Quasirelativistic systems of interacting particles

R. P. Gaĭda

Institute of Applied Problems of Mechanics and Mathematics, Ukrainian Academy of Sciences
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On the basis of the general principles of the relativistic theory of direct interactions (causality, Poincaré invariance, and separability), the elements of a quasirelativistic (classical and quantum) mechanics of a system of particles are presented. This makes it possible to take into account small relativistic effects in objects possessing internal structure. The relationship between the different formalisms of the theory is analyzed, the physical meaning of the variables is discussed, the general form of the post-Newtonian interaction Lagrangians and Hamiltonians is found, and their physical interpretation and relation to field approaches are considered. The introduction of quasirelativistic center-of-mass variables solves the problem of separating the motion of a weakly relativistic system of particles as a whole from its internal motion in the case of an arbitrary interaction.

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INTRODUCTION

In different branches of physics, the need for a consistent relativistic theory of many-particle systems, differing from local field theory, is felt more and more strongly. This is due, on the one hand, to the development of experimental techniques which have significantly raised the accuracy of the measurements and, on the other, to the fact that the present state of relativistic quantum field theory is such that it can serve as the basis of the description of only a limited number of physical objects. The existence of the splendidly working formalism of nonrelativistic classical and quantum mechanics of systems of interacting particles stimulates many authors to attempt the construction of an analogous relativistic or at least approximately relativistic theory. The quasipotential approach,[1,2] which is intimately related to the field-theory description, belongs in this category. Recently, great successes have been achieved in the construction of a relativistic theory of direct interactions of particles, which is an alternative to the field description. Although its direct application to calculations of nuclear systems is in an initial stage, there are grounds for expecting genuine successes in this direction.[3]

There exist phenomena for whose description nonrelativistic theories are clearly inadequate but for which the required accuracy makes it possible to take into account relativistic effects approximately by means of expansions in c^{-2}. A well-known example of this type is the fine structure of the hydrogen spectrum, which can be calculated either on the basis of the rigorous relativistic Dirac equation or in the framework of a corresponding approximate method.[4] In this simple example (it is simple because we are dealing with a one-body relativistic problem) the choice of the exact or the approximate relativistic equation is dictated by the required accuracy of the results, but in more complicated cases of systems of several weakly relativistic particles the situation is different, since the absence of an exact theory or its inadequate (from the point of view of physical interpretation or computational possibilities) development frequently lead to the need to use approximate approaches. This is particularly important for systems with nuclear interactions, which at present are not amenable to description by quantum field theories; however, even for objects in which the well-studied electromagnetic interactions play the main part, general methods of approximately relativistic (or, as we shall call them, quasirelativistic) theories are very helpful. If, for example, we consider systems of many particles with two-body interactions, then for given laws of interaction for any pair of particles, expressed in their center-of-mass system, theoretical study of the system as a whole is possible only if we can make calculations in an arbitrary inertial system, i.e., use interaction Hamiltonians that depend on the center-of-mass motion of the two-particle subsystems. This dependence, which is one of the aspects of the problem of adding relativistic (or quasirelativistic interactions, is determined by the requirements of Lorentz invariance of the theory and is not related to the nature of the interaction.[5–8] We encounter an analogous situation in the problem of separating the motion of a complicated system as a whole from the internal motion of the particles that constitute it.[9–11]

The aim of the present paper is to give a unified exposition of the elements of a quasirelativistic theory of direct interactions of particles and thereby generalize the nonrelativistic (classical and quantum) mechanics of a system of particles to phenomena in which relativisitc effects can be regarded as small corrections to the nonrelativistic results. We proceed from the general principles and equations of the relativistic theory of the direct interactions of particles, but the actual results will be restricted to the quasirelativisitc region, basically the so-called post-Newtonian approximation, by which we mean allowance for the corrections of order c^{-2} to nonrelativistic potentials that depend only on the interparticle distances.

A very interesting aspect of quasirelativistic mechanics is the circumstance that, like nonrelativistic mechanics, it can be presented in both classical and quantum versions, the connection between them being realized in the traditional manner. This is important both in principle and in view of the fact that the classical treatment is helpful for the physical interpretation of quasirelativistic quantum mechanics. In this connection, we can point out the problem of the physical

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meaning of the canonical variables, and also the problem of comparing the results of the theory of direct interactions with field-theoretical approaches. It should be borne in mind that the classical variant of the theory is also of independent interest on account of its applicability, for example, to objects consisting of massive gravitating bodies.[12]

The application of approximately relativistic equations to the description of the structure and properties of nuclei has a long history, though until recently results were sparse. The pioneer in this field was Breit[13] as long ago as 1937. Shirokov *et al.*[5] studied the problem of finding the relativistic corrections to the phenomenological potentials of the nucleon–nucleon interaction (see also Refs. 6, 7, and 9). The influence of relativistic effects in the motion of nucleons in nuclei on the scattering of electrons by nuclei was investigated in Ref. 14, and the relativistic corrections to the deuteron form factor in Ref. 15. Further references to some other examples of processes in which relativistic effects in complicated systems of particles are important can be found in Ref. 9. Without pretending to any degree of completeness of these historical comments, we mention that recently interest in relativistic effects in nuclei and other particle systems has considerably quickened. As examples, we mention some topical problems that have been considered in recent publications and require a relativistic (or, at least, approximately relativistic) treatment: the contribution of the relativistic corrections to the binding energy of few-nucleon systems (for example, the triton) and to the cross section of elastic pd scattering through large angles at intermediate energies,[16] the calculation of the levels and widths for systems such as positronium, muonium, baryonium, and also bound states in the quark–antiquark system[17]; the μ-mesic atoms and μ-mesic molecules of the hydrogen isotopes[18]; and the relativistic treatment of some processes in $NN\pi$ systems.[19]

The main aim of the review is to formulate the fundamentals and results of quasirelativistic mechanics; these provide a formalism for investigating relativistic effects of the types mentioned above in objects consisting of a fixed number of particles at not too high energies. The results presented in the paper are also of interest in that they may be helpful for developing a rigorously relativistic theory of direct interactions in the sense of the correspondence principle, i.e., in the region of sufficiently low energies the conclusions of the exact and approximate theories must be almost identical.

Such a review is necessary, since the literature contains many results relating to the quasirelativistic mechanics of particle systems obtained in the most varied ways—on the basis of classical or quantum field theory (in particular, electrodynamics), using Fokker-type action integrals, in the framework of the classical and quantum relativistic Hamiltonian theory of direct interactions, and by means of the Lagrangian and Newtonian formalisms. The relationship between these approaches and the corresponding results has not yet been well studied, so that at the present time it is difficult to regard them as the individual fragments or different variants of a common physical theory constructed on a foundation of clear physical ideas, a well-developed mathematical formalism, and a well-defined region of applicability. On the other hand, there is no doubting the value of the development of such a theory. The present review is an attempt to make a step in this direction.

1. PHYSICAL FOUNDATIONS OF RELATIVISTIC THEORIES OF DIRECT PARTICLE INTERACTIONS

Causality and action at a distance. The creation of the special theory of relativity and the subsequent successes of the field approach to the investigation of physical reality led to the opinion, generally accepted up to the middle of our century, that the description of systems of interacting particles based on the idea of action at a distance is admissible only in nonrelativistic (Galileo-invariant) physics. From this it was concluded that the interaction between particles can be realized only by the propagation of a field, the carrier of the interaction. It was found, however, that this is a misunderstanding, which arises from an insufficiently careful analysis of all aspects of the causality principle of the special theory of relativity, according to which events A and B separated by a spacelike four-dimensional interval cannot (by virtue of the finite propagation velocity of signals) exert any influence on each other.

The first attempts to construct a relativistic (based on Maxwell's electrodynamics) theory of direct interactions as an alternative to the field description were made by Schwarzschild[20] and by Tetrode and Fokker[21] as early as the beginning of this century; they showed that the electromagnetic interaction of charged particles can be described in terms of the variables of the particles themselves without the concept of the electromagnetic field (by means of the so-called Fokker-type action integrals; see also Ref. 22). This approach became widely known after the studies of Wheeler and Feynman.[23] The systematic study of the possibilities of a nonfield approach to the construction of a Lorentz-invariant dynamics of a system of particles, which has developed very strongly over the last two decades, was initiated by a classical paper in 1949 by Dirac.[24] The most detailed and complete analysis of the relationship between the causality concept and the relativistic description of direct particle interactions is due to Havas[25] (see also Ref. 26). Let us formulate the main conclusion of this analysis.

It is necessary to distinguish two causality concepts. The first, which arose in nonrelativistic mechanics and was generalized later to other fields of physics, relates solely to closed systems and consists of the following assertion: Knowledge of the initial conditions at the time t_0 in a closed system σ makes it possible to determine by means of the laws of motion its subsequent evolution in time. Such description of closed systems is frequently called *causal* or *predictive*. The second concept of causality, which is the causality

principle of the special theory of relativity mentioned above, is always associated with the consideration of systems that are not closed, namely, the propagation of a signal from event A (the "cause") to event B (the "effect") presupposes that event A is a definite interaction event of the given system with some external perturbation. Since the relativistic theory of direct interactions pretends to an adequate description of only closed systems σ, the causality principle of special relativity cannot be used as an argument against its validity.

The problem of Lorentz invariance in relativistic theories of direct interactions. The traditional method used to construct Lorentz-invariant equations of mechanics or field theory is to attempt to cast these equations in a four-dimensional tensor form. This method is very effective in the mechanical problem of a single body interacting (locally) with an external field (for example, the electromagnetic field) and in local field theory. Its main advantage is the manifest Lorentz (or Poincaré) invariance. The generalization of such an approach to the description of a system of directly interacting particles, which is nonlocal by its very nature, provides the basis of a number of directions of the relativistic theory of action at a distance. Included here are theories in which the point of departure is provided by manifestly Poincaré-invariant Fokker-type action integrals (Refs. 20–23 and 27–32), the integro-differential equations of motion of Van Dam and Wigner[33] or the differential–difference equations of Havas and Plebański[34]; the four-dimensional formulation of the second-order differential equations of motion for a system of interacting particles[35-42]; the four-dimensional Lagrangian formalism using singular Lagrangians[43-46] or individual variation principles with a Lorentz-invariant evolution parameter for each particle[47]; the four-dimensional Hamiltonian approach[48-54] based on Dirac's canonical formalism with constraints,[55,56] and some other approaches.[57-60] In theories of this kind, Poincaré invariance of the description is achieved by the form adopted for the basic expressions or equations and, thus, does not present problems.

However, in these theories other difficulties do arise precisely on account of their manifest Poincaré invariance. The situation here is analogous to that characteristic of equations of the Bethe–Salpeter type. The main difficulty is the many-parameter description of the evolution of the system (as a rule, the proper times of the individual particles are the parameters), which leads to not only mathematical complications (for example, in Fokker-type theories the equations of motion are integro-differential equations or differential equations with a retarded argument), but also difficulties in the physical interpretation of the theory. We note also that the many-time relativistic description has a form very different from the nonrelativistic description and requires the development of new methods at almost every stage in the construction of the theory.

In the light of these comments, there is considerable interest, especially from the point of view of our problem of describing weakly relativistic systems, in three-dimensional approaches to the relativistic theory of direct interactions using a single evolution parameter for the system of particles; in the instantaneous form of dynamics,[24] to the exposition of which we shall restrict ourselves in the present paper, the coordinate time t serves as this parameter. Such approaches, which we shall call *one-time* approaches, are comparatively close in their mathematical form a physical interpretation to the nonrelativistic (classical or quantum) mechanics of a system of particles. Because the one-time treatment is not invariant with respect to Lorentz transformations, such approaches do not possess manifest (four-dimensional) Poincaré invariance. However, this does not mean that, under definite conditions, they cannot satisfy the Poincaré–Einstein principle of relativity (see, for example, Ref. 56). It is the study of these conditions that constitutes the main problem in one-time theories of direct interactions. The problem can be formulated and solved by means of group-theoretical methods, which will be described in Secs. 2 and 3. These conditions impose definite restrictions on the functions that describe the direct interactions of the particles. In the different formalisms (Newtonian, Lagrangian, and Hamiltonian) the sets of these functions and the methods of realizing the conditions of invariance of the theory are different. The connection between the three formalisms is much more complicated than in nonrelativistic mechanics, and at the present time has hardly been studied in the exact theory. However, in the quasirelativistic approximation the connection is fairly simple, and this circumstance also makes it possible to solve the problem of quantization by the methods developed in nonrelativistic mechanics. We note also that there is an intimate connection between the one-time approach and the four-dimensional Fokker approach (see Sec. 4).

Physical meaning of the variables and separability of interactions. Since Dirac's paper,[24] the main direction in the development of the theory of direct interactions has been the construction of a relativistic Hamiltonian theory for a system with a given number of particles (Dirac's problem).[1)] The solution of Dirac's problem in the framework of classical mechanics encountered a serious difficulty associated with the no-interaction theorem,[26,63-66] according to which the physical positions of the particles (the covariant coordinates) of only noninteracting particles can serve as canonical coordinates. Since the concept of a world line does not occur in quantum mechanics and the requirement of covariance of the canonical coordinates, which represents the condition of invariance of a world line should not apparently play an important part, the main efforts were directed toward the development of a quantum relativistic Hamiltonian theory of direct interactions. Its development is associated above all with the names of Bakamjian and Thomas,[67] Foldy,[68] Fong

[1)] S. N. Sokolov[61,62] has also proposed a second-quantized variant of relativistic Hamiltonian theory, which describes particle creation and annihilation processes. At the comparatively low energies to which we restrict ourselves here, these processes are unimportant.

and Sucher,[69] Coester,[70] and Sokolov (Refs. 3, 8, 61, 62, and 71–75), in whose papers the quantum Dirac problem has received its most complete solution.

However, this contrasting of the quantum and classical variants of the theory cannot be regarded as satisfactory, being logically unjustified and in view of the fact that for the physical interpretation of the quantum theory and for the possibility of comparing it with the results of experiments great importance attaches to the physical meaning of the variables employed in the theory, and a deep understanding of them can hardly be possible without a corresponding analysis at the classical level.

It is therefore of interest to make a parallel investigation of the problem of constructing the classical and quantum theories of direct interactions, especially since, as was noted in Ref. 3, all the successful attempts at the construction of a relativistic description of a closed system of particles have, despite different forms, gradually developed into physically equivalent versions of a common theory.

Besides the problems due to the need to combine interaction of particles and ordinary transformation properties of their spatial coordinates, the problem of the choice of the variables (especially in the Hamiltonian formalism, in which freedom in the choice of the canonical variables is achieved at the price of loss of their clear physical meaning) is associated with a further (after Poincaré invariance) fundamental requirement of the theory of direct interactions, namely (*cluster*) *separability* of the interactions (Refs. 68, 6, 70, and 71). This condition will be discussed in more detail below (see Sec. 2), and we shall here make only some preliminary comments.

Suppose a system of particles σ is divided into two subsystems $\sigma_{\rm I}$ and $\sigma_{\rm II}$, and in a limit these are separated by an infinite distance. Since there is no interaction between particles with infinite separation, all the characteristics of, for example, subsystem $\sigma_{\rm I}$ must be independent in this limit of the variables of subsystem $\sigma_{\rm II}$. It is clear that the very possibility of quantitative formulation of this condition presupposes that among the variables that characterize the system there are some whose asymptotic behavior is capable of reflecting the considered limiting process, i.e., variables which have the meaning of spatial separation.

In this connection, it is very desirable to consider the possibility of constructing a Hamiltonian formulation of the relativistic theory of direct interactions (both quantum and classical) on the basis of a preliminary solution of this problem in the framework of the Newtonian or Lagrangian formalism, in which from the very start one can use the physical coordinates of the particles with their known transformation properties under Lorentz transformations. The problem of going over to a Hamiltonian form from relativistic equations of Newtonian type on the basis of the Lie–Königs theorem was considered by Kerner and Hill.[76-78] Another possibility is to make a transition from the Lagrangian to the Hamiltonian formalism. In the general case, this needs more than an ordinary Legendre transformation and warrants deep study. However, in the problem that is at the center of attention in the present paper, namely, the construction of the first post-Newtonian approximation of the relativistic mechanics of a system of particles, the transition from the Lagrangian to the Hamiltonian description can be made, as we shall see below, by the traditional method.[2)] Therefore, we shall proceed from the generalized Lagrange–Ostrogradskiĭ formalism,[81] which was used by the present author in collaboration with Klyuchkovskii and Tretyak[82] as the basis for a relativistic Lagrangian theory of direct interactions of particles. Moreover, in the post-Newtonian approximation it will be easy to follow the equivalence of such an approach to the results obtained directly in the Hamiltonian formalism.

We note also one important advantage of a classical relativistic Lagrangian theory of direct interactions, namely, the possibility of establishing its connection with field approaches by means of a Fokker formulation of the theory of action at a distance. We shall use this in Sec. 4 for the field-theoretical interpretation of approximately relativistic Lagrangians and Hamiltonians.

2. ONE-TIME FORMULATIONS OF POINCARÉ-INVARIANT THEORIES OF DIRECT INTERACTIONS

Lie algebra of the Poincaré group. Let G_r by an r-parameter Lie group of point transformations of Minkowski space M_4 with parameters λ^α,

$$\left.\begin{aligned} &x'^\mu = \varphi^\mu(x, \lambda);\\ &x = \{x^\mu\} \equiv \{ct, \mathbf{r}\};\ \mu = 0, 1, 2, 3;\\ &\lambda = \{\lambda^\alpha\},\ \alpha = 1, \ldots, r, \end{aligned}\right\} \tag{1}$$

and $\varphi^\mu(x, 0) = x^\mu$. The corresponding infinitesimal transformations[3)]

$$x'^\mu = x^\mu + \delta x^\mu = x^\mu + \xi^\mu_\alpha(x)\,\delta\lambda^\alpha + o(\delta\lambda) \tag{2}$$

are determined by the tangent vector fields[83] [the generators of the transformations (1)]

$$\mathscr{X}_\alpha = \xi^\mu_\alpha(x)\frac{\partial}{\partial x^\mu};\quad \xi^\mu_\alpha(x) = \left.\frac{\partial \varphi^\mu(x, \lambda)}{\partial \lambda^\alpha}\right|_{\lambda=0}. \tag{3}$$

They satisfy the commutation relations

$$[\mathscr{X}_\alpha, \mathscr{X}_\beta] = c^\gamma_{\alpha\beta}\mathscr{X}_\gamma, \tag{4}$$

or, equivalently,

$$\xi^\mu_\alpha \frac{\partial \xi^\nu_\beta}{\partial x^\mu} - \xi^\mu_\beta \frac{\partial \xi^\nu_\alpha}{\partial x^\mu} = c^\gamma_{\alpha\beta}\xi^\nu_\gamma, \tag{5}$$

where $c^\gamma_{\alpha\beta}$ is the tensor of the structure constants of the group G_r. Thus, the operators $\mathscr{X}_\alpha$ generate the Lie algebra AG_r of G_r.

In the case of the 10-parameter Poincaré group[4)] $\mathscr{P}$,

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[2)] Such a transition can also be readily made in the linear approximation in the interaction.[79,80]

[3)] Summation over repeated indices is understood.

[4)] By $\mathscr{P}$ we understand the main connectivity component $\mathscr{P}^\uparrow_+$,[84] which contains the identity transformation.

the generators of time $\mathscr{X}_i^{\mathrm{T}}$ and space ($\mathscr{X}_j^{\mathrm{T}}$) translations and of space ($\mathscr{X}_j^{\mathrm{R}}$) and Lorentz ($\mathscr{X}_j^{\mathrm{L}}$) rotations have the form[5)]

$$\mathscr{X}_0^T = -\frac{\partial}{\partial t};\ \mathscr{X}_j^T = -\frac{\partial}{\partial x^j};\tag{6}$$

$$\mathscr{X}_j^R = -\varepsilon_{jki}x^k\frac{\partial}{\partial x_i};\tag{7}$$

$$\mathscr{X}_j^L = -\frac{1}{c^2}x_j\frac{\partial}{\partial t} - t\frac{\partial}{\partial x^j},\tag{8}$$

where ε_{jki} is the Levi-Civita symbol with $\varepsilon_{123}=\pm 1$. Hence, for $A\mathscr{P}$ we obtain the commutation relations

$$[\mathscr{X}_i^T,\ \mathscr{X}_0^T]=0;\ [\mathscr{X}_j^R,\ \mathscr{X}_0^T]=0;\ [\mathscr{X}_i^T,\ \mathscr{X}_j^T]=0;\tag{9}$$

$$\left.\begin{aligned}&[\mathscr{X}_i^R,\ \mathscr{X}_j^T]=\varepsilon_{ijk}\mathscr{X}_k^T;\ [\mathscr{X}_i^R,\ \mathscr{X}_j^R]=\varepsilon_{ijk}\mathscr{X}_k^R;\\&[\mathscr{X}_i^R,\ \mathscr{X}_j^L]=\varepsilon_{ijk}\mathscr{X}_k^L;\end{aligned}\right\}\tag{10}$$

$$[\mathscr{X}_i^L,\ \mathscr{X}_0^T]=\mathscr{X}_i^T;\tag{11}$$

$$[\mathscr{X}_i^L,\ \mathscr{X}_j^T]=\delta_{ij}\mathscr{X}_0^T/c^2;\ \ [\mathscr{X}_i^L,\ \mathscr{X}_j^L]=-\varepsilon_{ijk}\mathscr{X}_k^R/c^2.\tag{12}$$

The Lie algebra $A\mathscr{G}$ of the Galileo group is determined by commutation relations that differ from those written down above only by replacement of (12) by the equations

$$[\mathscr{X}_i^G,\ \mathscr{X}_j^T]=0;\ [\mathscr{X}_i^G,\ \mathscr{X}_j^G]=0,\tag{13}$$

which can be obtained from (12) by means of the formal passage to the limit $c\to\infty$ ($\mathscr{X}_i^G$ is the generator of Galileo transformations).

