Nonrelativistic equations of motion for particles with arbitrary spin

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First- and second-order Galileo-invariant systems of differential equations which describe the motion of nonrelativistic particles of arbitrary spin are derived. The equations can be derived from a Lagrangian and describe the dipole, quadrupole, and spin-orbit interaction of the particles with an external field; these interactions have traditionally been regarded as purely relativistic effects. The problem of the motion of a nonrelativistic particle of arbitrary spin in a homogeneous magnetic field is solved exactly on the basis of the obtained equations. The generators of all classes of irreducible representations of the Galileo group are found.

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INTRODUCTION

The relativity principle and the Galileo transformation have been known for more than 300 years. However, it is only comparatively recently that the structure of the Galileo group G and its representations began to be studied. In 1952, Inönü and Wigner[1] described faithful representations of this group. Bargman[2] was the first to point out the fundamental part played by the projective representations of G in nonrelativistic quantum mechanics. It is interesting to note that projective representations of the Galileo group could have been discovered much earlier, since Lie[3] already established the algebra and invariance group of the diffusion equation, which apart from constant coefficients is identical to the one-dimensional Schrödinger equation for a noninteracting particle. On the basis of the invariance algebra of the diffusion equation (or the Schrödinger equation) and using methods known since the beginning of the century (Campbell's formula, Lie's equations), we necessarily arrive at the projective representations of G, as we shall show below.

Lévy-Leblond[4,5] began the systematic investigation of the representations of the Galileo group and equations invariant under it. Hagen and Hurley[6,7] obtained Galileo-invariant differential equations of first order describing the motion of a nonrelativistic particle with arbitrary spin. These equations do not give a complete description of the motion of a particle with spin in an external electromagnetic field, since they do not take into account such well-known physical effects as spin-orbit coupling and the Darwin interaction. In many books and papers, it is even asserted that such interactions are purely relativistic effects and can be adequately described only by means of equations invariant under the Poincaré group (for example, Dirac equations).

The present review, which is based on papers (Refs. 8–14) of the authors, is devoted to the derivation and systematic investigation of a new class of Galileo-invariant equations of motion for particles of arbitrary spin. By means of these equations one can, as in Dirac's relativistic theory for the electron, describe consistently the spin-orbit and Darwin interactions. The obtained equations, which are a special case of the equations of Lévy-Leblond and Hagen and Hurley (the LHH equations), form systems of first- and second-order partial differential equations of parabolic type.

To obtain and analyze Galileo-invariant equations, we use the algebraic approach developed in Refs. 15–17. This approach is based on a general form of the generators of G and the commutation relations of the Lie algebra of the Galileo group, and one uses it to find the explicit form of the time-displacement generator (the Hamiltonian) H, by means of which an invariant equation of the Schrödinger type is determined.

In principle, it is possible to construct a great many different Galileo-invariant equations for particles of arbitrary spin (which is also true of realistic equations) which are equivalent in the absence of an interaction. Therefore, it is important to have criteria to distinguish those that permit the most complete description of physical reality as, for example, in the case of the interaction of a particle with an external electromagnetic field. In the paper, we find a necessary condition which must be satisfied by first-order equations that describe the spin-orbit coupling of a particle to the field. This condition can be formulated as the requirement (not satisfied by the LHH equations) that the generators of the representation of the homogeneous Galileo group, realized on the solution set of the invariant equation, be nilpotent matrices with nilpotency exponent $N>2$.

In many cases, the structure of the equations obtained in the present paper permits the finding of their solutions directly for arbitrary spin. Using these equations, we shall solve exactly the problem of the energy spectrum of a charged nonrelativistic particle of arbitrary spin in a homogeneous magnetic field.

In the paper, we obtain the generators of all classes of irreducible representations of the extended Galileo group. The obtained realization is distinguished by the comparatively simple (symmetric) form of the generators, which is universal for all unitary representations of this group. We also investigate representations of the complete Galileo group, which includes the discrete transformations P, T, and C.

The decompositions of the tensor product of two or

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three irreducible representations of the Galileo group is beyond the scope of the present paper. This and many other questions relating to the theory of representations of G are well discussed in Refs. 4 and 18–20.

1. GALILEO INVARIANCE

In this section, we discuss the invariance algebra of the Schrödinger equation for a noninteracting particle and the discrete transformations P, T, and C in nonrelativistic quantum mechanics. On the basis of the invariance algebra and the Campbell-Hausdorff formula, we construct a representation of the extended Galileo group and calculate the multiplier which characterizes the projective representations of this group.

Invariance algebra of the Schrödinger equation. We investigate the symmetry properties of the basic equation of nonrelativistic quantum mechanics:

$$L\Psi=0,\ L=i\partial/\partial t-p^2/2m, \tag{1}$$

where

$$p^2=p_1^2+p_2^2+p_3^2,\ p_a=-i\partial/\partial x_a,\ \Psi=\Psi(t,\mathbf{x})\subset L_2.$$

We denote by $\{Q_A\}, A=1,2,\ldots,N, N<\infty$ a set of generators defined on a set which is everywhere dense in the space L_2 and form a Lie algebra. By definition, Eq. (1) is invariant under the algebra $\{Q_A\}$ if

$$[Q_A, L]\equiv Q_AL-LQ_A=f_AL, \tag{2}$$

where $\{f_A\}$ is a set of operators defined on L_2. Indeed, if (2) is satisfied, the transformation $\Psi\to Q_A\Psi$ carries a solution of Eq. (1) into another solution of this equation.

We consider the problem of finding the invariance algebra of Eq. (1) in the class of first-order differential operators. This problem reduces to the determination of all possible operators of the form

$$Q_A=B_A(t,\mathbf{x})+C_A^i(t,\mathbf{x})\frac{\partial}{\partial x_i}+D_A(t,\mathbf{x})\frac{\partial}{\partial t},\ i=1,2,3, \tag{3}$$

[where $B_A(t,\mathbf{x})$, $C_A^i(t,\mathbf{x})$, and $D_A(t,\mathbf{x})$ are functions of t and $\mathbf{x}$] that satisfy the conditions (2) and form a finite-dimensional Lie algebra. As was noted above, this problem for the one-dimensional Schrödinger equation was solved for the first time by Lie.[3] The solution is given in Ovsyannikov's book,[21] and it was recently obtained for the three-dimensional case in Refs. 22 and 23, the result of which can be formulated as a theorem.

Theorem 1. The maximal invariance algebra of Eq. (1) in the class of first-order differential operators is the 13-dimensional Lie algebra whose basis elements are given by

$$\left.\begin{aligned}&P_0=i\partial/\partial t,\ P_a=p_a,\ M=m;\\&J_a=(\mathbf{x}\times\mathbf{p})_a,\ G_a=tp_a-mx_a,\end{aligned}\right\} \tag{4}$$

$$D=2tP_0-x_ap_a+3i/2,\ A=t^2P_0-tD-mx^2/2. \tag{5}$$

This theorem will not be proved here (see Refs. 22 and 23). We merely note that the invariance of Eq. (1) under the algebra (4)–(5) can be readily verified directly. The operators (4) satisfy the condition (2) for $f_A\equiv 0$, and the operators (5) satisfy

$$[D, L]=-2iL;\ [A, L]=2itL.$$

The operators (4)–(5) satisfy the commutation relations

$$[P_a, P_b]=0;\ [P_a, J_b]=i\varepsilon_{abc}P_c; \tag{6}$$

$$[G_a, G_b]=0;\ [G_a, J_b]=i\varepsilon_{abc}G_c; \tag{7}$$

$$[P_a, G_b]=i\delta_{ab}M;\ [P_\mu, M]=[G_a, M]=[J_a, M]=0; \tag{8}$$

$$[P_0, P_a]=[P_0, J_a]=0; \tag{9}$$

$$[P_0, G_a]=iP_a,\ a, b, c=1, 2, 3,\ \mu=0, 1, 2, 3; \tag{10}$$

$$\left.\begin{aligned}&[D, P_a]=-iP_a;\ [D, G_a]=iG_a;\ [D, P_0]=-2iP_0;\\&[D, J_a]=[D, M]=[A, G_a]=[A, M]=[A, J_a]=0;\\&[A, P_a]=iG_a;\ [A, P_0]=iD;\ [A, D]=2iA,\end{aligned}\right\} \tag{11}$$

i.e., they form a Lie algebra, which is called the Lie algebra, of the Schrödinger group.

We note that on the solution set of Eq. (1) the operators (5) can be expressed in terms of the generators (4):

$$D=(2M)^{-1}(P_aG_a+G_aP_a);\ A=(2M)^{-1}G_aG_a \tag{12}$$

so that the symmetry under the transformations generated by the operators D and A does not lead to new conservation laws. Thus, we shall be primarily interested in the symmetry of Eq. (1) under the invariance algebra (4) (the Lie algebra of the extended Galileo group).

The algebra (4) has three fundamental invariant operators (Casimir operators):

$$\left.\begin{aligned}&C_1=2MP_0-P_aP_a;\ C_2=M;\\&C_3=W_aW_a=[MJ_a-\varepsilon_{abc}P_bG_c]\,[MJ_a-\varepsilon_{abc}P_bG_c],\end{aligned}\right\} \tag{13}$$

their eigenvalues being associated with the internal energy, mass, and spin of the nonrelativistic particle. Substituting (4) in (13), we see that the Schrödinger equation (1) describes a particle with spin $s=0$, internal energy $\varepsilon_0=0$, and mass m.

Thus, we have considered the symmetry properties of the Schrödinger equation with respect to the invariance algebra whose basis elements belong to the class of first-order differential operators. Note that if we consider the invariance algebra in the class of integro-differential operators, we can show that Eq. (1) is invariant under the algebras O(1,3) (Ref. 24) and O(2,4) (Ref. 25).

It is natural to ask whether there exist other differential equations besides (1) with the same symmetry [the same invariance algebra (4)–(5)] as the Schrödinger equation. An affirmative answer to this question is given in Secs. 2 and 3.

To conclude this section, we formulate the following lemma, which can be directly verified.

Lemma 1. Let $\{P_0, P_a, J_a, G_a, M\}$ be an arbitrary set of operators satisfying the algebra (6)–(10) and the additional requirement that M has an inverse. Then the operators (12) together with P_a, G_a, J_a, M, and $\hat{P}_0 = P_0 - (2M)^{-1}C_1$ satisfy the Lie algebra (6)–(11).

This lemma means that an arbitrary representation of the Galileo algebra (6)–(10) (corresponding to $c_2 \neq 0$) can be extended to a representation of the Lie algebra of the Schrödinger group (6)–(12) (just as an arbitrary representation of the group $P(1,3)$ corresponding to zero mass and a discrete spin can be extended to a representation of the conformal group).[26]

Finite transformations. If the invariance algebra of a differential equation is known, it is usually easy to find its symmetry group. Thus, on the basis of (4), we can obtain in explicit form the Galileo transformations for the coordinates x_a, the time t, and the wave function $\Psi(t, \mathbf{x})$:

$$\left.\begin{aligned} &x_a \to x_a' = U(\theta, \mathbf{v}, \mathbf{a}, a_0, b)\, x_a U^{-1}(\theta, \mathbf{v}, \mathbf{a}, a_0, b) \\ &= R_{ab}x_b + v_a t + a_a; \\ &t \to t' = U(\theta, \mathbf{v}, \mathbf{a}, a_0, b)\, t U^{-1}(\theta, \mathbf{v}, \mathbf{a}, a_0, b) = t + a_0; \end{aligned}\right\} \quad (14a)$$

$$\left.\begin{aligned} &x_a \to x_a'' = U^{-1}(\theta, \mathbf{v}, \mathbf{a}, a_0, b)\, x_a U(\theta, \mathbf{v}, \mathbf{a}, a_0, b) \\ &= R^{-1}_{ab}(x_b - v_b t - a_b); \\ &t \to t'' = U^{-1}(\theta, \mathbf{v}, \mathbf{a}, a_0, b)\, t U(\theta, \mathbf{v}, \mathbf{a}, a_0, b) = t - a_0; \end{aligned}\right\} \quad (14b)$$

$$\begin{aligned} \Psi(t, \mathbf{x}) \to \Psi'(t, \mathbf{x}) &= U^{-1}(\theta, \mathbf{v}, \mathbf{a}, a_0, b)\, \Psi(t, \mathbf{x}) \\ &= \exp[\mathrm{i}f(t', \mathbf{x}') - \mathrm{i}mb]\, \Psi(t', \mathbf{x}'), \end{aligned} \quad (15)$$

where

$$U(\theta, \mathbf{v}, \mathbf{a}, a_0, b) = \exp(\mathrm{i}J_c\theta_c)\exp(\mathrm{i}G_c v_c)\exp[\mathrm{i}P_\mu a^\mu + \mathrm{i}mb]; \quad (16)$$

$\theta_c, v_c, a_c, a_0, b$ are arbitrary real parameters; R_{ab} is the operator of a three-dimensional rotation:

$$R_{ab} = \delta_{ab}\cos\theta + \frac{\varepsilon_{abc}\theta_c}{\theta}\sin\theta + \frac{\theta_a\theta_b}{\theta^2}(1 - \cos\theta); \quad (17)$$

$\theta = (\theta_1^2 + \theta_2^2 + \theta_3^2)^{1/2}$; and $f(t', \mathbf{x}')$ is a phase factor[2]:

$$f(t', \mathbf{x}') = m\mathbf{v}\cdot\mathbf{x} + (1/2)\, m\mathbf{v}^2 t. \quad (18)$$

To prove the relations (14)–(18), it is sufficient to use the Campbell-Hausdorff formula

$$\exp(A)\exp(B) = \exp(A + B + (1/2)[A, B] + \ldots). \quad (19)$$

In accordance with (4) and (19),

$$\exp(-\mathrm{i}G_a v_a) = \exp(-\mathrm{i}tp_a v_a)\exp[\mathrm{i}f(t, \mathbf{x})]$$

and

$$\begin{aligned} U^{-1}(\theta, \mathbf{v}, \mathbf{a}, a_0, b) &= \exp[\mathrm{i}f(t', \mathbf{x}') - \mathrm{i}mb] \\ &\times \exp(-\mathrm{i}J_c\theta_c)\exp[\mathrm{i}p_a(a_a - v_a t) - \mathrm{i}P_0 a_0], \end{aligned} \quad (20)$$

from which the fulfillment of (14)–(18) follows directly.

Thus, a representation of the extended Galileo group on the solution set of Eq. (1) is given by the operators (20), whose action on the wave function and on the independent variables x_a and t is given by Eqs. (14)–(18).

By direct calculation, it is readily shown that the operators (20) satisfy the group law

$$\left.\begin{aligned} &U(\theta^{(2)}, \mathbf{v}^{(2)}, \mathbf{a}^{(2)}, a_0^{(2)}, b^{(2)})\, U(\theta^{(1)}, \mathbf{v}^{(1)}, \mathbf{a}^{(1)}, a_0^{(1)}, b^{(1)}) \\ &= U\Big(\theta^{(1)} + \theta^{(2)}, \mathbf{v}^{(1)} + R^{(1)}\mathbf{v}^{(2)}, \mathbf{a}^{(1)} + R^{(1)}\mathbf{a}^{(2)} + \mathbf{v}^{(1)}a_0^{(2)}; \\ &a_0^{(1)} + a_0^{(2)}, b_0^{(1)} + b_0^{(2)} + v_a^{(1)}R_{ab}^{(1)}a_b^{(2)} + \frac{1}{2}a_0^{(2)}(v_a^{(1)})^2\Big); \\ &U^{-1}(\theta, \mathbf{v}, \mathbf{a}, a_0, b) = U\Big(-\theta, -R^{-1}\mathbf{v}, -R^{-1}(\mathbf{a} - \mathbf{v}a_0), \\ &-a_0, -b_0 + a_c v_c - \frac{1}{2}a_0 v_a v_a\Big), \end{aligned}\right\} \quad (21)$$

where $R\mathbf{a} = \mathbf{a}'$, $R\mathbf{v} = \mathbf{v}'$, $a_a' = R_{ab}a_b$; $v_a' = R_{ab}V_b$, which can be taken as the abstract definition of the extended Galileo group.

Setting $b \equiv 0$ in (15), we arrive at the subgroup of the extended Galileo group known as the Galileo group. Formulas (15) do not define a faithful but only a projective representation of this group. Indeed, the group law for the Galileo transformations (14a) has the form

$$\begin{aligned} &g(\theta^{(1)}, \mathbf{v}^{(1)}, \mathbf{a}^{(1)}, \mathbf{a}_0^{(1)})\, g(\theta^{(2)}, \mathbf{v}^{(2)}, \mathbf{a}^{(2)}, a_0^{(2)}) \\ &= g(\theta^{(1)} + \theta^{(2)}, \mathbf{v}^{(1)} + R^{(1)}\mathbf{v}^{(2)}, \mathbf{a}^{(1)} + R^{(1)}\mathbf{a}^{(2)} + \mathbf{v}^{(1)}a_0^{(2)}, a_0^{(1)} + a_0^{(2)}). \end{aligned} \quad (22)$$

But it follows from (21) that

$$\begin{aligned} &U(\theta^{(2)}, \mathbf{v}^{(2)}, \mathbf{a}^{(2)}, a_0^{(2)}, 0)\, U(\theta^{(1)}, \mathbf{v}^{(1)}, \mathbf{a}^{(1)}, a_0^{(1)}, 0) \\ &= \exp(\mathrm{i}\omega_{12})\, U(\theta^{(1)} + \theta^{(2)}, \mathbf{v}^{(1)} + R^{(1)}\mathbf{v}^{(2)}, \mathbf{a}^{(1)} + R^{(1)}\mathbf{a}^{(2)} \\ &+ \mathbf{v}^{(1)}a_0^{(2)}, a_0^{(1)} + a_0^{(2)} 0), \end{aligned} \quad (23)$$

with phase factor

$$\omega_{12} = v_a^{(1)}R_{ab}^{(1)}a_b^{(2)} + (1/2)\, a_0^{(2)}v_a^{(1)}v_a^{(1)}. \quad (24)$$

In other words, for $b \equiv 0$ the operators (20) satisfy the group composition law (22) only up to multiplication by the factor $\exp(i\omega_{12})$, which does not change the norm of the wave function.

We have shown that the invariance algebra of Eq. (1) given by the operators (4) contains complete information about the symmetry properties of this equation under the continuous transformations. One can also consider discrete transformations of the form

$$\left.\begin{aligned} &x_a \to -x_a, \quad t \to t; \\ &x_a \to x_a, \quad t \to t; \end{aligned}\right\} \quad (25)$$

$$\Psi(t, \mathbf{x}) \to P\Psi(t, \mathbf{x}) = \eta_1\Psi(t, -\mathbf{x}); \quad (26)$$

$$\Psi(t, \mathbf{x}) \to T\Psi(t, \mathbf{x}) = \eta_2\Psi(-t, \mathbf{x}); \quad (27)$$

$$\Psi(t, \mathbf{x}) \to C\Psi(t, \mathbf{x}) = \eta_3\Psi^*(t, \mathbf{x}), \quad (28)$$

where $\eta_a = \pm 1$. By definition, the operators (25)–(28) satisfy the conditions $P^2 = T^2 = C^2 = I$, where I is the identity operator.

It is easy to show that the Schrödinger equation (1) is invariant under P and CT but not under C and T. Of course, this does not mean that there do not exist Galileo-invariant equations with other symmetry properties under the transformations (25)–(28). Therefore, we consider the complete Galileo group, which is defined as the set of transformations (14)–(18) and (25)–(28). An arbitrary operator of such transformations in the space of square-integrable functions can be represented in the form

$$U(\theta, \mathbf{v}, \mathbf{a}, a_0, b, \varepsilon_P, \varepsilon_T, \varepsilon_C) = U(\theta, \mathbf{v}, \mathbf{a}, a_0, b)\, P^{\frac{(1-\varepsilon_P)}{2}} T^{\frac{(1-\varepsilon_T)}{2}} C^{\frac{(1-\varepsilon_C)}{2}}, \tag{29}$$

where $U(\theta, \mathbf{v}, \mathbf{a}, a_0, b)$ is defined in (16) and (20), and ε_p, ε_T, and ε_C are parameters which independently take the values +1 or −1.

The operators (29) satisfy the group law

$$\begin{aligned} &U(\theta^{(2)}, \mathbf{v}^{(2)}, \mathbf{a}^{(2)}, a_0^{(2)}, b^{(2)}, \varepsilon_P^{(2)}, \varepsilon_T^{(2)}, \varepsilon_C^{(2)}) \\ &\quad \times U(\theta^{(1)}, \mathbf{v}^{(1)}, \mathbf{a}^{(1)}, a_0^{(1)}, b^{(1)}, \varepsilon_P^{(1)}, \varepsilon_T^{(1)}, \varepsilon_C^{(1)}) \\ &\quad = U(\theta^{(1)} + \varepsilon_C^{(1)}\theta^{(2)}, \mathbf{v}^{(1)} + \varepsilon_P^{(1)}\varepsilon_T^{(1)}\varepsilon_C^{(1)} R^{(1)}\mathbf{v}^{(2)}, \mathbf{a}^{(1)} \\ &\qquad + \varepsilon_P^{(1)}\varepsilon_C^{(1)} R^{(1)}\mathbf{a}^{(2)} + \varepsilon_T^{(1)}\varepsilon_C^{(1)}\mathbf{v}^{(1)} a_0^{(2)}, a_0^{(1)} \\ &\qquad + \varepsilon_T^{(1)}\varepsilon_C^{(1)} a_0^{(2)}, b^{(1)} + \varepsilon_C^{(1)} b^{(2)} + \varepsilon_P^{(1)}\varepsilon_C^{(1)} v_a^{(1)} R_{ab}^{(1)} a_b^{(2)} \\ &\qquad + (1/2)\,\varepsilon_T^{(1)}\varepsilon_C^{(1)} a_0^{(2)} v_a^{(1)} v_a^{(1)}, \varepsilon_P^{(1)}\varepsilon_P^{(2)}, \varepsilon_T^{(1)}\varepsilon_T^{(2)}, \varepsilon_C^{(1)}\varepsilon_C^{(2)}); \\ &U^{-1}(\theta, \mathbf{v}, \mathbf{a}, a_0, b, \varepsilon_P, \varepsilon_T, \varepsilon_C) = U(-\varepsilon_C\theta, -\varepsilon_P\varepsilon_T\varepsilon_C R^{-1}\mathbf{v}, \\ &\qquad - R^{-1}(\varepsilon_P\varepsilon_C\mathbf{a} - \varepsilon_P\mathbf{v}a_0); -\varepsilon_C\varepsilon_T a_0, \\ &\qquad - \varepsilon_C b + \varepsilon_T a_a v_a - (1/2)\,\varepsilon_C\varepsilon_T a_0 v_a v_a, \varepsilon_P, \varepsilon_T, \varepsilon_C). \end{aligned} \tag{30}$$

We shall regard the group composition law (30) as the definition of the 14-parameter group $\tilde{G}$. Below, we describe the class of irreducible projective representations of the group $\tilde{G}$ (30) corresponding to nonzero values of the invariant operator C_2 (13).

Irreducible representations of the algebra (6)–(10) *in the configuration space.* The irreducible representations of (6)–(10) can be divided into three classes corresponding to the following values of the invariant operators C_2 and C_3 (see Ref. 4):

I. $c_2 = m \neq 0,\ c_3 = m^2 s(s+1),\ s = 0, 1/2, 1, \ldots;$

II. $c_2 = 0,\ c_3 = 0;$

III. $c_2 = 0,\ c_3 = r^2 > 0.$

In Appendix 1, we find explicitly all inequivalent representations of the algebra (6)–(10) in the momentum space.

