# Methods of determining resonances in phase-shift analysis

F. Nichitiu

*Institute of Physics and Nuclear Technology, Bucharest*
Fiz. Elem. Chastits At. Yadra **12**, 805–847 (July–August 1981)

In elementary-particle and nuclear physics, two-particle reactions leading to the production of resonances are usually investigated by means of phase-shift analysis. The main problem here is to establish whether a resonance exists in a given partial wave. To solve this problem, it is necessary to consider in detail the energy dependence of the partial-wave amplitude. The review is devoted to methods of carrying out phase-shift analysis and the problems associated with the existence of a resonance and the determination of its parameters. Various examples of resonance behavior in nuclear and elementary-particle physics are considered. Particular attention is devoted to the problem of the existence of resonances in the pion–nucleus system.

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## INTRODUCTION

The investigation of scattering processes by means of phase-shift analysis can elucidate many important aspects of particle interaction. Phase-shift analysis is based solely on the most general conservation laws and is intimately related to experimental data; it enables one to obtain additional information permitting a better choice of a particular model for describing the interaction processes. Phase-shift analysis is important in the search for resonances and the determination of their quantum numbers.

In recent years, there has been a sharp increase in the interest in phase-shift analysis and several new methods have been proposed that make maximal use of the general principles of analyticity and unitarity for the partial-wave amplitudes (for example, the method of accelerated-convergence expansions, analysis of the zeros of the scattering amplitude, and analysis at fixed momentum transfer).

In the present review, we shall discuss problems that arise in phase-shift analysis. We pay particular attention to the problem of finding resonances by means of phase-shift analysis and the determination of their parameters. We begin by considering the method of phase-shift analysis, and we then analyze a problem that is very important in it—the problem of ambiguities. We then consider criteria for the existence of resonances and the influence of a nonresonance background on the determination of resonance parameters, and we consider some aspects of the application of phase-shift analysis in a relatively new field, namely, the investigation of pion scattering by nuclei at intermediate energies. In addition, we present some interesting results on the existence of new types of excited states in $\pi A$ systems.

## 1. METHODS OF PHASE-SHIFT ANALYSIS

*The formalism of phase-shift analysis.* On the basis of the asymptotic form of the wave function after scattering,

$$\Psi \approx \exp(ikz) + f(\theta)\exp(ikr)/r, \tag{1}$$

and expanding it and the incident plane wave in Legendre polynomials,

$$\exp(ikz) = \frac{1}{kr}\sum i^l (2l+1)\sin(kr - l\pi/2)\,P_l(\cos\theta), \tag{2}$$

we obtain the expression for the scattering amplitude of two particles with zero spin:

$$f(\theta) = \frac{1}{k}\sum (2l+1) f_l P_l(\cos\theta), \tag{3}$$

where $k$ is the c.m.s. momentum, and $f_l$ is the partial-wave amplitude, which can be expressed in terms of the phase shift $\delta_l$ and the inelasticity parameter $\eta_l$:

$$f_l = [\eta_l \exp(2i\delta_l) - 1]/2i. \tag{4}$$

The independent observable quantities for the elastic scattering of two particles with zero spin are the differential and total scattering cross sections:

$$\left.\begin{aligned} &d\sigma/d\Omega = |f(\theta)|^2; \\ &\sigma_{\text{tot}} = (4\pi/k)\,\text{Im}\, f(\theta = 0). \end{aligned}\right\} \tag{5}$$

The total elastic cross section $\sigma_{\text{el}}$ and the reaction cross section $\sigma_r$ can be expressed in terms of $\delta_l$ and $\eta_l$ as follows:

$$\left.\begin{aligned} &\sigma_{\text{el}} = \frac{4\pi}{k^2}\sum (2l+1)\,|f_l|^2 = \int \left(\frac{d\sigma}{d\Omega}\right) d\Omega; \\ &\sigma_r = \frac{4\pi}{k^2}\sum (2l+1)(1-\eta_l^2) = \sigma_{\text{tot}} - \sigma_{el}. \end{aligned}\right\} \tag{6}$$

In a collision of two particles with nonzero spins in the final state, the axial symmetry with respect to the direction of the incident particle is not preserved. The wave function in the asymptotic region will now also have a spin-dependent part. We write down expressions like (1) and (3) for the collision of two particles with spins 0 and $\frac{1}{2}$:

$$\Psi_{\pm 1/2} = \Psi_{\pm 1/2}^{\text{inc}} + \frac{\exp(ikr)}{r}\left[f_{--}^{++}(\theta,\varphi)\,\chi_{1/2\pm 1/2} + f_{+-}^{+-}(\theta,\varphi)\,\chi_{1/2\mp 1/2}\right], \tag{7}$$

in which the amplitudes $f^{++}(\theta,\varphi)$ and $f_{--}(\theta,\varphi)$ describe scattering without spin flip, while the amplitudes $f^{+-}(\theta,\varphi)$ and $f_{-+}(\theta,\varphi)$ correspond to scattering with reversal of the spin projection. They have the form

$$f^{++}(\theta,\varphi) = f_{--}(\theta,\varphi) = \frac{1}{k}\sum \left\{\sqrt{\frac{4\pi}{2l+1}}\left[(l+1)f_{l+} + l f_{l-}\right] Y_{l,0}(\theta,\varphi)\right\}; \tag{8}$$

$$f_{+-}^{+-}(\theta,\varphi) = \pm\frac{1}{k}\sum \sqrt{\frac{4\pi}{2l+1}}\left[f_{l+} - f_{l-}\right] Y_{l\pm 1}(\theta,\varphi), \tag{9}$$

where

$$f_{l\pm} = [\eta_{l\pm}\exp(2i\delta_{l\pm}) - 1]/2i$$

is the partial-wave amplitude in the state with total angular momentum $j = l \pm \frac{1}{2}$.

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The independent observable quantities for the scattering of two particles with spin 0 and $\frac{1}{2}$ can be expressed in terms of the scattering amplitude,

$$F(\cos\theta,\ \varphi) = f(\cos\theta) + i\sigma\cdot n g(\cos\theta), \tag{10}$$

where $\mathbf{n}=\mathbf{k}_i\cdot\mathbf{k}_f/|\mathbf{k}_i\cdot\mathbf{k}_f|$, $\sigma$ are the Pauli matrices, and

$$\begin{aligned} f(\cos\theta) &= \frac{1}{k}\sum [(l+1)f_{l+}+lf_{l-}]P_l(\cos\theta);\\ g(\cos\theta) &= \frac{1}{k}\sum (f_{l+}-f_{l-})P_l^1(\cos\theta), \end{aligned} \tag{11}$$

so that

$$d\sigma/d\Omega = |f|^2 + |g|^2 + 2\,\mathrm{Im}\,(fg^*)\,\mathbf{P}_i\mathbf{n}, \tag{12}$$

where $\mathbf{P}_i$ is the polarization of the initial state of the spin-$\frac{1}{2}$ particle. It is clear that the differential cross section for scattering without initial polarization contains only the first two terms of the expression (12):

$$(d\sigma/d\Omega)_0 = |f|^2 + |g|^2. \tag{13}$$

The total cross section can be obtained in accordance with the optical theorem and depends on $\mathrm{Im}\,f(0)$, since $g(0)=0$, and the total elastic cross section has the form

$$\sigma_{el}=\frac{4\pi}{k^2}\sum [(l+1)|f_{l+}|^2+l|f_{l-}|^2]. \tag{14}$$

The polarization vector of the final state depends on 16 physical quantities, but not all of these are independent. The expression for the polarization of the final state is

$$\begin{aligned} (d\sigma/d\Omega)\mathbf{P}_f &= 2\,\mathrm{Im}\,(fg^*)\,\mathbf{n} + 2\,\mathrm{Re}\,(fg^*)\,[\mathbf{P}_i\times\mathbf{n}]\\ &+ 2|g|^2(\mathbf{P}_i\mathbf{n})\,\mathbf{n} + (|f|^2-|g|^2)\,\mathbf{P}_i. \end{aligned} \tag{15}$$

We define the parameters $\alpha$, $\beta$, and $\gamma$ by

$$\left.\begin{aligned} &(d\sigma/d\Omega)_0\alpha = 2\,\mathrm{Im}\,(fg^*);\quad (d\sigma/d\Omega)_0\beta = 2\,\mathrm{Re}\,(fg^*);\\ &(d\sigma/d\Omega)_0\gamma = |f|^2-|g|^2. \end{aligned}\right\} \tag{16}$$

One can show that they are related by $\alpha^2+\beta^2+\gamma^2=1$. The polarization parameters $\beta$ and $\gamma$ can be related to the Wolfenstein polarization parameters $A$ and $R$.

If one is considering the scattering of particles whose spins are not equal to 0 or $\frac{1}{2}$, it is necessary to introduce a large number of independent amplitudes and express them in terms of different forms of differential cross sections and polarization parameters. For such cases, the Blatt–Biedenharn approach[1] (which is sometimes called the *ls* formalism) is most frequently used in practical phase-shift analysis. However, despite the basic simplicity of this method it has the shortcoming of leading to a large number of calculations as soon as the spin of the particles is different from $\frac{1}{2}$. A more elegant approach has been developed by Jacob and Wick.[2]

*Isotopic spin and Coulomb corrections.* In strong interactions the total isospin $I$ of the system is conserved, the strong interactions being independent of the charge, which is related to the isospin projection $I_3$. This has the consequence that each partial-wave amplitude in an expansion of the type (11) becomes a superposition of scattering amplitudes with definite values of the total isospin. For example, for pion–nucleon or pion–$^3$He ($I_\tau=1$, $I_{N(^3\mathrm{He})}=\frac{1}{2}$) scattering we obtain the following decompositions of the partial-wave amplitudes with respect to the states with total isospin $\frac{1}{2}$ and $\frac{3}{2}$:

$$\left.\begin{aligned} f_{l\pm}^{(-)} &= \frac{1}{3}(f_{l\pm}^{3/2}+2f_{l\pm}^{1/2}) && \text{for}\quad \pi^-p\to\pi^-p;\\ f_{l\pm}^{(+)} &= f_{l\pm}^{3/2} && \text{for}\quad \pi^+p\to\pi^+p;\\ f_{l\pm}^{\mathrm{CEX}} &= \frac{\sqrt{2}}{3}(f_{l\pm}^{3/2}-f_{l\pm}^{1/2}) && \text{for}\quad \pi^-p\to\pi^0n, \end{aligned}\right\} \tag{17}$$

and for $\pi\pi$ scattering the decompositions with respect to the states with total isospin 0, 1, and 2:

$$\left.\begin{aligned} f_l &= f_l^2, && \pi^\pm\pi^\pm\to\pi^\pm\pi^\pm;\\ f_l &= (f_l^2+f_l^1)/2, && \pi^\pm\pi^0\to\pi^\pm\pi^0;\\ f_l &= f_l^2/6+f_l^1/2+f_l^0/3, && \pi^+\pi^-\to\pi^+\pi^-;\\ f_l &= (f_l^2-f_l^0)/3 && \pi^+\pi^-\to\pi^0\pi^0. \end{aligned}\right\} \tag{18}$$

In the analysis of the scattering of two identical particles, care is required to take into account correctly the symmetry of the wave function. Thus, the summation over the orbital angular momentum $l$ for a system of two pions is over odd $l$ in the state with $I=1$ and over even $l$ in the states with $I=0$ and 2, i.e.,

$$f^I = \frac{1}{2k}\sum (1+(-1)^{I+l})(2l+1)f_l^I P_l(\cos\theta). \tag{19}$$

For $\pi^+\pi^-\to\pi^+\pi^-$ scattering we obtain

$$\begin{aligned} f(\theta) &= \frac{1}{k}\Big[\frac{1}{6}\sum_{0,2}(2l+1)f_l^2P_l(\cos\theta)\\ &+\frac{1}{2}\sum_{1,3}(2l+1)f_l^1P_l(\cos\theta)+\frac{1}{3}\sum_{0,2}(2l+1)f_l^0P_l(\cos\theta)\Big]. \end{aligned} \tag{20}$$

