\\
&\quad+\frac{M_1M_2}{R^2}\left[-n^{\alpha}P_{(2)}+2n^{\alpha}\Omega_{(1)}+n_{\varepsilon}\Omega^{\varepsilon\alpha}_{(2)}+2n_{\varepsilon}Q^{\varepsilon\alpha}_{(2)}-n^{\alpha}\frac{M_1}{R}\right];\\
&\int\rho_1\frac{\partial W^{\alpha}}{\partial t}\,dx=\int\rho_1[\partial^{\alpha}\widetilde{\Phi}_4-\widetilde{U}^{\alpha\varepsilon}\partial_{\varepsilon}\widetilde{U}-\partial^{\alpha}\widetilde{A}+\partial^{\alpha}\widetilde{\Phi}_1\\
&\quad-v^{\alpha}v^{\beta}\partial_{\beta}\widetilde{U}]\,dx+\frac{M_1M_2}{R^2}\left\{n^{\alpha}\left[P_{(2)}-\Omega_{(2)}-3n_{\varepsilon}n_{\beta}\Omega^{\varepsilon\beta}_{(2)}\right.\right.\\
&\quad\left.\left.-2Q_{(2)}-6n_{\varepsilon}n_{\beta}Q^{\varepsilon\beta}_{(2)}-\frac{M_1}{R}\right]-n_{\varepsilon}\Omega^{\alpha\varepsilon}_{(2)}-2n_{\varepsilon}Q^{\alpha\varepsilon}_{(2)}+2n_{\varepsilon}\Omega^{\alpha\varepsilon}_{(1)}\right\};\\
&\int\rho_1 v^{\beta}\partial_{\beta}V^{\alpha}\,dx=\int\rho_1 v^{\beta}\partial_{\beta}\widetilde{V}^{\alpha}\,dx-\frac{M_1M_2}{R^2}n_{\beta}V^{\beta}_{(1)}V^{\alpha}_{(2)};\\
&\int\rho_1 v^{\beta}\partial^{\alpha}V_{\beta}\,dx=-\frac{M_1M_2}{R^2}n^{\alpha}V^{\beta}_{(1)}V_{(2)\beta};\\
&\int\rho_1\frac{\partial U}{\partial t}\,dx=\int\rho_1\frac{\partial\widetilde{V}^{\beta}}{\partial x^{\beta}}\,dx-\frac{M_1M_2}{R^2}n_{\beta}V^{\beta}_{(2)};\\
&\int\rho_1\frac{\partial U^{\alpha\beta}}{\partial t}\,dx=\int\rho_1 v^{\varepsilon}\partial_{\varepsilon}\widetilde{U}^{\alpha\beta}\,dx\\
&\quad-\frac{M_1M_2}{R^2}[n^{\alpha}V^{\beta}_{(2)}+n^{\beta}V^{\alpha}_{(2)}+3n^{\alpha}n^{\beta}n_{\varepsilon}V^{\varepsilon}_{(2)}];\\
&\int\rho_1 v^{\beta}\partial_{\beta}U\,dx=\int\rho_1 v^{\beta}\partial_{\beta}\widetilde{U}\,dx+\frac{M_1M_2}{R^2}n_{\beta}V^{\beta}_{(1)};\\
&\int\rho_1 v^{\beta}\partial^{\alpha}U\,dx=\int\rho_1 v^{\beta}\partial^{\alpha}\widetilde{U}\,dx+\frac{M_1M_2}{R^2}n^{\alpha}V^{\beta}_{(1)};\\
&\int\rho_1\partial^{\alpha}\Phi\,dx=\int\rho_1\partial^{\alpha}\widetilde{\Phi}\,dx+\frac{M_1M_2}{R^2}\left\{n^{\alpha}\left[(1+\xi_3)\Pi_{(2)}\right.\right.\\
&\quad+2(3\gamma+1-2\beta+\xi_2)\Omega_{(2)}+(3\gamma+1-2\beta+\xi_2)\frac{M_1}{R}\\
&\quad-(2\gamma+2+\alpha_3+\xi_1)Q_{(2)}-3\xi_1 n_{\beta}n_{\delta}Q^{\beta\delta}_{(2)}+3(\gamma+\xi_4)P_{(2)}\\
&\quad-\frac{3}{2}\alpha_2(n_{\beta}w^{\beta})^2+\frac{1}{2}(\alpha_1-\alpha_2-\alpha_3)w_{\beta}w^{\beta}\\
&\quad\left.-\frac{1}{2}(2\alpha_3-\alpha_1)w_{\beta}V^{\beta}_{(2)}\right]+(3\gamma+1-2\beta+\xi_2)n_{\beta}\Omega^{\alpha\beta}_{(1)}\\
&\quad\left.-2\xi_1 n_{\beta}Q^{\alpha\beta}_{(2)}-\alpha_2 w^{\alpha}n_{\beta}w^{\beta}\right\};\\
&\int\rho_1\partial^{\alpha}\Phi_w\,dx=\int\rho_1[2\partial^{\alpha}\widetilde{\Phi}_2-\widetilde{U}^{\alpha\beta}\partial_{\beta}\widetilde{U}]\,dx\\
&\quad+\frac{M_1M_2}{R^2}\left\{5n_{\beta}\Omega^{\alpha\beta}_{(1)}+n^{\alpha}\left[3n_{\beta}n_{\delta}\Omega^{\beta\delta}_{(1)}+\frac{M_1}{R}-5\Omega_{(1)}-2\Omega_{(2)}\right]\right\}.
\end{aligned}\right\}\quad(308)$$

From the expression (303) for the gravitational self-potentials, we readily establish the relations

$$\left.\begin{aligned}
&\int\rho_1[\Pi\partial_{\beta}\widetilde{U}+\partial_{\beta}\widetilde{\Phi}_3]\,dx=0;\quad \int\rho_1[\widetilde{U}\partial_{\beta}\widetilde{U}+\partial_{\beta}\widetilde{\Phi}_2]\,dx=0;\\
&\int[p_1\partial_{\beta}\widetilde{U}+\rho_1\partial_{\beta}\widetilde{\Phi}_4]\,dx=0;\quad \int\rho_1[v_{\alpha}v^{\alpha}\partial_{\beta}\widetilde{U}-\partial_{\beta}\widetilde{\Phi}_1]\,dx=0;\\
&\int\rho_1\left[v^{\alpha}\frac{\partial\widetilde{V}^{\beta}}{\partial x^{\beta}}+v^{\beta}\partial_{\beta}\widetilde{V}^{\alpha}\right]dx=0;\\
&\int\rho_1\left[\frac{\partial\widetilde{V}^{\beta}}{\partial x^{\beta}}-v^{\beta}\partial_{\beta}\widetilde{U}\right]dx=0;\\
&\int\rho_1[v^{\beta}\partial^{\alpha}\widetilde{U}-\partial^{\alpha}\widetilde{V}^{\beta}]\,dx=0;\\
&\int\rho_1\partial^{\alpha}\widetilde{U}_{\beta\delta}\,dx=0;\quad M_1V^{\alpha}_{(1)}+M_2V^{\alpha}_{(2)}=0.
\end{aligned}\right\}\quad(309)$$

Substituting the expansions (308) in the equations of motion of the first body and using the relations (309), we obtain

$$M_1 a^{\alpha}_{(1)} = -(M_1 M_2/R^2) f^{\alpha} - \Phi^{\alpha}. \tag{310}$$

For the vector f^{α}, we have

$$\begin{aligned} f^{\alpha} = {} & n^{\alpha}[1 + (\gamma-2) P_{(1)} + (3\xi_4 - \xi_1 + \alpha_2 - \alpha_1/2 + \gamma - 1) P_{(2)} \\ & + (1+\xi_3)\Pi_{(2)} - 2\gamma Q_{(1)} - (2\gamma + \alpha_2 + \alpha_3 + 2) Q_{(2)} - \\ & - (2\beta + 2\gamma + 1 + \xi_w - \xi_2 + \alpha_1/2)(M_1/R) - 2(\beta+\gamma)(M_2/R) \\ & + (\alpha_1 - \alpha_2 + \xi_1 + 5\xi_w + 3 - 4\beta)\Omega_{(1)} + (3/2 - \alpha_2/2 + \xi_1/2 \\ & - 4\beta + 2\xi_2 + 2\xi_w)\Omega_{(2)} - (3/2)(1 + \alpha_2 - \xi_1) n_\beta n_\delta \Omega^{\beta\delta}_{(2)} \\ & - 3\xi_w n_\beta n_\delta \Omega^{\beta\delta}_{(1)} - 3(1+\alpha_2) n_\beta n_\delta Q^{\beta\delta}_{(2)} \\ & + (4\gamma + 4 + \alpha_1) V^{\beta}_{(1)} V_{(2)\beta}/2 + (\alpha_1 - \alpha_2 - \alpha_3) w_\beta w^\beta/2 \\ & - (3/2)\alpha_2 (n_\beta w^\beta)^2 - (2\alpha_3 + 2\alpha_2 - \alpha_1) w_\beta V^{\beta}_{(2)}/2 \\ & - 3\alpha_2 n_\beta w^\beta n_\delta V^{\delta}_{(2)} + \alpha_1 w_\beta V^{\beta}_{(1)}/2] + 4(\gamma+1) n_\beta Q^{\alpha\beta}_{(1)} \\ & + (4\gamma + 2 + \alpha_1 - 2\alpha_2) n_\beta Q^{\alpha\beta}_{(2)} + (2\gamma + 2 + \alpha_2 - \xi_1 \\ & + \xi_2 - 5\xi_w) n_\beta \Omega^{\alpha\beta}_{(1)} + (2\gamma + 1 + \alpha_1/2 - \alpha_2 + \xi_1) n_\beta \Omega^{\alpha\beta}_{(2)} \\ & - \alpha_2 V^{\alpha}_{(2)} n_\beta w^\beta - \alpha_2 w^\alpha n_\beta w^\beta - (8\gamma + 6 + \alpha_1)[V^{\alpha}_{(1)} V^{\beta}_{(2)} \\ & + V^{\alpha}_{(2)} V^{\beta}_{(1)}] n_\beta/4 + (\alpha_1 - 2\alpha_2) w^\alpha n_\beta V^{\beta}_{(2)}/2 \\ & - \alpha_1 w^\alpha n_\beta V^{\beta}_{(1)}/2 + O(\varepsilon^4). \end{aligned} \tag{311}$$

The vector Φ^{α}, which characterizes the "internal" force, has the form

$$\begin{aligned} \Phi^{\alpha} = {} & \int \rho_1\, dx \Big\{ \frac{v^\alpha}{\rho_1}\frac{\partial p_1}{\partial t} + \frac{1}{\rho_1}\partial^\alpha p_1 \left(\Pi - \frac{v_\varepsilon v^\varepsilon}{2} + \frac{p_1}{p_1}\right) \\ & + \left(1 + \frac{\alpha_1}{2}\right) v^\beta \partial_\beta \widetilde{V}^\alpha + \frac{1}{2}(\alpha_2 - \alpha_1 - \xi_1 - 1 \\ & + 2\xi_2 - 4\xi_w)\partial^\alpha \widetilde{\Phi}_2 + \left(3\xi_4 - \xi_1 + \alpha_2 - \frac{\alpha_1}{2} + 1\right)\partial^\alpha \widetilde{\Phi}_4 \\ & + \frac{1}{2}(2 + \alpha_2 + \alpha_3)\partial^\alpha \widetilde{\Phi}_1 + (1+\xi_3)\partial^\alpha \widetilde{\Phi}_3 - \frac{\alpha_1}{2} v^\alpha v^\beta \partial_\beta \widetilde{U} \\ & - \frac{1}{2}(1+\alpha_2)\partial^\alpha \widetilde{A} - \frac{1}{2}(1 + \alpha_2 - \xi_1 - 2\xi_w)\widetilde{U}^{\alpha\beta}\partial_\beta \widetilde{U} \\ & + \alpha_3 w_\beta v^\beta \partial^\alpha \widetilde{U} - (\alpha_1 - \alpha_2) w^\alpha v^\beta \partial_\beta \widetilde{U} + \alpha_2 w_\beta v_\delta \partial^\delta \widetilde{U}^{\alpha\beta} \Big\}. \end{aligned} \tag{312}$$

The integrand in (312) is the force exerted on an element of volume of the extended body by the other elements of the same body. For theories of gravitation possessing energy-momentum conservation laws for the matter and the gravitational field, this vector must, when allowance is made for the equations of the internal motion of the body, be equal to zero, since it is only in this case that the momentum of an isolated extended body, i.e., for $\rho_2 = 0, M_2 = 0$, will be conserved. Since our calculation applies only to such theories of gravitation, it must be assumed that by virtue of the equations of the internal motion of the matter of the extended body $\Phi^{\alpha} = 0$, and therefore the equations of motion of the center of mass of the extended body take the form

$$M_1 a^{\alpha}_{(1)} = -\frac{M_1 M_2}{R^2} f^{\alpha}. \tag{313}$$

As follows from the expressions (311) and (313), the acceleration of the center of mass of the extended body has the form

$$\begin{aligned} a^{\alpha}_{(1)} = {} & -\frac{M_2}{R^2}\Big\{ n^\alpha \Big[1 - 2\gamma Q_{(1)} - (2\gamma + \alpha_2 + \alpha_3 + 2) Q_{(2)} + (\gamma - 2) P_{(1)} \\ & + (1+\xi_3)\Pi_{(2)} + \left(3\xi_4 - \xi_1 + \alpha_2 - \frac{\alpha_1}{2} + \gamma - 1\right) P_{(2)} \\ & + (\alpha_1 - \alpha_2 + \xi_1 + 5\xi_w + 3 - 4\beta)\Omega_{(1)} - 3\xi_w n_\beta n_\delta \Omega^{\beta\delta}_{(1)} \\ & - \left(2\beta + 2\gamma + 1 + \xi_w - \xi_2 + \frac{\alpha_1}{2}\right)\frac{M_1}{R} - 2(\beta+\gamma)\frac{M_2}{R} \\ & + \left(\frac{3}{2} - \frac{\alpha_2}{2} + \frac{\xi_1}{2} - 4\beta + 2\xi_2 + 2\xi_w\right)\Omega_{(2)} \\ & - \frac{3}{2}(1 + \alpha_2 - \xi_1) n_\beta n_\delta \Omega^{\beta\delta}_{(2)} - \frac{3}{2}\alpha_2 (n_\beta w^\beta)^2 - 3(1+\alpha_2) n_\beta n_\delta Q^{\beta\delta}_{(2)} \\ & + \frac{1}{2}(4\gamma + 4 + \alpha_1) V^{\beta}_{(1)} V_{(2)\beta} + \frac{1}{2}(\alpha_1 - \alpha_2 - \alpha_3) w_\beta w^\beta \\ & - \frac{1}{2}(2\alpha_3 + 2\alpha_2 - \alpha_1) w_\beta V^{\beta}_{(2)} - 3\alpha_2 n_\beta w^\beta n_\delta V^{\delta}_{(2)} + \frac{1}{2}\alpha_1 w_\beta V^{\beta}_{(1)} \Big] \\ & + 4(\gamma+1) n_\beta Q^{\alpha\beta}_{(1)} + (4\gamma + 2 + \alpha_1 - 2\alpha_2) n_\beta Q^{\alpha\beta}_{(2)} \\ & - \frac{1}{4}(8\gamma + 6 + \alpha_1)[V^{\alpha}_{(1)} V^{\beta}_{(2)} + V^{\alpha}_{(2)} V^{\beta}_{(1)}] n_\beta \\ & + \left(2\gamma + 1 + \frac{\alpha_1}{2} - \alpha_2 + \xi_1\right) n_\beta \Omega^{\alpha\beta}_{(2)} - \alpha_2 V^{\alpha}_{(2)} n_\beta w^\beta \\ & + (2\gamma + 2 + \alpha_2 - \xi_1 + \xi_2 - 5\xi_w) n_\beta \Omega^{\alpha\beta}_{(1)} - \alpha_2 w^\alpha n_\beta w^\beta \\ & + \frac{1}{2}(\alpha_1 - 2\alpha_2) w^\alpha n_\beta V^{\beta}_{(2)} - \frac{1}{2}\alpha_1 w^\alpha n_\beta V^{\beta}_{(1)} \Big\} + O(\varepsilon^6)/R. \end{aligned} \tag{314}$$

We now consider the structure of the vector f^{α} (311). In the general case of an arbitrary ratio of the masses of the bodies, the velocity $V^{\alpha}_{(2)} = -(M_1/M_2)V^{\alpha}_{(1)}$ of the second body cannot be ignored. Then because the expression (311) contains terms of the form $n^{\alpha} V^{\beta}_{(1)} V_{(2)\beta}$, $[V^{\alpha}_{(1)} V^{\beta}_{(2)} + V^{\alpha}_{(2)} V^{\beta}_{(1)}] n_\beta$, it is impossible to separate the characteristics of the first and the second body, and therefore the introduction of a tensor of the passive gravitational mass is not possible in the general case.

It should be noted that Will suggests that the equations of motion (313) of the center of mass of an extended body should be written in the form

$$M_1 a^{\alpha}_{(1)} = m^{\alpha\beta}_{p} \partial_\beta \widetilde{U} + a^{\alpha}_{N}, \tag{315}$$

where he calls the tensor $m^{\alpha\beta}_{p}$, which depends on the characteristics of the first body, the tensor of the passive gravitational mass and determines it by means of the expression (219). He calls all the remaining terms on the right-hand side of the equations of motion, the N-body accelerations and includes them in a^{α}_{N}. However, this separation is arbitrary, since a^{α}_{N}, as is readily seen, contains terms of the same structure as the first term on the right-hand side of (315):

$$a^{\alpha}_{N} = L^{\alpha\beta}_{(1)} \partial_\beta \widetilde{U} + q^{\alpha},$$

where the tensor $L^{\alpha\beta}_{(1)}$ depends only on the characteristics of the first body (in particular, on the square of its velocity) and therefore can be combined in the expression (315) with the tensor $m^{\alpha\beta}_{p}$.