To investigate differential equations invariant under the Galileo group, we use representations of the Lie algebra of the extended Galileo group of the first class in the space of square-integrable functions $\Psi(t, \mathbf{x})$. Irreducible representations of the first class are specified by three numbers: ε_0 [the eigenvalue of the Casimir operator C_1 (13)], m and s. The explicit form of the corresponding generators P_0, P_a, J_a, G_a, and M is given by

$$\begin{aligned} &P_0 = p^2/2m + \varepsilon_0;\ P_a = p_a = -i\partial/\partial x_a,\ M = m; \\ &J_a = (\mathbf{x} \times \mathbf{p})_a + S_a,\ G_a = t p_a - m x_a, \end{aligned} \tag{31}$$

where S_a are $(2S+1) \times (2S+1)$ matrices forming the representation $D(s)$ of the Lie algebra of the group $O(3)$, and ε_0 and m are arbitrary real numbers. It can be shown by direct calculation that the operators (31) satisfy the commutation relations (6)–(10). The invariant operators (13) for the generators (31) take the form

$$C_1 = 2m\varepsilon_0,\ C_2 = m,\ C_3 = m^2 S^2 = m^2 s(s+1).$$

Finally, the operators (31) are Hermitian with respect to the scalar product

$$(\Psi_1, \Psi_2) = \int d^3x\, \Psi_1^\dagger(t, \mathbf{x})\, \Psi_2(t, \mathbf{x}), \tag{32}$$

where $\Psi(t, \mathbf{x})$ are $(2s+1)$-component functions:

Ψ is the column $(\Psi_1, \Psi_2, \ldots, \Psi_{2s+1})$, $\Psi_a \subset L_2$.

In other words, the operators (31) form Hermitian irreducible representations of the algebra (6)–(10).

In the case $s = 0$, the generators (31) reduce to the representation (4), which is realized on the solution set of the Schrödinger equation. In the general case, the space of the irreducible representation (31) can be associated with the state space of a free nonrelativistic particle with mass m, spin s, and internal energy ε_0.

Using (19), we can readily find the transformations of the wave function $\Psi(t, \mathbf{x})$ generated by the operators (31):

$$\begin{aligned} \Psi(t, \mathbf{x}) \to \Psi'(t, \mathbf{x}) &= \exp(-iP_\mu a^\mu - imb) \\ &\times \exp(-iG_a v_a)\exp(-iJ_a\theta_a)\,\Psi(t, \mathbf{x}) \\ &= \exp[if(t', \mathbf{x}') - imb]\, D^s(\theta)\, \Psi(t', \mathbf{x}'), \end{aligned} \tag{33}$$

where $D^s(\theta)$ are numerical matrices forming the representation $D(s)$ of $O(3)$:

$$D^s(\theta) = \exp(-i\mathbf{S}\cdot\theta), \tag{34}$$

and $\mathbf{x}'$, t' and $f(t', \mathbf{x}')$ are given by (14) and (18).

By the Galileo transformations for the wave function one sometimes understands the transition from $\Psi(t, \mathbf{x})$ to $\Psi''(t'', \mathbf{x}'')$, where $\Psi''(t'', \mathbf{x}'')$ is the function obtained from (33) by the substitution $\mathbf{x} \to \mathbf{x}''$, $t \to t''$ [see (14b)]. Making such a substitution on the right-hand side of (33), we arrive at the transformation

$$\Psi(t, \mathbf{x}) \to \Psi''(t'', \mathbf{x}'') = \exp[if(t, \mathbf{x}) - imb]\, D^s(\theta)\, \Psi(t, \mathbf{x}). \tag{35}$$

Formulas (14) and (33) [or (14) and (35)] determine the irreducible representation $D(m, \varepsilon_0, s)$ of the Galileo group in the configuration space.

Below, we also use reducible representations of the algebra (6)–(10) with the basis elements

$$\left.\begin{aligned} &P_0 = i\frac{\partial}{\partial t};\ P_a = p_a = -i\frac{\partial}{\partial x_a}; \\ &J_a = (\mathbf{x} \times \mathbf{p})_a + S_a;\ G_a = t p_a - m x_a + \eta_a, \end{aligned}\right\} \tag{36}$$

where η_a are numerical matrices that satisfy in conjunction with S_a the Lie algebra of the homogeneous Galileo group:

$$[\eta_a, \eta_b] = 0;\ [\eta_a, S_b] = i\varepsilon_{abc}\eta_c;\ [S_a, S_b] = i\varepsilon_{abc}S_c. \tag{37}$$

Formulas (36) give the general form of the generators of the Galileo group in the space of the square-integrable functions $\Psi(t, \mathbf{x})$ = the column $(\Psi_1, \Psi_2, \ldots, \Psi_n)$ generated by the local transformations

$$\Psi(t, \mathbf{x}) \to \Psi''(t'', \mathbf{x}'') = \exp[if(t, \mathbf{x}) - imb]\, D(\theta, \mathbf{v})\, \Psi(t, \mathbf{x}), \tag{38}$$

where $\mathbf{x}''$, t'', and $f(t, \mathbf{x})$ are defined in (14) and (18), and $D(\theta, \mathbf{v})$ are numerical matrices forming a representation of the homogeneous Galileo group:

$$D(\theta, \mathbf{v}) = \exp(-i\mathbf{S}\theta)\exp(-i\eta\mathbf{v}). \tag{39}$$

Discrete-symmetry operators. We now consider the representations of the complete Galileo group $\tilde{G}$ defined by the group law (30).

If $\varepsilon_p = \varepsilon_T = \varepsilon_C \equiv 1$, then the group $\tilde{G}$ reduces to the extended Galileo group G (21), which is a subgroup of $\tilde{G}$. It follows from (30) that the factor group $\tilde{G}/G$ contains the eight elements $\{I, P, C, T, PC, PT, CT, CPT\}$, corresponding to the parameter values $\varepsilon_T, \varepsilon_P, \varepsilon_C, \varepsilon_{PC} = \varepsilon_P\varepsilon_C$,

$\varepsilon_{PT}=\varepsilon_P\varepsilon_T$, $\varepsilon_{CT}=\varepsilon_C\varepsilon_T$, $\varepsilon_{CPT}=\varepsilon_P\varepsilon_C\varepsilon_T$, where ε_P, ε_C, $\varepsilon_T=\pm 1$. The group transformation law for the elements of the group $\tilde{G}/G$ can be represented in the form

Elements	I	P	T	C	PT	CP	CT	CPT
I	I	P	T	C	PT	CP	CT	CPT
P	P	I	PT	PC	T	C	CPT	CT
T	T	PT	I	CT	P	CPT	C	CP
C	C	CP	CT	I	CPT	P	T	PT
PT	PT	T	P	CPT	I	CT	CP	C
CP	CP	C	CPT	P	CT	I	PT	T
CT	CT	CPT	C	T	CP	PT	I	P
CPT	CPT	CT	CP	PT	C	T	P	I

We shall seek representations of the group (30) in the space of square-integrable functions $\Psi(t, \mathbf{x})$ with the scalar product (32). We shall consider only the representations of $\tilde{G}$ which reduce on reduction with respect to G to representations of the first class (when $m\neq 0$). The generators of G for representations of the first class are given by (31), and it therefore remains only to determine the explicit form of the operators P, T, and C, which generate the representation of the factor group $\tilde{G}/G$.

We conclude from (29) and (30) that the operators P, T, and C must satisfy the following commutation relations with the generators P_μ, J_a, G_a, and M:

$$PJ_a=J_aP;\ PP_0=P_0P;\ PM=MP; \tag{40}$$

$$PP_a=-P_aP;\ PG_a=-G_aP; \tag{41}$$

$$TJ_a=J_aT;\ TP_a=P_aT; \tag{42}$$

$$TP_0=-P_0T,\ TG_a=-G_aT,\ TM=-MT; \tag{43}$$

$$CJ_a=-J_aC;\ CP_a=-P_aC;\ CG_a=-G_aC; \tag{44}$$

$$CP_0=-P_0C;\ CM=-MC. \tag{45}$$

It also follows from (30) that the operators P, T, and C satisfy the conditions

$$C^2=T^2=P^2=1;\ CP=PC;\ CT=TC;\ PT=TP. \tag{46}$$

Since we are interested in not only faithful but also projective representations of the group (30), the relations (46) must be satisfied up to a phase factor[27]:

$$C^2=\exp(i\varphi_3);\ T^2=\exp(i\varphi_2);\ P^2=\exp(i\varphi_1);$$
$$CP=PC\exp(i\varphi_4);\ CT=TC\exp(i\varphi_5);\ PT=TP\exp(i\varphi_6), \tag{47}$$

where φ_n are real numbers, and we can set $\varphi_1=\varphi_2=0$.

It can be seen from (40)–(45) that the operators P, T, and C do not commute with the invariant operators (13) (for example, T does not commute with C_2), and therefore only the space of a reducible representation of the algebra (6)–(10) can serve as the domain of definition of the operators P, T, and C. The generators of such a representation for $m\neq 0$ can be chosen in the form of a direct sum of the generators (31) (to simplify the calculations, we set $\varepsilon_0=0$):

$$P_0=\mathbf{P}^2(2M)^{-1};\ P_a=p_a=-i\frac{\partial}{\partial x_a};\ M=\tilde{M};$$
$$J_a=(\mathbf{x}\times\mathbf{p})_a+S_a;\ G_a=tp_a-Mx_a, \tag{48}$$

where S_a are the generators of the reducible representation of the group $O(3)$, and M is a numerical matrix which commutes with S_a.

It follows from (40)–(45) and (48) that the operators P, T, and C satisfy the conditions

$$Px_a=-x_aP;\ Pp_a=-p_aP;\ Pt=tP;$$
$$Tx_a=x_aT;\ Tp_a=p_aT;\ Tt=-tT; \tag{49}$$
$$Cx_a=x_aC;\ Cp_a=-p_aC;\ Ct=tC,$$

where x_a and t are operators of multiplication by the independent variables. The general form of the operators satisfying (49) can be specified by the formulas

$$\left.\begin{aligned}P\Psi(t,\mathbf{x})&=r_1\Psi(t,-\mathbf{x});\\ T\Psi(t,\mathbf{x})&=r_2\Psi(-t,\mathbf{x});\\ C\Psi(t,\mathbf{x})&=r_3\Psi^*(t,\mathbf{x}),\end{aligned}\right\} \tag{50}$$

where r_a are numerical matrices.

From (40)–(45), (48), and (49) we find that the matrices r_a must satisfy the conditions

$$r_0r_2=-r_2r_0;\ r_1^2=1;\ r_2^2=1; \tag{51}$$

$$r_0S_a=S_ar_0;\ r_2S_a=S_ar_2; \tag{52}$$

$$r_0r_1=r_1r_0;\ r_1^2=1,\ r_1r_2=r_2r_1\exp(i\varphi_6);\ r_1S_a=S_ar^1; \tag{53}$$

$$r_3r_1=r_1^*r_3\exp(i\varphi_5);\ r_3r_2=r_2^*r_3\exp(i\varphi_4);$$
$$r_3^2=\exp(i\varphi_3);\ r_3r_0=r_0^*r_3, \tag{54}$$

where r_0 is the operator of the sign of the mass.

$$r_0=M\cdot|M|^{-1}. \tag{55}$$

Thus, the problem of describing the representations of the group $\tilde{G}$ for $m\neq 0$ reduces to the solution of the system of equations (51)–(54) for the matrices r_μ and S_a.

We give the solution of the system (51)–(54) in the form of the following theorem.

Theorem 2.: All possible (up to equivalence) irreducible matrices satisfying the system of relations (51)–(55) can be labeled by the set of numbers $(s, \varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \eta)$, where $s=0, 1, \frac{1}{2}, \ldots, \varepsilon_\mu$ and $\eta=\pm 1$.

The explicit form of the corresponding matrices is given by the following formulas: for $\varepsilon_0=1$; $\varepsilon_2=1$; ε_1, $\varepsilon_3, \eta=\pm 1$

$$r_1=\eta\begin{pmatrix}I&0\\0&\varepsilon_3 I\end{pmatrix};\ r_2=\begin{pmatrix}0&I\\I&0\end{pmatrix};\ r_3=\begin{pmatrix}I&0\\0&\varepsilon_1 I\end{pmatrix}\Delta_2;$$
$$r_0=\begin{pmatrix}I&0\\0&-I\end{pmatrix};\ S_a=\begin{pmatrix}S_a&0\\0&S_a\end{pmatrix};\ \Delta_2=\begin{pmatrix}\Delta&0\\0&\Delta\end{pmatrix}; \tag{56}$$

for $\varepsilon_0=-1$; $\varepsilon_2=1$; $\varepsilon_1, \varepsilon_3, \eta=\pm 1$

$$r_1=\eta\begin{pmatrix} I & 0 & 0 & 0\\ 0 & \varepsilon_3 I & 0 & 0\\ 0 & 0 & I & 0\\ 0 & 0 & 0 & \varepsilon_3 I\end{pmatrix};\quad r_2=\begin{pmatrix} 0 & I & 0 & 0\\ I & 0 & 0 & 0\\ 0 & 0 & 0 & I\\ 0 & 0 & I & 0\end{pmatrix};$$

$$r_3=\begin{pmatrix} 0 & 0 & -I & 0\\ 0 & 0 & 0 & -\varepsilon_1 I\\ I & 0 & 0 & 0\\ 0 & \varepsilon_1 I & 0 & 0\end{pmatrix};\quad r_0=\begin{pmatrix} I & 0 & 0 & 0\\ 0 & -I & 0 & 0\\ 0 & 0 & I & 0\\ 0 & 0 & 0 & -I\end{pmatrix}; \tag{57}$$

$$S_a=\begin{pmatrix} S_a & 0 & 0 & 0\\ 0 & S_a & 0 & 0\\ 0 & 0 & S_a & 0\\ 0 & 0 & 0 & S_a\end{pmatrix};\quad \Delta_4=\begin{pmatrix} \Delta & 0 & 0 & 0\\ 0 & \Delta & 0 & 0\\ 0 & 0 & \Delta & 0\\ 0 & 0 & 0 & \Delta\end{pmatrix};$$

for $\varepsilon_2=-1$; $\varepsilon_0, \varepsilon_1, \varepsilon_3=\pm 1$

$$r_1=\begin{pmatrix} I & 0 & 0 & 0\\ 0 & \varepsilon_3 I & 0 & 0\\ 0 & 0 & -I & 0\\ 0 & 0 & 0 & -\varepsilon_3 I\end{pmatrix};\quad r_2=\begin{pmatrix} 0 & I & 0 & 0\\ I & 0 & 0 & 0\\ 0 & 0 & 0 & I\\ 0 & 0 & I & 0\end{pmatrix};$$

$$r_3=\begin{pmatrix} 0 & 0 & \varepsilon_0 I & 0\\ 0 & 0 & 0 & \varepsilon_0\varepsilon_1 I\\ I & 0 & 0 & 0\\ 0 & \varepsilon_1 I & 0 & 0\end{pmatrix}; \tag{58}$$

$$r_0=\begin{pmatrix} I & 0 & 0 & 0\\ 0 & -I & 0 & 0\\ 0 & 0 & I & 0\\ 0 & 0 & 0 & -I\end{pmatrix};\quad S_a=\begin{pmatrix} S_a & 0 & 0 & 0\\ 0 & S_a & 0 & 0\\ 0 & 0 & S_a & 0\\ 0 & 0 & 0 & S_a\end{pmatrix},$$

where S_a are the generators of the irreducible representation $D(s)$ of the group O, and I and 0 are the $(2s+1)$-row unit matrix and matrix with components zero; Δ are matrices determined up to the sign by the relations[27]

$$\Delta s_a=-s_a^*\Delta;\quad \Delta^2=(-1)^{2s}. \tag{59}$$

Proof. Consider the relations (51) and (52). Using Schur's lemma and bearing in mind that the representations of the algebra (51) can be expressed as a direct sum of Pauli matrices, we obtain irreducible representations of the relations (51) and (52) in the form

$$r_0=\begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix};\quad r_2=\begin{pmatrix} 0 & I\\ I & 0\end{pmatrix};\quad S_a=\begin{pmatrix} s_a & 0\\ 0 & s_a\end{pmatrix}, \tag{60}$$

where s_a, I, and 0 are the matrices defined in the statement of the theorem.

We now consider the relations (51)–(53). Since the operators S_a, r_0, and r_2 can always be represented as a direct sum of the matrices (60), we can readily show that $\varphi_6=\pm\pi$ and that the irreducible solutions of the system of relations (51)–(53) are given by (60) and (61):

$$r_1=\eta\begin{pmatrix} I & 0\\ 0 & \varepsilon_3 I\end{pmatrix};\quad \eta=\pm 1. \tag{61}$$

Finally, representing r_1, r_2 and r_0, S_a as direct sums of the matrices (56) and (57), taking into account the relations

$$r_1^*=r_1;\quad r_2^*=r_2;\quad r_0^*=r_0$$

and using the equivalence transformations

$$r_k\to r_k'=Vr_kV^{-1}\ (k\neq 3);\quad r_3\to r_3'=Vr_3(V^{-1})^*,$$

we obtain irreducible solutions of the relations (51)–(54) in the form (56)–(59).

Note that the values of the parameters ε_μ are related as follows to the properties of the matrices r_μ:

$$r_1r_2=\varepsilon_3r_2r_1;\quad r_1r_3=\varepsilon_2r_3r_1;$$
$$r_2r_3=\varepsilon_1r_3r_2;\quad r_3^2=\varepsilon_0(-1)^{2s}. \tag{62}$$

Formulas (46) and (52)–(54) define all possible (up to equivalence) representations of the operators P, T, and C satisfying the conditions (40)–(45), (47), and (48). From (40)–(45), (47), and (48), we conclude that

$$M=r_0m. \tag{63}$$

Therefore, *the irreducible projective representations of the complete Galileo group* (30) *corresponding to* $m\neq 0$ *can be labeled by the numbers* ε, s, m, ε_μ, *and* η *and are given by* (29), (48), (50), (56)–(58), and (63).

Note that the representations of a subgroup of $\tilde{G}$ [including the transformations (14), (15), and (26) and the product CT, where C and T are defined in (27) and (28)] are found in Ref. 19.

2. SECOND-ORDER DIFFERENTIAL EQUATIONS FOR PARTICLES WITH ARBITRARY SPIN

In this section, we obtain two classes of Galileo-invariant systems of second-order differential equations for particles of arbitrary spin and give their Lagrangian formulation.

Statement of the problem. The Schrödinger equation (1) is invariant under the extended Galileo group and describes the motion of a free spinless particle. It is natural to consider whether there exist equations of the form

$$i\frac{\partial}{\partial t}\Psi(t,\mathbf{x})=H_s(\mathbf{p})\Psi(t,\mathbf{x}), \tag{64}$$

where $H_s(\mathbf{p})$ is some differential operator, which, like Eq. (1), have Galilean symmetry but describe particles with arbitrary spin. The present section is devoted to the derivation of such equations.

Definition 1. Equation (64) is Galileo-invariant if the operator $L=i(\partial/\partial t)-H_s(\mathbf{p})$ satisfies the conditions (2), where $\{Q_a\}$ is a set of operators $\{P_0, P_a, J_a, G_a, M\}$ satisfying the algebra (6)–(10).

We shall seek the invariant equations (64) in the space of $2(2s+1)$-component square-integrable functions

$$\Psi(t,\mathbf{x})=\begin{pmatrix}\Psi_1(t,x)\\ \Psi_2(t,x)\\ \vdots\\ \Psi_{2(2s+1)}\end{pmatrix},\quad \Psi_n\subset L_2. \tag{65}$$

We solve the problem of describing such equations in two approaches, which are in general inequivalent. In the first approach (I), the problem is formulated as follows: To find all possible (up to equivalence) operators H_s^{I} satisfying the condition (2) of Galileo invariance if the generators of the Galileo group have the form (36), where

$$S_a=\begin{pmatrix} s_a & 0\\ 0 & s_a\end{pmatrix};\quad \eta_a=k(\sigma_1+i\sigma_2)S_a, \tag{66}$$

s_a are generators of the irreducible representation $D(s)$ of the group $O(3)$, σ_1 and σ_2 are $2(2s+1)$-row Pauli matrices,

$$\sigma_0=\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix};\quad \sigma_1=\begin{pmatrix} 0 & I\\ I & 0\end{pmatrix};\quad \sigma_2=i\begin{pmatrix} 0 & -I\\ I & 0\end{pmatrix};\quad \sigma_3=i\begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix}, \tag{67}$$

I and 0 are the $(2s+1)$-row unit matrix and matrix with zeros as elements, and k is an arbitrary complex parameter.

Formulas (36) and (66) give the most general (up to equivalence) form of the generators of the Galileo group corresponding to the local transformations (38) of the wave function (65).

We shall show below that the operators $H_s^{\rm I}$ can always be chosen to make Eq. (64) invariant under the product $P\cdot T\cdot C$, where the operators of the transformations P, T, and C are given in (50) and (56).

Substituting (36) and (64) in (2), we see that Eq. (64) satisfies the condition of Galileo invariance if the Hamiltonian $H_s^{\rm I}$ satisfies the conditions

$$[H_s^{\rm I}, P_a]=[H_s^{\rm I}, J_a]=0; \tag{68}$$

$$[H_s^{\rm I}, G_a]=iP_a, \tag{69}$$

where P_a, J_a, and G_a are the generators (36).

In the second approach (II), the problem reduces to finding all possible differential operators $H_s^{\rm II}$ such that the operators

$$\left.\begin{array}{l} P_0^{\rm II}=H_s^{\rm II};\ P_s^{\rm II}=p_a=-i\frac{\partial}{\partial x_a};\ M^{\rm II}=\sigma_1 m;\\ J_a^{\rm II}=(\mathbf{x}\times\mathbf{p})_a+S_a;\\ G_a^{\rm II}=tp_a-Mx_a+\eta_a^{\rm II}\end{array}\right\} \tag{70}$$

satisfy the algebra (6)–(10). Here, σ_1 is one of the Pauli matrices (67), $\eta_a^{\rm II}$ are certain operators (whose explicit form must be found), and S_a are the matrices (66).

It can be shown that formulas (70) give the most general (up to equivalence) form of the generators of G when the invariant operators (13) have the values $|c_2|=m$ and $c_3=m^2s(s+1)$, and Eq. (60) is invariant under the transformation $P\cdot T$, where P and T are defined in (50) and (56) for $\varepsilon_3=-1$.

We require the generators (70) to be Hermitian under the scalar product (32) [where $\Psi(t,\mathbf{x})$ are the $2(2s+1)$-component functions (65)], which is usually adopted in quantum mechanics. An important difference between the representations (36) and (70) is that the generators $H_s^{\rm I}$ and $G_a^{\rm I}$ are not Hermitian under (32), but they are Hermitian in the Hilbert space with the scalar product

$$(\Psi_1, \Psi_2)=\int d^3x\Psi_1^\dagger\hat{M}\Psi_2, \tag{71}$$

where $\hat{M}$ is some positive-definite differential operator, or with respect to the indefinite metric (71) when $\hat{M}$ is some positive-indefinite numerical matrix. The explicit form of $\hat{M}$ is given below. The more complicated metric is a consequence of the local transformation properties (38) of the function $\Psi(t,\mathbf{x})$.

We require the Hamiltonian $H_s^{\rm II}$ to satisfy the condition

$$(H_s^{\rm II})^2=(m+p^2/2m)^2. \tag{72}$$

This is equivalent to the requirement that the internal energy of the particle [the eigenvalue of the invariant operator $G_1(13)$] be equal to its mass.

Thus, the problem of describing Galileo-invariant equations of the form (64) reduces to the solution of the system of commutation relations (68)–(69) for the operators $H_s^{\rm I}$ and $P_a^{\rm I}$, $G_a^{\rm I}$, $J_a^{\rm I}$, $M^{\rm I}$ (36), (66) and the relations (6)–(10) for the generators (70).