To formulate the conditions of Poincaré invariance of the one-time description of a relativistic system of directly interaction particles with world lines $\mathbf{x}_a(t)$, it is necessary to construct a representation of the Poincaré group that acts on some state space of this system (configuration space, phase space, etc.). The first step is to particularize the representation space; the three main formalisms of the theory (Lagrangian, Newtonian, and Hamiltonian) correspond to three possibilities: 1) an infinite extension $J^\infty(\mathbf{R}\times\mathbf{E}^{3N})\equiv\mathbf{E}$ (see Refs. 82, 85, and 91) of the extended configuration space of the system of particles with standard coordinates $(t,\mathbf{x},\mathbf{x}^1,\ldots,\mathbf{x}^\sigma,\ldots)$, where $\mathbf{x}^\sigma=\{\mathbf{x}_a^\sigma\},a=1,\ldots,N;\mathbf{x}_a^1\equiv\mathbf{x}_a;\mathbf{x}_a^\sigma=d^\sigma\mathbf{x}_a/dt^\sigma$; 2) the first extension $\mathbf{R}\times\mathbf{TE}^{3N}$ of the extended configuration space with coordinates $(t,\mathbf{x},\dot{\mathbf{x}})$; 3) the phase space $\mathbf{P}$ of the system with coordinates (q,p), where $q=\{q_a^i\},p=\{p_{ai}\},(a=1,\ldots,N;i=1,2,3)$ are the canonical coordinates and momenta of the particles; the connection between the canonical variables and the configuration variables will be discussed below. As canonical variables, we shall also use center-of-mass variables, which have a collective nature.

The next step is to construct for the group G_r the set of r generators X_α that act on the corresponding space and satisfy the commutation relations (4).

Representation of the Poincaré group in the configuration variables of the system of particles. We write the generators of the group of transformations G_r in E in the form[82]

$$X_\alpha=\omega_\alpha\frac{\partial}{\partial t}+\sum_{a=1}^{N}\sum_{\sigma=0}^{\infty}\xi_{\alpha a}^{(\sigma)i}\frac{\partial}{\partial\overset{\sigma}{x}{}_a^i},\quad \alpha=1,\ldots,r.\tag{14}$$

[5)]Since we shall use three-dimensional notation in what follows, we shall regard the subscripts and superscripts i, j, k as equivalent.

The vector fields ω_α and $\xi_{a\alpha}^{(\sigma)}$ determine the infinitesimal transformations of the coordinate of a point in E:

$$t'=t+\omega_\alpha\delta\lambda^\alpha;\ \overset{\sigma}{x}{}_a'^i(t')=\overset{\sigma}{x}{}_a^i(t)+\xi_{\alpha a}^{(\sigma)i}\delta\lambda^\alpha.\tag{15}$$

For the groups of spatial translations ($\alpha\equiv\overset{\mathrm{T}}{j}$) and rotations ($\alpha\equiv\overset{\mathrm{R}}{j}$), which do not affect the variable $t(\omega_\alpha=0)$,

$$X_j^T=-\sum_a\frac{\partial}{\partial x_a^j};\ X_j^R=-\varepsilon_{ijk}\sum_a\sum_\sigma\overset{\sigma}{x}{}_a^k\frac{\partial}{\partial\overset{\sigma}{x}{}_{ai}}.\tag{16}$$

Going over to boost transformations (Lorentz rotations) Λ with parameters δV^j (the components of the relative velocity V of the frames of reference S' and S), we write the first equation of (15) as

$$t'=t+\omega_j^L\delta V^j\equiv t-R_j\delta V^j/c^2,\tag{17}$$

where R can be interpreted as the position of an observer who relates the "new" simultaneity $t'=\text{const}$ to the "old" $t=\text{const}$ (see Refs. 26, 63, 82, and 86–88). Specification of $\boldsymbol{\omega}^{\mathrm{L}}$ is equivalent to establishing a one-to-one correspondence between the families of hypersurfaces $t=\text{const}$ and $t'=\text{const}$. As is shown in Ref. 82, it can be assumed without loss of generality that $\omega_j^{\mathrm{L}}=0$, i.e., $t'=t$. To obtain the second formula (15) for $\sigma=0$, we note that particle positions $\mathbf{x}_a'(t')$ that are simultaneous in the frame S' correspond in accordance with the Lorentz transformations to positions $\mathbf{x}_a(t_a)$ that are not simultaneous in S:

$$x_{ai}'(t')=\{\Lambda[t_a,\ \mathbf{x}_a(t_a)]\}_i;\ t_a=\{\Lambda^{-1}[t',\ \mathbf{x}_a'(t')]\}_0/c.\tag{18}$$

Substituting the second equation in (18) in the first and noting that $t'=t$ to terms linear in δV^j, we obtain[26,63,82]

$$\delta x_a^i\equiv x_a'^i(t')-x_a^i(t)=(-\delta_j^i t+v_a^i x_{aj}/c^2)\,\delta V^j\equiv\xi_{ja}^{Li}\delta V^j,\tag{19}$$

where $v_a^i=dx_a^i/dt=\dot{x}_a^i$. Since Eqs. (19) are a consequence of ordinary Lorentz transformations for the points of Minkowski space that lie on the world lines of the particles, we shall call them the *conditions of covariance of the coordinates* x_a^i, and the coordinates that satisfy them *covariant coordinates*. We emphasize that $\mathbf{x}_a(t)$ and $\mathbf{x}_a'(t)$ correspond to *different* points of the world line of particle a, so that Eqs. (19) are not ordinary Lorentz transformations; they may be called Lorentz transformations with conversion to a new simultaneity.[36]

From (19), we find the corresponding transformations of the derivatives:

$$\frac{d^\sigma x_a'^i}{dt'^\sigma}\equiv\overset{\sigma}{x}{}_a'^i(t')=\overset{\sigma}{x}{}_a^i(t)+\xi_{ja}^{(\sigma)Li}\delta V^j,\ \sigma=0,\ 1,\ \ldots,\tag{20}$$

where

$$\xi_{ja}^{(\sigma)Li}=\frac{d}{dt}\xi_{ja}^{(\sigma-1)Li}=\frac{d^\sigma}{dt^\sigma}\xi_{ja}^{Li},\ \xi_{ja}^{(0)Li}\equiv\xi_{ja}^{Li}.\tag{21}$$

Thus, for the boost generators we find

$$X_j^L=\sum_a\sum_{\sigma=0}^{\infty}\left[\frac{d^\sigma}{dt^\sigma}\left(-t\delta_{jk}+\frac{1}{c^2}v_{ak}x_{aj}\right)\right]\frac{\partial}{\partial\overset{\sigma}{x}{}_{ak}}.\tag{22}$$

It remains to determine the generator X_0^{T} of time displacements. If we are interested in only the Aristotle group (the direct product of the Euclid group and the group of time translations; see Refs. 36 and 82), then for the system of particles it could be chosen as in (6) in the form $\tilde{X}_0^{\mathrm{T}}=-\partial/\partial t$. However, such a choice

does not agree with the structure of the Poincaré group if the generators of Lorentz rotations are defined by (22), i.e., if $\omega_j^L=0$. To obtain $A\mathscr{P}$, we must take X_0^T in the form

$$X_0^T=\sum_{a=1}^{N}\sum_{\sigma=0}^{\infty}\dot{x}_a^{\sigma+1}\frac{\partial}{\partial x_a^k}=\frac{d}{dt}-\frac{\partial}{\partial t}. \tag{23}$$

This corresponds to representing the time translations $t'=t-\delta\tau$ by a shift along the particle trajectory[89]: $x_a'^i(t)=x_a^i(t+\delta\tau)=x_a^i(t)+\dot{x}_a^i(t)\delta\tau+o(\delta\tau)$.

By direct calculations it is readily verified that the generators (16), (22), and (23) form a basis of the algebra $A\mathscr{P}$, i.e., they satisfy the relations (9)–(12). It should be noted that representations of the Poincaré group by transformations that do not affect the time arise in the analysis of the symmetries of relativistic field equations by Fushchich.[90]

In the construction of a one-time theory of direct interactions, an important part is played by the presence in (22) of the velocities $\mathbf{v}_a$; this presence is due to the circumstance that the transition to the new simultaneity (on the new hypersurface $t'=\text{const}$) is accompanied by a shift along the world lines of the particles which cannot be eliminated when $N\geq 2$.[6)] Thus, we encounter the necessity of considering nonpoint coordinate transformations. It is this fact in conjunction with the condition of a contact transformation, which consists of the requirement that the transformed velocities satisfy the equation $v_a'^i(t')=dx_a'^i(t')/dt'$, that leads to an important consequence, namely, for the formulation of Poincaré invariance of a Lagrangian description of relativistic interactions, it is necessary to construct a representation of the group $\mathscr{P}$ by contact transformations of infinite order, which are also called Lie–Bäcklund transformations (see Ref. 91), acting on the infinite-dimensional space $\mathbf{E}=\mathbf{R}\times\mathbf{E}_\infty^{3N}$. This conclusion, which reflects the rigorous results obtained in the quoted papers, can be explained by means of the following simple arguments. If we consider the composition of two, three, etc., Lorentz transformations, i.e., a transition $S\to S'\to S''\to S'''\to\ldots$, then, as can be seen from (19)–(21), there appear in the transition formulas $\mathbf{x}_a(t)\to\mathbf{x}_a''(t'')$, $\mathbf{x}_a(t)\to\mathbf{x}_a'''(t''')$, etc., acceleration, third derivatives, etc. Taking into account the group structure of the transformations, we arrive at the proposition formulated above.

We note the following important circumstance. If the functions $\mathbf{x}_a(t)$ satisfy equations of motion of the second order, then in the sequence of transformations considered above these equations can be used to eliminate the accelerations. As a result, we arrive at transformations of the space $\mathbf{R}\times\mathbf{TE}^{3N}$ which are contact transformations with respect to the given equations of motion.[92,93] They will be used below to establish the conditions of Poincaré invariance of equations of motion of Newtonian type. However, in the investigation of the invariance of the Lagrangian formalism one must use only transformations that are contact transformations irrespective of the equations of motion, since the Lagrange function is defined not only on the real but also on the virtual trajectories of the particles.[94]

Canonical representations of the Poincaré group. As we have already noted, the problem of finding a canonical realization of the algebra $A\mathscr{P}$ on the phase space $\mathbf{P}$ of a dynamical system was first formulated by Dirac.[24] In the same paper, he proposed three different forms of relativistic dynamics (instant, point, and front), which are distinguished in canonical theory by which of the ten generators of the group $\mathscr{P}$ (the time and space translations H and P_i, and space and Lorentz rotations J_i and K_i) contain terms describing the interaction of the particles. To investigate the quasirelativistic approximation, the most convenient of these is the instant form, in which H and $\mathbf{K}$ contain the interaction, and $\mathbf{P}$ and $\mathbf{J}$ have the free-particle form. The connection between the different forms of dynamics in relativistic Hamiltonian theory was investigated in Refs. 62, 72, 75, and 95.

Canonical representations of Lie groups (including the Poincaré and Galileo groups) are studied in detail in Refs. 96 and 97. We give briefly the results needed to formulate the classical variant of Dirac's problem.

Suppose the Lie group G_r is realized by certain transformations of the phase space generated by a set of operators of the form

$$X_\alpha=\sum_m\left(\xi_\alpha^m\frac{\partial}{\partial q^m}+\eta_\alpha^m\frac{\partial}{\partial p^m}\right)\quad\alpha=1,\ \ldots,\ r. \tag{24}$$

If these transformations are canonical, i.e., they preserve the Hamiltonian structure of the equations of motion

$$\dot{q}^m=\partial H/\partial p_m;\ \dot{p}_m=-\partial H/\partial q^m, \tag{25}$$

then there exist functions $Y_\alpha(q,p)$, called *canonical generators*, such that

$$\xi_\alpha^m=-\partial Y_\alpha/\partial p_m;\ \eta_\alpha^m=\partial Y_\alpha/\partial q^m. \tag{26}$$

Then the operators X_α can be written in the form

$$X_\alpha=\sum_m\left(\frac{\partial Y_\alpha}{\partial q^m}\frac{\partial}{\partial p_m}-\frac{\partial Y_\alpha}{\partial p_m}\frac{\partial}{\partial q^m}\right)\equiv\{Y_\alpha,\ \ldots\}, \tag{27}$$

where for any pair of functions $f(q,p)$ and $g(q,p)$

$$\{f,\ g\}=\sum_m\left(\frac{\partial f}{\partial q^m}\frac{\partial g}{\partial p_m}-\frac{\partial f}{\partial p_m}\frac{\partial g}{\partial q^m}\right) \tag{28}$$

is the Poisson bracket. From the relations (4) for the canonical generators Y_α there follow the classical commutation relations (in terms of the Poisson brackets)

$$\{Y_\alpha,\ Y_\beta\}=c_{\alpha\beta}^\gamma Y_\gamma+d_{\alpha\beta},\ \alpha,\ \beta=1,\ \ldots,\ r, \tag{29}$$

where $d_{\alpha\beta}$ is a set of constants that satisfy a number of relations.

If the transformations generated by Y_α are to be symmetries of the Hamilton equations (25), it is sufficient (and, with certain reservations, necessary) for Y_α to be integrals of the motion,[98] i.e., to satisfy the relations

$$\partial Y_\alpha/\partial t+\{Y_\alpha,\ H\}=0. \tag{30}$$

In the case of the Poincaré group, we can without loss

[6)] In the case of a single particle, to obtain the ordinary Lorentz transformations of points of the world line γ_a we must in (17) set $\omega_j^L=-c^{-2}x_{aj}$, i.e., $R_j=x_{aj}$.

of generality set $d_{\alpha\beta}=0$ for all α and β.[56,63,97] Then the relations (29) for the group $\mathscr{P}$ take the form

$$\{P_i,\ H\}=0;\ \{J_i,\ H\}=0;\ \{P_i,\ P_j\}=0; \tag{31}$$

$$\{J_i,\ P_j\}=\varepsilon_{ijk}P_k;\ \{J_i,\ J_j\}=\varepsilon_{ijk}J_k;\ \{J_i,\ K_j\}=\varepsilon_{ijk}K_k; \tag{32}$$

$$\{K_i,\ H\}=P_i;\ \{K_i,\ P_j\}=\delta_{ij}H/c^2;\ \{K_i,\ K_j\}=-\varepsilon_{ijk}J_k/c^2, \tag{33}$$

where we have used the notation

$$H=-Y_0^T;\ P_i=Y_i^T;\ J_i=Y_i^R;\ K_i=-Y_i^L. \tag{34}$$

If in (31)–(33) the canonical generators are replaced by Hermitian operators—the generators of unitary transformations—and the Poisson brackets by commutators (multiplied by $-i/\hbar$), then we obtain the commutation relations that form the basis of quantum relativistic Hamiltonian theory.

The problem of constructing canonical representations of the Poincaré group (the classical variant of Dirac's problem) consists of finding ten functions H, P_i, J_i, K_i (identified with the energy, momentum, angular momentum, and center-of-mass integral of the motion) that satisfy the commutation relations (31)–(33), i.e., it consists of finding solutions of this nonlinear system of first-order differential equations.

We note here an important difference between the Hamiltonian and Lagrangian formalisms. In the latter, the transformation properties of the configuration variables $\overset{g}{x}{}_a^i$ are purely kinematic, which makes it possible to construct a representation of the group $\mathscr{P}$ before one considers the dynamics of the system, but in the Hamiltonian approach the situation is fundamentally different. The physical meaning and, therefore, the transformation properties of the canonical coordinates cannot be specified *a priori*, since they are determined by the generating functions Y_α, which are simultaneously dynamical characteristics (integrals of the motion) of the system and essentially dependent on the presence and form of the interaction.

For a system of N noninteracting particles, the representation of $\mathscr{P}$ in $\mathbf{P}$ is the direct product of irreducible representations, and its generators are sums of single-particle generators. Such representations were studied and classified in quantum mechanics by Wigner, Bargmann, and Shirokov,[99,100] and a corresponding analysis in terms of canonical representations is given in Ref. 97. The classical results given below can also be interpreted (after symmetrization of products of noncommuting operators) in terms of quantum mechanics.

A system of noninteracting spinless particles corresponds to the solution[24,63,68,97]

$$\mathbf{P}=\sum_a \mathbf{p}_a,\ \text{(a)};\ \mathbf{J}=\sum_a \mathbf{J}_a=\sum_a \mathbf{q}_a\times\mathbf{p}_a;\ \text{(b)} \tag{35}$$

$$H_f=\sum_a H_a=\sum_a \sqrt{m_a^2c^4+c^2p_a^2}; \tag{36}$$

$$\mathbf{K}_f=\sum_a \mathbf{K}_a=\sum_a(-t\mathbf{p}_a+H_a\mathbf{q}_a/c^2). \tag{37}$$

The subscript f, which indicates the absence of interactions between the particles, is omitted from the generators $\mathbf{P}$ and $\mathbf{J}$ because these generators preserve their form irrespective of the presence of an interaction in the instant form of dynamics that we are using.[24,68,63] If the particles have spins, which can be introduced phenomenologically in classical mechanics, the expressions (35b) and (37) are modified as follows[67,68,97]:

$$\mathbf{J}=\sum_a(\mathbf{q}_a\times\mathbf{p}_a+\mathbf{s}_a); \tag{38}$$

$$\mathbf{K}_f=\sum_a\left(-t\mathbf{p}_a+\frac{1}{c^2}H_a\mathbf{q}_a-\frac{\mathbf{s}_a\times\mathbf{p}_a}{m_ac^2+H_a}\right). \tag{39}$$

The important question of the connection between the Poincaré and Galileo groups, and also their representations, was investigated by Inönü and Wigner.[101] Without going into the mathematical aspects of the transition from representations of the group $\mathscr{P}$ to the Galileo group $\mathscr{G}$, we note that the commutation relations for canonical (Hermitian) generators of the group $\mathscr{G}$ can be formally obtained from (31)–(33) by setting $c\to\infty$. At the same time, for H and $\mathbf{K}$ we must write

$$H\approx Mc^2+H^{(0)};\ \mathbf{K}\approx\mathbf{K}^{(0)}, \tag{40}$$

where $M=\sum_a m_a$ is the rest mass of the nonrelativistic system of particles, and the expressions for $\mathbf{P}$ and $\mathbf{J}$ are kept in the form (35) [or (35a) and (38)]. The relations (31) and (32) do not change their form as $c\to\infty$ (except for the replacement of H by $H^{(0)}$ and $\mathbf{K}$ by $\mathbf{K}^{(0)}$), and instead of (33) we have

$$\{K_i^{(0)},\ H^{(0)}\}=P_i,\ \{K_i^{(0)},\ P_j\}=\delta_{ij}M,\ \{K_i^{(0)},\ K_j^{(0)}\}=0. \tag{41}$$

We can arrive at the same results by investigating the Galileo group itself, in the framework of which M is a neutral element of the Lie algebra of the group $\mathscr{G}$.[96,56] Note that in quantum mechanics the need to consider projective representations of the Galileo group is related to the presence in the commutation relations of terms of this type (see Refs. 102–104).

