Stochastic mechanics

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A systematic exposition is given of the results of investigations which develop Blokhintsev's idea of stochastic properties of space on a small scale. On the basis of this idea and in the framework of Kershaw's stochastic model as developed by the present author for the relativistic case the basic equations of the stochastic mechanics of Nelson, Kershaw, and de la Peña-Auerbach are derived. In this stochastic theory, Sivashinsky's equations are obtained for the self-turbulent motion of a free particle, and the two-body problem is studied in detail. The obtained results are given a physical interpretation.

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INTRODUCTION

In recent years, interest has significantly increased in the investigation of stochastic processes and fields; this is due in the first place to the fact that it has been possible to establish an intimate connection between the theory of stochastic processes, quantum mechanics,[1-7] and Euclidean quantum field theory,[8,9] which is known under the general name of stochastic quantization of systems (or stochastic mechanics). Studies are made on the generalization of the ideas of the stochastic quantization of Nelson and Fényes for continuous systems[9,10] (i.e., for systems with infinitely many degrees of freedom), and also for particles with spin,[10,12] and relativistic mechanics[11-16] (see also the other approaches of Ref. 17).

From the mathematical point of view, the most interesting of the results obtained must be seen in the fact that the dynamical equations of stochastic mechanics are nonlinear partial differential equations which admit linearization and that the obtained linear equations are formally identical with the Schrödinger equation if the diffusion coefficient D is taken equal to $\hbar/(2m)$.

This commonality of the mathematical formalism of the two theories suggests the existence of a deep connection between the theory of stochastic processes and quantum mechanics. This problem requires further detailed investigation (in this connection, see Refs. 18–21). With regard to the problem of the connection between stochastic (Markov) processes and Euclidean quantum field theory, Nelson[8] has finally formulated Euclidean quantum field theory in the language of random processes, and we now effectively possess the solution to the problem of the unique correspondence between the Euclidean and pseudo-Euclidean Green's functions,[22] the investigation of which was already begun by Schwinger,[23] Nakano,[24] Fradkin,[25] Symanzik,[26] Taylor,[27] and Efimov.[28]

Besides this approach, other directions are also being developed in the investigation of stochastic processes and fields. Some of them take as their point of departure the hypothesis of stochastic properties of the electromagnetic vacuum (stochastic electrodynamics). The reviews of Ref. 29 present the basic elements of stochastic electrodynamics and its connection with quantum theory. Stochastic electrodynamics is constructed as classical electrodynamics with radiation damping of charged particles and interaction of them with a background electromagnetic field with spectral density

$$\rho(\omega) = (\hbar/2\pi^2 c^3)\,\omega^3.$$

Other directions are based in some manner or other on the concept of stochastic space.[30-36] In this case, it is assumed that the origin of the random behavior of particles (like Brownian motion) is the stochastic nature of physical space. In other words, the stochastic nature of the processes is regarded as the result of the effect (or influence) of space itself on the physical system. The present review is concerned with this direction in the theory of stochastic processes, and the main attention is devoted to a generalization of Nelson's stochastic mechanics to the relativistic case.

Stochastic space was considered in connection with elementary-particle physics for the first time in Refs. 30–33 (see the review of Ref. 34). Frederick[36] and Roy[37] considered mathematical spaces with stochastic metric and quantized domain, respectively. Reference 38 is devoted to the construction of the relativistic kinematics of massive and massless particles in a stochastic phase space. The electrodynamics of particles with spins 0, $\frac{1}{2}$, and 1 and weak interactions were considered in Refs. 39 in the framework of stochastic theory.

In Ref. 16, which develops Blokhintsev's idea on the influence on a physical system of space with a small stochastic component, the present author considered the motion of a particle whose coordinates in the stochastic space $R_4(\hat{x}_\mu)$ are determined by two parts:

$$\hat{x}_\mu = x_\mu + b_\mu,$$

where x_μ is the regular part of the coordinate, and b_μ is a random vector with distribution $\tau(b_\mu)$ satisfying the conditions

$$\int d\tau(b_\mu) = 1, \quad d\tau(b_\mu) \geqslant 0;$$

nonlinear equations of motion were obtained for a stochastic particle[1-7] in the nonrelativistic and relativistic cases.

The attraction of the approach based on the hypothesis of a stochastic space to the description of stochastic processes is that we have succeeded in generalizing

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stochastic mechanics to the relativistic case and defining rigorously, in the mathematical sense, relativistic integrals of Feynman type.[40] In addition, in this scheme, as one would expect, we encounter the self-turbulent phenomenon,[43] which is characteristic of a nonlinear system.

Hitherto, we have not particularized the form of the space $R_4(\hat{x}_\mu)$, which will evidently depend on the operation of arithmetization of events. When we are speaking of the nonrelativistic motion of particles, it is sufficient to assume that space is stochastic with respect to the component

$$\mathbf{x}\rightarrow\hat{\mathbf{x}}=\mathbf{x}+\mathbf{b},$$

where b is a universal stochastic variable subject to a probability distribution with $\tau(\mathbf{b})=\tau(|\mathbf{b}|^2)$, for example,

$$\tau(\mathbf{b})=(2\pi l^2)^{-3/2}\exp\left(-\frac{|\mathbf{b}|^2}{2l^2}\right), \tag{1}$$

where l is a universal length whose significance is elucidated in Sec. 6. This form of the distribution follows from the homogeneity and isotropy of space.

In the relativistic case, such an operation requires some explanation. In relativistic theory, the space $R_4(\hat{x}_\mu)$ must be a Minkowski space. The indefinite metric of this space leads to a number of specific problems, which are not encountered in Euclidean space.

These difficulties peculiar to the physical space are associated with the requirements of invariance and the condition of normalization of the probability of some value of the interval $b^2=b_0^2-\mathbf{b}^2$ in a space with indefinite metric (for more detail, see Ref. 35). Thus, the invariance requirement means that the distribution $\tau(b_\mu)$ of the vector b_μ must be a function of the interval $b^2=b_\mu b^\mu$, and the normalization condition gives the equation

$$\int d\tau\,(b_\mu b^\mu)=1.$$

The simultaneous fulfillment of these two requirements for $\tau(b_\mu)$ is actually impossible in Minkowski space. The above difficulties can be avoided by making the following assumptions[16]:

1. The stochastic nature of the space $R_4(\hat{x}_\mu)$ is manifested in the Euclidean domain of the variables $\hat{x}_E^\mu=x_E^\mu+b_E^\mu$.

2. Physical quantities are regarded as functions of the complex time $t+i\tau$ in the limit $\tau=0$ (τ is a stochastic variable), which ensures the hypothesis of a stochastic nature of the Euclidean space $E_4(\hat{\mathbf{x}},\tau)$ instead of the Minkowski space $R_4(\hat{\mathbf{x}},t)$. A justification of the importance of the method with $t\rightarrow t+i\tau$ in quantum field theory and quantum mechanics can be found in Refs. 41 and 42, respectively.

In the framework of these assumptions, the present author[16] has succeeded in constructing an equation of Smoluchowski type for the probability density $\rho(x_\mu,s)$ in the relativistic case; for $\rho(x_\mu,s)$ there is a representation of the form

$$\rho(x_\mu,\ s+\Delta s)=\int d^4y_E a(y_E^2,\ \Delta s)\,\rho(x_0+iy_4,\ \mathbf{x}-\mathbf{y},\ s), \tag{2}$$

where s is some invariant parameter (proper time), which is interpreted in Refs. 16 and 53. The integration in (2) is over a four-dimensional Euclidean space and $a(y_E^2,\Delta s)$ is some integrable function of the variable $y_E^2=y_4^2+y_1^2+y_2^2+y_3^2$. Relativistic invariance is ensured by the fact that the algebra of the Lorentz group and the algebra of the group of four-dimensional Euclidean rotations are identical in the complex domain (for more details, see Ref. 28).

Thus, if we begin the construction of a theory in a stochastic Euclidean space, a relativistic invariant description of the motion of particles in stochastic space can be realized.

Our exposition is as follows. In Sec. 1, the stochastic nature of space is treated from a random-walk point of view. In the next section, we study the nonrelativistic motion of a particle in a force field. We then investigate the motion of a relativistic particle in a four-dimensional stochastic space and obtain equations of motion for a particle that are formally equivalent to the Klein–Gordon equation. Section 4 studies the two-body problem in the nonrelativistic and relativistic cases. In Secs. 5 and 6, Sivashinsky's equation is derived in the framework of stochastic theory[43] for the self-turbulent motion of a free particle and the obtained results are given a physical interpretation. Some appendices are given at the end of the paper.

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1. STOCHASTIC SPACE AND RANDOM WALK

We consider the motion of a particle in the space $R_3(\hat{\mathbf{x}})$. We assume that the particle executes N displacements; then its position $\mathbf{R}_0$ after N displacements is given by the expression

$$\mathbf{R}_0=\sum_{j=1}^{N}\mathbf{r}_j^0.$$

If the motion is uniform and rectilinear, then $\mathbf{r}_j^0=\mathbf{v}^0t_j$. We shall now assume that in each displacement $\mathbf{r}_j^0$ the particle makes a random displacement because of the stochastic nature of space: $\mathbf{b}_j=\alpha_j\mathbf{b}$, where α_j is an arbitrary sequence of real numbers, and b is a universal stochastic variable subject to the probability distribution (1). Then after N displacements in the stochastic space, the position of the particle is determined by

$$\Phi=\mathbf{R}_0+\mathbf{B}=\sum_{j=1}^{N}(\mathbf{r}_j^0+\alpha_j\mathbf{b}). \tag{3}$$

What is the probability $W_N(\Phi)d\Phi$ that the particle's coordinates after N displacements lie in the interval between Φ and $\Phi+d\Phi$ where Φ is determined by Eq. (3)? In such a formulation, the problem can be solved by the method proposed by Markov. The application of Markov's method to the random-walk problem can be found in Ref. 44.

The probability $W_N(\mathbf{B})d\mathbf{B}$ that after N displacements the particle is in the interval $(\mathbf{B},\mathbf{B}+d\mathbf{B})$

$$\mathbf{B}=\mathbf{\Phi}-\mathbf{R}_0,\ \mathbf{B}=\sum_{j=1}^{N}\alpha_j\mathbf{b} \tag{4}$$

is given by

$$W_N(\mathbf{B})=\frac{1}{8\pi^3}\iiint d\boldsymbol{\rho}\exp(-i\boldsymbol{\rho}\mathbf{B})A_N(\boldsymbol{\rho}), \tag{5}$$

where

$$A_N(\boldsymbol{\rho})=\prod_{j=1}^{N}\int d\mathbf{b}\tau(\mathbf{b})\exp(i\boldsymbol{\rho}\alpha_j\mathbf{b})$$

$$=\prod_{j=1}^{N}(2\pi l^2)^{-3/2}\int d\mathbf{b}\exp(i\boldsymbol{\rho}\alpha_j\mathbf{b}-|\mathbf{b}|^2/2l^2).$$

To estimate $A_N(\rho)$, we calculate the value of the typical integral

$$I=(2\pi l^2)^{-3/2}\int d\mathbf{b}\exp(i\boldsymbol{\rho}\alpha_j\mathbf{b}-|\mathbf{b}|^2/2l^2)=\exp(-l^2\alpha_j^2\rho^2/2),$$

and we then have

$$A_N(\boldsymbol{\rho})=\exp\left(-l^2\sum_{j=1}^{N}\alpha_j^2\rho^2/2\right)$$

$$=\exp\left(-\frac{P_N}{2}|\boldsymbol{\rho}|^2\right),\quad P_N=l^2\sum_{j=1}^{N}\alpha_j^2. \tag{6}$$

Substituting (6) in (5), we obtain

$$W_N(\mathbf{B})=(2\pi P_N)^{-3/2}\exp(-|\mathbf{B}|^2/2P_N). \tag{7}$$

We now determine D, which is called the dispersion. This vanishes, $D=0$, when $l=0$, i.e., when the space is not stochastic. Setting

$$\overline{(B_x)^2}=\overline{(B_y)^2}=\overline{(B_z)^2}=\lim_{N\to\infty}P_N$$

$$=\lim_{N\to\infty}l^2\sum_{j=1}^{N}\alpha_j^2=\lim_{N\to\infty}Nl^2\frac{1}{N}\sum_{j=1}^{N}\alpha_j^2$$

$$=\lim_{N\to\infty}ntl^2\frac{1}{N}\sum_{j=1}^{N}\alpha_j^2=t\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}(\Delta B_x^j)^2$$

and denoting by D the quantity

$$\lim_{N\to\infty}\frac{1}{2N}\sum_{j=1}^{N}(\Delta B_x^j)^2$$

where $\Delta B_x^j=l\sqrt{n}\alpha_j$, in which n is the number of displacements per unit time, we obtain

$$2Dt=\overline{(B_x)^2}.$$

Thus,

$$D=\frac{1}{2}\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}(\Delta B_x^j)^2. \tag{8}$$

After the determination of D, the expression (7) can be written in the form

$$\rho(\mathbf{B})=\lim_{N\to\infty}W_N(\mathbf{B})=(4\pi Dt)^{-3/2}\exp(-|\mathbf{B}|^2/4Dt). \tag{9}$$

Since $\mathbf{B}=\mathbf{\Phi}-\mathbf{R}_0$,

$$\rho(\mathbf{\Phi})=(4\pi Dt)^{-3/2}\exp(-|\mathbf{\Phi}-\mathbf{R}_0|^2/4Dt). \tag{10}$$

As one would expect, our problem has been reduced to a random-walk problem. In accordance with (9), a particle which makes n displacements per unit time with each of these displacements $\mathbf{b}_j=\mathbf{b}\alpha_j$ subject to the probability distribution $\tau(|\mathbf{b}|^2)$ is situated after time t in the element of volume between $\mathbf{B}$ and $\mathbf{B}+d\mathbf{B}$ with probability $\rho(\mathbf{B})d\mathbf{B}$ or $\rho(\mathbf{\Phi})d\mathbf{\Phi}$ (when $\mathbf{R}_0\neq0$). It is known from the theory of Brownian motion that the distribution $\rho(\mathbf{r},t;\mathbf{r}_0,\mathbf{u}_0)$, which determines the probability of obtaining the position $\mathbf{r}$ of the Brownian particle at the time t if $\mathbf{r}=\mathbf{r}_0$ and $\mathbf{u}=\mathbf{u}_0$ at $t=0$, is determined by an expression just like (10):

$$\rho(\mathbf{r},\ t;\ \mathbf{r}_0,\ \mathbf{u}_0)=(4\pi D_0t)^{-3/2}\exp\left(-\frac{|\mathbf{r}-\mathbf{r}_0|^2}{4D_0t}\right)$$

for $t\gg\beta^{-1}$, where $\beta=6\pi a\eta/m$, $D_0=kT/m\beta=kT/6\pi a\eta$, a is the radius of the Brownian particle, η is the coefficient of viscosity of the surrounding medium (the fluid), k is Boltzmann's constant, and T is the absolute temperature. Thus, the introduction of the stochastic nature of space leads to a Brownian type of motion of a particle which in the absence of the stochastic properties would move along a definite preassigned trajectory.