Thus, the decomposition of the expression (315) proposed by Will is in no way justified except by the desire to obtain at any price formal equality of the passive gravitational mass and the inertial mass of an extended body in general relativity. But in Einstein's theory there are no conservation laws of the matter and the gravitational field taken together. As is shown in the Introduction, a special consequence of this is the assertion that the inertial mass defined in general relativity has no physical meaning. Therefore, in Einstein's theory it is physically meaningless to compare it with the passive gravitational mass. However, in metric theories of gravitation that possess conservation laws of the matter and the gravitational field taken together, the concept of the inertial mass $m = \int dV[t^{00}_{M} + t^{00}_{g}]$ has a rigorous physical meaning. It follows from the given definition that the inertial mass of a body depends not only on its internal characteristics but also on the square of the velocity. It is to be expected that the passive gravitational mass of an extended body will also depend not only on the internal structure of the body but also on its velocity. Thus, in metric theories of gravitation that possess conservation laws of the matter and the gravitational field taken together, comparison of the passive and active gravitational masses of an extended body with its inertial mass is of undoubted interest.

How can one define in this case a passive gravitational mass? A tensor of the passive gravitational mass of an extended body can be defined naturally, though somewhat formally, directly on the basis of the equations of motion if in the post-Newtonian approximation it is possible to represent them in the quasi-Newtonian form

$$M_1 a^{\alpha}_{(1)} = m_p^{\alpha\beta} \partial_\beta \mathcal{U}, \tag{316}$$

where the tensor $p^{\alpha\beta}$ must depend only on the characteristics of the first body, and the generalized Newtonian potential $\mathcal{U}$ only on the characteristics of the second body and on the distance between the bodies. If these conditions are not satisfied, as in the general case of an arbitrary post-Newtonian system, the concept of a passive gravitational mass tensor becomes pointless. A different, more sensible definition of a passive gravitational mass tensor cannot be found.

Thus, such a tensor can be introduced only for a small class of binary systems. However, to answer the question we have posed relating to the nature of the center-of-mass motion of an extended body there is no need to invoke the concept of a tensor of the passive gravitational mass. All that we need to do is compare the motion of the center of mass of an extended body with some idealized picture, namely, with the motion of a test (point) body in a Riemannian space-time whose metric is formally equivalent to the metric produced by the two moving extended bodies.

Then identity of the expression for the acceleration $a^{\alpha}_{(1)}$ of the center of mass of the extended body with the expression for the acceleration a^{α}_0 of the point body will mean that for the same initial conditions the center of mass of the extended body and the point body will move along the same trajectory and have the same law of motion. Since by definition the point body moves along a geodesic of the Riemannian space-time, the center of mass of the extended body will also move along a geodesic in such a case. If the expressions for $a^{\alpha}_{(1)}$ and a^{α}_0 differ by post-Newtonian corrections, the center of mass of the extended body will not move along a geodesic of the Riemannian space-time in the general case.

Apart from everything else, such an approach makes it possible to take into account naturally the contribution of the gravitational self-field of the extended body to the space-time curvature.

Equations of geodesic motion. We consider a Riemannian space-time whose metric is identical to the metric of the two moving extended bodies considered earlier. We study the motion of the point body in a neighborhood of the point corresponding to the center of mass of the first body. The expression for the acceleration a^{α}_0 of the point body can be studied in two ways, namely, either by using the equations of geodesics of the Riemannian space-time

$$du^i/ds + \Gamma^i_{nm} u^n u^m = 0, \tag{317}$$

or by making calculations similar to those in calculating the acceleration of the center of mass of an extended body. In the latter case, it must be borne in mind that all the quantities that characterize the internal structure and the gravitational self-field of the point body are negligibly small. The two cases lead to the same result.

We write down the equation of geodesics with post-Newtonian accuracy. For $i=\alpha$,

$$du^\alpha/ds + \Gamma^{\alpha}_{00} u^0 u^0 + 2\Gamma^{\alpha}_{0\beta} u^0 u^\beta + \Gamma^{\alpha}_{\beta\delta} u^\delta u^\beta = O(\varepsilon^6).$$

Using the expressions (297) and (299), we reduce these equations to the form

$$\begin{aligned}
&\frac{d}{dt}\left[V^{\alpha}_{(0)}\left(1+U-\frac{1}{2}V^{\beta}_{(0)}V_{(0)\beta}\right)\right] + \partial^\alpha U\left[1-\frac{2\gamma+1}{2}V^{\beta}_{(0)}V_{(0)\beta}\right.\\
&\qquad\left. -(2\beta+2\gamma-1)U\right] + \frac{\partial U}{\partial t}\left[2\gamma V^{\alpha}_{(0)} - \frac{1}{2}(\alpha_1-2\alpha_2)w^\alpha\right]\\
&\qquad -\xi_w \partial^\alpha \Phi_w + \partial^\alpha \Phi + \frac{1}{2}(4\gamma+3+\alpha_1-\alpha_2+\xi_1)\frac{\partial V^\alpha}{\partial t}\\
&+\frac{1}{2}(1+\alpha_2-\xi_1)\frac{\partial W^\alpha}{\partial t} + \frac{1}{2}(4\gamma+4+\alpha_1)V^{\beta}_{(0)}[\partial_\beta V^\alpha - \partial^\alpha V_\beta]\\
&+2\gamma V^{\alpha}_{(0)}V^{\beta}_{(0)}\partial_\beta U + \alpha_2 w_\beta \frac{\partial U^{\alpha\beta}}{\partial t}\\
&-\frac{\alpha_1}{2}V^{\beta}_{(0)}[w^\alpha \partial_\beta U - w_\beta \partial^\alpha U] = O(\varepsilon^6),
\end{aligned} \tag{318}$$

where $V^{\alpha}_{(0)} = dx^\alpha/dt$ are the velocity components of the point body.

Using the Newtonian equations of motion of the point body,

$$dV^{\alpha}_{(0)}/dt + \partial^\alpha U = O(\varepsilon^4),$$

we transform the expression

$$\frac{d}{dt}\left[V^{\alpha}_{(0)}\left(1+U-\frac{1}{2}V^{0}_{(\beta)}V_{(0)\beta}\right)\right]$$

to the form

$$\begin{aligned}
&\frac{d}{dt}\left[V^{\alpha}_{(0)}\left(1+U-\frac{1}{2}V^{\beta}_{(0)}V_{(0)\beta}\right)\right] = a^{\alpha}_0 - U\partial^\alpha U\\
&+\Gamma^{\alpha}_{(0)}\left[\frac{\partial U}{\partial t} + 2V^{\beta}_{(0)}\partial_\beta U\right] + \frac{1}{2}V^{\beta}_{(0)}V_{(0)\beta}\partial^\alpha U + O(\varepsilon^6).
\end{aligned} \tag{319}$$

Substituting the relation (319) in the equation of motion (318), we obtain the following expression for the acceleration a^{α}_0 of the point body:

$$\begin{aligned}
a^{\alpha}_0 &= -\partial^\alpha U[1-\gamma V^{\beta}_{(0)}V_{(0)\beta} - 2(\beta+\gamma)U] + \xi_w \partial^\alpha \Phi_w\\
&-\frac{\partial U}{\partial t}\left[(2\gamma+1)V^{\alpha}_{(0)} - \frac{1}{2}(\alpha_1-2\alpha_2)w^\alpha\right] - 2(\gamma+1)V^{\alpha}_{(0)}V^{\beta}_{(0)}\partial_\beta U\\
&-\partial^\alpha\Phi - \frac{1}{2}(4\gamma+3+\alpha_1-\alpha_2+\xi_1)\frac{\partial V^\alpha}{\partial t} - \frac{1}{2}(1+\alpha_2-\xi_1)\frac{\partial W^\alpha}{\partial t}\\
&-\alpha_2 w_\beta \frac{\partial U^{\alpha\beta}}{\partial t} - \frac{1}{2}(4\gamma+4+\alpha_1)V^{\beta}_{(0)}[\partial_\beta V^\alpha - \partial^\alpha V_\beta]\\
&+\frac{1}{2}\alpha_1 V^{\beta}_{(0)}[w^\alpha \partial_\beta U - w_\beta \partial^\alpha U] + O(\varepsilon^6).
\end{aligned} \tag{320}$$

We study the motion of the point body in the neighborhood of the point corresponding to the center of mass of the extended body. For this, we expand all potentials in the expression (320) in powers of $1/R$ to accuracy $\sim O(1/R^4)$. We obtain

$$\begin{aligned}
a^{\alpha}_0 = \tilde{a}^{\alpha}_0 + (M_2/R^2)&\left\{n^\alpha\left[-1+\gamma V^{\beta}_{(0)}V_{(0)\beta} + (2\beta+\alpha_2/2-\xi_1/2\right.\right.\\
&-3/2-\alpha_1/2-2\xi_w)\tilde{U} + (2\gamma+\alpha_3+\alpha_2+2)Q_{(2)}\\
&+\left(2\beta+2\gamma+1+\frac{\alpha_1}{2}+\xi_w-\xi_2\right)\frac{M_1}{R} + 2(\beta+\gamma)(M_2/R)\cdot\\
&+(4\beta-2\xi_2-2\xi_w-3/2+\alpha_2/2-\xi_1/2)\Omega_{(2)} - (1+\xi_3)\Pi_{(2)}\\
&+(1+\alpha_1/2-\alpha_2+\xi_1-\gamma-3\xi_4)P_{(2)} + 3(1+\alpha_2)n_\beta n_\delta Q^{\beta\delta}_{(2)}\\
&+\frac{3}{2}(1+\alpha_2-\xi_1)n_\beta n_\delta \Omega^{\beta\delta}_{(2)} - \frac{1}{2}(\alpha_1-\alpha_2-\alpha_3)w_\beta w^\beta
\end{aligned}$$

$$
\begin{aligned}
&+\frac{3}{2}\alpha_2(n_\beta w^\beta)^2+(\alpha_2+\alpha_3-\alpha_1/2)V^\beta_{(2)}w_\beta-(2\gamma+2+\alpha_1/2)V^\beta_{(2)}V_{(0)\beta}\\
&+3\alpha_2 n_\beta V^\beta_{(2)}n_\delta w^\delta-\frac{1}{2}\alpha_1 w_\beta V^\beta_{(0)}+\frac{\xi_w}{2}\int\frac{\rho_1'\,dx'}{|x-x'|^3}(3x'_\beta x'^\beta-2x'_\beta x^\beta\\
&\qquad -x_\beta x^\beta+3(n_\beta x'^\beta)^2-9(n_\beta x^\beta)^2+6n_\beta n_\delta x'^\beta x^\delta)\Big]\\
&+(2\alpha_2-2-\alpha_1-4\gamma)n_\beta Q^{\alpha\beta}_{(2)}+\Big(\alpha_2-1-\xi_1-\frac{\alpha_1}{2}-2\gamma\Big)n_\beta\Omega^{\alpha\beta}_{(2)}\\
&-2(\gamma+1)V^\alpha_{(0)}n_\beta V^\beta_{(0)}+n_\beta V^\beta_{(2)}\Big[(2\gamma+1)V^\alpha_{(0)}-\frac{1}{2}(\alpha_1-2\alpha_2)w^\alpha\Big]\\
&+\Big(2\gamma+2+\frac{\alpha_1}{2}\Big)V^\alpha_{(2)}n_\beta V^\beta_{(0)}+\frac{\alpha_1}{2}w^\alpha n_\beta V^\beta_{(0)}+\alpha_2 V^\alpha_{(2)}n_\beta w^\beta\\
&+\alpha_2 w^\alpha n_\beta w^\beta-\Big(2\beta-\xi_2+2\xi_w-\frac{3}{2}-\frac{\alpha_2}{2}+\frac{\xi_1}{2}\Big)n_\beta\tilde U^{\alpha\beta}\\
&-(1-2\gamma-4\beta+\xi_2)n_\beta x^\beta\partial^\alpha\tilde U+\frac{3}{2}\xi_w n_\beta\int\frac{\rho_1'(x^\alpha-x'^\alpha)(x^\beta-x'^\beta)}{|x-x'|^5}\\
&\qquad\times[x'_\delta x'^\delta-x_\delta x^\delta+3(n_\delta x'^\delta)^2-3(n_\delta x^\delta)^2]\,dx'+E^{\alpha\beta}x_\beta\\
&+\xi_w n_\beta\int\frac{\rho_1'(2x^\alpha x'^\beta-x'^\alpha x^\beta-x^\alpha x^\beta)}{|x-x'|^3}\,dx'\\
&+3\xi_w n_\beta\int\frac{\rho_1'(x^\alpha-x'^\alpha)(x_\delta-x'_\delta)(x^\delta x'^\beta-x'^\delta x^\beta)}{|x-x'|^5}\,dx'\Big\}\\
&+\frac{M_2}{R}\{(4\beta+2\gamma-1-3\xi_w-\xi_2)\partial^\alpha\tilde U-\xi_w n_\beta n_\delta\partial^\beta\tilde U^{\alpha\delta}-\xi_w n^\alpha n_\beta\partial^\beta\tilde U\}\\
&\qquad-\partial^\alpha\tilde U+\partial^\alpha O(\varepsilon^6),
\end{aligned}
\tag{321}
$$

where the acceleration $\tilde a_0^\alpha$ is entirely due to the gravitational field produced by the first body:

$$
\begin{aligned}
\tilde a_0^\alpha&=\partial^\alpha\tilde U[\gamma V^\beta_{(0)}V_{(0)\beta}+2(\beta+\gamma)\tilde U]+3\gamma\partial^\alpha\tilde\Phi_2-\partial^\alpha\tilde\Phi\\
&+\frac{1}{2}(1+\alpha_2-\xi_1)\partial^\alpha\tilde A+\xi_w\partial^\alpha\tilde\Phi_w+\frac{1}{2}(\xi_1-\alpha_2)\partial^\alpha\tilde\Phi_1\\
&+\frac{1}{2}(4\gamma+2+\alpha_1-2\alpha_2+2\xi_1)\partial^\alpha\tilde\Phi_4-2(\gamma+1)V^\alpha_{(0)}V^\beta_{(0)}\partial_\beta\tilde U\\
&-\frac{1}{2}(4\gamma+3+\alpha_1-\alpha_2+\xi_1)\int\frac{\rho_1'\partial'^\alpha\tilde U\,dx'}{|x-x'|}\\
&-\frac{\partial\tilde U}{\partial t}\Big[(2\gamma+1)V^\alpha_{(0)}-\frac{1}{2}(\alpha_1-2\alpha_2)w^\alpha\Big]+\frac{\alpha_1}{2}V^\beta_{(0)}[w^\alpha\partial_\beta\tilde U-w_\beta\partial^\alpha\tilde U]\\
&-\alpha_2 w_\beta\frac{\partial\tilde U^{\alpha\beta}}{\partial t}-\frac{1}{2}(4\gamma+4+\alpha_1)V^\beta_{(0)}[\partial_\beta\tilde V^\alpha-\partial^\alpha\tilde V_\beta]\\
&-\frac{1}{2}(4\gamma+4+\alpha_1)\int\frac{\rho_1'v'^\alpha v'^\beta(x_\beta-x'_\beta)}{|x-x'|^3}\,dx'\\
&+\frac{1}{2}(1+\alpha_2-\xi_1)\int\frac{\rho_1'\rho_1''(x^\alpha-x'^\alpha)(x'_\beta-x''_\beta)(x^\beta-x'^\beta)}{|x-x'|^3|x'-x''|^3}\,dx'\,dx'',
\end{aligned}
\tag{322}
$$

and for what follows the form of the tensor $E^{\alpha\beta}\sim O(\varepsilon^2)/L$ is unimportant. Suppose the center of mass of the extended body and the point body are at a certain initial time at the same point of space and have equal velocities. We shall call the geodesic of the Riemannina space-time along which this point body moves the reference geodesic. We now compare the accelerations (314) of the center of mass of the extended body and (321) of the point body at the initial time. For this, we must set $V^\alpha_{(0)}=V^\alpha_{(1)}$, $x^\alpha=0$ in the expression (321). Then it follows from the definition of the center of mass of the first body that $\int\rho_1'x'^\alpha\,dx'=0$, and therefore $\partial^\alpha\tilde U=0$ as well for $x^\alpha=0$.