Relativistic equations of motion of Newtonian type. The Newtonian formalism in the relativistic theory of direct interactions consists of postulating equations of motion of the system of particles in the form

$$\ddot{x}_a^i-\mu_a^i(x,\ \dot{x},\ t)=0;\ x=\{x_b^i(t)\};\ \dot{x}=\{\dot{x}_b^i(t)\}. \tag{42}$$

Poincaré invariance of such an approach requires that the system of equations (42) admit a representation of the group $\mathscr{P}$ in the configuration space of the system of particles with the generators (14). This condition is expressed by the equation

$$X_\alpha[\ddot{x}_a^i-\mu_a^i(x,\ \dot{x},\ t)]|_\mu=0, \tag{43}$$

where the symbol $|_\mu$ indicates that the expression (43) must be calculated with allowance for (42). Substituting in (43) the expressions (16), (22), and (23), we find for the functions μ_a^i the system of equations

$$\frac{\partial\mu_a^i}{\partial t}=0;\ \sum_b\frac{\partial\mu_a^i}{\partial x_b^j}=0; \tag{44}$$

$$\sum_b\varepsilon_{jkl}\left(x_b^k\frac{\partial\mu_a^i}{\partial x_b^l}+\dot{x}_b^k\frac{\partial\mu_a^i}{\partial\dot{x}_b^l}\right)=\varepsilon_{jnl}\mu_a^n; \tag{45}$$

$$\sum_b\frac{\partial\mu_a^i}{\partial\dot{x}_b^j}+\frac{1}{c^2}\left\{\sum_b\left[r_{ab}^j\left(\dot{x}_b^k\frac{\partial\mu_a^i}{\partial x_b^k}+\mu_b^k\frac{\partial\mu_a^i}{\partial\dot{x}_b^k}\right)-\dot{x}_b^k\dot{x}_b^j\frac{\partial\mu_a^i}{\partial\dot{x}_b^k}\right]+2\mu_a^i\dot{x}_a^j+\mu_a^j\dot{x}_a^j\right\}=0. \tag{46}$$

In accordance with (44) and (45), μ_a^i must be translationally invariant (containing only the relative coordinates $r_{ab}^i=x_a^i-x_b^i$ of the particles) components of 3-

vectors with no explicit time dependence. These conditions apply equally to relativistic and nonrelativistic equations of motion. Equations (46), which are called the Currie–Hill conditions, express the conditions of form invariance of (42) with respect to Lorentz transformations (with conversion to the new simultaneity). They were first found in Refs. 105 and 106 as necessary conditions of Poincaré invariance; Bel[107] established their sufficiency (see also the reviews of Refs. 108 and 36). The main difficulties of this approach are due to the nonlinearity of the system (46). We shall not consider approximate methods of solving (46) (see Refs. 37, 38, and 109–112), since it appears to us more convenient to use the Lagrangian formalism, which makes it possible to find not only approximately Lorentz-covariant equations of motion but also the corresponding conservation laws. With regard to exact solutions of these equations, all that is known is the general solution for a system of two particles in the one-dimensional case.[113]

Relativistic Lagrange theory. The Lagrangian formalism is based on the variational principle[82]

$$\delta S=\delta\int_{t_1}^{t_2}L(t,\ x,\ \dot{x},\ \ldots,\ \overset{s}{x})\,dt=0, \tag{47}$$

in which *a priori* no restrictions are placed on the highest order s of the derivatives contained in the Lagrangian. The equations of the extremals of the functional (47) (the Euler–Lagrange equations) have the form

$$\mathscr{L}_{at}L\equiv\sum_{\sigma=0}^{\infty}\left(-\frac{d}{dt}\right)^{\sigma}\frac{\partial L}{\partial\overset{\sigma}{x}{}_a^i}=0. \tag{48}$$

The condition of invariance of a relativistic Lagrange theory with respect to the group G_r leads to the system of equations (see Refs. 85 and 82)[7)]

$$X_\alpha L+d\Omega_\alpha/dt=0, \tag{49}$$

where the operators X_α are defined in (14). In (49), Ω_α are certain functions that satisfy the relations

$$X_\alpha\Omega_\beta-X_\beta\Omega_\alpha=c_{\alpha\beta}^{\gamma}\Omega_\gamma, \tag{50}$$

which express the integrability conditions for the system (49).

The problem of constructing the general form of a Lagrangian L compatible with the requirement of covariance of the system of equations (48) with respect to G_r consists of integrating the system of equations (49) and (50).

In Ref. 82, the following important fact is established: To obtain the general solution L of the system (49) it is sufficient to take as the set of functions Ω_α any particular solution of the system (50), since any other solution leads to a Lagrangian which differs from L by a total derivative, i.e., gives the same Euler equations. In particular, one can use the trivial solution of (50), $\Omega_\alpha=0,\alpha=1,\ldots,r$, though this is not always convenient.

We write down the system of equations (49) for the group $\mathscr{P}$. Setting $\Omega_i^{\mathrm{T}}=0,\Omega_i^{\mathrm{R}}=0$ $(i=1,2,3)$, for the Euclid group we obtain the equations

$$-X_i^{\mathrm{T}}L\equiv\sum_a\frac{\partial L}{\partial x_a^i}=0; \tag{51}$$

$$-X_i^{\mathrm{R}}L\equiv\sum_a\sum_{\sigma=0}^{\infty}\varepsilon_{ijk}\overset{\sigma}{x}{}_a^j\frac{\partial L}{\partial\overset{\sigma}{x}{}_{ak}}=0, \tag{52}$$

which express the conditions of translational and rotational invariance of the Lagrangian L. For time shifts and boosts,

$$X_0^{\mathrm{T}}L+d\Omega_0^{\mathrm{T}}/dt\equiv dL/dt-\partial L/\partial t+d\Omega_0^{\mathrm{T}}/dt=0; \tag{53}$$

$$X_i^{L}L+\frac{d\Omega_i^{L}}{dt}\equiv\sum_a\sum_{\sigma=0}^{\infty}\left\{\left[\frac{d^\sigma}{dt^\sigma}\left(-t\delta_i^j+\frac{1}{c^2}\,x_{ai}v_a^j\right)\right]\frac{\partial L}{\partial\overset{\sigma}{x}{}_a^j}+\overset{\sigma+1}{x}{}_a^j\frac{\partial\Omega_i^L}{\partial\overset{\sigma}{x}{}_a^j}\right\}=0. \tag{54}$$

The choice $\Omega_0^{\mathrm{T}}=-L$ in (53) leads to the ordinary condition

$$\partial L/\partial t=0, \tag{55}$$

which expresses the invariance of the Lagrangian description of closed systems with respect to time translations.

The integrability conditions (50) reduce to the following system of equations for the three functions Ω_i^{L}:

$$\frac{\partial\Omega_i^L}{\partial t}=0;\ \sum_a\sum_{\sigma=0}^{\infty}\varepsilon_{ihl}\overset{\sigma}{x}{}_a^h\frac{\partial\Omega_j^L}{\partial\overset{\sigma}{x}{}_a^i}=-\varepsilon_{ijn}\Omega_n^L; \tag{56}$$

$$\sum_a\frac{\partial\Omega_j^L}{\partial x_a^i}=-\frac{1}{c^2}\,\delta_{ij}L;\ X_i^L\Omega_j^L-X_j^L\Omega_i^L=0. \tag{57}$$

Equations (56) mean that Ω_i^{L} must be the components of a 3-vector that does not depend explicitly on t. The first of the relations (57) precludes the choice $\Omega_i^{\mathrm{L}}=0$.

Thus, the problem of finding relativistic Lagrangians L reduces to the solution of the system of equations (54) and (57), in which L is a time-independent translationally invariant (i.e., containing the coordinates of the particles only in the form $r_{ab}^i=x_a^i-x_b^i$) scalar function, and Ω_i^{L} are functions which do not depend explicitly on the time and are the components of a three-dimensional vector.

This system of equations has the solution

$$L_f=\sum_a L_a=-\sum_a m_ac^2\sqrt{1-v_a^2/c^2}; \tag{58}$$

$$\Omega_{if}^L=\sum_a\Omega_{ia}^L=\sum_a m_ax_{ai}\sqrt{1-v_a^2/c^2}, \tag{59}$$

which describes a system of noninteracting particles.

If we introduce functions U and Ψ_i which vanish in the absence of interaction, setting

$$L=L_f-U;\ \Omega_i^L=\Omega_{if}^L+\Psi_i, \tag{60}$$

then from (51)–(57) we obtain for them the system of equations

$$\frac{\partial U}{\partial t}=0;\ \sum_a\frac{\partial U}{\partial x_a^i}=0;\ \sum_a\sum_{\sigma=0}^{\infty}\varepsilon_{ijk}\overset{\sigma}{x}{}_a^j\frac{\partial U}{\partial\overset{\sigma}{x}{}_{ak}}=0; \tag{61}$$

$$\frac{\partial\Psi_i}{\partial t}=0;\ \sum_a\sum_{\sigma=0}^{\infty}\varepsilon_{ijk}\overset{\sigma}{x}{}_a^j\frac{\partial\Psi_n}{\partial\overset{\sigma}{x}{}_{ak}}=-\varepsilon_{inl}\Psi^l; \tag{62}$$

$$X_i^LU=d\Psi_i/dt; \tag{63}$$

[7)] We take $t'=t$, as was assumed in (22) and (23) for the Poincaré group.

$$\sum_a \frac{\partial \Psi_i}{\partial x_a^j} = -\frac{1}{c^2}\delta_{ij} U; \tag{64}$$

$$X_i^L \Psi_j - X_j^L \Psi_i = 0. \tag{65}$$

The main difficulty in solving the system (61)–(65) is due to the fact that the space **E**, which is the domain of definition of U and Ψ_i, is infinite dimensional. However, the linearity of this system should be noted.

In accordance with Noether's theorem,[81,114-116] there are r conservation laws associated with the symmetry group G_r of the variational principle; for Lagrangians with higher derivatives, they have the form[117]

$$\frac{dG_\alpha}{dt} \equiv \frac{d}{dt}\Big\{\sum_a \sum_{\sigma=0}^{\infty} \sum_{\lambda=1}^{\infty} \xi_{\alpha a}^{(\lambda-1)i} \Big(-\frac{d}{dt}\Big)^\sigma \frac{\partial L}{\partial x_{ai}^{\sigma+\lambda}} + \Omega_\alpha\Big\} = 0. \tag{66}$$

For the ten integrals of the motion (the energy E, momentum **P**, angular momentum **J**, and center-of-mass integral of the motion **K**) corresponding to Poincaré invariance, we obtain

$$E \equiv G_0^T = \sum_a \sum_{\sigma=0}^{\infty} \sum_{\lambda=1}^{\infty} x_{ai}^{\lambda} \Big(-\frac{d}{dt}\Big)^\sigma \frac{\partial L}{\partial x_{ai}^{\sigma+\lambda}} - L; \tag{67}$$

$$P_i \equiv -G_i^T = \sum_a \sum_{\sigma=0}^{\infty} \Big(-\frac{d}{dt}\Big)^\sigma \frac{\partial L}{\partial x_{ai}^{\sigma+1}}; \tag{68}$$

$$J_i \equiv -G_i^R = \sum_a \sum_{\sigma=0}^{\infty} \sum_{\lambda=1}^{\infty} \varepsilon_{ijk} x_{aj}^{\lambda-1} \Big(-\frac{d}{dt}\Big)^\sigma \frac{\partial L}{\partial x_{ak}^{\sigma+\lambda}}, \tag{69}$$

$$K_i \equiv G_i^L = \sum_a \sum_{\sigma=0}^{\infty} \sum_{\lambda=1}^{\infty} \Big[\frac{d^{\lambda-1}}{dt^{\lambda-1}}\Big(-t\delta_i^j + \frac{1}{c^2} x_{ai} v_a^j\Big)\Big] \times \Big(-\frac{d}{dt}\Big)^\sigma \frac{\partial L}{\partial x_{aj}^{\sigma+\lambda}} + \Omega_i^L. \tag{70}$$

Since the free-particle Lagrangian (58) does not depend on the higher derivatives, the transition to the Hamiltonian formalism for systems without interaction can be realized in the usual manner (by a Legendre transformation). The corresponding integrals of the motion $E, \mathbf{P}_t, \mathbf{J}_t, \mathbf{K}_t$ in the canonical variables take the form (35)–(37), as is readily verified.

Relativistic Hamiltonian theory. No-interaction theorem. In the Hamiltonian formalism, as we have already said above, the main problem is the finding of ten canonical generators $H, \mathbf{P}, \mathbf{J}$, and **K** that satisfy the relations (31)–(33) and are integrals of the motion [Eq. (30)].

If we introduce in the Hamiltonian H and the boost generator **K** an interaction in the form of the functions U and Ψ, setting[8)]

$$H = H_f + U; \quad \mathbf{K} = \mathbf{K}_f + \Psi, \tag{71}$$

then from the commutation relations (31)–(33) for the functions U and Ψ_i we obtain the system of equations

$$\{P_i, U\} = 0; \ \{J_i, U\} = 0, \tag{72}$$

$$\{J_i, \Psi_j\} = \varepsilon_{ijk}\Psi^k, \tag{73}$$

$$\{K_{if}, U\} + \{\Psi_i, H_f\} + \{\Psi_i, U\} = 0; \tag{74}$$

$$\{\Psi_i, P_j\} = \delta_{ij} U/c^2; \tag{75}$$

$$\{K_{if}, \Psi_j\} + \{\Psi_i, K_{jf}\} + \{\Psi_i, \Psi_j\} = 0. \tag{76}$$

We note the following circumstance. It was established in Ref. 9 that three of the commutation relations, namely, those that contain H in the Poisson bracket [the first and second in (31) and the first in (33)] can be regarded as a consequence of the remainder. However, in the method of successive approximations based on expansions in c^{-2}, this conclusion, as was pointed out in Ref. 118, ceases to hold in each fixed approximation, since its proof would require consideration of the following approximation. Therefore, when finding approximate expressions for U and Ψ_i, we must use the complete set of equations (72)–(76).

Using (35a), we can rewrite the commutation relations containing P_i in the form

$$\sum_a \frac{\partial U}{\partial q_a^i} = 0; \ \sum_a \frac{\partial \Psi_i}{\partial q_a^j} = \frac{1}{c^2}\delta_{ij} U. \tag{77}$$

The first of them means that the canonical coordinates can occur in U only through $q_{ab}^i = q_a^i - q_b^i$. It also follows from (72) and (73) that U is a 3-scalar, and Ψ_i are the components of a 3-vector.

If we bear in mind that for the functions U and Ψ_i in the canonical formalism the equations $\partial U/\partial t = 0$ and $\partial \Psi_i/\partial t = 0$ hold by virtue of (30), then by comparing the systems of equations (61)–(65) and (72)–(76) we readily establish their mutual correspondence and formal analogy. However, this analogy, being a reflection of the conditions of symmetry with respect to the same group $\mathscr{P}$, is only superficial, since from the point of view of mathematical structure there is a profound difference between these equations: In the Lagrangian formalism, the equations are linear but expressed by means of operators that act on the infinite-dimensional space **E**; in the Hamiltonian approach, the domain of definition of the unknown functions is the $6N$-dimensional phase space **P**, but the equations are nonlinear. This is related to the difference discussed above between the physical meaning and transformation properties of the variables used in the two formalisms. We shall see below that in the post-Newtonian approximation, in which we are primarily interested, these systems of equations are actually identical.

We now consider the additional conditions on the system (72)–(76), which give physical meaning to the solutions of Dirac's problem. The strongest of them, satisfied only in the post-Newtonian approximation (if we discount very special cases in the higher approximations; see Ref. 111), is the canonical condition of invariance of a world line. In the Hamiltonian formalism it was written down for the first time by Pryce,[119] discussed by Thomas,[120] and analyzed in detail in connection with the problem of constructing a relativistic Hamiltonian theory by Currie, Jordan, and Sudarshan.[26,63]

We assume that the canonical variables q_a^i are the covariant coordinates of the particle positions, i.e., that under an infinitesimal Lorentz transformation Λ they transform in accordance with (19). Requiring that this transformation correspond to a canonical representation of the group $\mathscr{P}$,

$$\delta q_a^i = \{q_a^i, K_j\}\delta V^j, \tag{78}$$

[8)] Although the functions U and Ψ_i are defined on **P**, we shall use for them the same notation as in (60), in which **E** is the domain of definition. In the general case, this is justified by their mutual correspondence, and in the post-Newtonian approximation they actually coincide, as will be seen in Sec. 3.

we obtain the equation

$$\{q_{ai}, K_j\} = -t\delta_{ij} + \frac{1}{c^2} q_{aj}\{q_{ai}, H\}, \tag{79}$$

which expresses the condition of invariance of a world line in the canonical formalism. Using the equation $\{q_{ai}, K_j\} = \partial K_j/\partial p_{ai}$, we readily obtain from (79) (under the assumption $q_{ai} \neq q_{bi}, a \neq b$) the equation

$$\partial^2 H/\partial p_{ai}\partial p_{bj} = 0 \quad (b \neq a), \tag{80}$$

from which it can be seen that the Hamiltonian must have the structure

$$H = \sum_a H_a(q, p_a); \tag{81}$$

here, each term of the sum depends on the momentum of only one particle. As is shown in Refs. 26 and 63 for $N=2$, in Ref. 64 for $N=3$, and in Ref. 65 for arbitrary N, Eq. (81) in conjunction with the commutation relations of the algebra $A\mathscr{P}$ leads to the conclusion that $H = H_t$, i.e., $U = 0$. This is the well-known *no-interaction theorem*, which states that if the canonical coordinates are covariant and if canonical representations of the Poincaré group correspond to transitions between inertial frames of reference, then the only solution of Dirac's problem is the set of generators $H_t, \mathbf{K}_t, \mathbf{P}, \mathbf{J}$ corresponding to noninteracting particles. A detailed discussion of the various formulations of this theorem can be found in Ref. 121.

Since one can go over from the Hamiltonian formalism by means of a Legendre transformation to the standard (without higher derivatives) Lagrangian method, this result also indicates that there do not exist Lagrangians which depend only on the covariant coordinates of the particles and their first derivatives capable of describing relativistic systems with interaction.[122,10,82]

Thus, in a relativistic Hamiltonian theory one cannot use covariant particle coordinates as canonical variables. One can only require asymptotic covariance of the canonical coordinates, i.e., covariance in the limiting case when the particles are infinitely far from one another. This property of the coordinates q_a^i is intimately related to the condition of *separately of interactions* (see Sec. 1). In terms of Hamiltonian theory, it can be expressed by the following requirement[68,70,71]: For any division of the system σ into nonintersecting subsystems $\sigma_\alpha, \alpha = 1, \ldots, n, n \leq N$, the limits of the generators of canonical (in quantum mechanics, unitary) representations of the group $\mathscr{P}$ for the system σ go over into the sums of the generators of the subsystems σ_α in the limit when the distances between these subsystems tend to infinity. In the instant form of dynamics, separability requires that in this limit

$$U \to \sum_\alpha U_\alpha; \quad \Psi \to \sum_\alpha \Psi_\alpha, \tag{82}$$

where the quantities with subscript α contain only the variables of the particles in the subsystem σ_α. It follows from this in particular that when all the particles of the system are infinitely far from each other the functions U and Ψ must vanish. The precise mathematical formulation of the conditions of separability, long a problem in the relativistic theory of direct interactions (see Refs. 24, 68, 70, and 123), was given by Sokolov.[71]

The conditions (82) are very important restrictions on the functions (in quantum theory, the operators) U and Ψ, which describe the interactions of the relativistic particles. They make it impossible to obtain a direct generalization of the well-known Bakamjiam–Thomas model,[67] the solution of the two-particle Dirac problem, to systems with a number of particles $N > 2$. In the approximate approach based on an expansion in powers of c^{-2}, separability of the quantum interaction Hamiltonian was obtained for the first time by Shirokov *et al.*,[5] although in their papers the concept of separability was not formulated. In the exact theory, quantum-mechanical generators of the group $\mathscr{P}$ with separable interaction for $N=3$ were constructed by Coester[70] and Sokolov[74] and for a system with an arbitrary number of particles by Sokolov.[73,62]

It is important to emphasize that it is the condition of separability of the interaction, combined with the commutation relations of the algebra $A\mathscr{P}$, that ensures relativistic invariance of the quantum description of scattering processes.[71]

To the additional conditions on Dirac's problem, we can add the requirement that a nonrelativistic limit $U^{(0)}$ of the function U exist. It is not a necessary consequence of any physical considerations, and interaction Hamiltonians which vanish in the nonrelativistic limit are also admissible. The condition formulates an *a priori* restriction on the class of solutions of Dirac's problem. In what follows, we shall also use expansions of the generators of $\mathscr{P}$ in c^{-2}. As we shall see below, they give entirely sensible results, although the problem of the existence of exact solutions analytic in c^{-2} is mathematically very tricky, and we shall not discuss it.

3. APPROXIMATELY RELATIVISTIC INTERACTIONS OF PARTICLES

We shall here use the conditions formulated above for Poincaré invariance of the description of a system of particles to find the general form of quasirelativistic interaction Lagrangians and Hamiltonians (we shall also use for them the general expression "quasirelativistic potentials"); more precisely, we find the general form of the relativistic corrections of order c^{-2} to a given nonrelativistic potential. We shall concentrate our main attention on the post-Newtonian approximation, for which the mathematical formalism and physical interpretation are very close to nonrelativistic theory. First, we discuss the following general question: What must be the transformation properties of approximately relativistic equations of motion?

Approximate Lorentz invariance. The concept of approximate Lorentz invariance was used long ago (see in particular Refs. 13, 86, 68, and 124), though without a discussion of its content. It was formulated and justified physically in Refs. 125 and 126 and considered in connection with quantum mechanics in Refs. 127 and 128.

Let σ be a physical object that is a system of N inter-

acting particles whose velocities $\mathbf{v}_a$ $(a=1,\ldots,N)$ in a fixed inertial frame of reference S satisfy the condition $v_a \equiv |\mathbf{v}_a| \ll c$; this condition makes it possible to use an expansion in v_a^2/c^2 in the equations of mechanics and for a given degree of accuracy to restrict oneself in the series to a finite number of terms. We denote by n the maximal power of the relativistic corrections $(v_a^2/c^2)^n$ taken into account in the frame S in the mechanical equations. Further, let Σ be the class of inertial frames of reference S′ with velocities $\mathbf{V}$ for which $|\mathbf{V}| \leq v_a$. Then in an arbitrary system S′ belonging to Σ, the order of magnitude of the ratios $v_a'^2/c^2$ is equal to the order of magnitude of the ratios v_a^2/c^2, so that the power of the relativistic corrections that must be taken into account in S′ is also n. This means that all frames of reference $S' \in \Sigma$ are equivalent not only in the sense of the relativity principle but also from the point of view of the description of the physical object σ. In other words, the approximately relativistic equations of motion must have the same form in all these frames of reference. Since we ignore the terms $(v_a^2/c^2)^k, k>n$ in the equations of motion of the particles of σ, it follows from the condition $V \leq v_a$ that in the transformation formulas of the physical quantities corresponding to the transitions $S \to S'$ we must use the expansions in powers of $\beta^2 \equiv V^2/c^2$, ignoring terms of order $\beta^{2k}, k>n$. Thus, the approximately relativistic equations of motion taking into account relativistic effects of order n must, by virtue of the relativity principle, be invariant with respect to approximate Lorentz transformations of the same order.