We now turn to the derivation of a differential equation for the probability density $\rho(\mathbf{r},t)$. The derivation of the equation is the same as in random-walk theory. Therefore, we shall here give only the results. As usual, we make important assumptions:

1) The time interval Δt is chosen sufficiently large for the particle to make a large number of displacements in this interval but such that during this interval the mean-square increment of $\mathbf{r}$, i.e., $\langle|\Delta\mathbf{r}|^2\rangle$, remains small; then under these conditions the probability that the particle is displaced by $\Delta\mathbf{r}$ in the time Δt is given by

$$\Psi(\Delta\mathbf{r},\ \Delta t)=(4\pi D\Delta t)^{-3/2}\exp(-|\Delta\mathbf{r}|^2/4D\Delta t) \tag{11}$$

and does not depend on $\mathbf{r}$.

2) It is assumed that the displacement of the particle in space at a given time does not depend on the previous displacements; then

$$\rho(\mathbf{r},\ t+\Delta t)=\int d(\Delta\mathbf{r})\rho(\mathbf{r}-\Delta\mathbf{r},\ t)\Psi(\Delta\mathbf{r},\ \Delta t). \tag{12}$$

An equation of the type (12) is called a Smoluchowski equation. Since $\langle|\Delta\mathbf{r}|^2\rangle$ is small by hypothesis, we can expand $\rho(\mathbf{r}-\Delta\mathbf{r},t)$ in the integrand in a Taylor series and integrate the obtained expression term by term. Going to the limit $\Delta t\to0$, we obtain

$$\partial\rho/\partial t=D\Delta\rho. \tag{13}$$

2. EQUATIONS OF MOTION OF A NONRELATIVISTIC PARTICLE

In constructing the dynamics of stochastic particles, it is customary to use the mathematical concepts of left and right derivatives, by means of which the stochastic and systematic velocities of the particle are constructed.[1,5,7,9] Newton's law is used as the dynamical equation.

However, Kershaw's approach[3] (see also Ref. 13) was based on equations of Smoluchowski type for the probability density $\rho(\mathbf{x},t)$ and the mean particle velocity $\mathbf{v}(\mathbf{x},t)$.

Despite the different interpretation of the stochastic behavior of the system (the particle), the mathematical method of the description of the stochastic processes in our scheme will be the same as in the studies of Kershaw[3] and Lehr and Park[13] which are based on the theory of Bohm[56] and de Broglie[57] with hidden parameters.

In Sec. 1, we considered the random-walk problem from the point of view of a stochastic space. The greatest interest attaches to the study of the motion of a particle in a space whose properties are assumed to be stochastic. We now turn to this question.

If it is assumed that the displacement of the particle in space at a given time does not depend on the previous displacements, then the following relation holds for the probability density $\rho(\mathbf{x}, t)$:

$$\rho(\mathbf{x},\ t+\Delta t)=\int \rho(\mathbf{x}-\delta\mathbf{x}_+,\ t)\,\Psi_+(\mathbf{x}-\delta\mathbf{x}_+,\ t;\ \delta\mathbf{x}_+,\ \Delta t)\,d(\delta\mathbf{x}_+), \quad (14)$$

where $\Psi_+(\mathbf{x}-\delta\mathbf{x}_+, t; \delta\mathbf{x}_+, \Delta t)$ can be interpreted as the probability that a particle at the position $\mathbf{x}-\delta\mathbf{x}_+$ at the time t is displaced through $\delta\mathbf{x}_+$ during the time Δt and, therefore, reaches the point $\mathbf{x}$ at the time $t+\Delta t$. In the stochastic theory of Refs. 1, 13, and 16 with twice the number of transition probabilities, one uses $\Psi_-(\mathbf{x}+\delta\mathbf{x}_-, t; \delta\mathbf{x}_-, \Delta t)$, which is the probability that a particle has moved from the position $\mathbf{x}+\delta\mathbf{x}_-$ by $\delta\mathbf{x}_-$ in the time interval Δt prior to the time t and thus occupied the position at the point $\mathbf{x}$ at the earlier time $t-\Delta t$. Thus, the equation analogous to (14) takes in this case the form

$$\rho(\mathbf{x},\ t-\Delta t)=\int \rho(\mathbf{x}+\delta\mathbf{x}_-,\ t)\,\Psi_-(\mathbf{x}+\delta\mathbf{x}_-,\ t;\ \delta\mathbf{x}_-,\ \Delta t)\,d(\delta\mathbf{x}_-). \quad (15)$$

For $\Psi_\pm$, we can here choose an expression of the form

$$\Psi_\pm=(4\pi D_\pm\Delta t)^{-3/2}\exp\left[-\frac{(\delta\mathbf{x}_\pm-\mathbf{v}_\pm(\mathbf{x},t)\,\Delta t)^2}{4D_\pm\Delta t}\right],$$

where $\delta\mathbf{x}_\pm=\mathbf{v}_\pm(\mathbf{x},t)\Delta t+\Delta\mathbf{x}_\pm$ is the total displacement of the particle in the time Δt, and $D_\pm$ are certain constants of the type of a diffusion coefficient. Setting $D_+=D_-=D$, expanding ρ and $\Phi_\pm$ in Taylor series, integrating over $\delta\mathbf{x}_\pm$, and retaining terms of order Δt, we obtain Fokker–Planck equations for $\rho(\mathbf{x}, t)$.

$$\left.\begin{aligned}\partial\rho/\partial t&=-\nabla(\rho\mathbf{v}_+)+D\nabla^2\rho;\\ \partial\rho/\partial t&=-\nabla(\rho\mathbf{v}_-)-D\nabla^2\rho,\end{aligned}\right\} \quad (16)$$

or

$$\left.\begin{aligned}\partial\rho/\partial t&=-\nabla(\rho\mathbf{v});\ \mathbf{u}=D\nabla\ln\rho;\\ \mathbf{v}&=(\mathbf{v}_++\mathbf{v}_-)/2;\ \mathbf{u}=(\mathbf{v}_+-\mathbf{v}_-)/2.\end{aligned}\right\} \quad (17)$$

In stochastic theory, $\mathbf{v}$ and $\mathbf{u}$ are called the ordinary (or regular) and the stochastic velocity of the particle, respectively, and $\mathbf{v}_+$ and $\mathbf{v}_-$ the forward and backward velocity.

We now consider the motion of a particle in an external force field $\mathbf{F}=-\nabla U$. Following Kershaw,[3] we can derive equations of Smoluchowski type for the mean velocities $\mathbf{v}_+$ and $\mathbf{v}_-$ of particles in accordance with the formulas

$$\begin{aligned}\mathbf{v}_\pm(\mathbf{x},\ t\pm\Delta t)&=\frac{1}{N^\mp}\int\left[\mathbf{v}_\pm(\mathbf{x}\mp\delta\mathbf{x}_\pm,\ t)\pm\frac{\Delta t}{m}\mathbf{f}_\pm(\mathbf{x}\mp\delta\mathbf{x}_\pm,\ t)\right]\\ &\quad\times\rho(\mathbf{x}\mp\delta\mathbf{x}_\pm,\ t)\,\Psi_\pm(\mathbf{x}\mp\delta_\pm,\ t;\ \delta\mathbf{x}_\pm,\ \Delta t)\,d(\delta\mathbf{x}_\pm);\end{aligned} \quad (18)$$

$$\begin{aligned}\mathbf{v}_\pm(\mathbf{x},\ t\mp\Delta t)&=\frac{1}{N^\mp}\int\left[\mathbf{v}_\pm(\mathbf{x}\pm\delta\mathbf{x}_\mp,\ t)\mp\frac{\Delta t}{m}\mathbf{f}'_\pm(\mathbf{x}\pm\delta\mathbf{x}_\mp,\ t)\right]\\ &\quad\times\rho(\mathbf{x}\pm\delta\mathbf{x}_\mp,\ t)\,\Psi_\mp(\mathbf{x}\pm\delta\mathbf{x}_\mp,\ t;\ \delta\mathbf{x}_\mp,\ \Delta t)\,d(\delta\mathbf{x}_\mp),\end{aligned} \quad (19)$$

where $\mathbf{f}_\pm$ and $\mathbf{f}'_\pm$ are certain external forces, and

$$N^\pm=\int\rho(\mathbf{x}\mp\delta\mathbf{x}_\pm,\ t)\,\Psi_\pm(\mathbf{x}\mp\delta\mathbf{x}_\pm,\ t;\ \delta\mathbf{x}_\pm,\ \Delta t)\,d(\delta\mathbf{x}_\pm)$$

are normalization factors. The upper and lower signs correspond to $\mathbf{v}_+$ and $\mathbf{v}_-$, respectively.

Expanding $\mathbf{v}_\pm, \rho, \Psi_\pm, \mathbf{f}_\pm$, and $\mathbf{f}'_\pm$ in Taylor series, integrating over $\delta\mathbf{x}_\pm$, and going to the limit $\Delta t\to 0$, we obtain from (18) and (19) four possible equations:

$$\left.\begin{aligned}m\left(\frac{\partial\mathbf{v}_\pm}{\partial t}+(\mathbf{v}_\pm\nabla)\,\mathbf{v}_\pm\right)&=\mathbf{f}_\pm\pm mD\left(2\,\frac{(\nabla\rho\nabla)}{\rho}\,\mathbf{v}_\pm+\nabla^2\mathbf{v}_\pm\right);\\ m\left(\frac{\partial\mathbf{v}_\pm}{\partial t}+(\mathbf{v}_\mp\nabla)\,\mathbf{v}_\pm\right)&=\mathbf{f}'_\pm\mp mD\left(\frac{2}{\rho}\,(\nabla\rho\nabla)\,\mathbf{v}_\pm+\nabla^2\mathbf{v}_\pm\right).\end{aligned}\right\} \quad (20)$$

Going over to the variables $\mathbf{v}$ and $\mathbf{u}$ and adding and subtracting Eqs. (20), we obtain the following equations, which describe entirely different processes:

$$d_c\mathbf{v}-\lambda\,d_s\mathbf{u}=\frac{1}{m}\,\mathbf{F}^+_\lambda;\ \ d_c\mathbf{u}+\lambda\,d_s\mathbf{v}=\frac{1}{m}\,\mathbf{F}^-_\lambda \quad (21)$$

and

$$d_c\mathbf{v}-\lambda\,d_s\mathbf{v}=\frac{1}{m}\,\mathbf{F}_\lambda;\ \ d_c\mathbf{u}+\lambda\,d_s\mathbf{u}=\frac{1}{m}\,\mathbf{F}'_\lambda, \quad (22)$$

where

$$d_c=\partial/\partial t+(\mathbf{v}\nabla);\ d_s=(\mathbf{u}\nabla)+D\nabla^2;\ \lambda=\pm 1;$$

$$\mathbf{F}^+_1=\frac{1}{2}(\mathbf{f}_++\mathbf{f}_-);\ \mathbf{F}^+_{(-1)}=\frac{1}{2}(\mathbf{f}'_++\mathbf{f}'_-);\ \mathbf{F}^-_1=\frac{1}{2}(\mathbf{f}'_--\mathbf{f}'_-);$$

$$\mathbf{F}^-_{(-1)}=\frac{1}{2}(\mathbf{f}_+-\mathbf{f}_-);\ \mathbf{F}_1=\frac{1}{2}(\mathbf{f}_++\mathbf{f}'_-);\ \mathbf{F}_{(-1)}=\frac{1}{2}(\mathbf{f}'_++\mathbf{f}_-);$$

$$\mathbf{F}'_1=\frac{1}{2}(\mathbf{f}'_+-\mathbf{f}_-);\ \mathbf{F}'_{(-1)}=\frac{1}{2}(\mathbf{f}_+-\mathbf{f}'_-).$$

The left-hand sides of the obtained equations have a definite parity under the time-reversal transformation. Indeed, under $t\to -t$

$$\mathbf{v}\to -\mathbf{v};\ d_c\to -d_c;$$
$$\mathbf{u}\to\mathbf{u};\ d_s\to d_s,$$

and we readily conclude that the expression $d_c\mathbf{v}-\lambda d_s\mathbf{u}$ does not change, while $d_c\mathbf{u}+\lambda d_s\mathbf{v}$ ($\lambda=\pm 1$) changes sign under the transformation $t\to -t$. Therefore, the right-hand side of the corresponding equations, which is the force, must be chosen such that the individual equation as a whole remains invariant under $t\to -t$. This requirement is satisfied if we assume that

$$\mathbf{f}_+\to\mathbf{f}_-,\ \mathbf{f}'_+\to\mathbf{f}'_-\ \text{as}\ \ t\to -t.$$

Then $\mathbf{F}^+_\lambda$ does not change, while $\mathbf{F}^-_\lambda$ reverses its sign, and $\mathbf{F}_1\to\mathbf{F}_{(-1)}$ and $\mathbf{F}'_1\to -\mathbf{F}'_{(-1)}$ as $t\to -t$ and, therefore, the four equations (22) actually reduce to two equations. We also write down the equation

$$\partial\mathbf{u}/\partial t=-\nabla(\mathbf{v}\mathbf{u})-D\nabla(\nabla\mathbf{v}), \quad (23)$$

which follows from the continuity equation and the expression (17) for the velocity $\mathbf{u}$.

We see that on the basis of the hypothesis of a stochastic space and Smoluchowski's equations we have obtained in the framework of Kershaw's approach the same fundamental equations that were obtained by Nelson[1] and de la Peña-Auerbach and Cetto[4] (see also the review of Ref. 5) by different routes. Note that Kershaw could not obtain these equations, since he considered only one transition probability, for example $\Psi_+(\Delta\mathbf{x}, \Delta t)$.