Then from the expressions (314) and (321) for the difference between the accelerations of the point body and the center of mass of the extended body we obtain

$$
\begin{aligned}
\delta a^\alpha=a_0^\alpha-a^\alpha_{(1)}&=\frac{M_2}{R^2}\Big\{n^\alpha\Big[\gamma(V^\beta_{(1)}V_{(1)\beta}-2Q_{(1)})+(\gamma-2)P_{(1)}\\
&+(\alpha_1-\alpha_2+\xi_1+5\xi_w+3-4\beta)\Omega_{(1)}-3\xi_w n_\beta n_\delta\Omega^{\beta\delta}_{(1)}\\
&+\frac{1}{2}(4\beta+\alpha_2-\xi_1-3-\alpha_1-4\xi_w)\tilde U\\
+\frac{3}{2}\xi_w\int\frac{\rho_1'\,dx'}{|x'|^3}&((n_\beta x'^\beta)^2+x'_\beta x'^\beta)\Big]+2(\gamma+1)[2Q^{\alpha\beta}_{(1)}-V^\alpha_{(1)}V^\beta_{(1)}]n_\beta\\
&+(2\gamma+2+\alpha_2-\xi_1+\xi_2-5\xi_w)n_\beta\Omega^{\alpha\beta}_{(1)}\\
&-\Big(2\beta-\xi_2+\frac{7}{2}\xi_w-\frac{3}{2}-\frac{\alpha_2}{2}+\frac{1}{2}\xi_1\Big)n_\beta\tilde U^{\alpha\beta}\\
&+\frac{9}{2}\xi_w n_\beta\int\frac{\rho_1'x'^\alpha x'^\beta(n_\delta x'^\delta)^2\,dx'}{|x'|^5}\Big\}-\xi_w\frac{M_2}{R}n_\beta n_\delta\partial^\delta\tilde U^{\alpha\beta}+3\gamma\partial^\alpha\tilde\Phi_2\\
&-\partial^\alpha\tilde\Phi+\frac{1}{2}(1+\alpha_2-\xi_1)\partial^\alpha\tilde A+\xi_w\partial^\alpha\tilde\Phi_w+\frac{1}{2}(\xi_1-\alpha_2)\partial^\alpha\tilde\Phi_1\\
&+\frac{1}{2}(4\gamma+2+\alpha_1-2\alpha_2+2\xi_1)\partial^\alpha\tilde\Phi_4\\
&-\frac{\partial\tilde U}{\partial t}\Big[(2\gamma+1)V^\alpha_{(1)}-\frac{1}{2}(\alpha_1-2\alpha_2)w^\alpha\Big]\\
&-\frac{1}{4}(4\gamma+3+\alpha_1-\alpha_2+\xi_1)\int\frac{\rho_1'\partial'^\alpha\tilde U}{|x'|}\,dx'-\alpha_2 w_\beta\frac{\partial\tilde U^{\alpha\beta}}{\partial t}\\
&-\frac{1}{2}(4\gamma+4+\alpha_1)V^\beta_{(1)}[\partial_\beta\tilde V^\alpha-\partial^\alpha\tilde V_\beta]\\
&+\frac{1}{2}(4\gamma+4+\alpha_1)\int\frac{\rho_1'v'^\alpha v'^\beta x'_\beta\,dx'}{|x'|^3}+\frac{1}{2}(1+\alpha_2-\xi_1)\\
&\times\int\frac{\rho_1'\rho_1''x'^\alpha x'^\beta(x'_\beta-x''_\beta)}{|x'|^3|x'-x''|^3}\,dx'\,dx''+\frac{O(\varepsilon^6)}{R}.
\end{aligned}
\tag{323}
$$

It is remarkable that the subtraction of the accelerations $a^\alpha_{(0)}$ and $a^\alpha_{(1)}$ has resulted in the disappearance of all the terms ($P_{(2)}$, $\Omega^{\alpha\beta}_{(2)}$, $\Pi_{(2)}$, $Q^{\alpha\beta}_{(2)}$ that characterize the internal structure of the second body.

We are interested in the question of how the center of mass of an arbitrary extended body moves: along a geodesic of the Riemannian space-time or not? Since different extended bodies differ in their composition, matter distribution, pressure distribution, velocities of the internal motion, shape, etc., the quantities $\Omega^{\alpha\beta}_{(1)}$, $Q^{\alpha\beta}_{(1)}$, $P_{(1)}$, $\tilde U$, $\tilde U^{\alpha\beta}$, $\tilde\Phi$ for different extended bodies are different and on the transition from one body to another change relative to each other. If we solved the question of the center-of-mass motion in principle for all extended bodies, we must assume that all these quantities are independent at each time. This is the rub in the general formulation of the problem of the center-of-mass motion, since by an appropriate choice of the extended body (by changing the shape and matter distribution in such a way that some multipole moments of the masses appear and others disappear, by making the body rotate, by exciting in it pressure and velocity waves, etc.) we can vary the values of (307) in wide ranges.

When this circumstance is taken into account, it is readily seen that the difference δa^α (323) between the accelerations of the point body and the center of mass of an arbitrary extended body does not vanish in any metric theory of gravitation. Therefore, in the post-Newtonian approximation the center of mass of an arbitrary extended body does not move along a geodesic of the Riemannian space-time in any metric theory of gravitation possessing energy-momentum conservation laws for the matter and the gravitational field taken together.

In this connection the following question arises: What is the nature of the motion of the center of mass of an extended body relative to the reference geodesic of the Riemannian space-time on the average over a fairly long time interval? However, to answer this question we need a tensor virial theorem.

Tensor virial theorem for an extended body moving in an external gravitational field. Since the formulation of this theorem has often been given in the literature in a not entirely correct form, we shall derive it here in some detail.

We decompose the motion of each element of volume of the first body into a sum of two motions: the motion due to the gravitational field of the second body, and the motion due to all the other elements of the first body (the influence of the gravitational self-field, the influence of pressure, etc.). We shall also assume that the variations of all quantities in time due to the internal forces take place fairly rapidly, so that the characteristic time τ during which these variations take place is short compared with the period T of revolution of the body in orbit, i.e., $\tau \ll T$.

For the velocity of an element of volume we have

$$v_1^\alpha = \tilde{v}^\alpha + \tilde{\tilde{v}}^\alpha. \tag{324}$$

where $\tilde{v}^\alpha$ and $\tilde{\tilde{v}}^\alpha$ are the velocities due to the internal and external forces, respectively. By definition, these velocities satisfy the Newtonian equations of motion

$$\left.\begin{aligned} &\rho_1 d\tilde{v}^\alpha/dt = -\rho_1 \partial^\alpha \tilde{U} + \partial^\alpha p_1 + \rho_1 O(\varepsilon^4); \\ &\rho_1 d\tilde{\tilde{v}}^\alpha/dt = -\rho_1 \partial^\alpha U_2 + \rho_1 O(\varepsilon^4). \end{aligned}\right\} \tag{325}$$

where

$$U_2 = \int \frac{\rho_2\, dy}{|x-y|}.$$

It follows from Eqs. (313) and (325) that in the first approximation the velocity $\tilde{\tilde{v}}^\alpha$ is equal to the center-of-mass velocity of the body:

$$V_{(1)}^\alpha = dY_{(1)}^\alpha/dt = \tilde{\tilde{v}}^\alpha + O(\varepsilon^3).$$

Therefore, the expression (324) takes the form $\tilde{v}^\alpha = V_{(1)}^\alpha + \tilde{v}^\alpha + O(\varepsilon^3)$. Since $v_1^\alpha = dX_{(1)}^\alpha/dt$, where $X_{(1)}^\alpha = Y_{(1)}^\alpha + x_1^\alpha$, we have $\tilde{v}^\alpha = dx_1^\alpha/dt$.

Using the first of the equations in (325), we find

$$\rho_1 \frac{d^2}{dt^2} x_1^\alpha x_1^\beta = 2\rho_1 \tilde{v}^\alpha \tilde{v}^\beta - \rho_1 [x_1^\beta \partial^\alpha \tilde{U} + x_1^\alpha \partial^\beta \tilde{U}] + x_1^\alpha \partial^\beta p_1 + x_1^\beta \partial^\alpha p_1 + \rho_1 O(\varepsilon^4).$$

Integrating this expression over the volume of the first body, we obtain

$$\frac{1}{2M_1}\frac{d^2}{dt^2} I_{(1)}^{\alpha\beta} = 2q_{(1)}^{\alpha\beta} + \Omega_{(1)}^{\alpha\beta} - \gamma^{\alpha\beta} P_{(1)}, \tag{326}$$

where

$$I_{(1)}^{\alpha\beta} = \int \rho_1 x_1^\alpha x_1^\beta\, dx; \quad q_{(1)}^{\alpha\beta} = \frac{1}{2M_1}\int \rho_1 \tilde{v}_1^\alpha \tilde{v}_1^\beta\, dx;$$
$$\Omega_{(1)}^{\alpha\beta} = -\frac{1}{2M_1}\int \frac{\rho_1 \rho_1' (x^\alpha - x'^\alpha)(x^\beta - x'^\beta)}{|x-x'|^3}\, dx\, dx';$$
$$Q_{(1)}^{\alpha\beta} = \frac{1}{2M_1}\int \rho_1 v_1^\alpha v_1^\beta\, dx; \quad P_{(1)} = \frac{1}{M_1}\int p_1\, dx.$$

By virtue of the definition of the center-of-mass velocity of an extended body,

$$\int \rho_1 \tilde{v}_1^\alpha\, dx = 0.$$

It follows that

$$2q_{(1)}^{\alpha\beta} + V_{(1)}^\alpha V_{(1)}^\beta = 2Q_{(1)}^{\alpha\beta}.$$

Therefore, the expression (326) can be written in the form

$$\frac{1}{2M_1}\frac{d^2}{dt^2} I_{(1)}^{\alpha\beta} = 2Q_{(1)}^{\alpha\beta} + \Omega_{(1)}^{\alpha\beta} - \gamma^{\alpha\beta} P_{(1)} - V_{(1)}^\alpha V_{(1)}^\beta. \tag{327}$$

We now average the expression (327) over an interval of time τ_0 appreciably longer than the characteristic time τ but appreciably less than the orbital period T: $T \gg \tau_0 \gg \tau$. It follows from what we have said above that the mean value of $d^2/dt^2\, I_{(1)}^{\alpha\beta}$, as the time derivative of the bounded quantity $d/dt\, I_{(1)}^{\alpha\beta}$, is equal to zero. Using the relation

$$\overline{V}_{(1)}^\alpha = V_{(1)}^\alpha + O(\varepsilon^3).$$

we obtain the tensor virial theorem for an extended body moving in an external gravitational field:

$$\overline{Q}_{(1)}^{\alpha\beta} = \frac{1}{2}\gamma^{\alpha\beta}\overline{P}_{(1)} - \frac{1}{2}\overline{\Omega}_{(1)}^{\alpha\beta} + \frac{1}{2}V_{(1)}^\alpha V_{(1)}^\beta. \tag{328}$$

Contraction of the tensor indices in the expression (328) gives

$$\overline{Q}_{(1)} = (3/2)\,\overline{P}_{(1)} - \overline{\Omega}_{(1)} 2 + V_{(1)}^{(\beta)} V_{(1)\beta}/2. \tag{329}$$

Similar relations hold for the second body.

Averaging of the center-of-mass motion of an extended body with respect to the reference geodesic of the riemannian space-time. As follows from the expression (323), the difference between the accelerations of a point body and the center of mass of an extended body at the initial instant, when they coincide and have equal velocities, is very small—it has post-Newtonian order of magnitude. Since this difference does not depend on the displacement δx^α of the point body from the center of mass, one could expect that with the passage of time the point body would move quite far outside the extended body. However, as the point body moves away from the center of mass of the extended body, restoring forces, due to the gravitational field of the extended body, come into play. These are also small forces. Therefore, the relative motion of the point body and the center of mass of the extended body takes place fairly slowly, so that the characteristic time of motion is appreciably greater than τ_0. This makes it possible to simplify the investigation of the evolution of the relative motion, since one can average the expressions for the acceleration a_0^α of the point body and the acceleration $a_{(1)}^\alpha$ of the center of mass of the extended body over the time interval τ_0, thereby eliminating the small short-period oscillations of the orbit of the extended body and the reference geodesic due to the short-period internal motions of the matter of the extended bodies that produce the Riemannian space-time metric.

Applying the tensor virial theorem (328) and (329) to both bodies, for the averaged acceleration of the center of mass of the extended body we obtain

$$\begin{aligned}\overline{a_{(1)}^\alpha} = -\frac{M_2}{R^2}\Big\{ n^\alpha \Big[&1 - \gamma V_{(1)}^\beta V_{(1)\beta} - \frac{1}{2}(2\gamma + \alpha_2 + \alpha_3 + 2) V_{(2)}^\beta V_{(2)\beta} \\ &- \left(2\beta + 2\gamma + 1 + \xi_W - \xi_2 + \frac{\alpha_1}{2}\right)\frac{M_1}{R} - 2(\beta+\gamma)\frac{M_2}{R} \\ &+ (\alpha_1 - \alpha_2 + \xi_1 + 5\xi_W + 3 + \gamma - 4\beta)\overline{\Omega}_{(1)} - 3\xi_W n_\delta n_\beta \overline{\Omega}_{(1)}^{\beta\delta} \\ &+ \left(\frac{5}{2} + \frac{\xi_1}{2} + \frac{\alpha_3}{2} + 2\xi_2 + 2\xi_W + \gamma - 4\beta\right)\overline{\Omega}_{(2)} \\ &+ (1+\xi_3)\overline{\Pi}_{(2)} + \left(3\xi_4 - \xi_1 - \frac{3}{2} - \frac{3}{2}\alpha_3\right)\overline{P}_{(2)} \\ &+ \frac{3}{2}\xi_1 n_\beta n_\delta \overline{\Omega}_{(2)}^{\beta\delta} - \frac{3}{2}\alpha_2 (n_\beta w^\beta)^2 - \frac{3}{2}(1+\alpha_2)(n_\beta V_{(2)}^\beta)^2 \\ &+ \frac{1}{2}(4\gamma + 4 + \alpha_1) V_{(1)}^\beta V_{(2)\beta} + \frac{1}{2}(\alpha_1 - \alpha_2 - \alpha_3) w_\beta w^\beta \\ &- \frac{1}{2}(2\alpha_3 + 2\alpha_2 - \alpha_1) w_\beta V_{(2)}^\beta + \frac{1}{2}\alpha_1 w_\beta V_{(1)}^\beta - 3\alpha_2 n_\beta n_\delta w^\beta V_{(2)}^\delta \Big] \\ &+ 2(\gamma+1) V_{(1)}^\alpha n_\beta V_{(1)}^\beta + \frac{1}{2}(4\gamma + 2 + \alpha_1 - 2\alpha_2) V_{(2)}^\alpha n_\beta V_{(2)}^\beta\end{aligned}$$

$$+\xi_1 n_\beta \overline{\Omega}^{\alpha\beta}_{(2)}-\frac{1}{4}(8\gamma+6+\alpha_1)[V^\alpha_{(1)}V^\beta_{(2)}+V^\alpha_{(2)}V^\beta_{(1)}]\,n_\beta$$
$$+(\alpha_2-\xi_1+\xi_2-5\xi_W)\,n_\beta\overline{\Omega}^{\alpha\beta}_{(1)}-\alpha_2 V^\alpha_{(2)}n_\beta w^\beta-\alpha_2 w^\alpha n_\beta w^\beta$$
$$+\frac{1}{2}(\alpha_1-2\alpha_2)\,w^\alpha n_\beta V^\beta_{(2)}-\frac{\alpha_1}{2}\,w^\alpha n_\beta V^\beta_{(1)}\Big\}. \qquad (330)$$

Proceeding similarly, for the averaged difference between the acceleration of the point body and the acceleration of the center of mass of the extended body we obtain

$$\overline{\delta a^\alpha}=\overline{a}^\alpha_0-\overline{a}^\alpha_{(1)}=-\overline{\partial^\alpha\widetilde{U}}+\frac{M_2}{R^2}\Big\{n^\alpha\Big[-3\xi_W n_\beta n_\delta\overline{\Omega}^{\beta\delta}_{(1)}$$
$$+\Big(2\beta+\frac{\alpha_2}{2}-\frac{\xi_1}{2}-\frac{3}{2}-\frac{\alpha_1}{2}-2\xi_W\Big)\,\overline{\widetilde{U}}$$
$$+(\alpha_1+\xi_1-\alpha_2+3+\gamma-4\beta+5\xi_W)\,\overline{\Omega}_{(1)}$$
$$+\frac{\xi_W}{2}\int\frac{\overline{\rho}'_1\,dx'}{|x-x'|^3}\,[3x'_\beta x'^\beta-2x^\beta x'_\beta-x_\beta x^\beta+3(n_\beta x'^\beta)^2$$
$$-9\,(n_\beta x^\beta)^2+6n_\beta n_\delta x'^\beta x^\delta]\Big]+(\alpha_2-\xi_1+\xi_2-5\xi_W)\,n_\beta\overline{\Omega}^{\alpha\beta}_{(1)}$$
$$-\Big(2\beta+2\xi_W-\xi_2-\frac{3}{2}-\frac{\alpha_2}{2}+\frac{1}{2}\,\xi_1\Big)\,n_\beta\overline{\widetilde{U}}^{\alpha\beta}$$
$$-(1-4\beta-2\gamma+\xi_2)\,n_\beta x^\beta\overline{\partial^\alpha\widetilde{U}}+\frac{3}{2}\,\xi_W n_\beta\int\frac{\overline{\rho}'_1\,(x^\alpha-x'^\alpha)}{|x-x'|^5}$$
$$\times(x^\beta-x'^\beta)\,[x'_\delta x'^\delta-x_\delta x^\delta+3\,(n_\delta x'^\delta)^2-3\,(n_\delta x^\delta)^2]\,dx'+3\xi_W n_\beta$$
$$\times\int\frac{\overline{\rho}'_1\,(x^\alpha-x'^\alpha)\,(x_\delta-x'_\delta)\,(x^\delta x'^\beta-x'^\delta x^\beta)}{|x-x'|^5}\,dx'$$
$$+\xi_W n_\beta\int\frac{\overline{\rho}'_1\,(2x^\alpha x'^\beta-x'^\alpha x^\beta-x^\alpha x^\beta)}{|x-x'|^3}\,dx'\Big\}$$
$$+\frac{M_2}{R}\{-\xi_W n_\beta n_\eta\partial^\beta\overline{\widetilde{U}}^{\alpha\eta}-\xi_W n^\alpha n_\beta\partial^\beta\overline{\widetilde{U}}$$
$$+(4\beta+2\gamma-1-3\xi_W-\xi_2)\,\partial^\alpha\overline{\widetilde{U}}\}+\overline{\widetilde{a}}^\alpha_0+\partial^\alpha O\,(\varepsilon^6). \qquad (331)$$

Here we have used the fact that in the considered case $V^\alpha_{(1)}=V^\alpha_{(0)}+O(\varepsilon^3)$.

This result shows that the acceleration of the center of mass of the extended body differs in the post-Newtonian approximation from the acceleration of a test body for any metric theory of gravitation possessing energy–momentum conservation laws for the matter and the gravitational field taken toegther. This disagrees with the result of Nordtvedt,[130] who concluded that in the post-Newtonian approximation these accelerations could be equal for certain metric theories of gravitation. The difference arises because Nordtvedt did not take into account the influence of the gravitational self-field of the extended body on the motion of the test body. But this influence must be taken into account, since it has post-Newtonian order of magnitude.