If the frame of reference S* moves relative to S with velocity $V^* \approx c$, then the velocities v_a^* of the particles of σ with respect to S* will be near the velocity of light $(v_a^* \approx V^* \approx c)$. Therefore, in the equations of motion of these particles written down in the frame S*, we cannot restrict ourselves to the approximation used in S (or in $S' \in \Sigma$), i.e., the form of the equations of motion must change on the transition $S \to S^*$. Since such a transition must be described by exact Lorentz transformations, we conclude that approximately relativistic equations (in particular, nonrelativistic equations) cannot be invariant with respect to exact Lorentz transformations. This does not contradict the relativity principle, since one can imagine an object σ^* (identical in its intrinsic nature to the object σ) whose particles move with respect to the system S* with velocities v_a^{**} equal in order of magnitude to the velocities v_a with respect to S considered above. The equations of motion of the particles of σ^* in the frame S* will have the same form as the equations for σ in S.

It follows from our arguments that the transformation of the equations of the approximately relativistic theory corresponding to the transition $S \to S' \in \Sigma$ must be made by means of a one-time expansion in powers of c^{-2} of both these equations and the transformation formulas. The order of magnitude of the individual terms in the transformed equations is determined by the total power of the parameter c^{-2} (Refs. 86, 124, 129, and 130).

In the problem of finding the general form of quasi-relativistic potentials, the above arguments do not have fundamental significance, since the problem is solved by integrating a system of differential equations corresponding to an *infinitesimal* Lorentz transformation which contains the parameters V_i only linearly. However, in all cases of finite Lorentz transformations (for example, for the transition between the center-of-mass frame and the laboratory frame) they must be taken into account.

Conditions of approximate Poincaré invariance of an interaction Lagrangian. As we have already noted in Sec. 2, the finding of a relativistic interaction Lagrangian U reduces to the solution of the system of equations (63)–(65), in which U is a translationally invariant 3-scalar which does not depend explicitly on the time, and Ψ_i are the components of a 3-vector, also independent of the time. Assuming that there exist solutions of this system analytic in c^{-2}, we represent these functions in the form of the series[9)]

$$U = \sum_{n=0}^{\infty} c^{-2n} \overline{U}^{(n)} \equiv \sum_{n=0}^{\infty} U^{(n)}; \tag{83}$$

$$\Psi_i = \sum_{n=0}^{\infty} c^{-2n} \overline{\Psi}_i^{(n)} \equiv \sum_{n=0}^{\infty} \Psi_i^{(n)}, \tag{84}$$

where $U^{(n)}$ are translationally invariant 3-scalars. Then from (63)–(65) for the functions $U^{(n)}$ and $\Psi_i^{(n)}$ we obtain [using the second equation of (61)] the infinite hierarchy of equations

$$\sum_a \left\{ \frac{\partial U^{(n)}}{\partial v_a^i} + \sum_{s=0}^{\infty} \left[\overset{s+1}{x}_{aj} \frac{\partial \Psi_i^{(n)}}{\partial \overset{s}{x}_{aj}} - \frac{1}{c^2} \sum_{l=0}^{s} \binom{s}{l} \overset{l}{x}_{ai} \overset{s-l+1}{x}_{aj} \frac{\partial U^{(n-1)}}{\partial \overset{s}{x}_{aj}} \right] \right\} = 0; \tag{85}$$

$$\sum_a \frac{\partial \Psi_i^{(n)}}{\partial x_a^j} = \frac{1}{c^2} \delta_{ij} U^{(n-1)}; \tag{86}$$

$$\sum_a \left[\frac{\partial \Psi_j^{(n)}}{\partial v_a^i} - \frac{\partial \Psi_i^{(n)}}{\partial v_a^j} + \frac{1}{c^2} \sum_{s=0}^{\infty} \sum_{l=0}^{s} \binom{s}{l} \overset{s-l+1}{x}_{ak} \left(\overset{l}{x}_{aj} \frac{\partial \Psi_i^{(n-1)}}{\partial \overset{s}{x}_{ak}} - \overset{l}{x}_{ai} \frac{\partial \Psi_j^{(n-1)}}{\partial \overset{s}{x}_{ak}} \right) \right] = 0$$

$$n = 0, 1, \ldots; \; U^{(-1)} \equiv 0; \; \Psi_i^{(-1)} \equiv 0; \; \binom{s}{l} \equiv \frac{s!}{l!\,(s-l)!}. \tag{87}$$

In the successive solution of the system of equations (85)–(87), we shall use the result of Ref. 82 that we have already mentioned in Sec. 2; this will enable us, without loss of generality, to substitute in Eq. (85) for $U^{(n)}$ an arbitrary particular solution $\Psi_i^{(n)}$ of Eqs. (86) and (87).

In the zeroth approximation $(n=0)$, we have the equations

$$\sum_a \left(\frac{\partial U^{(0)}}{\partial v_{ai}} + \sum_{s=0}^{\infty} \overset{s+1}{x}_{aj} \frac{\partial \Psi_i^{(0)}}{\partial \overset{s}{x}_{aj}} \right) = 0; \tag{88}$$

$$\sum_a \frac{\partial \Psi_i^{(0)}}{\partial x_{aj}} = 0; \tag{89}$$

$$\sum_a \left(\frac{\partial \Psi_i^{(0)}}{\partial v_a^j} - \frac{\partial \Psi_i^{(0)}}{\partial v_a^i} \right) = 0. \tag{90}$$

Choosing as a particular solution of the homogeneous

[9)] The powers of the parameter c^{-1} are even because of the condition of invariance with respect to time reversal, which in its turn is due to the absence in the theory of direct interactions of real radiation processes.

system (89) and (90)

$$\Psi_i^{(0)}=0, \tag{91}$$

we obtain for $U^{(0)}$ the well-known condition of Galileo invariance:

$$\sum_a \frac{\partial U^{(0)}}{\partial v_a^i}=0. \tag{92}$$

Its general solution with allowance for the fact that the nonrelativistic equations of motion must be second-order equations has the form

$$U^{(0)}=U^{(0)}(\mathbf{r},\ \mathbf{v});\ \mathbf{r}=\{\mathbf{r}_{ab}\};\ v=\{\mathbf{v}_{ab}\}, \tag{93}$$

where $\mathbf{v}_{ab}=d\mathbf{r}_{ab}/dt=\mathbf{v}_a-\mathbf{v}_b$. As a rule, $U^{(0)}$ depends only on the distances between the particles:

$$U^{(0)}=u_0(r). \tag{93a}$$

However, in individual cases (in particular, in some phenomenological models in nuclear physics[131,132]) one uses nonrelativistic potentials that also depend on the velocities, which is equivalent to nonlocal potentials in quantum mechanics (see, for example, Ref. 133).

Using (91) and (93), we obtain from (85)–(87) in the first approximation the system of equations

$$\sum_a \Big\{\frac{\partial U^{(1)}}{\partial v_a^i}+\sum_{s=0}^{\infty} x_{aj}^{s+1}\frac{\partial \Psi_i^{(1)}}{\partial x_{aj}^s}-\frac{1}{c^2}\Big[x_{ai}v_{aj}\frac{\partial U^{(0)}}{\partial x_{aj}}+(v_{ai}v_{aj}+x_{ai}\dot{v}_{aj})\frac{\partial U^{(0)}}{\partial v_{aj}}\Big]\Big\}=0; \tag{94}$$

$$\sum_a \frac{\partial \Psi_i^{(1)}}{\partial x_a^j}=\frac{1}{c^2}\delta_{ij}U^{(0)}; \tag{95}$$

$$\sum_a\Big(\frac{\partial \Psi_i^{(1)}}{\partial v_a^j}-\frac{\partial \Psi_j^{(1)}}{\partial v_a^i}\Big)=0, \tag{96}$$

for which the scheme of solution is as follows. From (95) and (96) we find (for given $U^{(0)}$) an arbitrary particular solution for $\Psi_i^{(1)}$, after which the general solution of the system (94) can be written in the form $U^{(1)}=U_{\mathrm{P}}^{(1)}+u_1$, where $U_{\mathrm{P}}^{(1)}$ is a particular solution of this system and u_1 is the general solution of an equation of the form (92) containing the factor c^{-2}.

Suppose that in the nonrelativistic limit the interaction is described (as is usually assumed) by a superposition of two-body symmetric potentials:

$$U^{(0)}=\sum_{a<b}\sum U_{ab}^{(0)}(\mathbf{r}_{ab},\ \mathbf{v}_{ab});\ U_{ab}^{(0)}=U_{ba}^{(0)}. \tag{97}$$

We denote by $\Psi_{ab\,i}^{(1)}$ an arbitrary particular solution of the system of equations (95) and (96) for the two-particle system σ_{ab} consisting of particles a and b. Then the functions

$$\Psi_i^{(1)}=\sum_{a<b}\sum \Psi_{abi}^{(1)} \tag{98}$$

are solutions of the given system of equations for arbitrary N, and the corresponding general solution of the system (94) will have the form

$$U^{(1)}=\sum_{a<b}\sum U_{ab}^{(1)}+u_1. \tag{99}$$

Since u_1 is an arbitrary Galileo-invariant function, by $U_{ab}^{(1)}$ one can understand a particular solution of the system (94) for σ_{ab}. However, it is more convenient to regard $U_{ab}^{(1)}$ as a general solution, i.e., as a solution containing an arbitrary two-particle Galileo-invariant function u_{1ab}. Then in (99) we must regard u_1 as representing essentially many-particle expressions.

The question of many-particle interactions, i.e., interactions that do not reduce to two-body interactions, warrants further investigation. We emphasize right away (below, we shall discuss this question in more detail) that the very concepts of "two-particle and many-particle interactions" (or "forces") can be used only within a definite approach (Newtonian, Lagrangian, or Hamiltonian), since the manner in which an interaction is a "many-particle" interaction changes in the general case on transitions between these formalisms.

In the more general case when the nonrelativistic potential has the form of a cluster expansion (superposition of n-particle potentials),

$$U^{(0)}=\sum_{a_1<a_2}\sum U_{a_1a_2}^{(0)}+\sum_{a_1<a_2<a_3}\sum U_{a_1a_2a_3}^{(0)}+\ldots+\sum_{a_1<a_2<\ldots<a_n}\sum U_{a_1a_2\ldots a_n}^{(0)}, \tag{100}$$

its general solution can, by virtue of the linearity of the system (94)–(96), be represented by the similar expansions

$$\Psi_i^{(1)}=\sum_{k=2}^{n}\sum_{a_1<\ldots<a_k}\sum \Psi_{ia_1\ldots a_k}^{(1)}; \tag{101}$$

$$U^{(1)}=\sum_{k=2}^{n}\sum_{a_1<\ldots<a_k}\sum U_{a_1\ldots a_k}^{(1)}, \tag{102}$$

where $\Psi_{ia_1\ldots a_k}^{(1)}$ and $U_{a_1\ldots a_k}^{(1)}$ are solutions of the system (94)–(96) for $N=k$.

In what follows, we shall restrict ourselves to two-body (in terms of the Lagrangian formalism) interactions (97)–(99), when the problem of finding general expressions for $U^{(1)}$ in a system of N bodies reduces to such a two-particle problem.

General form of post-Newtonian interaction Lagrangians. We investigate in detail in the first quasirelativistic approximation the important class of interactions that in the nonrelativistic limit are described by a velocity-independent potential (93a) (post-Newtonian approximation). This case is particularly interesting in that it admits interaction Lagrangians which depend only on the coordinates and velocities. A large class of such interactions for a two-particle system was given for the first time by Breit[13] on the basis of the requirement of approximate Lorentz invariance of the action differential; this case was also considered from different points of view in Refs. 134, 135, 87, and 10 and in the framework of the general Lagrangian approach in Ref. 136.

To obtain a general solution $U^{(1)}$ of the system (94) that depends only on the coordinates and velocities, it is expedient to find first a *general* solution of the system (95) that depends *only on the coordinates*, when Eqs. (96) are satisfied trivially.[10] For a system of two particles a and b, the solution has the form

$$\Psi_{iab}^{(1)}=\frac{1}{c^2}\Big[\frac{1}{2}(x_{ai}+x_{bi})U_{ab}^{(0)}+r_{abi}\varphi_{ab}\Big], \tag{103}$$

[10]Velocity-dependent functions $\Psi_{iab}^{(1)}$ are associated with solutions of the system (9) containing an essential dependence on the accelerations, i.e., they lead to equations of motion of higher orders.

where $\varphi_{ab}=\varphi_{ab}(r_{ab})$ are arbitrary functions of the distance r_{ab}. From the natural condition $U_{ab}^{(0)}=U_{ba}^{(0)}\Psi_{iab}=\Psi_{iba}$ of symmetry of the interaction[11] it follows that $\varphi_{ab}=-\varphi_{ba}$. This last relation can be satisfied (for nonidentical particles) by means of an antisymmetric (with respect to interchange of the particles) factor constructed from the parameters of the particles (for example, their masses).

Substituting (103) in (94), we obtain the equation

$$\frac{\partial U_{ab}^{(1)}}{\partial \mathbf{v}_a}+\frac{\partial U_{ab}^{(1)}}{\partial \mathbf{v}_b}=\frac{1}{c^2}\left\{\frac{1}{2}\left[\mathbf{r}_{ab}(\mathbf{v}_a+\mathbf{v}_b)\cdot\mathbf{r}_{ab}\frac{1}{r_{ab}}\frac{dU_{ab}^{(0)}}{dr_{ab}}-(\mathbf{v}_a+\mathbf{v}_b)U_{ab}^{(0)}\right]-\mathbf{v}_{ab}\varphi_{ab}-\mathbf{r}_{ab}(\mathbf{r}_{ab}\mathbf{v}_{ab})\frac{1}{r_{ab}}\frac{d\varphi_{ab}}{dr_{ab}}\right\}. \quad (104)$$

Its general solution has the form[12]

$$U_{ab}^{(1)}=-\frac{1}{2c^2}\left\{\mathbf{v}_a\cdot\mathbf{v}_bU_{ab}^{(0)}-(\mathbf{r}_{ab}\mathbf{v}_a)(\mathbf{r}_{ab}\mathbf{v}_b)\frac{1}{r_{ab}}\frac{dU_{ab}}{dr_{ab}}+\mathbf{v}_{ab}\cdot(\mathbf{v}_a+\mathbf{v}_b)\varphi_{ab}+(\mathbf{r}_{ab}\mathbf{v}_{ab})[\mathbf{r}_{ab}\cdot(\mathbf{v}_a+\mathbf{v}_b)\frac{1}{r_{ab}}\frac{d\varphi_{ab}}{dr_{ab}}\right\}+\frac{1}{c^2}u_{1ab}, \quad (105)$$

where u_{1ab} is an arbitrary scalar Galileo-invariant function.

We note the following circumstance. In (105), we can use the identity

$$(\mathbf{v}_a+\mathbf{v}_b)\mathbf{v}_{ab}\varphi_{ab}+\mathbf{r}_{ab}\cdot(\mathbf{v}_a+\mathbf{v}_b)(\mathbf{r}_{ab}\mathbf{v}_{ab})\frac{1}{r_{ab}}\frac{d\varphi_{ab}}{dr_{ab}}=\frac{d}{dt}[\mathbf{v}_a+\mathbf{v}_b)\cdot\mathbf{r}_{ab}\varphi_{ab}]-(\dot{\mathbf{v}}_a+\dot{\mathbf{v}}_b)\cdot\mathbf{r}_{ab}\varphi_{ab}. \quad (106)$$

If we now go beyond the framework of the standard Lagrangian formalism, in which the Lagrangian depends on derivatives of not higher than first order, then the total derivative in (106), which does not affect the Euler–Lagrange equations (48), can be omitted, and the last term in (106), which contains accelerations and is Galileo-invariant, can be included in the function u_{1ab}. As a result, we arrive at a function $U^{(1)}$ of the form $U^{(1)}=U^{(1)}(\mathbf{r},\mathbf{v},\dot{\mathbf{v}})$, which requires the formalism with higher derivatives. At the same time, in view of the linear dependence on the accelerations with velocity-independent factors multiplying $\dot{\mathbf{v}}_a$, the Euler equations are of the second order and are identical to the Euler–Lagrange equations for the expression (105). Bearing in mind that the absence in the curly brackets in (105) of terms with φ_{ab} corresponds to the particular solution (103) of the system (95) for $\varphi_{ab}=0$, we conclude that, restricting ourselves to this particular solution for $\Psi_{iab}^{(1)}$ and allowing a dependence of the Lagrangians on the accelerations, we lose no generality in the solution of the system (94). This remark serves as an illustration of theorem 2 of Ref. 82, which we have already mentioned, according to which the possibility of adding to the Lagrangian the total derivative of an arbitrary function means that any particular solution of the system of equations (64) and (65) can be used for Ψ.

This circumstance must be borne in mind on the transition from the two-particle system to the case of N particles ($N\geq 3$). If it is assumed that the Galileo-invariant terms u_{1ab} contain a dependence on the accelerations of the form (106), then the expression (99) with the function $U_{ab}^{(1)}$ obtained from (105) for $\varphi_{ab}=0$ with a many-particle Galileo-invariant function u_1 (also containing an analogous dependence on the accelerations that does not lead to higher orders in the equations of motion) will be the general solution to the problem of finding the post-Newtonian corrections to the many-particle interaction potential $U^{(0)}(r)$. But if we require the function $U^{(1)}$ to be independent of the accelerations, then, as in the case of two particles, we cannot restrict ourselves to a particular solution of the system (95) when finding the general solution of the system (94). The general solution of the system (95) that depends only on the coordinates will contain not only the superposition (98) with two-particle functions of the form (103) but also analogous three-, four-, etc., particle terms with arbitrary translationally invariant many-particle functions φ_{abc}, φ_{abcd}, etc. Corresponding many-particle Galileo-noninvariant terms also appear in the general solution of the system (94), so that in such an approach we must write instead of (99) the more general expansion

$$U^{(1)}=\sum_{a<b}\sum U_{ab}^{(1)}+\sum_{a<b<c}\sum\sum U_{abc}^{(1)}+\ldots+u_1.$$

Without going into a more detailed discussion of the poorly investigated problems of many-particle interactions, we merely mention that the condition of separability of the interactions imposes very important restrictions on functions of this type. We shall return to this question in discussing the problem of the addition of interactions in the Hamiltonian formalism.

Post-Newtonian equations of motion and conservation laws. In the first quasirelativistic approximation, we write the total Lagrangian of the system in the form

$$L=L_f-U\approx\sum_a\left(-m_ac^2+\frac{m_av_a^2}{2}+\frac{m_av_a^4}{8c^2}\right)-\sum_{a<b}\sum(U_{ab}^{(0)}+U_{ab}^{(1)}). \quad (107)$$

If $U^{(0)}=u_0(r)$, the Euler equations (48) are transformed into the ordinary Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial v_{ai}}-\frac{\partial L}{\partial x_{ai}}=0, \quad (108)$$

which can be reduced to the Newtonian form (42). This can be done in two equivalent ways. The first is by solving (to terms $\sim c^{-2}$) the system of equations obtained by substituting (107) in (108) for the accelerations $\dot{v}_{ai}$. In the second method, the equations of motion of zeroth order, $m_a\dot{v}_{ai}=-\partial U^{(0)}/\partial x_a^i$, are used in the terms of Eqs. (108) of order c^{-2}. Either method leads to the post-Newtonian equations of motion

$$m_a\dot{v}_a^i=-\sum_b{}'\frac{r_{ab}^i}{r_{ab}}\frac{dU_{ab}^{(0)}}{dr_{ab}}-\sum_b{}'\frac{\partial U_{ab}^{(1)}}{\partial r_{ab}^i}+\sum_b{}'\left(\mathbf{v}_{ab}\frac{\partial}{\partial\mathbf{r}_{ab}}\right)\frac{\partial U_{ab}^{(1)}}{\partial v_a^i}+\frac{1}{2c^2}\sum_b{}'\frac{v_a^2r_{ab}^i}{r_{ab}}\frac{dU_{ab}^{(0)}}{dr_{ab}}+\frac{1}{c^2}\sum_b{}'\frac{v_a^i(\mathbf{r}_{ab}\cdot\mathbf{v}_a)}{r_{ab}}\frac{dU_{ab}^{(0)}}{dr_{ab}}-\frac{1}{m_a}\sum_b{}'\frac{1}{r_{ab}}\frac{dU_{ab}^{(0)}}{dr_{ab}}\left(\mathbf{r}_{ab}\frac{\partial}{\partial\mathbf{v}_a}\right)\sum_c{}'\frac{\partial U_{ac}^{(1)}}{\partial v_{ai}}-\sum_{b\neq c}{}'\sum\left[\left(\mathbf{r}_{bc}\frac{\partial}{\partial\mathbf{v}_b}\right)\frac{\partial U_{ab}^{(1)}}{\partial v_{ai}}\frac{1}{m_br_{bc}}\frac{dU_{bc}^{(0)}}{dr_{bc}}\right]. \quad (109)$$

We make two remarks concerning the equations of motion (109). First, because $U^{(1)}$ is a solution of the system (94)–(96), it is readily verified that these

[11] In Ref. 134, Woodcock and Havas also give corresponding expressions that do not have the property of symmetry with respect to the interacting particles.