In the case of the scattering of charged particles, it is necessary to take into account not only the short-range nuclear potential but also the long-range Coulomb potential. In this case, the scattering amplitude can be written in the form

$$f(\theta) = f_c(\theta)+\frac{1}{k}\sum (2l+1)\exp(2i\sigma_l)f_lP_l(\cos\theta). \tag{21}$$

where $f_C(\theta)$ is the Coulomb amplitude, $\sigma_l$ is the Coulomb phase,

$$f_c(\theta) = -\frac{\eta}{2k\sin^2\theta/2}\exp[2i\sigma_0-i\eta\ln\sin^2\theta/2]; \tag{22}$$

$$\sigma_l = \arg\Gamma(l+1-i\eta); \tag{23}$$

$\eta$ is the Coulomb parameter, and

$$\eta = z_1z_2\alpha/v. \tag{24}$$

These relations are obtained in the nonrelativistic limit, and Eq. (21) holds for the collision of two point charges. Allowance for the fact that the particles have certain characteristic dimensions of their charge distributions and allowance for the correct relativistic kinematics lead to some corrections that become very important when the experimental data are sufficiently accurate. A more detailed study of nuclear scattering in the presence of the Coulomb potential requires the introduction of additional corrections to the nuclear phase shifts. A discussion of the so-called outer and inner Coulomb corrections can be found in Ref. 3, in which the Coulomb corrections for the phase-shift analysis of the pion–nucleon interaction at low energies[4] are considered. Together with the increase in the accuracy of the experimental data, and especially because of Sternheim and Hofstadter's idea[5] for determining the electromagnetic pion radius by investigating elastic $\pi^\pm$ $^4$He scattering, the problem of the Coulomb corrections has become very important in phase-shift analysis. This question has been much discussed in experimental and theoretical papers.[6-9] The Coulomb corrections necessary for phase-shift analysis include the

relativistic corrections of the Coulomb factors,[10-12] and it is also necessary to take into account the mass differences of particles belonging to an isotopic multiplet. These last corrections are particularly necessary in multichannel analysis.[13-17]

*Classical methods of phase-shift analysis.* Not only in elementary-particle physics but also in many branches of nuclear physics phase-shift analysis is widely used to obtain information on the scattering amplitude and, especially, to determine the energy dependence of the partial-wave amplitudes. In practice, the scattering amplitudes, which reproduce the physical observable quantities, are expanded with respect to the partial waves up to a given maximal $l$. The problem is to determine partial-wave amplitudes yielding optimal agreement with the experimental data. To implement this aim of phase-shift analysis, i.e., to determine the scattering amplitude as a complex function of two variables (the angle and energy), there exist two general methods—energy-independent and energy-dependent phase-shift analysis.

To carry out the first, it is necessary to have several complete sets of experimental data (differential and total cross sections, polarization parameters, etc.) at different but nearly equal energies. Then for each given energy one finds sets of parameters for which the functional

$$\chi^2=\sum_{i=1}^{N}\left[\frac{E_i-F\left(f\left(x_i,\sqrt{s}\right)\right)}{\Delta\left(E\right)_i}\right]^2 \tag{25}$$

has minimal value. Here, $E_i$ is the value of the experimental quantity with error $\Delta(E)_i$ at the point $i$ with coordinate $x_i$ at fixed energy $\sqrt{s}$, $F[f(x_i,\sqrt{s})]$ is the expression for $E$ in terms of the scattering amplitude $f(x,\sqrt{s})$ at the point $x_i$, and $N$ is the number of experimental data. Generally, one obtains several sets of partial-wave amplitudes at each energy (see Sec. 2). It is assumed that they include the sought, "physical" set. A number of methods are used to find it and determine the energy dependence of the partial-wave amplitudes. The most common of them is the *shortest-path* method. One considers all possible trajectories of the partial-wave amplitudes in the complex plane that join the solutions at different energies. The shortest path is defined as the minimum of a functional which depends on these curves.[18] As an example, for the "path function" one can take the following expression, which is used in Ref. 19:

$$d_{m,n}=\sum_{k=k_0}^{k_N-1}\left\{\sum_{l,j,I}W_{l,j}\left|f_{l,j,I}\left(m,\ k+1\right)-f_{l,j,I}\left(n,\ k\right)\right|^2\right\}^{1/2}, \tag{26}$$

where $W_{l,j}$ is a weight, equal to $j+\frac{1}{2}$, or, emphasizing the effect of the centrifugal barrier, $(l+j+\frac{1}{2})/2$. The expression (26) is the sum of the distances for the solutions $m$ and $n$ obtained as a result of the phase-shift analysis. The summation is over all energies and over all partial waves (over $k$ and $l,j,I$). The unique solution and, hence, the unique energy dependence is established by seeking the minimum of the expression (26) among all possible combinations of the solutions $m$ and $n$:

$$d_{\min}=\min_{m,n}\left(d_{m,n}\right). \tag{27}$$

Another way of obtaining the shortest path is based on the mathematical method of determining the minimum of a functional in the process of fitting experimental data.[10] For this, one begins the phase-shift analysis at a relatively low energy, where there exists only one solution, and one then continues it at the following energy, using the previously obtained parameters as initial parameters for the following energy. In this way, one ensures that the values of the parameters for the new solution are "near" the previous values in the parameter space. In this sense, we could be said to construct the shortest path on the $\chi^2$ hypersurface, whereas earlier [see (26) and (27)] we sought the shortest path in a space of $2L$ dimensions, where $L$ is the number of partial-wave amplitudes.

In other more complicated methods, continuity of the parameters with respect to the energy for the different solutions is ensured by means of dispersion relations for the partial-wave amplitudes[21,22] or by analyzing the trajectories of the zeros of the scattering amplitude.[23,24] This last method is very effective for determining a unique solution even when the corresponding partial-wave amplitudes differ little.[25,26]

The characteristic feature of the second type of phase-shift analysis is that from the beginning a definite energy dependence of the partial-wave amplitudes (or of the phase shifts and inelasticity parameters) is specified. One then carries out a simultaneous fitting of all the existing data at all energies. Energy-dependent phase-shift analysis works well at low energies, where the energy dependence of the partial-wave amplitudes is determined by a few terms in the effective-range expansion. At higher energies, this expansion ceases to be valid. Therefore, for the energy dependence one either uses some "neutral" parametrization or chooses it in accordance with a model. In the first group, we have polynomial parametrization for the phase shifts and the inelasticity parameters.[7,11] In this case, the parameters of resonances are determined not directly from the experimental data but from subsequent analysis of the energy dependences $\delta_l(E)$ and $\eta_l(E)$. Another example is provided by a parametrization of the type[12,27]

$$\cot\delta_l=\frac{1}{k^{2l+1}}\sum_{0}^{N}\alpha_n T^n, \tag{28}$$

where $\alpha_n$ are complex numbers. This parametrization preserves unitarity of the partial-wave amplitude and has the correct behavior at the threshold.

Because of the special properties of Padé approximants, they can be used to parametrize the energy dependence of the partial-wave $S$ matrix. This reduces the number of free parameters and results in a more correct allowance for the analytic properties of the partial-wave amplitudes.

For the second, model-dependent method of energy parametrization one typically introduces terms of Breit-Wigner type and a nonresonance background.[28] One also uses dispersion relations for the partial-wave amplitudes with energy-dependent discontinuities across the

unphysical cuts. In all these methods, it is assumed that the partial-wave amplitudes with large $l$ are zero. Information about the high partial waves is usually taken from some model. In this case, the scattering amplitude is written as

$$f(x, E) = [f^{(M)}(x, E) - \sum f_l^{(M)}(E) P_l(x)] + \sum_{l=0}^{L} f_l(E) P_l(x),$$

where $f^{(M)}$ is the contribution of the model (for example, the OBEP model for the nucleon–nucleon interaction). The energy dependence of the amplitudes with small $l$ is determined by the sum of the model term and a phenomenological expression: $f_l(E) = f_l^{(M)} + F(E)$. Of course, the introduction of models of the interaction dynamics reduces the subjectivity in the choice of the functions for the energy dependence, but the results of the phase-shift analysis become model dependent.

In both forms of phase-shift analysis, the same general criteria (see Sec. 3) are used to find the resonances. The parameters of the resonances and of the nonresonance background are determined either directly from the experimental data (in the case of energy-dependent analysis) or by an additional analysis of the energy dependence of the partial-wave amplitudes (in the case of energy-independent analysis). An additional analysis is also needed to determine the parameters of the poles that characterize the resonances.

*Modern methods of phase-shift analysis.* The main task of phenomenological analysis is, using only the most general theoretical principles such as analyticity, crossing symmetry, and unitarity of the scattering amplitude, to extract the maximal information from the experimental data with the minimal number of parameters. These requirements are met by the method of optimal conformal mappings, in which the polynomial expansion converges very rapidly.[30,31] Hence the name: ACE (accelerated-convergence expansion) method. To check and estimate the error which arises from the use of a finite number of terms in the expansion of the scattering amplitude, Cutkosky's test function is employed.[32]

The most stable and rapidly converging polynomial expansion for the scattering amplitude is obtained by a conformal mapping of the complex plane of $x = \cos\theta$ into an ellipse with foci $\pm 1$. The physical region $x \in (-1, +1)$, is carried into the same interval $(-1, +1)$, and the cuts in the $x$ plane are mapped onto the boundary of the ellipse. The scattering amplitude

$$kf(x) = \sum_{l=0}^{L} a_l P_l(x) \tag{29}$$

is expanded in the $z = z(x)$ plane as follows:

$$kf(z(x)) = \sum_{l=0}^{L} a_l P_l(z(x)),$$

the inequality $N \leq L$ always holding here. However, the inequality becomes more pronounced when $L$ is large. The partial-wave amplitudes $a_l$ in (30) are determined from the expression

$$a_l = \sum_{n,j}^{N} C_{l,n} C_{j,n}^{-1} a_j, \tag{30}$$

where

$$C_{l,n} = \frac{2l+1}{2} \int_{-1}^{+1} P_l(x) P_n(z(x))\, dx.$$

It can be seen from (30) that the partial-wave amplitudes are not independent, since the higher partial waves are determined by a linear combination of the lower ones. In principle, this fact helps in resolving the ambiguity of phase-shift analysis.[8,33]

The use of analyticity of the scattering amplitudes significantly improves the convergence in the ACE method. If expressions are known from other sources for the contributions of some singularities (for example, the contributions from diagrams with exchange of particles), they can be introduced exactly into a phase-shift analysis of the ACE type, improving the convergence of the expansion even more. However, the model dependence of the analysis is then greater. The ACE method was used in the phase-shift analysis of processes such as $K^+p$ (Refs. 34–37), $\pi N$ (Refs. 38 and 39), or $\pi^4$He scattering.[33]

In the following section, it will be shown that it is convenient to represent the different sets of solutions in terms of the positions of the zeros of the scattering amplitude in the complex plane of $x = \cos\theta$. In addition, study of the behavior of the trajectories of the zeros of the amplitude as a function of the energy can give information about the dynamics of the process,[40] for example, information about the existence of resonances.[41] These circumstances strongly stimulated the development of the analysis of the zeros of the scattering amplitude. The advantages of such a method are that in it one does not use any energy parametrization and one can rapidly obtain all possible solutions. Such an analysis was made in Ref. 25 for the $K^-p \to \Lambda\pi^0$ interaction.

Another modern method is phase-shift analysis at fixed momentum transfer. Essentially, it is an analysis of the amplitude in which one requires not only agreement with experiment and fulfillment of the unitarity condition for the scattering amplitude but also analyticity in $s$ and $t$. The "at fixed transfer" is explained by the fact that the analyticity of the scattering amplitude is checked by means of dispersion relations at fixed $t$ or $u$,[42] or by a series expansion of the amplitude, also at fixed $t$ or $u$.[43–47] In this method of phase-shift analysis, one uses simultaneously all sets of experimental data with respect to $s$ and $t$ (as in energy-dependent analysis). The behavior at high energies is taken from theoretical models.

In both variants of the phase-shift analysis at fixed momentum transfer the experimental information must be augmented by model-dependent quantities to ensure the correct analytic structure of the amplitude. In the first variant, the total amplitude is "forced" to be analytic by means of dispersion relations at fixed $t$. In the second variant, this is achieved by means of such a representation of only one part of the total amplitude—the part that is analytic in the plane with one cut and has the correct behavior at the threshold and at infinity. The intermediate result for both approaches is the same, i.e., one obtains a certain finite set of amplitudes as functions of $s$ for different values of $\cos\theta$. The

next step is to obtain estimates of the partial waves. This is done as in ordinary phase-shift analysis by imposing additional constraints obtained from dispersion relations for the scattering amplitude at fixed $t$.