If in the expression (331) for the difference between the accelerations of the test body and the extended body we ignore the influence of the gravitational self-field of the extended body on the motion of the test body and also set $\xi_w=0$, we arrive at the result

$$\overline{\delta a^\alpha}=(M_2/R^2)\,\{n^\alpha\,(\alpha_1+\xi_1-\alpha_2+3+\gamma-4\beta)\,\overline{\Omega}_1$$
$$+(\alpha_2-\xi_1+\xi_2)\,n_\beta\overline{\Omega}^{\alpha\beta}_{(1)}\},$$

which is essentially analogous to the results obtained by Nordtvedt[130] and Will.[131] It follows directly from this formula that in metric theories of gravitation with parameters satisfying the equations

$$\alpha_1+\xi_1-\alpha_2+3+\gamma-4\beta=0;\quad \alpha_2-\xi_1+\xi_2=0,$$

the difference between the accelerations $\overline{\delta a^\alpha}$ is equal to zero and the center-of-mass motion takes place along a geodesic of the Riemannian space-time. But such an approximation is invalid, since in it the center-of-mass motion takes place in the Riemannian space-time whose metric is produced by the two moving bodies but is compared with a geodesic of the different Riemannian space-time whose metric is produced by the second body alone.

As follows from the expression (331), the difference $\overline{\delta a^\alpha}$ between the accelerations of the center of mass of the extended body and the test body depends on the matter distribution of the extended body and on the nature of its internal motion (rotation of the body as a whole, etc.). Therefore, to estimate the difference $\overline{\delta a^\alpha}$ between the accelerations, one needs detailed knowledge of the structure of the extended body. For qualitative analysis of the center-of-mass motion of the extended body with respect to the geodesic of the Riemannian space-time, we note that the right-hand side of (331) is an expansion in the small parameter ε^2 to order ε^4. Since the first term in this expansion has order of magnitude $O(\varepsilon^2)$, it is to be expected that the deviation δx^α of the point body from the center of mass of the extended body will be small. This justifies our solution of the equations (331) for δx^α in successive stages corresponding to expansion of (331) in powers of ε^2.

As initial time $t=0$ for our study, we must choose a time when the point body is situated at the center of mass of the extended body and has the same velocity as it:

$$\delta x^\alpha(0)=0;\quad \delta\dot{x}^\alpha(0)=0. \qquad (332)$$

Then the solution of our problem tells us how much the center of mass of the extended body deviates in its motion from the reference geodesic of the Riemannian space-time.

We represent the deviation δx^α as an expansion in the parameter ε^2:

$$\left.\begin{aligned}&\delta x^\alpha=\overset{(0)}{\delta x^\alpha}+\overset{(2)}{\delta x^\alpha}+\ldots;\\&\overset{(0)}{\delta x^\alpha}\sim O(1)\,L;\ \ \overset{(2)}{\delta x^\alpha}\sim O(\varepsilon^2)\,L,\end{aligned}\right\} \qquad (333)$$

where L is the characteristic dimension of the extended body.

It follows from the expression (332) that for each of the terms of the expansion (333) we have null initial conditions:

$$\left.\begin{aligned}&\overset{(0)}{\delta x^\alpha}(0)=\overset{(2)}{\delta x^\alpha}(0)=\ldots=0;\\&\overset{(0)}{\delta\dot{x}^\alpha}(0)=\overset{(2)}{\delta\dot{x}^\alpha}(0)=\ldots=0.\end{aligned}\right\} \qquad (334)$$

For the following, it is necessary to expand each term on the right-hand side of Eq. (331) in a series in the small parameter δx^α.

We construct, for example, the expansion of the first term. In a spherical coordinate system with origin at the center of mass of the extended body, the conserved density ρ of the ideal fluid is a continuous function of the variables t, r, θ, φ:

$$\rho=\rho\,(t,\,r,\,\theta,\,\varphi)\geqslant 0.$$

Like every function, we expand the density ρ in an infinite series in spherical harmonics on each of the spherical surfaces with center at the center of mass of the extended body, the coefficients of the series depending on r and t:

$$\rho(r, t) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} r^n \rho_{n,m}(r, t) Y^*_{n,m}(\theta, \varphi), \tag{335}$$

where $Y_{n,m}(\theta, \varphi)$ is a spherical harmonic: for $m \geq 0$

$$Y_{n,m}(\theta, \varphi) = (-1)^m i^n \sqrt{\frac{(2n+1)}{4\pi} \frac{(n-m)!}{(n+m)!}} P_n^m(\theta) \exp(im\varphi);$$

for $m < 0$

$$Y_{n,-|m|}(\theta, \varphi) = (-1)^{n-m} Y^*_{n,|m|}(\theta, \varphi),$$

and the coefficients of the series (335) are determined by the relations

$$\left.\begin{array}{l} \rho_{n,m}(r, t) = \frac{1}{r^n} \int d\Omega \rho(\mathbf{r}, t) Y_{n,m}(\theta, \varphi) \\ (d\Omega = \sin\theta \, d\theta \, d\varphi). \end{array}\right\} \tag{336}$$

It is readily seen that in the expansion (335) the dipole term is absent. Indeed, it follows from the definition of the center of mass of the extended body, $\int \rho_1 x^\alpha dx = 0$, and the expressions (335) that in a coordinate system with origin at the center of mass of the body

$$\rho_{1,m}(r, t) = 0.$$

For the potential $\tilde{U}$, we have the expression

$$\tilde{U} = \int \frac{\rho_1(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'.$$

Since the point of observation $\mathbf{r}$ is within the region of integration, this integral is improper, with singular point $\mathbf{r}' = \mathbf{r}$. However, this integral also converges absolutely. By the definition of absolute convergence of an improper integral, we have

$$\int_{(V)} f d^3\mathbf{r}' = \lim_{V_\delta \to 0} \int_{(V - V_\delta)} f d^3\mathbf{r}',$$

and this limit does not depend on the method by which the region V_δ is contracted to the singular point $\mathbf{r}' = \mathbf{r}$. For our purposes, it is convenient to take as the region V_δ the region between two spheres of radii $r - \delta$ and $r + \delta$ and then go to the limit $\delta \to 0$. Therefore, we rewrite the expression for $U(\mathbf{r}, t)$ in the form

$$\tilde{U}(\mathbf{r}, t) = \lim_{\delta \to 0} \left\{ \int_0^{r-\delta} (r')^2 dr' \int \frac{d\Omega \rho_1(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} + \int_{r+\delta}^{L} (r')^2 dr' \int \frac{d\Omega \rho_1(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} \right\}, \tag{337}$$

where L is the distance from the center of mass of the first body to its most distant point.

In each of the integrals in the expression (337) we replace $\rho_1(\mathbf{r}, t)$ by its series (335), and we also use the well-known expansion $(r_2 > r_1)$

$$\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{r_1^n}{r_2^{n+1}} \frac{Y^*_{n,m}(\theta_2, \varphi_2) Y_{n,m}(\theta_1, \varphi_1)}{(2n+1)}. \tag{338}$$

Thus, in the integrands in the expression (337) we shall have products of two infinite series in spherical harmonics. But by virtue of the orthogonality of sperical harmonics on a sphere,

$$\int Y^*_{n,m}(\theta, \varphi) Y_{l,p}(\theta, \varphi) d\Omega = \delta_{nl} \delta_{mp},$$

we obtain after integration over $d\Omega$ an expression for $\tilde{U}$ in the form of an ordinary series:

$$\tilde{U}(\mathbf{r}, t) = 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{n,m}(r, t) Y^*_{n,m}(\theta, \varphi), \tag{339}$$

where

$$A_{n,m}(r, t) = \frac{1}{2n+1} \left\{ \frac{1}{r^{n+1}} \int_0^r (r')^{2n+2} dr' \rho_{n,m}(r', t) + r^n \int_r^L (r')^{1-n} dr' \rho_{n,m}(r', t) \right\}. \tag{340}$$

We now transform the expression (339) to a form convenient for further investigation. We use the circumstance that at small r the coefficients (340) of the series (339) decrease fairly rapidly with increasing r. Therefore, for our purposes we can restrict ourselves to just a few terms of the series (339):

$$\tilde{U} = 4\pi \{A_{0,0}(r, t) + \sum_{m=-2}^{2} A_{2,m}(r, t) Y_{2,m}(\theta, \varphi) + \ldots\}.$$

Using the relations (335), we can write this expression in the form

$$\tilde{U} = \frac{M(r, t)}{r} + \frac{1}{2} D_{\alpha\beta} \frac{x^\alpha x^\beta}{r^5} + \ldots, \tag{341}$$

where we have introduced the notation

$$\left.\begin{array}{l} M(r, t) = \int_0^r (r')^2 dr' \int \rho_1(\mathbf{r}', t) d\Omega \\ \quad + r \int_r^L r' dr' \int \rho_1(\mathbf{r}', t) d\Omega; \\ D_{\alpha\beta}(r, t) = \int_0^r (r')^4 dr' \int d\Omega \rho_1(\mathbf{r}', t) [3n'_\alpha n'_\beta + \gamma_{\alpha\beta}] \\ \quad + r^5 \int_r^L \frac{dr'}{r'} \int d\Omega \rho_1(\mathbf{r}', t) [3n'_\alpha n'_\beta + \gamma_{\alpha\beta}]; \; n'_\alpha = x'_\alpha / r' \end{array}\right\} \tag{342}$$

Note that by virtue of the continuity of the function ρ as $r \to 0$

$$\int d\Omega \, Y^*_{n,m}(\theta, \varphi) \rho_1(r, t) \sim r^n,$$

and therefore none of the integrals in the expressions (342) have singularities at $r = 0$. Then from the expression (341) we obtain the expansion needed for what follows:

$$\partial^\alpha \tilde{U} = \Phi^\alpha_\beta \delta x^\beta + \tilde{U} O((\delta x^\alpha)^2), \tag{343}$$

where

$$\Phi^\alpha_\beta = \frac{4\pi}{3} \rho(0, t) \delta^\alpha_\beta + \int_0^L \frac{dr'}{r'} \int d\Omega \rho_1(\mathbf{r}', t) [3n'_\alpha n'_\beta + \gamma_{\alpha\beta}]. \tag{344}$$

We shall assume further that the matter in the extended body is distributed fairly uniformly, $\rho(0, t) \neq 0$, and the second term in the expression (344) is definitely smaller than the first. Expanding the remaining terms of Eq. (331) similarly, and using the expressions (333), we obtain in the first approximation

$$\frac{d^2}{dt^2} \overset{(0)}{\delta x^\alpha} + \Phi^\alpha_\beta \overset{(0)}{\delta x^\beta} = 0.$$

It is obvious that the solution of this equation is $\overset{(0)}{\delta x^\alpha} \equiv 0$ by virtue of the initial conditions (334).

In the second approximation, we have

$$\frac{d^2}{dt^2} \overset{(2)}{\delta x^\alpha} + \overline{\Phi}^\alpha_\beta \overset{(2)}{\delta x^\beta} = b^\alpha = \text{const}, \tag{345}$$

where

$$b^\alpha = \frac{M_2}{R^2}\Big\{n^\alpha\,[(\alpha_1-\alpha_2+\xi_1+3+\gamma-4\beta+5\xi_W)\,\bar{\Omega}_{(1)}$$

$$-3\xi_W n_\beta n_\delta \bar{\Omega}^{\beta\delta}_{(1)} + (2\beta+\alpha_2/2-\xi_1/2-3/2-\alpha_1/2-(7/2)\,\xi_W)\overline{\tilde{U}(0)}$$

$$+(3/2)\,\xi_W n_\beta n_\delta \bar{\tilde{U}}^{\beta\delta}(0)] + (\alpha_2-\xi_1+\xi_2-5\xi_W)\,n_\beta \bar{\Omega}^{\alpha\beta}_{(1)}$$

$$-(2\beta-\xi_2+(7/2)\,\xi_W-3/2-\alpha_2/2+\xi_1/2)\,n_\beta \bar{\tilde{U}}^{\alpha\beta}(0)$$

$$+\frac{9}{2}\,\xi_W\int\frac{\bar{\rho}_1' x'^\alpha (n_\beta x'^\beta)^3}{|x'|^5}\,dx'\Big\} - \xi_W\,\frac{M_2}{R}\int\frac{\bar{\rho}_1' x'^\alpha (n_\beta x'^\beta)^2}{|x'|^5}\,dx'$$

$$+\frac{1}{2}(1+\alpha_2-\xi_1)\int\frac{\overline{\rho_1'\rho_1''}x'^\alpha x'^\beta (x_\beta'-x_\beta'')}{|x'|^3\,|x'-x''|^3}\,dx'\,dx''$$

$$-(2\beta-1-\xi_2+\xi_W)\int\frac{\overline{\rho_1'\rho_1''}x'^\alpha\,dx'\,dx''}{|x'|^3\,|x'-x''|}$$

$$-\Big(1-\gamma-\alpha_2+\frac{\alpha_1}{2}+\xi_1-3\xi_4\Big)\int\frac{\overline{p_1'}x'^\alpha\,dx'}{|x'|^3} + (1+\xi_3)\int\frac{\overline{\rho_1'\Pi}\,x'^\alpha}{|x'|^3}\,dx'$$

$$-\Big(1+\gamma+\frac{\alpha_3}{2}+\frac{\alpha_2}{2}\Big)\int\frac{\overline{\rho_1' v_\beta' v'^\beta}}{|x'|^3}\,x'^\alpha\,dx'$$

$$+\frac{1}{2}(4\gamma+2+\alpha_1-2\alpha_2)\int\frac{\overline{\rho_1' v'^\alpha v'^\beta}}{|x'|^3}\,x_\beta'\,dx'$$

$$-\frac{3}{2}\alpha_2 w_\beta w_\eta\int\frac{\overline{\rho_1'}x'^\alpha x'^\beta x'^\eta}{|x'|^5}\,dx' - \frac{1}{2}(2\alpha_3+2\alpha_2-\alpha_1)\,w_\beta$$

$$\times\int\frac{\overline{\rho_1' v'^\beta}x'^\alpha dx'}{|x'|^3} - \Big[(1+2\gamma)\,V^\alpha_{(1)} - \frac{1}{2}(\alpha_1-2\alpha_2)\,w^\alpha\Big]$$

$$\times\int\frac{\overline{\rho_1' v_\beta'}x'^\beta}{|x'|^3}\,dx' - \frac{3}{2}(1+\alpha_2)\int\frac{\overline{\rho_1'(v_\beta' x'^\beta)^2 x'^\alpha}}{|x'|^5}\,dx'$$

$$+\frac{1}{2}(4\gamma+4+\alpha_1)\,V^\beta_{(1)}\int\frac{\overline{\rho_1'(v_\beta' x'^\alpha - v'^\alpha x_\beta')}}{|x'|^3}\,dx'$$

$$-\frac{1}{2}(4\gamma+3+\alpha_1+\xi_1-\alpha_2)\int\frac{\overline{\rho_1'\rho_1''}(x'^\alpha-x''^\alpha)}{|x'|\,|x'-x''|^3}\,dx'\,dx''$$

$$+\alpha_2 w_\beta\int\frac{\overline{\rho_1'(v'^\alpha x_\delta' x'^\delta - 3v_\delta' x'^\delta x'^\alpha)x'^\beta}}{|x'|^5}\,dx'$$

$$+\xi_W\int\frac{\overline{\rho_1'\rho_1''}(x''^\alpha x_\delta' x'^\delta - 3x'^\alpha x_\beta'' x'^\beta)}{|x'|^5\,|x'-x''|}\,dx'\,dx'', \qquad (346)$$

and $v'^\alpha = V^\alpha_{(1)} + \tilde{v}^\alpha$.

Equations (345) describe oscillatory motion of a point with respect to an equilibrium position situated about $\varepsilon^2 L$ from the center of mass of the extended body. Under the assumptions made above concerning the terms in the expression (344) for Φ^α_β, Eq. (345) simplifies to

$$\frac{d^2}{dt^2}\overset{(2)}{\delta x^\alpha} + \omega^2 \overset{(2)}{\delta x^\alpha} = b^\alpha,$$

where

$$\omega^2 = (4\pi/3)\,\rho\,(0,\,t). \qquad (347)$$

The solution of this equation using the initial conditions (334) is

$$\overset{(2)}{\delta x^\alpha} = (b^\alpha/\omega^2)\,[1-\cos\omega t]. \qquad (348)$$

Since the right-hand side of the expression (331) is defined only to accuracy $O(\varepsilon^6)$, further expansion of it with respect to the small parameter δx^α will lead to an increased accuracy in the determination of δa^α.

As we have already pointed out, the difference between the accelerations of the point body and the center of mass of the extended body depends strongly on the internal structure of the body. However, for simplicity we shall study the obtained solution for the following special cases.