[12] In Ref. 136, the expression for $U_{ab}^{(1)}$ contains two functions φ_a and φ_b, but it can be readily reduced to the form (105) by setting $\varphi_a+\varphi_b=2\varphi_{ab}$ and redefining u_{1ab}.

equations of motion satisfy the Currie–Hill conditions (46) to accuracy c^{-2}.[13] Second, they contain three-particle terms nonlinear in the interaction. In other words, in terms of the equations of motion or the interaction "forces," if these last are taken to be the right-hand sides of equations of motion of the type (109), the principle of superposition of interactions is not satisfied in the post-Newtonian approximation. This corresponds to the general conclusion drawn in Ref. 137, namely, an acceleration dependence of the forces is not compatible with the principle of superposition of the forces. A similar situation also obtains in nonrelativistic mechanics if the potential $U^{(0)}$ depends on the velocities.

The general form of approximately relativistic equations of motion of the type (109) can also be obtained by solving the Currie–Hill equations (46) by an expansion of the functions μ_{ai} in powers of c^{-2}; in Ref. 111, this method was used to find the general form of μ_{ai} to terms of order c^{-4}. A similar approach based on the Droz–Vincent equations was used in Ref. 110.

For the integrals of the motion (67)–(70), we obtain by means of (107) and (105) the quasirelativistic expressions

$$\mathbf{P}=\sum_a m_a\mathbf{v}_a\left(1+\frac{v_a^2}{2c^2}\right)+\frac{1}{2c^2}\sum_{a<b}\sum\left\{(\mathbf{v}_a+\mathbf{v}_b)U_{ab}^{(0)}-\mathbf{r}_{ab}(\mathbf{r}_{ab}\cdot(\mathbf{v}_a+\mathbf{v}_b))\frac{1}{r_{ab}}\frac{dU_{ab}^{(0)}}{dr_{ab}}-\mathbf{v}_{ab}\varphi_{ab}-\mathbf{r}_{ab}(\mathbf{r}_{ab}\cdot\mathbf{v}_{ab})\frac{1}{r_{ab}}\frac{d\varphi_{ab}}{dr_{ab}}\right\}+o(c^{-2}); \quad (110)$$

$$E=\sum_a\left(m_ac^2+\frac{m_av_a^2}{2}+\frac{3m_av_a^4}{8c^2}\right)+\sum_{a<b}\sum U_{ab}^{(0)}+\frac{1}{2c^2}\sum_{a<b}\sum\left\{\mathbf{v}_a\cdot\mathbf{v}_bU_{ab}^{(0)}-(\mathbf{r}_{ab}\cdot\mathbf{v}_a)(\mathbf{r}_{ab}\cdot\mathbf{v}_b)\frac{1}{r_{ab}}\frac{dU_{ab}^{(0)}}{dr_{ab}}-\frac{1}{2}\mathbf{v}_{ab}\cdot(\mathbf{v}_a+\mathbf{v}_b)\varphi_{ab}-\frac{1}{2}(\mathbf{r}_{ab}\cdot\mathbf{v}_{ab})\mathbf{r}_{ab}\cdot(\mathbf{v}_a+\mathbf{v}_b)\frac{1}{r_{ab}}\frac{d\varphi_{ab}}{dr_{ab}}\right\}+\frac{1}{c^2}\sum_{a<b}\sum\left(u_{1ab}-\mathbf{v}_{ab}\cdot\frac{\partial u_{1ab}}{\partial\mathbf{v}_{ab}}\right)+o(c^{-2}); \quad (111)$$

$$\mathbf{J}=\sum_a m_a\mathbf{x}_a\times\mathbf{v}_a\left(1+\frac{v_a^2}{2c^2}\right)+\frac{1}{2c^2}\sum_{a<b}\sum\left\{(\mathbf{x}_a\times\mathbf{v}_b+\mathbf{x}_b\times\mathbf{v}_a)U_{ab}^{(0)}+\mathbf{x}_a\times\mathbf{x}_b(\mathbf{v}_a+\mathbf{v}_b)\mathbf{r}_{ab}\frac{1}{r_{ab}}\frac{dU_{ab}^{(0)}}{dr_{ab}}-(\mathbf{x}_a\times\mathbf{v}_a-\mathbf{x}_b\times\mathbf{v}_b)\varphi_{ab}+\mathbf{x}_a\times\mathbf{x}_b(\mathbf{r}_{ab}\cdot\mathbf{v}_{ab})\frac{1}{r_{ab}}\frac{d\varphi_{ab}}{dr_{ab}}\right\}-\frac{1}{c^2}\sum_{a<b}\sum\mathbf{r}_{ab}\frac{\partial u_{1ab}}{\partial\mathbf{v}_{ab}}+o(c^{-2}); \quad (112)$$

$$\mathbf{K}=-t\mathbf{P}+\sum_a m_a\mathbf{x}_a\left(1+\frac{v_a^2}{2c^2}\right)+\frac{1}{2c^2}\sum_{a<b}\sum[(\mathbf{x}_a+\mathbf{x}_b)U_{ab}^{(0)}+\mathbf{r}_{ab}\varphi_{ab}]+o(c^{-2}). \quad (113)$$

As is shown in Ref. 88, the quantities determined by the expressions (110)–(113) have the transformation properties characteristic of the energy, momentum, angular momentum, and center-of-mass integral of the motion with respect to finite approximate Lorentz transformations due to Eq. (104) for $U^{(1)}$. In Ref. 136, a similar property was established for the exact integrals of the motion.

General form of post-Newtonian interaction Hamiltonians. We formulate the problem considered above in the framework of the Hamiltonian formalism, restricting ourselves for the time being to spinless particles. In view of (35) and (38), the expansion in powers of c^{-2} should be applied to only the generators H and K_i [see (36), (37), and (71)]:

$$H=Mc^2+H^{(0)}+H^{(1)}+O(c^{-4})\approx Mc^2+\sum_a\left(\frac{p_a^2}{2m_a}-\frac{p_a^4}{8m_a^3c^2}\right)+U^{(0)}+U^{(1)}; \quad (114)$$

$$K_i=K_i^{(0)}+K_i^{(1)}+O(c^{-4})\approx-t\sum_a p_{ai}+\sum_a\left(m_a+\frac{p_a^2}{2m_ac^2}\right)q_{ai}+\Psi_i^{(1)}. \quad (115)$$

Here, we have used the circumstance that, as is shown in Ref. 68 and in agreement with the conclusion drawn above in the Lagrangian approach, we can without loss of generality set $\Psi_i^0=0$, i.e., for the canonical generator of Galileo transformations we can take the expression

$$K_i^{(0)}=-tP_i+\sum_a m_aq_{ai}\equiv-tP_i-MR_i; \quad (116)$$

where

$$R_i=\frac{1}{M}\sum_a m_aq_{ai} \quad (117)$$

are the coordinates of the nonrelativistic center of mass.

It follows from (72) and (73) that $U^{(0)}$ and $U^{(1)}$ are translationally invariant 3-scalars, and $\Psi_i^{(1)}$ form a 3-vector. Since $U^{(0)}$ is assumed to be a known function, and we ignore terms of order c^{-4}, the system of equations (74)–(76) for $U^{(1)}$ and $\Psi_i^{(1)}$ becomes linear and its solution does not present difficulties. We shall not write out explicitly this system in the variables (q,p),[14] since for $U^{(0)}=u_0(r)$ it differs from Eqs. (94)–(96) for this case [for $N=2$, see Eq. (104)] only by the use of the nonrelativistic relation $\mathbf{p}_a=m_a\mathbf{v}_a$. Therefore assuming two-body nonrelativistic potentials and that $\Psi_i^{(1)}$ are independent of the momenta [we shall discuss this last assumption below; see (120)], we can immediately write down the general solution of this system in the form (98), (99) with the expression (103) preserving its form and (105) replaced by (see Ref. 138, in which this result is obtained in the Hamiltonian approach)

$$U_{ab}^{(1)}=-\frac{1}{2c^2}\left\{\frac{\mathbf{p}_a}{m_a}\cdot\frac{\mathbf{p}_b}{m_b}U_{ab}^{(0)}-\frac{\mathbf{r}_{ab}\cdot\mathbf{p}_a}{m_a}\frac{\mathbf{r}_{ab}\cdot\mathbf{p}_b}{m_b}\frac{1}{r_{ab}}\frac{dU_{ab}^{(0)}}{dr_{ab}}+\left(\frac{p_a^2}{m_a^2}-\frac{p_b^2}{m_b^2}\right)\varphi_{ab}+\left[\mathbf{r}_{ab}\left(\frac{\mathbf{p}_a}{m_a}-\frac{\mathbf{p}_b}{m_b}\right)\right]\left[\mathbf{r}_{ab}\left(\frac{\mathbf{p}_a}{m_a}+\frac{\mathbf{p}_b}{m_b}\right)\right]\times\frac{1}{r_{ab}}\frac{d\varphi_{ab}}{dr_{ab}}\right\}+u_{1ab}\mathbf{r}_{ab}\left(\frac{\mathbf{p}_a}{m_a}-\frac{\mathbf{p}_b}{m_b}\right). \quad (118)$$

The corresponding expression (114) for the total Hamiltonian H can be obtained from the Lagrangian (107) by an ordinary Legendre transformation using the equation

$$\mathbf{p}_a=\partial L/\partial\mathbf{v}_a. \quad (119)$$

The Hamiltonian H, like the expressions (35) and (115) for the remaining generators of the Poincaré group,

[13] As is shown in Ref. 136, this holds not only in the considered case $U^{(0)}=u_0(r)$ but also in the more general case when the nonrelativistic potentials depend on the velocities.

[14] Below, this system will be written down in a form that makes it possible to consider the corrections $U^{(1)}$ and $\Psi_i^{(1)}$ in the case of a nonstatic nonrelativistic limit as well; see Eqs. (128)–(130).

can also be regarded as the result of a transition in the expressions (110)–(113) for the ten integrals of the motion from the variables (x,v) to the canonical variables $(q=x,p)$ on the basis of the relation (119).

In the expressions (114)–(118), the physical coordinates of the particles x_a^i, which satisfy the condition (79) of world-line invariance, serve as the canonical coordinates q_a^i. Such a possibility is due to the circumstance that Eq. (79) is satisfied trivially in the zeroth approximation when the expansions (114) and (115) are substituted in it, while for the terms of order c^{-2} Eq. (79) gives the equation

$$\frac{\partial K_j^{(1)}}{\partial p_a^i}=\frac{x_{aj}p_{ai}}{m_a c^2},$$

from which we find

$$K_j^{(1)}=\sum_a \frac{p_a^2}{2m_a c^2}x_{aj}+\Psi_j^{(1)}(\mathbf{x}), \tag{120}$$

where $\Psi^{(1)}$ is an arbitrary vector function of the coordinates. Comparing (120), (115), and (103), we conclude that the integration of the equations that in the Hamiltonian formalism express the condition of world-line invariance gives in the considered approximation the same result as integration of the equations for $\mathbf{K}^{(1)}$ which follow from the commutation relations for the algebra $A\mathscr{P}$ if in the latter case we restrict ourselves to functions $\Psi^{(1)}$ that do not depend on the momenta.

Thus, the first post-Newtonian approximation is remarkable in that the no-interaction theorem does not appear in it. This circumstance was first established in Refs. 139 and 126, and it is also discussed in Refs. 10 and 140. Recalling Eqs. (80) and (81), which are a consequence of the condition (79), we note that when the expansions of the generators H and $\mathbf{K}$ in powers of c^{-2} are used these equations must be satisfied for $H^{(n-1)}$ if we require (79) to be satisfied to order n. In particular, for $n=1$ Eqs. (80) and (79) are satisfied trivially for a nonrelativistic Hamiltonian with $U^{(0)}=u_0(r)$.

From the point of view of the connection between the Hamiltonian and the Lagrangian formalism, this conclusion reflects the fact that in the post-Newtonian approximation there exist acceleration-independent Lagrangians, which permit the usual transition to the canonical method. As is noted in Ref. 111, Lagrangians of the form $L(x,\dot{x})$ also exist in the second approximation in c^{-2} if $L^{(1)}$ has a structure analogous to H in Eq. (80).

We also give two other forms of expression (118), in which we use the nonrelativistic two-particle center-of-mass variables

$$\mathbf{P}_{ab}=\mathbf{p}_a+\mathbf{p}_b;\ \boldsymbol{\pi}_{ab}=\frac{m_b\mathbf{p}_a-m_a\mathbf{p}_b}{M_{ab}}. \tag{121}$$

Substituting in (118) the expressions

$$\mathbf{p}_a=\frac{m_a}{M_{ab}}\mathbf{P}_{ab}+\boldsymbol{\pi}_{ab};\ \mathbf{p}_b=\frac{m_b}{M_{ab}}\mathbf{P}_{ab}-\boldsymbol{\pi}_{ab}, \tag{122}$$

which are the inverses of (121), we obtain

$$U_{ab}^{(1)}=-\frac{1}{2c^2}\left\{\left(\frac{\mathbf{P}_{ab}^2}{M_{ab}^2}+\frac{\varkappa_{ab}}{M_{ab}\mu_{ab}}\mathbf{P}_{ab}\cdot\boldsymbol{\pi}_{ab}\right)U_{ab}^{(0)}\right.$$
$$-\left[\frac{(\mathbf{r}_{ab}\mathbf{P}_{ab})^2}{M_{ab}^2}+\frac{\varkappa_{ab}}{M_{ab}\mu_{ab}}(\mathbf{r}_{ab}\mathbf{P}_{ab})(\mathbf{r}_{ab}\boldsymbol{\pi}_{ab})\right]\frac{1}{r_{ab}}\frac{dU_{ab}^{(0)}}{dr_{ab}}$$
$$\left.+\frac{2}{M_{ab}\mu_{ab}}\left[\mathbf{P}_{ab}\cdot\boldsymbol{\pi}_{ab}\varphi_{ab}+(\mathbf{r}_{ab}\mathbf{P}_{ab})(\mathbf{r}_{ab}\boldsymbol{\pi}_{ab})\frac{1}{r_{ab}}\frac{d\varphi_{ab}}{dr_{ab}}\right]\right\}+\tilde{u}_{1ab}, \tag{123}$$

where

$$\mu_{ab}=m_am_b/M_{ab};\ \varkappa_{ab}=(m_b-m_a)/M_{ab}, \tag{124}$$

$$\tilde{u}_{1ab}=u_1(\mathbf{r}_{ab},\mu_{ab}^{-1}\boldsymbol{\pi}_{ab})$$
$$+\frac{1}{2c^2}\left\{\frac{1}{M_{ab}\mu_{ab}}\left[\pi_{ab}^2U_{ab}^{(0)}-(\mathbf{r}_{ab}\cdot\boldsymbol{\pi}_{ab})^2\frac{1}{r_{ab}}\frac{dU_{ab}^{(0)}}{dr_{ab}}\right]\right.$$
$$\left.-\frac{\varkappa_{ab}}{\mu_{ab}^2}\left[\pi_{ab}^2\varphi_{ab}+(\mathbf{r}_{ab}\cdot\boldsymbol{\pi}_{ab})^2\frac{1}{r_{ab}}\frac{d\varphi_{ab}}{dr_{ab}}\right]\right\}. \tag{125}$$

In Eq. (123), the function $\tilde{u}_{1ab}(\mathbf{r}_{ab},\pi_{ab})$ combines all the Galileo-invariant terms, i.e., the terms that do not contain the total momentum $\mathbf{P}_{ab}$ of the two-particle system.

The expressions (123) can also be represented in a different form by using the Poisson brackets:

$$U_{ab}^{(1)}=-\frac{1}{2c^2}\left(\frac{P_{ab}^2}{M_{ab}^2}U_{ab}^{(0)}+\frac{1}{M_{ab}^2}\{(\mathbf{r}_{ab}\cdot\mathbf{P}_{ab})(\boldsymbol{\pi}_{ab}\cdot\mathbf{P}_{ab}),U_{ab}^{(0)}\}\right.$$
$$\left.+\frac{\varkappa_{ab}}{M_{ab}\mu_{ab}}\{(\mathbf{r}_{ab}\cdot\mathbf{P}_{ab})\pi_{ab}^2,U_{ab}^{(0)}\}+\frac{1}{M_{ab}\mu_{ab}}\{\pi_{ab}^2,\theta_{ab}\})+\tilde{u}_{1ab}, \tag{126}$$

where

$$\theta_{ab}(\mathbf{r}_{ab},\mathbf{P}_{ab})=-\mathbf{r}_{ab}\cdot\mathbf{P}_{ab}\left(\frac{\varkappa_{ab}}{2}U_{ab}^{(0)}+\varphi_{ab}\right). \tag{127}$$

This last expression for the function $U_{ab}^{(1)}$ can be obtained by direct solution of the commutation relations for the canonical generators of the Poincaré group by the commutator-algebra method developed (in the quantum variant) in Refs. 68 and 6. In this way, a general solution of Dirac's problem in the c^{-2} approximation can also be obtained in the case when the nonrelativistic potential depends on the momenta and, thus, the canonical coordinates cannot have the property of covariance in the quasirelativistic approximation. We shall return to this somewhat later, and write down here only the corresponding system of equations for $U^{(1)}$ and $\Psi^{(1)}$ that follow from (74)–(76) with allowance for the first relation (72). If for $K_i^{(0)}$ we use the second expression (116), we obtain

$$M\{R_i,U^{(1)}\}+\{K_{ij},U^{(0)}\}+\{\Psi_i^{(1)},H^{(0)}\}=0; \tag{128}$$

$$\{\Psi_i^{(1)},P_j\}=\frac{1}{c^2}\delta_{ij}U^{(0)}; \tag{129}$$

$$\{P_i,K_j^{(1)}\}-\{P_j,\Psi_i^{(1)}\}=0, \tag{130}$$

In the case $U_{ab}^{(0)}=U_{ab}^{(0)}(r_{ab})$, the system of equations (128)–(130) is equivalent to the system (94)–(96) for the same case, and Eq. (126) gives its general solution (118), which satisfies to accuracy c^{-2} the additional condition (79) of covariance of the canonical coordinates.

We now consider *many-particle* post-Newtonian interaction Hamiltonians, i.e., the problem of adding the interactions in the Hamiltonian approach in the c^{-2} approximation for $U^{(0)}=u_0(r)$. In solving this problem, we must satisfy the conditions of approximate Poincaré invariance and separability. Since the equations that express the conditions of Poincaré invariance in the Hamiltonian and Lagrangian formalisms are identical in the post-Newtonian case (apart from the replacement of $\mathbf{p}_a$ by $m_a\mathbf{v}_a$), all the results from the standard Lagrangian approach in which $U^{(1)}$ does not contain accelerations can be applied here. Thus, a superposition principle operates, so that the correction $U^{(1)}$ to the many-particle nonrelativistic potential $U^{(0)}(r)=\sum\sum_{a<b}u_{ab}^{(0)}(r_{ab})$ is determined by Eq. (99), in which $U_{ab}^{(1)}$ can be taken to be any of the expressions (118), (123), or (126).

The requirement of separability of the interactions plays a more important role in many-particle ($N \geqslant 3$) systems than in two-particle systems. In the latter case, it is sufficient for its fulfillment that the following conditions hold:

$$u_{1ab} \xrightarrow[r_{ab}\to\infty]{} 0;\ r_{ab}U_{ab}^{(0)} \xrightarrow[r_{ab}\to\infty]{} 0;\ r_{ab}\varphi_{ab} \xrightarrow[r_{ab}\to\infty]{} 0. \tag{131}$$

For $N \geqslant 3$, it is easy to construct solutions of the system (129) that do not satisfy the separability condition. Take, for example, a particular solution of Eqs. (129) of the form

$$\Psi^{(1)} = \frac{1}{c^2} U^{(0)} \sum_{d=1}^{N} \alpha_d \mathbf{x}_d = \frac{1}{c^2} \left(\sum_{d=1}^{N} \alpha_d \mathbf{x}_d \right) \sum\sum_{a<b} U_{ab}^{(0)};\ \sum_{a=1}^{N} \alpha_a = 1. \tag{132}$$

If one of the particles (say, the first) is removed to infinity, $|\mathbf{x}_1 - \mathbf{x}_a| \to \infty, a = 2, \ldots, N$, then the separability condition is expressed by the equation

$$\Psi^{(1)} \to \Psi_1^{(1)} + \overline{\Psi}_1^{(1)},$$

in which $\Psi_1^{(1)}$ depends only on $\mathbf{x}_1$, and $\overline{\Psi}_1^{(1)}$, in contrast, does not depend on $\mathbf{x}_1$. The function (132) does not satisfy this requirement.