Knowing the special properties of the Padé approximants for representing analytic functions, one can propose their use instead of a polynomial expansion of the amplitude. It is to be expected that the number of parameters then needed to describe the amplitude will be reduced.

## 2. AMBIGUITIES IN PHASE-SHIFT ANALYSIS

*Classical ambiguities.* A characteristic feature of phase-shift analysis is its ambiguity, i.e., the set of parameters found in some manner for the partial-wave amplitude is not unique. There always exist transformations that give another set of parameters and leave the calculated values of the scattering cross sections unchanged. The study of these transformations and the finding of methods to reduce the number of acceptable solutions is an important problem in phase-shift analysis.

The simplest type of ambiguity is the so-called *trivial ambiguity*:

$$f(\theta) \to -f^*(\theta) \quad (\mathrm{R}), \tag{31}$$

i.e., $f_l \to -f_l^*$. This transformation does not change the calculated cross sections (5) and merely changes the sign of the real part of the forward scattering amplitude. For the scattering of particles with spin 0 and $\frac{1}{2}$, the trivial ambiguity is

$$\left.\begin{aligned} &f(\theta) \to -f^*(\theta);\\ &g(\theta) \to -g^*(\theta);\\ &f_{l\pm} \to -f_{l\pm}^*. \end{aligned}\right\} \tag{32}$$

However, as can be seen from (16), this transformation changes the sign of the polarization $P$.

Another classical ambiguity for the scattering of particles with spins 0 and $\frac{1}{2}$ is the Minami transformation[48]

$$f_{l\pm} \to f_{(l+1)\mp} \quad (\mathrm{M}) \tag{33}$$

or

$$f(\theta) \to f(\theta)\cos\theta - g(\theta)\sin\theta;$$
$$g(\theta) \to -f(\theta)\sin\theta - g(\theta)\cos\theta,$$

which changes only the sign of the polarization $P$. At the same time, the polarization parameters $\beta$ and $\gamma$ in (16) become linear combinations of their initial values. In contrast to the transformations (R) and (M) or combinations of them, which preserve the unitarity of the partial-wave amplitudes, the Fermi–Yang transformation (or the Yang ambiguity)[49] can lead to nonunitary partial-wave amplitudes:

$$f_{l+} \to \frac{1}{2l+1}[f_{l+} + 2l f_{l-}]; \quad f_{l-} \to \frac{1}{2l+1}[2(l+1)f_{l+} - f_{l-}] \quad (\mathrm{Y}) \tag{34}$$

or

$$f(\theta) \to f(\theta); \quad g(\theta) \to -g(\theta).$$

The transformation (Y) leaves unchanged the differential cross section, the parameter $\gamma$, and $\mathrm{Re}\,f(0)$, changing only the sign of the polarization and the parameter $\beta$. It is hardly probable that the transformation (Y) gives unitary solutions in the region of energies up to the threshold of the inelastic processes.

*Ambiguities in the region of energies up to the threshold of inelastic processes.* The first example of nontrivial ambiguity in this region of energies, which initiated the intensive study of the problem of ambiguities in phase-shift analysis, was given by Crichton in 1966.[50] To discuss necessary and sufficient conditions for the existence and uniqueness of the scattering amplitude, it is necessary to use unitarity relations in integral form. In the purely elastic case ($\eta_l = 1$), we have

$$\mathrm{Im}\,F(x_{12}) = \frac{1}{4\pi}\int d\Omega_3 F^*(x_{13})\,F(x_{23}), \tag{35}$$

where $F(x_{12})$ is the amplitude for scattering from the direction 1 into the direction 2. The modulus of the scattering amplitude is determined from the differential cross section, and then, using (35) for

$$F(x) = |F(x)|\exp(i\varphi(x)), \tag{36}$$

we obtain a nonlinear integral equation for the phase of the amplitude:

$$\sin\varphi(x_{12}) = \frac{1}{4\pi}\Big[\int d\Omega_3 |F(x_{13})|\,|F(x_{32})| \times \cos[\varphi(x_{13}) - \varphi(x_{32})]/|F(x_{12})|\Big]. \tag{37}$$

A sufficient condition for the existence of a solution of this equation was determined in Ref. 51,

$$\sin\mu = \max\int \frac{d\Omega_3}{4\pi}\frac{|F(x_{13})|\,|F(x_{32})|}{|F(x_{12})|} < 1, \tag{38}$$

where the maximum is taken over all possible values of $x_{12} = \cos\theta_{12}$. From the existence of a solution for $\varphi(x)$ there follows the existence of a unitary solution of the phase-shift analysis in the elastic region.

Uniqueness of the solution is ensured by the stronger condition

$$\sin\mu < 0{,}79. \tag{39}$$

Because of this, the phase-shift analysis of elastic scattering has a single unique unitary solution for the scattering amplitude if the condition (39) is satisfied. However, if this is not the case one can say nothing about the uniqueness of the solution. If $\sin\mu < 1$, there exists only one solution.[52]

In particular, from the condition (38) there follows automatically a restriction on the values of the phase shifts for all partial-wave amplitudes except the amplitude in the $S$ wave,[52,53] i.e., $|\delta_l| < \pi/6$. This has the consequence that the condition (38) is never satisfied for resonance amplitudes for which $l \neq 0$, i.e., one must be careful with elastic resonances.

For amplitudes that differ by a polynomial, it was shown in Ref. 54 that there exists a condition for uniqueness of the phase-shift analysis which is fundamentally different from (38) or (39). It takes the form that if $k^2\sigma_{\mathrm{tot}}/4\pi < 1.388$, then the scattering amplitude can be determined uniquely.

If the scattering amplitudes have normal behavior at the threshold, then at energies sufficiently close to the threshold the condition (38) is always satisfied. Since $\delta_l \sim a_l k^{2l+1}$ near the threshold, it can be assumed with confidence that there exists a region in which the $S$-

wave amplitude is dominant and the amplitudes with $l \neq 0$ can be ignored. Then $\sin\mu < 1$ and the solution is unique. This fact has important consequences for phase-shift analysis. Since the scattering amplitude is an analytic function of the energy, the unique solution at the threshold can be continued to the entire energy region. Hence, there exists a unique unitary solution for the phase-shift analysis, the so-called *physical solution* in the complete energy region.

*Discrete ambiguities.* If the scattering amplitude is written in the form[55]

$$F(x) = F(1) \prod_{i=1}^{L} \frac{x - x_i}{1 - x_i}, \qquad x = \cos\theta. \tag{40}$$

where $x_i$ are the zeros of the amplitude, it can be immediately seen that when $x_i$ is replaced by $x_i^*$ the differential and total cross sections remain unaltered:

$$\left.\begin{aligned} k^2 \frac{d\sigma}{d\Omega} &= |F(1)|^2 \prod \frac{(x - x_i)(x - x_i^*)}{(1 - x_i)(1 - x_i^*)}; \\ \frac{k^2}{4\pi} \sigma_{\text{tot}} &= \operatorname{Im} F(1). \end{aligned}\right\} \tag{41}$$

Thus, one can obtain $2L$ solutions, not counting the trivial ambiguity. The same solutions are obtained by projecting onto the partial waves the scattering amplitude with all possible $x_i \rightleftharpoons x_i^*$ combinations. From the $2L$ solutions, one can eliminate the ones that do not satisfy the unitarity condition. In the inelastic region, i.e., where the inelastic channels are open, it is impossible to determine the number of unitary solutions in advance.

The discrete ambiguities were analyzed for the scattering of particles with spins 0 and $\frac{1}{2}$ by Gersten[55] and Barrelet.[23] In the formalism of Gersten, one introduces the variable $t = \tan(\theta/2)$, the differential cross section and the polarization being expressed in terms of the function $G(t)$:

$$G(t) = \frac{f(0)}{[1 + t^2]^L} \prod_{i=1}^{2L} \frac{t_i - t}{t_i}, \tag{42}$$

where $t_i$ are the zeros of the function $G(t)$ in the complex plane of $\tan(\theta/2)$. Thus

$$\left.\begin{aligned} d\sigma/d\Omega &= (|G(t)|^2 + |G(-t)|^2)/2; \\ P\, d\sigma/d\Omega &= (|G(t)|^2 - |G(-t)|^2)/2. \end{aligned}\right\} \tag{43}$$

As in the case of the scattering of spinless particles, the replacement of one or several of the zeros $t_i$ by their complex conjugates $t_i^*$ in (42) leads to different amplitudes but leaves the differential and the total cross section and the polarization unchanged.

The classical examples of ambiguities are obtained by the substitution $G(t) \to -G^*(-t)$, which gives the trivial ambiguity (R), and the substitution $G(t) \to 1 - it/1 + itG(-t)$, giving the Minami ambiguity (M). Using this formalism, Berends[56] verified the uniqueness of the solution in the phase-shift analysis of elastic pion–nucleon scattering. The final conclusion is that such a solution exists only up to $\sqrt{s} = 2.025$ GeV.

In the formalism of Barrelet, one uses a conformal mapping of the complex $x = \cos\theta$ plane onto the new $z$ plane:

$$x \to z = x + (x^2 - 1)^{1/2}, \tag{44}$$

which carries the Lehmann–Martin ellipse—the analyticity ellipse of the scattering amplitude—into the ring $r(z_0)$, where $z_0$ is the image of the nearest singularity in the $x$ plane. The amplitudes $f(x)$ and $g(x) = (1 - x^2)^{1/2}\tilde{g}(x)$ are analytic in the complex $x$ plane, i.e., in the ellipse determined by the position of the nearest singularity and with foci $\pm 1$. Therefore, the function

$$\tilde{F}(x) = f(x) + (x^2 - 1)^{1/2}\, \tilde{g}(x),$$

is also analytic in the given region with a cut between $x = 1$ and $x = -1$. Above and below this cut, the function $\tilde{F}(x)$ has the value $f(x) \pm ig(x)$. The relation (44) transforms the physical interval $(-1 \leq x \leq 1)$ into the unit circle $r(1)$, so that in the new plane the upper edge of the cut $-1 \leq x \leq 1$ is carried into $z = \exp(i\theta)$ with $0 \leq \theta \leq \pi$, and the lower edge into $z = \exp(i\theta)$ with $-\pi \leq \theta \leq 0$. Hence, the function is analytic in the ring $r(z_0)$. The function $\varphi(z) = F(z)F^*(1/z^*)$ is also analytic in $r(z_0)$. In the unit circle, it has the values

$$\xi(\exp[+i\theta]) = \frac{d\sigma}{d\Omega}(x)\,[1 \mp P(x)]; \quad 0 \leqslant \theta \leqslant \pi. \tag{45}$$

As in Gersten's method, one obtains a compact representation of the physical observables by means of a single function. However, this function has simpler analytic properties, and its expansion in partial-wave amplitudes is readily found:

$$kF(z) = \sum_l \left(j_{l\pm} + \frac{1}{2}\right) F_{l\pm} P_{l\pm}(z), \quad j_{l\pm} = l \pm 1/2,$$

where $P_{l\pm}(z)$ are known as Barrelet polynomials:

$$\left.\begin{aligned} P_{l\pm}(z) &= P_l(z) \pm \frac{2(z^2-1)^{1/2}}{2l+1 \pm 1} P_l'(z); \\ Q_{l\pm}(z) &= Q_l(z) \mp \frac{2(z^2-1)^{1/2}}{2l+1 \pm 1} Q_l'(z); \\ F_{l\pm} &= \frac{1}{8\pi i k} \int dz\,(1 - z^{-2})\, Q_{l\pm} F(z). \end{aligned}\right\} \tag{46}$$