1. Suppose the extended body is a homogeneous, spherically symmetric nonrotating sphere of radius L. In this case, the oscillation frequency of the center of mass of the extended body relative to the Riemannian space-time geodesic is

$$\omega = \sqrt{M_1/L^3} \sim \varepsilon.$$

To calculate the difference between the accelerations of the point body and the center of mass of the extended body we note that

$$\int d\Omega\,\frac{x^\alpha x^\beta}{x^2} = -\frac{4\pi}{3}\,\gamma^{\alpha\beta};$$

$$\int d\Omega\,\frac{x^\alpha x^\beta x^\delta x^\eta}{x^4} = \frac{4\pi}{15}\,[\gamma^{\alpha\beta}\gamma^{\delta\eta} + \gamma^{\alpha\delta}\gamma^{\beta\eta} + \gamma^{\alpha\eta}\gamma^{\beta\delta}];$$

$$\int d\Omega\,\frac{x^\alpha}{x} = \int d\Omega\,\frac{x^\alpha x^\beta x^\delta}{x^3} = 0.$$

Then for the potentials in the expression (345), we obtain

$$\tilde{U}(0) = \frac{3}{2}\,\frac{M_1}{L};\quad \tilde{U}^{\alpha\beta}(0) = -\frac{1}{2}\,\frac{M_1}{L}\,\gamma^{\alpha\beta};$$

$$\Omega_{(1)} = \frac{3}{5}\,\frac{M_1}{L};\quad \Omega^{\alpha\beta}_{(1)} = \frac{1}{5}\,\frac{M_1}{L}\,\gamma^{\alpha\beta}.$$

Substituting these relations in the expression (346), we obtain

$$b^\alpha = \frac{M_1 M_2}{R^2 L}\,n^\alpha\,\frac{1}{10}\Big[6\gamma+16\beta-12-3\xi_2-\frac{3}{2}\,\alpha_1-\xi_1+\alpha_2-15\xi_W\Big].$$

It follows from this equation that the difference between the accelerations of the point body and the center of mass of the sphere has post-Newtonian order: $|b^\alpha| \sim \varepsilon^4$. In this case, the amplitude of the oscillations is

$$|A^\alpha| = \frac{|b^\alpha|}{\omega^2} = \frac{1}{10}\,\frac{M_2}{R}\Big(\frac{L}{R}\Big)\,L\Big[6\gamma+16\beta-12$$

$$-3\xi_2-\frac{3}{2}\,\alpha_1-\xi_1+\alpha_2-15\xi_W\Big].$$

2. Suppose the extended body is a homogeneous spherically symmetric sphere of radius L rotating with angular frequency ω_0 about an axis passing through the center of mass of the sphere. In this case, the velocity with which a volume element of the sphere moves is

$$v'^\alpha = V^\alpha_{(1)} + \omega^\alpha_\beta x^\beta,$$

where $\omega^{\alpha\beta} = -\omega^{\beta\alpha}$ is the three-dimensional tensor of the angular velocity.

Substituting this relation in the expression (346), we obtain

$$b^\alpha = \frac{M_2 M_1}{10R^2 L}\,n^\alpha\Big[6\gamma+16\beta-12-3\xi_2-\frac{3}{2}\,\alpha_1-\xi_1+\alpha_2-15\xi_w\Big]$$

$$+\frac{M_1}{4L}\,[(\alpha_1-2\alpha_3)\,\omega^{\alpha\beta}{}_{w\beta} + (2+\alpha_1-2\alpha_3)\,\omega^\alpha_\beta V^\beta_{(1)}].$$

The amplitude of the oscillations is

$$A^\alpha = \frac{M_2}{10R}\Big(\frac{L}{R}\Big)\,Ln^\alpha\Big[6\gamma+16\beta-12-3\xi_2-\frac{3}{2}\,\alpha_1-\xi_1+\alpha_2-15\xi_w\Big]$$

$$+\frac{L}{4}\,[(\alpha_1-2\alpha_3)\,\omega^{\alpha\beta}w_\beta L + (2+\alpha_1-2\alpha_3)\,\omega^\alpha_\beta V^\beta_{(1)}L] \sim \varepsilon^2 L. \qquad (349)$$

3. We now consider an extended body with spherically symmetric matter distribution. Suppose this body rotates with angular frequency ω_0 about an axis passing through its center of mass. Then for the difference between the accelerations of the center of mass of the extended body and the test body we obtain from the expression (346)

$$b^\alpha = \frac{M_2 M_1 n^\alpha}{10R^2 L}\,\{2q_1\,(3\alpha_1+2\xi_1-2\alpha_2+9+3\gamma-12\beta+13\xi_w+\xi_2)$$

$$+q_2\Big(40\beta-30+5\alpha_2-5\xi_1-\frac{15}{2}\,\alpha_1-5\xi_2-41\xi_w\Big)\Big\}$$

$$+\frac{M_1}{4L}\,q_2\,[(\alpha_1-2\alpha_3)\,\omega^{\alpha\beta}w_\beta + (2+\alpha_1-2\alpha_3)\,\omega^\alpha_\beta V^\beta_{(1)}\},$$

where

$$q_1=\frac{80\pi^2 L}{3M_1^2}\int_0^L r'\,dr'\rho_1(r')\int_0^{r'} r^2\,dr\rho_1(r);\quad q_2=\frac{8\pi L}{3M_1}\int_0^L r\,dr\rho_1(r)$$

are coefficients which characterize the inhomogeneity of the matter distribution along the radius of the sphere. For a homogeneous matter distribution in the sphere, these coefficients are equal to unit, but in the general case of an extended body with spherically symmetric matter distribution the coefficients q_1 and q_2 and q_2 are positive numbers.

The amplitude of the oscillations of the center of mass of the extended body with respect to the reference geodisic is

$$A^\alpha=\frac{3M_2M_1n^\alpha}{40\pi R^2L\rho(0)}\Big\{2q_1(3\alpha_1+2\xi_1-2\alpha_2+9+3\gamma-12\beta+13\xi_w+\xi_2)$$
$$+q_2\Big(40\beta-30+5\alpha_2-5\xi_1-\frac{15}{2}\alpha_1-5\xi_2-41\xi_w\Big)\Big\}$$
$$+\frac{3M_1q_2}{16\pi\rho(0)L}\Big[(\alpha_1-2\alpha_3)\,\omega^{\alpha\beta}w_\beta+(2+\alpha_1-2\alpha_3)\,\omega^\alpha_\beta V^\beta_{(1)}\Big]\sim\varepsilon^2L.$$

If we now substitute in the above expressions the values of the post-Newtonian parameters of the field theory of gravitation, $\beta=\gamma=1$, $\alpha_1=\alpha_2=\alpha_3=0$, $\xi_1=\xi_2=\xi_3=\xi_4=\xi_w=0$, it is easy to see that the center-of-mass motion of an extended body with spherically symmetric matter distribution will not occur along a geodesic of the Riemannian space-time.

If the extended body is not spherically symmetric, then the expression for the difference between the acceleration of its center of mass and the acceleration of the test body contains additional terms due to the presence of the multipole mass moments of the extended body, and the motion of such an arbitrary extended body will not occur along a geodesic of the Riemannian space-space-time for any metric theory of gravitation possessing energy-momentum conservation laws for the matter and the gravitational field taken together. The actual difference between the accelerations will depend on the post-Newtonian parameters of the theory, on the multipole mass moments of the extended body, and on the nature of its motion.

Thus, in any metric theory of gravitation possessing energy-momentum conservation laws for the matter and the gravitational field taken together the center of mass of an arbitrary extended body moving in an orbit executes small oscillations about the reference geodesic of the Riemannian space-time. The frequency $\omega\sim\sqrt{M_1/L^3}$ of these oscillations is small, of order ε, and the amplitude $|b^\alpha|/\omega^2$ is of order ε^2L.

Application of the results to the sun-earth system. To compare our results with Will' Will's,[131] we apply our general expressions to a particular post-Newtonian system—the Sun-Earth system. This system is characterized by the following quantities.

1. The mean value of the gravitational potential of the Sun in the neighborhood of the Earth's orbit[72]:

$$U_\odot\approx 10^{-8}.$$

2. The mean value of the gravitational self-potential of the Earth[72]:

$$\tilde U_\oplus\approx 2\cdot 10^{-9}.$$

3. The mean velocity of the Earth in orbit around the Sun[72]:

$$v_\oplus\approx 10^{-4}\,c.$$

4. The ratio of the mass of the Sun to the mass of the Earth[72]:

$$M_\odot/M_\oplus\approx 3\cdot 10^5.$$

5. The ratio of the gravitational self-energy of the Earth to its total energy[102]:

$$\Omega_\oplus\approx 5\cdot 10^{-10}.$$

6. The radius of the Earth[72]:

$$R_\oplus\approx 6\cdot 10^6\ \text{m}.$$

7. The frequency of the Earth's rotation about its axis:

$$\omega_\oplus\approx 7\cdot 10^{-5}\ \text{rad/sec}.$$

8. The mean distance between the Sun and the Earth:

$$R\approx 1.5\cdot 10^{11}\ \text{m}.$$

9. The matter density in the neighborhood of the Earth's center of mass[136]:

$$\overline{\rho_\oplus}(0,\,t)\approx 13\ \text{g/cm}^3.$$

10. The eccentricity of the Earth's orbit:

$$e_\oplus\approx 1.6\cdot 10^{-2}.$$

It should also be noted that for all bodies of the solar system[72] the shear stresses are appreciably less than the isotropic pressure, and therefore all bodies in the solar system can be assumed to consist of an ideal fluid.

It follows from these estimates that the characteristic small parameter ε for the description of the motion of the Sun-Earth system must be taken to be $\varepsilon\approx 10^{-5}$. Then $\tilde U_\oplus$, $U_\odot$, $\Omega_\oplus$ will be of order ε^2. The velocity of the Earth in the frame of reference attached to the center of mass of the Sun-Earth system is of order ε, and the velocity of the Sun is of order ε^2:

$$v_\odot\approx\frac{M_\oplus}{M_\odot}v_\oplus\approx 3\cdot 10^{-10}c.$$

The mean value of the Earth's gravitational potential in the neighborhood of the Sun,

$$M_\oplus/R=\tilde U_\oplus R_\oplus/R\sim 10^{-5}\tilde U_\oplus,$$

is of order ε^3.

For generality, we shall assume that the velocity of the solar system w^α with respect to the hypothetical universal rest frame is of order ε. In addition, following Will, we shall assume that the Sun is a massive point body. Under these conditions, the general expressions obtained above simplify appreciably. In this case, from the expression (330) for the averaged center-of-mass acceleration of the Earth we have

$$a^\alpha_\oplus=-\frac{M_\odot}{R^2}\Big\{n^\alpha\Big[1+\gamma v^2_\oplus+(\alpha_1-\alpha_2+\xi_1+5\xi_w+3+\gamma-4\beta)$$
$$\times\Omega_\oplus-3\xi_w n_\beta n_\delta\Omega^{\beta\delta}_\oplus-2(\beta+\gamma)\frac{M_\odot}{R}-\frac{1}{2}(\alpha_1-\alpha_2-\alpha_3)\,w^2$$
$$-\frac{3}{2}\alpha_2(n_\beta w^\beta)^2+\frac{\alpha_1}{2}w_\beta v^\beta_\oplus\Big]+$$

$$+2(\gamma+1)v_{\oplus}^{\alpha}n_{\beta}v_{\oplus}^{\beta}+(\alpha_2-\xi_1+\xi_2-5\xi_w)n_{\beta}\Omega_{\oplus}^{\alpha\beta}$$
$$-\alpha_2 w^{\alpha}n_{\beta}w^{\beta}-\frac{\alpha_1}{2}w^{\alpha}n_{\beta}v_{\oplus}^{\beta}+O(\varepsilon^3)\Big\}. \quad (350)$$

As we have seen, separation of the characteristics of the bodies on the right-hand side of Eqs. (314) and (330) into active and passive gravitational masses is impossible in the general case, and therefore the introduction of a tensor of the passive gravitational mass is then not possible. But in the case of the Sun-Earth system, such a separation is possible.

Indeed, in the given case, the magnitude of the reduced multipole mass moments of the Sun can be ignored, as a result of which the definition (316) of the tensor of the passive mass of a body takes the form

$$M_{\oplus}a_{\oplus}^{\alpha}=\frac{M_{\odot}m_p^{\alpha\beta}n_{\beta}}{R^2}.$$

Thus, we need to reduce the expression (350) for the acceleration of the Earth's center of mass to quasi-Newtonian form. For this, we must transform the right-hand side of the expression (350) to ensure that it does not contain terms that decrease faster than $1/R^2$ with increasing distance between the bodies. Using the equation of the orbit,

$$r=p/(1-e_{\oplus}\cos\varphi),$$

we find the relation between the radial and transversal components of the Earth's velocity:

$$v_r/v_{\varphi}=(1/r)(dr/d\varphi)=e_{\oplus}\sin\varphi/(1-e_{\oplus}\cos\varphi)=e_{\oplus}\sin\varphi+O(e_{\oplus}^2).$$

Since $e_{\oplus}^2=O(\varepsilon)$, for the square of the Earth's velocity we have

$$v_{\oplus}^2=v_r^2+v_{\varphi}^2=v_{\varphi}^2[1+O(\varepsilon)].$$

Then the equations of motion of the Earth's center of mass in the Newtonian approximation take the form

$$(v_{\oplus}^2/R)[1+O(e_{\oplus})]=M_{\odot}/R^2.$$

It follows that

$$M_{\odot}/R=v_{\oplus}^2+O(\varepsilon^2 e_{\oplus}).$$

Using this relation, we reduce the right-hand side of the expression (350) to quasi-Newtonian form:

$$a_{\oplus}^{\alpha}=-(M_{\odot}/R^2)\Big\{n^{\alpha}\Big[1-(2\beta+\gamma)v_{\oplus}^2$$
$$+(\alpha_1-\alpha_2+\xi_1+5\xi_w+3+\gamma-4\beta)\Omega_{\oplus}$$
$$-3\xi_w n_{\beta}n_{\delta}\Omega_{\oplus}^{\beta\delta}-\frac{1}{2}(\alpha_1-\alpha_2-\alpha_3)w^2-\frac{3}{2}\alpha_2(n_{\beta}w^{\beta})^2$$
$$+\frac{\alpha_1}{2}w_{\beta}v_{\oplus}^{\beta}\Big]+2(\gamma+1)v_{\oplus}^{\alpha}n_{\beta}v_{\oplus}^{\beta}+(\alpha_2-\xi_1+\xi_2-5\xi_w)$$
$$\times n_{\beta}\Omega_{\oplus}^{\alpha\beta}-\alpha_2 w^{\alpha}n_{\beta}w^{\beta}-\frac{\alpha_1}{2}w^{\alpha}n_{\beta}v_{\oplus}^{\beta}+O^{\alpha}(\varepsilon^2 e_{\oplus})\Big\}.$$

In this case, we can introduce a tensor of the passive gravitational mass of the Earth in accordance with the expression

$$m_p^{\alpha\beta}/M_{\oplus}=-\gamma^{\alpha\beta}[1-(2\beta+\gamma)v_{\oplus}^2$$
$$+(\alpha_1-\alpha_2+\xi_1+5\xi_w+3+\gamma-4\beta)\Omega_{\oplus}-3\xi_w n_{\beta}n_{\delta}\Omega_{\oplus}^{\beta\delta}$$
$$+(\alpha_1/2)w_{\beta}v_{\oplus}^{\beta}-(3/2)\alpha_2(n_{\beta}w^{\beta})^2-(\alpha_1-\alpha_2-\alpha_3)w^2/2]$$
$$-2(\gamma+1)v_{\oplus}^{\alpha}v_{\oplus}^{\beta}-(\alpha_2-\xi_1+\xi_2-5\xi_w)\Omega_{\oplus}^{\alpha\beta}$$
$$+\alpha_2 w^{\alpha}w^{\beta}+(\alpha_1/2)w^{\alpha}v_{\oplus}^{\beta}+O(\varepsilon^2 e_{\oplus}). \quad (351)$$

If in the expression (351) we now set $w^{\alpha}=0$ and $\xi_w=0$, we obtain

$$m_p^{\alpha\beta}/M_{\oplus}=-\gamma^{\alpha\beta}[1-(2\beta+\gamma)v_{\oplus}^2+(\alpha_1-\alpha_2+\xi_1+3+\gamma-4\beta)\Omega_{\oplus}]$$
$$-2(\gamma+1)v_{\oplus}^{\alpha}v_{\oplus}^{\beta}-(\alpha_2-\xi_1+\xi_2)\Omega_{\oplus}^{\alpha\beta}+O(\varepsilon^3). \quad (352)$$

We now compare this expression with (219), which was obtained by Will. The main difference between these expressions is the absence of the terms $v_{\oplus}^2$ and $v_{\oplus}^{\alpha}v_{\oplus}^{\beta}$ in (219). However, it is necessary to take into account the terms $v_{\oplus}^2$ and $v_{\oplus}^{\alpha}v_{\oplus}^{\beta}$, since they have post-Newtonian order of magnitude, $v_{\oplus}^2\sim\varepsilon^2$, and exceed by more than one order of magnitude the remaining post-Newtonian corrections in the expressions (219) and (352). The data obtained from the lunar laser ranging in conjunction with other experiments have made it possible to determine fairly accurately the values of all the post-Newtonian parameters in the expression (352). Let us estimate the deviation from unity of the ratio of the passive mass to the inertial mass.

The results of the experiments[101,102] on the lunar laser ranging in conjunction with other expreiments showed that

$$\left.\begin{aligned}&|(\alpha_2-\xi_1+\xi_2)\Omega_{\oplus}^{\alpha\beta}+\gamma^{\alpha\beta}(\alpha_1-\alpha_2+\xi_1+3+\\&+\gamma-4\beta)\Omega_{\oplus}|<2\cdot10^{-11};\\&|\gamma-1|<0{,}005;\ |\beta-1|\leqslant 0{,}01.\end{aligned}\right\} \quad (353)$$

These bounds on the values of the post-Newtonian parameters are a serious barrier for a number of metric theories of gravitation, but nevertheless a number of theories do satisfy the conditions (353). One such theory is the field theory of gravitation.[30] Substituting the estimates (353) in (352), we obtain

$$m_p^{\alpha\beta}/M_{\oplus}=-\gamma^{\alpha\beta}[1-3v_{\oplus}^2+O(10^{-10})]-4v_{\oplus}^{\alpha}v_{\oplus}^{\beta}+O^{\alpha\beta}(10^{-10}). \quad (354)$$

Thus, in any metric theory of gravitation which satisfies the conditions (353) and possesses energy-momentum conservation laws for the matter and the gravitational field taken together the ratio of the passive gravitational mass to the inertial mass in the post-Newtonian approximation is not equal to unity but differs from it approximately by 10^{-8} in accordance with (354), which contradicts the claims made in Refs. 101 and 102 on the basis of the lunar laser ranging data that the passive gravitational mass and the inertial mass of the Earth are equal in the post-Newtonian approximation. However, such an interpretation of the lunar laser ranging results in Refs. 101 and 102 is incorrect, since they are based on Will's expression (219) for the passive gravitational mass. As we have shown in this section, such a definition of the passive gravitational mass is quite arbitrary and has no physical meaning.