Detailed investigations in Eqs. (21) and (23) are made in Refs. 1–5. In Appendix 1, we consider a question associated with the requirements imposed on the equations of stochastic mechanics. In particular, on the basis of some mathematical assumptions about stochastic processes, Nelson[1] obtained the first equation in (21) with $\lambda=1$ and (23) and showed that if one considers a charged particle and takes the forces

$$\mathbf{F}_1^+=e\mathbf{E}+\frac{e}{c}\mathbf{v}\times\mathbf{H}\ \text{and}\ \mathbf{F}_1^+=-\nabla U,$$

then these equations are equivalent to the Schrödinger equations

$$\frac{\partial\psi}{\partial t}=-\frac{i}{2m\hbar}\left(-i\hbar\nabla-\frac{e}{c}\mathbf{A}\right)^2\psi-\frac{ie}{\hbar}U\psi$$

and

$$i\hbar\frac{\partial\psi}{\partial t}=\left(-\frac{\hbar^2}{2m}\nabla^2+U\right)\psi$$

respectively. Here $D=\hbar/2m$, $\psi=\exp(R+iS)$,

$$R=\frac{1}{2}\ln\rho,\quad \operatorname{grad}S=\frac{m}{\hbar}\left(\mathbf{v}+\frac{e}{mc}\mathbf{A}\right),$$

and $\mathbf{A}$, the vector potential, is related to $\mathbf{E}$ and $\mathbf{H}$ by

$$\mathbf{H}=\operatorname{curl}\mathbf{A},\quad \mathbf{E}=-\frac{1}{c}\frac{\partial A}{\partial t}-\nabla\Phi.$$

Further, on the basis of the description of the mean local motion of the elements of continuous media (ensembles), which leads to two different kinds of motion during a short time interval Δt, de la Peña-Auerback and Cetto[4] obtained Eqs. (21) with $\lambda=1$. These equations can also be reduced to Schrödinger equations for the wave functions ψ and ψ^* if the Lorentz force is chosen as follows:

$$\mathbf{F}_1^+=-e\nabla\Phi-\frac{e}{c}\frac{\partial\mathbf{A}}{\partial t}+\frac{e}{c}\mathbf{v}\times\mathbf{H}$$

and

$$\mathbf{F}_1^-=\frac{e}{c}\mathbf{u}\times\mathbf{H}+\frac{e}{c}D\nabla\times\mathbf{H}.$$

They also showed[4] that the second equation of (21) with $\lambda=-1$ describes an Einstein process. Skagerstam[5] investigated Eqs. (21) and (23) from the point of view of the theory of stochastic processes defined in a classical configuration space.

Recently, Davidson[7] showed that in the Fényes–Nelson stochastic model there is an entire class of different dynamical schemes which lead to a Schrödinger equation as the solution of a Markov diffusion problem. As a result, the dispersion constant D in the Markov theory is an arbitrary positive parameter.

Note that the first equation of (21) with $\lambda=1$ agrees with the continuity equation (17) or (23), and thus it gives the correct classical equation of motion of a particle with velocity $\mathbf{u}+\mathbf{v}$ in the external electromagnetic field $\mathbf{F}=\mathbf{F}_1^++\mathbf{F}_1^-$ (in this connection, see Ref. 4).

The remaining equations (22) have the same form as the Navier–Stokes equations for "fluids" with velocities $\mathbf{v}$ and $\mathbf{u}$ if D is formally identified with the coefficient of viscosity. They are studied in Sec. 5.

Finally, we determine

$$\lim_{\Delta t\to 0}\frac{1}{2}\frac{\overline{(\Delta\mathbf{x})^2}}{\Delta t}.$$

By hypothesis,

$$\overline{(\Delta x_+)^2}=\overline{(\Delta x_-)^2}=\lim_{N\to\infty}l^2\sum_{j=1}^{N}\alpha_j^2=\Delta t\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}(\Delta x_j)^2,$$

where $\Delta x_j=l\sqrt{n}\alpha_j$, and n is the number of displacements per unit time. We also have

$$\overline{(\Delta x_+)^2}=2D_+\Delta t=\overline{(\Delta x_-)^2}=2D_-\Delta t,$$

and hence

$$\lim_{\Delta t\to 0}\frac{1}{2}\frac{\overline{(\Delta\mathbf{x})^2}}{\Delta t}=\frac{3}{2}\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}(\Delta x_j)^2=\frac{3}{2}\frac{\hbar}{m}.$$

We have arrived at Nelson's relation.

Thus, from the hypothesis of the stochastic nature of space we have obtained Nelson's mechanics, which is identical with quantum mechanics. The connection between the Schrödinger equation and the stochastic theory based on the hypothesis of a stochastic space can be demonstrated by the following simple example. Suppose that at $t=0$ the wave function has the form

$$\varphi(\mathbf{x},\ t=0)=N\exp(-\mathbf{x}^2/2a^2),\tag{24}$$

where N is a normalization constant, and a is a positive number; then for $t>0$ the wave function is determined by means of the Green's function of the Schrödinger equation:

$$\varphi(\mathbf{x},\ t)=\int K(\mathbf{x}-\mathbf{x}',\ t)\,\varphi(\mathbf{x}',\ 0)\,d\mathbf{x}',$$

where

$$K(\mathbf{x},\ t)=\frac{1}{i}(4\pi Dt)^{-3/2}\exp(i\mathbf{x}^2/4Dt).$$

After simple calculations, we obtain

$$|\varphi(\mathbf{x},\ t)|^2=N^2\left[1+\frac{\hbar^2t^2}{m^2a^4}\right]^{-3/2}\exp\left\{-\frac{\mathbf{x}^2}{a^2}\frac{1}{(1+\hbar^2t^2/m^2a^4)}\right\}.\tag{25}$$

We now consider the probability density corresponding to the function (24), i.e.,

$$\rho(\mathbf{x},\ t=0)=N^2\exp(-\mathbf{x}^2/a^2),\tag{26}$$

and we attempt to determine $\rho(\mathbf{x},t)$ by means of the stochastic mechanics considered above. Since

$$\mathbf{u}=D\nabla\ln\rho,$$

the velocity corresponding to the expression (26) is

$$\mathbf{u}^0(\mathbf{x},\ t=0)=-\frac{\hbar}{ma^2}\mathbf{x},$$

and the solution of an equation of the type (16) for $\mathbf{u}^0=\mathbf{v}_+$ $(\mathbf{v}_-=0)$ is determined by the relation

$$\begin{aligned}\rho(\mathbf{x},\ t)&=N^2/(4\pi Dt)^{3/2}\int d\mathbf{x}'\exp\left[-\frac{(\mathbf{x}-\mathbf{x}'-\mathbf{u}^0(\mathbf{x}')\,t)^2}{4Dt}-\frac{\mathbf{x}'^2}{a^2}\right]\\&=\frac{N^2}{(1+\hbar^2t^2/m^2a^4)^{3/2}}\exp\left[-\frac{\mathbf{x}^2}{a^2}\Big/\left(1+\frac{\hbar^2t^2}{m^2a^4}\right)\right],\end{aligned}\tag{27}$$

which is identical to the quantum-mechanical quantity (25). The kernel

$$G(\mathbf{x},\ t,\ \mathbf{x}',\ t'=0)=(4\pi Dt)^{-3/2}\exp\left[-\frac{(\mathbf{x}-\mathbf{x}'-\mathbf{u}^0(\mathbf{x}')\,t)^2}{4Dt}\right]$$

of this equation satisfies the Fokker–Planck equation

$$\frac{\partial G}{\partial t}=\beta\frac{\partial(\mathbf{x}G)}{\partial\mathbf{x}}+D\frac{\partial^2}{\partial\mathbf{x}^2}G,$$

where

$$\mathbf{x}\beta=-\mathbf{u}^0(\mathbf{x}),\quad \beta=\hbar/(ma^2).$$

3. RELATIVISTIC DESCRIPTION OF THE MOTION OF A PARTICLE IN STOCHASTIC SPACE

As was pointed out in the Introduction, in the relativistic case we shall formally consider the motion of a particle which executes a random walk due to the stochastic nature of the four-dimensional Euclidean space $E_4(\hat{x},\tau)$. Suppose the particle executes N displacements; then its coordinates are determined by the expression

$$B_\mu=\sum_{j=1}^{N}\beta_j b_\mu,$$

where β_j is a sequence of N numbers, and the vector $b_\mu=(b_4,\mathbf{b})$ is distributed with the probability

$$\tau(b_\mu)\,d^4b=\tau(b_\mu b^\mu)\,d^4b.$$

Here

$$b^2=b_\mu b^\mu=b_4^2+\mathbf{b}^2.$$

The probability $W_N(B_\mu)d^4B$ that B_μ lies in the interval between B_μ and $B_\mu+dB_\mu$ is given by

$$W_N(B_\mu)\,d^4B=d^4B\,(2\pi P_N)^{-2}\exp(-B^2/2P_N),\quad P_N=\sum_{j=1}^{N}l^2\beta_j^2.$$

Defining

$$2Ds=\overline{B_x^2}=\overline{B_y^2}=\overline{B_z^2}=\overline{B_4^2}$$

or

$$D=\frac{1}{2}\lim_{N\to\infty}\frac{1}{N}\,l^2\sum_{j=1}^{N}n\beta_j^2\ (n=N/s),$$

we obtain

$$\rho(B_\mu)=\lim_{N\to\infty}W_N(B_\mu)=(4\pi Ds)^{-2}\exp(-B^2/4Ds),$$

where s is some positive scalar whose meaning will be explained in Sec. 6. For the time being, it can be understood in the present case as the proper time of the particle. Further, as in the three-dimensional case, we define the "transition probability"

$$\Psi(\Delta y_E,\ \Delta s)=(4\pi D\Delta s)^{-2}\exp\left(-\frac{|\Delta y_E|^2}{4D\Delta s}\right),$$

and then the Smoluchowski equation takes the form

$$\rho_E(x_\mu^E,\ s+\Delta s)=\int d^4y_E\rho_E(x_\mu^E-y_\mu^E,\ s)\,\Psi(y_E,\ \Delta s),\tag{28}$$

where for convenience we have set $y_E=\Delta y_E$, and the diffusion equation becomes

$$\frac{\partial\rho_E}{\partial s}=D\Box_E\rho_E,\quad \Box_E=\frac{\partial^2}{\partial x_4^2}+\frac{\partial^2}{\partial \mathbf{x}^2}.\tag{29}$$

If $\Psi(y_E,\Delta s)$ depends on the point x_μ^E, for example,

$$\Psi(x_\mu^E;\ y_E,\ \Delta s)=(4\pi D\Delta s)^{-2}\exp\left[-\frac{(y_E-y_E^0(x_\mu^E))^2}{4D\Delta s}\right],$$

where $(y_E^0)^\mu=u^\mu\Delta s$ and $u_E^\mu=(u_4,\mathbf{u})$ is the four-dimensional Euclidean velocity of the particle, then instead of (29) we obtain an equation of the Fokker–Planck type in the Euclidean space:

$$\partial\rho_E/\partial s=-\partial_\mu(\rho u_E^\mu)+D\Box_E\rho_E,\ \partial_\mu=(\partial/\partial x_4,\ \nabla).$$

The question of the transition to pseudo-Euclidean space (Minkowski space) now arises. In his monograph on nonlocal interactions of quantized fields,[28] Efimov shows that when field theory is constructed from a Euclidean basis the transition to the pseudo-Euclidean domain can be realized by a displacement of the coordinates such that x_0 acquires a purely imaginary addition, while the coordinates $\mathbf{x}$ remain real. It is found that the procedure for displacing the coordinate x_0 is deeply connected to a fundamental problem such as causality, and it evidently also has a direct bearing on the relativistically invariant description of extended objects (for more details, see Ref. 28). Using this idea, we can write Eq. (28) in the form

$$\rho(x_\mu,\ s\pm\Delta s)=\int d^4y_E\Psi_\pm(y_E,\ \Delta s;\ \mathbf{x}\mp\mathbf{y},\ x_0+iy_4)\,\rho(\mathbf{x}\mp\mathbf{y},\ x_0+iy_4,\ s),$$

where the variables $x_\mu=(x_0,\mathbf{x})$ are pseudo-Euclidean and $\Psi_\pm$ can be chosen in the form

$$\Psi_\pm=(4\pi D_\pm\Delta s)^{-2}\exp\left[-\frac{(y_E-y_\pm^{\Pi})^2}{4D_\pm\Delta s}\right],$$

$$y_\pm^{\Pi}=(\pm iu_\pm^0\Delta s,\ \mathbf{u}_\pm\Delta s),\ D_-=D_+=D,$$

where $u_\pm^\mu$ are four-dimensional velocity vectors. From this we readily obtain the two Fokker–Planck equations

$$\partial\rho/\partial s=-\partial_\mu(\rho u_+^\mu)+D\Box\rho;\quad \partial\rho/\partial s=-\partial_\mu(\rho u_-^\mu)-D\Box\rho,\tag{30}$$

or, in terms of v^μ and u^μ,

$$\partial\rho/\partial s=-\partial_\mu(\rho v^\mu);\quad u^\mu=\frac{1}{2}(u_+^\mu-u_-^\mu)=-D\partial^\mu\ln\rho,\tag{31}$$

where

$$v^\mu=\frac{1}{2}(u_+^\mu+u_-^\mu);\quad \partial_\mu=\left(\frac{\partial}{\partial x_0},\ \nabla\right);\quad \Box=-\frac{\partial^2}{\partial x_0^2}+\frac{\partial^2}{\partial \mathbf{x}^2}.$$

We now attempt to obtain equations of motion of a particle, which in accordance with the correspondence principle must take the form (20) depending on the choice of the force field in the nonrelativistic limit. In the framework of our scheme, relativistic equations for $u_\pm^\mu$ are obtain from the integral equations

$$u_\pm^\mu(x_\nu,\ s+\varepsilon\Delta s)=\frac{1}{N^\pm}\int\Big[u_\pm^\mu(\mathbf{x}-\varepsilon\mathbf{y},\ x_0+iy_4,\ s)+\varepsilon\frac{\Delta s}{m}F_\pm^\mu(\mathbf{x}-\varepsilon\mathbf{y},\ x_0+iy_4)\Big]\times\rho(\mathbf{x}-\varepsilon\mathbf{y},\ x_0+iy_4,\ s)\,\Psi_\pm(\mathbf{x}-\varepsilon\mathbf{y},\ x_0+iy_4,\ s;\ y_E,\ \Delta s)\,d^4y_E;\tag{32}$$

$$u_\pm^\mu(x_\nu,\ s-\varepsilon\Delta s)=\frac{1}{N^\mp}\int\Big[u_\pm^\mu(\mathbf{x}+\varepsilon\mathbf{y},\ x_0+iy_4,\ s)-\frac{\varepsilon\Delta s}{m}F_\pm'^\mu(\mathbf{x}+\varepsilon\mathbf{y},\ x_0+iy_4)\Big]\rho(\mathbf{x}+\varepsilon\mathbf{y},\ x_0+iy_4,\ s)\times\Psi_\mp(\mathbf{x}+\varepsilon\mathbf{y},\ x_0+iy_4,\ s;\ y_E,\ \Delta s)\,d^4y_E,\tag{33}$$

where

$$N^\pm=\int d^4y_E\Psi_\pm(\mathbf{x}-\varepsilon\mathbf{y},\ x_0+iy_4,\ s;\ y_E,\ \Delta s)\,\rho(\mathbf{x}-\varepsilon\mathbf{y},\ x_0+iy_4,\ s),$$

$$\varepsilon=\begin{cases}1 & \text{for } u_+^\mu;\\ -1 & \text{for } u_-^\mu,\end{cases}$$

and $F_\pm^\mu$ and $F_\pm'^\mu$ are certain forces.