Explicit Form of the Hamiltonians $H_s^{\rm I}$. We give the solution of Problem 1 in the form of the following theorem.

Theorem 3. All possible (up to equivalence) Hamiltonians $H_s^{\rm I}$ that satisfy together with the generators (36) and (66) the commutation relations (68) and (69) are given by

$$H_s^{\rm I}=\sigma_1 am+2iak\sigma_3 S_a p_a+(1/2m)C_{ab}p_a p_b; \tag{73}$$

$$\tilde{H}_s^{\rm I}=(a/2)(\sigma_1-i\sigma_2)m+\sigma_3\tilde{a}m-2a\tilde{k}(i\sigma_1-\sigma_2)S_a p_a+(1/2m)C_{ab}p_a p_b, \tag{74}$$

where

$$C_{ab}=\delta_{ab}-2ak^2(\sigma_1+i\sigma_2)(S_aS_b+S_bS_a); \tag{75}$$

a, $\tilde{a}$, and k are arbitrary parameters.

Proof. The general form of the Hamiltonian $H_s^{\rm I}$ is most readily found in the representation in which $\eta_a^{\rm I}=0$. The transition to such a representation is realized by means of the transformation

$$\begin{aligned} &H_s^{\rm I}\to(H_s^{\rm I})'=VH_s^{\rm I}V^{-1},\ P_a^{\rm I}\to(P_a^{\rm I})'\\ &=VP_a^{\rm I}V^{-1}=p_a,\ J_a^{\rm I}\to(J_a^{\rm I})'=VJ_a^{\rm I}V^{-1}\\ &=J_a^{\rm I},\ G_a^{\rm I}\to(G_a^{\rm I})'=VG_a^{\rm I}V^{-1}=tp_a-mx_a,\end{aligned} \tag{76}$$

where

$$V=\exp[(i/m)\eta_a p_a]=1+(i/m)\eta_a p_a. \tag{77}$$

The transformation (76), (77) reduces the generators (36) to the direct sum of the generators (31).

From (68), (69), and (76), we find the general form of the operator $(H_s^{\rm I})'$:

$$(H_s^{\rm I})'=p^2/2m+A,\ A=\sigma_\mu a^\mu m, \tag{78}$$

where σ_μ are the matrices (67), and a^μ are arbitrary complex coefficients.

Thus, in the representation (76) Eq. (64) takes the form

$$i\frac{\partial}{\partial t}\Psi'=(p^2/2m+\sigma_\mu a^\mu m)\Psi',\ \Psi'=V\Psi, \tag{79}$$

and the equation in the original Ψ representation can be obtained from (79) by means of the transformation which is the inverse of (76) and (77).

We show that the matrix A in (78) can be reduced to one of the following forms:

$$A=\sigma_3\tilde{a}m+(a/2)(\sigma_1-i\sigma_2)m \tag{80}$$

or

$$A=\sigma_1 am, \tag{81}$$

where a and $\tilde{a}$ are arbitrary coefficients.

Indeed, the coefficient a_0 can always be made to vanish by means of the unitary transformation

$$(H_s^{\mathrm{I}})' \to \exp(\mathrm{i}a_0mt)\,(H_s^{\mathrm{I}})'\exp(-\mathrm{i}a_0mt)$$
$$+\mathrm{i}\exp(\mathrm{i}a_0mt)\frac{\partial}{t}\exp(-\mathrm{i}a_0mt) = (H_s^{\mathrm{I}})' - a_0m. \quad (82)$$

Further, we have the three possibilities

$$A \equiv 0,\ a_b = 0;\ b = 1, 2, 3; \quad (83)$$

$$A^2 = a_1^2 + a_2^2 + a_3^2 = 0,\ a_b \neq 0; \quad (84)$$

$$A^2 = a_1^2 + a_2^2 + a_3^2 = a^2 \neq 0. \quad (85)$$

Equation (83) is identical to (80) for $a = \tilde{a} = 0$. The case (84) corresponds to a nonunitary representation of the extended Galileo group (the invariant operator $C_1 = 2mP_0 - P_aP_a = 2m^2A$ is given by a nilpotent matrix) and must therefore be rejected.

We consider the third possibility, i.e., (85). Suppose $\tilde{a}_1^2 + \tilde{a}_2^2 \neq 0$; then the transformation

$$A \to V_1AV_1^{-1};$$
$$V_1 = b + \mathrm{i}\sigma_3c + (\sigma_1 + \mathrm{i}\sigma_2)\,d;\ V^{-1} = b - \mathrm{i}\sigma_3c$$
$$-(\sigma_1 + \mathrm{i}\sigma_2)\,d;\ b = \cos\varphi;\ c = \sin\varphi; \quad (86)$$
$$\varphi = \frac{1}{2}\operatorname{arctg}\left(\frac{a_1 + 2d^2}{a_2 - 2\mathrm{i}d^2}\right);\quad d = \left[\frac{a_3^2(a_1 - \mathrm{i}a^2)}{4a\,(a_1^2 + a_2^2)}\right]^{1/2}, \quad (87)$$

carries the matrix A (78) into the form (81). But if $a_1^2 + a_2^2 = 0$, then by means of the transformation $A \to V_2AV_2^{-1}$, where

$$V_2 = 1 + (\sigma_1 + \mathrm{i}\sigma_2)\,(f/2);\ V_2^{-1} = 1 - (\sigma_1 + \mathrm{i}\sigma_2)\,(f/2),$$
$$f = \begin{cases} a_1/a_3, & \text{if}\quad a_2 = \mathrm{i}a_1, \\ 0, & \text{if}\quad a_2 = -\mathrm{i}a_1, \end{cases} \quad (88)$$

the matrix (82) is reduced to the form (80).

The operators (86) and (88) satisfy the conditions

$$V_\alpha\eta_a V_\alpha^{-1} = \varkappa_\alpha\eta_a,\quad \alpha = 1, 2, \quad (89)$$

where η_a are the matrices (66), κ_α are numerical coefficients ($\kappa_1 = \exp(2i\varphi)$, $\kappa_2 = 1$), and the parameter φ is given in (87). It is easy to show that no operator exists which satisfies one of the conditions (89) and transforms (80) to the form (81).

Subjecting the operators (78), (80), and (81) to the transformation that is the inverse of (76), we arrive at the Hamiltonians (73) and (74). The operators (73) and (74) obviously satisfy the conditions (68) and (69). In addition, these operators exhaust all possible solutions of the relations (36), (66), (68), and (69) up to the equivalence transformations $H_s^{\mathrm{I}} \to V_\alpha H_s^{\mathrm{I}} V_\alpha^{-1} + (\partial V_\alpha/\partial t)V_\alpha^{-1}$, where V_α are the numerical matrices (88), which in accordance with (89) do not change the general form of the generators (36) and (66). This proves the theorem.

Thus, we have obtained Galileo-invariant differential equations in the form (64), (73), and (74). The invariance of these equations under the Galileo transformations (14) and (38) can be verified directly. Indeed, using the relation [see (66)]

$$\exp(-\mathrm{i}\eta_a v_a) = 1 - \mathrm{i}\eta_a v_a, \quad (90)$$

we find that the operators (73) and (74) satisfy the conditions

$$\exp\,[\mathrm{i}f(t, \mathbf{x})]\,D(\theta, \mathbf{v})\left[\mathrm{i}\frac{\partial}{\partial t} - H_s^{\mathrm{I}}(p)\right]D^{-1}(\theta, \mathbf{v})$$
$$\times\exp\,[-\mathrm{i}f(t, \mathbf{x})] = \mathrm{i}\frac{\partial}{dt''} - H_s^{\mathrm{I}}(\mathbf{p}''), \quad (91)$$

where $H_s^{\mathrm{I}}(\mathrm{p}'')$ are the Hamiltonians obtained from (73) and (74) by the substitution $p_a \to p_a'' = -i(\partial/\partial x_a'')$. It follows from (91) that $\Psi''(t'', \mathbf{x}'')$ (38) satisfies the same equation as the untransformed function $\Psi(t, \mathbf{x})$:

$$\mathrm{i}\frac{\partial}{\partial t''}\Psi''(t'', \mathbf{x}'') = H_s^{\mathrm{I}}(\mathbf{p}'')\,\Psi''(t'', \mathbf{x}'').$$

It is easy to show [and this is most readily done in the representation (76)] that the Casimir operators (13), constructed from the generators (36), have the eigenvalues $c_1 = \pm m$, $c_2 = m$, and $c_3 = m^2s(s+1)$. Thus, Eqs. (64), (73), and (74) can be interpreted as the equations of motion of a free nonrelativistic particle with mass m, spin s, and internal energy $\pm m$.

Lagrangian formulation. Formulas (73) and (74) give nonrelativistic Hamiltonians for particles with arbitrary spin. These Hamiltonians are determined up to the arbitrary parameters a, $\tilde{a}$, and k, which can be chosen to make Eqs. (64), (73), and (74) invariant under the antiunitary transformation $\Theta^* = P\cdot T\cdot C$, where the operators P, T, and C are given by Eqs. (50):

$$\Psi(t, \mathbf{x}) \to \Theta^*\Psi(t, \mathbf{x}) = r\Psi^*(-t, -\mathbf{x}),\ r = r_1r_2r_3.$$

A necessary and sufficient condition for such invariance of H_s^{I} or $\tilde{H}_s^{\mathrm{I}}$ is the simultaneous fulfillment of the relations

$$a^* = \pm a,\ k^* = \pm k\ \text{ or }\ a = 0,\ \tilde{a}^* = \tilde{a},\ k^* = k. \quad (92)$$

At the same time

$$r = \begin{cases} \sigma_3\Delta_2, & \text{if}\quad a^* = -a,\ k^* = -k\ \text{ or }\ a = 0, \tilde{a}^* = \tilde{a},\ k^* = k; \\ \Delta_2, & \text{if}\quad a^* = a,\ k^* = k, \end{cases}$$

where Δ_2 are the matrices given in (56) and (59).

Under the restrictions on the parameters a, $\tilde{a}$, and k given by formulas (92), Eqs. (64), (73), and (74) are invariant under the group $G \times F$, where F is the group consisting of the two elements $F = \{\Theta^*, I\}$, where I is an identity transformation. Note, however, that these equations are not invariant under the transformations (50) taken separately.

By Lemma #1, Eqs. (64), (73), and (74) are also invariant under the Lie algebra of the Schrodinger group, whose basis elements include not only the generators P_a, G_a, J_a, and M (36) but also the operators

$$\hat{P}_0 = p^2/2m;\ D = t\hat{P}_0 - x_ap_a + 3\mathrm{i}/2 + (1/m)\,\eta_ap_a;$$
$$A = t^2\hat{P}_0 - tD - m\mathbf{x}^2/2 + \eta_ax_a. \quad (93)$$

The operators P_a, J_a, G_a, M (36) and P_0, D, A (93) satisfy the algebra (6)–(11).

The Hamiltonians (73) and (74) and the generators (36) and (66) are non-Hermitian in the scalar product (32). However, the Hamiltonian (73) is Hermitian in the metric (71), where the positive-definite operator $\hat{M}$ is equal to

$$\hat{M} = V^\dagger V = 1 + [\mathrm{i}(k^* - k)\,\sigma_1 - (k^* + k)\,\sigma_2]\,\frac{S_ap_a}{m}$$
$$+ 2k^*k\,(1 - \sigma_3)\left(\frac{S_ap_a}{m}\right)^2.$$

In addition, if the parameters a, $\tilde{a}$, and k satisfy the conditions (92), the Hamiltonians (73) and (74) are also Hermitian in the indefinite metric when $\hat{M}$ in (71) has the form

$$\hat{M}=\xi=\begin{cases}\sigma_1, & \text{if} \quad a^*=a,\ k^*=k \ \text{or} \ k^*=k,\ \tilde{a}^*=\tilde{a},\ a=0,\\ \sigma_2, & \text{if} \quad a^*=-a,\ k^*=-k.\end{cases} \quad (94)$$

If (94) is satisfied, then Eqs. (64), (73), and (74) can be obtained from a variational principle. Indeed, choosing the Lagrangian $L_0(t, \mathbf{x})$ in the form

$$L_0(t,\mathbf{x})=\frac{i}{2}\left(\overline{\Psi}\frac{\partial\Psi}{\partial t}-\frac{\partial\overline{\Psi}}{\partial t}\Psi\right)-am\overline{\Psi}\sigma_1\Psi$$
$$+a\tilde{k}\left(\overline{\Psi}\sigma_3S_a\frac{\partial\Psi}{\partial x_a}-\frac{\partial\overline{\Psi}}{\partial x_a}\sigma_3S_a\Psi\right)-\frac{1}{2m}\frac{\partial\overline{\Psi}}{\partial x_b}C_{ab}\frac{\partial\Psi}{\partial x_b},\quad \overline{\Psi}=\Psi^{\dagger}\xi, \quad (95)$$

if H_s^{I} is given by (73), and

$$L_0(t,\mathbf{x})=\frac{i}{2}\left\{\overline{\Psi}\frac{\partial\Psi}{\partial t}-\frac{\partial\overline{\Psi}}{\partial t}\Psi\right\}+m\overline{\Psi}\left[\frac{a}{2}(\sigma_1-i\sigma_2)+a\sigma_3\right]\Psi$$
$$+2\tilde{a}k\left[\overline{\Psi}(\sigma_2-i\sigma_1)S_a\frac{\partial\Psi}{\partial x_a}-\frac{\partial\overline{\Psi}}{\partial x_a}(\sigma_2-i\sigma_1)S_a\Psi\right]$$
$$-\frac{1}{2m}\frac{\partial\overline{\Psi}}{\partial x_a}C_{ab}\frac{\partial\Psi}{\partial x_b}, \quad (96)$$

if the Hamiltonian has the form (74), we can show that the Euler–Lagrange equations for the functions (95) and (96) are identical to Eqs. (64), (73), and (74) for $\overline{\Psi}(t, \mathbf{x})$. For $\Psi(t, \mathbf{x})$, we obtain the equation

$$i\frac{\partial}{\partial t}\overline{\Psi}=[H_s^{\mathrm{I}}]^{\mathrm{T}}\overline{\Psi},$$

where $[H_s^{\mathrm{I}}]^{\mathrm{T}}$ are the operators obtained from (73) and (74) by the operation of transposition of all the matrices in H_s^{I} and $\tilde{H}_s^{\mathrm{I}}$.

In addition, it is easy to show that Eqs. (95) and (96) define real functions of Ψ and $\overline{\Psi}$ and their first derivatives which are invariant under the Galileo transformations (38) and (66) and, therefore, $L_0(t, \mathbf{x})$ can be interpreted as the Lagrangian of a free nonrelativistic particle with arbitrary spin s.

Explicit form of the Hamiltonians H_s^{II}. We now find differential operators H_s^{II} that satisfy together with the generators (70) the relations (6)–(10) and (72).

Theorem 4. All possible (up to equivalence) differential operators H_s^{II} which are Hermitian in the metric (32) and satisfy the conditions (6)–(10), (70), and (72) are given by

$$H_s^{\mathrm{II}}=\sigma_1\left[m+\frac{p^2}{2m}-\frac{(\mathbf{S}\cdot\mathbf{p})^2}{ms^2}\sin^2\theta_s\right]$$
$$+\sigma_2\frac{\sqrt{2}\sin\theta_s}{s}\mathbf{S}\cdot\mathbf{p}-\sigma_3\left[a_s\frac{p^2}{2m}+b_s\frac{(\mathbf{S}\cdot\mathbf{p})^2}{2ms^2}\right], \quad (97)$$

where

$$a_{1/2}=\sin 2\theta_{1/2};\ b_{1/2}=0;\ a_1=1;\ b_1=\sin 2\theta_1;\ a_{3/2}=b_{3/2}$$
$$-\frac{5}{4}\sin 2\theta_{3/2}=-\frac{1}{8}\sin 2\theta_{3/2}-\frac{3}{4}\sin\theta_{3/2}\left(1-\frac{1}{9}\sin\theta_{3/2}\right)^{1/2};$$
$$a_s=b_s=\theta_s=0,\ s>3/2, \quad (98)$$

and $\theta_{1/2}, \theta_1, \theta_{3/2}$ are arbitrary real parameters.

Proof. We show first that the operators H_s^{II} can contain derivatives of order not higher than the second.

Indeed, suppose $H_s^{\mathrm{II}}=\sum_{i=0}^{N}H_i$, where H_i contains derivatives of only i-th order. Then from (72), we obtain

$$H_NH_N=H_N^{\dagger}H_N=0 \quad \text{or} \quad H_N=0, \quad \text{if} \quad N>2. \quad (99)$$

We represent the required differential operators H_s^{II} in the form of expansions in the spin matrices and the $2(2s+1)$-row Pauli matrices (67):

$$H_s^{\mathrm{II}}=\sum_{\mu=0}^{3}\left(a_\mu m+b_\mu\frac{p^2}{2m}+c_\mu\mathbf{S}\cdot\mathbf{p}+d_\mu\frac{(\mathbf{S}\cdot\mathbf{p})^2}{2m}\right)\sigma_\mu,$$

where $a_\mu, b_\mu, c_\mu, d_\mu$ are arbitrary real coefficients. Using the orthogonal projection operators[16,17]

$$\Lambda_r=\prod_{r'\neq r}\frac{(\mathbf{S}\cdot\mathbf{p})\,p^{-1}-r'}{r-r'},\quad r, r'=-s, -s+1, \ldots, s,$$

which satisfy the orthogonality and completeness conditions

$$\Lambda_r\Lambda_{r'}=\delta_{rr'}\Lambda_r,\ \sum_r\Lambda_r=1,\ \sum_r r^k\Lambda_r=\left(\frac{\mathbf{S}\cdot\mathbf{p}}{p}\right)^k,$$

we can rewrite Eq. (100) in the form

$$H_s^{\mathrm{II}}=\sum_{\mu=0}^{3}\sum_{r=-s''}^{s'}\left[a_\mu m+(b_\mu+r^2d_\mu)\frac{p^2}{2m}+rpc_\mu\right]\sigma_\mu\Lambda_r. \quad (101)$$

The operators (101) obviously satisfy the conditions (68). We require fulfillment of (77). Substituting (101) in (72), using the orthogonality of the projection operators Λ_r, and equating the independent terms, we find that a_r, b_r, c_r, and d_r must satisfy one of the following systems of equations:

$$\sum_{i=1}^{3}a_i^2=1;\ \sum_{i=1}^{3}[r^2c_i^2+a_i(b_i+r^2d_i)]=1;$$
$$\sum_{i=1}^{3}c_ir(b_i+r^2d_i)=0;\ \sum_{i=1}^{3}rc_ia_i=0;\ \sum_{i=1}^{3}(b_i+r^2d_i)^2=1 \quad (102)$$

or

$$a_0=b_0=1;\ d_0=c_0=a_i=b_i=c_i=d_i=0;\ i=1, 2, 3. \quad (103)$$

The general solution of the system of algebraic equations (102) (up to linear equivalence transformations) is given by the formulas

$$\left.\begin{array}{l}a_1=1;\ a_0=a_2=a_3=0;\\ b_1=1;\ b_3=a_s,\ b_0=b_2=0;\\ c_2=\dfrac{\sqrt{2}\sin\theta_s}{s^2};\ c_0=c_1=c_3=0;\\ d_1=-(c_2)^2;\ d_3=\dfrac{b_s}{s^2},\ d_0=d_2=0,\end{array}\right\} \quad (104)$$

where a_s, b_s, and θ_s are given in (98). We can show that Eqs. (103) are incompatible with (6)–(10).

Substituting (104) in (100), we arrive at the Hamiltonians (97). To complete the proof of the theorem, it is now sufficient to find the explicit form of the operators η_a^{II} for which the operators (70) satisfy the relations (10) and (57). It is easy to show that η_a^{II} can be chosen in the form

$$\eta_a^{\mathrm{II}}=[U, \sigma_1x_am]U^{\dagger}, \quad (105)$$

where

$$U=\frac{E+H_s^{\mathrm{II}}\sigma_1}{\sqrt{2E\left[2E-\left(\frac{pr}{ms}\sin\theta_s\right)^2\right]}}\Lambda_r,\ E=m+\frac{p^2}{2m} \quad (106)$$

is the operator which diagonalizes the Hamiltonians (97) and the generators (70):

$$\left.\begin{array}{l}U^{+}H_s^{\mathrm{II}}U=\sigma_1E;\ U^{+}G_a^{\mathrm{II}}U=tp_a-\sigma_1mx_a;\\ U^{+}J_a^{\mathrm{II}}U=J_a^{\mathrm{II}};\ UP_a^{\mathrm{II}}U=P_a^{\mathrm{II}}.\end{array}\right\} \quad (107)$$

The theorem is proved.

Thus, we have obtained Hamiltonians of nonrelativistic particles with spin s in the form (97). Equations

(64) with the Hamiltonians H_s^{II} are invariant with respect to the Lie algebra of the extended Galileo group (70), and hence, with respect to the Lie algebra on the Schrödinger group, whose basis elements are given by (12) and (70). These equations are also invariant under the discrete transformation $\Theta = P\cdot T$, where P and T are defined in (50):

$$\Psi(t, \mathbf{x}) \to \Theta\Psi(t, \mathbf{x}) = \sigma_2\Psi(-t, -\mathbf{x}). \qquad (108)$$

Here, σ_2 is one of the matrices (67). We can show that Eqs. (64) and (97) are invariant under each of the transformations (50), but only if r_a are given by certain integro-differential operators rather than numerical matrices.

Note that in the case $s=\frac{1}{2}$, $\theta_{1,2}=\pi/4$, $k=-i$, $a=1$ Eqs. (64), (73) and (64, (97) can be written in the compact form

$$(\gamma_\mu p^\mu - m)\Psi(t, \mathbf{x}) = \mathrm{i}\gamma_4 \frac{p^2}{2m}\Psi(t, \mathbf{x}) \qquad (109)$$

and

$$(\gamma_\mu p^\mu - m)\Psi(t, \mathbf{x}) = (1+\gamma_4-\gamma_0)\frac{p^2}{2m}\Psi(t, \mathbf{x}), \qquad (110)$$

where

$$\gamma_0=\sigma_1,\ \gamma_a=-2\mathrm{i}\sigma_2 S_a,\ \gamma_4=\mathrm{i}\,\gamma_0\gamma_1\gamma_2\gamma_3$$

are the Dirac matrices.

Equations (109) and (110) differ from the relativistic Dirac equation only by the presence of the terms on the right-hand side, which obviously destroy the invariance under the Poincaré group but ensure invariance of the equations under the Galileo group.

To solve specific problems using the above equations, we may need the explicit form of the matrices S_a in the Hamiltonians H_s^I and H_s^{II}. The matrix elements of the generators S_a in the Gel'fand–Tsetlin basis[28] are given in Eqs. (149) and (150). We also give Eqs. (64) and (73) for $s=\frac{1}{2}$ and 1 in component form:

1. $s=\frac{1}{2}$; $\Psi(t, \mathbf{x})$ is the column $(\Phi_1, \Phi_2, \chi_1, \chi_2)$;

$$\left.\begin{aligned}&\left(\mathrm{i}\frac{\partial}{\partial t}-\frac{p^2}{2m}\right)\Phi = am\chi + \mathrm{i}ak\sigma\cdot\mathbf{p}\Phi - \frac{a^2k^2}{2m}p^2\chi;\\ &\left(\mathrm{i}\frac{\partial}{\partial t}-\frac{p^2}{2m}\right)\chi = am\Phi - \mathrm{i}ak\sigma\cdot\mathbf{p}\chi,\end{aligned}\right\} \qquad (111)$$

where Φ and χ are the two-component spinors

$$\Phi=\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix},\quad \chi=\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix};$$

σ_a are the Pauli matrices.