But nevertheless it is not the terms that are responsible for the inequality between the passive gravitational mass and the inertial mass of the Earth which are the cause of its nongeodesic motion. As follows from the expression (331), the averaged differences between the accelerations of a point body and the Earth's center of mass does not contain terms of the form $(M_{\odot}/R^2)v_{\oplus}^2 n^{\alpha}$, $(M_{\odot}/R^2)v_{\oplus}^{\alpha}v_{\oplus}^{\beta}n_{\beta}$. This indicates that the difference between the passive mass of the Earth and its inertial mass is not the cause of the nongeodesic motion of its center of mass.

To emphasize the quantities that characterize the oscillatory motion of the Earth's center of mass with respect to the reference geodesic, we use Eqs. (346)–(348), setting $\gamma=\beta=1$, $\alpha_1=\alpha_2=\alpha_3=0$, $\xi_1=\xi_2=\xi_3=\xi_4=\xi_w=0$.

It should be noted that the estimates obtained here will be approximate in nature, since to find their exact values it would be necessary to make a more detailed analysis taking into account the internal structure of the Earth. If the Earth is assumed to be a homogeneous, spherically symmetric, nonrotating sphere, then, as follows from the expressions given at the end of the preceeding section, its center of mass will execute oscillations with respect to the reference geodesic with period

$$T=2\pi\Big/\sqrt{\frac{4\pi G}{3}\rho_{\oplus}}\approx 3.3\cdot 10^3 \text{ sec}$$

and amplitude

$$|\mathbf{A}|=(M_{\odot}/R^2)\,R_{\oplus}^2\approx 3\cdot 10^{-4}\text{ cm}.$$

Rotation of the Earth leads to an appreciable increase in the amplitude of the oscillations:

$$\mathbf{A}=\frac{1}{2}R_{\oplus}^2[\omega_{\oplus}\mathbf{v}_{\oplus}]+\frac{M_{\odot}\mathbf{R}}{R^3}R_{\oplus}^2. \tag{355}$$

Since $\omega_{\oplus}\approx 7\times 10^{-5}$ rad/sec, for the amplitude of the oscillations we have in this case

$$|\mathbf{A}|\approx 4{,}6\cdot 10^{-2}\text{ cm}\sim \varepsilon^2 R_{\oplus}$$

Since the rotation axis of the Earth makes an angle $\alpha\approx 66°33'$ with the plane of the ecliptic, this amplitude will be subject to seasonal variations, changing from about 4.6×10^{-2} cm (in the winter and summer) to 4.2×10^{-2} cm (at the vernal and autumnal equinoxes). In this case, the ratio of the difference between the accelerations of the Earth's center of mass and a test body to the Earth's acceleration is

$$\frac{|\delta a^{\alpha}|}{|a^{\alpha}|}\approx\frac{1}{2}\frac{M_{\oplus}}{M_{\odot}}\frac{R^2}{R_{\oplus}}\,|[\omega_{\oplus}\mathbf{v}_{\oplus}]|\approx 10^{-7}. \tag{356}$$

If we bear in mind that the Earth is not a sphere but a spheroid, then in the expression for the amplitude of the oscillations ther are additional terms due to the presence of the multipole mass moments of the Earth, these giving only corrections to the amplitude (355).

For example, for the ratio of the difference between the accelerations of the Earth's center of mass and a test body to the Earth's acceleration we obtain among the other corrections to the expression (356)

$$|\delta a^{\alpha}|/|a^{\alpha}|\sim J_3 v_{\oplus}^2(M_{\oplus}/M_{\odot})(R/R_{\oplus})^2,$$

where J_3 is the coefficient which multiplies $P_3(\cos\theta)$ in the expansion of the Newtonian potential of the Earth in spherical harmonics. According to the data in Ref. 137, we have $J_3\approx -2.5\times 10^{-6}$. It follows that this correction is $|\delta a^{\alpha}|/|a^{\alpha}|\approx 10^{-10}\approx\varepsilon^2$.

Thus, we have shown that in any metric theory of gravitation possessing energy-momentum conservation laws of the matter and the gravitational field taken together the center of mass of an extended body does not move along a geodesic of the Riemannian space-time.

The averaging of the center-of-mass motion made to eliminate the short-period oscillations of the orbit due to the internal motion of the matter has enabled us to show that the center of mass of an extended body moving in orbit executes an oscillatory motion with respect to the reference geodesic.

Application of our general expressions to the Sun-Earth system and the use of the lunar laser ranging results in conjunction with other experiments show to high accuracy $O(10^{-10})$ that the ratio of the Earth's passive mass to its inertial mass is not equal to unity but differs from it by about 10^{-8}. During its motion in orbit, the Earth executes an oscillatory motion with respect to the reference geodesic with a period of order 1 h and an amplitude not less than 10^{-3} cm. Although this amplitude is small, it does have the post-Newtonian order $A\sim\varepsilon^2 R_{\oplus}$, and therefore the deviation of the motion of the Earth's center of mass from geodesic motion can be detected in an appropriate experiment with post-Newtonian degree of accuracy.

In the present section, we have considered only one of the consequences of the deviation of the Earth's center-of-mass motion from motion along a geodesic of the Riemannian space-time, namely, the existence of a motion of the Earth's center of mass with respect to a test body placed in a sufficiently small cavity in the neighborhood of the Earth's center of mass. However, the fact that the motion of the Earth's center of mass is nongeodesic will, of course, lead to observable consequences in other post-Newtonian experiments, in particular, in experiments made with test bodies on the surface of the Earth. Therefore, theoretical analysis of various gravimetric experiments and further specifically designed experiments will make it possible to determine more accurately the numerical values of the post-Newtonian parameters.

18. INTERACTION OF WEAK GRAVITATIONAL WAVES WITH ELECTROMAGNETIC FIELDS IN THE FIELD THEORY OF GRAVITATION

The interaction of weak gravitational waves with the electromagnetic fields of different objects has recently been often studied. Such processes are of interest mainly because the results obtained can provide the theoretical basis for the design of detectors of high-frequency gravitational radiation.

The scheme for calculating the interaction of weak gravitational waves with electromagnetic fields in the field theory of gravitation is identical to the scheme for calculating this interaction in any other metric theory of gravitation and is based on the application of perturbation theory to the generally covariant Maxwell equations.

Maxwell's equations in this case can be written in the form

$$\left.\begin{aligned}&F^{ik}{}_{;k}=\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^k}[\sqrt{-g}\,F^{ik}]=-\frac{4\pi}{c}j^i;\\&F_{ik;l}+F_{kl;i}+F_{li;k}=0.\end{aligned}\right\} \tag{357}$$

If we introduce the four-dimensional vector potential A_i by means of

$$F_{ik}=A_{k;i}-A_{i;k}, \tag{358}$$

then when the condition $A^{i}_{;i}=0$ is satisfied Eqs. (357) can be written as

$$A_{i}{}^{;k}{}_{;k}-R_{ik}A^{k}=\frac{4\pi}{c}j_{i}.$$

In the presence of weak gravitational waves, the metric tensor of the effective Riemannian space-time can be expanded in the small parameter $h\ll 1$, the amplitude of a weak gravitational wave:

$$g^{ni}=\gamma^{ni}-h\overset{(1)}{f}{}^{ni}-h^{2}\overset{(2)}{f}{}^{ni}-\ldots, \tag{359}$$

where $f^{0i}=0$ by virtue of the TT gauge of the gravitational wave. The current 4-vector is expanded similarly:

$$j^{n}=\overset{(0)}{j}{}^{n}+h\overset{(1)}{j}{}^{n}+h^{2}\overset{(2)}{j}{}^{n}+\ldots, \tag{360}$$

where $\overset{(0)}{j}{}^{n}$ is the original unperturbed current 4-vector, and $\overset{(p)}{j}{}^{n}h^{p}$ are the corrections due to the influence of the gravitational waves on the motion of the source.

In this case, the 4-potential should be sought in the form

$$A_{i}=\overset{(0)}{A}_{i}+h\overset{(1)}{A}_{i}+h^{2}\overset{(2)}{A}_{i}+\ldots, \tag{361}$$

where $\overset{(0)}{A}_{n}$ is the 4-potential of the original unperturbed electromagnetic field, and $h^{p}\overset{(p)}{A}_{n}$ are the corrections to the 4-potential in the p-th order of perturbation theory due to the influence of the gravitational waves.

Substituting the expansions (359)–(361) in the Maxwell equations (357), expanding the obtained relation in a series in powers of h, and equating to zero the expressions that play the parts of the coefficients of this series, we obtain Maxwell's equations in the approximation of p-th order.

Restricting ourselves to the approximation of first order in h, we obtain

$$\Box\overset{(0)}{A}{}^{n}=\frac{4\pi}{c}\overset{(0)}{j}{}^{n};\quad \Box\overset{(1)}{A}{}^{n}=\frac{4\pi}{c}[j^{n}_{\text{int}}+\overset{(1)}{j}{}^{n}], \tag{362}$$

where

$$\left.\begin{array}{c} j^{0}_{\text{int}}=-\frac{c}{4\pi}\operatorname{div}\mathbf{D};\quad \mathbf{j}_{\text{int}}=-\frac{c}{4\pi}\left[\frac{\partial\mathbf{D}}{\partial x^{0}}+\operatorname{curl}\mathbf{B}\right]; \\ D^{\alpha}=\overset{(1)}{f}{}^{\alpha\beta}\overset{(0)}{E}_{\beta};\quad B^{\alpha}=\overset{(1)}{f}{}^{\alpha\beta}\overset{0}{H}_{\beta}. \end{array}\right\} \tag{363}$$

Thus, to calculate the interaction of the weak gravitational waves with the electromagnetic field in the first order of perturbation theory, we must first, given the distribution of the charges and the currents in the absence of gravitational waves, determine the original unperturbed electromagnetic field [the first of the equations in (362)]. Then, using the obtained solution, and also the correction to the current 4-vector linear in h, we find from Eq. (362) the vector potential in the first order of perturbation theory due to the effect of the gravitational waves on the system. It can be seen from the expressions (363) that in the first approximation the influence of the gravitational waves on the original electromagnetic field is analogous to the introduction of coordinate- and time-dependent permittivity and magnetic permeability tensors of the vacuum.

The efficiency of the interaction process is measured by the transformation coefficient α, which is equal to the ratio of the energy flux density of the resulting electromagnetic wave to the energy flux density of the original gravitational wave: $\alpha=S_{\text{emw}}/S_{\text{gw}}$.

It should be noted that in the general case it is necessary to consider Maxwell's equations in conjunction with the gravitational field equations (113) of the field theory of gravitation, since in the external field there is, besides the transition of energy of the gravitational waves into energy of the electromagnetic waves, the reverse process as well. If $\alpha\ll 1$, the reverse process can be ignored. But if α is near to or greater than unity, it is necessary to consider the two processes simultaneously, and also to take into account perturbations of other types which render the transition of the gravitational wave energy into electromagnetic wave energy incomplete. As an example of a calculation of the interaction of weak gravitational waves with electromagnetic fields in the field theory of gravitation, we study, for example, the possibility of using neutron stars as space detectors of gravitational waves.

As is well known, the solution to the problem of detecting gravitational waves is at present usually attacked on the basis of detection of such waves under laboratory conditions. However, gravitational waves can also be detected from the existence of characteristic features in an electromagnetic wave produced by the interaction of gravitational waves with the fields of astrophysical objects, for example, neutron stars.

This new method of detecting gravitational waves that we propose can in individual cases be much more effective than the traditional use of laboratory detectors, since neutron stars have electromagnetic fields with a strength unattainable under laboratory conditions ($H\approx 10^{14}$–10^{16} A/m), and fields of this intensity extend appreciable distances ($r\approx 10^{4}$–10^{6} m), forming in this way a cosmic gravitational wave detector.

Another advantage of this method is that the gravitational wave generated by the interaction can be detected using present-day radio telescopes, whose collecting surface reaches 10^{4} m^{2}. The construction of laboratory gravitational wave detectors with such a surface area in the near future is improbable.

Therefore, in individual cases it is expedient to use the cosmic gravitational wave detectors placed at our disposal by nature. Although such detectors do not permit the experimentalist to adapt them to obtain comprehensive and complete information about gravitational waves, the detection of gravitational waves by means of a cosmic detector is nevertheless important to settle the question of the existence of gravitational waves and for the correct and purposeful improvement of laboratory detectors.

In this connection, it must be especially emphasized that it will only be possible to say that gravitational waves have been detected by a cosmic detector if the generated electromagnetic wave has features that enable one to distinguish it with a high degree of confidence from an electromagnetic wave generated in any other process. The processes considered below meet this condition to a sufficient extent.

Interaction of a plane gravitational wave with the field of a magnetic dipole. We consider the field of a magnetic dipole with magnetic moment m at the coordinate origin:

$$\mathbf{H} = [3(\mathbf{rm})\,\mathbf{r} - r^2\mathbf{m}]/r^5. \tag{364}$$

Examples of such fields are provided by the magnetic fields of particles, planets, and neutron stars.

Suppose that on the z axis and far from the magnetic dipole there is a source which radiates monochromatic spherical gravitational waves in accordance with Eq. (127). In a region of space whose linear dimensions are appreciably smaller than the distance from the center of this region to the source of the gravitational waves, a spherical gravitational wave can be regarded as an elliptically polarized plane wave. Therefore, in our case the expression (127) in the neighborhood of the magnetic dipole ($n^3 \approx 1, n^2 = n^2 \approx 0$) takes the form

$$\left.\begin{array}{l} f^{11} = -f^{22} = h_0 \cos 2\beta \exp[i(kz - \omega t)]; \\ f^{12} = ih_0 \sin 2\beta \exp[i(kz - \omega t)], \end{array}\right\} \tag{365}$$

where h_0 is the amplitude of the gravitational wave, and $h_0 \ll 1$. Using the expressions (365) and (100) and averaging over the polarizations, we can express the amplitude h_0 in terms of the energy flux density F of the gravitational wave:

$$h_0^2 = 32\pi GF/(k^2c^3).$$

The degree of ellipticity of the polarization of the gravitational wave (365) is measured by $\tan 2\beta$. If $\tan 2\beta = 0$ or ∞, then the wave is linearly polarized; if $|\tan 2\beta| = 1$, the wave is circularly polarized. For other values of $\tan 2\beta$, the wave is elliptically polarized. If $\tan 2\beta > 0$, the wave has right-handed polarization; if $\tan 2\beta < 0$, left-handed polarization. We shall study the interaction of a plane elliptically polarized gravitational wave (365) with the magnetic dipole field (364). From Maxwell's equations (362) after linearization with respect to the small parameter h_0 we obtain the fairly simple equations

$$\Box \overset{(1)}{\mathbf{A}} = \operatorname{curl} \mathbf{B}; \quad \Box \overset{(1)}{A^0} = 0. \tag{366}$$

It should be noted that if we regard the field (364) as the field of a particle with magnetic moment m we shall assume that the gravitational wave does not change m. But if the field (364) is the field of a planet or a star, we shall assume that it is produced by currents circulating within the conducting planet or star. In this case, the initial gravitational wave influences the circulating currents, and in the expression (360) we have

$$\overset{(1)}{j^n} = f^n(r) \exp[i(kz - \omega t)]. \tag{367}$$

It is easy to see that the contribution of these currents to the radiation is small compared with the radiation generated by the interaction of the gravitational wave with the magnetic field outside the star, since the radiation due to the currents (367) arises within the conducting planets or stars and is therefore effectively absorbed over a distance equal to a few wavelengths from the point of emission, being transformed into the Joule heat of the planet or star. Assuming that $\Box \overset{(1)}{\mathbf{A}} = \operatorname{curl} \mathbf{L}$, we rewrite (366) in the form $\Box \mathbf{L} = \mathbf{B}$.

It is easy to see that

$$\left.\begin{array}{l} L^1 = \frac{\partial\psi}{\partial y} i \sin 2\beta + \frac{\partial\psi}{\partial x} \cos 2\beta; \\ L^2 = \frac{\partial\psi}{\partial x} i \sin 2\beta - \frac{\partial\psi}{\partial y} \cos 2\beta, \end{array}\right\} \tag{368}$$

and that the function ψ satisfies the equation

$$\Box \psi = (\mathbf{mr}/r^3) \exp[i(kz - \omega t)]. \tag{369}$$

We write the solution of Eq. (369) in the form of retarded potentials. Since the source is defined in the whole of space in (369), it is necessary to take into account retardation accurately. After integration, we sum the resulting infinite series, obtaining

$$\psi = \frac{i}{2k}\left[\frac{m_1x + m_2y}{r(r-z)} - \frac{m_3}{r}\right][\exp(ikr) - \exp(ikz)]\exp(-i\omega t). \tag{370}$$

Using the expressions (368) and (370), we find the expression for the radial component of the energy flux density of the electromagnetic wave generated by the interaction. Omitting the terms that decrease faster than $1/r^2$ for all values of the angles θ and φ, we obtain

$$\begin{aligned} S_r = \frac{k^2GF}{2c^4r^2}\Big\{&[(1+\cos\theta)(m_1\cos\varphi + m_2\sin\varphi) \\ &- m_3\sin\theta]^2[1 + \cos^2\theta - \sin^2\theta\cos 4\varphi\cos 4\beta] \\ &- \frac{2(m_1\cos\varphi + m_2\sin\varphi)^2\cos^3\theta(1+\cos\theta)^2}{kr(1-\cos\theta)}\sin k(r-z) \\ &+ \frac{2(m_1^2 + m_2^2)\cos^2\theta(1+\cos\theta)}{k^2r^2(1-\cos\theta)^2}[1 - \cos k(r-z)]\Big\}. \end{aligned} \tag{371}$$

We investigate this expression in more detail, since measurement of the energy flux density of the electromagnetic wave (371) and elucidation of all the features of this radiation give the observer the possibility of obtaining information about the initial gravitational wave (365).