The generalized Minami ambiguity is obtained by the substitution

$$F(z) \to \frac{1}{z} F^*(1/z^*), \quad \text{(M)}$$

and the Yang ambiguity and all combinations of (M), (Y), and (R) arise under the substitution $F(z) \to z^n F(z)$, where $n = \pm 1, \pm 2$. To analyze the discrete ambiguities it is convenient to represent the amplitude $F(z)$ in the form

$$F(z) = \frac{F(1)}{z^L} \prod_{i}^{N} \frac{z - z_i}{1 - z_i}, \tag{47}$$

where $N = L + j_{\max} - \frac{1}{2}$. As is shown in Refs. 57–60, the phase-shift analysis has at most two unitary discrete solutions in the case of the scattering of spinless particles below the threshold. At energies above the threshold of the inelastic channel the restrictions imposed by the unitarity condition become weaker, since instead of the strict equality $\eta_l = 1$ we have in this region of energies only the inequality $0 \leq \eta_l < 1$. This gives rise to the so-called continuous ambiguity. It turns out that the number of unitary solutions obtained by taking the complex conjugate of the zeros of the amplitude increases with increasing number of partial waves. Such behavior is characteristic of, for example, $^4$He–$^4$He, $^3$He–$^4$He, and $\pi^4$He elastic scattering.[62]

*Continuous ambiguity.* Continuous ambiguity of the phase-shift analysis can be obtained by the transformation

$$F'(x) \to F(x) \exp[i\varphi(x)], \quad (48)$$

where $\varphi(x)$ is a phase which depends on the angle. If $\varphi(x=1)=0$, then the transformation (48) leaves the differential and total cross sections unchanged. It is then possible to have an infinite number of partial-wave expansions of the scattering amplitude, which arise from the infinite number of possible forms of the angular dependence of the phase $\varphi(x)$. A natural additional condition is the fulfillment of unitarity for any possible transformation (48). Therefore, the continuous ambiguity corresponds to a small zone (or zones) on the Argand diagram, for which each point is related to a definite point of such a zone on a different Argand diagram. However, it is clear from the relation (48) that the continuous ambiguity occurs only in the case when an infinite number of waves is taken in the partial-wave expansion of the scattering amplitude. Since the partial-wave amplitudes decrease exponentially with increasing $l$, the zones of continuous ambiguity decrease, tending to a point when $l$ tends to infinity. For large values of $l$, the ambiguity in the partial-wave amplitudes becomes appreciable, and therefore in practical phase-shift analysis there is no point in the direct determination of the amplitudes from the experimental data.

The first paper devoted to the study of continuous ambiguity was Ref. 63. This direction was subsequently developed strongly and various specific cases were analyzed in Refs. 59, 61, and 64–66.

*Methods of solving the ambiguity problem.* The aim of phase-shift analysis is to determine a scattering amplitude that describes the experimental data and satisfies the general conditions of Lorentz invariance, unitarity, and analyticity. In other words, this means that it is necessary to find functions of two variables $s$ and $t$ that can be expanded with respect to unitary partial-wave amplitudes. A unique scattering amplitude makes it possible to determine correctly the parameters of resonances. This is one of the important results of phase-shift analysis. It is well known that if an analytic function is given in some interval of the analyticity domain, then it is known in its complete domain of definition. This property of analytic functions makes it possible to find a unique solution for the scattering amplitude by using the circumstance that near the elastic threshold there exists a unique solution. The analytic continuation of this solution from the region near the threshold, where the Martin condition $\sin\mu < 0.79$ can be satisfied, ensures the existence of the so-called physical solution at all energies.

From these considerations in phase-shift analysis comes the idea of continuous, or smooth, continuation with respect to the energy of the partial-wave amplitudes as a method of obtaining a unique solution. In the set of solutions found at a given energy, the one closest to the physical solution will be the one that together with the corresponding solutions at other energies determines a smooth function of the energy, i.e., ensures the shortest path between the energies. Besides this method, which is characteristic for energy-independent phase-shift analysis, one also uses the method of specific parametrizations of the partial-wave amplitudes (energy-dependent analysis), and also several variants of dispersion relations, which are used in both types of phase-shift analysis.

To reduce the number of ambiguities, it is helpful to use the ACE method or Padé approximants, since they are sensitive to the analytic structure of the scattering amplitude. For this purpose, one can also take the higher partial waves from a model.

In a certain sense this is an indirect consequence of Burkhardt's theorem[67] on the uniqueness of phase-shift analysis. This theorem states that knowledge of the differential cross section in the physical region $-1 \leqslant x \leqslant 1$ and of the analytic structure of the scattering amplitude is insufficient for finding a unique representation of the amplitude. However, by determining the discontinuity of the amplitude across the final segment of the cut of the complex $x$ plane, one can achieve uniqueness. But it is difficult to use this result in practice to obtain a unique solution.

## 3. CRITERIA FOR THE EXISTENCE OF A RESONANCE

*Resonance states.* Resonance is one of the most interesting phenomena in scattering experiments. The first indication of the production of a resonance state of two colliding particles (nuclei or elementary particles) is the appearance of a maximum in the interaction cross section. The position and width of the peak in the cross section are related to the mass and lifetime of the resonance. Like bound and virtual states, resonances are associated with zeros of the Jost function. We write down the solution of the Schrödinger equation for a zero of the Jost function ($k_R = k_0 + ik_1$):

$$\Psi(r, t) = \Phi(r) \exp(-iEt) \sim \exp(ik_0 r) \exp(-k_1 r) \exp(iE_R t).$$

We obtain an expression for the number of particles leaving a sphere of radius $r = A$ per unit time:

$$N \sim \exp(2k_1 A) \exp(2k_0 k_1 t).$$

This expression will describe a decay if the zero of the Jost function is in the negative half-plane ($k_1 < 0$ and $k_0 > 0$ because scattering processes are being studied). In the region around this point (for not very large $k_1$) we can write $f(k)$ as

$$f(k) \approx \left(\frac{df}{dk}\right)_{k_R} (k - k_R),$$

from which it follows that the phase of the Jost function, i.e., the ordinary phase shift in the phase-shift analysis, is determined by the expression

$$\delta(k) \approx -\arg(df/dk)_R - \arg(k - k_R) \equiv \delta_F + \delta_R.$$

Here, $\delta_F$ is the background phase shift, and $\delta_R$ is the resonance phase shift. Using the expression for the $S$ matrix in terms of the Jost function

$$S(k) = f(-k)/f(k)$$

and the analytic properties of the Jost function [in the present case, the property $f(-k) = f^*(k)$], we find that

$$S(k) = S_F(k) \frac{k - k_R^*}{k - k_R} = S_F S_R. \quad (49)$$

It can be seen from this that to every pole of the $S$ ma-

trix in the lower half-plane ($\mathrm{Im}k<0$) there corresponds a zero in the upper half-plane situated symmetrically with respect to the pole. This phenomenon is called *unitary unipolar approximation of the resonance*. It almost completely determines the behavior of the $S$ matrix when $k_0$ is sufficiently large or, in other words, when $\mathrm{Im}k_R=k_1\ll 1/r_0$, where $r$ is the range of the nuclear forces.

Further, from the symmetry property of the $S$ matrix $S(k)S^*(k^*)=1$, $S(k)S(-k)=1$ it follows that $S(k)=S^*(-k^*)\sim(k+k_R)/(k+k_R^*)$. This corresponds to a zero–pole pair at negative values of the momentum (for $k_0<0$). i.e. far from the physical region.

Both pole-zero pairs together with the background give the so-called bipolar approximation of the resonance in the complex $k$ plane:

$$S(k)=S_F(k)\frac{(k-k_R^*)(k+k_R)}{(k-k_R)(k+k_R^*)}=S_FS_R. \tag{50}$$

In formation reactions, i.e.,

$$A+B\to R\to A'+B',$$

where $A,B,A',B'$ are elementary particles or nuclei, and $R$ is a resonance (or compound nucleus), the mass of the latter is determined by the total c.m.s. energy. Because of this, the partial-wave amplitudes or elements of the $S$ matrix are usually expressed as functions of the energy. Because of the connection between $E$ and $k$ it is found that the $S$ matrix in the $E$ plane has a square-root branch point at $E=0$. To each resonance there corresponds a zero–pole pair of the $S$ matrix in the complex $k$ plane or two zero–pole pairs in the $E$ plane. In the latter case, the zeros are on the first sheet and the poles on the second. Bound states correspond to poles on the first sheet and zeros on the second, both points being situated on the real axis below the threshold of the elastic channel. Figures 1a and 1b show the singularities of the $S$ matrix in the complex $E$ and $k$ planes.

The polar approximation of the $S$ matrix in the complex $E$ plane is written in the form

$$S(E)=(E_R^*-E)/(E_R-E)=(E_0-E+i\Gamma/2)/(E_0-E-i\Gamma/2) \tag{51}$$

(here, for simplicity a zero background, $\delta_F=0$, is taken). In contrast to the complex $k$ plane, in the complex $E$ plane one can write down only a unipolar approximation.

The real part of the position of the pole $E_0$ is related to the mass of the resonance and is the total c.m.s. energy, and the half-width $\Gamma/2$ corresponds to the distance of the pole to the real axis. This form of the $S$ matrix is called the *Breit–Wigner form*.

*Consequence of the Breit–Wigner approximation.* In the Breit–Wigner approximation,[68] the partial-wave amplitude is written in the form

$$f_l=(\Gamma/2)/(E_0-E-i\Gamma/2)=1/(\cot\delta-i). \tag{52}$$

The partial-wave cross section corresponding to the resonance partial wave has a well-defined maximum at $E=E_0$:

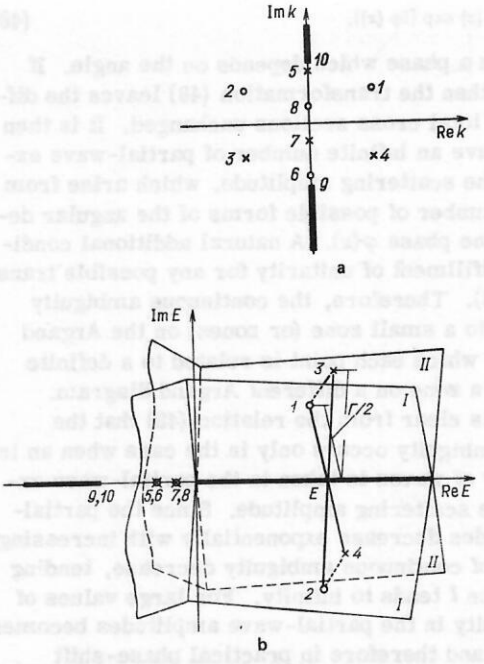


FIG. 1. Schematic representation of analyticity domains of the matrices $S_l(k)$ and $S_l(E)$. The Riemann sheets are denoted by I and II, on which the zeros (open circles) and poles (crosses) are indicated. The positions 1–4 correspond to a resonance state; 5 and 6, to a bound state; 7 and 8, to a virtual state; and 9 and 10, to the start of a dynamical cut.

$$\sigma_{l_R}=\frac{4\pi}{k^2}\frac{\Gamma^2/4}{(E_0-E)^2+\Gamma^2/4}(2l_R+1). \tag{53}$$

When one studies a collision between two particles at near-resonance energies and the remaining partial waves are relatively small, only the resonance partial wave makes a significant contribution to the total cross section. Therefore, a clear maximum is also observed in the experimental cross section. This maximum is approximately symmetric and, as we have already said, is the first indication of the existence of a resonance.

The energy dependence of the real and imaginary parts of the amplitude for a pure backgroundless elastic resonance is characterized by a zero of $\mathrm{Re}f_l$ and a maximum of $\mathrm{Im}f_l$ at the energy corresponding to the mass of the resonance:

$$\left.\begin{aligned}\mathrm{Re}\,f_l&=\frac{\Gamma/2}{(E_0-E)^2+\Gamma^2/4}(E_0-E);\\ \mathrm{Im}\,f_l&=\frac{\Gamma/2}{(E_0-E)^2+\Gamma^2/4}.\end{aligned}\right\} \tag{54}$$

We denote the energy-dependent factor in (52) by $\varepsilon=\cot\delta$. It can be seen directly that the partial-wave amplitude in the complex plane, i.e., on the Argand diagram, describes a circle with center at $(0,\frac{1}{2})$ and radius $\frac{1}{2}$, the so-called *unitarity circle* (Fig. 2):

$$x^2+(y^2-1/2)^2=1/4,$$

where $x=\mathrm{Re}f_l=\varepsilon/(\varepsilon^2+1)$, $y=\mathrm{Im}f_l=1/(\varepsilon^2+1)$.