2. $s=1$; $\Psi(t, \mathbf{x})$ is the column $(\Phi_1, \Phi_2, \Phi_3, \chi_1, \chi_2, \chi_3)$;

$$\left.\begin{aligned}&\left(\mathrm{i}\frac{\partial}{\partial t}-\frac{p^2}{2m}\right)\Phi = am\chi - 2ak\mathbf{p}\times\Phi - \frac{ak^2}{m}[p^2\chi - \mathbf{p}(\mathbf{p}\cdot\chi)];\\ &\left(\mathrm{i}\frac{\partial}{\partial t}-\frac{p^2}{2m}\right)\chi = am\Phi + 2ak\mathbf{p}\times\chi,\end{aligned}\right\} \qquad (112)$$

where χ and Φ are the vectors with components χ_1, χ_2, χ_3 and Φ_1, Φ_2, Φ_3. Equations (112) are identical to (64) and (73) if the matrix elements of the generators S_a are chosen in the form

$$(S_a)_{bc} = -\mathrm{i}\varepsilon_{abc}, \qquad (113)$$

where ε_{abc} is the absolutely antisymmetric tensor of third rank.

3. FIRST-ORDER EQUATIONS

Here, we consider systems of first-order partial differential equations with constant coefficients, i.e., equations of the form

$$L\Psi(t, \mathbf{x}) = 0,\ L=\beta_\mu p^\mu + \beta_5 m, \qquad (114)$$

where $\Psi(t, \mathbf{x})$ is the column vector

$$\Psi(t, \mathbf{x}) = \begin{pmatrix}\Psi_1(t, \mathbf{x})\\ \Psi_2(t, \mathbf{x})\\ \vdots\\ \Psi_n(t, \mathbf{x})\end{pmatrix},\quad n<\infty;$$

β_μ and β_5 are certain numerical matrices.

There is a well-developed theory of equations of the form (114) invariant under transformations of the group $O(3)$ and the Lorentz group,[28] but systems of first-order differential equations invariant under the Galileo group have been comparatively little studied.

The present section is devoted to the construction of Galileo-invariant systems of differential equations of first order for particles with arbitrary spin. Among the equations obtained below, there are some which, in contrast to the LHH equations, describe the spin-orbit and Darwin interactions of particles with a field.

Basic definitions and formulation of the problem. We consider the systems of differential equations (114) and determine the conditions which the matrices β_μ and β_5 must satisfy if these equations are to be invariant under Galileo transformations.

Generalizing the symmetry properties of the Schrödinger equation (1), we shall say that Eq. (114) is invariant under the Galileo group if the operator L (114) satisfies the conditions (2), where Q_A are the generators of an arbitrary representation of the extended Galileo group. We require the generators of the Galileo group on the set of solutions of Eq. (114) to have the locally covariant form (36), and the operators f_A to be specified by numerical matrices [in this, and only this, case the finite Galileo transformations for $\Psi(t, \mathbf{x})$ have the local form (38)]. Then the condition of Galileo invariance can be formulated as follows.

Definition 2. Equation (114) is locally invariant under the Galileo group if the operator L (114) satisfies the conditions

$$\tilde{J}_a L - LJ_a = 0;\ \tilde{G}_a L - LG_a = 0, \qquad (115)$$

where J_a, G_a and $\tilde{J}_a, \tilde{G}_a$ are the generators of the extended Galileo group in the representation

$$\left.\begin{aligned}&J_a = (\mathbf{x}\times\mathbf{p})_a + S_a;\ G_a = tp_a - mx_a + \eta_a;\\ &\tilde{J}_a = (\mathbf{x}\times\mathbf{p})_a + \tilde{S}_a;\ \tilde{G}_a = tp_a - mx_a + \tilde{\eta}_a,\end{aligned}\right\} \qquad (116)$$

and S_a, η_a (and $\tilde{S}_a, \tilde{\eta}_a$) are matrices that form representations (in the general case inequivalent) of the Lie algebra of the homogeneous Galileo group (37).

Substituting the generators (116) and the operator L from (114) in (115), we obtain the condition of local Galileo invariance of Eqs. (114) in the form of the following equations for the matrices β_k $(k=0, 1, 2, 3, 5)$:

$$\left.\begin{array}{l}\tilde{S}_a\beta_0-\beta_0 S_a=0;\ \tilde{S}_a\beta_5-\beta_5 S_a=0;\\ \tilde{\eta}_a\beta_0-\beta_0\eta_a=0;\ \tilde{\eta}_a\beta_5-\beta_5\eta_a=-i\beta_a;\\ \tilde{\eta}_a\beta_b-\beta_b\eta_a=-i\delta_{ab}\beta_0,\end{array}\right| \quad (117)$$

where S_a, η_a and $\tilde{S}_a, \tilde{\eta}_a$ are matrices satisfying the algebra (37).

Thus, the problem of describing first-order Galileo-invariant equations of the form (114) can be reduced to the solution of the matrix equations (37) and (117).

Of particular interest for physics are equations which admit a Lagrangian formulation. We investigate the additional restrictions imposed on the matrices β_k, S_a, η_a, $\tilde{S}_a$, and $\tilde{\eta}_a$ by the requirement that Eq. (114) be derivable by the principle of least action from an appropriately chosen Lagrangian. For this, we begin by formulating the following assertion, which makes it possible to separate classes of equivalent Galileo-invariant equations (114).

Lemma 2. Let $\eta_a, S_a, \tilde{\eta}_a, \tilde{S}_a, \beta_k$ be a set of matrices satisfying the conditions (37) and (117). Then the matrices

$$\left.\begin{array}{l}\beta'_k=B\beta_k;\ S'_a=S_a;\ \eta'_a=\eta_a;\\ \tilde{\eta}'_a=B\eta_a B^{-1};\ \tilde{S}'_a=B\tilde{S}_a B^{-1},\end{array}\right\} \quad (118)$$

where B is an arbitrary nondegenerate matrix, also satisfy the equations (37) and (117).

To prove the lemma, it is sufficient to multiply each of Eqs. (117) from the left by the matrix B and use the relation $B^{-1}B=1$.

By Lemma 2, each solution of the system of relations (117) determines an entire class of equations that are locally invariant under the Galileo group:

$$B(\beta_\mu p^\mu+\beta_5 m)\Psi(t,\mathbf{x})=0, \quad (119)$$

where B is an arbitrary nondegenerate matrix.

We require that Eqs. (114) admit a Lagrangian formulation. The general form of the Lagrangian corresponding to Eq. (114) (up to terms which do not contribute to the equations of motion) is given by

$$L=\frac{i}{2}\left(\overline{\Psi}\beta_\mu\frac{\partial\Psi}{\partial x_\mu}-\frac{\partial\overline{\Psi}}{\partial x_\mu}\beta_\mu\Psi\right)-\overline{\Psi}\beta_5 m\Psi, \quad (120)$$

where $\overline{\Psi}=\Psi^\dagger B$, and B is some nonsingular matrix. From the condition of reality of the Lagrangian, we obtain

$$(B\beta_\mu)^\dagger=B\beta_\mu;\ (B\beta_5)^\dagger=B\beta_5. \quad (121)$$

By Lemma 2 and using (121) we conclude that the matrices β_μ and β_5 can be taken to be Hermitian [which corresponds to a Lagrangian (120) with $\overline{\Psi}=\Psi^+$]. But then from the requirement that the Lagrangian be invariant under the infinitesimal transformations

$$\Psi\to(1+iG_a v_a)\Psi;\ \Psi\to(1+iJ_a\theta_a)\Psi, \quad (122)$$

where G_a and J_a are given by Eqs. (116), we obtain the following conditions for the matrices $\tilde{S}_a$ and $\tilde{\eta}_a$ from (117):

$$\tilde{S}_a=S_a^\dagger;\ \tilde{\eta}_a=\eta_a^\dagger. \quad (123)$$

Substituting (123) in (117) and using the Hermiticity of the matrices S_a, we arrive at the system of equations

$$S_a\beta_0-\beta_0 S_a=0; \quad (124)$$

$$S_a\beta_5-\beta_5 S_a=0; \quad (125)$$

$$\eta_a^\dagger\beta_0-\beta_0\eta_a=0; \quad (126)$$

$$\eta_a^\dagger\beta_5-\beta_5\eta_a=-i\beta_a; \quad (127)$$

$$\eta_a^\dagger\beta_b-\beta_b\eta_a=-i\delta_{ab}\beta_0; \quad (128)$$

$$\beta_k^\dagger=\beta_k,\ k=0,1,2,3,5. \quad (129)$$

Thus, we have found that the problem of describing Galileo-invariant equations of the form (114) that admit a Lagrangian formulation is equivalent to solving the system of equations (37) and (124)–(129) for the matrices β_k, S_a, and η_a.

We formulate one further assertion, which will be used below to calculate the explicit form of the matrices β_k.

Lemma 3. Suppose the matrices S_a, η_a, and β_k satisfy the relations (37) and (124)–(129). Then the set of matrices $\{S_a, \eta_a, \beta'_k\}$, where

$$\beta'_k=V^\dagger\beta_k V, \quad (130)$$

and V is an arbitrary matrix which commutes with S_a and η_a,

$$[V,S_a]=[V,\eta_a]=0, \quad (131)$$

also satisfies Eqs. (37) and (124)–(129).

Proof. We multiply each of the relations (124)–(129) from the left by V^+ and from the right by V. As a result, using (131), we arrive at the equations (124)–(129) for the β'_k determined by Eq. (130).

If the matrix V can be inverted, then β_k and β'_k are equivalent solutions of the system of relations (124)–(129). We shall seek solutions of these relations up to the equivalence transformations given by (130) and (131).

Canonical form of Eqs. (114). Before we turn to the solution of the system of relations (37) and (124)–(129), we investigate some properties of Eqs. (114); these can be derived without using the explicit form of the matrices β_k.

It is well known (see Ref. 29) that relativistic wave equations of the form (114) (where β_5 is an invertible matrix) can be reduced to a standard (canonical) form (including only a single matrix $\beta'_0=\beta_5^{-1}\beta_0$):

$$\beta'_0\sqrt{p_\mu p^\mu}\Psi=m\Psi.$$

We shall show that the Galileo-invariant equations (114) can also be reduced to a canonical form (in which the operator L is expressed solely in terms of the two matrices β_0 and β_5). For this, we subject the function $\Psi(t,\mathbf{x})$ in (114) to the transformation $\Psi(t,\mathbf{x})\to\Psi'(t,\mathbf{x})=V^{-1}\Psi(t,\mathbf{x})$, where

$$V=\exp(-i\boldsymbol{\eta}\cdot\mathbf{p}/m). \quad (132)$$

The function $\Psi'(t,\mathbf{x})$ obviously satisfies the equation

$$L'\Psi'(t, \mathbf{x})=0,\ L'=V^{\dagger}LV, \quad (133)$$

where L is the operator (114).

Using the Campbell–Hausdorff formula

$$\left.\begin{array}{l}\exp(A^{\dagger})B\exp(A)=\sum\{A, B\}^n\frac{1}{n!};\\ \{A, B\}^n=[A^{\dagger}\{A, B\}^{n-1}],\ \{A, B\}^0=B\end{array}\right\} \quad (134)$$

and the commutation relations (124)–(128), we obtain

$$\left.\begin{array}{l}V^{\dagger}\beta_0V=\beta_0,\ V^{\dagger}\beta_ap_aV=\beta_ap_a+\beta_0\frac{p^2}{m},\\ V^{\dagger}\beta_5V=\beta_5+\frac{1}{m}\beta_ap_a+\frac{1}{2m}\beta_0p^2,\end{array}\right\} \quad (135)$$

whence

$$L'=V^{\dagger}(\beta_0p_0-\beta_ap_a+\beta_5m)V=\beta_0\left(p_0-\frac{p^2}{2m}\right)+\beta_5m. \quad (136)$$

Substituting (136) in (133), we arrive at the equation in canonical form:

$$\left[\beta_0\left(p_0-\frac{p^2}{2m}\right)+\beta_5m\right]\Psi'(t, \mathbf{x})=0. \quad (137)$$

Thus, an arbitrary first-order Galileo-invariant equation (114) can be transformed to a system of second-order equations in the form (137) (for the LHH equations[5-7] this fact was established in Ref. 11). However, Eqs. (114) and (137) become inequivalent after the introduction of the minimal interaction with an external electromagnetic field (i.e., after the substitution $p_\mu \to p_\mu - eA_\mu$, where A_μ is the 4-vector potential of the electromagnetic field).

Equations (137) have a form which is manifestly invariant under Galileo transformations. On the solutions of Eqs. (137), the generators of the Galileo group have a quasidiagonal form given by Eqs. (31), in which S_a are the generators of some reducible representation of the group $O(3)$ that commute with the matrices β_0 and β_5.

Using the representation (137), we formulate one additional requirement which we impose on the matrices β_k. Namely, we require fulfillment of the condition

$$\det(\beta_\mu p^\mu+\beta_5m)\equiv\det\left[\beta_0\left(p_0-\frac{p^2}{2m}\right)+\beta_5m\right]=c\left(p_0-\frac{p^2}{2m}\right)^n, \quad (138)$$

where c is some nonvanishing constant, and n is an integer ($n\neq 0$).

The condition (138) means that Eq. (114) is of parabolic type.

Finite-dimensional representations of the Lie algebra of the homogeneous Galileo group. We turn to the solution of the system of relations (37) and (124)–(129).

We consider first the relations (37), which determine the Lie algebra of the homogeneous Galileo group, which is locally isomorphic to the Euclidean group $E(3)$. Each solution of the relations (37), i.e., each set of finite-dimensional matrices S_a and η_a satisfying (37) give a representation of this algebra.

The group $E(3)$ is not semisimple, and therefore its finite-dimensional representations are nonunitary and are not completely reducible. The paper of Ref. 30 is devoted to the description of the nondecomposable finite-dimensional representations of the Lie algebra of $E(3)$. However, the results obtained in Ref. 30 have a very general and, apparently, not very constructive form. In addition, not all of the representations given in Ref. 30 are inequivalent.

Below, we describe constructively a class of finite-dimensional representations of the algebra (37). The algebra (37) has two invariant operators:

$$D_1=S_a\eta_a;\ D_2=\eta_a\eta_a, \quad (139)$$

which are nilpotent matrices in the case of finite-dimensional representations, i.e., they satisfy the conditions

$$D_1^N=D_2^{N'}=0, \quad (140)$$

where N and N' are positive integers.

The algebra (37) includes the $O(3)$ subalgebra formed by the matrices S_a. The representations of this subalgebra are well known and are given by an integral or half-integral number s.

We consider only representations of the algebra (37) that on reduction with respect to the $O(3)$ algebra include not more than two inequivalent representations. In this case, the matrices S_a can be chosen in the form

$$S_a=S_a^{nm}=\left(\begin{array}{c|c}I_n\otimes\hat{S}_a & \hat{0}^{\dagger}\\ \hline \hat{0} & I_m\otimes\Sigma_a\end{array}\right), \quad (141)$$

where $\hat{S}_a$ and Σ_a are the generators of the irreducible representations $D(s)$ and $D(s')$ of the group $O(3)$, i.e., matrices satisfying the relations

$$\left.\begin{array}{l}[\hat{S}_a, \hat{S}_b]=i\varepsilon_{abc}s_c,\ s_as_a=s(s+1);\\ [\Sigma_a, \Sigma_b]=i\varepsilon_{abc}\Sigma_c,\ \Sigma_a\Sigma_a=s'(s'+1);\end{array}\right\} \quad (142)$$

I_n and I_m are the $n\times n$ and $m\times m$ unit matrices, $\hat{0}$ are $m\times n$ matrices with zeros as elements, and the symbol $A\otimes B$ denotes the direct (Kronecker) product of matrices. We also restrict ourselves to the case when the representation space of the $E(3)$ algebra includes not more than two orthogonal subspaces invariant under the action of the operator (139). Such representations are described by the following theorem.

Theorem 5. All possible (up to equivalence) nondecomposable finite-dimensional representations of the $E(3)$ algebra (37) which include not more than two inequivalent representations of the $O(3)$ subalgebra and satisfy the additional requirement that the invariant operator D_1 (139) has not more than two orthogonal invariant subspaces can be labeled by the set of integers (n, m, α), where

$$\alpha=1, 2;\ n\leqslant 4;\ m\leqslant 4;\ |n-m|\leqslant 2;\ nm\neq 9.$$

The explicit form of the corresponding matrices S_a and η_a is given by

$$S_a=S_a^{(nm\alpha)}=S_a^{nm};\ \eta_a=\eta_a^{(nm\alpha)};$$

$$\eta_a^{nm1}=\frac{1}{2s}\left(\begin{array}{c|c}a_1^{nm}\otimes\hat{S}_a & a_2^{nm}\otimes K_a\\ \hline a_3^{nm}\otimes K_a^{\dagger} & a_4^{nm}\otimes\Sigma_a\end{array}\right);\ \eta_a^{(nm2)}=[\eta_a^{(nm1)}]^{\dagger}, \quad (143)$$

where S_a^{nm} are the matrices (141), $s'=s-1$, K_a are $(2s-1)\times(2s+1)$ matrices determined by the relations

$$\left.\begin{array}{l}K_a\hat{S}_b-\Sigma_bK_a=i\varepsilon_{abc}K_c;\\ \hat{S}_a\hat{S}_b+K_a^{\dagger}K_b=is\varepsilon_{abc}\hat{S}_c+s^2\delta_{ab};\end{array}\right\} \quad (144)$$

a_1^{nm} are the matrices with elements

$$[a_1^{nm}]_{ij}=\begin{cases}\delta_{i-1,\,j},\ n\geqslant m,\ i,j\leqslant n,\\ \frac{s-1}{s+1}\delta_{i-1,\,j},\ n<m;\end{cases}$$

$$[a_2^{nm}]_{ij}=\begin{cases}-(2s-1)^{-1/2}\delta_{i-2,\,j},\ n>m,\ i\leqslant n,\ j\leqslant m\\ (2s+1)^{-1/2}\delta_{i,\,j},\ n<m,\\ k\delta_{i,\,j+1},\ n=m;\end{cases}$$

$$[a_3^{nm}]_{ij}=\begin{cases}(2s-1)^{-1/2}\delta_{i,\,j},\ n>m,\ i\leqslant m,\ j\leqslant n,\\ (2s+1)^{-1/2}\delta_{i-2,\,j},\ n\leqslant m;\end{cases}\tag{145}$$

$$[a_4^{nm}]_{ij}=\begin{cases}\frac{s+1}{s-1}\delta_{i-1,\,j},\ n\geqslant m,\\ \delta_{i-1,\,j},\ n<m,\end{cases}$$

where k is an arbitrary parameter.

The representations $D(n, m, \alpha=1)$ and $D(n, m, \alpha=2)$ are equivalent if and only if $|n-m|=1$.

Proof. The complete proof of the theorem will not be given here. However, it should be noted that Eqs. (143) determine the general form of matrices η_a satisfying the commutation relations $[\eta_a, S_b]=i\varepsilon_{abc}\eta_c$ with the generators (141). We require that the matrices η_a (143) commute with one another, and we use the relations[7]

$$\left.\begin{aligned}&K_a\hat{S}_b-K_b\hat{S}_a=\mathrm{i}\,(s+1)\,\varepsilon_{abc}K_c;\\ &\Sigma_aK_b-\Sigma_bK_a=\mathrm{i}\,(1-s)\,\varepsilon_{abc}K_c;\\ &K_aK_q^+-K_bK_a^+=-\mathrm{i}\,(2s+1)\,\varepsilon_{abc}\Sigma_c;\\ &K_a^+K_b-K_b^+K_a=\mathrm{i}\,(2s-1)\,\varepsilon_{abc}\hat{S}_c;\end{aligned}\right\}\tag{146}$$

and we then obtain the following system of equations for the matrices a_l^{nm}:

$$\begin{aligned}&(a_1^{nm})^2+(2s-1)\,a_2^{nm}a_3^{nm}=0;\\ &(s+1)\,a_1^{nm}a_2^{nm}-(s-1)\,a_2^{nm}a_4^{nm}=0;\\ &(s+1)\,a_3^{nm}a_1^{nm}-(s-1)\,a_4^{nm}a_3^{nm}=0;\\ &(a_4^{nm})^2-(2s+1)\,a_3^{nm}a_2^{nm}=0.\end{aligned}\tag{147}$$

The condition that the invariant operator D_1 (139) have not more than two invariant subspaces reduces to the requirement that the matrices a_1^{nm} and a_3^{nm} should not be completely reducible.

All inequivalent solutions of the relations (147) are given by Eqs. (142) and (145). Further, the larger of the numbers (n, m) is equal to the nilpotency exponent of the invariant operator D_1 (139).

For completeness, we give the explicit form of the matrices S_a and K_a in the basis $|s, s_3\rangle$, in which the operators $\mathbf{S}^2$ and S_3 are diagonal[7]:

$$\begin{aligned}&\mathbf{S}^2\,|s, s_3\rangle=s\,(s+1)\,|s, s_3\rangle;\\ &S_3\,|s, s_3\rangle=s_3\,|s, s_3\rangle;\\ &S_1\,|s, s_3\rangle=a^s_{s_3,\,s_3+1}\,|s, s_3+1\rangle+a^s_{s_3,\,s_3-1}\,|s, s_3-1\rangle;\\ &S_2\,|s, s_3\rangle=\mathrm{i}a^s_{s_3,\,s_3+1}\,|s, s_3+1\rangle-\mathrm{i}a^s_{s_3,\,s_3-1}\,|s, s_3-1\rangle;\\ &K_1\,|s, s_3\rangle=C^{s,\,s-1}_{s_3}\,|s-1, s_3\rangle+C^{s,\,s-1}_{s_3,\,s_3-2}\,|s-1, s_3-2\rangle;\\ &K_2\,|s, s_3\rangle=\mathrm{i}C^{s,\,s-1}_{s_3}\,|s-1, s_3\rangle-\mathrm{i}C^{s,\,s-1}_{s_3,\,s_3-2}\,|s-1, s_3-2\rangle;\\ &K_3\,|s, s_3\rangle=f^{s,\,s-1}_{s_3}\,|s-1, s_3\rangle,\end{aligned}\tag{148}$$

where

$$s_3=-s,\ -s+1,\ \ldots,\ s;$$

$$\left.\begin{aligned}&a^{s'}_{s_3,\,s_3\pm1}=\frac{1}{2}\sqrt{s_3\,(s_3\pm1)-s\,(s+1)};\\ &f^{s,\,s-1}_{s_3}=\sqrt{s_3\,(2s-s_3)};\\ &C^{s,\,s-1}_{s_3}=\frac{1}{2}\sqrt{(2s-s_3)\,(2s+1-s_3)};\\ &C^{s,\,s-1}_{s_3,\,s_3-2}=\frac{1}{2}\sqrt{s_3\,(s_3+1)}.\end{aligned}\right\}\tag{149}$$

The expressions for the matrices Σ_a can be obtained from Eqs. (148) and (149) by the substitution $s\to s'$, $s'=s-1$.

Explicit form of the matrices β_k. In solving the system of equations (37) and (124)–(129), we restrict ourselves to the case when the matrices S_a and η_a realize the nondecomposable representations of the algebra (37) (37) described in Theorem 5.

We consider first the case of the representations of the algebra (37) corresponding to $N\leqslant 3$, where N is the nilpotency exponent of the invariant operator D_1 (139). We require that the matrices β_k satisfy the condition (138).

We give the solution to the problem in the form of the following assertion.