Expanding the expressions in (32) and (33) in Taylor series, integrating over d^4y_E, going to the limit $\Delta s\to 0$, and making some calculations, we obtain the equations

$$m\left(\frac{\partial u_\pm^\mu}{\partial s}+u_\pm^\nu\partial_\nu u_\pm^\mu\right)=F_\pm^\mu\pm mD\left(2\frac{u^\nu}{D}\partial_\nu u_\pm^\mu+\Box u_\pm^\mu\right);$$

$$m\left(\frac{\partial u_\pm^\mu}{\partial s}+u_\mp^\nu\partial_\nu u_\pm^\mu\right)=F_\pm'^\mu\mp mD\left(2\frac{u^\nu}{D}\partial_\nu u_\pm^\mu+\Box u_\pm^\mu\right).\tag{34}$$

From this we obtain relativistic equations for v^μ and u^μ:

$$D_cv^\mu-\lambda D_su^\mu=\frac{1}{m}\Phi_\lambda^{(+)\mu};\quad D_cu^\mu+\lambda D_sv^\mu=\frac{1}{m}\Phi_\lambda^{(-)\mu}\tag{35}$$

and

$$D_cv^\mu-\lambda D_sv^\mu=\frac{1}{m}\Phi_\lambda^\mu;\quad D_cu^\mu+\lambda D_su^\mu=\frac{1}{m}\Phi_\lambda'^\mu,\tag{36}$$

where $D_c=\partial/\partial s+v^\lambda\partial_\lambda; D_s=u^\lambda\partial_\lambda+D\Box, \lambda=\pm 1$. The functions $\Phi_\lambda^{(+)\mu}, \ldots, \Phi_\lambda'^{\mu}$ can be expressed in terms of $F_+^\mu, \ldots, F_-'^\mu$ in the same way as in the nonrelativistic case. $\Phi_\lambda^{(+)\mu}(\Phi_\lambda^{(-)\mu})$ does not (does) change sign, and

$$\Phi_1^\mu\to\Phi_{-1}^\mu \text{ and } \Phi_1'^\mu\to-\Phi_{-1}'^\mu \text{ under } s\to-s.$$

Equations (35) and (36) in conjunction with Eq. (31) are the covariant analog of (21), (22), and (17) in the relativistic case.

Note that the left-hand side of the first equation in (35) for $\lambda=1$ is identical to the expression for the acceleration obtained on the basis of some assumptions in the framework of the mathematical approach of Nelson (see Ref. 14 and also Ref. 13).

We can consider the special case when only the Lorentz force $\Phi_{\lambda=1}^{(+)\mu}$ is present; this is related to the electromagnetic field $F^{\mu\nu}$ by

$$\Phi_1^{(+)\mu}=\frac{e}{c}F^{\mu\nu}v_\nu=\frac{e}{c}(\partial^\mu A^\lambda-\partial^\lambda A^\mu)v_\lambda.$$

Here, A^μ is the electromagnetic potential, for which the Lorentz condition is satisfied:

$$\partial_\mu A^\mu=0.$$

In this case, it is usually assumed that the generalized momentum is a 4-gradient of a world scalar S:

$$\partial^\mu S=mv^\mu+\frac{e}{c}A^\mu,$$

and then the first equation in (35) with $\lambda=1$ and (31) are equivalent to the Klein–Gordon equation

$$\left(\partial^\mu-\frac{e}{c}iA^\mu\right)^2\varphi-\frac{m^2c^2}{\hbar^2}\varphi=0. \quad (37)$$

The proof is given in Refs. 1 and 13, and therefore we shall not give it here.

The equations of (35) with $\lambda=1$ were also investigated by Vigier[15] in the framework of the approach of Ref. 4. The external field in this case was chosen in the form

$$\Phi_{\lambda=1}^{(+)\mu}=\frac{e}{c}F^{\mu\nu}v_\nu;\quad \Phi_{\lambda=1}^{(-)\mu}=\frac{e}{c}F^{\mu\nu}u_\nu+\frac{e}{c}D\partial^\nu F^{\nu\mu}.$$

Such a choice of the field $\Phi_1^{(-)\mu}$ ensures consistency of the second equation in (35) with (31) for $\lambda=1$, as a result of which there is no overdetermination of the physical quantities ρ and v^μ (see Appendix 1).

The covariant analog of (22) is provided by Eqs. (36), which will be considered in Sec. 5.

4. THE TWO-BODY PROBLEM IN STOCHASTIC THEORY

We consider first two interacting nonrelativistic particles. We shall investigate the problem in the framework of a stochastic theory based on Smoluchowski equations for the probability density $\rho(\mathbf{x}_1,\mathbf{x}_2,t)$ of finding the first particle at the point $\mathbf{x}_1$ and the second at the point $\mathbf{x}_2$ at the time t with relative velocities $\mathbf{v}_1(\mathbf{x}_1,\mathbf{x}_2,t)$ and $\mathbf{v}_2(\mathbf{x}_1,\mathbf{x}_2,t)$.[45] It is assumed that the interaction potential $U(\mathbf{x}_1,\mathbf{x}_2)$ between two particles will depend only on the difference $\mathbf{x}_1-\mathbf{x}_2$ between their coordinates, i.e., $U(\mathbf{x}_1,\mathbf{x}_2)=U(\mathbf{x}_1-\mathbf{x}_2)$.

The problem we consider was first studied by Kershaw[3] for the probability density $\Psi_+(\Delta x,\Delta t)$, and he obtained equations that describe the motion of the center of mass of the two particles and their relative motion. As above, we shall investigate the problem of two bodies in the framework of two transition probabilities Ψ_+ and Ψ_-. Here, for the first particle

$$\Psi_\pm^1(\Delta\mathbf{x}_1^\pm,\ \Delta t)=(2\pi\tau_1\Delta t/m_1)^{-3/2}\exp\left[-\frac{m_1(\Delta\mathbf{x}_1^\pm)^2}{2\tau_1\Delta t}\right],$$

and for the second

$$\Psi_\pm^2(\Delta\mathbf{x}_2^\pm,\ \Delta t)=(2\pi\tau_2\Delta t/m_2)^{-3/2}\exp\left[-\frac{m_2(\Delta\mathbf{x}_2^\pm)^2}{2\tau_2\Delta t}\right],$$

where $\tau_1=2D_1m_1$ and $\tau_2=2D_2m_2$.

Without loss of generality, we take $\tau_1=\tau_2=\tau$. In terms of the variables

$$\mathbf{r}^+=\mathbf{x}_1^+-\mathbf{x}_2^+;\quad \mathbf{R}^+=(m_1\mathbf{x}_1^++m_2\mathbf{x}_2^+)/(m_1+m_2)$$

Ψ_+^1 and Ψ_-^2 take the form

$$\Psi_+(\Delta\mathbf{r}^+,\ \Delta t)=\frac{\partial(x_2^+)}{\partial(\mathbf{r}^+)}\int\Psi_+^1(\Delta\mathbf{x}_1^+,\ \Delta t)\,\Psi_+^2(\Delta\mathbf{x}_1^+-\Delta\mathbf{r}^+,\ \Delta t)\,d^3(\Delta x_1^+)$$
$$=(2\pi\tau\Delta t/\mu)^{-3/2}\exp\left[-\frac{\mu(\Delta\mathbf{r}^+)^2}{2\tau\Delta t}\right],$$

where $\partial(\mathbf{x}_2^+)/\partial(\mathbf{r}^+)$ is the Jacobian of the transition from $\mathbf{x}_2^+$ to $\mathbf{r}^+$, and $\mu=m_1m_2/(m_1+m_2)$. Similarly,

$$\Psi_+(\Delta\mathbf{R}^+,\ \Delta t)$$
$$=\frac{\partial(x_2^+)}{\partial(\mathbf{R}^+)}\int\Psi_+^1(\Delta\mathbf{x},\ \Delta t)\,\Psi_+^2\left(\Delta t,\ \frac{M}{m_2}\Delta\mathbf{R}^+-\frac{m_1}{m_2}\Delta\mathbf{x}\right)d^3(\Delta x)$$
$$=(2\pi\tau\Delta t/M)^{-3/2}\exp\left[-\frac{M(\Delta\mathbf{R}^+)^2}{2\tau\Delta t}\right],$$

where $M=m_1+m_2$ is the total mass of the particles. Suppose

$$\mathbf{V}^+(\mathbf{r},\ \mathbf{R},\ t)=(m_1\mathbf{v}_1^++m_2\mathbf{v}_2^+)/M;$$
$$\mathbf{C}^+(\mathbf{r},\ \mathbf{R},\ t)=\mathbf{v}_1^+-\mathbf{v}_2^+;$$

here

$$\mathbf{r}=\mathbf{x}_1^++\mathbf{x}_1^--\mathbf{x}_2^+-\mathbf{x}_2^-=\mathbf{x}_1-\mathbf{x}_2;$$
$$\mathbf{R}=[m_1(\mathbf{x}_1^++\mathbf{x}_1^-)+m_2(\mathbf{x}_2^++\mathbf{x}_2^-)]/M=(m_1\mathbf{x}_1+m_2\mathbf{x}_2)/M.$$

Then the total variations $\delta\mathbf{R}^+$ and $\delta\mathbf{r}^+$ are determined by the equations

$$\delta\mathbf{R}^+=\mathbf{V}^+\Delta t+\Delta\mathbf{R}^+ \text{ and } \delta\mathbf{r}^+=\mathbf{C}^+\Delta t+\Delta\mathbf{r}^+$$

respectively. By analogy with the earlier investigations,

$$\Psi_R^+(\delta\mathbf{R}^+,\ \Delta t,\ \mathbf{r},\ \mathbf{R},\ t)=(2\pi\tau\Delta t/M)^{-3/2}\exp\left[-\frac{M(\delta\mathbf{R}^+-\mathbf{V}^+\Delta t)^2}{2\tau\Delta t}\right]$$

and

$$\Psi_r^+(\delta\mathbf{r}^+,\ \Delta t,\ \mathbf{r},\ \mathbf{R},\ t)=(2\pi\tau\Delta t/\mu)^{-3/2}\exp\left[-\frac{\mu(\delta\mathbf{r}^+-\mathbf{C}^+\Delta t)^2}{2\tau\Delta t}\right]$$

and at the same time $\rho(\mathbf{r},\mathbf{R},t)=\rho(\mathbf{x}_1,\mathbf{x}_2,t)\partial(\mathbf{x}_1,\mathbf{x}_2)/\partial(\mathbf{r},\mathbf{R})$ satisfies the equation

$$\rho(\mathbf{r},\ \mathbf{R},\ t+\Delta t)=\int\rho(\mathbf{r}-\delta\mathbf{r}^+,\ \mathbf{R}-\delta\mathbf{R}^+,\ t)$$
$$\times\Psi_r^+(\delta\mathbf{r}^+,\ \Delta t,\ \mathbf{r}-\delta\mathbf{r}^+,\ \mathbf{R}-\delta\mathbf{R}^+,\ t)$$
$$\times\Psi_R^+(\delta\mathbf{R}^+,\ \Delta t,\ \mathbf{r}-\delta\mathbf{r}^+,\ \mathbf{R}-\delta\mathbf{R}^+,\ t)\,d^3(\delta r^+)\,d^3(\delta R^+)$$
$$=\rho-\Delta t\,[\nabla_r(\rho\mathbf{C}^+)+\nabla_R(\rho\mathbf{V}^+)-D_\mu\nabla_r^2\rho-D_M\nabla_R^2\rho],$$

whence

$$\frac{\partial\rho}{\partial t}=-\nabla_r(\rho\mathbf{C}^+)-\nabla_R(\rho\mathbf{V}^+)+D_\mu\nabla_r^2\rho+D_M\nabla_R^2\rho, \quad (38)$$

where $D_\mu=\tau/(2\mu), D_M=\tau/(2M)$.

The corresponding equations for ρ obtained on the basis of the concept of the transition probabilities Ψ_-^1

and Ψ^2_- take the form

$$\rho(\mathbf{r}, \mathbf{R}, t-\Delta t) = \int \rho(\mathbf{r}+\delta\mathbf{r}^-, \mathbf{R}+\delta\mathbf{R}^-, t)\, \Psi_r^- \Psi_R^- d^3(\delta r^-)\, d^3(\delta R^-) \quad (39)$$

or

$$\partial\rho/\partial t = -\nabla_r(\rho\mathbf{C}^-) - \nabla_R(\rho\mathbf{V}^-) - D_\mu \nabla_r^2\rho - D_M \nabla_R^2\rho,$$

where

$$\mathbf{C}^- = \mathbf{v}_1^- - \mathbf{v}_2^-;\quad \mathbf{V}^- = (m_1\mathbf{v}_1^- + m_2\mathbf{v}_2^-)/M.$$

From Eqs. (38) and (39), we obtain

$$\left.\begin{aligned} &\partial\rho/\partial t = -\nabla_r(\rho\mathbf{V}_r) - \nabla_R(\rho\mathbf{V}_c);\\ &\mathbf{u}_c = D_M \nabla_R \ln\rho\ \ \mathbf{u}_r = D_\mu \nabla_r \ln\rho. \end{aligned}\right\} \quad (40)$$

Here

$$\mathbf{V}_c = \frac{1}{2}(\mathbf{V}^+ + \mathbf{V}^-) = (m_1\mathbf{v}_1 + m_2\mathbf{v}_2)/M;$$
$$\mathbf{V}_r = \frac{1}{2}(\mathbf{C}^+ + \mathbf{C}^-) = \mathbf{v}_1 - \mathbf{v}_2;$$
$$\mathbf{u}_c = \frac{1}{2}(\mathbf{V}^+ - \mathbf{V}^-) = (m_1\mathbf{u}_1 + m_2\mathbf{u}_2)/M;$$
$$\mathbf{u}_r = \frac{1}{2}(\mathbf{C}^+ - \mathbf{C}^-) = \mathbf{u}_1 - \mathbf{u}_2.$$