On the other hand, if we substitute in (98) the two-particle expressions (103), which satisfy the separability condition, then we obtain a many-particle function $\Psi^{(1)}$ which also satisfies this condition. The same conclusion can be drawn with regard to the expression (99) with separable two-particle functions $U_{ab}^{(1)}$. Many-particle relativistic corrections to a nonrelativistic potential that possess the separability property were first obtained by Shirokov *et al.*,[5] although the actual concept of separability was formulated by Foldy somewhat later.[68]

As we have already mentioned in connection with the Lagrangian formalism, one can allow the presence in $\Psi^{(1)}$ of many-particle terms with arbitrary functions φ_{abc} and corresponding many-particle terms in $U^{(1)}$. The question of the compatibility of such expressions with the condition of separability of the interactions remains open. In what follows, we shall restrict ourselves in the post-Newtonian approximation to superpositions of two-particle terms, as is usually done in the literature.

We discuss one further important aspect of the relation between two-particle and many-particle interaction Hamiltonians in the post-Newtonian approximation. Since theoretical calculations for a closed system of particles are made, as a rule, in its center-of-mass system ($\mathbf{P} = 0$), it can be seen from (123) or (126) that for $N = 2$ all the terms in $U^{(1)}$ that are not Galileo-invariant vanish in this system. As was already noted in 1959 by Shirokov,[5] it follows from this that for studying the question of the relativistic corrections to the phenomenological potentials of nucleon–nucleon interactions a two-particle system is unsuited, since the Galileo-invariant terms in $U^{(1)}$ cannot be distinguished from a nonrelativistic potential.

The situation is changed significantly if the two-particle expressions (123) or (126) occur in many-particle interactions. Since in this case there does not exist a frame of reference in which all $\mathbf{P}_{ab} = 0$, the many-particle Hamiltonian will contain Galileo-noninvariant, i.e., $\mathbf{P}_{ab}$-dependent, terms of the expressions (123) or (126), these reflecting the effects associated with the center-of-mass motion of the two-particle subsystem. It is these terms that are responsible for the specifically relativistic corrections to the phenomenological potentials. Thus, the concept of a two-particle interaction potential in quasirelativistic (and, of course, relativistic) mechanics differs from this concept in nonrelativistic theory, in which, because of the condition of Galileo invariance, i.e., because $U_{ab}^{(0)}$ does not depend on $\mathbf{P}_{ab}$, $U_{ab}^{(0)}(\mathbf{r}_{ab}, \boldsymbol{\pi}_{ab})$ determines the interaction of particles a and b not only in the center-of-mass system of the subsystem σ_{ab} but also in any other inertial frame of reference. Approximately Lorentz-invariant interaction potentials of two particles that are present in many-particle Hamiltonians must be considered with allowance for the center-of-mass motion of the two-particle subsystems. This circumstance means that particular value attaches to the approach based on a Poincaré-invariant theory of direct interactions, making it possible to combine the general requirements applied to any physical system with the specific properties of particular interactions in the two-particle subsystems. These subsystems may be studied by means of other methods (for example, field-theoretical, quasipotential, analysis of experimental data through two-particle scattering, etc.).

Allowance for spins. In the framework of classical Hamiltonian mechanics, spins $\mathbf{s}_a$ of the particles can be introduced phenomenologically [see (38) and (39)]. In the first approximation in c^{-2}, the last term in (39) takes the form

$$\mathbf{K}_{fs}^{(1)} = -\frac{1}{2c^2} \sum_a \frac{\mathbf{s}_a \times \mathbf{p}_a}{m_a}. \tag{133}$$

At the same time, the property of covariance of the canonical coordinates [see (19) and (79)] is lost even for free particles.[179,180] We find the corresponding correction to the two-particle potential $U_{ab}^{(1)}$. To compensate the additional term in the second Poisson bracket of Eq. (128), the correction $U_{ab}^{(1)}$ must contain a term which depends on the particle spins and, as is readily verified, has the form

$$U_{sab}^{(1)} = \frac{1}{2M_{ab}c^2} \left\{ \left(\frac{\mathbf{s}_a}{m_a} - \frac{\mathbf{s}_b}{m_b} \right) \times \boldsymbol{\pi}_{ab} \cdot \mathbf{P}_{ab},\ U_{ab}^{(0)} \right\}. \tag{134}$$

In addition, a spin dependence may also be contained in the Galileo-invariant functions φ_{ab} and u_{1ab}, and also (for $N \geqslant 3$) in the corresponding many-particle functions.

We note that systems of particles with spins have been studied little in the Newtonian and Lagrangian approaches; first results were obtained recently in Refs. 141 and 142.

Quantization. The transition from the classical quasirelativistic Hamiltonian H to the quantum $\hat{H}$ can be made in the same way as in nonrelativistic theory,[143] i.e., by means of the substitution

$$\mathbf{p} \to -i\hbar \frac{\partial}{\partial \mathbf{q}} = \hat{\mathbf{p}},\ \{f, g\} \to -\frac{i}{\hbar}[f, g], \tag{135}$$

where p is the momentum canonically conjugate to q. A nonuniqueness arises on the symmetrization of ex-

pressions containing products of functions of the coordinates and functions of the momenta. (Such symmetrization ensures Hermiticity of the operator $\hat{H}$.) Without going into an analysis of this problem in the case of expressions bilinear in the momenta, we use the method of symmetrization adopted in Refs. 6 and 68. Applying it to the expression (118), we obtain the Hermitian operator

$$\hat{U}_{ab}^{(1)} = -\frac{1}{4c^2}\left\{\frac{\hat{\mathbf{p}}_a}{m_a}\cdot\frac{\hat{\mathbf{p}}_b}{m_b}U_{ab}^{(0)} + U_{ab}^{(0)}\frac{\hat{\mathbf{p}}_a}{m_a}\cdot\frac{\hat{\mathbf{p}}_b}{m_b} - \frac{1}{m_a m_b}\hat{\mathbf{p}}_a(\hat{\mathbf{p}}_i\mathbf{r}_{ab})\frac{1}{r_{ab}}\frac{dU_{ab}^{(0)}}{dr_{ab}} - \frac{1}{m_a m_b r_{ab}}\times\frac{dU^{(0)}}{dr_{ab}}\mathbf{r}_{ab}(\mathbf{r}_{ab}\hat{\mathbf{p}}_a)\hat{\mathbf{p}}_b + \left(\frac{\hat{p}_a^2}{m_a^2} - \frac{\hat{p}_b^2}{m_b^2}\right)\varphi_{ab} + \varphi_{ab}\left(\frac{\hat{p}_a^2}{m_a^2} - \frac{\hat{p}_b^2}{m_b^2}\right) + \left(\frac{\hat{\mathbf{p}}_a}{m_a} + \frac{\hat{\mathbf{p}}_b}{m_b}\right)\left(\left(\frac{\hat{\mathbf{p}}_a}{m_a} - \frac{\hat{\mathbf{p}}_b}{m_b}\right)\mathbf{r}_{ab}\right)\frac{1}{r_{ab}}\frac{d\varphi_{ab}}{dr_{ab}} + \frac{1}{r_{ab}}\frac{d\varphi_{ab}}{dr_{ab}}\mathbf{r}_{ab}\left(\mathbf{r}_{ab}\left(\frac{\hat{\mathbf{p}}_a}{m_a} - \frac{\hat{\mathbf{p}}_b}{m_b}\right)\right)\left(\frac{\hat{\mathbf{p}}_a}{m_a} + \frac{\hat{\mathbf{p}}_b}{m_b}\right) + \hat{u}_{1ab}, \quad (136)$$

where $\hat{u}_{1ab}$ is obtained from u_{1ab} by the same quantization procedure. If for u_{1ab} we write, as is usual,

$$u_{1ab} = \frac{1}{2c^2}\left\{\left(\frac{\mathbf{p}_a}{m_a} - \frac{\mathbf{p}_b}{m_b}\right)^2\alpha_{ab}(r_{ab}) + \left[\mathbf{r}_{ab}\left(\frac{\mathbf{p}_a}{m_a} - \frac{\mathbf{p}_b}{m_b}\right)\right]^2\beta_{ab}(r_{ab}) + \gamma_{ab}(r_{ab})\right\}, \quad (137)$$

where α_{ab}, β_{ab}, and γ_{ab} are arbitrary functions of r_{ab}, then

$$\hat{u}_{1ab} = \frac{1}{4c^2}\left\{\left(\frac{\hat{\mathbf{p}}_a}{m_a} - \frac{\hat{\mathbf{p}}_b}{m_b}\right)^2\alpha_{ab}(r_{ab}) + \alpha_{ab}(r_{ab})\left(\frac{\hat{\mathbf{p}}_a}{m_a} - \frac{\hat{\mathbf{p}}_b}{m_b}\right)^2 + \left(\frac{\hat{\mathbf{p}}_a}{m_a} - \frac{\hat{\mathbf{p}}_b}{m_b}\right)\cdot\left(\left(\frac{\hat{\mathbf{p}}_a}{m_a} - \frac{\hat{\mathbf{p}}_b}{m_b}\right)\mathbf{r}_{ab}\right)\mathbf{r}_{ab}\beta_{ab}(r_{ab}) + \beta_{ab}(r_{ab})\mathbf{r}_{ab}\left(\mathbf{r}_{ab}\left(\frac{\hat{\mathbf{p}}_a}{m_a} - \frac{\hat{\mathbf{p}}_b}{m_b}\right)\right)\left(\frac{\hat{\mathbf{p}}_a}{m_a} - \frac{\hat{\mathbf{p}}_b}{m_b}\right)\right) + \gamma_{ab}(r_{ab})\right\}. \quad (138)$$

If we proceed from the classical expression (126), to which we add the correction (134) to take into account the presence of spins of the particles, then by means of the substitution (135) we obtain the operator

$$\hat{U}_{ab}^{(1)} = -\frac{1}{2c^2}\left\{\frac{P_{ab}^2}{M_{ab}^2}U_{ab}^{(0)} - \frac{\mathrm{i}}{2M_{ab}^2\hbar}[(\mathbf{r}_{ab}\hat{\mathbf{P}}_{ab})(\hat{\mathbf{P}}_{ab}\hat{\boldsymbol{\pi}}_{ab}) + (\hat{\boldsymbol{\pi}}_{ab}\hat{\mathbf{P}}_{ab})(\hat{\mathbf{P}}_{ab}\mathbf{r}_{ab}), U_{ab}^{(0)}] - \frac{\mathrm{i}\varkappa_{ab}}{2M_{ab}\mu_{ab}\hbar}[(\mathbf{r}_{ab}\mathbf{P}_{ab})\hat{\pi}_{ab}^2 + \hat{\pi}_{ab}^2(\mathbf{r}_{ab}\hat{\mathbf{P}}_{ab}), U_{ab}^{(0)}] - \frac{\mathrm{i}}{M_{ab}\mu_{ab}\hbar}[\hat{\pi}_{ab}^2, \theta_{ab}] - \frac{\mathrm{i}}{M_{ab}\hbar}\left[\left(\frac{\mathbf{s}_a}{m_a} - \frac{\mathbf{s}_b}{m_b}\right)\times\hat{\boldsymbol{\pi}}_{ab}\hat{\mathbf{P}}_{ab}, U_{ab}^{(0)}\right]\right\} + \hat{u}_{1ab}, \quad (139)$$

which is in complete agreement with the results of Foldy and Krajcik[6] obtained by solving the quantum Dirac problem in the c^{-2} approximation.

The transition from the two-particle system to the many-particle Hamiltonian is made in the post-Newtonian approximation of quantum mechanics in the same way as in the classical theory, i.e., on the basis of the superposition principle.

General case of the quasirelativistic approximation. We mention briefly the complications encountered by an approximately relativistic theory of direct interactions if one goes beyond the framework of the post-Newtonian approximation considered above. We begin with the first approximation in c^{-2} in the case of interactions described in the nonrelativistic limit by velocity-dependent potentials $U^{(0)}$. Then it can be seen from Eqs. (94) and (95) that in the Lagrangian formalism $\Psi^{(1)}$ contains velocities, and the function $U^{(1)}$ depends essentially on the accelerations, i.e., the terms with the accelerations cannot be eliminated by adding a total derivative in the time, so that the Euler equations are of higher than the second order. Since the higher derivatives will occur in them only in small terms, they can be eliminated by means of the nonrelativistic equations of motion, as a result of which we arrive at second-order equations. Such an elimination can also be made in the integrals of motion (67)–(70), but, as we have already noted, it is not admissible in the Lagrangian which occurs in the variational principle.

The general solution of the system of equations (94)–(96) in the considered case containing derivatives of not higher than the second order was obtained in Ref. 136. For a system of two particles, $U_{ab}^{(1)}$ has the form

$$U_{ab}^{(1)} = \frac{1}{4c^2}\left\{-2\mathbf{v}_a\mathbf{v}_b U_{ab}^{(0)} + [\mathbf{r}_{ab}\mathbf{v}_a)\mathbf{v}_b + (\mathbf{r}_{ab}\mathbf{v}_b)\mathbf{v}_a]\frac{\partial U_{ab}^{(0)}}{\partial r_{ab}} + [v_a^2\mathbf{v}_b - v_b^2\mathbf{v}_a + 3(\mathbf{v}_a\mathbf{v}_b)\mathbf{v}_{ab} + (\mathbf{r}_{ab}\cdot\mathbf{v}_a + \mathbf{r}_{ab}\cdot\mathbf{v}_b)(\dot{\mathbf{v}}_a + \dot{\mathbf{v}}_b)]\cdot\frac{\partial U_{ab}^{(0)}}{\partial \mathbf{v}_{ab}}\right\} + u_{1ab}. \quad (140)$$

We shall not write out the corresponding equations of motion of Newtonian type or the conservation laws, which can be found in Ref. 136. Also given in Ref. 136 are the analogous results for the post-post-Newtonian (or second post-Newtonian) approximation, which takes into account corrections of order c^{-4} in the case of a nonrelativistic potential $U^{(0)} = u_0(r)$.

In the Hamiltonian approach, Eqs. (128)–(130) for $U^{(1)}(q,p)$ and $\Psi^{(1)}(q,p)$ differ from the post-Newtonian equations in that $U^{(0)} = U^{(0)}(q,p)$ and the canonical coordinates q_a^i cannot be identical to the physical coordinates.

In the case of two particles, the general solution of the system (129)–(130) has the form (in Ref. 6, these results are obtained in the quantum variant of the theory)

$$\Psi_{ab}^{(1)} = \frac{1}{c^2}\mathbf{R}_{ab}U_{ab}^{(0)} + \Phi_{ab}^{(1)}, \quad (141)$$

where

$$\Phi_{ab}^{(1)} = -\frac{1}{c^2}\frac{\partial}{\partial \mathbf{P}_{ab}}\theta_{ab}(\mathbf{P}_{ab}, \mathbf{r}_{ab}, \boldsymbol{\pi}_{ab}) = -\frac{1}{c^2}\{\mathbf{R}_{ab}, \theta_{ab}\}, \quad (142)$$

and θ_{ab} is an arbitrary scalar function of the indicated variables that is symmetric with respect to interchange of the indices a and b. [In (127), we had a special form of this function.] The corresponding general solution of the system of equations (128) has the form (126) except that the function θ_{ab} is not determined by Eq. (127). All these results can be transferred in the standard manner to the quantum theory, in which the operator $\hat{U}_{ab}^{(1)}$ is given by (139).

It is clear that the transition from the Lagrangian to the Hamiltonian formalism cannot be made in the considered case by means of a standard Legendre transformation, because of the essential dependence of the Lagrangian on the accelerations. The investigation of these very interesting questions goes beyond the scope of the present paper.

For $U^{(0)} = U^{(0)}(\mathbf{r}, \mathbf{v})$, an important difference arises between the Lagrangian and Hamiltonian descriptions in the quasirelativistic approximation on the transition to N-particle systems ($N \geq 3$). In the Lagrangian formal-

ism, two-body nonrelativistic potentials of the form (97) correspond, in accordance with (99), to solutions describing two-body quasirelativistic potentials, but in the Hamiltonian approach the situation is different, since, substituting the superpositions (97) and (98) with functions $\Psi_{iab}^{(1)}$ of the form (141) in Eq. (128), we obtain in the last Poisson bracket triple sums of the form

$$\sum_{a<b<c}\sum\sum \{\mathbf{R}_{ab}U_{ab}^{(\omega)}, U_{ac}^{(\omega)}\}.$$

Therefore, for many-particle systems the correction $U^{(1)}$ to the two-body nonrelativistic potential necessarily contains three-particle terms.[6]

From the point of view of the Lagrangian formalism, the presence of many-particle corrections to the interaction Hamiltonians in the general case of the quasirelativistic approximation is due to the circumstance that the Hamiltonization of the theory is more complicated because of the presence of higher derivatives in the Lagrangians, which, in particular, leads to the appearance of many-particle interaction Hamiltonians corresponding to two-particle interaction Lagrangians.

4. RELATIONSHIP TO FOKKER-TYPE ACTION INTEGRALS AND THE FIELD APPROACH

A basic feature of the approach presented here to the construction of a relativistic theory of direct interactions is the striving to find the general structure of the corresponding expressions (Lagrange functions, canonical generators, equations of motion, conservation laws) on the basis of certain natural requirements (Poincaré invariance, separability of the interactions, etc.). This generality is reflected in the presence of certain arbitrary functions, in particular, the functions u_{1ab} and φ_{ab} in the expressions for the quasirelativistic interaction Lagrangians and Hamiltonians obtained in the previous section. In other words, our approach establishes a general framework within which the descriptions of particular systems and interactions must be realized but does not give indications of the concrete choice of the arbitrary functions that must be made for each particular object. Ultimately, only experiment can answer this question; however, here too the theoretical analysis must serve as a basis of certain recommendations.

In the development of such recommendations, an important part can be played by the solution of the relativistic (or quasirelativistic) inverse scattering problem (the recovery of the potential from the S matrix); in the framework of quantum relativistic Hamiltonian theory, this problem was investigated by Sokolov.[62,144] Another possibility is to compare the "forces" or potentials of the theory of direct interactions with the corresponding expressions obtained in the framework of the field-theoretical approach by means of systematic elimination of the field variables. Such an elimination (for example, by means of the Lienard–Wiechert potentials in electrodynamics) can be made in the equations of motion of the particles, in the action integrals, in the Lagrangians, or in the Hamiltonians. For systems of particles with electromagnetic interaction, this procedure was used in classical theory in Refs. 145–151 (see also Ref. 152), and in quantum theory in Ref. 153 and many other papers (see the reviews of Ref. 2, and also Ref. 154); for systems with arbitrary vector or scalar interaction, it was done in Refs. 155 and 156, respectively. For systems with gravitational interaction, the transition from the description of the motion of bodies in the framework of the general theory of relativity to a description in terms of direct interactions was investigated in the first and second post-Newtonian approximations in Refs. 86 and 157–162. The case of a combined gravitational and electromagnetic interaction was considered in Ref. 163. As examples of the use of quantum field theory for a similar purpose, we mention the review of Ref. 164, which is devoted to a discussion of the one-boson exchange potential of the nucleon–nucleon interaction, and Ref. 165, in which the elimination of the field variables leads (in the framework of a simple model) to a set of generators of unitary representations of the Poincaré group expressed in terms of the variables of the particles.

A convenient intermediate link for comparing the results of the theory of direct interactions and field approaches is the formalism of Fokker-type action integrals mentioned in Sec. 2. This formalism is based on an action of the form

$$S = -\sum_a m_a c \int d\tau_a \sqrt{u_a^2} - \frac{1}{c}\sum\sum_{a<b} \int d\tau_a \int d\tau_b \Lambda_{ab}[x_a(\tau_a) - x_b(\tau_b), u_a(\tau_a), u_b(\tau_b)], \quad (143)$$

where $x_a^\nu(\tau_a)$ describes in four-dimensional form the parametric equations of the world lines of the particles, $u_a^\nu \equiv dx_a^\nu/d\tau_a$ is the 4-velocity, and Λ_{ab} are arbitrary Poincaré-invariant functions constructed from scalar products of the 4-vectors indicated in (143).

Use of the four-dimensional, manifestly Lorentz-invariant expression (143) is the basis of a different approach to the construction of a one-time Lagrangian formulation of the relativistic mechanics of a system of particles developed in Refs. 134, 135, and 166–169 (see also Ref. 32). The most general structure of the post-Newtonian interaction Lagrangian corresponding to (143) was first obtained by Woodcock and Havas.[134] By means of a somewhat different method proposed in Ref. 135, and proceeding from more general expressions for the function Λ_{ab}, Klyuchkovskiĭ[166,167] found the general form of a quasirelativistic (to accuracy c^{-2}) Lagrangian for interactions described in the nonrelativistic limit by velocity-independent potentials, the corrections of order c^{-4} corresponding to the static nonrelativistic limit, and also the interaction Lagrangian in the linear approximation in the coupling constant. In Refs. 168 and 169, Tretyak and the present author formulated a general procedure for finding a one-time relativistic interaction Lagrangian, the equations of motion, and the conservation laws from a given function Λ_{ab}. They also showed that the results obtained in this manner satisfy the conditions of Poincaré invariance of a Lagrangian description of a relativistic system of particles as considered in Secs. 2 and 3 of the present review.