From the expression (371) for the forward radiation ($\theta = 0$), we have

$$S_r = S_z = \frac{k^2GF}{c^4r^2}(m_1^2 + m_2^2). \tag{372}$$

Therefore, if a gravitational wave propagates through a cloud containing a large number N of randomly distributed but polarized particles of one species, the forward radiation at a large distance from the cloud is, in accordance with Ref. 138,

$$S_{z\,\mathrm{tot}} = \frac{k^2GF}{c^4r^2}(m_1^2 + m_2^2)N^2.$$

The intensity of the electromagnetic waves radiated when a gravitational wave interacts with a polarized particle is

$$I = \frac{\pi k^2GF}{30c^4}[101(m_1^2 + m_2^2) + 48m_3^2].$$

If the particle is not polarized, then

$$I = \frac{25\pi k^2GF}{9c^4}|\mathbf{m}|^2.$$

It can be seen from Eqs. (371) and (372) that S_r and S_z vary not only with a change in the point of observation (θ, φ) but also with a change in the orientation of the magnetic moment m. We assume that the magnetic dipole executes quasisteady rotation around an axis in the xz plane with frequency $\omega_0 \ll \omega$.

Quasisteady in this case means that the distance from the magnetic dipole to the point of observation satisfies $r \ll c/\omega_0$. Then at this distance the interaction of the gravitational wave with the resulting magnetic-dipole radiation can be ignored compared with the interaction of the gravitational wave with the field (364). When $r > c/\omega_0$, it is also necessary to take into account the interaction of the gravitational wave with the resulting magnetic-dipole radiation. To be specific, we shall assume that the rotation axis is perpendicular to the vector m and makes an angle β_1 with the z axis. In this case,

$$\mathbf{m} = |\mathbf{m}|\{-\cos\beta_1\cos\omega_0 t;\ -\sin\omega_0 t;\ \sin\beta_1\cos\omega_0 t\}. \quad (373)$$

Then as a result of the interaction of the gravitational wave (365) with the magnetic field of the rotating dipole, an electromagnetic wave is generated with frequency ω that is modulated in amplitude at frequency ω_0 and has modulation depth which depends on the angles β_1, θ, φ:

$$S_z = \frac{k^2 GF\mathbf{m}^2}{2c^4 r^2}[1+\cos^2\beta_1 - \sin^2\beta_1\cos 2\omega_0 t] \quad (374)$$

and for $\theta^2 \gg \pi/kr$

$$S_r = \frac{k^2 GF\mathbf{m}^2}{2c^4 r^2}[(1+\cos\theta)(\cos\varphi\cos\beta_1\cos\omega_0 t + \sin\varphi\sin\omega_0 t) + \sin\theta\sin\beta_1\cos\omega_0 t]^2[1+\cos^2\theta - \sin^2\theta\cos 4\varphi\cos 4\beta_1]. \quad (375)$$

Therefore, the high-frequency electromagnetic wave will arrive at the observer in pulses with period $T = 2\pi/\omega_0$ (like pulsar radiation). Averaging the expressions (374) and (375) over the period $T = 2\pi/\omega_0$, we find the amount of energy of the electromagnetic wave which passes through an area of cross section 1 cm^2 in each pulse:

$$\overline{S}_r = \frac{\pi k^2 GF\mathbf{m}^2}{2c^4 r^2\omega_0}[(1+\cos\theta)\sin\theta\cos\varphi\sin 2\beta_1 + \sin^2\theta\sin^2\beta_1 + (1+\cos\theta)^2(\sin^2\varphi + \cos^2\varphi\cos^2\beta_1)][1+\cos^2\theta - \sin^2\theta\cos 4\varphi\cos 4\beta_1]. \quad (376)$$

In the direction of the z axis, we have

$$\overline{S}_z = \frac{2\pi k^2 GF\mathbf{m}^2}{c^4 r^2\omega_0}\left[1 - \frac{\sin^2\beta_1}{2}\right]. \quad (377)$$

The energy of the electromagnetic waves emitted in one pulse over all directions is

$$\overline{I} = \frac{\pi k^2 GF\mathbf{m}^2}{15c^4\omega_0}\left[101 - \frac{53}{2}\sin^2\beta_1\right]. \quad (378)$$

Thus, as a result of the interaction of the plane gravitational wave with the field of the rotating magnetic dipole, an electromagnetic wave is generated with frequency equal to that of the gravitational wave and amplitude modulated by the rotation frequency of the dipole.

Let us estimate the distances from a neutron star at which the electromagnetic wave produced by such an interaction can be detected. Following Ref. 120, we assume that in the radio range one can detect an electromagnetic wave with energy density

$$S_{\min} = 10^{-19}\ \text{W/m}^2. \quad (379)$$

Using the parameters of neutron stars,[120,139] we find that the magnetic moment of the star is

$$|\mathbf{m}| \approx 10^{28} - 10^{29}\ \text{A}\cdot\text{m}^2 \quad (380)$$

From the expression (377),

$$r = (\omega\,|\mathbf{m}|/c^3)\sqrt{GF/S}.$$

Using the estimates (379) and (380), we find that

$$r \leqslant 2(10^{15} - 10^{16})(\omega/c)\sqrt{F}\ \text{m}.$$

Since the majority of the known neutron stars[130-142] are at distances $r \approx 10^{17} - 10^{19}$ m from the Earth, an observer on the Earth will detect the electromagnetic wave produced by the interaction if the energy flux density of the gravitational wave incident on the neutron star is $F \approx 10^5 - 10^7$ W/m^2.

We also estimate the minimal value of the energy flux of gravitational waves that can be detected as a result of interaction of the plane gravitational wave (365) with the magnetic moments of particles forming a cloud of astrophysical size. At distances much greater than the cloud diameter, the expression (372) for the forward radiation is valid, and therefore, assuming that the radius L of the cloud is of order 10^{14} m, the concentration of particles in the cloud is $n \approx 10^9$ m^{-3}, and the distance from the cloud to the point of observation is $r \approx 10^{17}$ m, we find that the minimal energy flux of a gravitational wave that can be detected as a result of this interaction is $F = 10^{-3}$ W/m^2.

Interaction of a weak gravitational wave with the field of a rotating magnetic dipole. Another cosmic detector is realized when a source of weak gravitational waves is situated inside a rotating neutron star. According to Refs. 121 and 122, there is photoproduction of gravitons in the interior of stars due to the Coulomb and magnetic dipole fields of the particles that constitute the matter of the star, and also in the magnetic field of the complete star. Thus, stars are sources of weak gravitational waves, and, dependeing on the spectrum of the initial photons, the resulting gravitational radiation can belong to any frequency range. Many stars have a field which coincides with the field of a rotating magnetic dipole for which the magnetic axis is at an angle to the rotation axis. It is not our aim to give a detailed description of the actual processes which occur within stars and lead to the appearance of a magnetic dipole field, and we shall therefore consider the simplest model of a rotating magnetic dipole.

We assume that the star is a conducting sphere of radius b which rotates with constant frequency $\omega_0 \ll c/b$ about an axis passing through the center of the sphere. We assume that within the sphere closed currents circulate, their magnetic moment m being rigidly attached to the sphere and making angle β with the rotation axis. After simple calculations, we obtain the following expression for the electromagnetic field outside the sphere ($r \geqslant b$):

$$\left.\begin{aligned}\mathbf{E} &= [\mathbf{r}\dot{\mathbf{m}}(t')]/(cr^3) + [\mathbf{r}\ddot{\mathbf{m}}(t')]/(c^2r^2);\\ \mathbf{H} &= [3\mathbf{r}(\mathbf{m}(t')\mathbf{r}) - r^2\mathbf{m}(t')]/r^5\\ &+ 3\mathbf{r}(\dot{\mathbf{m}}(t')\mathbf{r})/(cr^4) - \dot{\mathbf{m}}(t')/(cr^2)\\ &+ [\mathbf{r}(\ddot{\mathbf{m}}(t')\mathbf{r}) - r^2\ddot{\mathbf{m}}(t')]/(c^2r^3),\end{aligned}\right\} \quad (381)$$

where $\mathbf{m} = |\mathbf{m}|\{\sin\beta\cos\omega_0 t';\ \sin\beta\sin\omega_0 t';\ \cos\beta\}$, $t' = t - r/c$.

We assume that the source of the gravitational waves within the star radiates gravitational waves of frequency $\omega \gg \omega_0$ on a flat space-time background.

We consider the simplest, weak ($h \ll 1$) gravitational wave with components outside the source in the form

$$\left.\begin{aligned} &f^{12} = -\frac{1}{2}\frac{\partial}{\partial\varphi} f^{11}; \quad f^{23} = -\frac{\partial}{\partial\varphi} f^{13}; \quad f^{33} = 0; \\ &f^{11} = -f^{22} = hkb \frac{H^{(1)}_{7/2}(kr)}{\sqrt{kr}} P^2_3(\theta) \sin 2\varphi \exp(-i\omega t); \\ &f^{13} = -\frac{hkb}{\sqrt{kr}} [2H^{(1)}_{7/2}(kr) P^1_3(\theta) + 12H^{(1)}_{3/2}(kr) P^1_1(\theta)] \\ &\quad \times \sin\varphi \exp(-i\omega t). \end{aligned}\right\} \tag{382}$$

After averaging over the wave period, the intensity of the gravitational radiation is

$$dI/d\Omega = 225h^2k^2b^2c^5 \sin^4\theta/(16\pi G). \tag{383}$$

Expressing the amplitude h of the gravitational wave in terms of the source power F, we obtain $h = \sqrt{\pi FG/(30k^2b^2c^5)}$. We shall assume that the interaction of the gravitational wave (382) with the field of the rotating magnetic dipole (381) takes place in vacuum.

The solution of (362) will be unique if we require the fulfillment of Sommerfeld's radiation condition and continuity of the tangential components of the fields E and H on the surface of the sphere. Substituting the expressions (381) and (382) in (363), we find

$$\left.\begin{aligned} &q_{\rm int} = q_1(\mathbf{r}) \exp[-i(\omega + \omega_0)t] + q_2(\mathbf{r}) \exp[-i(\omega - \omega_0)t]; \\ &\mathbf{j}_{\rm int} = \mathbf{j}_0(\mathbf{r}) \exp(-i\omega t) + \mathbf{j}_1(\mathbf{r}) \exp]-i(\omega + \omega_0)t] \\ &\quad + \mathbf{j}_2(\mathbf{r}) \exp[-i(\omega - \omega_0)t]. \end{aligned}\right\} \tag{384}$$

It is readily seen that an interaction between the wave part of the gravitational wave (382) with the wave part of the electromagnetic wave (381) does not occur, and the contribution to the expressions (384) corresponding to this interaction is equal to zero. We note also that the expressions (384) can be expanded in series with a finite number of terms in spherical harmonics of the first kind.

We write down the solution of (362) in the form of retarded potentials. The right-hand side of (362) is defined in the whole of space outside the sphere of radius b, and therefore the point of observation is within the region of integration. Thus, it is very important to take into account retardation exactly. Using Gegenbauer's theorem, which makes it possible to do this, for the field intensities E and H of the generated electromagnetic wave at $r \geqslant b$ we obtain (retaining only the leading terms)

$$\left.\begin{aligned} &E_\theta = \frac{15ihkb\,|\mathbf{m}|}{4r}\sqrt{\frac{2}{\pi}}\left[\frac{1}{b^2} - \frac{1}{r^2}\right]\Big[\sin\beta\cos\theta \\ &\quad \times \cos\left(\varphi - \omega_0 t + \frac{\omega_0 r}{c}\right) - \cos\beta\sin\theta\Big]\sin^2\theta \exp[-i(\omega t - kr)]; \\ &E_\varphi = \frac{15ihkb\,|\mathbf{m}|}{4r}\sqrt{\frac{2}{\pi}}\left[\frac{1}{b^2} - \frac{1}{r^2}\right]\sin\beta\sin^2\theta \\ &\quad \times \sin\left(\varphi - \omega_0 t + \frac{\omega_0 r}{c}\right)\exp[-i(\omega t - kr)]; \; H_\varphi = E_\theta; \; H_\theta = -E_\varphi. \end{aligned}\right\} \tag{385}$$

It can be seen from (385) that as a result of the interaction of the gravitational field (382) with the field (381) of the rotating magnetic dipole an electromagnetic wave is generated with frequency ω and amplitude modulated at frequency $\omega_0 \ll \omega$, the modulation depth depending on the coordinates of the point of observation and the angle β. It is interesting to note that the main contribution to the result of the interaction is made by the interaction of the wave part of the gravitational wave with the nonwave part $H_0 = m/r^3$ of the magnetic field.

The amplitude-modulation effect can be understood on the basis of the following simple considerations. The gravitational wave interacts not with the wave part of the electromagnetic field of the rotating neutron star but only with the nonwave parts. The field described by these terms decreases fairly rapidly with increasing distance from the star, so that the main part of the resulting radiation is generated near the surface of the star. But at these distances the field can be regarded as the field of a magnetic dipole in quasisteady rotation, i.e., its instantaneous value at a certain point is determined by the position of the magnetic dipole at the same instant of time. In the first order of perturbation theory, the gravitational wave interacts only with the magnetic-field components at right angles to the direction of its propagation, so that in the direction of the magnetic axis there is no radiation, while the intensity of the electromagnetic wave is maximal in the directions perpendicular to the direction of the magnetic axis. Thus, the amplitude of the electromagnetic wave is different for different directions. If the magnetic dipole rotates about an axis which does not coincide with the magnetic axis, this entire directional diagram of the electromagnetic radiation rotates with the rotation frequency of the dipole, and as a result the amplitude of the high-frequency electromagnetic wave at a fixed observer will vary periodically, i.e., amplitude modulation will occur. The modulation depth and its frequency are determined not only by the angle between the magnetic axis and the rotation axis but also by the position of the observer.

For the energy flux density of the electromagnetic wave radiated into an element of solid angle we obtain, after averaging over the period $T = 2\pi/\omega$,

$$\begin{aligned} \frac{dI}{d\Omega} = \frac{15FG\mathrm{m}^2}{256c^4}\left[\frac{1}{b^2} - \frac{1}{r^2}\right]^2 \sin^4\theta\,[1 - \cos^2\beta\cos^2\theta \\ - \sin^2\theta\sin^2\beta\cos 2(\varphi - \omega_0 t + \omega_0 r/c) \\ - \sin 2\beta\sin\theta\cos\theta\cos(\varphi - \omega_0 t + \omega_0 r/c)]. \end{aligned} \tag{386}$$

Therefore, the high-frequency electromagnetic wave will reach the observer in pulses (like pulsar radiation) whose shape and frequency depend on $r, \theta, \varphi, \beta$. For a nonrotating magnetic dipole, it is necessary to set $\omega_0 = 0$ in (386).

The total radiation flux through a sphere of radius r_0 is

$$I = \frac{FG\mathrm{m}^2}{8c^4}\left[1 - \frac{\cos^2\beta}{7}\right]\left[\frac{1}{b^2} - \frac{1}{r_0^2}\right]^2. \tag{387}$$

It follows from the expression (387) that most of the high-frequency electromagnetic radiation is generated in the region $b < r < 10b$; outside it, only 2×10^{-2} of the radiation power is generated.

We determine the coefficient of transformation of the energy of the gravitational wave into the electromagnetic wave:

$$\begin{aligned} \alpha = \frac{\pi G\mathrm{m}^2}{8c^4}\left[\frac{1}{b^2} - \frac{1}{r^2}\right]^2 [1 - \cos^2\beta\cos^2\theta - \sin^2\theta\sin^2\beta\cos 2 \\ \times (\varphi - \omega_0 t + \omega_0 r/c) - \sin 2\beta\sin\theta\cos\theta\cos(\varphi - \omega_0 t + \omega_0 r/c)]. \end{aligned} \tag{388}$$

The coefficient α reflects the properties of the field transformer (381). Hitherto, it has been assumed that the source radiates strictly coherent gravitational waves. However, it is more sensible to assume that the gravitational waves are produced by a large number of individual emitters, each of which radiates trains of gravitational waves with random phase and arbitrarily oriented directional diagram (383), this resulting in the generation of a noncoherent gravitational wave with spherical directional diagram. Then the energy flux density of the electromagnetic waves is

$$S = F\alpha/(4\pi r^2). \quad (389)$$

It should be noted that we have throughout ignored the influence on the interaction process of the refractive index of the medium, the static gravitational field of the star itself, and other perturbations. But all these perturbations merely decrease the flux density of the generated electromagnetic wave, and the amplitude-modulation effect is not qualitatively changed.

It follows from (388) that if $2|\cot\beta\cot\theta|$ is appreciably greater than or appreciably less than unit the pulses will arrive with frequencies ω_0 or $2\omega_0$, respectively. If $2|\cot\gamma\cot\theta|$ is comparable with unity, the pulses will have a complicated shape, since pulses of frequencies ω_0 and $2\omega_0$ are added, and the amplitude and phase of each of them will depend on the coordinates of the point of observation and on the angle β. It should be noted that the amplitude modulation gives only a "window," in which one can observe the high-frequency electromagnetic wave. The source of the gravitational waves may generate gravitational waves of irregular amplitude, and therefore an observer on the Earth will detect a generated electromagnetic wave only when there is a fairly powerful burst of high-frequency gravitational waves in the "window."