We recall that the potential $U(r)$ acts only on the velocity $\mathbf{C}$; then the equations for $\mathbf{C}^\pm$ and $\mathbf{V}^\pm$ have the form

$$\mathbf{V}^\pm(\mathbf{r}, \mathbf{R}, t\pm\Delta t) = \frac{1}{N^\pm}\int \mathbf{V}^\pm(\mathbf{r}\mp\delta\mathbf{r}^\pm, \mathbf{R}\mp\delta\mathbf{R}^\pm, t) \times \rho(\mathbf{r}\mp\delta\mathbf{r}^\pm, \mathbf{R}\mp\delta\mathbf{R}^\pm, t)\, \Psi_r^\mp \Psi_R^\mp d^3(\delta r^\pm)\, d^3(\delta R^\pm);$$

$$\mathbf{C}^\pm(\mathbf{r}, \mathbf{R}, t\pm\Delta t) = \frac{1}{N^\pm}\int \Big[\mathbf{C}^\pm(\mathbf{r}\mp\delta\mathbf{r}^\pm, \mathbf{R}\mp\delta\mathbf{R}^\pm, t) \mp \Delta t \frac{1}{\mu}\nabla_r U(\mathbf{r}\mp\delta\mathbf{r}^\pm)\Big]\rho(\mathbf{r}\mp\delta\mathbf{r}^\pm, \mathbf{R}\mp\delta\mathbf{R}^\pm, t) \times \Psi_r^\mp \Psi_R^\mp d^3(\delta r^\pm)\, d^3(\delta R^\pm),$$

where

$$N^\pm = \int d^3(\delta R^\pm)\, d^3(\delta r^\pm)\, \rho(\mathbf{r}\mp\delta\mathbf{r}^\pm, \mathbf{R}\mp\delta\mathbf{R}^\pm, t)\, \Psi_r^\mp \Psi_R^\mp.$$

After some calculations, we obtain the equations

$$d_s'\mathbf{V}_c - d_u'\mathbf{u}_c = 0;\quad d_s'\mathbf{V}_r - d_u'\mathbf{u}_r = -\nabla_r U/\mu, \quad (41)$$

where

$$d_s' = \frac{\partial}{\partial t} + (\mathbf{V}_c\nabla_R) + (\mathbf{V}_r\nabla_r);\quad d_u' = (\mathbf{u}_c\nabla_R) + (\mathbf{u}_r\nabla_r) + D_\mu\nabla_r^2 + D_M\nabla_R^2.$$

Thus, the operators d_s' and d_u' decompose into a sum of two independent parts. Accordingly, we can seek $\rho(\mathbf{r},\mathbf{R},t)$ in the form of the product

$$\rho(\mathbf{r}, \mathbf{R}, t) = \rho_r(\mathbf{r}, t)\, \rho_R(\mathbf{R}, t),$$

and the variables $\mathbf{V}_{c,r}$ and $\mathbf{u}_{c,r}$ can be set equal to

$$\mathbf{V}_r(\mathbf{r}, \mathbf{R}, t) = \mathbf{V}_r(\mathbf{r}, t);\quad \mathbf{u}_r(\mathbf{r}, \mathbf{R}, t) = \mathbf{u}_r(\mathbf{r}, t);$$
$$\mathbf{V}_c(\mathbf{r}, \mathbf{R}, t) = \mathbf{V}_c(\mathbf{R}, t);\quad \mathbf{u}_c(\mathbf{r}, \mathbf{R}, t) = \mathbf{u}_c(\mathbf{R}, t),$$

where ρ_R, $\mathbf{V}_c$ and $\mathbf{u}_c$ describe the motion of the center of mass (as the free motion of a particle with mass m_1+m_2) and ρ_r, $\mathbf{V}_r$, and $\mathbf{u}_r$ describe the relative motion of the particles [as the motion of a particle of mass μ in the centrally symmetric field $U=U(r)$]; then Eqs. (40) and (41) take the form

$$\partial\rho_R/\partial t = -\mathrm{div}(\rho_R\mathbf{V}_c);\quad d_R'\mathbf{V}_c - d_R''\mathbf{u}_c = 0 \quad (42)$$

and

$$\partial\rho_r/\partial t = -\mathrm{div}(\rho_r\mathbf{V}_r);\quad d_r'\mathbf{V}_r - d_r''\mathbf{u}_r = -\nabla_r U(r)/\mu, \quad (43)$$

where

$$d_R' = \frac{\partial}{\partial t} + (\mathbf{V}_c\nabla_R);\quad d_R'' = (\mathbf{u}_c\nabla_R) + D_M\nabla_R^2;\quad d_r' = \frac{\partial}{\partial t} + (\mathbf{V}_r\nabla_r);$$
$$d_r'' = (\mathbf{u}_r\nabla_r) + D_\mu\nabla_r^2.$$

Following Nelson, Eqs. (42) and (43) can be linearized by the substitutions

$$\varkappa_r = \frac{1}{2}\ln\rho_r;\quad \varkappa_R = \frac{1}{2}\ln\rho_R;\quad \mathbf{u}_r = 2D_\mu\nabla_r\varkappa_r;$$
$$\mathbf{u}_c = 2D_M\nabla_R\varkappa_R;\quad \mathbf{V}_r = 2D_\mu\nabla_r\delta_r;$$
$$\mathbf{V}_c = 2D_M\nabla_R\delta_R;$$
$$\varphi_r = \varphi_r(\mathbf{r}, t) = \exp(\varkappa_r + i\delta_r);\quad \varphi_R = \varphi_R(\mathbf{R}, t) = \exp(\varkappa_R + i\delta_R).$$

We then obtain the following two equations for φ_r and φ_R:

$$\frac{\partial\varphi_r}{\partial t} = iD_\mu\nabla_r^2\varphi_r - i\frac{1}{2\mu D_\mu}U(r)\,\varphi_r \quad (44)$$

and

$$\partial\varphi_R/\partial t = iD_M\nabla_R^2\varphi_R$$

respectively. The last two equations are formally identical with Schrödinger equations for φ_r and φ_R if we set $\tau = \hbar$, i.e.,

$$D_M = \hbar/2M;\quad D_\mu = \hbar/2\mu.$$

We now turn to the problem of two relativistic particles. In this paper, we consider the stochastic behavior of two identical and correlated relativistic scalar particles, since this problem has physical interest. Our investigation[46] is based on equations of Smoluchowski type for the probability density $\rho(x_1^\nu, x_2^\mu, s_1, s_2)$ for finding the first particle at the point x_1^ν and the second at the point x_2^μ at the "times" s_1 and s_2 and for their relative velocities $v_1^\nu(x_1^\lambda, x_2^\mu, s_1, s_2)$ and $v_2^\mu(x_1^\nu, x_2^\nu, s_1, s_2)$, respectively.

For the direct generalization of the results obtained above for the single-particle model to the two-particle case, it was found to be mathematically more convenient to introduce the eight-dimensional configuration space considered in Ref. 47.

In this space, the positions of two particles and their relative velocities are determined by the eight-component vectors X^i and v^i $(i=1,\dots,8)$, respectively, where

$$\{X^i\}_{i=1,\dots,8} \equiv \{x_1^\mu, x_2^\nu\}_{\mu,\nu=0,\dots,3};$$
$$\{v^i(X^j, s_1, s_2)\} \equiv \{v_1^\mu, v_2^\nu\}.$$

Here, x_1^μ and x_2^ν are the coordinate vectors of each particle. The metric tensor g_{ij} in this space is defined as in Ref. 47:

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

and

$$X^2 = X_i X^i = g_{ij}X^i X^j = (x_1)^2 + (x_2)^2.$$

If $x_1^\mu(s_1)$ and $x_2^\nu(s_2)$ are the trajectories of each particle, then in the eight-dimensional space their common trajectory will be $X^j(s_1, s_2)$.

As in the three- and four-dimensional cases, we introduce the two-particle analog $\Psi(X^j, s_1, s_2, \Delta s_1, \Delta s_2)$ of the single-particle transition probabilities $\Psi(\mathbf{x}, t,$

Δt) and $\Psi(x^\mu, s, \Delta s)$ used above. If the particles are not correlated, $\Psi(X^j, s_1, s_2, \Delta s_1, \Delta s_2)$ can be factorized:

$$\Psi(X^j, s_1, s_2, \Delta s_1, \Delta s_2) = \Psi_1(x_1^\nu, s_1, \Delta s_1)\, \Psi_2(x_2^\mu, s_2, \Delta s_2).$$

Without loss of generality, we choose the gauge $\Delta s_1 = \Delta s_2 = \Delta s$; then the equation of Smoluchowski type for $\rho(X^j, s_1, s_2)$ takes the form

$$\begin{aligned} &\rho(X^j, s_1 \pm \Delta s, s_2 \pm \Delta s) \\ &= \int d^8Y_E \rho(X^I \mp Y^I, X^1 + iY_E^1, X^5 + iY_E^5, s_1, s_2) \\ &\times \Psi^\pm(X^I \mp Y^I, X^1 + iY_E^1, X^5 + iY_E^5, s_1, s_2, \Delta s, Y_E^i), \end{aligned} \tag{45}$$

where X^I and Y^I ($I = 2, 3, 4, 6, 7, 8$) are the spatial parts of the vectors X^i and Y^i, respectively.

Taking into account the explicit form of $\Psi^\pm$,

$$\left.\begin{aligned} &\Psi^\pm = (4\pi D_\pm \Delta s)^{-4} \exp\left[-\frac{(Y_E^i - Y_\pm^i)^2}{4D_\pm \Delta s}\right]; \\ &Y_\pm^i = (\pm i v_\pm^1 \Delta s, \pm i v_\pm^5 \Delta s, v_\pm^I \Delta s), \end{aligned}\right\} \tag{46}$$

we obtain from (45) the differential equations

$$\left.\begin{aligned} &\partial\rho/\partial s_1 + \partial\rho/\partial s_2 + \partial_i(\rho v_+^i) - D_+\Box\rho = 0; \\ &\partial\rho/\partial s_1 + \partial\rho/\partial s_2 + \partial_i(\rho v_-^i) + D_-\Box\rho = 0; \\ &\partial_i = \partial/\partial X^i, \quad -\partial_i\partial^i = \Box = \Box_1 + \Box_2. \end{aligned}\right\} \tag{47}$$

Here, we have set $D_- = D_+ = D$, where D is the dispersion, and v_+^i and v_-^i are the forward and backward velocities. Going over to the variables

$$v^i = \frac{1}{2}(v_+^i + v_-^i); \quad u^i = \frac{1}{2}(v_+^i - v_-^i)$$

and adding and subtracting Eqs. (47), we obtain

$$\frac{\partial\rho}{\partial s_1} + \frac{\partial\rho}{\partial s_2} + \partial_i(\rho v^i) = 0; \quad u^i = -D\partial^i \ln \rho, \tag{48}$$

where v^i is the ordinary (regular) and u^i the stochastic velocity of the two-particle system.

In our scheme (see Sec. 6) conservation of mass (of the probability density multiplied by the volume) means that there is no loss of mass through any hypersurfaces characterized by the vectors v_1^μ and v_2^ν; in this connection, we adopt the physical hypothesis that the total number of particles (i.e., the pair in the real space-time) is conserved. Then we can write

$$\begin{aligned} &\frac{\partial\rho}{\partial s_1} + \frac{\partial\rho}{\partial s_2} = (\mathrm{d}\rho \cdot \mathbf{v}_1) + (\mathrm{d}\rho \cdot \mathbf{v}_2) \\ &= \frac{\partial\rho}{\partial x_1^0} v_1^0 + \frac{\mathbf{v}_1}{\sqrt{1-\beta_1^2}} \frac{\partial\rho}{\partial \mathbf{x}_1} + \frac{\partial\rho}{\partial x_2^0} v_2^0 \\ &+ \frac{\mathbf{v}_2}{\sqrt{1-\beta_2^2}} \frac{\partial\rho}{\partial \mathbf{x}_2} = 0 \quad (\beta_i^2 = v_i^2/c^2, \ i = 1, 2) \end{aligned}$$

and our continuity equation in the configuration space becomes

$$\partial_i(\rho v^i) = 0, \tag{49}$$

or, in terms of v^i and u^i,

$$-u^i v_i + D\partial^i v_i = 0. \tag{50}$$

In the case of a two-particle system, following Kershaw,[3] we can derive equations of the type (45) for the mean velocities $v_\pm^i(X^j, s_1, s_2)$ in some external field $F_\pm^i(X^j, s_1, s_2)$:

$$\begin{aligned} &v_\pm^i(X^j, s_1 + \varepsilon\Delta s, s_2 + \varepsilon\Delta s) \\ &= \frac{1}{N^\pm} \int \Big[v_\pm^i(X^I - \varepsilon Y^I, X^1 + iY_E^1, X^5 + iY_E^5, s_1, s_2) \\ &+ \frac{\varepsilon\Delta s}{M} F_\pm^i(X^I - \varepsilon Y^I, X^1 + iY_E^1, X^5 + iY_E^5, s_1, s_2) \Big] \\ &\times \Psi^\pm(X^I - \varepsilon Y^I, X^1 + iY_E^1, X^5 + iY_E^5, s_1, s_2, \Delta s, Y_E^i) \\ &\times \rho(X^I - \varepsilon Y^I, X^1 + iY_E^1, X^5 + iY_E^5, s_1, s_2)\, d^8Y_E, \end{aligned} \tag{51}$$

where

$$\begin{aligned} N^\pm = \int d^8Y_E \Psi^\pm(X^I - \varepsilon Y^I, X^1 + iY_E^1, \ldots, Y_E^i) \\ \times \rho(X^I - \varepsilon Y^I, \ldots, s_2) \end{aligned}$$

are normalization factors; M is some effective mass of our two-particle system; and

$$\varepsilon = \begin{cases} 1 & \text{for } v_+^i; \\ -1 & \text{for } v_-^i. \end{cases}$$

In our case, Eqs. (51) lead to the differential equations

$$\frac{\partial u_\pm^i}{\partial s_1} + \frac{\partial u_\pm^i}{\partial s_2} + u_\pm^j \partial_j u_\pm^i = F_\pm^i/M \pm D\left(\frac{2}{D} u^j \partial_j u_\pm^i + \Box u_\pm^i\right). \tag{52}$$

Adding Eqs. (52), we obtain

$$\left.\begin{aligned} &D_c v^i - D_s u^i = \frac{1}{2M}(F_+^i + F_-^i) = F^i/M, \\ &D_c = \partial/\partial s_1 + \partial/\partial s_2 + v^i\partial_i, \\ &D_s = u^i\partial_i + D\Box. \end{aligned}\right\} \tag{53}$$

Equation (53) in conjunction with the continuity equation (49) is the covariant analog of the single-particle case for a two-particle system.