Theorem 6. All possible (up to equivalence) matrices β_k, S_a, and η_a satisfying the relations (124)–(129) and (138)–(140) (with $N\leqslant 3$) and the conditions of Theorem 5 are given by the following formulas (150)–(153):

$$\beta_0=\left(\begin{array}{c|c|c}I&\cdot&\cdot\\\hline\cdot&\cdot&\cdot\\\hline\cdot&\cdot&\cdot\end{array}\right);\ \beta_5=2\left(\begin{array}{c|c|c}\cdot&\cdot&\cdot\\\hline\cdot&I&\cdot\\\hline\cdot&\cdot&c^{-2}1\end{array}\right);$$

$$\beta_a=\frac{\mathrm{i}}{s}\left(\begin{array}{c|c|c}\cdot&\hat{S}_a&c^{-1}K_a^+\\\hline-\hat{S}_a&\cdot&\cdot\\\hline-c^{-1}K_a&\cdot&\cdot\end{array}\right);\tag{150}$$

$$S_a=\left(\begin{array}{c|c|c}\hat{S}_a&\cdot&\cdot\\\hline\cdot&\hat{S}_a&\cdot\\\hline\cdot&\cdot&\Sigma_a\end{array}\right);\ \eta_a=\frac{1}{2s}\left(\begin{array}{c|c|c}\cdot&\cdot&\cdot\\\hline\hat{S}_a&\cdot&\cdot\\\hline cK_a&\cdot&\cdot\end{array}\right);\ c=(2s-1)^{-1/2};$$

$$\beta_0=\left(\begin{array}{c|c|c}\cdot&\cdot&\cdot\\\hline\cdot&1&\cdot\\\hline\cdot&\cdot&\cdot\end{array}\right);\ \beta_5=2\left(\begin{array}{c|c|c}d^{-2}I&\cdot&\cdot\\\hline\cdot&\cdot&\cdot\\\hline\cdot&\cdot&1\end{array}\right);$$

$$\beta_a=\frac{\mathrm{i}}{s}\left(\begin{array}{c|c|c}\cdot&-d^{-1}K_a^+&\cdot\\\hline d^{-1}K_a&\cdot&\Sigma_a\\\hline\cdot&-\Sigma_a&\cdot\end{array}\right);\tag{151}$$

$$S_a=\left(\begin{array}{c|c|c}\hat{S}_a&\cdot&\cdot\\\hline\cdot&\Sigma_a&\cdot\\\hline\cdot&\cdot&\Sigma_a\end{array}\right);\ \eta_a=\frac{1}{2s}\left(\begin{array}{c|c|c}\cdot&dK_a^+&\cdot\\\hline\cdot&\cdot&\cdot\\\hline\cdot&\Sigma_a&\cdot\end{array}\right);\ d=(2s+1)^{-1/2};$$

$$\beta_0=\left(\begin{array}{c|c|c|c|c}\cdot&I&\cdot&\cdot&\cdot\\\hline I&\cdot&\cdot&\cdot&\cdot\\\hline\cdot&\cdot&\cdot&\cdot&\cdot\\\hline\cdot&\cdot&\cdot&\cdot&\cdot\\\hline\cdot&\cdot&\cdot&\cdot&\cdot\end{array}\right);$$

$$\beta_5=2\left(\begin{array}{c|c|c|c|c}\cdot&\frac{a^2}{2}I&aI&\cdot&\cdot\\\hline\frac{a^2}{2}I&aI&I&\cdot&\cdot\\\hline aI&I&\cdot&\cdot&\cdot\\\hline\cdot&\cdot&\cdot&-a\hat{1}&(s-1)\,\hat{1}\\\hline\cdot&\cdot&\cdot&(s-1)\,\hat{1}&\cdot\end{array}\right);$$

$$S_a=\left(\begin{array}{c|c|c|c|c}S_a&\cdot&\cdot&\cdot&\cdot\\\hline\cdot&S_a&\cdot&\cdot&\cdot\\\hline\cdot&\cdot&S_a&\cdot&\cdot\\\hline\cdot&\cdot&\cdot&\Sigma_a&\cdot\\\hline\cdot&\cdot&\cdot&\cdot&\Sigma_a\end{array}\right);$$

$$\beta_a=\frac{\mathrm{i}}{s}\left(\begin{array}{c|c|c|c|c} \cdot & \cdot & S_a & \cdot & c(s-1)K_a^{\dagger} \\ \hline \cdot & \cdot & \cdot & csK_a^{\dagger} & \cdot \\ \hline -S_a & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & -csK_a & \cdot & \cdot & \cdot \\ \hline -c(s-1)K_a & \cdot & \cdot & \cdot & \cdot \end{array}\right);$$

$$\eta_a=\frac{1}{2s}\left(\begin{array}{c|c|c|c|c} \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline S_a & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & S_a & \cdot & -cK_a^{\dagger} & \cdot \\ \hline cK_a & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & cK_a & \cdot & \frac{s+1}{s-1}\Sigma_a & \cdot \end{array}\right); \qquad (152)$$

$$\beta_0=\left(\begin{array}{c|c|c|c|c} \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \hat{1} & \cdot \\ \hline \cdot & \cdot & \hat{1} & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right);$$

$$\beta_5=2\left(\begin{array}{c|c|c|c|c} bI & (s+1)I & \cdot & \cdot & \cdot \\ \hline (s+1)I & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \frac{b^2}{2}\hat{1} & b\hat{1} \\ \hline \cdot & \cdot & \frac{b^2}{2}\hat{1} & b\hat{1} & \hat{1} \\ \hline \cdot & \cdot & b\hat{1} & \hat{1} & \cdot \end{array}\right);$$

$$S_a=\left(\begin{array}{c|c|c|c|c} S_a & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & S_a & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \Sigma_a & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \Sigma_a & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \Sigma_a \end{array}\right);$$

$$\beta_a=\frac{\mathrm{i}}{s}\left(\begin{array}{c|c|c|c|c} \cdot & \cdot & \cdot & -dsK_a^{\dagger} & \cdot \\ \hline \cdot & \cdot & -d(s+1)K_a^{\dagger} & \cdot & \cdot \\ \hline \cdot & d(s+1)K_a & \cdot & \cdot & \Sigma_a \\ \hline dsK_a & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & -\Sigma_a & \cdot & \cdot \end{array}\right);$$

$$\eta_a=\frac{1}{2s}\left(\begin{array}{c|c|c|c|c} \cdot & \cdot & dK_a^{\dagger} & \cdot & \cdot \\ \hline \frac{s-1}{s+1}S_a & \cdot & \cdot & dK_a^{\dagger} & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \Sigma_a & \cdot & \cdot \\ \hline dK_a & \cdot & \cdot & \Sigma_a & \cdot \end{array}\right), \qquad (153)$$

where the symbols S_a, Σ_a, and K_a denote the $(2s+1)\times(2s+1)$, $(2s-1)\times(2s-1)$, and $(2s-1)\times(2s+1)$ matrices determined by the relations (144) and (146), I and 1 are the $(2s+1)$- and $(2s-1)$-row unit matrices, and the dots are matrices of the corresponding dimensions with zeros as elements; a and b are arbitrary numbers.

Proof. Using Schur's lemma, we conclude that the general form of the matrices β_0 and β_5 satisfying the relations (124), (125), (129), and (141) is given by

$$\beta_0=\left(\begin{array}{c|c} y_1\otimes I & 0 \\ \hline 0 & y_2\otimes\hat{1} \end{array}\right);\ \beta_5=2\left(\begin{array}{c|c} x_1\otimes I & 0 \\ \hline 0 & x_2\otimes 1 \end{array}\right), \qquad (154)$$

where x_1 and y_1 (and x_2 and y_2) are unknown $n\times n$ and $m\times m$ Hermitian matrices.

We require that the matrix β_0 satisfies the conditions (126). Using (143) and (154), we obtain the following equations for y_1 and y_2:

$$(a_1^{nm})^{\dagger}y_1-y_1a_1^{nm}=(a_3^{nm})^{\dagger}y_2-y_1a_2^{nm}=(a_4^{nm})^{\dagger}y_2-y_2a_4^{nm}=0. \qquad (155)$$

Substitititing β_5 (154) and η_a (143) in (127), we find β_a:

$$\beta_a=\frac{\mathrm{i}}{s}\left(\begin{array}{c|c} A\otimes S_a & B^{\dagger}\otimes K_a^{\dagger} \\ \hline -B\otimes K_a & D\otimes\Sigma_a \end{array}\right), \qquad (156)$$

where

$$A=(a_1^{nm})^{\dagger}x_1-x_1a_1^{nm};\ B=(a_2^{nm})^{\dagger}x_1-x_2a_3^{nm};$$
$$D=(a_4^{nm})^{\dagger}x_2-x_2a_4^{nm}. \qquad (157)$$

Finally, substituting (143), (154), and (156) in (128), we arrive at the system of equations

$$\begin{gathered} Aa_1^{nm}\otimes S_aS_b+B^{\dagger}a_3^{nm}\otimes K_a^{\dagger}K_b-(a_1^{nm})^{\dagger}A\otimes S_bS_a \\ +(a_3^{nm})^{\dagger}BK_b^{\dagger}K_a=2s^2y_1\otimes I\delta_{ab}; \\ Aa_2^{nm}\otimes S_aK_s^{\dagger}+B^{\dagger}a_4^{nm}\otimes K_a\Sigma_b \\ -(a_1^{nm})^{\dagger}B^{\dagger}\otimes S_bK_a^{\dagger}-(a_3^{nm})^{\dagger}D\otimes K_b^{\dagger}\Sigma_a=0; \\ Da_4^{nm}\otimes\Sigma_a\Sigma_b-Ba_2^{nm}\otimes K_aK_b^{\dagger}-(a_2^{nm})^{\dagger}B^{\dagger}\otimes K_bK_a^{\dagger} \\ -(a_4^{nm})^{\dagger}D\otimes\Sigma_b\Sigma_a=2s^2y_2\otimes I\delta_{ab}. \end{gathered} \qquad (158)$$

Expressing $K_a^{\dagger}K_{br}S_bS_a, K_b^{\dagger}K_a, \Sigma_b\Sigma_a$ in terms of $S_aS_b, \Sigma_a\Sigma_b, S_c, \Sigma_c$ and $\delta_{ab}I, \delta_{ab}\hat{1}$ in (158) by means of (144) and (146), and equating the matrix coefficients of these linearly independent operators, we arrive at the equations

$$\begin{gathered} \{(a_1^{\dagger}x_1a_1-x_1a_1a_1)(2s-1)+x_1a_3a_2-a_3^{\dagger}x_2a_3\}+\text{h.c.}=0; \\ (2s-1)(a_1^{\dagger}a_1^{\dagger}x_1-a_1^{\dagger}x_1a_1+a_3^{\dagger}x_2a_3)+(s-1)a_3^{\dagger}a_2^{\dagger}x_1-sx_1a_2a_3=0; \\ (a_3^{\dagger}x_2a_3-x_1a_2a_3)+\text{h.c.}=2(2s-1)y_1; \\ \{(s-1)(x_2a_3a_2-a_2^{\dagger}x_1a_2)+(s+1)(2s-1)(a_4^{\dagger}x_2a_4-x_2a_4a_4)\}+\text{h.c.}=0; \\ (2s+1)(s-1)a_2^{\dagger}x_1a_2+s(s-1)x_2a_3a_2+(s^2-1)a_2^{\dagger}a_3^{\dagger}x_2 \\ +(s+1)(2s-1)(a_4^{\dagger}a_4^{\dagger}x_2-a_4^{\dagger}x_2a_4)=0; \\ (a_2^{\dagger}x_1a_2-x_2a_3a_2)+\text{h.c.}=2(2s-1)y_2; \\ 2a_2^{\dagger}x_1a_1-a_2^{\dagger}a_1^{\dagger}x_1-x_2a_3a_1+2a_4^{\dagger}x_2a_3-a_4^{\dagger}a_2^{\dagger}x_1-x_2a_4a_3=0; \\ 2a_2^{\dagger}x_1a_1-a_2^{\dagger}a_1^{\dagger}x_1-x_2a_3a_1-(s-1)(a_4^{\dagger}x_2a_3-x_2a_4a_3) \\ +(s+1)(a_2^{\dagger}a_1^{\dagger}x_1-a_2^{\dagger}x_1a_1)=0, \end{gathered} \qquad (159)$$

where

$$A+\text{h.c.}=A+A^{\dagger},\ a_k=a_k^{nm},\ k=1,2,3,4.$$

For given a_1, a_2, a_3, and a_4, Eqs. (155) and (159) determine a system of linear homogeneous equations for the matrix elements $(x_\alpha)_{ij}$ and $(y_\alpha)_{ij}$. Solving this system for the matrices $a_k=a_k^{nm}$ given by (145), we arrive at the results formulated in Theorem 6.

We have obtained four inequivalent classes of Galileo-invariant equations of the form (114), the matrices β_k being given by (150)–(153). If the matrices β_k have the form (150), then the function $\Psi(t,\mathbf{x})$ has $6s+1$ components, and Eq. (114) describes the motion of a free nonrelativistic particle with arbitrary spin s. Such equations are equivalent to the Hagen-Hurley equations[6,7] and in the case $s=\frac{1}{2}$ are identical to the Lévy-Leblond equations.[5]

Equations (114) and (151) have $6s-1$ components and describe a Galileo particle with spin $s'=s-1$. It will be shown below that these equations lead to a constant of the dipole interaction different from the one predicted by Hurley's equations.

Equations (114) are of greatest interest in the case when the matrices β_k have the form (152) and (153). These equations can also be interpreted as the equations of motion of a nonrelativistic particle with arbitrary spin. As will be shown below, it is these equations (in contrast to the LHH equations) which permit a Galileo-invariant description of the spin-orbit coupling of a charged particle to an external electromagnetic field (in the framework of the minimal-coupling principle).

We give the solutions of the relations (124)–(129) for the case when the representation of the algebra (37) corresponds to $N=4$, where N is the nilpotency exponent of the invariant operator D_1 (139). Such a problem has nontrivial solutions only for $n=2$, $m=4$, $s=\frac{1}{2}$. The matrices β_5 are given by Eqs. (154), in which the nonvanishing elements of the matrices x_1 and x_2 have the form

$$\begin{aligned}(x_1)_{13}=(x_1)_{14}=(x_1)_{22}=(x_1)_{23}=\frac{1}{s+1}(x_1)_{24}\\ =(x_1)_{31}=(x_1)_{32}=s(x_1)_{33}=(x_1)_{41}\\ =\frac{1}{s+1}(x_1)_{42}=(x_2)_{11}=(x_2)_{12}=(x_2)_{21}=\frac{s}{(s+1)^2}(x_2)_{22}=1.\end{aligned} \tag{160}$$

Using Eqs. (154) and (160) and the relations (143), (145), and (124)–(128), we can readily find the explicit form of the corresponding matrices β_0 and β_a.

We give some of the equations (114) and (150)–(153) for $s=0$, $\frac{1}{2}$, and 1 in component form:

$$s=0;\ \begin{cases} p_0\psi_0+\mathrm{i}\mathbf{p}\cdot\boldsymbol{\psi}=0,\\ 2m\boldsymbol{\psi}-\mathrm{i}\mathbf{p}\cdot\psi_0=0,\end{cases} \tag{161}$$

$$s=\frac{1}{2};\ \begin{cases} p_0\varphi+\mathrm{i}\boldsymbol{\sigma}\cdot\mathbf{p}\chi=0,\\ 2m\chi-\mathrm{i}\boldsymbol{\sigma}\cdot\mathbf{p}\varphi=0,\end{cases}\quad \varphi=\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix},\quad \chi=\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}, \tag{162}$$

$$s=1;\ \begin{cases} p_0\boldsymbol{\chi}+\mathbf{p}\times\boldsymbol{\Phi}+a^2m\boldsymbol{\chi}+2am\boldsymbol{\varphi}=0,\\ p_0\boldsymbol{\varphi}+\mathbf{p}\Phi_0+a^2m\boldsymbol{\varphi}+2am(\boldsymbol{\chi}+\boldsymbol{\Phi})=0,\\ \mathbf{p}\cdot\boldsymbol{\chi}+2am\Phi_0=0,\\ -\mathbf{p}\times\boldsymbol{\chi}+2am(\boldsymbol{\chi}+\boldsymbol{\varphi})=0,\end{cases} \tag{163}$$

where σ_a are the Pauli matrices; $\psi_\mu, \chi_\alpha, \varphi_\alpha, \chi_a, \varphi_a, \Phi_\mu$ ($\mu=0,1,2,3$; $\alpha=1,2$, $a=1,2,3$) are single-component wave functions.

The systems (114) with the matrices (151), (150), and (152), respectively, reduce to Eqs. (163)–(165) (for the given values of s). Equations (114) and (152) for $s=1$ [or, which is the same thing, the system of equations (163)] can also be written in the form

$$\left\{\frac{1}{2}(\hat\beta_0+\hat\beta_4)\,p_0+\left[\hat\beta_0-\hat\beta_4+2aI+\frac{a^2}{2}(\hat\beta_0+\hat\beta_4)\right]m-\hat\beta_a p_a\right\}\Psi=0,$$

where $\hat\beta_l$ ($l=0,1,2,3,4$) are the Kemmer–Duffin matrices.

Equations (114) *for representations with arbitrary nilpotency exponent*. Above, we have obtained all possible (up to equivalence) solutions of Eqs. (124)–(129) for the representations of the algebra (37) listed in Theorem 5. However, this theorem describes only a certain class of representations of the algebra (37) corresponding to $N\leqslant 4$, where N is the nilpotency exponent of the invariant operator D_1 (139).

Below, we obtain a class of equations corresponding to arbitrary N. The equations are derived with allowance for the circumstance that the extended Galileo group G is a subgroup of the generalized Poincaré group $P(1,4)$ (the group of rotations and displacements in five-dimensional Minkowski space), and, therefore, every equation invariant under $P(1,4)$ is automatically invariant under G as well.

We consider the system of partial differential equations

$$(\tilde\beta_\mu p^\mu+\varkappa)\,\Psi(x_0,x_1,x_2,x_3,x_4)=0, \tag{164}$$

where $p_\mu=-i\dfrac{\partial}{\partial x_\mu}$, $\mu=0,1,2,3,4$, $\tilde\beta_\mu$ are numerical matrices, and κ is an arbitrary parameter.

We require that the matrices $\tilde\beta_\mu$ satisfy the relations

$$[\tilde\beta_\mu, S_{\nu\lambda}]=\mathrm{i}(g_{\mu\lambda}\tilde\beta_\nu-g_{\mu\nu}\tilde\beta_\lambda), \tag{165}$$

where $S_{\mu\nu}$ are the generators of the group $O(1,4)$; $g_{\mu\nu}$ is the metric tensor: $g_{\mu\nu}=\mathrm{diag}\,(1,-1,-1,-1)$. If (165) is satisfied, Eq. (167) is invariant under the group $P(1,4)$.[31]

As was pointed out above, in this case Eq. (167) is also invariant under the Galileo group. Indeed, making the change of variables

$$x_0=\frac{1}{2}(2\hat x_0+\hat x_4),\quad x_4=\frac{1}{2}(2\hat x_0-\hat x_4) \tag{166}$$

in (164), so that

$$\frac{\partial}{\partial x_0}=\frac{1}{2}\frac{\partial}{\partial\hat x_0}+\frac{\partial}{\partial\hat x_4};\quad \frac{\partial}{\partial x_4}=\frac{1}{2}\frac{\partial}{\partial\hat x_0}-\frac{\partial}{\partial\hat x_4}, \tag{167}$$

we obtain the equation

$$(\beta_0\hat p_0-\beta_4\hat p_4-\tilde\beta_a p_a+\varkappa)\,\Psi(\hat x_0,\hat x_4,\mathbf{x})=0, \tag{168}$$

where

$$\begin{aligned}&\hat p_0=\mathrm{i}\frac{\partial}{\partial\hat x_0},\quad \hat p_4=-\mathrm{i}\frac{\partial}{\partial\hat x_4},\\ &\beta_0=\frac{1}{2}(\tilde\beta_0+\tilde\beta_4),\quad \beta_4=\tilde\beta_0-\tilde\beta_4.\end{aligned} \tag{169}$$

The Galileo invariance of Eq. (168) can be readily verified directly by using the following realization of the extended Galileo group:

$$\begin{aligned}&P_0=-\mathrm{i}\frac{\partial}{\partial\hat x_0};\quad P_a=p_a=-\mathrm{i}\frac{\partial}{\partial x_a};\quad M=\hat p_4=-\mathrm{i}\frac{\partial}{\partial\hat x_4};\\ &J_a=(\mathbf{x}\times\mathbf{p})_a+S_a,\ G_a=\hat x_0p_a-x_aM+\eta_a,\end{aligned} \tag{170}$$

where

$$S_a=(1/2)\,\varepsilon_{abc}S_{bc},\quad \eta_a=(1/2)(S_{0a}+S_{4a}).$$

The operator $L=\beta_0\hat p_0-\beta_4\hat p_4-\tilde\beta_a p_a$ and the generators (170) satisfy the condition (2).

Imposing on the solutions of Eq. (168) the Galileo-invariant subsidiary condition

$$\hat p_4\Psi(\hat x_0,\hat x_4;\mathbf{x})=-\lambda\varkappa\Psi(\hat x_0,\hat x_4,\mathbf{x}),$$

we obtain from (168) a Galileo-invariant equation in the form (114), where

$$\beta_a=\tilde\beta_a,\ \beta_5=\beta_4+\lambda I;\quad m=\varkappa/\lambda, \tag{171}$$

and I is the unit matrix.

Thus, with every solution of the system of commutation relations (165) we can associate a Galileo-invariant equation (114), in which the matrices β_μ and β_5 are given by Eqs. (169) and (171).

In contrast to the analogous relations for the matrices that determine Poincaré-invariant equations,[28] the complete solution of the relations (165) has not hitherto been obtained. We use the special solution of Eqs. (165) proposed in Ref. 31. We denote by $\hat S_{kl}$ the generators of a finite-dimensional irreducible representation of $O(1,5)$. Then the matrices

$$S_{\mu\nu}=\hat{S}_{\mu\nu};\quad \tilde{\beta}_\mu=\hat{S}_{5\mu};\quad \mu=0,1,2,3,4, \tag{172}$$

satisfy the relations (165). Substituting (175) in (169) and (171), we obtain the matrices β_k in the form

$$\beta_a=\hat{S}_{5a};\quad \beta_0=(\hat{S}_{40}+\hat{S}_{50})/2;\quad \beta_5=S_{40}-S_{50}+\lambda I. \tag{173}$$

Therefore, with each irreducible representation of $O(1,5)$ we can associate the Galileo-invariant equation (114), (173). The generators of the Galileo group on the solution set of (114), (173) have the locally covariant form (36), where

$$M=m=\varkappa/\lambda;\quad \eta_a=(\hat{S}_{4a}+\hat{S}_{0a})/2;\quad S_a=\varepsilon_{abc}\hat{S}_{bc}/2. \tag{174}$$

The finite-dimensional representations of the group $O(1,5)$, which is locally isomorphic to the group $O(6)$, are specified by three numbers (n_1, n_2, and n_3), which are simultaneously integral or half-integral.[28] If the matrices $\hat{S}_{kl}$ from (173) form the representation $D(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ of the Lie algebra of the group $O(1,5)$, then the equations (114), (173) are equivalent to the Lévy-Leblond equation[5] for a nonrelativistic particle with spin $s=\frac{1}{2}$. The representation $D(1,1,0)$ leads to Eqs. (114), (152) for particles with spin $s=1$. The matrices (172) are identical to the Kemmer-Duffin matrices. In the general case, Eqs. (114), (173) describe a multiplet of nonrelativistic particles with spins $s_1, s_2, \ldots$, where the numbers s_i characterize the representations of the group $O(3)$ in the irreducible representation $D(n_1,n_2,n_3)$ of $O(1,5)$.

One can show that the matrices (174) satisfy the conditions

$$(\eta_a S_a)^{2s+1}=0;\quad (\eta_a S_a)^{2s}\neq 0, \tag{175}$$

where s is the maximal spin of the particles described by Eqs. (114),(173). Thus, the equations found here correspond to representations of the algebra (37) with arbitrary nilpotency exponent of the invariant operator D_1 (139).