For the present purposes, the most interesting fact is that if the function Λ_{ab} in the integrand in (143) has a certain structure the Fokker formalism corresponds in a certain sense to field-theoretical descriptions. More precisely, it is equivalent to theories of "adjunct fields,"[23,134] in which there is no self-interaction or radiative reaction. (See Refs. 170 and 171 for the connection between these theories.) Such theories are based on an action (143) with a function Λ_{ab} of the form[29,134,31]

$$\Lambda_{ab}=\Lambda_{ab}^{(n,l)}=g_ag_b\,(u_au_b)^{n-l}\,(u_a^2u_b^2)^{(1-n)/2}\times[(u_a\,(x_a-x_b))\,(u_b\,(x_a-x_b))]^l\,G_{ab}^{n,\,l}\,[(x_a-x_b)^2], \quad (144)$$

where g_a are constant numbers, the coupling constants, $l=0,1,\ldots,n$, the number n characterizes the spin (rank) of the carrier field of the interaction, and G is the Green's function of the corresponding field equation. The values $n=0$ and 1 correspond to scalar and vector interactions[27]; a special case of the latter is the electromagnetic interaction described by the function[20,21]

$$\Lambda_{ab}=e_ae_b\,(u_a,\,u_b)\,\delta\,[(x_a-x_b)^2], \quad (145)$$

(where e_a are the charges of the particles), which corresponds in the field description to the half-sum of the retarded and advanced Lienard–Wiechert potentials. (For the so-called total absorption conditions,[23] which ensure equivalence between this and the ordinary approach containing purely retarded fields and the Dirac radiative reaction,[172,173] see Refs. 174, 22, and 175.) Similar theories were also constructed for arbitrary scalar and vector interactions,[27] the linear approximation of general relativity,[28,176,177] which can be regarded in terms of the theory of direct interactions as a scalar–tensor linear combination,

$$\Lambda_{ab}^{gr}=2\Lambda_{ab}^{(2,\,0)}-\Lambda_{ab}^{(0,\,0)}, \quad (146)$$

and also for one of the models of the weak interaction[178] and a number of other systems.

We shall restrict ourselves below to the field-theoretical interpretation of the post-Newtonian interaction Lagrangians and Hamiltonians considered in Sec. 3. First, we find the connection between the expression (105), (137) obtained there and the result of Ref. 134 found on the basis of Fokker action integrals. The connection is expressed by the relations

$$\left.\begin{aligned}&U_{ab}^{(0)}=g_ag_bV_{ab};\ \varphi_{ab}=g_ag_bW_{ab}/2;\\&\alpha_{ab}=-g_ag_b\,(V_{ab}+X_{ab}-W_{ab}/2);\\&\beta_{ab}=g_ag_b\left(-Y_{ab}+\frac{1}{2r_{ab}}\frac{dW_{ab}}{dr_{ab}}\right);\ \gamma_{ab}=0.\end{aligned}\right\} \quad (147)$$

The functions $V_{ab}(r_{ab}), W_{ab}(r_{ab}), X_{ab}(r_{ab}), Y_{ab}(r_{ab})$ can be expressed in accordance with the formulas found in Refs. 134 in terms of the function Λ_{ab}. If the latter has the form (144), then the nonrelativistic potential is determined by the equation

$$U_{ab\,(n,\,l)}^{(0)}=g_ag_b\int\limits_{-\infty}^{\infty}d\theta G_{ab}^{(n,\,l)}\,(\theta^2-r_{ab}^2) \quad (148)$$

and is identical to the Coulomb (or Newtonian gravitational) potential if $G_{ab}^{(n,l)}$ is the symmetric Green's function of the d'Alembert equation, and to the Yukawa potential in the case of the field described by the Klein–Gordon equation. For the correction $U_{ab}^{(1)}$, the following expression is obtained:

$$U_{ab\,(n,\,l)}^{(1)}=-\frac{1}{2c^2}\{[\mathbf{v}_a\cdot\mathbf{v}_b-(n-1)\,v_{ab}^2]\,U_{ab\,(n,\,l)}^{(0)}-\Big[(\mathbf{r}_{ab}\cdot\mathbf{v}_a)\,(\mathbf{r}_{ab}\cdot\mathbf{v}_b)+\frac{l\,(l-1)}{2l-1}\,(\mathbf{r}_{ab}\cdot\mathbf{v}_{ab})^2\Big]\frac{1}{r_{ab}}\frac{dU_{ab\,(n,\,l)}^{(0)}}{dr_{ab}}. \quad (149)$$

Note that, as follows from comparison of (149) and (105), $\varphi_{ab}(r_{ab})=0$; therefore, to all interactions that admit a field interpretation there corresponds in (103) a function $\Psi_{ab}^{(1)}$ of the form $\Psi_{ab}^{(1)}=\frac{1}{2}\,(\mathbf{x}_a+\mathbf{x}_b)U_{ab}^{(0)}$; in connection with this, the terms $U_{ab\,(n,\,l)}^{(1)}$ that are noninvariant with respect to Galileo transformations have the same form for all interactions. The Galilieo-invariant terms $U_{ab\,(n,\,l)}^{(1)}$ corresponding in (105) to an arbitrary function $u_{1ab}(\mathbf{r}_{ab},\mathbf{v}_{ab})$ are uniquely determined by the Newtonian potential and the pair of numbers n and l. In particular, for scalar, vector, and tensor (second-rank) interactions, we have, respectively,

$$u_{1ab}^{(0,\,0)}=-\frac{v_{ab}^2}{2c^2}\,U_{ab}^{(0)};\quad u_{1ab}^{(1,\,0)}=u_{1ab}^{(1,\,1)}=0; \quad (150)$$

$$u_{1ab}^{(2,\,0)}=u_{1ab}^{(2,\,1)}=\frac{v_{ab}^2}{2c^2}\,U_{ab}^{(0)}; \quad (151)$$

$$u_{1ab}^{(2,\,2)}=\frac{1}{2c^2}\Big[v_{ab}^2U_{ab}^{(0)}+\frac{2}{3}\,(\mathbf{r}_{ab}\cdot\mathbf{v}_{ab})^2\,\frac{1}{r_{ab}}\,\frac{dU_{ab}^{(0)}}{dr_{ab}}\Big]. \quad (152)$$

We consider the more general case of direct interactions which can be associated with a superposition of different fields. Suppose

$$\Lambda_{ab}=\sum_{n,\,l}a_{ab}^{(n,\,l)}\Lambda_{ab}^{(n,\,l)};\quad \sum_{n,\,l}a_{ab}^{(n,\,l)}=1, \quad (153)$$

where $a_{ab}^{(n,l)}$ are constants. If all terms $\Lambda_{ab}^{(n,l)}$ of the sum (153) are associated with the same functional form of the nonrelativistic potential $U_{ab}^{(0)}$, then from (149) and (153) we obtain

$$U^{(1)}=\sum_{n,\,l}a_{ab}^{(n,\,l)}U_{ab\,(n,\,l)}^{(1)}=-\frac{1}{2c^2}\Big\{(\mathbf{v}_a\cdot\mathbf{v}_b+A_{ab}v_{ab}^2)\,U_{ab}^{(0)}-[(\mathbf{r}_{ab}\cdot\mathbf{v}_a)\,(\mathbf{r}_{ab}\cdot\mathbf{v}_b)+B_{ab}\,(\mathbf{r}_{ab}\cdot\mathbf{v}_{ab})^2]\,\frac{1}{r_{ab}}\,\frac{dU_{ab}^{(0)}}{dr_{ab}}\Big\}, \quad (154)$$

where A_{ab} and B_{ab} are constants (not necessarily integral) determined by the formulas

$$A_{ab}=\sum_{n,\,l}(1-n)\,a_{ab}^{(n,\,l)};\quad B_{ab}=\sum_{n,\,l}\frac{l\,(l-1)}{2l-1}\,a_{ab}^{(n,\,l)}. \quad (155)$$

It follows from (154) that the field-theoretical interpretation of a given post-Newtonian interaction potential is not unique, since the fixed pair of numbers A_{ab} and B_{ab} can be obtained in accordance with (155) in different ways. We illustrate this assertion by the example of the part of the Einstein–Infeld–Hoffmann Lagrangian linear in the interaction (see for example, Ref. 152), when $A_{ab}=-3, B_{ab}=0$.

We give the two simplest methods of realizing this case:

1) $a_{ab}^{(0,\,0)}=-1,\ a_{ab}^{(2,\,0)}=2;\ a_{ab}^{(n,\,l)}=0$ for $n\neq 0,\ 2,\ l\neq 0;$

2) $a^{(0,\,0)}=-3,\ a_{ab}^{(1,\,0)}=4;\ a_{ab}^{(n,\,l)}=0$ for $n\neq 0,\ 1,\ l\neq 0.$

Thus, the post-Newtonian gravitational interaction can be interpreted in the linear approximation as a superposition of either scalar and tensor (second-rank)[177] or scalar and vector interactions.[135]

Our results can be transferred in the standard man-

ner from the Lagrangian to the Hamiltonian formalism and to quantum mechanics. If we use the expressions (126), (127), and (139) for the post-Newtonian correction to the potential, then it follows from the equation $\varphi_{ab}=0$ that

$$\theta_{ab}=-\varkappa_{ab}(\mathbf{r}_{ab}\cdot\mathbf{P}_{ab})U_{ab}^{(0)}/2, \tag{156}$$

and for the functions $\tilde{u}_{1ab}$ in the special cases considered above we find in accordance with (125) and (150)–(152) and using the equation $\pi_{ab}=\mu_{ab}\mathbf{v}_{ab}$

$$\tilde{u}_{1ab}^{(0,0)}=\frac{1}{2c^2\mu_{ab}}\left[\left(\frac{1}{M_{ab}}-\frac{1}{\mu_{ab}}\right)\pi_{ab}^2U_{ab}^{(0)}-\frac{1}{M_{ab}}(\mathbf{r}_{ab}\cdot\pi_{ab})^2\frac{1}{r_{ab}}\frac{dU_{ab}^{(0)}}{dr_{ab}}\right], \tag{157}$$

$$\tilde{u}_{1ab}^{(1,0)}=\tilde{u}_{1ab}^{(1,1)}=\frac{1}{2c^2M_{ab}\mu_{ab}}\left[\pi_{ab}^2U_{ab}^{(0)}-(\mathbf{r}_{ab}\cdot\pi_{ab})^2\frac{1}{r_{ab}}\frac{dU_{ab}^{(0)}}{dr_{ab}}\right]; \tag{158}$$

$$\tilde{u}_{1ab}^{(2,0)}=\tilde{u}_{1ab}^{(2,1)}=\frac{1}{2c^2\mu_{ab}}\left[\left(\frac{1}{M_{ab}}+\frac{1}{\mu_{ab}}\right)\pi_{ab}^2U_{ab}^{(0)}-\frac{1}{M_{ab}}(\mathbf{r}_{ab}\cdot\pi_{ab})^2\frac{1}{r_{ab}}\frac{dU_{ab}^{(0)}}{dr_{ab}}\right]; \tag{159}$$

$$\tilde{u}_{1ab}^{(2,2)}=\frac{1}{2c^2\mu_{ab}}\left[\left(\frac{1}{M_{ab}}+\frac{1}{\mu_{ab}}\right)\pi_{ab}^2U_{ab}^{(0)}+\left(\frac{2}{3\mu_{ab}}-\frac{1}{M_{ab}}\right)(\mathbf{r}_{ab}\cdot\pi_{ab})^2\frac{1}{r_{ab}}\frac{dU_{ab}^{(0)}}{dr_{ab}}\right]. \tag{160}$$

Corresponding expressions can also be written down for the quantum-mechanical operators $\hat{u}_{1ab}$.

We consider as an example the important special case of the interaction of two point charges, when $U_{ab}^{(0)}=e_ae_b/r_{ab}$. In the operator (139), we use the expression (156) and choose the Galileo-invariant operator $\hat{\tilde{u}}_{1ab}$ in the form

$$\hat{\tilde{u}}_{1ab}=\frac{\pi e_ae_b}{2c^2}\left(\frac{6}{m_am_b}-\frac{1}{m_a^2}-\frac{1}{m_b^2}\right)\delta(\mathbf{r}_{ab})-\frac{e_ae_b}{2c^2}\frac{\mathbf{r}_{ab}}{r_{ab}^3}\times\pi_{ab}\cdot\left[\frac{1}{m_a}\left(\frac{1}{m_a}+\frac{2}{m_b}\right)\mathbf{s}_a+\frac{1}{m_b}\left(\frac{1}{m_b}+\frac{2}{m_a}\right)\mathbf{s}_b\right]+\frac{e_ae_b}{m_am_bc^2}\left[\frac{\mathbf{s}_a\cdot\mathbf{s}_b}{r_{ab}^3}-\frac{3(\mathbf{s}_a\cdot\mathbf{r}_{ab})(\mathbf{s}_b\cdot\mathbf{r}_{ab})}{r_{ab}^5}-\frac{8\pi}{3}\mathbf{s}_a\cdot\mathbf{s}_b\,\delta(\mathbf{r}_{ab})\right].$$

Then after transition in (139) to the momenta $\hat{\mathbf{p}}_a$ and $\hat{\mathbf{p}}_b$, we obtain the expression

$$\hat{U}_{ab}^{(1)}=-\frac{e_ae_b}{2c^2}\left\{\frac{1}{m_am_b}\frac{1}{r_{ab}}\left[\hat{\mathbf{p}}_a\cdot\hat{\mathbf{p}}_b+\frac{1}{r_{ab}^2}\mathbf{r}_{ab}\cdot(\mathbf{r}_{ab}\cdot\hat{\mathbf{p}}_a)\,\hat{\mathbf{p}}_b\right]+\frac{\mathbf{r}_{ab}\times\hat{\mathbf{p}}_a\cdot\mathbf{s}_a}{m_a^2}-\frac{\mathbf{r}_{ab}\times\hat{\mathbf{p}}_b\cdot\mathbf{s}_b}{m_b^2}+\frac{2}{m_am_br_{ab}^3}[\mathbf{r}_{ab}\times\hat{\mathbf{p}}_a\cdot\hat{\mathbf{s}}_b-\mathbf{r}_{ab}\times\hat{\mathbf{p}}_b\cdot\mathbf{s}_a]+\pi\hbar^2\left(\frac{1}{m_a^2}+\frac{1}{m_b^2}\right)\delta(\mathbf{r}_{ab})-\frac{2}{m_am_b}\times\left[\frac{\mathbf{s}_a\cdot\mathbf{s}_b}{r_{ab}^3}-\frac{3(\mathbf{s}_a\cdot\mathbf{r}_{ab})(\mathbf{s}_b\cdot\mathbf{r}_{ab})}{r_{ab}^5}-\frac{8\pi}{3}\mathbf{s}_a\cdot\mathbf{s}_b\delta(\mathbf{r}_{ab})\right]\right\}.$$

Setting here $\mathbf{s}_a=1/2\hbar\sigma_a$ (σ_{ai} are Pauli matrices), we arrive at the expression for $\hat{U}_{ab}^{(1)}$ which determines the correction of order c^{-2} to the Coulomb potential in the Breit Hamiltonian.[153,154] We note that the other method of symmetrizing $U_{ab}^{(1)}$ in accordance with

$$p_{ai}p_{bj}f(\mathbf{r}_{ab})\to\frac{1}{2}[p_{ai}f(\mathbf{r}_{ab})p_{bj}+p_{bj}f(\mathbf{r}_{ab})p_{aj}]$$

also leads in the considered case to the Breit Hamiltonian if in the brackets of the first term of the operator $\hat{\tilde{u}}$ the term $6/(m_am_b)$ is omitted.

We should emphasize once more that for all interactions that have a field interpretation the function θ_{ab} is determined by (156) and, thus, is nonzero when $\mathbf{P}_{ab}\neq0$. Therefore, in many-particle Hamiltonians H, the corresponding term of the two-particle potentials U_{ab} always makes a contribution to the dependence of H on the center-of-mass motion of the two-particle subsystems with $m_a\neq m_b$. This circumstance was not noted in Ref. 18, in which the terms of this type were incorrectly set equal to zero in a study of electromagnetic interactions.

5. QUASIRELATIVISTIC CENTER-OF-MASS VARIABLES

The theoretical investigation of a system of interacting particles very often requires a separate study of the motion of the system as a whole in which one abstracts its physical properties from its complicated structure and internal motion, which together with the interaction of the particles determines the structure of the system. For such separation of the internal and external motion, it is necessary to associate the system with a set of quantities that characterize the system "as a whole," i.e., quantities (which may be called external variables) in terms of which the description of the motion of the system as a whole has the same form as for a single particle. In nonrelativistic mechanics, such a set is provided by the radius vector R of the center of mass and its velocity $\dot{\mathbf{R}}$ (in the Newtonian and Lagrangian formalism) or momentum P (in the Hamiltonian approach). In addition, a nonrelativistic system as a whole is associated with a mass M and intrinsic angular momentum $\mathbf{S}=\mathbf{J}-\mathbf{R}\times\mathbf{P}$, which are determined by its internal structure.

We mention three important properties of the coordinates R_i of the nonrelativistic center of mass that it is desirable to preserve in a relativistic theory:

1) for a closed system of particles, the point with coordinates R_i moves uniformly and rectilinearly in any inertial frame of reference;

2) the same point of space corresponds to the quantities **R** determined in different frames of reference, i.e., the coordinates R_i transform in accordance with the Galilean formulas;

3) the set of quantities R_i, P_i can be used as canonical variables because they satisfy the corresponding commutation relations.

As was shown by the analysis made by Pryce[119] (see also Ref. 179), there does not exist a relativistic generalization of the center-of-mass concept possessing the three properties (when the Galileo transformations are replaced by Lorentz transformations) for systems with nonvanishing intrinsic angular momentum S.[15)] We shall restrict ourselves to three generalizations which preserve property 1, the very essence of the concept. In the terminology of Fleming,[179] the generalizations are the center of mass, the center of inertia (in Ref. 181, it is called the proper center of mass), and the center of spin; they coincide when $\mathbf{S}=0$. The *center of mass* $\mathbf{Q}_{\mathrm{CM}}$ (it would be better to call it the center of energy) is determined in an arbitrary inertial frame by analogy with the Newtonian concept and with allowance for the Einstein relation between energy and mass. The coordinates Q_{CM}^i do not (when $\mathbf{S}\neq0$) have the proper-

[15)] The same difficulty arises in the description of a single particle with spin.[179,180]

ties 2 and 3 (see also Ref. 152). The *center of inertia* Q_{CI} is a point that coincides with the center of mass in the center-of-mass frame ($\mathbf{P}=0$); by definition, Q^i_{CI} are obtained in an arbitrary frame of reference by means of a corresponding Lorentz transformation, i.e., the coordinates Q^i_{CI} are covariant. Although it has the property 2, this concept violates (for $\mathbf{S}\neq 0$) condition 3. Finally, the *center of spin* Q_{CS} or *canonical center of mass*, whose definition will be given below, is characterized (for $\mathbf{S}\neq 0$) by noncovariant coordinates (violation of condition 2) that possess property 3. In quantum mechanics, the coordinate Q^i_{CS} corresponds to the so-called Newton–Wigner position operator (Ref. 180; see also Ref. 182). The form of the equations that determine Q_i depends on the form of the dynamics; as above, we restrict ourselves to the instant form.

The relativistic (or quasirelativistic) problem of separating internal and external motion has been investigated (by various authors) only in the Hamiltonian formalism (see Refs. 9–11, 48, 52, 67, 183, and 184), and therefore we shall use below the center of spin Q_{CS}. Bearing in mind that this expression is not generally used, we shall call it by the traditional term "center of mass" and denote it by Q. This problem has hardly been investigated at all in the Lagrangian formalism of relativistic mechanics.

From the formal point of view, the problem of separating the motion of the system as a whole from its internal motion consists of finding a set of center-of-mass variables, some of them being the external variables Q_i, P_i, S_i, while the remainder describe the internal motion. We denote the internal variables by (ρ, π, σ), where ρ and π correspond to canonical coordinates and momenta and σ to the spins. If we proceed from the individual variables of individual particles $(\mathbf{q}, \mathbf{p}, \mathbf{s})$, which were used above, then to solve the problem it is necessary to find the transformation formulas

$$F\colon (\mathbf{q},\ \mathbf{p},\ \mathbf{s}) \to (\mathbf{Q},\ \mathbf{P},\ \mathbf{S};\ \rho,\ \pi,\ \sigma). \tag{161}$$

On the other hand, the solution of Dirac's problem can be sought from the very beginning by postulating certain relativistic center-of-mass variables, as is done, for example, in the Bakamjian–Thomas model.[67] However, because the physical meaning of these variables is not clear, even in this case it is important to establish their connection with the particle variables in the form of the transformation F^{-1} that is the inverse of (161). In the literature, both methods—the direct and the inverse—are used. In a strictly relativistic theory, the transformations are known for arbitrary N only for noninteracting particles.[185] We shall restrict ourselves here to the quasirelativistic approximation, in which center-of-mass variables for an N-particle system can be constructed for an arbitrary interaction of the particles. For some special types of interaction, the corresponding transformations were found in Ref. 186, and for the general case in Ref. 9. We shall discuss briefly the basic assumptions and results of Ref. 9, which is a study in the framework of quantum mechanics.