Note that the left-hand side of Eq. (53) is identical to the expression for the "velocity" obtained by Cufaro Petroni and Vigier[47] on the basis of some assumptions in the framework of the mathematical approach of Nelson[1] and Guerra and Ruggiero.[14]

The coupled pair of nonlinear differential equations (50) and (53) can be linearized if we set

$$v^i = \frac{1}{m} \partial^i \Phi, \tag{54}$$

as in the previous papers of Refs. 1, 16, and 47, where $\Phi(X^i, s_1, s_2)$ is the phase function determined by the equation

$$\Phi(X^i, s_1, s_2) = \frac{mc^2}{2}(s_1 + s_2) + S(X^i). \tag{55}$$

Using the expressions (48), (54), and (55), on the basis of Eqs. (49) and (53) we obtain the equation of Hamilton–Jacobi type[47]

$$(\partial_i\partial^i - \partial_i S \partial^i S/\hbar - 2m^2c^2/\hbar^2) R = 0 \tag{56}$$

for the two-particle system in the case when there is no external force: $F^i \equiv 0$. Here, we have set

$$R = \rho^{1/2}, \quad D = \hbar/2m.$$

From Eq. (56) there follows a continuity equation of the form

$$2\partial_i R \partial^i S + R \partial_i \partial^i S = 0.$$

Finally, we have a formal equation for $\psi = R\exp(iS/\hbar)$:

$$(\Box - 2m^2c^2/\hbar^2)\psi = 0. \tag{57}$$

In the nonrelativistic limit, Eq. (57) leads to an ordinary two-particle Schrödinger equation for $\psi(\mathbf{x}_1, \mathbf{x}_2, t) = R(\mathbf{x}_1, \mathbf{x}_2, t)\exp(iS/\hbar)$, which decomposes into two equations:

$$\frac{\partial P}{\partial t} + \nabla_1\left(P\frac{\nabla_1 S}{m}\right) + \nabla_2\left(P\frac{\nabla_2 S}{m}\right) = 0,$$

where $P = R^2 = \psi^*\psi$, and

$$\frac{\partial S}{\partial t} + \frac{(\nabla_1 S)^2}{m} + \frac{(\nabla_2 S)^2}{m} + Q = 0.$$

Here,

$$Q=-\frac{\hbar^2}{2m}(\nabla_1^2R/R+\nabla_2^2R/R)$$

is some potential, and it is usually called the nonlocal quantum potential of the two bodies (see, for example, Ref. 47).

We note in conclusion that the physical consequences of the results obtained above are discussed in Ref. 47.

5. DERIVATION OF SIVASHINSKY'S EQUATION FOR THE SELF-TURBULENT MOTION OF A FREE PARTICLE IN STOCHASTIC MECHANICS

Sivashinsky[48] noted a formal analogy between the equation of a flame front and the Hamilton–Jacobi equation for the motion of a free particle. He showed that if in the equation for the flame front one introduces terms with higher derivatives describing the structural perturbations of the front, the front is unstable with respect to long-wavelength perturbations. As a result, the original deterministic equation can generate solutions of stochastic type. An attempt was made to interpret the equation with higher derivatives as a Hamilton–Jacobi equation describing the motion of a "quantum" particle.

However, in the framework of Sivashinsky's approach the choice of the potential (the self-interaction potential of the particle) which " generates the turbulence" in the Hamilton–Jacobi equation is not unique and does not have a clear physical justification.

This section of the review is devoted to the derivation of a Sivashinsky equation for the self-turbulent motion of a free particle in the framework of stochastic theory on the basis of the hypothesis of a stochastic space.[43] We are therefore attempting to justify the occurrence of the potential which generates the turbulence in the equation of motion of the particle.

We consider first a scalar particle in the stochastic space $R_3(\hat{\mathbf{x}})$ with coordinates

$$\hat{\mathbf{x}}=\mathbf{x}+\mathbf{b}.$$

Since in our model the real points of the space $R_3(\hat{\mathbf{x}})$ are stochastic, these points cannot be used as the basis for a coordinate system, and for the same reason derivatives with respect to them cannot be formulated. The stochastic nature of space is manifested only in the microscopic world. Therefore, one can take a macroscale of nonstochastic space mathematically continued from a stochastic microregion. Such a mathematical construction ensures a nonstochastic space to which the stochastic physical space can be referred (see Ref. 36). In our case, this mathematical construction reduces to averaging with respect to the measure $\tau(\mathbf{b})$ at each point $\hat{\mathbf{x}}=\mathbf{x}+\mathbf{b}$ of the space $R_3(\hat{\mathbf{x}})$. Thus, by the physical quantity $f(\mathbf{x},t)$ we shall understand $\langle f(\hat{\mathbf{x}},t)\rangle$, averaged with respect to the measure $\tau(\mathbf{b})$ at a given time t. The assumption that the stochastic component of the space $R_3(\hat{\mathbf{x}})$ is small means that

$$F(\mathbf{x},t)=\langle f(\mathbf{x}+\mathbf{b},t)\rangle=\int d\mathbf{b}\tau(\mathbf{b})f(\mathbf{x}+\mathbf{b},t)$$
$$=\Big\langle f(\mathbf{x},t)+b_i\frac{\partial}{\partial x_i}f(\mathbf{x},t)+\frac{1}{2}b_ib_j\frac{\partial^2}{\partial x_i\,\partial x_j}f(\mathbf{x},t)+\ldots\Big\rangle\approx f(\mathbf{x},t)+l^2\Delta f. \qquad (58)$$

It is assumed that $\tau(\mathbf{b})=\tau(-\mathbf{b})$.

It should be noted that with a suitable choice of the function $\tau(\mathbf{b})$ the method of averaging (58), which ensures the transition from the small to the large scale, leads to an entirely new object[1)]:

$$F_n(\mathbf{x},t)=\sum_{n=0}^{\infty}\frac{c_n}{(2n)!}(l^2\nabla^2)^n f(\mathbf{x},t)=\int d\mathbf{y}K(\mathbf{x}-\mathbf{y})f(\mathbf{y},t),$$

and in the relativistic case

$$F_n(x_\mu,s)=\sum_{n=0}^{\infty}\frac{c_n'}{(2n)!}(l^2\Box)^n f(x_\mu,s)=\int d^4yK(x-y)f(y,s),$$

where

$$K(\mathbf{x})=\sum_{n=0}^{\infty}\frac{c_n}{(2n)!}(l^2\nabla^2)^n\delta^3(\mathbf{x});$$

$$K(x_\mu)=\sum_{n=0}^{\infty}\frac{c_n'}{(2n)!}(l^2\Box)^n\delta^4(x),$$

and the coefficients c_n (c_n') depend on the particular choice of the distribution $\tau(\mathbf{b})$ $[\tau(y_\mu^E)]$. In quantum field theory, such an object has been investigated carefully by Efimov[28] from the point of view of generalized functions $K(x)$ whose space–time properties depend essentially on different sequences of coefficients $\{c_n\}$. Efimov showed that an object constructed by means of such generalized functions $K(x)$ is smeared (nonlocal) in space.

Thus, the procedure for averaging (58) in the framework of our hypothesis leads to a nonlocal object. However, such averaging cannot change the physical nature of the object (for example, if it was stochastic before the averaging, it still is so after it) but merely changes the spatial structure of the object, which is smeared in a certain region determined by l. In calculations of local and nonlocal objects such as the velocity $\mathbf{v}(\mathbf{x},t)$, the randomness effect which arises from the fluctuation of the spatial coordinates accumulates with the course of time, is manifested only in the dynamical aspect, and does not depend on the intermediate averaging procedure (58). Then in our case, the local, $f(\mathbf{x},t)$, and the nonlocal, $F_n(\mathbf{x},t)$, objects can be assumed to be stochastic quantities. At the present time, it is not clear how one should write down the probability equation (an equation of Chapman–Kolmogorov type) for a nonlocal stochastic quantity $F_n(\mathbf{x},t)$. Therefore, in a rough approximation, when the parameter l is infinitesimally small, we shall assume that $F_n(\mathbf{x},t)$ is determined by a finite sum of the form (58). The object

$$F_n(\mathbf{x},t)=f(\mathbf{x},t)+f_l(\mathbf{x},t). \qquad (59)$$

when $f_l(\mathbf{x},t)$ is determined by a finite sum of a series in the parameter l, possesses,[28] like $f(\mathbf{x},t)$, a local property and, therefore, for $F_n(\mathbf{x},t)$ we can write an equation of Chapman–Kolmogorov type.

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1) An object of this kind was constructed for the first time by Blokhintsev[34] in quantum field theory for the special case when $b_\mu=c\gamma_\mu$, where c is a stochastic variable and γ_μ are the ordinary Dirac matrices.

As a first approximation in the parameter l, we shall ignore the second term in (59), i.e.,

$$\langle f(\mathbf{x}, t)\rangle \equiv F(\mathbf{x}, t) \approx f(\mathbf{x}, t).$$

It is to this approximation that our previous results apply, and in this section we shall not ignore the second term in the expression (59).

We now attempt to obtain the general form of the dynamical equations of a scalar particle in the stochastic space when a term of order l^2 is present in the expressions for the velocity and the force in accordance with Eq. (59):

$$\mathbf{v}_\pm = \mathbf{v}_\pm + \mathbf{v}_\pm^l; \quad \mathbf{f}_\pm = \mathbf{f}_\pm + \mathbf{f}_\pm^l.$$

It is assumed that in Eqs. (18) and (19) [(32) and (33)] of the Smoluchowski type for $\mathbf{v}_\pm$ ($v_\pm^\mu$) the small corrections $\mathbf{v}_\pm^l$ and $\mathbf{f}_\pm^l$ ($v_\pm^{\mu,l}$ and $F_\pm^{\mu,l}$) occur only in a symmetric combination with respect to Δt and $\delta\mathbf{x}$ (Δs and $\delta x_E^\mu = y_E^\mu$), i.e., they are even functions under the transformations $\Delta t \to -\Delta t$ and $\delta\mathbf{x} \to -\delta\mathbf{x}$ ($\Delta s \to -\Delta s$ and $\delta x_E^\mu \to -\delta x_E^\mu$). In this case, $\mathbf{v}_\pm$ and $\mathbf{f}_\pm$ ($v_\pm^\mu$ and $F_\pm^\mu$), which occur in Eqs. (18) and (19) [(33) and (34)], are replaced by the expressions

$$\mathbf{v}_\pm + \sum_{\{\Delta t, \delta\mathbf{x}\}} \mathbf{v}_\pm^l \text{ and } \mathbf{f}_\pm + \sum_{\{\delta\mathbf{x}\}} \mathbf{f}_\pm^l$$

$$(v_\pm^\mu + \sum_{\{\Delta s, y_E\}} v_\pm^{\mu, l} \text{ and } F_\pm^\mu + \sum_{\{y_E\}} F_\pm^{\mu, l}),$$

where the symbol $\sum_{\{\dots\}}$ denotes the operation of symmetrization with respect to the variables $\{\dots\}$; for example,

$$\sum_{\{\mathbf{y}\}} \mathbf{v}_\pm^l(\mathbf{x}-\mathbf{y}) = \sum_{\{\mathbf{y}\}} \mathbf{v}_\pm^l(\mathbf{x}+\mathbf{y}) \equiv \frac{1}{2}[\mathbf{v}_\pm^l(\mathbf{x}+\mathbf{y}) + \mathbf{v}_\pm^l(\mathbf{x}-\mathbf{y})].$$

As a result, we obtain equations analogous to (21), (22) and (35), (36):

$$\left.\begin{aligned} &d_c\mathbf{v} - \lambda d_s'\mathbf{u} = \frac{1}{m}\mathbf{F}_\lambda^+;\\ &d_c\mathbf{u} + \lambda d_s'\mathbf{v} = \frac{1}{m}\mathbf{F}_\lambda^-;\end{aligned}\right\} \qquad (21a)$$

$$\left.\begin{aligned} &d_c\mathbf{v} - \lambda d_s'\mathbf{v} = \frac{1}{m}\mathbf{F}_\lambda;\\ &d_c\mathbf{u} + \lambda d_s'\mathbf{u} = \frac{1}{m}\mathbf{F}_\lambda'\end{aligned}\right\} \qquad (22a)$$

and

$$\left.\begin{aligned} &D_c v^\mu - \lambda D_s' u^\mu = \frac{1}{m}\Phi_\lambda^{(+)\mu};\\ &D_c u^\mu + \lambda D_s' v^\mu = \frac{1}{m}\Phi_\lambda^{(-)\mu};\end{aligned}\right\} \qquad (35a)$$

$$\left.\begin{aligned} &D_c v^\mu - \lambda D_s' v^\mu = \frac{1}{m}\Phi_\lambda^\mu;\\ &D_c u^\mu + \lambda D_s' u^\mu = \frac{1}{m}\Phi_\lambda'^\mu\end{aligned}\right\} \qquad (35b)$$

in the nonrelativistic and relativistic cases, respectively.

Here

$$\left.\begin{aligned} &d_s' = (\mathbf{u}\nabla) + D(\nabla^2 + l^2\nabla^4);\\ &D_s' = u^\lambda\partial_\lambda + D(\Box + l^2\Box^2).\end{aligned}\right\} \qquad (60)$$

Naturally, if we ignore the terms of order $Dl^2\nabla^4$ ($Dl^2\Box^2$) in the expressions for d_s' (D_s'), then, as one would expect, we obtain the old equations (21) and (22) [(35) and (36)].

In the limit $\mathbf{v}^+ \equiv \mathbf{v}^-$, i.e., $\mathbf{u} \equiv 0$, we obtain from (21a) and (22a) Newton's equation

$$d_c\mathbf{v} = \mathbf{F}/m$$

and also the equation

$$d_c\mathbf{v} - \lambda D(\nabla^2 + l^2\nabla^4)\mathbf{v} = \mathbf{F}_\lambda/m$$

for the particle.

For $\mathbf{F}_\lambda = 0$, these last equations are equations of Sivashinsky type for a free particle. For example, setting $\lambda = -1$, we obtain

$$d_c\mathbf{v} + D\nabla^2\mathbf{v} + Dl^2\nabla^4\mathbf{v} = 0, \qquad (61)$$

which is invariant under a Galileo transformation.

Note that for the system of equations (21a) [or (21)] the condition $D \to 0$ in the "classical" limit is consistent, since then the first equation of (21a) goes over into Newton's equation, and both sides of Eq. (21a) vanish in the limit when $D \to 0$. We recall that $\mathbf{F}^+$ is the external force, which does not contain a stochastic component, and the second equation of (21a) for $\lambda = 1$ agrees with the continuity equation (see also Appendix 1)

$$\frac{\partial\rho}{\partial t} + \nabla(\rho\mathbf{v}) = 0.$$

Therefore, if the Lorentz force is introduced in a natural manner in our formalism, we can write

$$\mathbf{F}_{\bar{1}} = -m(\mathbf{u} \times (\nabla \times \mathbf{v}) + D\nabla \times (\nabla \times \mathbf{v})).$$

In the case of the electromagnetic field, it takes the form

$$\mathbf{F}_{\bar{1}} = \frac{e}{c}\mathbf{u} \times \mathbf{H} + \frac{e}{c}D\nabla \times \mathbf{H}.$$

Conversely, in the system of equations (22a) it is impossible to go over formally to the condition $D \to 0$ in the "classical" limit, since in the expressions for the forces $\mathbf{F}_\lambda$ and $\mathbf{F}_\lambda'$ the ordinary and stochastic components of the force are already present. It is entirely possible that the stochastic part of the force does not necessarily vanish even as $D \to 0$, and on the other hand the stochastic velocity $\mathbf{u} = D\nabla\ln\rho$ vanishes in the limit $D \to 0$. Then the stochastic terms on both sides of Eq. (22a) do not vanish as $D \to 0$.