4. NONRELATIVISTIC PARTICLE WITH ARBITRARY SPIN IN AN EXTERNAL ELECTROMAGNETIC FIELD

The equations of motion of free nonrelativistic particles can be of real interest for physics only if they are a first step in the description of particles which participate in various interactions. One such interaction is that of a charged particle of nonzero spin in an external electromagnetic field.

It will be shown below that the equations obtained in the framework of the minimal-coupling principle describe the dipole, spin-orbit, and Darwin interactions of a charged nonrelativistic particle with an external field, i.e., they take into account all the physical effects predicted in the order $1/m^2$ by the relativistic Dirac equation. The problem of the motion of a charged particle with arbitrary spin in a constant magnetic field is solved exactly.

Second-order equations for a particle with spin interacting with an external electromagnetic field. We obtained above the Schrodinger equations (64), (73), (74) and (64), (97), which describe the motion of free nonrelativistic particles with arbitrary spin. To go over to the description of the motion of a charged particle in an external electromagnetic field, we make in these equations the usual substitution

$$p_\mu \to \pi_\mu=p_\mu-eA_\mu, \tag{176}$$

where A_μ is the 4-vector potential of the electromagnetic field. As a result, we arrive at the equations

$$L(\pi)\Psi(t,\mathbf{x})=0,\quad L(\pi)=i\frac{\partial}{\partial t}-H_s^\alpha(\pi,A_0), \tag{177}$$

where $H_s^\alpha(\pi,A_0)$ is one of the Hamiltonians obtained from (73), (74), or (97) by means of the substitution (176):

$$H_s^{\mathrm{I}}(\pi,A_0)=\sigma_1 am+\frac{\pi^2}{2m}+eA_0+2iak\sigma_3\mathbf{S}\cdot\pi$$
$$-(\sigma_1+i\sigma_2)\frac{2ak^2}{m}\left[(\mathbf{S}\cdot\pi)^2-\frac{e}{2}\mathbf{S}\cdot\mathbf{H}\right]; \tag{178}$$

$$\tilde{H}_s^{\mathrm{I}}(\pi,A_0)=\sigma_3\tilde{a}m+\frac{a}{2}(\sigma_1-i\sigma_2)m-2\tilde{a}k(i\sigma_1-\sigma_2)\mathbf{S}\cdot\pi$$
$$-(\sigma_1+i\sigma_2)\frac{2ak^2}{m}\left[(\mathbf{S}\cdot\pi)^2-\frac{e}{2}\mathbf{S}\cdot\mathbf{H}\right]+\frac{\pi^2}{2m}+eA_0; \tag{179}$$

$$H_s^{\mathrm{II}}(\pi,A_0)=\sigma_1\left\{m+\frac{\pi^2}{2m}-\frac{\sin^2\theta_s}{ms^2}\left[(\mathbf{S}\cdot\pi)^2-\frac{e}{2}\mathbf{S}\cdot\mathbf{H}\right]\right\}$$
$$-\sigma_3\left\{a_s\frac{\pi^2}{2m}+\frac{b_s}{2ms^2}\left[(\mathbf{S}\cdot\pi)^2-\frac{e}{2}\mathbf{S}\cdot\mathbf{H}\right]\right\}$$
$$+\sigma_2\frac{\sqrt{2}\sin\theta_s}{s}\mathbf{S}\cdot\pi. \tag{180}$$

In (178)–(180), the symbol $\mathbf{H}$ denotes the magnetic field vector: $\mathbf{H}=-i\pi\times\pi$.

Equations (177)–(180) are obviously invariant under the gauge transformations

$$\Psi(t,\mathbf{x})\to\Psi(t,\mathbf{x})\exp[ie\varphi(t,\mathbf{x})],\quad A_\mu\to A_\mu+\frac{\partial\varphi(t,\mathbf{x})}{\partial x_\mu}. \tag{181}$$

We show that Eqs. (177), (178) and (177), (179) are invariant under transformations of the Galileo group (14),(38) if the potential A_μ transforms in accordance with the Galileo law[5]

$$A_0\to A_0''=A_0+\mathbf{v}\cdot\mathbf{A},\quad A_a\to A_a''=R_{ab}A_b, \tag{182}$$

where R_{ab} is the operator of the three-dimensional rotation (17). The invariance of the equation

$$L(\pi)\Psi(t,\mathbf{x})=0, \tag{183}$$

where $L(\pi)$ is some linear operator which depends functionally on π_μ in (176), with respect to the Galileo transformations (14), (38), and (182) means that the transformed function $\Psi''(t'',\mathbf{x}'')$ (38) satisfies the same equation as the original function:

$$L(\pi'')\Psi''(t'',\mathbf{x}'')=0, \tag{184}$$

where $L(\pi'')$ is the operator obtained from $L(\pi)$ by the substitution $\pi_\mu\to\pi_\mu''=-i\frac{\partial}{\partial x''}-eA_\mu''$, and x_a'', t'', and A_μ'' are given by Eqs. (14) and (182).

The condition of Galileo invariance of Eq. (185) can be formulated as a requirement imposed on the operator $L(\pi)$ [cf. (91)].

Definition 3. Equation (183) is invariant under the transformations of the Lorentz group (14), (38) if the operator $L(\pi)$ satisfies the conditions

$$\exp[if(t,\mathbf{x})]\hat{D}(\theta,\mathbf{v})L(\pi)D^{-1}(\theta,\mathbf{v})\exp[-if(t,\mathbf{x})]=L(\pi''), \tag{185}$$

where $f(t,\mathbf{x})$ is the phase factor (18),

$$\tilde{D}(\theta, \mathbf{v}) = \exp(-i\tilde{\mathbf{S}}\cdot\tilde{\theta})\exp(-i\tilde{\eta}\cdot\mathbf{v});$$
$$D(\theta, \mathbf{v}) = \exp(-i\mathbf{S}\cdot\theta)\exp(-i\eta\cdot\mathbf{v}), \tag{186}$$

and S_a, η_a and $\tilde{S}_a$, $\tilde{\eta}_a$ are the matrices realizing representations (in the general case inequivalent) of the Lie algebra of the homogenous Galileo group (37).

If (185) is satisfied, then (184) follows directly from (14), (38), (182), and (183).

Using the relations

$$\left.\begin{array}{l} \pi_0'' = \pi_0 + \mathbf{v}\cdot\pi; \quad \pi_a'' = R_{ab}\pi_b; \\ \exp[if(t, \mathbf{x})]\,\pi_a \exp[-if(t, \mathbf{x})] = \pi_a + mv_a \end{array}\right\} \tag{187}$$

and Eq. (90), we can directly verify that the operators (177) satisfy the condition (185) and (186), where $\tilde{D}(\theta,\mathbf{v}) = D(\theta,\mathbf{v})$, and $D(\theta, \mathbf{v})$ are defined in (39) and (66), and; therefore, Eqs. (177), (178) and (177), (179) are Galileo invariant.

It is convenient to analyze Eqs. (177) in a representation in which the operators (180)–(182) are quasidiagonal, i.e., they commute with the matrix σ_1 or σ_3 (67). As in the case of the Dirac equation, the Hamiltonians (178)–(180) can be diagonalized only approximately. We show below that up to terms of order $1/m^2$ the operators (178), (179) for $a=0$, and (180) can be reduced to the form

$$\begin{aligned} [H_s^{\mathrm{I}}(\pi, A_0)]' &= \sigma_1 am + \frac{\pi^2}{2m} + eA_0 - eB\sigma_1\frac{\mathbf{S}\cdot\mathbf{H}}{m} \\ &+ \frac{eD^2}{2m^2}\Big[-\frac{1}{2}\mathbf{S}\cdot(\pi\times\mathbf{E} - \mathbf{E}\times\pi) + \frac{1}{3}Q_{ab}\frac{\partial E_a}{\partial x_b} \\ &+ \frac{1}{3}s(s+1)\,\mathrm{div}\,\mathbf{E}\Big] + \frac{eBD}{m^2}\Big[\mathbf{S}\cdot(\pi\times\mathbf{H} - \mathbf{H}\times\pi) \\ &- \frac{2}{3}Q_{ab}\frac{\partial H_a}{\partial x_b}\Big] + o\left(\frac{1}{m^3}\right); \end{aligned} \tag{188}$$

$$[\tilde{H}_s^{\mathrm{I}}(\pi, A_0)]' = \sigma_3\tilde{a}m + \frac{\pi^2}{2m} + eA_0 + o\left(\frac{1}{m^3}\right); \tag{189}$$

$$\begin{aligned} [H_s^{\mathrm{II}}(\pi, A_0)]' &= \sigma_1\left(m + \frac{\pi^2}{2m} + eA_0 + \frac{e\sin^2\theta_s}{2ms^2}\mathbf{S}\cdot\mathbf{H}\right) \\ &+ \frac{e\sin^2\theta_s}{4m^2s^2}\Big[\frac{1}{2}\mathbf{S}\cdot(\pi\times\mathbf{E} - \mathbf{E}\times\pi) \\ &- \frac{1}{3}Q_{ab}\frac{\partial E_a}{\partial x_b} - s(s+1)\,\mathrm{div}\,\mathbf{E}\Big] \\ &+ \frac{e\sqrt{2}\sin\theta_s}{4m^2s}\left(\frac{b_s}{4s^2} - a_s\right)\mathbf{S}\cdot(\pi\times\mathbf{H} - \mathbf{H}\times\pi) \\ &+ \frac{e\sqrt{2}\,b_s\sin\theta_s}{24m^2s^3}Q_{ab}\frac{\partial H_a}{\partial x_b} + o\left(\frac{1}{m^3}\right), \end{aligned} \tag{190}$$

where $\mathbf{E} = i[\overline{\pi}, \pi_0]$ is the electric field vector, Q_{ab} is the quadrupole-interaction tensor

$$Q_{ab} = \frac{1}{3}\{3[S_a, S_b]_+ - 2\delta_{ab}s(s+1)\}; \tag{191}$$

and B and D are arbitrary coefficients which can be expressed as follows by means of the parameters a and k:

$$B = ak^2, \quad D = k. \tag{192}$$

But if $a \neq 0$ in (179), the approximate Hamiltonian $[\tilde{H}^{\mathrm{I}}(\pi, A_0)]$ has a structure analogous to (188).

The Hamiltonians (188) and (190) contain terms corresponding to the interaction of a point charged particle with an external electromagnetic field $\left(\sim\frac{\pi^2}{2m} + eA_0\right)$, and also dipole ($\sim\mathbf{S}\cdot\mathbf{H}$), spin–orbit [$\sim\mathbf{S}\cdot(\pi\times\mathbf{E} - \mathbf{E}\times\pi)$], Darwin ($\sim$div $\mathbf{E}$), and quadrupole ($\sim Q_{ab}\partial E_a/\partial x_b$) interactions. The last two terms (which are P noninvariant and can be made arbitrarily small in the limit $D\to 0$, $\theta_s \to 0$) can be interpreted as magnetic spin–orbit and magnetic quadrupole interactions. The approximate Hamiltonians obtained by diagonalization of the relativistic equations for particles of arbitrary spin[16,17] have a similar structure.

We emphasize that, in contrast to the Dirac equation, the Galileo-invariant equations (177), (178) and (177), (180) [and the Hamiltonians (188) and (190)] are defined up to the arbitrary parameters a, k, and θ_s, which can be chosen, say, to make the constants of the dipole and spin–orbit interactions correspond to the experimental data. If

$$\theta_s = \pi/4, \ a=1, \ k=2, \ s=1/2, \tag{193}$$

then the first six terms in (188) and (190) are identical to the Hamiltonian obtained by diagonalization of the Dirac equation.[32] However, the operators (188) and (190) contain additional terms (which depend on the magnetic field intensity), which can be obtained from the generalized Dirac equation that takes into account the anomalous interaction of a particle with a field.

Note that Eqs. (177), (178) and (177), (179) can be obtained in the framework of a Lagrangian formalism if the parameters a, $\tilde{a}$, and k satisfy the conditions (92). Indeed, making the minimal substitution $\partial/\partial x_\mu \to \partial/\partial x_\mu + ieA_\mu$ in the Lagrangians (95) and (96), we obtain the operators[14]

$$\left.\begin{array}{l} L(t, \mathbf{x}) = L_0(t, \mathbf{x}) + L^{\mathrm{int}}(t, \mathbf{x}); \\ \tilde{L}(t, \mathbf{x}) = \tilde{L}_0(t, \mathbf{x}) + \tilde{L}^{\mathrm{int}}(t, \mathbf{x}), \end{array}\right\} \tag{194}$$

where $L_0(t,\mathbf{x})$ and $\tilde{L}_0(t,\mathbf{x})$ are given by (95) and (96), and $L^{\mathrm{int}}(t,\mathbf{x})$ and $\tilde{L}^{\mathrm{int}}(t,\mathbf{x})$ are, respectively,

$$\begin{aligned} L^{\mathrm{int}}(t, \mathbf{x}) &= e\{\overline{\Psi}A_0\Psi + 2ik\overline{\Psi}\sigma_3 S_a' A_a\Psi \\ &+ \frac{1}{2m}\Big[i\frac{\partial\overline{\Psi}}{\partial x_a}C_{ab}A_b\Psi - i\overline{\Psi}A_aC_{ab}\frac{\partial\Psi}{\partial x_b} \\ &- e\overline{\Psi}C_{ab}A_aA_b\Psi\Big]\}; \end{aligned} \tag{195}$$

$$\begin{aligned} \tilde{L}^{\mathrm{int}}(t, \mathbf{x}) &= e\{\overline{\Psi}A_0\Psi - 2\tilde{a}k\overline{\Psi}(\sigma_1 + i\sigma_2)S_aA_a\Psi \\ &+ \frac{i}{2m}\left(\frac{\partial\overline{\Psi}}{\partial x_a}A_a\Psi - \overline{\Psi}A_a\frac{\partial\Psi}{\partial x_a}\right) - \frac{e}{2m}A_bA_b\overline{\Psi}\Psi\}. \end{aligned} \tag{196}$$

It is easy to show that the Euler–Lagrange equations for the functions (194)–(196) lead to (177), (178) and (177), (179).

We give the explicit form of the operators which transform the Hamiltonians (178), (179) for $a=0$, and (180) to the form (188), (189), and (190), respectively:

$$V^{\mathrm{I}} = \exp(iC_s^{\mathrm{I}})\exp(iB_s^{\mathrm{I}})\exp(iA_s^{\mathrm{I}}); \tag{197}$$

$$\tilde{V}^{\mathrm{I}} = \exp(i\tilde{C}_s^{\mathrm{I}})\exp(i\tilde{B}_s^{\mathrm{I}})\exp(i\tilde{A}_s^{\mathrm{I}}); \tag{198}$$

$$V^{\mathrm{II}} = \exp(iC_s^{\mathrm{II}})\exp(iB_s^{\mathrm{II}})\exp(iA_s^{\mathrm{II}}), \tag{199}$$

where

$$A_s^{\mathrm{I}} = -i\sigma_2 k\frac{\mathbf{S}\cdot\pi}{m}; \quad A_s^{\mathrm{II}} = \sigma_3\frac{\sqrt{2}\sin\theta_s}{2ms}\mathbf{S}\cdot\pi;$$

$$\begin{aligned} B_s^{\mathrm{I}} &= \sigma_3\frac{k}{2m^2}\Big\{\frac{1}{2a}[\mathbf{S}\cdot\pi, \pi^2]_+ \\ &+ ik[2(\mathbf{S}\cdot\pi)^2 + e\mathbf{S}\cdot\mathbf{H}] + \frac{1}{a}\mathbf{S}\cdot\mathbf{E}\Big\}, \end{aligned}$$

$$\begin{aligned} C_I^{\mathrm{I}} &= \sigma_2\frac{k^2}{m^3}\Big\{-\frac{2ik}{3}(\mathbf{S}\cdot\pi)^3 + ik[\mathbf{S}\cdot\pi, e\mathbf{S}\cdot\mathbf{H}]_+ \\ &+ [(\mathbf{S}\cdot\pi)^2, eA_0]\Big\} + \frac{i}{2m}\sigma_1\frac{\partial B_s^{\mathrm{I}}}{\partial t}; \end{aligned}$$

$$B_s^{\mathrm{II}}=\sigma_2 \frac{1}{4m^2}\Big\{a_s\pi^2+\frac{b_s}{2s^2}[2(\mathbf{S}\cdot\boldsymbol{\pi})^2-e\mathbf{S}\cdot\mathbf{H}] + \frac{e\sqrt{2}\sin\theta_s}{s}\mathbf{S}\cdot\mathbf{E}\Big\};$$

$$C_s^{\mathrm{II}}=\sigma_1 \frac{1}{8m^3}\Big\{\frac{\sqrt{2}\sin\theta_s}{s}\Big[\mathbf{S}\cdot\boldsymbol{\pi}, \pi^2+\Big(\frac{e\sin\theta_s}{s}\Big)^2 e\mathbf{S}\cdot\mathbf{H}\Big]_+ + iea_s[\pi^2, A_0]-\frac{4\sqrt{2}\sin^3\theta_s}{s^3}(\mathbf{S}\cdot\boldsymbol{\pi})^3 - \frac{iebs}{s^2}[(\mathbf{S}\cdot\boldsymbol{\pi})^2, A_0]\Big\}+\frac{i}{2m}\sigma_1\frac{\partial B_s^{\mathrm{II}}}{\partial t};$$

$$A_s^{\mathrm{I}}=\frac{ik}{m}(\sigma_2-i\sigma_1)\mathbf{S}\cdot\boldsymbol{\pi};\quad \tilde{B}_s^{\mathrm{I}}=\frac{k}{2\tilde{a}m^2}(\sigma_2-i\sigma_1)\mathbf{S}\cdot\mathbf{E};$$

$$\tilde{C}_s^{\mathrm{I}}=\frac{ik}{4m^3}(\sigma_2-i\sigma_1)[\pi^2, \mathbf{S}\cdot\boldsymbol{\pi}]+\frac{i}{2m}\sigma_3\frac{\partial\tilde{B}_s^{\mathrm{I}}}{\partial t}.$$

The Hamiltonians (178)–(180) and (188)–(190) are related by

$$H'=VHV^{-1}+i\frac{\partial V}{\partial t}V^{-1},$$

where H is one of the Hamiltonians (178), (179), or (180), and V is one of the operators (197), (198), or (199), respectively.

Introduction of minimal coupling in a first-order equation. Making the substitution (176) in Eq. (114), we arrive at systems of equations of the form

$$L(\pi)\Psi(t, \mathbf{x})=0;\quad L(\pi)=\beta_\mu\pi^\mu+\beta_5 m, \tag{200}$$

which can also be interpreted as the equations of motion of a particle with spin in an external electromagnetic field.

Let us briefly discuss the properties of Eqs. (200), which can be obtained without using the explicit form of the β matrices. It is easy to show that these equations are invariant under the gauge transformations (181). From the relations (124)–(128) we find by direct calculation that the operator $L(\pi)$ (200) satisfies the conditions (185) of Galilean invariance (where $\tilde{D}(\theta,\mathbf{v}) = [D^{-1}(\theta,\mathbf{v})]^\dagger$. Finally, Eqs. (200) can be obtained using the variational principle from the Lagrangian

$$\hat{L}(t, \mathbf{x})=L(t, \mathbf{x})+ie\overline{\Psi}\beta_\mu A_\mu\Psi,$$

where $L(t,\mathbf{x})$ is given by (120).

For the further analysis of Eqs. (200) and the physical interpretation of their solutions, we subject the function $\Psi(t,\mathbf{x})$ and the operator $L(\pi)$ to the transformations

$$\Psi\to\Psi'=V\Psi;\quad L(\pi)\to L'(\pi)=V^\dagger L(\pi)V, \tag{201}$$

where

$$V=\exp\Big(i\frac{\boldsymbol{\eta}\cdot\boldsymbol{\pi}}{m}\Big). \tag{202}$$

We then arrive at the equivalent equation

$$L'(\pi)\Psi'(t, \mathbf{x})=0. \tag{203}$$

We find the explicit form of the operator $L'(\pi)$. Using (134) and the commutation relations (124)–(128), we obtain

$$V^\dagger\beta_0\pi_0 V=\beta_0\Big(\pi_0+\frac{e}{m}\boldsymbol{\eta}\cdot\mathbf{E}+\frac{e}{2m^2}\eta_a\eta_b\frac{\partial E_a}{\partial x_b}\Big);$$

$$V^\dagger\beta_a\pi_a V=\beta_a\pi_a+\beta_0\Big[\frac{\pi^2}{2m}+\frac{3e}{4m^2}\boldsymbol{\eta}\cdot(\boldsymbol{\pi}\times\mathbf{H}-\mathbf{H}\times\boldsymbol{\pi})\Big]+\frac{e}{m}\boldsymbol{\beta}\times\boldsymbol{\eta}\cdot\mathbf{H};$$

$$V^\dagger\beta_5 mV=\beta_5 m+\beta_a\pi_a+\frac{1}{2}\beta_0\Big[\frac{\pi^2}{m}-\frac{e}{2m^2}\boldsymbol{\eta}\cdot(\boldsymbol{\pi}\times\mathbf{H}-\mathbf{H}\times\boldsymbol{\pi})\Big]+\frac{e}{2m}\boldsymbol{\beta}\times\boldsymbol{\eta}\cdot\mathbf{H},$$

from which it follows directly that

$$L'(\pi)=\beta_0\Big(\pi_0-\frac{\pi^2}{2m}\Big)+\beta_5 m+\frac{e}{m}\Big(\beta_0\boldsymbol{\eta}\cdot\mathbf{E}+\frac{1}{2}\boldsymbol{\beta}\times\boldsymbol{\eta}\cdot\mathbf{H}\Big)+\frac{e}{m^2}\beta_0\Big[\eta_a\eta_b\frac{\partial E_a}{\partial x_b}+\boldsymbol{\eta}\cdot(\boldsymbol{\pi}\times\mathbf{H}-\mathbf{H}\times\boldsymbol{\pi})\Big]. \tag{204}$$

Equations (203) and (204) cannot be obtained from (137) by the substitution $p_\mu\to\pi_\mu$, but they contain additional terms which depend on the intensity of the electromagnetic field. It will be shown below that these terms describe interactions due to the particle's spin.

Equations (203) and (204) can describe different physical effects depending on which representation of the algebra (37) is realized by the matrices η_a. A simple analysis shows that if these matrices form representations corresponding to $N\leq 2$, where N is the nilpotency exponent of the invariant operator D_1 (139), then the operator (204) does not include terms which depend on the electric field intensity. Indeed, in this case $\eta_a\eta_b = 0$, whence [and from the relations (128)]

$$\beta_0\eta_a\equiv 0. \tag{205}$$

Substituting (205) in (204), we obtain the operator $L'(\pi)$ in the form

$$L'(\pi)=\beta_0\Big(\pi_0-\frac{\pi^2}{2m}\Big)+\beta_5 m+\frac{e}{2m}\boldsymbol{\beta}\times\boldsymbol{\eta}\cdot\mathbf{H}. \tag{206}$$

If $\mathbf{H}\equiv 0$, and the electric field is nonzero, $\mathbf{E}\neq 0$, then the operator (206) commutes with the spin matrices S_a and, therefore, Eq. (203) does not describe coupling of the particle's spin to the electric field. This result can be formulated as follows,

Lemma 4. A necessary condition for Eq. (200), where β_k are matrices satisfying Eqs. (124)–(129) and (37), to describe the coupling of the particle's spin to the electric field is

$$(S_a\eta_a)^2\neq 0. \tag{207}$$

The matrices (150) and (151) do not satisfy the condition (207). Therefore, Eqs. (200), (150) and (200), (151) (which include the LHH equations) do not describe coupling (spin-orbit, quadrupole, etc.) of the spin to the external electric field.