For the resonance amplitude, the phase shift corresponding to orbital angular momentum $l_R$ is

$$\delta_{l_R}(k)=\mathrm{arctg}\{\Gamma/[2(E_0-E)]\}. \tag{55}$$

At the resonance, when $E=E_0$, the phase shift passes

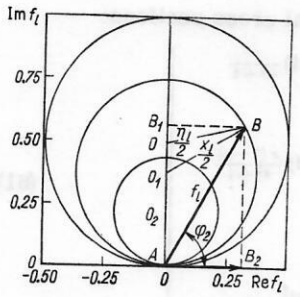


FIG. 2. Argand diagram for the partial-wave elastic-scattering amplitude in the presence of open inelastic channels and under the assumption of constant partial-wave elasticity, $x_l$ = const.

through $\pi/2$ and the effective cross section reaches its maximal value. Phase shifts $\delta_l = 45$ and 135° are obtained at the energies $E = E_0 \mp \Gamma/2$ where the total cross section $\mathrm{Im} f_l$ reaches half the maximum, and $\mathrm{Re} f_l$ takes the maximal and minimal values permitted by the unitarity circle.

Another test for the determination of resonances used in phase-shift analysis is to study the rate of change of the partial-wave amplitudes on the Argand diagram.

Suppose the resonance is described as before by a simple form of the Breit–Wigner type. Then the rate of change of the amplitude, i.e., the derivative with respect to the energy

$$v = |df_l(k)/d\varepsilon| = 1/(\varepsilon^2 + 1),$$

has a clear maximum at $E = E_0$. In practical phase-shift analysis the rate of change of the amplitude at the point $i$ of the Argand diagram is calculated by means of the expression

$$v(i) = \frac{1}{2}\left\{\frac{|f(i+1) - f(i)|}{E_{i+1} - E_i} + \frac{|f(i) - f(i-1)|}{E_i - E_{i-1}}\right\}.$$

Direct application of the tests described above permits easy determination of the parameters of a resonance only in the very rare case of purely elastic and backgroundless resonances. Of this kind is the noted resonance $\Delta_{33}$ in elastic $\pi p$ scattering, which is shown in Fig. 3. In the general case, it is necessary to take into account the influence of decays into other channels and the nonresonance background.

*Nonresonance scattering in the inelastic case.* In the inelastic case, i.e., at energies above the threshold of the inelastic channel, the intermediate state which is formed may decay into the original state or into one of the open channels. The ratios of the probabilities of decay into the different final states are characteristic properties of the resonance. For example,[69] the resonance $\Lambda(1520)$ observed in $K^-p$ scattering decays with probability 45% into the elastic $K^-p$ channel, with probability 45% into the $\pi\Sigma$ channel, and with probability 10% into $\pi\pi\Lambda$.

If we ignore the influence of the background, then it follows from the constancy of the ratios of the decay probabilities into the different channels that the elasticities $x_l = \sigma_{\mathrm{el},l}/\sigma_{\mathrm{tot},l}$ do not depend on the energy. This is observed experimentally. The curves of constant

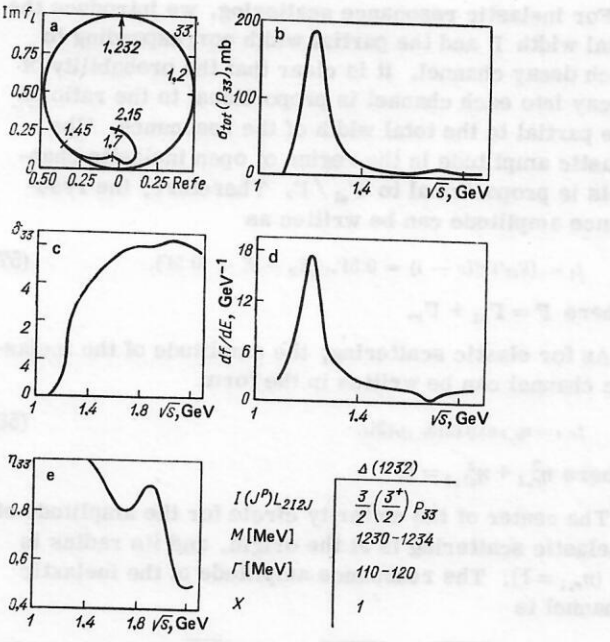


FIG. 3. Partial-wave amplitude $P_{33}$ for elastic scattering in the region of the first baryon resonance $\Delta_{33}(1232)$. a) The Argand diagram; b) the contribution of the partial-wave amplitude $P_{33}$ to the total cross section; c), d), and e) the energy dependences of the phase shift, the rate of change of the partial-wave amplitude, and the inelasticity parameter $\eta_{33}$, respectively.

elasticity in the Argand diagrams are circles of diameter $x_l$:

$$\mathrm{Re} f_l^2 + (\mathrm{Im} f_l - x_l/2)^2 = x_l^2/4. \qquad (56)$$

These circles touch the real axis at the origin. At the resonance energy $E_R$, as in the elastic case, the phase shift passes through $\pi/2$ for $x_l > \frac{1}{2}$ or 0 for $x_l < \frac{1}{2}$. The energy dependence of the parameter has a minimum at $E = E_R$. The connection between $\eta$, $\delta$, and the partial-wave amplitude on the Argand diagram is shown in Fig. 4.

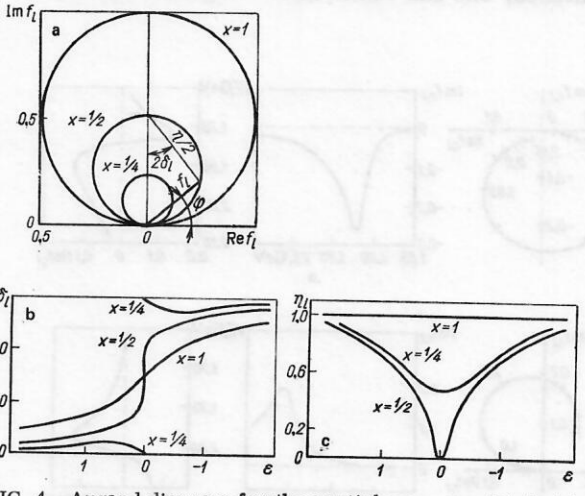


FIG. 4. Argand diagram for the partial-wave amplitude (a), energy dependence of the phase shift (b), and energy dependence of the inelasticity (c) for three values of the partial-wave elasticities $x$.

For inelastic resonance scattering, we introduce the total width $\Gamma$ and the partial width corresponding to each decay channel. It is clear that the probability of decay into each channel is proportional to the ratio of the partial to the total width of the resonance. The elastic amplitude in the region of open inelastic channels is proportional to $\Gamma_{el}/\Gamma$. Therefore, the resonance amplitude can be written as

$$f_l = (\Gamma_{el}/\Gamma)/(\varepsilon - i) = 0.5\Gamma_{el}/(E_0 - E - i0.5\Gamma). \quad (57)$$

where $\Gamma = \Gamma_{el} + \Gamma_r$.

As for elastic scattering, the amplitude of the inelastic channel can be written in the form

$$f_{r,l} = \eta_{r,l} \exp(2i\delta_{r,l})/(2i), \quad (58)$$

where $\eta_{r,l}^2 + \eta_{el,l}^2 = 1$.

The center of the unitarity circle for the amplitude of inelastic scattering is at the origin, and its radius is $\frac{1}{2}$ $(\eta_{r,l} = 1)$. The resonance amplitude of the inelastic channel is

$$f_{r,l} = \frac{\sqrt{\Gamma_{el}\Gamma_r/\Gamma^2}}{\varepsilon - i} = \frac{\sqrt{x(1-x)}}{\varepsilon - i} = \frac{0.5\sqrt{\Gamma_{el}\Gamma_r}}{E_0 - E - i0.5\Gamma}, \quad (59)$$

where $x_r = \Gamma_r/\Gamma = 1 - x$.

The rate of change of the partial-wave amplitude is maximal near the resonance: $v_{max}|_{E=E_R} = 2\sqrt{xx_r}/\Gamma$. A good approximation for the resonance in the case of a weak background is constancy of the decay probabilities: $x_{r,l} \approx$ const. In this case, we obtain for the trajectory of the resonance amplitude the same circle as in (56):

$$\mathrm{Re}\, f_{r,l}^2 + (\mathrm{Im}\, f_{r,l} - \sqrt{x_l x_{r,l}}/2)^2 = x_l x_{r,l}/4. \quad (60)$$

Because of the square root, the circle (60) can be situated in either the upper or the lower half-plane. The sign of $\sqrt{xx_r}$ depends on the quantum numbers of the incident particle and the resonance (see, for example, the standard convention established by Levi Setti[70] for $\Lambda$ and $\Sigma$ resonances observed in the inelastic $K^-p \to \Sigma\pi$ and $K^-p \to \Lambda\pi$ channels). Examples of such inelastic resonances are shown in Figs. 5a and 5b. Using the expression for the elastic and inelastic partial-wave amplitudes, one can obtain Breit–Wigner formulas for the elastic, inelastic, and total cross sections:

$$\left.\begin{aligned}
&\sigma_{el} = \frac{4\pi}{k^2}(2l+1)|f_{el}|^2 = \frac{4\pi}{k^2}(2l+1)\frac{x^2}{\varepsilon^2+1} \\
&= \frac{\pi}{k^2}(2l+1)\frac{\Gamma_{el}^2}{(E_0-E)^2+\Gamma^2/4}; \\
&\sigma_p = \frac{4\pi}{k^2}(2l+1)|f_r|^2 = \frac{4\pi}{k^2}(2l+1)\frac{x(1-x)}{\varepsilon^2+1} \\
&= \frac{\pi}{k^2}(2l+1)\frac{\Gamma_{el}\Gamma_r}{(E_0-E)^2+\Gamma^2/4}; \\
&\sigma_{tot} = \frac{4\pi}{k^2}(2l+1)(|f_{el}|^2 + |f_r|^2) \\
&= \frac{4\pi}{k^2}(2l+1)\frac{x}{\varepsilon^2+1} = \frac{\pi}{k^2}\frac{\Gamma_{el}\Gamma}{(E_0-E)^2+\Gamma^2/4}.
\end{aligned}\right\} \quad (61)$$

These cross sections bear the ratio $x^2 : x(1-x) : x$. Since $x < 1$, the resonance maximum in the elastic-scattering cross section is less clearly expressed than the one in the total cross section. All the relations for the amplitude of inelastic scattering obtained above are also valid for several interaction channels, when each channel is described by its own amplitude:

$$\begin{aligned}
&a+b \to a+b \quad f_{el}, \\
&a+b \to c+d \quad f_{r(1)}, \\
&a+b \to e+f \quad f_{r(2)}. \\
&\cdots\cdots \quad \cdots
\end{aligned}$$

Let $\alpha$ be the initial state and $\beta$ and $\gamma$ the final states; then the total width is $\Gamma = \Gamma_\alpha + \Gamma_\beta + \Gamma_\gamma + \ldots$ and the elasticity of each channel is $x_i = \Gamma_i/\Gamma$, where $i = \alpha, \beta, \gamma, \ldots$. As an example, we consider the resonance $\Sigma(2030)$ (see Fig. 5b):

$$K^-p \to \Sigma(2030) \to \begin{cases} K^-p & x_{el} = 0.2; \quad \Gamma = 160\ \text{MeV}; \\ \pi\Lambda & \sqrt{x_{el}x_\Lambda} = 0.2; \quad \Gamma_\Lambda/\Gamma = 0.2; \\ \pi\Sigma & \sqrt{x_{el}x_\Sigma} = -0.1; \quad \Gamma_\Sigma/\Gamma = 0.04. \end{cases}$$
$$l = 3,\ j = 3/2,\ I = 1$$

An important property of the Breit–Wigner approximation for the amplitude of resonance scattering is that on the Argand diagram the trajectory of the amplitude describes a circle in the counterclockwise direction with increasing energy. This property is known as the Wigner condition; it follows from the causality principle and has the consequence that the amplitude has the form $f \sim 1/(E_0 - E - i\Gamma/2)$. This eliminates the complex-conjugate amplitude $f^* \sim 1/(E_0 - E + i\Gamma/2)$; although it would give the same cross section, it would move clockwise on the Argand diagram.