A special method is needed to investigate the nonlinear equations (21a), (22a), and (61), whose solution goes beyond the scope of the present paper.

In Ref. 49, Eq. (61) was studied numerically and it was shown that its solution behaves like a turbulent process.

In this paper, we shall make only some general comments about these equations. First, we note that Eq. (61) is equivalent to the Hamilton–Jacobi equation only in the case when the "flow" $\mathbf{v} = \nabla S$ is irrotational, i.e., $\operatorname{curl}\mathbf{v} = 0$. Otherwise, the solution of Eq. (61) grows unboundedly with the time. For example, let us consider the field $\mathbf{v} = (u, v, w)$, where

$$\varphi_t + D\nabla^2\varphi + Dl^2\nabla^4\varphi = 0.$$

Note that $\operatorname{div}\mathbf{v} = 0$. For such a field, all the nonlinear terms in Eq. (61) vanish, and we obtain a purely linear equation for $\varphi(x,y,t)$:

$$u(x, y, z, t)=v(x, y, z, t)=0; \quad w(x, y, z, t)=\varphi(x, y, t).$$

This equation has spatially periodic solutions of the type

$$\varphi=A\exp(\sigma t+ik_1x+ik_2y),$$

where

$$\sigma=D(k_1^2+k_2^2)-Dl^2(k_1^2+k_2^2)^2.$$

If $k_1^2+k_2^2<l^{-2}$ then the amplitudes of these solutions grow exponentially.

The second question relates to the choice of the sign of the parameter λ. For example, if the terms of order Dl^2 are ignored, the second equation of (21a) for $\lambda=-1$ does not agree with the continuity equation, and, therefore, the case $\lambda=-1$ for (21a) is eliminated from consideration altogether.

When $\lambda=1$, the equation of type (61) obtained from Eq. (22a) is such that for it there is no scattering in the short-wavelength regions. It is well known that for such equations an important problem—the boundary condition—is not formulated sufficiently well. For this reason, the case $\lambda=1$ for (61) simply has no physical meaning if one does not assume the existence of terms with derivatives of higher than fourth order.

Since our approximation includes only terms with derivatives of fourth order, the case $\lambda=1$ for Eq. (61) is eliminated. Of course, the questions posed above remain valid in the relativistic case.

In the relativistic case, Sivashinsky's equation (61) now takes the form

$$D_c v^\mu=-D(\Box+l^2\Box^2)v^\mu, \tag{62}$$

which is obtained from Eq. (35b) for $u^\mu\equiv 0$ and $\Phi^\mu=0$.

Thus, adopting the hypothesis of a stochastic space, we have obtained Sivashinsky's equation. We have shown that the resulting instability of the uniform and rectilinear motion of a free particle leads to random fluctuations of its trajectory. Despite the purely classical nature of the original equation, typically quantum effects are imitated: the uncertainty relation, de Broglie waves and their interference, discrete energy levels, and zero-point fluctuations.

As we have seen above, the self-interaction potential of the particle [the right-hand side of Eqs. (61) and (62)] which generates the turbulence in the motion of the free particle has a stochastic origin. In other words, the stochasticity, which disappears in the limit $u\Rightarrow 0$, renders the motion for v unstable and preserving thus a memory of itself.

6. PHYSICAL INTERPRETATION OF THE RESULTS

In our approach, the following questions naturally arise:

1. What is the physical meaning of the stochastic nature of space in the nonrelativistic and relativistic cases?

2. What is the universal length and what is its order of magnitude?

3. How is one to interpret the invariant parameter s used in the construction of the relativistic dynamics, etc.? Some possibilities that lead to a stochastic space $R_4(\hat{x})$ were discussed by Blokhintsev.[35]

In the framework of our approach in the nonrelativistic case, the stochastic nature of space leads to the impossibility of determining the coordinates of a particle with an accuracy exceeding at the least the value of the particle's Compton wavelength. In the relativistic case, it is harder to give this property a physical interpretation. Formally, it can be interpreted as the presence of fluctuations of the four-dimensional coordinates of the particle in the Euclidean space $E_4(\hat{x},\tau)$ (i.e., as the presence of a random walk in the imaginary time). By itself, such an interpretation has no physical meaning, but the method has value as one method of relativistic description of random processes in terms of an equation of Smoluchowski type. It should be noted that Eqs. (32) and (33) obtained above do not in general bear any relation to the ordinary Smoluchowski equations, which describe a stochastic consequence and, therefore, the quantities $\Psi_\pm$ which occur in these equations cannot be interpreted as transition probabilities.

By a universal (or fundamental) length, we shall understand the following. Physically, the universal length l is some characteristic distance over which the corresponding notions about space and locality (causal connection) begin to break down; in particular, stochastic properties and nonlocality can be manifested if they exist. The estimates made in Refs. 39 and 50 show that $l\lesssim 10^{-15}$–10^{-16} cm. Other possibilities for introducing the concept of a fundamental length in physics have been discussed in Refs. 38, 31, 50, and 51.

A second important question is associated with the evolution of the system on the transition from one hyperplane σ_1 to another σ_2. Suppose the hyperplane σ is characterized by a single timelike vector v_μ. The hyperplane σ is orthogonal to the world line $\mathscr{P}(s)$ of the particle, and the 4-velocity $u^\mu=d\mathscr{P}/ds=dx^\mu/ds$ is the unit vector tangent to the world line of the particle.

Hitherto, we have considered the evolution of the system from the point of view of its proper time. For example, by means of this concept we have derived the probability distribution $\rho(x_\mu, s+\Delta s)$ for the time $s+\Delta s$ from the probability distribution $\rho(x_\mu, s)$ for the earlier time s, and we have also obtained a differential equation for $\rho(x_\mu, s)$. In reality, however, the proper time and the coordinate $x_0=ct$ are dependent and related by $dx^0/ds=c\sqrt{1-\beta^2}=u^0$, and in the rest frame of the particle they are simply equal.

Bearing this in mind, we attempt to derive an equation for $\rho(x_\mu)$ from a different point of view, namely, on the basis of a geometrical object—a "differential form" or "1-form" (for more detail, see Ref. 52); then the development of the probability distribution ρ is determined by means of an oriented family of flat surfaces $\tilde{k}=d\rho$. Using only surfaces, it is impossible to characterize fully the 1-form $\tilde{k}$; it is also necessary to specify an orientation. Suppose $\tilde{k}$ is composed of the surfaces of the constant values $\rho = 7, 8, 9, \ldots$; then the direction from surface to surface is assumed to be "positive" if ρ increases in this direction.

We now define the concept of the derivative along the direction of the vector v. We take a certain vector v, construct the curve $\mathscr{P}(\lambda)$, which is determined by the equation $\mathscr{P}(\lambda) - \mathscr{P}_0 = \lambda \mathrm{v}$, and differentiate the function ρ along this curve:

$$\partial_{\mathrm{v}}\rho = \left(\frac{d}{d\lambda}\right)_{\lambda=0} \rho(\mathscr{P}(\lambda)) = \left(\frac{d\rho}{d\lambda}\right)_{\mathscr{P}_0}.$$

The "differential operator"

$$\partial_{\mathrm{v}} = (d/d\lambda)$$

for $\lambda = 0$ along the curve $\mathscr{P}(\lambda) - \mathscr{P}_0 = \lambda \mathrm{v}$, which carries out this differentiation, is called the "operator of the derivative along the direction of the vector v." The derivative along the direction $\partial_{\mathrm{v}}\rho$ and the gradient $\mathbf{d}\rho$ are intimately related. This is immediately seen by applying ∂_{v} to the equation $\rho(\mathscr{P}) = \rho(\mathscr{P}_0) + \langle \mathbf{d}\rho, \mathscr{P} - \mathscr{P}_0 \rangle$ + nonlinear terms, which determines the "gradient," and calculating the result at the point $\mathscr{P}_0$, which gives

$$\partial_{\mathbf{v}}\rho = \langle \mathbf{d}\rho,\, d\mathscr{P}/d\lambda \rangle = \langle \mathbf{d}\rho \cdot \mathbf{v} \rangle.$$

In terms of Ref. 52, this result reads: $\mathbf{d}\rho$ is a linear machine for calculating the rate of change of ρ along an arbitrary vector v. Introducing $\mathbf{v}$ into $\mathbf{d}\rho$, we obtain at the output $\partial_{\mathbf{v}}\rho$, the "number of intersected planes," a number which for sufficiently small v is simply equal to the increment of ρ between the base and tip of the vector v.

In the coordinate representation, the operator ∂_{v} of the derivative along the direction of the vector v is determined by the expression

$$\partial_{\mathrm{v}} = v^\mu \partial/\partial x^\mu.$$

The notation and concepts are taken from Ref. 52.

Thus, the method based on the use of the concept of the derivative along a direction warrants attention. For example, when this method is used Eqs. (32) and (33) simplify significantly by virtue of the elimination of the dependence on $u_\mp^\mu$ in the expression for $\Psi_\pm$, which now takes the form

$$\Psi_\pm = (4\pi D_\pm \Delta s_\pm)^{-2} \exp\left(-(\Delta y)^2/4D\Delta s_\pm\right).$$

Here, s_+ and s_- are the parameters which characterize the derivative along the directions of the vectors u_+^μ and u_-^μ, respectively. In this case, Eqs. (34) take the form

$$\partial u_+^\mu/\partial s_+ = 2u^\nu\,\partial_\nu u_+^\mu + D\,\Box\, u_+^\mu + F_+^\mu/m;$$

$$\partial u_-^\mu/\partial s_- = -2u^\nu\,\partial_\nu u_-^\mu - D\,\Box\, u_-^\mu + F_-^\mu/m;$$

$$\partial u_-^\mu/\partial s_+ = 2u^\nu\,\partial_\nu u_-^\mu + D\,\Box\, u_-^\mu + F_-'^\mu/m;$$

$$\partial u_+^\mu/\partial s_- = -2u^\nu\,\partial_\nu u_+^\mu - D\,\Box\, u_+^\mu + F_+'^\mu/m.$$

By definition, the derivaties along the directions of u_+^μ and u_-^μ are

$$\partial u_\pm^\mu/\partial s_+ = u_+^\nu\,\partial_\nu u_\pm^\mu,\quad \partial u_\pm^\mu/\partial s_- = u_-^\nu\,\partial_\nu u_\pm^\mu,$$

and we therefore obtain Eqs. (35) and (36) with

$$D_c = v^\nu \partial_\nu;\quad D_s = u^\nu \partial_\nu + D\,\Box.$$

Equations of the form (35) with $\lambda = 1$ and $D_c = v^\lambda \partial_\lambda$ were obtained for the first time by Lehr and Park[13] (see also Ref. 12); they are, of course, equivalent to Eqs. (35) with $D_c = \partial/\partial s + v^\lambda \partial_\lambda$ and $D_s = u^\lambda \partial_\lambda + D\Box$.

Thus, the derivative with respect to s which occurs in this review can be understood as the derivative along the direction of some arbitrary vector v^μ. In the special case when the particle's velocity u^μ is taken as the vector v^μ, we can interpret s as the proper time of the particle. In our approach, conservation of mass (of the probability density multiplied by the volume) signifies the absence of mass loss through any hypersurface characterized by the vector v^μ; it follows from this in particular that the probability density for finding the particle along the world trajectory is constant, i.e.,

$$\frac{\partial\rho}{\partial s} = (\mathrm{d}\rho \cdot \mathrm{u}) = \frac{\partial\rho}{\partial x_0} u^0 + \frac{\mathrm{v}}{\sqrt{1-\beta^2}} \frac{\partial\rho}{\partial x} = 0. \tag{63}$$

This last equation expresses the conservation of the mass (the probability density of the current) of the particle.

Note that Eq. (63) recalls the equation that reflects the absence of heat transfer and requires constancy of the specific entropy (the entropy of unit mass) for a fluid particle (see, for example, Ref. 52):

$$d\mu/dt = 0 \quad \text{or} \quad \partial\mu/dt + (\mathbf{v}\nabla)\,\mu = 0,$$

and the last equation expresses the conservation of mass:

$$\partial\rho/\partial t + \nabla(\rho \mathrm{v}) = 0.$$

We now obtain the continuity equation for the probability density of the current in our case. We see from (30) that the condition (63) leads to the following two equations:

$$\partial_\mu(\rho u_+^\mu) = D\,\Box\,\rho$$

and

$$\partial_\mu(\rho u_-^\mu) = -D\,\Box\,\rho.$$

Further, we determine the vectors of the four-dimensional current:

$$\mathscr{Y}_\pm^\mu = \rho u_\pm^\mu;\ \mathscr{Y}^\mu = \frac{1}{2}(\mathscr{Y}_+^\mu + \mathscr{Y}_-^\mu);\ v^\mu = \mathscr{Y}^\mu/\rho,$$

and then $\mathscr{Y}^\mu$ satisfies the continuity equation

$$\partial_\mu \mathscr{Y}^\mu = 0. \tag{64}$$

Since $v^\mu v_\mu = c^2$, $|\rho|^2 = \mathscr{Y}^\mu \mathscr{Y}_\mu/c^2$ is a world scalar and, therefore, the equations

$$\partial_\mu \mathscr{Y}_+^\mu = D\,\Box\,|\rho|$$

and

$$\partial_\mu \mathscr{Y}_-^\mu = -D\,\Box\,|\rho|$$

are already expressed in covariant form.

APPENDIX 1. INVESTIGATION OF THE EQUATION OF STOCHASTIC MECHANICS

The description of a stochastic system is complete if its probability density $\rho(\mathrm{x}, t)$ and velocity $\mathrm{v}(\mathrm{x}, t)$ are known, since the stochastic velocity u is related to

$\rho(x,t)$ by

$$\mathbf{u}(x,\ t)=D\nabla\ln\rho(x,\ t).$$

For them, we have obtained the equations

$$\partial\rho/\partial t+\operatorname{div}(\rho\mathbf{v})=0;\ m(d_c\mathbf{v}-d_s\mathbf{u})=\mathbf{f}, \tag{A.1}$$

and in the relativistic case

$$\partial_\mu\mathcal{Y}^\mu=0;\ m(D_c v^\mu-D_s u^\mu)=F^\mu.$$

In addition, there are equations for $\mathbf{u}$ and u^μ:

$$\left.\begin{aligned} &m(d_c\mathbf{u}+d_s\mathbf{v})=\mathbf{f}_s\\ &m(D_c u^\mu+D_s v^\mu)=F^\mu_s.\end{aligned}\right\} \tag{A.2}$$

On physical grounds, it can be asserted that these basic equations must satisfy a number of requirements.