We note two further important features of the radiation. It follows from (385) that the radiation is linearly polarized but consists of two parts. The plane of polarization of one part is fixed, but that of the other rotates with the rotation frequency of the star. For $\cos\theta > 0$, the rotation of the plane of polarization is clockwise; for $\cos\theta < 0$, anticlockwise.

Thus, the electromagnetic radiation generated by the interaction of a weak gravitational wave with the field of a rotating magnetic dipole is unique. It should also be noted that the angles β and θ can be determined from the polarization of the electromagnetic wave and the direction of rotation of the plane of polarization.

It follows from the expression (388) that the motion of the Earth together with the observer relative to the star leads to a change in the amplitude and phase difference of the pulses of frequencies ω_0 and $2\omega_0$, which, in its turn, causes a change in the pulse shape—a drift of the subpulses forming the resulting pulse. If the motion of the Earth occurs along the direction r with velocity v_r, the drift will have a periodic nature, and $N = c/v_r$ pulses will pass in a time equal to the drift period. Assuming that $v_r = 380$ km/sec (the velocity with which the Earth moves relative to the neutron stars[143]), we obtain $N \approx 10^3$. This value can be reduced if the radial velocity of the Earth with respect to the stars is greater. It also follows from Eq. (388) that in the case of motion of the observer toward the rotation axis ($\sin\theta \to 0$) the amplitude-modulation effect steadily decreases.

Thus, in this case too the resulting electromagnetic wave has a number of features (amplitude modulation, unique polarization states, drift of the subpulses) that make it possible to attribute its generation to the interaction of a gravitational wave with the field of a rotating neutron star.

We now estimate the distance at which an electromagnetic wave produced by the interaction of the weak gravitational wave (382) with the field (381) of a rotating neutron star can be detected. Using the parameter of neutron stars (see Refs. 139–143), $b \approx 10^4$ m, $H \approx 10^{14}$–10^{16} A/m, we find that in this case the maximal value of the coefficient of transformation of the energy of the gravitational waves into the energy of electromagnetic waves is $\alpha_{\max} = 10^{-10}$. According to Ref. 122, neutron stars are potential sources of weak gravitational waves with a power that may reach $F = 6 \times 10^{27}$ W, which corresponds to a gravitational-wave amplitude $h \approx 10^{-20}$. Therefore, the electromagnetic wave generated as a result of the interaction can be detected at a distance $r \approx 10^{18}$ m from the neutron star.

Since the distances to the majority of the known neutron stars are $r \approx 10^{17}$–10^{19} m, the interaction we have considered means that, using the features of the electromagnetic waves generated by this interaction, one can not only look for a fairly powerful source of gravitational radiation but also make experiments with a view to studying the properties of gravitational waves and comparing them with the predictions of different theories of gravitation. Then the data obtained in such measurements, together with bounds on the post-Newtonian parameters and on the Peters-Mathews coeficients, will restrict still further the number of theories of gravitation capable of an adequate description of reality.

19. POSSIBLE EXPERIMENTS TO FIND DIFFERENCES BETWEEN THE PREDICTIONS OF THE FIELD THEORY OF GRAVITATION AND EINSTEIN'S GENERAL RELATIVITY

The field theory of gravitation and Einstein's general theory of relativity are entirely different theories of gravitation, since the fundamental principles of these theories and the gravitational field equations are different. Therefore, in the same physical situation these theories will give different predictions.

The difference between them must be manifested particularly clearly in the description by these theories of gravitational waves, and also the effects due to strong gravitational fields. It should be noted that since Einstein's quadrupole formula is not contained in general relativity and, quite generally, Einstein's theory contains no direct connection between the change in the energy of matter and the radiation of curvature waves, it is in principle impossible in general relativity to explain energy loss by matter on gravitational radiation.

Therefore, study of the motion of binary systems and determination of the possible energy losses by them through gravitational radiation would be a fundamental experimental test for the field theory of gravitation and general relativity. At the same time, the experimental observation of loss of energy by matter through gravitational radiation would uniquely refute general relativity and serve as a confirmation of the ideas of the field theory of gravitation.

As we pointed out above, the behavior of the Universe at the early stages of its evolution in a strong gravitational field differs qualitatively in the field theory of gravitation from the corresponding description of the Universe in general relativity. Since the early stages in the evolution of the Universe essentially determine the flux densities and spectral characteristics of the fossil (background) electromagnetic, neutrino, and gravitational radiations, measurement of these characteristics will make possible a qualitative comparison of the results of the measurements with the predictions of general relativity and the field theory of gravitation.

In addition, strong gravitational fields essentially determine the internal structure of superdense objects. Therefore, the difference between the descriptions of the internal structure of stars in the field theory of gravitation and in Einstein's general relativity must lead to different values for the limiting masses of stable stars. There are also important differences in the properties of gravitational waves in the field theory of gravitation and in Einstein's theory in the presence of external gravitational fields.

In general relativity, it is customary to call metric waves gravitational waves and base the theoretical investigation of these waves on energy-momentum pseudotensors.

But we regard such an approach as absolutely devoid of any physical meaning. The energy-momentum pseudotensors of general relativity have absolutely nothing to do with the existence of a gravitational field, so that none of the conclusions obtained using them reflect the essence of the problem. In Einstein's theory one can speak only of curvature waves, since the gravitational field in the theory is characterized by the curvature tensor. It is this tensor that occurs in the deviation equation (3), which is the basis of the principle of operation of any of the quadrupole mass detectors of gravitational waves.

The presence of curvature waves in some region of space-time is a clear indication of the presence in this region of gravitational waves radiated by some source. Moreover, curvature waves cannot be created or annihilated by a transformation of the coordinate system. But metric waves cannot serve as a criterion for the existence of gravitational waves emitted by a source, since metric waves can also be produced by a simple transformation of the coordinate system. Therefore, by gravitational waves in general relativity we shall always understand curvature waves described by the fourth-rank curvature tensor.

In general relativity, the natural geometry for electromagnetic waves and metric waves is the common Riemannian geometry. Since curvature waves are expressed in terms of the second derivatives of the transverse part of the metric waves, the natural geometry for curvature waves is also the Riemannian geometry. Therefore, in Einstein's theory electromagnetic waves and curvature waves will propagate in the same manner, and curvature waves will be as subject as electromagnetic waves to the gravitational frequency displacement $\delta\nu/\nu = U_1 - U_2$, they will undergo the same deflection $\delta\varphi = 4M/b$, and have equal propagation velocities and, therefore, equal delay times in external gravitational fields.

In the field theory of gravitation, the natural geometry for the gravitational field is the pseudo-Euclidean geometry, whereas the matter is described in the effective Riemannian geometry. Therefore, in the field theory of gravitation external gravitational fields affect only the propagation of electromagnetic waves. Gravitational waves in the field theory of gravitation propagate along geodesics of the pseudo-Euclidean space-time and they are not subject to the gravitational red shift of the frequency, ray bending, and time delay of a signal in external gravitational fields. In the field theory, the propagation velocity of gravitational waves does not depend on external gravitational fields.

These differences between the properties of gravitational waves in the field theory of gravitation and in Einstein's theory make it possible to propose a number of experiments using weak gravitational waves in which these theories give different predictions. In principle, the following two formulations of such experiments are possible.

Experiments using laboratory detectors of gravitational waves. It is expected[112-119] that in the near future it will be possible to detect under laboratory conditions gravitational radiation arriving from extraterrestrial sources. Then the use of two or more gravitational wave detectors will make it possible[32] to measure with sufficient accuracy the angle with which a gravitational ray is bent in the weak gravitational field of the Sun. The formulation of this experiment will depend on whether or not the extraterrestrial source of gravitational waves is also a source of electromagnetic radiation.

If the source of gravitational waves does not emit electromagnetic waves, the experiment will be analogous to the measurement of the bending of a light ray. Then comparison of the obtained deflection angle of the gravitational ray with the corresponding values predicted by the field theory of gravitation ($\delta\varphi = 0$) and by Einstein's general relativity ($\delta\varphi = 4M/b$) will show to what extent the predictions of these theories agree with the results of the experiments.

If the source of gravitational waves also radiates electromagnetic waves, the scheme of the experiment simplifies appreciably, since in accordance with Einstein's theory the "electromagnetic" and "gravitational" images of the source must always coincide.

According to the field theory of gravitation, the picture will be somewhat different. As the edge of the solar disk approaches the line joining the source of the gravitational waves and the observer, the gravitational and electromagnetic images of the source will begin to divide, and the electromagnetic image will be observed to be further from the center of the Sun than the gravitational image. When the edge of the solar disk touches the line joining the observer and the source of the gravtational waves, the angular distance between the electromagnetic and gravitational images of the source will be maximal and equal to $\delta\varphi = 4M/b$. When the solar disk covers the source–observer line, the electromagnetic image will disappear. After the solar disk has passed this line, the electromagnetic image will reappear, and it will again be observed further from the center of the Sun, by the angular distance $\delta\varphi = 4M/b$, than the gravitational image. With increasing distance of the solar disk from the source–observer line, the angular distance between the electromagnetic and gravitational images will decrease, and in the limit $b \to \infty$ the two images will coincide.

Experiments using laboratory and cosmic detectors of gravitational waves. It is, however, to be expected that sufficiently powerful sources of weak gravitational waves will be encountered extremely rarely under astrophysical conditions. It could be that the gravitational-wave sources that can be detected on the Earth are fairly far from the plane of the Earth's orbit and, therefore, are not covered by the solar disk. In this case, to formulate an experiment in which the field theory of gravitation and general relativity give different predictions, it is necessary to use cosmic detectors as well as a laboratory detector of gravitational waves. Depending on the disposition of the gravitational-wave source and the cosmic detector, it is possible to use the following two experimental arrangements.

The first of them is realized when the source of the weak gravitational waves is situated inside a rotating neutron star. Many neutron stars[120] have an electromagnetic field identical to the field of a rotating magnetic dipole whose magnetic axis makes a certain angle ψ_0 with the rotation axis. Then in accordance with Refs. 121 and 122 there is in the interior of these stars photoproduction of gravitons in the Coulomb and magnetic dipole fields of the particles that constitute the matter of the star, and also in the magnetic field of the complete star.

Thus, stars are a source of weak gravitational waves, and depending on the spectrum of the initial photons, the resulting gravitational radiation can belong to any frequency region.

The calculations show that in the field theory of gravitation, as in Einstein's theory, the interaction of weak gravitational waves with the electromagnetic field of a rotating neutron star generates an electromagnetic wave with a number of unique features (amplitude modulation, unusual polarization, drift of subpulses, etc.). Thus, on the basis of these features an observer on the Earth can conclude with a high degree of confidence that a given electromagnetic wave has been generated by the interaction of a gravitational wave with the electromagnetic field of a rotating star. The amplitude of the electromagnetic wave will be somewhat different in these theories because of the different influence of a static gravitational field on the interaction process, but basically this is not important. More important for us is the circumstance that in a static external gravitational field the subsequent propagation of the resulting electromagnetic wave and the initial gravitational wave will occur differently in the field theory of gravitation and in Einstein's theory.

According to Einstein's theory, the two waves will propagate along the same paths, undergoing the same gravitational red shifts of their frequencies and the same ray bending and having equal group velocities. Therefore, an observer on the Earth detecting them must find that they have the same frequencies, the same pulse shape within the window, no time delay between the arrival of the electromagnetic and the gravitational pulses, and also coincident electromagnetic and gravitational images of the source.

In the field theory of gravitation, the generated electromagnetic wave as it propagates in the external gravitational field will also be subject to the gravitational red shift, its rays will be bent, and its group velocity will depend on the potential of the external field. But in the field theory of gravitation gravitational waves are not subject to the influence of a gravitational field, so that they will propagate with constant velocity, without change of frequency and without deflection in external gravitational fields.

Therefore, an observer on the Earth, detecting both waves, must find a red shift of the electromagnetic spectrum relative to the gravitational, and also the existence of a time delay between the arrival of the gravitational and electromagnetic pulses within the window. In addition, the gravitational and electromagnetic images of the source will not coincide in the general case.

We note also that the results of this experiment will make it possible to obtain a number of important astrophysical data. Indeed, as is shown in Refs. 123 and 124, from the frequency and depth of the amplitude modulation, and also the polarization of the electromagnetic wave, it is possible to determine the rotation frequency ω_0 of the neutron star, the angle ψ_0 between the rotation axis and the magnetic moment of the star, and also the angle θ between the rotation axis and the direction to the Earth. Measuring the flux densities of the gravitational and electromagnetic waves on the Earth, one can determine the transformation coefficient α, and thus the product of the magnetic field intensity on the surface of the star and its radius.

In addition, in the field theory of gravitation this experiment makes it possible, using the red shift of the electromagnetic spectrum relative to the gravitational spectrum, to measure the difference of the gravitational potentials between the point of observation and the surface of the star:

$U_2 - U_1 = -\delta\nu/\nu.$

Measuring the time delay ΔT between the arrival of the gravitational and electromagnetic pulses, one can determine the mean gravitational potential U on the path of propagation of the electromagnetic wave:

$$\overline{U} = \frac{1}{L}\int U\,dl \approx \frac{\Delta T}{2L},$$

where L is the distance between the neutron star and the Earth.

If the source of the gravitational waves is outside the neutron star, the analysis of the results of the observations will be somewhat more complicated, since the source, the cosmic detector, and the laboratory detector will be situated at the points of a triangle. However, in this case too the results of the observations will permit a conclusion to be drawn about the properties of the gravitational waves and will also yield information about the astrophysical objects.

Thus, in the near future, after the creation of laboratory gravitational-wave detectors, there will be a real possibility of verifying the predictions of the field theory of gravitation and Einstein's general theory of relativity with regard to the properties of gravitational waves in external gravitational fields.

CONCLUSIONS

We have considered the formulation of the field theory of gravitation as the theory of a symmetric tensor field of second rank in flat space-time. In the theory, the customary ideas about energy transport by physical fields have a rigorous meaning, and the gravitational field, like all other physical fields, carries positive-definite energy and momentum. The equations of motion of matter are formulated in terms of an effective Riemannian space-time with metric tensor g_{ni}, which ensures that in the theory the inertial and gravitational masses of a point body are equal. By combining ideas about the gravitational field as a physical field that carries energy with the identity principle we arrive at new gravitational field equations and new notions of space and time. The gravitational field equations in matter are nonlinear because of the nonlinear dependence of the source on the components of the gravitational field. The source in the gravitational field equations is matter, and the gravitational field itself acts as a source only to the extent that the expression $T^{ki}\partial g_{ki}/\partial f_{nm}$ depends on the components of the gravitational field. Outside the matter, the field equations are linear. At the same time, because of the gauge invariance, the gravitational field equations, which are partial differential equations of fourth order, go over into equations of second order outside the matter.

The post-Newtonian approximation of the field theory of gravitation and analysis of the results of modern gravitational experiments show that the field theory of gravitational with minimal coupling can describe all the existing experimental facts. In the field theory of gravitation, there is no preferred rest frame, since the geometry of the pseudo-Euclidean space-time is not an *a priori* but a natural geometry for all physical fields, including the gravitational field. In contrast, the Riemannian space-time for the motion of matter is an effective space-time, reflecting merely the effect of the gravitational field on matter in the pseudo-Euclidean space-time. Therefore, neither in its significance nor in its field equations can the field theory of gravitation be classified as a bimetric theory of gravitation.

In the field theory of gravitation, the energy-momentum tensor concept is common to all physical fields, and therefore the existence of curvature waves in the Riemannian space-time reflects the transport of energy and momentum by the gravitational waves in the pseudo-Euclidean space-time. Therefore, in the field theory of gravitation it is possible to make different energy calculations. In the field theory of gravitation, the energy losses due to the emission of weak gravitational waves by a slowly moving source are determined by the expression

$$-dE/dt = (G/45c^5)\,\dddot{D}^2_{\alpha\beta}. \qquad (390)$$

In constrast to the field theory of gravitation, general relativity does not contain conservation laws in their usual sense, as a result of which Einstein's theory has only vanishing integrals of the motion. In general relativity, calculation of the energy loss by a source, and also determination of the energy fluxes of gravitational waves is impossible, since in Einstein's theory there are no conservation laws connecting the change in the energy-momentum tensor of the matter to the existence of curvature waves. Therefore, Eq. (390) is also in principle absent in general relativity.

The field equations of the field theory of gravitation differ from those of Einstein's theory, which leads to quite different descriptions in these theories of effects in strong strong gravitational fields, and also in the properties of gravitational waves. Among the differences, in the field theory of gravitation there is no bending of a gravitational wave beam passing near a massive body, with the consequence that massive bodies do not have a focusing effect on gravitational waves. In addition, in contrast to Einstein's theory, in the field theory a change in the frequency of free gravitational waves emitted by a source occurs only as a result of relative motion of the source and the observer (Doppler effect), since there is no gravitational red shift of free gravitational waves in vacuum.

As in Einstein's theory, in the field theory of gravitation the gravitational field of a nonstatic, spherically symmetric source is a static field outside the matter. Nonstationary homogeneous models of the Universe in the field theory of gravitation describe the cosmological red shift and allow monotonic and nonmonotonic behavior. In contrast to Einstein's theory, the deceleration parameter is determined not only by the mean density of matter in the Universe but also by the minimal-coupling parameters, and therefore the field theory of gravitation does not present the difficulties encountered in connection with the fact that the mean matter density is insufficient to obtain the observed deceleration parameter.

In the field theory of gravitation with minimal coupling, the forces of gravitational attraction go over into forces of gravitational repulsion with increasing gravitational potential. Therefore, instead of the gravitational collapse of astrophysical objects inherent in general relativity, the field theory has a new mechanism of release of energy, since small perturbations in the radius of an astrophysical object once the critical value of the mean gravitational potential has been attained necessarily lead to expansion of the matter, which may be accompanied by the ejection of a certain fraction of the mass of the object and the release of energy.

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Translated by Julian B. Barbour