Further, we see that Eqs. (200) with the matrices (152) and (153) [for which the relations (207) hold] describe the interactions listed above.

We now consider in detail Eqs. (203) and (204) for the cases when the matrices β_k are given by one of the formulas (150)–(153). Denoting

$$\Psi'(t, \mathbf{x})=\begin{pmatrix}\Phi_1(t, \mathbf{x})\\ \Phi_2(t, \mathbf{x})\\ \chi(t, \mathbf{x})\end{pmatrix},$$

where Φ_1 and Φ_2 are $(2s+1)$-component functions and χ is a $(2s-1)$-component function, and substituting the explicit form of the matrices β_k and η_a from (150) in (203) and (204), we arrive at the equations

$$i\frac{\partial}{\partial t}\Phi_1(t, \mathbf{x})=\Big[\frac{\pi^2}{2m}+eA_0-\frac{eg}{2m}\hat{\mathbf{S}}\cdot\mathbf{H}\Big]\Phi_1,\quad g=\frac{1}{s}. \tag{208}$$

Thus, Eqs. (150) and (200) (the LHH equations) reduce to the Pauli equation (208) for the $(2s+1)$-component function $\Phi_1(t,\mathbf{x})$.

Further, denoting

$$\Psi'(t, \mathbf{x}) = \begin{pmatrix} \Phi(t, \mathbf{x}) \\ X_1(t, \mathbf{x}) \\ X_2(t, \mathbf{x}) \end{pmatrix}, \tag{209}$$

and substituting (209) and (151) in (203) and (206), we obtain

$$i\frac{\partial}{\partial t}X_1 = HX_1; \quad X_2 = \Phi = 0,$$

$$H = \frac{\pi^2}{2m} - \frac{\Sigma\cdot\mathbf{H}}{2sm} + eA_0,$$

or, in the notation $s' = s-1, \Sigma = \mathbf{S}'$,

$$H = \frac{\pi^2}{2m} - g'\frac{e}{2m}\mathbf{S}'\cdot\mathbf{H} + eA_0; \quad g' = \frac{1}{s'+1}. \tag{210}$$

Therefore, Eqs. (151) and (200) also reduce to the Pauli equation for a particle with arbitrary spin but predict a different property (compared with the LHH equations) of the dipole moment of the particle (since for $s = s'$ the factors g and g' are not equal).

We now consider the case when the matrices β_k are given by formulas (152). We take

$$\Psi'(t, \mathbf{x}) \text{ to be the column } (\Phi_1, \Phi_2, \Phi_3, \chi_1, \chi_2), \tag{211}$$

and substitute (152) and (211) in (203) and (204). After simple calculations, we arrive at the expressions

$$L\Phi_1 \equiv \Big\{\pi_0 - \frac{\pi^2}{2m} + \frac{e}{4sm}\hat{\mathbf{S}}\cdot\mathbf{H} + \frac{e}{4sam}\Big[\hat{\mathbf{S}}\cdot\mathbf{E} + \frac{1}{2m}\hat{\mathbf{S}}\cdot(\pi\times\mathbf{H} - \mathbf{H}\times\pi)\Big] + \frac{e^2}{16am^2s(2s-1)}[s^2\mathbf{H}^2 - (\hat{\mathbf{S}}\cdot\mathbf{H})^2]\Big\}\Phi_1 = 0; \tag{212}$$

$$\left.\begin{aligned} &\Phi_2 = -a\Phi_1; \quad \Phi_3 = -\frac{1}{2m}\Big[\pi_0 - \frac{\pi^2}{2m} - a^2m + \frac{e\hat{\mathbf{S}}\cdot\mathbf{H}}{4sm}\Big]\Phi_1; \\ &\chi_1 = -\frac{(s+1)e\mathbf{K}\cdot\mathbf{H}}{8am^2s\sqrt{2s-1}}\Phi_1; \quad \chi_2 = 0. \end{aligned}\right\} \tag{213}$$

Thus, Eqs. (203) and (204) with the matrices (152) reduce to Eq. (212) for the $(2s+1)$-component function $\Phi_1(t, \mathbf{x})$ [the remaining components of $\Psi'(t, \mathbf{x})$ can be expressed in terms of Φ_1 in accordance with formulas (213)]. To find the physical meaning of the solutions of Eq. (212), we subject the function Φ_1 and the operator L to a transformation which makes it possible to eliminate the term $\dfrac{\mathbf{S}\cdot\mathbf{E}}{4sam}$ corresponding to the unphysical electric dipole interaction:

$$\Phi_1 \to \Phi_1' = \exp\Big(i\frac{\mathbf{S}\cdot\pi}{4sam}\Big)\Phi_1;$$

$$L \to L' = \exp\Big(i\frac{\mathbf{S}\cdot\pi}{4sam}\Big)L\exp\Big(-i\frac{\mathbf{S}\cdot\pi}{4sam}\Big).$$

Using the Campbell–Hausdorff formula (134) and the identities

$$i[\mathbf{S}\cdot\pi, \mathbf{S}\cdot\mathbf{H}] = \frac{1}{2}[S_a, S_b]_+\frac{\partial H_a}{\partial x_b} + \frac{1}{2}i[\mathbf{S}\cdot\pi, \pi^2];$$

$$i[\mathbf{S}\cdot\pi, \pi^2] = e\mathbf{S}(\pi\times\mathbf{H} - \mathbf{H}\times\pi), \quad i[\mathbf{S}\cdot\pi, \mathbf{S}\cdot\mathbf{E}] = -\frac{1}{2}\mathbf{S}\cdot(\pi\times\mathbf{E} - \mathbf{E}\times\pi) + \frac{1}{3}Q_{ab}\frac{\partial E_a}{\partial x_b} + s(s+1)\,\mathrm{div}\,\mathbf{E},$$

we obtain

$$L' = \pi_0 - \frac{\pi^2}{2m} + \frac{e}{4sm}\mathbf{S}\cdot\mathbf{H} + \frac{e}{16s^2m^2a^2}\Big[-\frac{1}{2}\mathbf{S}\cdot(\pi\times\mathbf{E} - \mathbf{E}\times\pi) + \frac{1}{3}Q_{ab}\frac{\partial E_a}{\partial x_b} + s(s+1)\,\mathrm{div}\,\mathbf{E} + \frac{a}{3}Q_{ab}\frac{\partial H_a}{\partial x_b} - \frac{a}{2}\mathbf{S}\cdot(\pi\times\mathbf{H} - \mathbf{H}\times\pi)\Big] + o\Big(\frac{1}{m^3}\Big) + o(e^2). \tag{214}$$

Like the approximate Hamiltonians (178)–(180), the operator (214) contains terms corresponding to dipole, quadrupole, and spin-orbit interaction of a charged particle with an external electromagnetic field.

Similarly, we can show that Eq. (203), (204) with the matrices (153) reduces to the equation

$$\pi_0\chi_1 = \Big\{\frac{\pi^2}{2m} - \frac{e\mathbf{S}\cdot\mathbf{H}}{4(s'+1)m} - \frac{e}{4b(s'+1)m} \times\Big[\mathbf{S}'\cdot\mathbf{E} + \frac{1}{2m}\mathbf{S}\cdot(\pi\times\mathbf{H} - \mathbf{H}\times\pi)\Big] + o(e^2)\Big\}\chi_1,$$

which can be reduced by the unitary transformation

$$\chi_1 \to \chi_1' = \exp\Big(i\frac{\mathbf{S}'\cdot\pi}{4(s'+1)mb}\Big)\chi$$

to the form $L'\chi_1 = 0$, where

$$L' = \pi_0 - \frac{\pi^2}{2m} + \frac{e\mathbf{S}'\cdot\mathbf{H}}{4(s'+1)m} + \frac{e}{16(s'+1)^2b^2m^2}\Big[-\frac{1}{2}\mathbf{S}'\cdot(\pi\times\mathbf{E} - \mathbf{E}\times\pi) + \frac{1}{3}Q_{ab}\frac{\partial E_a}{\partial x_b} + s(s+1)\,\mathrm{div}\,\mathbf{E} + \frac{b}{3}Q_{ab}\frac{\partial H_a}{\partial x_b} - \frac{b}{2}\mathbf{S}'\cdot(\pi\times\mathbf{H} - \mathbf{H}\times\pi)\Big] + o\Big(\frac{1}{m^3}\Big) + o(e^2). \tag{215}$$

The operator (215) can be obtained from (214) by the substitution $s \to s'+1, \mathbf{S} \to \mathbf{S}', a \to b$.

We note finally that all the approximate Hamiltonians (208), (210), (214), and (215) obtained by diagonalization of the first-order equations (200) can be obtained from the operator (188) by an appropriate choice of the coefficients B and D. In other words, the second-order equations (179) and (180) are more universal than Eqs. (200), since they include the latter as special cases in the $1/m^2$ approximation.

It should be noted that in the framework of the Galileo group equations in the Schrödinger form (177) are more natural than equations of the form (200), since in nonrelativistic quantum mechanics the time coordinate t is distinguished and, therefore, need not necessarily occur in the equations of motion on an equal footing with the spatial variables x_a.

Anomalous coupling in nonrelativistic quantum mechanics. The substitution $p_\mu \to \pi_\mu$ in the equation of motion is not the only possible way of describing coupling of a particle to an external electromagnetic field. A more general approach, widely used in relativistic quantum mechanics, takes into account the so-called anomalous coupling of a particle to a field. This is described mathematically by adding to the equation of motion terms which depend on the electromagnetic field intensity.

In the present paper, we restrict ourselves to a dipole anomalous coupling and consider equations of the form

$$i\frac{\partial}{\partial t}\Psi = \hat{H}_s^{\mathrm{I}}(\pi, A_0)\Psi;$$

$$H_s^{\mathrm{I}}(\pi, A_0) = H_s^{\mathrm{I}}(\pi, A_0) + \frac{e}{m}(A_a^sE_a + B_a^sH_a) \tag{216}$$

and

$$L\Psi \equiv \beta_\mu\pi^\mu + \beta_5 m + \frac{e}{m}(C_aH_a + D_aE_a), \tag{217}$$

where E_a and H_a are the components of the magnetic and electric fields, $H_s^{\mathrm{I}}(\pi, A_0)$ is the operator (178), β_μ are the matrices (150) and (151), and A_a^s, B_a^s, C_a, and

D_a are certain (as yet unknown) matrices which must be such that Eqs. (216) and (217) are invariant under the Galileo group.

We prove the following assertion.

Theorem 7. The operator $i\frac{\partial}{\partial t} - H_s^{\mathrm{I}}(\pi, A_0)$ satisfies the condition of Galilean invariance (185) if and only if the matrices A_a^s and B_a^s have the form

$$A_a^s = k_1\eta_a; \quad B_a^s = k_1 S_a + k_1'\eta_a, \tag{218}$$

where k_1 and k_1' are arbitrary numbers, and η_a and S_a are matrices given by (66).

Proof. A detailed proof of Theorem 7 is given in Ref. 14, and therefore we shall give only its outline. The invariance condition (185) reduces to the equations

$$D(\theta, \mathbf{v})(A_a^s E_a + B_a^s H_a) D^{-1}(\theta, \mathbf{v}) = A_a^s E_a'' + B_a^s H_a'', \tag{219}$$

where the matrices $D(\theta, \mathbf{v})$ are given by Eqs. (39) and (66), and

$$\left.\begin{aligned} H_a'' &= -i\varepsilon_{abc}\pi_b''\pi_c'' = R_{ab}H_b; \\ E_a'' &= i[\pi_0'', \pi_a''] = R_{ab}E_b - (\mathbf{v}\times\mathbf{H})_a. \end{aligned}\right\} \tag{220}$$

Substituting (39) and (66) in (219) and (220), we arrive at equations for A_a^s and B_a^s:

$$\left.\begin{aligned} &[B_a^s, S_b] = i\varepsilon_{abc}B_c^s, \ [A_a^s, S_b] = i\varepsilon_{abc}A_c^s; \\ &[\eta_a, A_b^s] = 0, \ [\eta_a, B_b^s] = i\varepsilon_{abc}A_c^s; \\ &\eta_a A_b^s \eta_c + \eta_c A_b^s \eta_a = \eta_a B_b^s \eta_c + \eta_c B_b^s \eta_a = 0. \end{aligned}\right\} \tag{221}$$

The general solution of Eqs. (221) is given by formulas (218).

Substituting (218) in (216) and subjecting the Hamiltonian $H_s^{\mathrm{I}}(\pi, A_0)$ to the transformation

$$\hat{H}_s^{\mathrm{I}}(\pi, A_0) \to [\hat{H}_s^{\mathrm{I}}(\pi, A_0)]' = V\hat{H}_s^{\mathrm{I}}V^{-1} + i\frac{\partial V}{\partial t}V^{-1},$$

where

$$V = \exp\left(iD\frac{\mathbf{S}\cdot\pi}{m}\right)\exp\left(\frac{1}{2m}\sigma_1\frac{\partial S}{\partial t}\right)\exp(iS)\exp\left(i\frac{\eta\cdot\pi}{m}\right);$$

$$D = \sigma_1 k(k'-1); \quad S = \frac{\sigma_2}{2m^2}\Big\{(k_1' - \eta k^2)\mathbf{S}\cdot\mathbf{H} - k(k'-1)\Big[\mathbf{S}\cdot\mathbf{E} + \frac{1}{2m}\mathbf{S}\cdot(\pi\times\mathbf{H} - \mathbf{H}\times\pi)\Big] + \frac{kk_1'}{2m}[S_a, S_b]_+\frac{\partial H_a}{\partial x_b}\Big\},$$

we obtain[14]

$$\hat{H}_s(\pi, A_0) = [H_s^{\mathrm{I}}(\pi, A_0)]' + \frac{kk_1'}{3m^2}Q_{ab}\frac{\partial H_a}{\partial x_b}, \tag{222}$$

where $[H_s^{\mathrm{I}}(\pi, A_0)]'$ is given by Eq. (186) for the following values of D and B:

$$B = k_1 + \sigma_1(k_1' - ak^2), \quad D = \sigma_1 k(k_1' - 1). \tag{223}$$

Comparing (186) and (223), we see that the introduction of the dipole anomalous coupling in the second-order equations (177) and (178) hardly changes the structure of the Hamiltonian in the $1/m^2$ approximation [essentially, all that changes is the coefficient of the term which represents the magnetic quadrupole interaction, since the coefficients (192) and (223) on the sets of functions $\Psi_\pm' = \frac{1}{2}(1 \mp \sigma_1)\Psi'$ can in equal measure be regarded as arbitrary parameters].

We now consider Eq. (217). Requiring that the operator L satisfies the condition of Galilean invariance (187), where $\tilde{D}(\theta, \mathbf{v}) = [D^{-1}(\theta, \mathbf{v})]^\dagger$, and taking into account the relations (220), we find by direct calculation that the general form of the matrices C_a and D_a is given by the formulas

$$C_a = \frac{k_2}{2}\varepsilon_{abc}\beta_0\beta_b\beta_c; \quad D_a = \frac{k_2'}{2}\varepsilon_{abc}\beta_0\beta_b\beta_c + \frac{ik_2}{2}(1 - 2\beta_0)\beta_a, \tag{224}$$

where k_2 and k_2' are arbitrary constants.

Substituting (224) in (217), using the explicit form of the β matrices [(150) and (151)], and making some simple calculations, we obtain equations for the $(2s+1)$-component function $\Phi_1 = \beta_0\Psi$ and the $(2s'+1)$-component function χ_1:

$$i\frac{\partial\Phi_1}{\partial t} = H_s\Phi_1, \quad i\frac{\partial\chi_1}{\partial t} = H_s'\chi_1, \tag{225}$$

where

$$H_s = \frac{\pi}{2m} + eA_0 - \frac{e(1+k_2')}{2ms}\mathbf{S}\cdot\mathbf{H} - \frac{ek_2}{2ms}\mathbf{S}\cdot\mathbf{H} - \frac{ek_2}{2m^2 s}[\mathbf{S}\cdot\pi, \mathbf{S}\cdot\mathbf{H}] + \frac{e^2k_2^2}{4m^2}H^2, \tag{226}$$

and H_s' can be obtained from (226) by the substitution $\mathbf{S} \to \mathbf{S}'$, $s \to s'+1$.

Subjecting the functions Φ_1 and χ_1 and the Hamiltonians H_s and H_s' to the transformations

$$\left.\begin{aligned} &\Phi_1 \to U\Phi_1; \quad H_s \to UH_sU^{-1} + i\frac{\partial U}{\partial t}U^{-1}; \\ &\chi_1 \to U'\chi_1; \quad H_s' \to U'H_s'(U')^{-1} + i\frac{\partial U}{\partial t}U^{-1}, \end{aligned}\right\} \tag{227}$$

where

$$U = \exp\left(\frac{ik_2\mathbf{S}\cdot\pi}{2sm}\right); \quad U' = \exp\left(\frac{ik_2\mathbf{S}'\cdot\pi}{2(s'+1)m}\right), \tag{228}$$

we arrive at the operator (188), where

$$B = (1+k_2')/2s; \quad D = k_2/2s. \tag{229}$$

Thus, the Hamiltonians of particles with arbitrary spin obtained from Eqs. (150), (151), and (217) in the $1/m^2$ approximation are identical, up to the coefficients D and B, with the Hamiltonian (188) obtained by diagonalizing Eqs. (177) and (178). Therefore, the LHH equations (150) and (217) and Eqs. (150) and (217), generalized to the case of dipole anomalous coupling of the parparticle to the external field, also describe dipole, quadrupole, and spin-orbit couplings.

Introduction of anomalous coupling in the Schrödinger equation. We have seen that the various Galileo-invariant equations (177), (178), (200), (216), and (217) lead in the $1/m^2$ approximation to the same (up to the coefficients) Hamiltonians of particles with arbitrary spin. To explain this fact, we prove the following assertion.

Lemma 5. Let $L(\pi)$ be an arbitrary linear operator which depends functionally on π_μ (176) and satisfies the condition of Galilean invariance (185). Then the operator

$$\hat{L}(\pi) = \exp\left(i\frac{\tilde{\eta}\cdot\pi}{m}\right)L(\pi)\exp\left(-i\frac{\eta\cdot\pi}{m}\right) \tag{230}$$

also satisfies the condition (185) with $\eta_a = \tilde{\eta}_a = 0$.

Proof. We apply to $\hat{L}(\pi)$ (230) from the left the operator $\exp[if(t,\mathbf{x})]\exp(-i\mathbf{S}\cdot\theta)$ and from the right the operator $\exp[-if(t,\mathbf{x})]\exp(i\mathbf{S}\cdot\theta)$. Using identities which can be readily obtained by means of the Campbell-Hausdorff formula,

$$\exp[if(t,\mathbf{x})]\exp\left(i\frac{\tilde{\eta}\cdot\pi}{m}\right) = \exp\left(i\frac{\eta\cdot\pi}{m}\right)\exp[if(t,\mathbf{x})]\exp(-i\eta\cdot\mathbf{v});$$

$$\exp\left(-i\frac{\eta\cdot\pi}{m}\right)\exp[-if(t,x)]$$
$$=\exp(i\eta\cdot v)\exp\left(-i\frac{\eta\cdot\pi}{m}\right)\exp[-if(t,x)]$$

and bearing in mind that by definition the operator $L(\pi)$ satisfies the relations (185), we obtain

$$\exp[if(t,x)]\exp(-i\mathbf{S}\cdot\theta)\hat{L}(\pi)\exp(i\mathbf{S}\cdot\theta)\exp[-if(t,x)]=\hat{L}(\pi''),$$

i.e., $\hat{L}(\pi)$ does indeed satisfy the condition of Galilean invariance (185) with $\eta_a=\tilde{\eta}_a=0$. The lemma is proved.

It follows from the lemma that an arbitrary Galileo-invariant equation (183) can be reduced, by means of the transition

$$\Psi(t,x)\to\hat{\Psi}(t,x)=\exp\left(i\frac{\eta\cdot\pi}{m}\right)\Psi(t,x) \quad (231)$$

to a new wave function, to an equation invariant under the Galileo group:

$$\hat{L}(\pi)\hat{\Psi}(t,x)=0, \quad (232)$$

where the function $\hat{\Psi}(t,\mathbf{x})$ has the simple transformation properties (34) and (35) [in this case, the representation of the homogeneous Galileo group realized on the solution set of the invariant equation reduces to a representation of the group $O(3)$].

By means of the transformations (230) and (231), the equations (177), (178), (200), (218), and (219) considered above reduce to (232), where the operator $\hat{L}(\pi)$ has the general form

$$\hat{L}(\pi)=A^1\left(\pi_0-\frac{\pi^2}{2m}\right)+A^2m+\frac{e}{m}B_a^1H_a$$
$$+\frac{e}{m}B_a^2\left[E_a+\frac{1}{2m}(\pi\times\mathbf{H})_a-\frac{1}{2m}(\mathbf{H}\times\pi)_a\right]$$
$$+\frac{e}{m^2}\left(Q_{ab}^1\frac{\partial E_a}{\partial x_b}+Q_{ab}^2\frac{\partial H_a}{\partial x_b}\right), \quad (233)$$

and $A^\alpha B_a^\alpha$, Q_{ab}^α $(\alpha=1,2)$ are matrices with the following commutation relations with the generators of the rotation group:

$$\left.\begin{array}{l}[A^\alpha,S_a]=0;\quad [B_a^\alpha,S_b]=i\varepsilon_{abc}B_c^\alpha;\\ [Q_{ab}^\alpha,S_c]=i(\varepsilon_{acd}Q_{bd}-\varepsilon_{bcd}Q_{ad}).\end{array}\right\} \quad (234)$$

Thus, instead of the different Galileo-invariant equations considered above, we can investigate an equation of the form (232), (233), which for an appropriate choice of the matrices A^α, B_a^α, and Q_{ab}^α is equivalent to (177), (178), (200), (218), or (219).

If $A^\alpha=I$, $B_a^\alpha=S_a$, $Q_{ab}^\alpha=Q_{ab}$, where S_a are the generators of the irreducible representation $D(s)$ of the group $O(3)$, and Q_{ab} is the tensor of the quadrupole coupling (191), then Eqs. (232) and (233) reduce to (212). These equations can be regarded as a Galileo-invariant generalization of the Schrodinger equation (1) for a $(2s+1)$-component function to take into account the minimal and anomalous couplings of the particle to the electromagnetic field (such generalizations were considered in a different approach in Ref. 32). Thus, the introduction of the minimal coupling in the first-order equations (152) and (200) has been shown to be equivalent to the introduction of anomalous coupling in the Schrödinger equation.

Nonrelativistic particle of arbitrary spin in a homogeneous magnetic field. We consider Eqs. (216) and (217) for the case of a constant homogeneous magnetic field and find the eigenvalues of the operator $\hat{H}_s^1(\pi,A_0)$. It can be assumed that the intensity vector of such a field is parallel to the third component of the momentum, i.e., in (217) it is sufficient to set

$$H_1=H_2=E_1=E_2=E_3=0;\quad H_3=H. \quad (235)$$

In accordance with (240), the vector potential A_μ can be chosen in the form

$$A_0=A_2=A_3=0;\quad A_1=-eHx_2. \quad (236)$$

Substituting (178) and (240) in (216) and (218) and simplifying the calculations by taking $k_1=1$, we arrive at the Hamiltonian

$$H_s=\sigma_1am+\frac{\pi^2}{2m}+\frac{e\mathbf{S}\cdot\mathbf{H}}{m}+2iak\mathbf{S}\cdot\pi$$
$$+\frac{1}{m}(\sigma_1+i\sigma_2)[2a(k\mathbf{S}\cdot\pi)^2-ek_0\mathbf{S}\cdot\mathbf{H}], \quad (237)$$

where $k_0=ak^2-k_1'$ can be regarded as an independent parameter.