1. To ensure that the physical quantities ρ and $\mathbf{v}$ are not overdetermined, it is necessary to require that the last two equations for $\mathbf{u}$ and u^μ be equivalent to the continuity equations for ρ and $\mathcal{Y}^\mu$, respectively.

2. We shall assume that in stochastic mechanics the work done by a field on a particle is associated only with the velocity $\mathbf{v}$ and does not depend on the velocity $\mathbf{u}$, i.e.,

$$\left\langle\frac{d}{dt}(mv^2)\right\rangle=(\mathbf{f}\cdot\mathbf{v});\quad\left(\frac{d}{dt}=\frac{\partial}{\partial t}+(\mathbf{v}\cdot\nabla)\right).$$

In the absence of an external force, this condition amounts to conservation of the kinetic energy in time.

3. In the relativistic case, we must have

$$v^\mu v_\mu=c^2.$$

Thus, on the basis of the first requirement we can choose the form of the external forces $\mathbf{f}_s$ and F^μ_s, and the requirements 2 and 3 impose the further conditions

$$(\mathbf{v}\cdot d_s\mathbf{u})=0,\ v_\mu D_s u^\mu=0. \tag{A.3}$$

Note that in the nonrelativistic limit the last condition goes over into $\mathbf{v}\cdot d_s\mathbf{u}=0$, so that these conditions are interrelated.

As external field, we now consider the electromagnetic field and show that Eqs. (A.2) with external force

$$\mathbf{f}_s=\frac{e}{c}\mathbf{u}\times\mathbf{H}+\frac{e}{c}D\nabla\times\mathbf{H};\ \mathbf{H}=\operatorname{curl}\mathbf{A};$$

$$F^\mu_s=\frac{e}{c}F^{\mu\nu}u_\nu+\frac{e}{c}D\partial^\nu F^{\nu\mu},\ F^{\mu\nu}=\left(\frac{\partial A_\nu}{\partial x_\mu}-\frac{\partial A_\mu}{\partial x_\nu}\right) \tag{A.4}$$

are equivalent to equations for ρ and $\mathcal{Y}^\mu$, respectively. In the nonrelativistic case, we set

$$m\mathbf{v}+\frac{e}{c}\mathbf{A}=\nabla S,$$

where S is the action. Then the first equation of (A.2) with allowance for (A.4) gives

$$d_c\mathbf{u}+d_s\mathbf{v}=-\mathbf{u}\times\operatorname{curl}\mathbf{v}-D\nabla\times\operatorname{curl}\mathbf{v}. \tag{A.5}$$

Applying the operator ∇ to the continuity equation for ρ and using the formula $\mathbf{u}=D\nabla\ln\rho$, we obtain

$$\frac{\partial\mathbf{u}}{\partial t}+\nabla(\mathbf{u}\cdot\mathbf{v})+D\nabla(\nabla\cdot\mathbf{v})=0. \tag{A.6}$$

Using the well-known formulas of vector analysis,

$$\nabla(\mathbf{u}\cdot\mathbf{v})=(\mathbf{u}\nabla)\mathbf{v}+(\mathbf{v}\nabla)\mathbf{u}+\mathbf{u}\times\operatorname{curl}\mathbf{v}+\mathbf{v}\times\operatorname{curl}\mathbf{u},$$
$$\nabla(\nabla\cdot\mathbf{v})=\Delta\mathbf{v}+\nabla\times(\nabla\times\mathbf{v}),\ \operatorname{curl}\operatorname{grad}\equiv 0,$$

we arrive at Eq. (A.5), which is what we wanted to prove.

As in the derivation of Eq. (A.5), the time component of the second equation in (A.2) can be reduced to the form

$$D_c u_0+D_s v_0=(\mathbf{u}\nabla)v_0-\mathbf{u}\partial_0\mathbf{v}-D\nabla\partial_0\mathbf{v}+D\Delta v_0. \tag{A.7}$$

Here, we have used Eqs. (A.4) and the equation (see, for example, Ref. 54)

$$mv^\mu+\frac{e}{c}A^\mu=-\partial S/\partial x^\mu.$$

Differentiating the continuity equation for $\mathcal{Y}^\mu$, we find

$$D_c u_0+D_s v_0=(\mathbf{u}\cdot\nabla)v_0-\mathbf{u}\partial_0\mathbf{v}-D\nabla\partial_0\mathbf{v}+D\Delta v_0+(\mathbf{v}\cdot\nabla)u_0-\mathbf{v}\partial_0\mathbf{u}.$$

Since $u_0=D\partial_0\ln\rho$ and $\mathbf{u}=D\nabla\ln\rho$, $(\mathbf{v}\nabla)u_0-\mathbf{v}\partial_0\mathbf{u}=0$, from which we obtain Eq. (A.7).

Similarly, we can easily show that the three spatial components of the second equation of (A.2) are identical to the vector equation

$$D_c\mathbf{u}+D_s\mathbf{v}=-v_0\nabla u_0+v_0\partial_0\mathbf{u}-u_0\nabla v_0+u_0\partial_0\mathbf{v}-\mathbf{u}\times\operatorname{curl}\mathbf{v}+D\partial_0\nabla v_0$$
$$-D\partial_0^2\mathbf{v}-D\nabla\times\operatorname{curl}\mathbf{v},$$

which is obtained by differentiating the continuity equation for $\mathcal{Y}^\mu$ by the operator ∇.

We now turn to the additional conditions (A.3). The equation

$$\mathbf{v}\cdot\mathbf{B}=0, \tag{A.8}$$

where

$$\mathbf{B}=d_s\mathbf{u}=(\mathbf{u}\cdot\nabla)\mathbf{u}+D\Delta\mathbf{u}$$

has a nondenumerable set of solutions (we assume that $\mathbf{B}$ is an unknown vector), since it determines only the component of $\mathbf{B}$ in the direction of the vector $\mathbf{v}$, the value of which is $B_v=0$; however, the component perpendicular to $\mathbf{v}$ remains entirely arbitrary. Thus, if $\mathbf{B}$ is regarded as the radius vector of some point M relative to the origin O, the locus of the ends of all vectors $\mathbf{B}$ satisfying Eq. (A.8) will be the plane perpendicular to $\mathbf{v}$ through O. In the relativistic case, this question was discussed in Ref. 15 from the geometrical point of view.

To conclude this Appendix, we note that our equations (A.1), (A.2), and (A.6) are nonlinear partial differential equations, so that their direct solution is an almost insoluble problem in concrete applications. Finally, we give some special solutions of Eqs. (A.1) for the simplest case of free and uniform motion of a particle with the initial conditions

$$\rho(x,\ t=0)=\frac{1}{\sqrt{\pi}\,l}\exp\left[-\frac{(x-x_0)^2}{l^2}\right];\ v(x,\ t=0)=v_0.$$

These solutions have the form

$$\rho(x,\ t)=\frac{1}{\sqrt{\pi}\,l\sqrt{1+bt^2}}\exp\left[-\frac{(x-x_0-v_0t)^2}{l^2(1+bt^2)}\right];\ b=4D^2/l^2;$$
$$v(x,\ t)=(v_0+bt(x-x_0))/(1+bt^2);$$
$$u(x,\ t)=-\sqrt{b}(x-x_0-v_0t)/(1+bt^2),$$

and their mean value and dispersion are

$$\langle v(x,\ t)\rangle=\int dx\,v(x,\ t)\,\rho(x,\ t)=v_0;$$

$$\langle u(x,\ t)\rangle=\int dx\,u(x,\ t)\,\rho(x,\ t)=0;$$

$$\langle x(t)\rangle=\int dx\,x\,\rho(x,\ t)=x_0+v_0t;$$

$$\text{Duc. } x^2(t)=\frac{1}{2}l^2(1+bt^2);\quad \text{Duc. } v^2(x,\ t)=\frac{l^2}{2}\frac{b^2t^2}{1+bt^2};$$

$$\langle v^2(x,\ t)\rangle=v_0^2+\frac{l^2}{2}\frac{b^2t^2}{1+bt^2}.$$

In the case of a uniform translational and rotational motion of particles (f = const), solutions similar to those above can be readily obtained.

APPENDIX 2. CAUCHY PROBLEM FOR THE DIFFUSION EQUATION

We consider the following equation in two-dimensional space–time (x_1 and x_0):

$$\frac{\partial\rho(x_0,\ x_1,\ s)}{\partial s}=D\Box_2\rho(x_0,\ x_1,\ s),\quad \Box_2=-\frac{\partial^2}{\partial x_0^2}+\frac{\partial^2}{\partial x_1^2}.$$

Since the variables s and x_0 are related, this equation can be written in the form

$$\left(\Box_2-a^2\frac{\partial}{\partial t}\right)\rho(x_0,\ x_1)=0,\ a^2=1/D,$$

$$x_0=ct.$$

We now investigate the Green's function of the equation

$$\left(\Box_2-a^2\frac{\partial}{\partial t}\right)G(x_1-x_{10},\ t-t_0)=\delta(x_1-x_{10})\,\delta(t-t_0),$$

where (x_1, t) is the point of observation and (x_{10}, t_0) is the source point. To construct the function G, we consider first the two functions

$$D_1(x_0,\ x_1)=\frac{1}{2\pi i}\int\exp(ip_0x_0-ip_1x_1)\,\delta\left(i\frac{p_0c}{D}-p^2\right)\theta(p^0)\,d^2p;$$

$$D_2(x_0,\ x_1)=\frac{1}{2\pi}\int\exp(-ip_0x_0+ip_1x_1)\,\delta\left(-i\frac{p_0c}{D}-p^2\right)\theta(p^0)\,d^2p.$$

After elementary calculations, we obtain

$$D_1(x_0,\ x_1)=\frac{1}{4\pi i}\exp\left[-\frac{c^2}{2D}(t-t_0)\right]\int dp\,\frac{1}{\omega}\exp(i\omega x_0+ipx_1);$$

$$D_2(x_0,\ x_1)=\frac{i}{4\pi}\exp\left[-\frac{c^2}{2D}(t-t_0)\right]\int dp\,\frac{1}{\omega}\exp(-i\omega x_0-ipx_1).$$

Here

$$c\tau=x_0=c(t-t_0);\ x_1=x_1-x_{10}=R;\ \omega=\sqrt{p^2-\frac{c^2}{4D^2}}.$$

The quantities D_1 and D_2 can be readily calculated by means of contour integrals (see, for example, Ref. 55):

$$D_1=-\frac{1}{2}\exp(-\mu c(t-t_0))\,\mathscr{Y}_0(\mu\sqrt{R^2-c^2\tau^2})\,(1-\theta(R+c\tau));$$

$$D_2=\frac{1}{2}\exp(-\mu c(t-t_0))\,\mathscr{Y}_0(\mu\sqrt{R^2-c^2\tau^2})\,\Theta(c\tau-R),$$

$$\mu=c/2D.$$

The Green's function is determined by

$$G(R,\ \tau)=-c(D_2-D_1)$$

$$=\frac{c}{2}\exp\left[-\frac{1}{2}a^2c^2(t-t_0)\right]\mathscr{Y}_0\left(\frac{a^2c}{2}\sqrt{R^2-c^2\tau^2}\right)\Theta(c\tau-|R|).$$

It is readily seen that, as one would expect,

$$\lim_{c\to\infty}a^2G(R,\ \tau)=\frac{1}{\sqrt{4\pi D\tau}}\exp\left(-\frac{R^2}{4D\tau}\right).$$

We now solve the one-dimensional initial-value problem

$$\rho(x,\ t)=a^2\int dx_{10}\,[\rho G]_{t_0=0}+\frac{1}{c^2}\int dx_{10}\left[G\frac{\partial\rho}{\partial t_0}-\rho\frac{\partial G}{\partial t_0}\right]_{t_0=0},$$

whence

$$\rho(x,\ t)=\frac{1}{2}\exp(-\mu ct)\Big\{f(x+ct)+f(x-ct)$$

$$+\int_{x-ct}^{x+ct}f(x_{10})\,dx_{10}\left[\frac{\mu}{2}I_0(\mu\sqrt{c^2t^2-(x_{10}-x)^2})\right.$$

$$\left.+\frac{1}{2c}\frac{\partial}{\partial t}I_0(\mu\sqrt{c^2t^2-(x_{10}-x)^2})\right]$$

$$+\frac{1}{2c}\int_{x-ct}^{x+ct}I_0(\mu\sqrt{c^2t^2-(x_{10}-x)^2})\,v(x_{10})\,dx_{10}\Big\},$$

where

$$f(x_{10})=\rho(x_{10},\ t)|_{t=0};\ v(x_{10})=\left.\frac{\partial\rho(x_{10},\ t)}{\partial t}\right|_{t=0}.$$

Here, $\mathscr{Y}_0(x)$ and $I_0(x)$ are the well-known cylindrical functions.

Finally, for comparison we write down the Green's function of the Schrödinger equation in the proper-time representation:

$$i\frac{\partial\psi}{\partial s}=-D\Box\psi,\ D=\hbar/2m=1/a^2,$$

or

$$\left(\Box_2+ia^2\frac{\partial}{\partial t}\right)\mathscr{K}(x_1-x_{10},\ t-t_0)=-\delta(t-t_0)\,\delta(x_1-x_{10}),$$

where

$$\mathscr{K}(R,\ \tau)=-\frac{ia^2}{4}c\exp\left[-i\frac{\pi}{4}+\frac{1}{2}ia^2c^2\tau\right]\left[\mathscr{Y}_0\left(\frac{a^2c}{2}\sqrt{c^2\tau^2-R^2}\right)\right.$$

$$\left.-iN\left(\frac{a^2c}{2}\sqrt{c^2\tau^2-R^2}\right)\right]\Theta(c\tau-R);$$

$$\tau=t-t_0;\ R=x_1-x_{10},$$

and in the limit $c\to\infty$ the function $\mathscr{K}(R, \tau)$ is equal to the Green's function

$$G=\frac{1}{i}(4\pi Dt)^{-1/2}\exp\left(i\frac{x^2}{4Dt}\right)$$

of the equation

$$i\frac{\partial\psi}{\partial t}=-D\frac{\partial^2}{\partial x^2}\psi.$$

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