It was shown in Ref. 72 that the retardation of the scattered wave relative to the incident wave packet has the consequence that the scattering amplitude has Breit–Wigner form. However, the connection between the maximum of the delay time and the existence of the resonance determined by the pole of the $S$ matrix at $E = E_0 - i\Gamma/2$ is not unique. This is proved in Refs. 73 and 74, in which it is shown that one can construct several models of the $S$ matrix having all the general properties, i.e., unitarity, analyticity, the correct behavior at the threshold, and so forth, for which the phase shift $\delta$ passes through $\pi/2 + n\pi$ at $E = E_0$ but without the simultaneous appearance of a pole of the $S$ matrix on the second energy sheet. At the same time, some threshold effects can give rise to a sharp increase in the phase shift as a function of the energy. However, the reason for this behavior is not the existence of resonances.

An important conclusion of what we have said above is that relatively narrow resonances lead to maxima of the

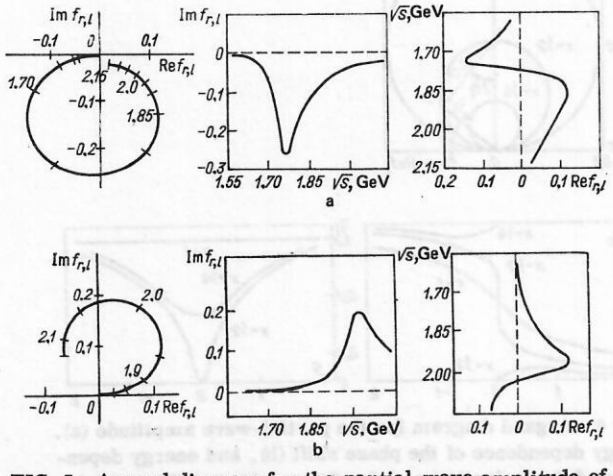


FIG. 5. Argand diagram for the partial-wave amplitude of $\bar{K}N \to \Lambda\pi$ inelastic scattering: a) $l = 2$ ($D15$), b) $l = 3$ ($F17$).

TABLE I. Criteria for the existence of "pure" resonances.

| Characteristic | Elastic channel, $ab \to ab$ | Inelastic channel, $ab \to cd$ |
|---|---|---|
| Unitarity | $0 \leqslant \eta_{el} \leqslant 1$<br>$0 \leqslant x_{el} \leqslant 1$ | $0 \leqslant \eta_r \leqslant 1$<br>$0 \leqslant x_{el} x_r \leqslant 0.25$ |
| At resonance | $\delta_{el} = \pi/2$ for $x_{el} > 0.5$<br>$\delta_{el} = 0$ for $x_{el} < 0.5$ | $\delta_r = \pi/2$ |
| At resonance | Minimum of<br>$\eta_{el} = \lvert 2 - 2x_{el} \rvert$ | Maximum of<br>$\eta_r = 2\sqrt{x_{el} x_r}$ |
| Wigner condition | Counterclockwise motion | |
| Rate of change of the amplitude on the Argand diagram | Maximal at resonance | |

delay time. But the opposite assertion is not always true. Hence, the Wigner condition is not only a necessary but also a sufficient criterion for the existence of a resonance.

Some characteristics of the partial-wave amplitudes for "pure" resonances are given in Table I. In phase-shift analysis, these characteristics are indicators of the existence of resonances.

## 4. INFLUENCE OF THE BACKGROUND ON THE RESONANCE BEHAVIOR OF A PARTIAL-WAVE AMPLITUDE AND ENERGY DEPENDENCE OF THE RESONANCE WIDTH

*Nonresonance background.* The problem of determining resonance states was posed simultaneously with the appearance of the compound-nucleus model for describing the interaction of neutrons with nuclei at low energies.[75] Resonances of the compound nucleus appear in the form of sharp peaks in the total cross sections. Such behavior can be described by the expression

$$\sigma = \frac{4\pi}{k^2} \sum (2l+1) |f_{l,P} + f_{l,R}|^2, \qquad (62)$$

where $f_{l,R}$ is the Breit–Wigner resonance amplitude of the channel: $f_{l,R} = 0.5\,\Gamma_a/(E_0 - E - i0.5\,\Gamma)$, and $f_{l,P}$ is the amplitude obtained from the nonresonance potential. At low energies, one usually chooses a "hard-sphere" potential, i.e., $f_{l,P} = [\exp(2ikR_0) - 1]/2i$.

At higher energies and especially in the case of interaction between elementary particles, the separation of resonance states becomes a complicated problem and requires special analysis. The partial-wave amplitudes corresponding to the background and the resonance must be summed in such a way as to make the total amplitude unitary. Such an amplitude can be obtained by writing the $S$ matrix in the form of the product

$$S = S_F S_R. \qquad (63)$$

Since $S_F$ and $S_R$ are separately unitary, $S$ will also be unitary. In this case, the partial-wave amplitude can be written in the form

$$f = f_F + \eta_F \exp(2i\delta_F) f_R. \qquad (64)$$

The idea of factorizing the $S$ matrix follows from the hypothesis that the background and resonance phase

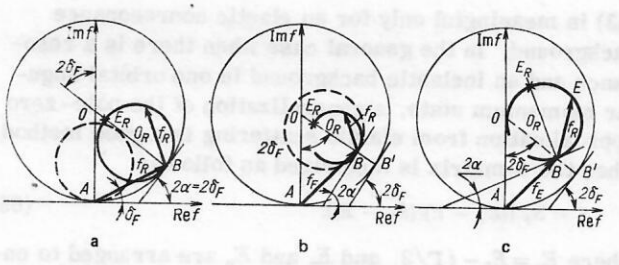


FIG. 6. Influence of constant nonresonance background on the resonance circle in the Argand diagram. a) Inelastic background, b) inelastic background with rotation of the resonance circle through the angle $\alpha = \delta_F$, c) the same case but with $\alpha \neq \delta_F$.

shifts are added, i.e., $\delta = \delta_F + \delta_R$, $\eta = \eta_F \eta_R$. From the physical point of view, this hypothesis is true only if the background and resonance interactions take place in different spatial regions, which is rather improbable; the $S$-matrix factorization method is quite widespread in the analysis of resonances and there are even some successes when it is used, particularly for the elastic background.[21,76]

In the ideal case when there is an energy-independent and purely elastic background, the center of the resonance circle is displaced on the Argand diagram through the angle $2\delta_F$, which makes the resonance circle touch the unitarity circle at one point. At the resonance energy, the inelasticity parameter reaches a minimum $\eta_{min}$. However, this is not true for an inelastic background because then the resonance circle does not touch the unitarity circle and the angle through which it is turned may differ from $2\delta_F$ (see Fig. 6).

A typical example of such a situation is provided by the $^4P$ wave in elastic $pd$ scattering.[77] Despite the fact that in any real situation a nonresonance background is not constant, it is frequently the case that near the resonance energy the background can be assumed to vary little. Thus, the resonance behavior of the $^4P$ wave shown in Fig. 7 can be assumed to be the result of superposition of a relatively constant nonresonance background in the energy range from 8 to 25 MeV on a resonance (an excited state of $^3$He) at about 13 MeV (the region where the amplitude vector changes rapidly on the Argand diagram). A resonance circle with $\Gamma_{el}/\Gamma \approx 0.13$, turned through an angle $2\varphi \neq 2\delta_F$, is obtained. Thus, it can be assumed that the factorization method

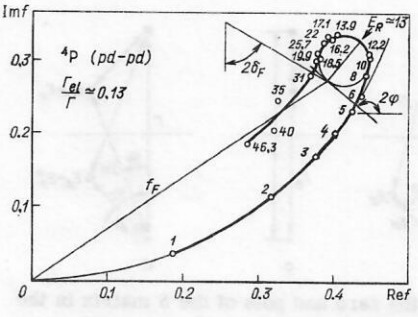


FIG. 7. Resonance behavior of the partial wave $l = 1$ and $s = 3(^4P)$ from elastic $pd$ scattering.

(63) is meaningful only for an elastic nonresonance background. In the general case when there is a resonance and an inelastic background in one orbital angular momentum state, a generalization of the pole–zero approximation from elastic scattering is a good method. Then the $S$ matrix is factorized as follows:

$$S = S_F\,[(E_z - E)/(E_p - E)], \tag{65}$$

where $E_p = E_0 - i\Gamma/2$, and $E_z$ and $E_p$ are arranged to ensure that not only $S_F$ but also the total $S$ matrix is unitary. Through its position, the zero determines the rotation angle of the resonance circle on the Argand diagram.[78,79]

We denote $i\gamma_{el} = E_z - E_p$ and $i\gamma_r = E_p^* - E_z$, where $\gamma$ is a complex number. Then the expression (65) becomes

$$S = S_F\,[1 - (E_p - E_z)/(E_p - E)] = S_F + i\exp(2i\alpha_{el})\,R_{el}/(E_p - E). \tag{66}$$

Here, $\alpha$ is the rotation angle and the residue of the pole corresponds to the width of the resonance. The partial-wave amplitude is

$$f = f_F + 0.5\Gamma_{el}\exp(2i\alpha_{el})/(E_p - E). \tag{67}$$

If $\mathrm{Re}E_z = \mathrm{Re}E_p$, then $\gamma_{el}$ and $\gamma_r$ go over into $\Gamma_{el}$ and $\Gamma_r$, and we again obtain the expression (64). We assume that the relative phase between the background and the resonance is zero, $\alpha = \delta_F$. But if $E_z = E_p^*$, we obtain the case of purely elastic scattering (Fig. 8, in which $\alpha = \varphi + \delta_F$). The $S$ matrix corresponding to inelastic scattering has the form

$$S_r = S_F\,[(E_p^* - E_z)\,(E_z - E_p)]^{1/2}/(E_p - E) = i\exp(2i\alpha_r)\,R_r/(E_p - E),$$

where $\alpha_r$ is the rotation angle of the resonance circle, which must differ from the background phase shift, and the residue at the pole corresponds to $\sqrt{\Gamma_{el}\Gamma_r}$. Then the partial-wave amplitude is

$$f = \exp(2i\alpha_r)\,0.5\sqrt{\Gamma_{el}\Gamma_r}/(E_p - E).$$

The generalization of these results for the multichannel case[79,80] preserves the basic features indicated above, i.e., the background amplitude (elastic or inelastic) is added to a resonance amplitude of Breit–Wigner type displaced relative to the background through a certain angle. In the multichannel case, unitarity imposes direct connections between the parameters of the resonance and background and the shift angles.[28] However, in practice it is impossible to verify these relations because of the absence of complete information on all interaction channels. Thus, in an actual phase-shift analysis the parameters of the background and the resonance and the shift angles are determined from the experimental data.

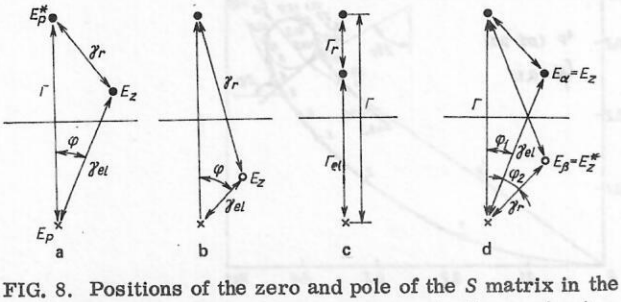


FIG. 8. Positions of the zero and pole of the $S$ matrix in the $E$ plane in the presence of a nonresonance background: a) $x_l < 0.5$, b) $x_l > 0.5$, c) $x_l = 1$, d) for two scattering channels.

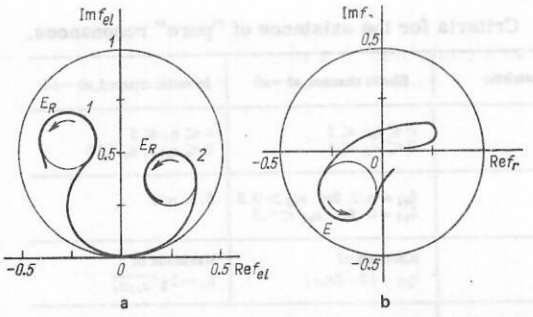


FIG. 9. Trajectories of partial resonance amplitudes in the presence of an energy-dependent nonresonance background. a) For elastic scattering: 1 and 2 correspond to repulsive and attractive backgrounds, respectively; b) for inelastic interaction.