We transform H_s to a form in which it depends only on commuting operators. This makes it possible to determine the eigenvalues of the Hamiltonian (235) without solving equations of motion.

Using the operator

$$U=\frac{1}{2}\left(1+\sigma_3\frac{h}{\sqrt{h^2}}\right)\left(1+\frac{i}{m}\eta\cdot\pi\right);$$
$$U^{-1}=\left(1-\frac{i}{m}\eta\cdot\pi\right)\left(1-\sigma_3\frac{h}{\sqrt{h^2}}\right),$$

where η_a are given by (66), and $h=\sigma_1am+\dfrac{ek_0}{m}\eta\cdot\mathbf{H}$, we obtain

$$H_s'=UH_sU^{-1}=\frac{\pi^2}{2m}+S_3H+\sigma_3(a^2m^2+2ak_0S_3H)^{1/2}. \quad (238)$$

All the quantities in the Hamiltonian (238) commute with one another and with H_s' and have eigenvalues as follows[34]:

$$\frac{\pi^2}{2m}\Phi=\frac{1}{2m}[(2n+1)eH+p_3^2],\quad n=0,1,2,\ldots;$$
$$S_3\Phi=s_3\Phi;\quad s_3=-s,\ -s+1,\ \ldots,\ s;\quad \sigma_3\Phi=\varepsilon\Phi,\ \varepsilon=\pm1,$$

from which we conclude that the eigenvalues of H_s' are

$$E_{\varepsilon n s_3 p_3}=(2n+1+2s_3)\frac{eH}{2m}+\frac{p_3^2}{2m}+\varepsilon(a^2m^2+2ak_0s_3H)^{1/2}. \quad (239)$$

Setting $k_0=0$, $s=\frac{1}{2}$ in (239), we obtain a formula that, apart from the unimportant term εam, gives the well-known energy spectrum of a nonrelativistic particle in a homogeneous magnetic field (the Landau levels). If $k_0\neq0$ but $k_0\ll am$, then

$$E_{\varepsilon n s_3 p_3}=[2n+1+2s_3(1+\varepsilon k_0)]\frac{eH}{2m}$$
$$+\varepsilon am+\frac{\varepsilon k_0^2e^2H^2S_3^2}{8am^3}+o\left(\frac{1}{m^5}\right). \quad (240)$$

In contrast to the case $k_0=0$, Eq. (240) includes a correction which takes into account the deviation of the particle's dipole moment from unity and a correction quadratic in the magnetic field intensity.

The explicit form of the eigenfunctions of the operator (237), which can be readily found using the results of Ref. 17, is not given here.

CONCLUSIONS

1. The systems of first- and second-order differential equations (177), (178), (200), (216)–(218), and (224) found above are invariant under Galileo and gauge

transformations and describe dipole, quadrupole, spin-orbit, and Darwin couplings of a particle of arbitrary spin to an external electromagnetic field. An alternative method of Galileo-invariant description of spin-orbit coupling was proposed in Ref. 10, in which the LHH equations were used in Hamiltonian form. Thus, the listed interactions are not purely relativistic effects and they can be consistently treated in the framework of nonrelativistic quantum mechanics. In Ref. 38, our conclusion[8,9] concerning the nonrelativistic nature of the spin-orbit interaction was discussed from a classical point of view.

2. Of course, equations of the form (64) and (114) do not exhaust all possible linear differential equations invariant under the Galileo group. Thus, to describe a nonrelativistic particle with spin $s=1$, we can use the Galileo-invariant analog of the Proca equations:

$$\left.\begin{aligned}&(2mp_0-\mathbf{p}^2)\Psi_\nu=0, \quad \nu=0,1,2,3;\\ &m\Psi_0-p_a\Psi_a=0, \quad a=1,2,3.\end{aligned}\right\} \tag{241}$$

Equations (241) are a special case of systems of equations of the form

$$C_1\Psi\equiv(2mp_0-\mathbf{p}^2)\Psi=0;\quad C_2\Psi\equiv W_aW_a\Psi$$
$$\equiv[m^2\mathbf{S}^2+m\mathbf{p}(\mathbf{s}\times\boldsymbol{\eta}-\boldsymbol{\eta}\times\mathbf{s})]\Psi=m^2s(s+1)\Psi,$$

where C_1 and C_2 are the Casimir operators (13) for the representations (36).

3. The generators (36) are non-Hermitian with respect to a scalar product of the type (32) because finite-dimensional representations of the homogeneous Galileo group are nonunitary. A similar situation obtains in relativistic theory, in which nonunitary representations of the homogeneous Lorentz group are realized on the solutions of finite systems of equations, and the requirement that these representations be unitary is equivalent to the transition to systems of equations for functions with infinitely many components. Therefore, it is of interest to consider infinite-component equations invariant under the Galileo group. As an example of such equations we can take the systems (114) and (173) in which $\hat{S}_{kl}$ are the generators of a unitary infinite-dimensional representation of the group $O(1,5)$.

4. After the present paper had been submitted, we were acquainted with Ref. 37, in which it is also shown that the requirement of Galileo invariance of the equations of motion of a charged particle in an external electromagnetic field does not lead unambiguously to minimal coupling of the particle to the field but admits other types of interaction. This result agrees well with the results of Refs. 8–14.

APPENDIX 1: IRREDUCIBLE REPRESENTATIONS OF THE LIE ALGEBRA OF THE EXTENDED GALILEO GROUP

We obtain irreducible representations of the algebra (6)–(10) in the orthogonal basis $|c,p,\lambda\rangle$, where $|\ldots\rangle$ are eigenvectors of the complete set of commuting operators C_a (13), P_μ, and $\Lambda=P_a\cdot J_a\cdot P^{-1}$, $P=(P_1^2+P_2^2+P_3^2)^{1/2}$:

$$\left.\begin{aligned}&C_a|c,\hat{p},\lambda\rangle=c_a|c,\hat{p},\lambda\rangle,\ a=1,2,3;\\ &P_\mu|c,\hat{p},\lambda\rangle=p_\mu|c,\hat{p},\lambda\rangle,\ \mu=0,1,2,3;\\ &\Lambda|c,\hat{p},\lambda\rangle=\lambda|c,\hat{p},\lambda\rangle.\end{aligned}\right\} \tag{A.1}$$

It is interesting to consider only those representations that do not reduce to representations of a subalgebra of the algebra (6)–(10).

We prove first the following assertion.

Lemma. The Lie algebra determined by the commutation relations

$$\left.\begin{aligned}&[\lambda_1,\lambda_2]=[\lambda_2,\lambda_3]=[\lambda_3,\lambda_1]=i\frac{c_2^2}{\sqrt{3}}\lambda_0;\\ &[\lambda_0,\lambda_a]=\frac{i}{2\sqrt{3}}\varepsilon_{abc}(\lambda_b-\lambda_c),\ a,b,c=1,2,3,\end{aligned}\right\} \tag{A.2}$$

where c_2^2 is an arbitrary real number, is isomorphic when $c_2^2>0$ to the Lie algebra of the group $O(3)$, when $c_2^2=0$ to the Lie algebra of the group $E(2)$, and when $c_2^2<0$ to the Lie algebra of the group $O(1,2)$.

Proof. We note first that among the four elements λ_μ of the algebra (A.1) only three are linearly independent, since we can always set

$$\lambda_1+\lambda_2+\lambda_3=0.$$

The isomorphism formulated in the lemma can be established by means of the relations

$$\left.\begin{aligned}&\lambda_0=K_3,\ \lambda_1=\frac{1}{2\sqrt{3}}[K_1(1+\sqrt{3})+K_2(1-\sqrt{3})];\\ &\lambda_2=\frac{1}{2\sqrt{3}}[K_1(1-\sqrt{3})+K_2(1+\sqrt{3})],\ \lambda_3=-\frac{1}{3}(K_1+K_2),\end{aligned}\right\} \tag{A.3}$$

where

$$\begin{aligned}&K_1=mS_1',\ K_2=mS_2,\ K_3=S_3,\ \text{if}\quad c_2^2=m^2>0;\\ &K_1=T_1,\ K_2=T_2,\ K_3=T_0,\ \text{if}\quad c_2^2=0;\\ &K_1=\eta S_{01},\ K_2=\eta S_{02},\ K_3=S_{12},\ \text{if}\quad c_2^2=-\eta^2<0,\end{aligned} \tag{A.4}$$

and $S_a, T_\alpha, S_{\alpha\beta}$ ($a=1,2,3,\ \alpha,\beta=0,1,2$) are the generators of the groups $O(3)$, $E(2)$, and $O(1,2)$, respectively, i.e., matrices satisfying the commutation relations

$$\left.\begin{aligned}&[S_a,S_b]=i\varepsilon_{abc}S_c;\\ &[T_1,T_0]=-iT_2,\ [T_2,T_0]=iT_1,\ [T_1,T_2]=0;\\ &[S_{01},S_{02}]=-iS_{12},\ [S_{01},S_{12}]=-iS_{02},\ [S_{02},S_{12}]=iS_{01}.\end{aligned}\right\} \tag{A.5}$$

By direct verification we can show that if the matrices S_a, T_α, and $S_{\alpha\beta}$ satisfy the relations (A.5), then the matrices λ_μ (A.3), (A.4) satisfy the algebra (A.2) and, conversely, (A.5) follows from (A.2)–(A.4). The lemma is proved.

One can show that the algebra (A.2) is satisfied by the operators

$$\lambda_0=W_0'p^{-1};\ \lambda_a=\frac{1}{2\sqrt{3}}\varepsilon_{abc}(W_b'-W_c'),$$

where W_μ^1 are the components of the nonrelativistic analog of the Lubanski-Pauli vector

$$W_0=P_aJ_a;\ W_a=mJ_a-(\mathbf{P}\times\mathbf{G})_a$$

in the frame with $p_1=p_2=p_3$. We now formulate the fundamental theorem.

Theorem. An arbitrary Hermitian representation of the Lie algebra of the extended Galileo group (6)–(10) can be realized by means of the operators

$$\left.\begin{aligned}&P_0=p_0,\ P_a=p_a,\ M=m;\\ &J_a=-i\left(\mathbf{p}\times\frac{\partial}{\partial\mathbf{p}}\right)_a+\lambda_0\frac{\sqrt{3}p_a+p}{\sqrt{3}p+p_1+p_2+p_3};\\ &G_a=-ip_0\frac{\partial}{\partial p_a}+\frac{(\lambda\times\mathbf{p})_a}{p^2}-\frac{\varepsilon_{abc}(p_b-p_c)\lambda_0mp-\lambda\cdot\mathbf{p}}{2p^2(\sqrt{3}p+p_1+p_2+p_3)},\end{aligned}\right\} \tag{A.6}$$

where λ_μ are matrices which satisfy the commutation relations (A.2).

Proof. It can be shown by direct verification that the operators (A.6) satisfy the commutation relations (6)–(10), i.e., realize a representation of the Lie algebra of the extended Galileo group.

We show that by choosing all irreducible representations of the algebra (A.2) we obtain in accordance with formulas (A.6) representations of the algebra (6)–(10) corresponding to all possible values of the invariant operators (13). Substituting (A.6) in (13), we obtain

$$C_1=2mp_0-\mathbf{p}^2,\ C_2=m,\ C_3=m^2\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2. \qquad \text{(A.7)}$$

Using the isomorphism (A.3), (A.4), we rewrite the operator C_3 (A.7) in the form

$$\left.\begin{array}{ll} C_3=m^2(S_1^2+S_2^2+S_3^2), & \text{if}\quad c_2^2=m^2>0; \\ C_3=T_1^2+T_2^2, & \text{if}\quad c_2=0. \end{array}\right\} \qquad \text{(A.8)}$$

It can be seen from (A.8) that the invariant operator C_3 can be expressed in terms of the Casimir operators of the groups $O(3)$ and $D(2)$, the little groups of the Galileo group whose eigenvalues (together with c_1 and c_2) label all irreducible Hermitian representations of the algebra (6)–(10). The theorem is proved.

Thus, the Hermitian irreducible representations $D(c_1, c_2, c_3)$ of the algebra (6)–(10) can be divided into three classes corresponding to the following values of the invariant operators (A.6):

$$\begin{array}{l} \text{I.}\ -\infty<c_1<\infty,\ -\infty<c_2<0,\ 0<c_2<\infty,\ c_3=m^2 s(s+1); \\ \text{II.}\ -\infty<c_1<0,\ c_2=c_3=0; \\ \text{III.}\ -\infty<c_1<\infty,\ c_2=0,\ c_3=r^2>0. \end{array} \qquad \text{(A.9)}$$

Using the explicit form of the matrices S_a and T_α (see Ref. 35) and the isomorphism (A.3), (A.4), we can readily calculate explicit expressions for the matrices λ_μ in the basis $|c, \hat{p}, \lambda\rangle$:

$$\begin{array}{c} \lambda_0|c,\hat{p},\lambda\rangle=\lambda^\alpha|c,\hat{p},\lambda\rangle,\ \alpha=\text{I, II, III}; \\ \lambda_1|c,\hat{p},\lambda\rangle=\frac{1}{4\sqrt{3}}\left(a^\alpha_{\lambda,\lambda+1}|c,p,\lambda+1\rangle+a^\alpha_{\lambda,\lambda-1}|c,\hat{p},\lambda-1\rangle\right); \\ \lambda_2|c,\hat{p},\lambda\rangle=\frac{1}{4\sqrt{3}}\left(b^\alpha_{\lambda,\lambda+1}|c,\hat{p},\lambda+1\rangle+b^\alpha_{\lambda,\lambda-1}|c,\hat{p},\lambda-1\rangle\right); \\ \lambda_3|c,\hat{p},\lambda\rangle=-(\lambda_1+\lambda_2)|c,\hat{p},\lambda\rangle, \end{array} \qquad \text{(A.10)}$$

where the values of the index α depend on the c_a [this dependence is given in (A.8)]. At the same time

$$\begin{array}{c} \lambda^{\text{I}}=-s,\ -s+1,\ -s+2,\ \ldots,\ s; \\ a^{\text{I}}_{\lambda,\lambda\pm1}=[(1+\sqrt{3})\mp(1\pm\sqrt{3})]\sqrt{s(s+1)-\lambda^{\text{I}}(\lambda^{\text{I}}\pm1)}; \\ b^{\text{I}}_{\lambda,\lambda\pm1}=[(1-\sqrt{3})\mp(1\mp\sqrt{3})]\sqrt{s(s+1)-\lambda^{\text{I}}(\lambda^{\text{I}}\pm1)}; \\ \lambda^{\text{II}}=\tilde{\lambda},\ a^{\text{II}}_{\lambda,\lambda\pm1}=b^{\text{II}}_{\lambda,\lambda\pm1}=0; \\ \lambda^{\text{III}}=n+\varphi,\ a^{\text{III}}_{\lambda,\lambda\pm1}=r(1\pm\sqrt{3})(1\pm i);\ b^{\text{III}}_{\lambda,\lambda\pm1}=r(1\mp\sqrt{3})(1\pm i), \end{array} \qquad \text{(A.11)}$$

where $n=0, 1, 2, \ldots,\ 0\leqslant\varphi\leqslant1$, and $\tilde{\lambda}$ and s are arbitrary integral or half-integral numbers.

Formulas (A.1), (A.6), and (A.9)–(A.11) (for fixed c_a) completely determine the explicit form of the generators of the Galileo group for all classes of irreducible representations.

The class-I representations (which are usually associated with a nonrelativistic particle with spin s, mass m, and internal energy $\varepsilon_0=c_1/2m$) were obtained in a different realization in Ref. 4, in which class-II representations of the algebra (6)–(10) were also found; these can be associated with a nonrelativistic massless particle. Such representations have the additional invariant operator $C_4=J_aP_aP^{-1}$ and are one-dimensional with respect to the index λ. The class-III representations are infinite-dimensional with respect to the spin index. The representations of this class of the Lie algebra of the extended Galileo group are apparently obtained for the first time in the present paper.

A distinctive feature of the realization (A.6) is the identical and symmetric form of the generators P_μ, J_a, and G_a for all classes of the irreducible representations (whereas usually[4] the irreducible representations of different classes have entirely different realizations). In the case $m=0$ and especially $c_3=0$, the analytic expressions for the operators G_a simplify considerably (at the same time, $m=\lambda_a\equiv0$).

The connection between the representations (A.6) and the realizations obtained in Ref. 4 is given by the formulas

$$\begin{array}{c} U_\alpha G_a^\alpha U_\alpha^\dagger=G_a;\ U_\alpha J_a^\alpha U_\alpha^\dagger=J_a;\ U_\alpha P_\mu^\alpha U_\alpha^\dagger=P_\mu,\ \alpha=\text{I, II}; \\ U_{\text{I}}=\exp\left(i\frac{\boldsymbol{\lambda}\cdot\mathbf{p}}{2m\tilde{p}}\operatorname{arctg}\frac{\tilde{p}}{p_1+p_2+p_2}\right); \\ \tilde{p}=[(p_1-p_2)^2+(p_3-p_1)^2+(p_2-p_3)^2]^{1/2}; \\ U_{\text{II}}=\exp\left(2i\lambda_0\operatorname{arctg}\frac{p_2-p_1}{(\sqrt{3}+1)(p+p_3)+p_1+p_2}\right), \end{array}$$

where $P_\mu^{\text{I}}, J_a^{\text{I}}, G_a^{\text{I}}$ and $P_\mu^{\text{II}}, J_a^{\text{II}}, G_a^{\text{II}}$ are the generators of the Galileo group of the classes I and II in the realization found in Ref. 4, and P_μ, J_a, and G_a are given by formulas (A.6).

Note that formulas (A.2) and (A.6) also determine class-IV representations, corresponding to $c_2^2<0$. These representations are non-Hermitian, although they are generated by Hermitian representations of the algebra $O(1,2)$. However, the operators that form the direct sum of such representations,

$$\hat{P}_\mu=\begin{pmatrix} P_\mu & 0 \\ 0 & P_\mu \end{pmatrix};\ \hat{M}=\begin{pmatrix} M & 0 \\ 0 & -M \end{pmatrix}=i\begin{pmatrix} \eta & 0 \\ 0 & -\eta \end{pmatrix};$$
$$\hat{J}_a=\begin{pmatrix} J_a & 0 \\ 0 & J_a \end{pmatrix},\ \hat{G}_a=\begin{pmatrix} G_a & 0 \\ 0 & G_a \end{pmatrix},$$

where P_μ, J_a, and G_a are given by the relations (A.2) and (A.6) with $c_2^2<0$, are Hermitian in the indefinite metric

$$(\varphi_1, \varphi_2)=\int d^3p\,\varphi_1^\dagger(\mathbf{p})\,\sigma_1\varphi_2(\mathbf{p}),$$

where

$$\varphi=\begin{pmatrix} \Psi \\ \chi \end{pmatrix},\ \sigma_1=\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix};$$

Ψ and χ are elements in the spaces of the representations $D(c_1, c_2, c_3)$ and $D(c_1^*, c_2, c_3)$, and I and 0 are the unit matrix and the null matrix of the corresponding dimension.

APPENDIX 2: ON THE CONNECTION BETWEEN REPRESENTATIONS OF THE EXTENDED GALILEO GROUP AND THE GENERALIZED POINCARE GROUP P(1, 4)

The extended Galileo group G is a subgroup of the generalized Poincaré group $P(1, 4)$, the group of rotations and displacements in five-dimensional Minkowski space. This means, in particular, that every representation of the group $P(1, 4)$ determines a representation of the group G, which in the general case is reducible.

The 15 generators $J_{\mu\nu}$ $(\mu, \nu = 0, 1, 2, 3, 4, J_{\nu\mu} = -J_{\mu\nu})$, satisfying the following commutation relations form the Lie algebra of the group $P(1,4)$[31]:

$$[P_\mu, P_\nu]=0;\ [P_\mu, J_{\nu\lambda}]=\mathrm{i}\,(g_{\mu\nu}P_\lambda-g_{\mu\lambda}P_\nu);$$
$$[J_{\mu\nu}, J_{\lambda\sigma}]=\mathrm{i}\,(g_{\mu\lambda}J_{\nu\sigma}+g_{\nu\sigma}J_{\mu\nu}-g_{\mu\sigma}J_{\nu\lambda}-g_{\nu\lambda}J_{\mu\sigma}), \qquad \text{(A.12)}$$

where $g_{\mu\nu}$ is the metric tensor, and $g_{00} = -g_{kk} = 1, k = 1, 2, 3, 4$; $g_{\mu\nu} = 0, \mu \neq \nu$.

Going over in (A.12) to the new basis

$$\hat{P}_0=P_0-P_4,\ M=\frac{1}{2}(P_0+P_4);\ \hat{P}_a=P_a;\ K=J_{04};$$
$$J_a=\frac{1}{2}\varepsilon_{abc}J_{bc},\ G_a^+=\frac{1}{2}(J_{0a}+J_{4a}),\ G_a^-=J_{0a}-J_{4a}, \qquad \text{(A.13)}$$

we obtain the algebra [isomorphic to (A.12)]

$$\left.\begin{array}{l}[\hat{P}_0, \hat{P}_a]=[\hat{P}_0, M]=[\hat{P}_a, M]=[\hat{P}_a, \hat{P}_b]=0;\\ [\hat{P}_0, J_a]=[M, J_a]=[G_a^+, G_b^+]=[M, G_a^+]=0;\\ [\hat{P}_a, J_b]=\mathrm{i}\varepsilon_{abc}\hat{P}_c,\ [\hat{P}_a, G_b^+]=\mathrm{i}\delta_{ab}M;\\ [J_a, J_b]=\mathrm{i}\varepsilon_{abc}J_c;\ [\hat{P}_0, G_b^+]=\mathrm{i}\hat{P}_b;\end{array}\right\} \qquad \text{(A.14)}$$

$$[\hat{P}_0, G_a^-]=[G_a^-, G_b^-]=0;\ [G_a^-, M]=-\mathrm{i}P_a;$$
$$[G_a^-, J_b]=\mathrm{i}\varepsilon_{abc}G_c^-;\ [G_a^-, \hat{P}_b]=-\mathrm{i}\delta_{ab}P_0; \qquad \text{(A.15)}$$
$$[G_a^-, G_b^+]=-\mathrm{i}\,(\varepsilon_{abc}J_c+\delta_{ab}K);\ [\hat{P}_a, K]=[J_a, K]=0;$$
$$[\hat{P}_0, K]=-\mathrm{i}\hat{P}_0;\ [M, K]=\mathrm{i}M;\ [G_a^\pm, K]=\pm\,\mathrm{i}G_a^\pm.$$

The commutation relations (A.13) are identical to (6)–(10), i.e., determine the Lie algebra of the extended Galileo group.

It follows from the above that any equation invariant under the group $P(1,4)$ is also invariant under the extended Galileo group. For example, the five-dimensional Klein-Gordon equation

$$P_\mu P^\mu \Psi=0;\ P_\mu=p_\mu=-\mathrm{i}\frac{\partial}{\partial x_\mu},\ \mu=0, 1, 2, 3, 4,$$

can be reduced by means of the substitution (A.13) to a form manifestly invariant under the Galileo group:

$$(2M\hat{P}_0-\hat{P}_a\hat{P}_a)\,\Psi=0. \qquad \text{(A.16)}$$

Equation (A.16) can be interpreted as the Schrödinger equation for a particle with variable mass.

In Ref. 36, an arbitrary irreducible representation of the group $P(1,4)$ is reduced with respect to representations of the group G, i.e., a complete investigation is made of the irreducible representations of G that occur in a given representation of $P(1,4)$ and explicit expressions are found for the unitary operators which couple the canonical basis of the representations of the group $P(1,4)$ to the G basis in which the Casimir operators (13) are diagonal.

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