Suppose the generators[16] $\mathbf{T}$, $\mathbf{J}$, H, and $\mathbf{K}$ are expressed in terms of the individual particle variables $\mathbf{q}_a$, $\mathbf{p}_a$, and $\mathbf{s}_a$ by Eqs. (35)–(39) and (71) or, in the quasirelativistic approximation, by Eqs. (35), (38), (114), and (115). The original requirement for finding center-of-mass variables formulated in Ref. 9 is that in the required variables the above generators have a "single-particle" form. In other words, they must be expressed in terms of $\mathbf{Q}$, $\mathbf{P}$, and $\mathbf{S}$ by the same formulas as the generators $\mathbf{T}_a, \mathbf{J}_a, H_a, \mathbf{K}_a$ in terms of $\mathbf{q}_a$, $\mathbf{p}_a$, and $\mathbf{s}_a$ [see (35)–(39)]. The part of the particle rest energy $m_a c^2$ must here be played by the internal Hamiltonian h of the complete system, i.e., the Hamiltonian H in the center-of-mass system ($\mathbf{P}=0$), which is the self-energy of the system, while the part of the spin of a particle must be played by the intrinsic angular momentum $\mathbf{S}=\mathbf{J}-\mathbf{Q}\times\mathbf{P}$ of the system, i.e., $\mathbf{J}$ in the center-of-mass system. Thus, in the required center-of-mass variables, these generators must be expressed by the equations[17]

$$\mathbf{T}=\mathbf{P}\ \text{(a)};\quad \mathbf{J}=\mathbf{Q}\times\mathbf{P}+\mathbf{S}\ \text{(b)}; \tag{162}$$

$$H=\sqrt{h^2+c^2P^2}\ \text{(a)};\quad \mathbf{K}=-t\mathbf{P}+\frac{1}{2c^2}(H\mathbf{Q}+\mathbf{Q}H)-\frac{\mathbf{S}\times\mathbf{P}}{H+h}\ \text{(b)}; \tag{163}$$

which can serve as the definition of the canonical center of mass Q; regarding (162) and (163) as a system of equations for $\mathbf{Q}$, $\mathbf{P}$, $\mathbf{S}$, and h, we can readily express the "external" variables in terms of the generators and show that property 3 of the coordinates Q_i is a consequence of the commutation relations of the algebra $A\mathscr{P}$. It also follows from these relations that h and $\mathbf{S}$ commute with $\mathbf{Q}$ and $\mathbf{P}$, i.e., they are functions of the internal variables alone. In addition, h commutes with all the generators of the group $\mathscr{P}$.

The generators $\mathbf{T}$ and $\mathbf{J}$, expressed in terms of the center-of-mass variables, preserve their form (162) in any approximation in c^{-2}, and the expressions (163) in the first approximation in c^{-2} can be represented in the form

$$H=Mc^2+h^{(0)}+\frac{P^2}{2M}+h^{(1)}-\frac{P^2}{2M^2c^2}\,h^{(0)}$$
$$-\frac{P^4}{8M^3c^2}+o(c^{-2})\equiv Mc^2+H^{(0)}+H^{(1)}+o(c^{-2}); \tag{164}$$

$$\mathbf{K}=-t\mathbf{P}+\left(M+\frac{h^{(0)}}{c^2}\right)\mathbf{Q}+\frac{P^2\mathbf{Q}+\mathbf{Q}P^2}{4Mc^2}-\frac{\mathbf{S}\times\mathbf{P}}{2Mc^2}$$
$$+o(c^{-2})=\mathbf{K}^{(0)}+\mathbf{K}^{(1)}+o(c^{-2}). \tag{165}$$

Thus, the problem reduces in the first quasirelativistic approximation to finding a transformation (161) as a result of which the expressions (35), (38), (114), and (115) take the form (162), (164), and (165), respectively.

The method of constructing relativistic (in explicit form, only quasirelativistic) center-of-mass variables proposed in Ref. 9 uses two successive steps. The

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[16] To distinguish the canonical generator of spatial translations from the variable $\mathbf{P}$, with which it is actually identical, we denote it by $\mathbf{T}$.

[17] The second term in Eq. (163.b) is given in a symmetric form, which makes it possible to interpret the generator $\mathbf{K}$ in terms of either classical or quantum mechanics. To simplify the expressions, the "hat" over the operators is omitted here.

first is the transition from the particle coordinates to nonrelativistic center-of-mass variables, this being made in accordance with the equations

$$\mathbf{P}=\sum_a \mathbf{p}_a;\quad \mathbf{Q}=\sum_a \frac{m_a\mathbf{q}_a}{M};\quad \mathbf{S}=\sum_a(\boldsymbol{\rho}_a\times\boldsymbol{\pi}_a+\boldsymbol{\sigma}_a); \tag{166}$$

$$\mathbf{q}_a=\boldsymbol{\rho}_a+\mathbf{Q};\quad \mathbf{p}_a=\boldsymbol{\pi}_a+\frac{m_a}{M}\mathbf{P};\quad \mathbf{s}_a=\boldsymbol{\sigma}_a, \tag{167}$$

in which the internal variables $\boldsymbol{\rho}_a$ and $\boldsymbol{\pi}_a$ (respectively, the positions and momenta of the particles with respect to the center of mass), which are not independent variables, satisfy the relations

$$\sum_a m_a\boldsymbol{\rho}_a=0;\qquad \sum_a \boldsymbol{\pi}_a=0. \tag{168}$$

The second step is to go over from the nonrelativistic to relativistic center-of-mass variables by means of a certain unitary (in classical mechanics, canonical) transformation:

$$\mathbf{q}_a=\exp(i\Phi)(\boldsymbol{\rho}_a+\mathbf{Q})\exp(-i\Phi); \tag{169}$$

$$\mathbf{p}_a=\exp(i\Phi)\left(\boldsymbol{\pi}_a+\frac{m_a}{M}\mathbf{P}\right)\exp(-i\Phi); \tag{170}$$

$$\mathbf{s}_a=\exp(i\Phi)\,\boldsymbol{\sigma}_a\exp(-i\Phi). \tag{171}$$

The operator Φ is found on the basis of the requirement that substitution of (169)–(171) in the generators of the group $\mathscr{P}$, expressed in terms of the particle variables, reduces the generators to the single-particle form (162), (163). Representing Φ in the form of a series in c^{-2}($\Phi=\Phi^{(0)}+\Phi^{(1)}+\cdots$), Krajcik and Foldy[9] show that by virtue of the commutation relations of the algebra $A\mathscr{P}$ the system of equations for Φ that follows from this requirement always has a solution. To terms of order c^{-2}, we have[9]

$$\Phi^{(0)}=1;$$

$$\Phi^{(1)}=-\frac{1}{2c^2}\sum_a\left\{\frac{1}{2M^2}[(\boldsymbol{\rho}_a\cdot\mathbf{P})(\boldsymbol{\pi}_a\cdot\mathbf{P})+(\mathbf{P}\cdot\boldsymbol{\pi}_a)(\mathbf{P}\cdot\boldsymbol{\rho}_a)]\right.$$
$$\left.-\frac{1}{2m_aM}[(\boldsymbol{\rho}_a\cdot\mathbf{P})\,\pi_a^2+\pi_a^2(\mathbf{P}\cdot\boldsymbol{\rho}_a)]-\frac{1}{2m_aM}\boldsymbol{\sigma}_a\times\boldsymbol{\pi}_a\cdot\mathbf{P}\right\}$$
$$-\frac{1}{Mc^2}\int_0^{\mathbf{P}}d\mathbf{P}\cdot\mathbf{w}^{(1)}+\frac{1}{c^2}\Pi^{(1)}, \tag{172}$$

where

$$\mathbf{w}^{(1)}=\Psi^{(1)}-U^{(0)}\mathbf{Q}/c^2, \tag{173}$$

where $\Pi^{(1)}$ is an arbitrary scalar function of only the internal variables. To the same accuracy, we find from (169)–(171)

$$\mathbf{q}_a=\boldsymbol{\rho}_a+\mathbf{Q}-\frac{1}{2c^2}\left\{\left(\boldsymbol{\rho}_a\cdot\frac{\mathbf{P}}{M}\right)\left(\frac{\boldsymbol{\pi}_a}{m_a}+\frac{\mathbf{P}}{2M}\right)\right.$$
$$+\left(\frac{\boldsymbol{\pi}_a}{m_a}+\frac{\mathbf{P}}{2M}\right)\left(\frac{\mathbf{P}}{M}\cdot\boldsymbol{\rho}_a\right)+\sum_b\left[\frac{\pi_b^2\boldsymbol{\rho}_b-\boldsymbol{\rho}_b\pi_b^2}{2m_bM}\right.$$
$$\left.\left.-\frac{(\boldsymbol{\rho}_b\times\boldsymbol{\pi}_b)\times\mathbf{P}}{2M^2}-\frac{\boldsymbol{\sigma}_b\times\boldsymbol{\pi}_b}{2m_bM}-\frac{\boldsymbol{\sigma}_b\times\mathbf{P}}{2M^2}\right]+\frac{\boldsymbol{\sigma}_a\times\mathbf{P}}{2m_aM}\right\}$$
$$-\frac{\mathbf{w}^{(1)}}{M}-i\left[\frac{1}{M}\int_0^{\mathbf{P}}d\mathbf{P}\cdot\mathbf{w}^{(1)}-\Pi^{(1)},\ \boldsymbol{\rho}_a\right]+o(c^{-2}); \tag{174}$$

$$\mathbf{p}_a=\boldsymbol{\pi}_a+\frac{m_a}{M}\mathbf{P}+\frac{1}{Mc^2}\left(\frac{\pi_a^2}{2m_a}-\frac{m_a}{M}\sum_b\frac{\pi_b^2}{2m_b}+\frac{\boldsymbol{\pi}_a\cdot\mathbf{P}}{2M}\right)\mathbf{P}$$
$$-i\left[\frac{1}{M}\int_0^{\mathbf{P}}d\mathbf{P}\cdot\mathbf{w}^{(1)}-\Pi^{(1)},\ \boldsymbol{\pi}_a\right]+o(c^{-2}); \tag{175}$$

$$\mathbf{s}_a=\boldsymbol{\sigma}_a-\frac{\boldsymbol{\sigma}_a\times(\boldsymbol{\pi}_a\times\mathbf{P})}{2m_aMc^2}-i\left[\frac{1}{M}\int_0^{\mathbf{P}}d\mathbf{P}\cdot\mathbf{w}^{(1)}-\Pi^{(1)},\ \boldsymbol{\sigma}_a\right]+o(c^{-2}). \tag{176}$$

The corresponding formulas that express $(\mathbf{Q},\mathbf{P},\mathbf{S},\boldsymbol{\rho},\boldsymbol{\pi},\boldsymbol{\sigma})$ in terms of $(\mathbf{q},\mathbf{p},\mathbf{s})$ can be readily obtained by inverting the transformations (174)–(176) using the relations (168).

Substitution of the expressions (174)–(176) in the generators (35), (38), (114), and (115) leads to Eqs. (162), (164), and (165), and for $\mathbf{S}$, as in the nonrelativistic limit, the expression (166) is obtained, and the first two terms in the expansion of the internal Hamiltonian $h=Mc^2+h^{(0)}+h^{(1)}+o(c^{-2})$ can be calculated in accordance with the formulas

$$h^{(0)}=\sum_a\frac{\pi_a^2}{2m_a}+U^{(0)}, \tag{177}$$

$$h^{(1)}=-\sum_a\frac{\pi_a^4}{8m_a^3c^2}-\sum_a\frac{(\boldsymbol{\pi}_a\cdot\mathbf{P})}{m_aMc^2}\left(\frac{\pi_a^2}{2m_a}+\frac{\boldsymbol{\pi}_a\cdot\mathbf{P}}{2M}\right)$$
$$-\frac{P^2U^{(0)}}{2M^2c^2}+U^{(1)}+i[\Phi^{(1)},\ h^{(0)}], \tag{178}$$

where

$$U^{(n)}=U^{(n)}\left(\mathbf{q}_a=\mathbf{p}_a+\mathbf{Q};\quad \mathbf{p}_a=\boldsymbol{\pi}_a+\frac{m_a}{M}\mathbf{P};\ \mathbf{s}_a=\boldsymbol{\sigma}_a\right). \tag{179}$$

Although (178) contains the external variable $\mathbf{P}$, the expression $h^{(1)}$ does not in reality depend on $\mathbf{P}$, as can be seen by calculating the commutator in (178). The relations (172)–(179) can also be interpreted classically if the quantum Poisson bracket is replaced by the classical one and the commutativity of all classical variables is taken into account.

The presence in the obtained expressions of the arbitrary function $\Pi^{(1)}$ of the internal variables should be noted. (It can include purely kinematic terms as well as terms that depend on the interaction.) As a result, these expressions determine an entire class of quasirelativistic center-of-mass variables; the choice of the function $\Pi^{(1)}$ can be exploited to simplify calculations in concrete problems.

A somewhat modified approach to the problem is of interest, in which independent nonrelativistic center-of-mass variables, namely, canonical Jacobi variables, are used as the zeroth approximation; the corresponding quasirelativistic center-of-mass variables also form a set of independent canonical variables. For $N=2$, they are constructed in the framework of classical mechanics in Ref. 11; for $N=3$, they were introduced in Ref. 187 by postulating in the center-of-mass system the same relations between the independent quasirelativistic center-of-mass variables and the dependent variables considered above as hold in nonrelativistic mechanics for the Jacobi variables. The systematic construction of quasirelativistic center-of-mass variables of Jacobi type for arbitrary N will be considered in a separate paper (see also Ref. 188). In a strictly relativistic theory, the introduction of collective center-of-mass variables is a very complicated problem.

We make two further comments about quasirelativistic center-of-mass variables. The first concerns the transformation formulas (174)–(176), and also the inverse transformations in the center-of-mass system. If we ignore the presence in them of the arbitrary function $\Pi^{(1)}$, which can be set equal to zero, then for $\mathbf{P}=0$ the relations (174)–(176) coincide with the nonrelativistic transformations (167) except for the term $-\mathbf{w}^{(1)}/M$ in Eq. (174). Since this term vanishes in $\mathbf{q}_{ab}=\mathbf{q}_a-\mathbf{q}_b$, the transformation of the translationally invariant

Hamiltonian of a closed system of particles from the particle coordinates to the center-of-mass variables in the center-of-mass system, $\mathbf{P}=0$, is actually made in accordance with the nonrelativistic formulas. Because of this circumstance, the widespread use in the literature of the nonrelativistic center-of-mass variables in quasirelativistic Hamiltonians to separate the internal motion does not lead to errors if it is done only in the center-of-mass system. Such an approach is no longer justified in the description of nonclosed systems, when $\mathbf{P}\neq\text{const}$, so that in such a case it is necessary to use the complete transformations (174)–(176). This leads to specific relativistic effects, which until recently were ignored (see, for example, Ref. 189, in which the transformations (174)–(176) are used to calculate the relativistic corrections to the amplitudes of electric and magnetic dipole single-photon transitions in systems of particles with electromagnetic interaction).

A second remark is associated with $\mathbf{w}^{(1)}$, which occurs in the formulas we are discussing and is expressed in accordance with (173) by the functions $\Psi^{(1)}$ and $U^{(0)}$, which describe the interactions of the particles. If $\mathbf{w}^{(1)}\neq 0$, then the formulas for the transition from the particle variables to the center-of-mass variables contain terms which depend on the interaction, i.e., in contrast to the nonrelativistic transformations, they are not purely kinematic. In particular, for a system of two particles the condition $\mathbf{w}^{(1)}=0$ is equivalent in the post-Newtonian approximation to the equation

$$\varphi_{ab}=\frac{m_a-m_b}{2M_{ab}}U^{(0)},$$

as can be seen by comparing (173) and (103). This condition is certainly not satisfied for all interactions which admit a field interpretation (in particular, electromagnetic interactions; see Ref. 118), since in this case $\varphi_{ab}=0$ (see Sec. 4).

If we consider many-particle systems ($N\geqslant 3$), then the equation $\mathrm{w}^{(1)}=0$ corresponds in the post-Newtonian approximation to a function $\Psi^{(1)}$ of the form (132) with coefficients $\alpha_a=m_a/M$. However, as we know, this form of the function $\Psi^{(1)}$ contradicts the condition of separability of the interactions. Thus, the transformation from the particle coordinates to the center-of-mass variables in the quasirelativistic approximation, and *a fortiori* in the higher approximations and in the exact theory, can be kinematic in nature only in very special cases. In other words, the solution to the quasirelativistic problem of separating the internal and external motion depends on the particular system of particles: The corresponding transformations to the center-of-mass variables contain functions which describe the interparticle interactions.

From the formal point of view, these transformations can always be reduced to a form in which the interaction in the system is not reflected. This is due to the possibility of using canonical (in quantum mechanics, unitary) transformations that describe the replacement of the original variables $\mathbf{q}_a, \mathbf{p}_a$ (in the post-Newtonian approximation, we took $\mathbf{q}_a=\mathbf{x}_a$) by new variables $\tilde{\mathbf{q}}_a, \tilde{\mathbf{p}}_a$, in which $\tilde{\mathbf{w}}^{(1)}=0$.[9] However, as was noted in Ref. 9, a transformation of this kind involves a loss of the clear physical meaning of the employed variables, and therefore use of such variables requires a certain care.

CONCLUSIONS

Summarizing, we can say that the various treatments in the literature of weakly relativistic systems of particles (classical and quantum) can be combined in the post-Newtonian approximation in the framework of a unified quasirelativistic (classical or quantum) mechanics of systems of directly interacting particles; this mechanics is based on clear physical concepts and well-defined basic propositions. The mathematical formalism of this theory is more complicated than the one used in nonrelativistic mechanics and requires further development. In particular, the quasirelativistic equation of Schrödinger type is an equation of fourth order (the Hamiltonian contains fourth powers of the momenta), which leads to certain difficulties.[18] In our view, it would be interesting to make a more detailed study of the class of canonical (unitary) transformations that do not violate the covariance of the canonical coordinates in the post-Newtonian approximation but nevertheless change the form of the potentials; this problem is intimately related to the physical interpretation of quasirelativistic Hamiltonians, i.e., the problem of finding definite expressions for the potentials for particular systems. At the present time, the relationship between the classical Hamiltonian approach and the quantum approach is not clear, i.e., further study of the quantization problem (the nonuniqueness of symmetrization) is required. In this connection, it would be interesting to eliminate systematically the field variables in quantum field theory by a procedure similar to that used in Ref. 165 in the case of a particular model. It is to be expected that this will yield in quantum theory results for the theory of direct interactions similar to those established in the classical approach on the basis of Fokker-type action integrals.

Sokolov[191] has proposed a different way to solve this problem, which consists of finding a classical limit of the quantum-mechanical generators that preserves the structure of the Poincaré group.

A separate problem that warrants attention is the use of the results presented in the present paper to study relativistic effects in systems that are not closed. Although, strictly speaking, the relativistic theory of direct interactions applies only to closed systems, the use of approximate approaches can also be regarded as justified for objects with internal structure subject to external interactions. If the external interaction is due to the presence of a massive particle, this case can be included in the general scheme for closed systems if the mass of one of the particles of the system can tend to infinity. A more careful analysis of the interaction with a radiation field is required.

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[18] For some problems in the quantum theory of scattering, they are considered in Refs. 138 and 190.

Besides the approximation in c^{-2} considered in the review, it is also of interest to investigate other approximations in the relativistic theory of direct interactions, namely, in the coupling constant (Refs. 37, 39, 76, 80, and 147) and in the ratio m_a/m_b when $m_a \ll m_b$.[38] The study of these approaches is currently in its initial stage.

We recall that our entire exposition has been based on the instant form of dynamics. Some successes have also been achieved in the development of the other forms of relativistic dynamics – the point form (Refs. 8, 62, 73, 74, 191, and 192) and the front form (Refs. 61, 191, 193–195, and 16) (in Refs. 193–195 and 16, the latter is used for calculations of specific systems, namely, quark models of hadrons and scattering of nucleons by light nuclei). It is therefore important to investigate the quasirelativistic approximation in the point and front forms of dynamics and to establish the correspondence between the quasirelativistic quantities in the different forms of dynamics. It is to be expected that this will lead to recommendations regarding the expediency of using one or other form of dynamics to study relativistic effects in different physical objects.

With regard to the concrete applications of the quasirelativistic mechanics of a system of interacting particles, these were mentioned in the Introduction and we shall not repeat them (see also Refs. 196 and 197). We shall merely add here the remark that a systematic study of relativistic effects in atoms and molecules has recently begun; these effects are frequently very important for explaining the chemical properties of many elements, especially heavy ones (see Ref. 198, where there are further references). It would also be interesting to develop quasirelativistic statistical physics,[199] which has been considered hitherto mainly in connection with systems of particles with electromagnetic interaction.[200-204] To construct the characteristics of a system (partition function, distribution functions, kinetic equations), one can here use not only the usual method based on knowledge of the Hamiltonian of the system but also the Feynman formalism of path integration, which makes it possible to proceed directly from the Lagrangian description.[202] Such an approach is worth using because in the limit $N \to \infty$ the construction of the Hamiltonian of a system from a given Lagrangian entails considerable difficulties.[200,201] (This emphasizes once more the important part that the Lagrangian formalism plays in quasirelativistic mechanics.) Finally, systems of gravitating bodies can be considered in terms of the quasirelativistic theory of direct interactions in the post-Newtonian approximation; as is shown in Ref. 205, this approach encompasses all "viable" theories of gravitational interactions of point particles considered in the framework of the so-called parametrized post-Newtonian formalism.

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