It can be concluded from the above that in the presence of a nonresonance background in the elastic channel a resonance need not necessarily give phase shifts which pass through $\pi/2$ or 0, and a minimum of the inelasticity parameter is also not necessary. Thus, these criteria are approximate. Nice circles on the Argand diagram can be obtained only in the ideal case when the background does not depend on the energy. However, in reality the trajectory of the partial-wave amplitudes on the Argand diagrams is always curved (Fig. 9), so that the determination of the parameters of resonances requires rather complicated analysis.

*Energy dependence of the width* $\Gamma$. Every partial width and, hence, the total width of a resonance depends on the internal dynamics of the resonance state and on various kinematic factors. The most important of these is the centrifugal barrier. In addition, it is necessary to take into account surface and Coulomb effects. In a compound nucleus, for example, the centrifugal barrier $B$ influences the probability $T$ of transition (the transfer coefficient) of a nucleon or group of nucleons through the surface of the nucleus: $T = QB$, where $Q$ is the surface effect. The probability $P$, which is uniformly distributed over the elements of the compound nucleus (the energy within the resonance was concentrated on the one nucleon or group of nucleons and caused the decay), depends on the dynamics of the resonance. The decay rate, i.e., the width of the resonance in one decay channel, is then proportional to these two factors: $\Gamma_i \sim TP$. To determine $T$, we analyze the solution of the Schrödinger equation for $r < r_0$ and $r > r_0$, where $r_0$ is the radius of the compound nucleus or the range of the interaction potential. Then $T \approx 4kV_l(Rr_0)/\varepsilon$, where $v_l = [F_l^2(Rr_0) + G_l^2(Rr_0)]^{-1}$. Here, $F_l(Rr_0)$ and $G_l(Rr_0)$ are the regular and irregular solutions of the Schrödinger equation, and $\varepsilon^2 = k^2 - V(r_0)$.

For resonances in the $S$ wave, the centrifugal barrier is absent, i.e., $B_0 = 1$. Since $T = BQ$ and $v_0 = 1$, $Q_0 \approx 4k/\varepsilon$. Considering the above relations, we see that the effect of the centrifugal barrier is proportional to $v_l$, and the surface effect is proportional to the momentum

$k$: $B_l \sim v_l$, $Q_l \sim k$. All the dynamical and structure effects are contained in a single coefficient, which is called the reduced decay width $\gamma_i^2$. It is this quantity that is determined from the experimental data. It is related to the width of the Breit–Wigner resonance in the $l$ wave by

$$\Gamma_i(k) = k v_l(kr_{0i})\,\gamma_i^2, \tag{68}$$

where $r_{0i}$ is the decay radius of channel $i$ and, like $\gamma_i^2$, is a free parameter.

For small $l = 0, 2, 3$, the centrifugal-barrier factor is given by

$$\left.\begin{aligned} &v_0(kr_0) = 1;\ v_1(kr_0) = (kr_0)^2\,[1 + (kr_0)^2]^{-1};\\ &v_2(kr_0) = (kr_0)^4\,[9 + 3(kr_0)^2 + (kr_0)^4]^{-1}. \end{aligned}\right\} \tag{69}$$

Therefore, the quantity that expresses the energy dependence of the width $\Gamma$ is the product of the centrifugal-barrier factor and a factor which depends on purely nuclear effects. As in nuclear physics, in elementary-particle physics one adopts this energy dependence (despite the fact that the determination of the centrifugal-barrier factor is nonunique[81]); (68) is widely used. To analyze the experimental data, one uses the following expression for $\Gamma$:

$$\Gamma = \Gamma_R \frac{(kr_0)^{2l+1}}{B_l(k)} \left[\frac{B_l(k)}{(kr_0)^{2l+1}}\right]_{k=k_R}, \tag{70}$$

where $B_l(k) = (kr_0)^{2l}[v_l(kr_0)]^{-1}$ and $k_R$ is the momentum at the resonance point. At high energies, where the relativistic kinematics is important, Layson[82] proposed making a correction by introducing the factor $2E_R/(E_R + E)$. Then the final expression for the width of the resonance becomes

$$\Gamma(E) = \Gamma_r \left(\frac{k}{k_R}\right)^{2l+1} \frac{2E_R}{E_R + E} \frac{B_l(k_R)}{B_l(k)}. \tag{71}$$

This form for $\Gamma(E)$ leads to an asymmetric energy dependence of the partial-wave cross sections and of the real and imaginary parts of the partial-wave amplitude. At the same time, $\Gamma(E_2) > \Gamma_R > \Gamma(E1)$, where $E_1 < E_R < E_2$. This asymmetry depends strongly on the interaction range. As can be seen from the energy dependence of the width $\Gamma(E)$, the centrifugal-barrier factor gives the correct behavior at the threshold: $\Gamma \underset{k\to 0}{\sim} \Gamma_0 k^{2l+1}$. Besides the effect of the centrifugal barrier, which basically determines the energy dependence of $\Gamma$, the resonance width is also sensitive to the Coulomb interaction between the colliding particles. For example, for the resonance $\Delta_{33}(1232)$ the Coulomb corrections in the first order in the Coulomb parameter[44] have the form

$$\Gamma(E, \eta) = \Gamma(E, \eta = 0) \times \{1 - \eta\pi + 2\eta kr_0\,[1 - (2/3)(kr_0)^2 \ln(kr_0)] + \ldots\}$$

for $kr_0 \ll 1$ and

$$\Gamma(E, \eta) = \Gamma(E, \eta = 0)\,[1 - \eta/(kr_0)]$$

for $kr_0 \gg 1$.

The Coulomb interactions can be expressed in terms of $\eta$ and depend strongly on $r_0$. For example, for $\Delta_{33}$ we obtain

$$\Gamma(\Delta^0) - \Gamma(\Delta^{++}) \approx 3.6 \text{ MeV for } r_0 = 0 \text{ F};$$
$$\Gamma(\Delta^0) - \Gamma(\Delta^{++}) \approx 1.5 \text{ MeV for } r_0 = 0.75 \text{ F}.$$

This difference between the widths and masses of the resonances belonging to one isomultiplet is very important in the verification of isospin conservation or interaction models. It can be obtained only when a phase-shift analysis is made with allowance for many Coulomb corrections.

To conclude this section, we recall that the theory of resonances can also be discussed by means of relativistic kinematics using the invariant variable $s$, the square of the total c.m.s. energy. In this case, we obtain the relativistic Breit–Wigner formulas[83,84]

$$T_{el} = 0.5 m_R \Gamma_{el}(s)/[m_R^2 - s - i0.5\Gamma(s) m_R];$$
$$f_{el} = 0.5\Gamma_{el}(\sqrt{s})/[m_R - \sqrt{s} - i0.5\Gamma(\sqrt{s})].$$

*Poles and resonances.* It is widely accepted that resonances can be associated with various effects, which we have already discussed above ($\delta_l = \pi/2$ or 0, maximum in $\sigma$, etc.). However, the physical existence of a resonance is intimately related to the existence of a pole of the $S$ matrix on an unphysical sheet of the $E$ plane (or, in the multichannel case, on several unphysical sheets). As we have seen, one can find examples of partial-wave amplitudes that satisfy the criteria of resonance behavior given above. However, these partial-wave amplitudes need not have a pole on the second sheet of the $E$ plane. Such cases occur in elastic[74] and in inelastic and multichannel scattering.[85]

The determination of the position of the pole on one or several unphysical sheets of the $E$ plane for a partial-wave amplitude is the most important test for the existence of resonances. This problem can be attacked by extrapolating the scattering amplitude specified on the real axis (which is experimentally accessible) to the unphysical region of the $E$ plane, i.e., to complex values of the energy. However, because of the existence of an error corridor, in which there can be several analytic functions which describe the experimental data equally well, the extrapolation becomes unreliable. The instability increases with increasing distance from the experimentally studied region. Numerous exact mathematical methods have been developed in recent years (see Refs. 30, 31, and 86–88) to reduce the uncertainties in extrapolation problems. These methods have been widely used in phase-shift analysis, particularly to determine poles corresponding to resonances.

In phenomenological phase-shift analysis, the most common approach is to use a Breit–Wigner parametrization or a polynomial expression with the correct behavior at the threshold. The corresponding formulas are known by the following names[69]:

*Standard,*

$$\tan\delta = \sqrt{s_R}\,\Gamma(k)/(s_R - s), \tag{72}$$

where $\Gamma$ for $\Delta_{33}$ is

$$\Gamma(k) = \Gamma_R (k/k)^3\,[1 + (k_R r)^2]/[1 + (kr)^2];$$

*Layson,*

$$\tan\delta = \Gamma(w)/[2(w_R - w)] \tag{73}$$

with $\Gamma(w) = \gamma^2[4m_p(kr)^3/(w + w_p)(1 + (kr)^2)]$; $w = (s + m_\pi^2 - m_p^2)/2\sqrt{s}$, where $m_p$ is the proton mass, and $\gamma^2$ is the

reduced width;

*Chew–Low*,

$$\tan\delta = \sqrt{s}\,\Gamma(k)/[2\sqrt{s}(\sqrt{s_R}-\sqrt{s})] \quad (74)$$

with $\Gamma(k)=\lambda k(kr)^2/[1+(kr)^2]$. In the expressions (72)–(74), $r$ is the channel radius.

Formulas of *polynomial* type are effective-range expansions:

$$k^3\cot\delta = \sum_{n=1}^{N} a_n k^{2n-2}. \quad (75)$$

The expressions (72)–(74) contain only three adjustable parameters. Using them to analyze the $\Delta_{33}$ resonance in $\pi p$ scattering, one can obtain good agreement with the experimental data if one introduces a fourth parameter, which characterizes the nonresonance background ($\tan\delta_F = ak^3$).[69,89] The mass and width of the $\Delta_{33}$ resonance as determined by means of the expressions (72)–(74) depend on the form of the parametrization which is used. Naturally, the ambiguity increases with increasing number of parameters. The mean values for the mass and width of the $\Delta_{33}$ resonance obtained in this manner are $M_{BW}=1232.4+2.2$ MeV and $\Gamma=116.4+7.2$ MeV.

In contrast to these phenomenological parameters, the position of the pole of the partial-wave amplitude on the second sheet of the $E$ plane is much more stable.[90] The position of the pole depends weakly on the employed parametrization and the extrapolation method. If the partial-wave amplitude is an analytic function of the energy and defined on the physical sheet by

$$f^{I}(s) = \{\eta(s)\exp[2i\delta(s)]-1\}/[2ik(s)] = [k(s)(\cot\Delta(s)-i)]^{-1}$$

with $\Delta=\delta-i(\ln\eta)/2$, then its extrapolation to the unphysical sheet is

$$f^{II}(s) = f^{I}(s)/[1+2ikf^{I}(s)].$$

The partial-wave amplitude $f^{II}(s)$ on the unphysical sheet has the same analytic properties as $f^{I}(s)$, except for possible poles associated with resonances. Thus, determination of a pole is equivalent to solving the equation $\cot\Delta(s)=i$.

An indirect method of determining a pole position is through polynomial parametrization of the trajectories of the zeros of the amplitude in the $x=\cos\theta$ plane.[41] This gives an analytic continuation of the function associated with the trajectory of a zero to the complex $E$ plane. An interesting example of the influence of a resonance on the trajectories of the zeros is provided by the phase-shift analysis of the elastic $p^4$He interaction at energies below 45 MeV. The resonance $3/2^+$ $^5$Li at $\sqrt{s}=21.94$ GeV$^2$ (excitation energy 16.66 MeV) is manifested in the trajectories of the zeros of the scattering amplitudes where they become linearly dependent on the energy.[91] This effect is shown in Fig. 10, in which we have plotted the trajectories of the real parts of the zeros of the scattering amplitude in the variables $s$ and $t$.

Another simple and rapid method of determining the position of a pole is based on the application of diagonal Padé approximants of the second kind, constructed directly from the values of the function, to the $S$ matrix obtained from the phase-shift analysis. We recall that almost all the results obtained for the position of the $\Delta_{33}$ resonance pole by means of the various methods of parametrization of the partial-wave amplitudes and extrapolation to the pole[88–90,92] lie in the range

$$M_{pole} = (1212 \pm 0.5) - i\,(49.0 \pm 1.5)\ \text{MeV}.$$

The residue at the pole of the amplitude, which contains all the dynamic information, the kinematic threshold effects, and the influence of the background, are also fairly stable in the various approaches,

$$R = (36 \pm 1) - i\,(38 \pm 1.5)\ \text{MeV}.$$

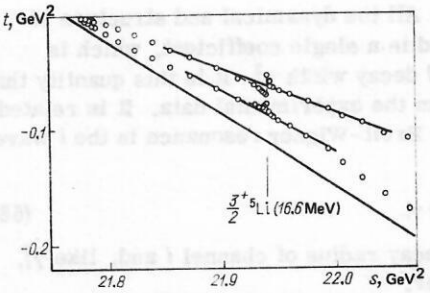


FIG. 10. Projections of two trajectories of the zeroes of the $p^4$He elastic scattering amplitude onto the $st$ plane. The continuous line that does not pass through the points shows the boundary of the physical region.

## 5. RESONANCES IN THE PION–NUCLEUS SYSTEM

*Phase-shift analysis of $\pi A$ scattering.* An extensive program of experimental and theoretical investigations of the pion–nucleus interaction has been carried out. The majority of modern theories of pion–nucleus scattering take information on the pion–nucleon interaction as their point of departure and compare their results with the experimental cross sections. However, as was shown in Ref. 93, the difference between the predictions of the various theories is most strongly manifested when they are compared with the behavior of the partial-wave amplitudes obtained from phase-shift analysis.

Thus, the analysis made in Ref. 93 showed that the optical model with Laplacian-type potential is inadequate for describing $\pi^4$He elastic scattering. If the results of optical-model calculations with the Kisslinger potential and with Laplacian-type potential are compared, the two approaches give the same degree of agreement with the experimental differential cross sections, but the Kisslinger potential gives the correct sign of the $S$-wave phase shift.

Another aspect of pion–nucleus phase-shift analysis is that one can use the obtained phase shifts to construct the scattering amplitudes for other nuclei. Such an attempt was made in Ref. 94, in which the $\pi^4$He scattering phase shifts were used instead of the $\pi N$ phase shifts to construct the $\pi^{12}$C scattering amplitude. An advantage of such an approach is that purely nuclear effects such as true pion absorption, charge exchange, and Fermi motion of the nucleons in the nuclei are already taken into account in the input phase shifts.

Finally, the most interesting question in the phase-shift analysis of pion–nucleus scattering at intermediate energies is the influence of the first baryon resonance $\Delta_{33}$ on the characteristics of the pion–nucleus system. It will be considered separately in the next section.

Also of interest is the possibility of obtaining information about the properties of elementary particles by studying their interactions with nuclei. For example, attempts were made to determine the pion charge radius in $\pi^4$He scattering[5] or to extract the pion–nucleus coupling constant.[95–98] The first phase-shift analysis of pion–nucleus scattering was in fact stimulated by the possibility of determining the pion charge radius (see Refs. 99–101, 12, 10, and 93). It was assumed that the $\pi^4$He scattering amplitude is the sum of the nuclear amplitude and the Coulomb amplitude, modified by the pion and nucleus form factors:

$$f^{\pm} = f_N \pm f_C F_\alpha F_\pi. \tag{76}$$

Then the mean of the $\pi^+$ and $\pi^-$ differential scattering cross sections must depend only on the nuclear amplitude, and the difference of the cross sections must depend linearly on the electromagnetic form factor of the pion:

$$\left.\begin{aligned}(d\sigma^+/d\Omega + d\sigma^-/d\Omega)/2 &= |f_N|^2;\\ (d\sigma^+/d\Omega - d\sigma^-/d\Omega)/2 &= -2\,\mathrm{Re}\,(f_C^* f_N F_\alpha F_\pi).\end{aligned}\right\} \tag{77}$$

The problem was to find the correct nuclear amplitude and take into account all possible Coulomb corrections. The first analysis of $\pi^{\pm 4}$He elastic-scattering experiments was made in Refs. 99–101. The nuclear amplitude was taken either from the optical model or simply from the phase-shift analysis. However, it was found that all the Coulomb corrections are small and their introduction cannot explain the fact that a value exceeding 2 F is obtained for the pion radius.[101,102] At the same time, the analysis with allowance for relativistic effects made in Ref. 103 showed that the pion radius is $r_\pi < 1$ F, which is comparable with the theoretical predictions. However, as was shown in Refs. 8–10 and 26, this result arose not from allowance for the relativistic effects but because Christensen[103] found the correct solution out of the two solutions to the phase-shift analysis in the case of $\pi^4$He scattering. The first of them leads to the pion radius $r_\pi \approx 2$ F, while the second always gives $r_\pi < 1$ F. Thus, allowance for the ambiguity of the phase-shift analysis leads to much stronger effects for the determination of $r_\pi$ than all the other Coulomb and relativistic corrections. In Refs. 10 and 12, the value $\langle r_\pi^2\rangle^{1/2} = 0.83 + 0.17$ F was obtained on the basis of an analysis of all the existing experimental data on $\pi^{\pm}{}^4$He scattering by means of energy-dependent phase-shift analysis. Phase-shift analyses have now been made for the interaction of pions with other nuclei: $^{12}$C (Ref. 104) and $^{16}$O (Refs. 105 and 106).

*Resonances in pion–nucleus scattering.* It is well known that the characteristics of pion–nucleus scattering at medium energies are largely determined by the existence of the $\Delta_{33}$ resonance. In this region, the total scattering cross sections have a clearly defined maximum at a lower energy than in $\pi N$ scattering. It is natural to consider whether this peak in the cross sections is due to the existence of resonances in the $\pi A$ system.

This problem has frequently been discussed in the literature,[107–110] and the most varied opinions have been voiced. It is not our aim to compare all these approaches. We shall consider this problem purely phenomenologically, i.e., from the point of view of phase-shift analysis.

An interesting feature in the phase-shift analysis of elastic $\pi A$ scattering was the discovery of resonance behavior of the partial-wave amplitudes. It was shown in Ref. 20 for the first time that the partial-wave amplitudes obtained from the phase-shift analysis of $\pi^4$He scattering describe counterclockwise circles on the Argand diagram. Moreover, judging from the energy dependences of the phase shifts in the $S$, $P$, and $D$ waves, the resonances in the $\pi^4$He system must be close to each other. Moreover, it was recognized that there is a fairly strong background, which is, moreover, energy dependent. A similar resonance behavior of the partial-wave amplitudes was discovered in elastic $\pi^{12}$C and $\pi^{16}$O scattering.[104,106]

The existence of excited states in the $\pi A$ system (isobaric nucleus,[111] collective resonance,[110] resonance[112]) can be explained in the models developed in Refs. 109 and 110.

In our opinion, the only correct way to establish whether there are resonances in the $\pi A$ system is to make an accurate investigation of the energy dependence of the partial-wave amplitude. The difficulty is that, as we noted above, the pion–nucleus resonances must overlap strongly, since $\Gamma$ is large and exceeds the distance between the resonances. Therefore, in some studies an attempt was made to treat this problem as one of the existence of "pseudoresonances" or as the effect of smearing of the $\Delta_{33}$ resonance.[113,114]

Our point of departure has been that to analyze strongly overlapping resonances with large background one cannot use a simple parametrization of the Breit–Wigner type, and we therefore analyzed the positions of the poles and zeros of the $S$ matrix for elastic $\pi^{-4}$He scattering by means of Padé approximants.

We began with an energy-dependent phase-shift analysis, in which the partial $S$ matrix has the form

$$S_l = \frac{1-2iak^{2l+1}}{1+2iak^{2l+1}}\,\frac{1+b_1k^{2l+2}+ck^{2l+4}\exp(-2id)\,(1+2iak^{2l+1})}{1+b_2k^{2l+2}+ck^{2l+4}\,(1-2iak^{2l+1})}; \tag{78}$$

$$S_l \xrightarrow[k\to 0]{} 1;$$

$$S_l \xrightarrow[k\to\infty]{} \eta_\infty \exp(2i\delta_\infty), \text{where } \eta_\infty = \exp(-2\,\mathrm{Im}\,d),\ \mathrm{Im}\,d > 0,\ \delta_\infty = \mathrm{Re}\,d,$$

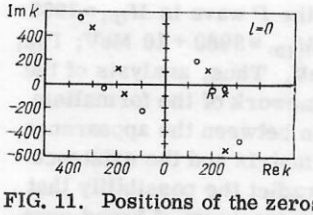


FIG. 11. Positions of the zeros (open circles) and poles (crosses) of the $S$ matrix in the complex $k$ plane for the $S$ wave in elastic $\pi^4$He scattering. The pole and zero associated with the resonance are indicated by the letter R.

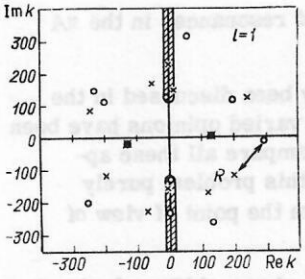


FIG. 12. The same as in Fig. 11 but for the $P$ wave.

which ensures correct behavior of the partial-wave amplitude at the threshold and at infinity. The expression (78) is a diagonal Padé approximant, and therefore it is easy to obtain the positions of the zeros and poles in the complex $k$ plane.

Analysis of the $S$ matrix in the form (78) led us to the conclusion that there is indeed a resonance in the $\pi^4$He system. The positions of the zeros and poles for the $S$ and $P$ waves are shown in Figs. 11 and 12. These figures clearly reveal a zero–pole pair, which we interpret as a manifestation of the resonance. We then consider the zeros and poles of the $S$ matrix in the $E$ plane and find a similar picture. The positions of the zeros and poles in the $E$ plane for the $S$ wave are shown in Fig. 13.

To investigate the stability of these zero–pole pairs with respect to different parametrizations, we use Padé approximants of the second kind to represent the $S$ matrix. The results of this analysis showed that the zero–pole pairs for the $S$ wave are in fact very stable and do not change their position when the degree of the Padé approximants is increased from [3/3] to [5/5]. For the $P$ wave, only the pole, but not the zero, is stable. This is due to the strong energy dependence of the background.

For greater certainty, we tested the method of finding a zero–pole pair in the $S$ matrix by means of Padé approximants for the $\Delta_{33}$ resonance in $\pi N$ scattering and for different functions with positions of the zeros and poles known in advance. In all cases, it was possible to find the expected position of the pole–zero pair with high accuracy.

We therefore regard it as proven that there is in the $\pi^4$He system at least one resonance, which has the parameters

$$M_{(\pi^4\mathrm{He})^*S} = 4020 + 2\ \mathrm{MeV};\ \Gamma_{\mathrm{tot}} = 123 + 10\ \mathrm{MeV};$$
$$\Gamma_{\mathrm{el}} = 40.5 + 3\ \mathrm{MeV};\ x_{\mathrm{el}} = 0.33.$$

The position of the pole in the $P$ wave is $M_{(P)} \approx 3965 + 8$ MeV, and in the $D$ wave $M_{(D)} \approx 3980 + 10$ MeV; $\Gamma_{(P)} = 120$ MeV and $\Gamma_{(D)} = 140$ MeV. Thus, analysis of the experimental data in the framework of the formalism in which there is a connection between the appearance of a pole–zero pair in the $S$ matrix and the existence of a resonance does not contradict the possibility that the $\pi^4$He system has an entire spectrum of broad overlapping resonances in the range of incident-pion energies from 100 to 170 MeV.

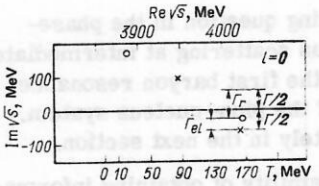


FIG. 13. Positions of the zeros (open circles) and poles (crosses) of the $S$ matrix in the complex energy plane for the $S$ wave in $\pi^4$He interactions. The widths of the resonance are also shown.

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