Exact solutions for cylindrically symmetric configurations of gauge fields. II

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A constructive proof is given of the complete integrability of a large class of two-dimensional nonlinear systems including, in particular, the duality equations for the Yang–Mills fields for arbitrary embedding of the subgroup $SU(2)$ in an arbitrary simple Lie group. General solutions are constructed explicitly for the generalized two-dimensional Toda chain, which corresponds in the case of fixed ends to simple Lie algebras and infinite-dimensional contragredient algebras for the corresponding periodic problem (multicomponent generalizations of the sine–Gordon equation). A number of other two-dimensional nonlinear systems are integrated, including Volterra equations (difference Korteweg–de Vries equations) and the supersymmetric generalization of Liouville's equation. The method developed for integrating the nonlinear equations is based on an explicit realization for these systems of a Lax-type representation by operators that take values in the corresponding algebra and the theory of representations of the corresponding groups.

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INTRODUCTION

The present paper continues our review in Ref. 1 devoted to the construction of exact solutions of cylindrically symmetric equations of gauge fields and it contains a further development of the group-theoretical method of integration of a large class of nonlinear two-dimensional equations of mathematical physics.

The main conclusion of constructive nature that follows from the results presented in the first part of the review consists of the proof of the complete integrability of the system of cylindrically symmetric duality equations in 4-dimensional Euclidean space R_4 for the classical Yang–Mills fields in the case of minimal embedding of the subgroup $SU(2)$ in an arbitrary simple gauge Lie group G. We obtained the following general solutions of the nonlinear system:

$$\partial^2 x_\alpha(z_+, z_-)/\partial z_+\partial z_- = \exp\sum_{\beta=1}^{r} k_{\alpha\beta}x_\beta(z_+, z_-),$$
$$1\leqslant\alpha\leqslant r,\ x_0=x_{r+1}=0, \qquad (1)$$

(k is the Cartan matrix of G of rank r), which describe the corresponding field configurations and are determined by $2r$ arbitrary functions; we also identified r-parameter subclasses of these solutions corresponding to instantons and nonsingular monopoles.

In the time which has elapsed since the writing of the review of Ref. 1, many of the questions listed in the conclusions to that review as the subjects of further investigations have been solved (see Refs. 2–6, 44, and 45), namely:

1) the construction of an invariant method of integrating the system (1) without recourse to information about the explicit form of the root systems of each type of simple Lie algebra separately;

2) the integration of the system (1) in the case of nonsemisimple Lie algebras, including infinite-dimensional contragredient algebras and superalgebras;

3) the finding of a Bäcklund transformation for systems of the type (1);

4) the generalization to the case of arbitrary embedding of $SU(2)$ in an arbitrary Lie group G (the construction of the corresponding duality equations of the gauge fields and their integration), etc.

The possibility of explicit realization for the considered equations of a pair of operators of a Lax-type representation that take values in the algebra of the corresponding group made it possible to reformulate the problem of integration of a large class of nonlinear systems, including (1), in terms of the fundamental concepts of the theory of group representations and to construct general solutions to them. As a result, the integration procedure presented in Ref. 1, which has a semi-intuitive nature and involves great computational difficulties for groups of the type E_7 and E_8 (which have cumbersome root systems), has acquired considerable generality and simplicity. These circumstances have prompted us to write a continuation to the review of Ref. 1, the final text of which has been based on lectures given at seminars at Serpukhov, Dubna, the P.N. Lebedev Physics Institute, the Leningrad Branch of the V.A. Steklov Mathematics Institute, and Moscow State University, and at Symposia on the theory of solitons (Kiev, 1979) and group-theoretical methods in physics (Zvenigorod, 1979).

The equations of the gauge fields considered in the review constitute an essentially nonlinear system, which, in particular, ensures a unified description for an arbitrary simple group of instanton, monopole (dyon), and meron configurations, which are the basic nonlinear objects in the theory of gauge fields. In addition, depending on the choice of an adequate algebraic structure and grading in it, the equations of this type describe a large class of nonlinear effects in other very different fields of theoretical physics and mechanics.

The system (1), in particular, is encountered not only in the theory of gauge fields but also, for particular choice of the matrix k, in the physics of the solid state and plasmas, the theory of electrolytes, aerodynamics, nonlinear optics, cosmological models, etc. In the one-dimensional case ($x_\alpha = x_\alpha(z_+ + z_-)$) for the Cartan matri-

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ces k of simple Lie algebras it is equivalent to the generalized (finite, nonperiodic) Toda chain (see, for example, Refs. 7 and 8), for which numerous studies (for example, Refs. 7–20) in recent years have been devoted to the study and construction of exact solutions. For the generalized Cartan matrices of the contragredient infinite-dimensional algebras,[21] the system (1), which can be regarded in this case as a multicomponent generalization of the sine–Gordon equation; describes corresponding Toda chains with periodic boundary conditions. When Z_2-graded Lie superalgebras are taken as the Cartan k matrices, one again obtains interesting systems, in particular, the equations of "N waves" of nonlinear optics. A large class of nonlinear equations, in particular the Korteweg–de Vries equation, is obtained as a special case of the system (1) corresponding to the continuous generalization of the matrix k to a continuous spectrum of indices α and β. Thus, many well-known two-dimensional equations of mathematical physics (see, for example, Refs. 22–25) that arise in different branches of physics and are treated in the framework of different mathematical methods are special cases of (1) for a definite choice of the matrix k or its continuous analog.

As was noted above, the system (1) arises as the realization of the duality condition of a gauge field that takes values in the algebra of the gauge group G in the case of minimal embedding in it of the three-dimensional subalgebra $su(2)$, which occurs in the definition of the diagonal group.[1] In the case of arbitrary embedding of $su(2)$ in $\mathfrak{g}$, a much more general system is obtained, and it will be seen from what follows that the developed method of integration applies not only to compact but also to noncompact, nonsemisimple Lie algebras and also to infinite-dimensioanl contragredient algebras and Lie superalgebras.

Another important feature of the system (1) is its intimate relation to numerous nonlinear differential–difference equations of first order in the derivatives. In particular, a finite system of the form

$$\left.\begin{array}{l}\partial N_{2\alpha-1}/\partial z_+ = -N_{2\alpha-1}(N_{2\alpha}-N_{2\alpha-2});\\ \partial N_{2\alpha}/\partial z_- = N_{2\alpha}(N_{2\alpha+1}-N_{2\alpha-1}),\\ 1\leqslant\alpha\leqslant r+1,\ N_0=N_{2r+2}=0,\end{array}\right\} \qquad (2)$$

(the functions N_a, $1\leqslant a\leqslant 2r+1$, depend on the two independent variables z_+ and z_-) is the Bäcklund transformation for the system (1) in the case of a Cartan matrix k of the series A_r.[1)] Because of this, one can construct general solutions of the system (2) that depend on $2r+1$ arbitrary functions if one knows the explicit form of the general solutions of Eqs. (1) for the series A_r. In what follows, we shall refer to the system (2) as two-dimensionalized Volterra equations because of the circumstance that in the one-dimensional case, $N_a = N_a(t)$, $t\equiv z_+ - z_-$, this system is a special variant of the equations which Volterra introduced in connection with problems of ecology[27] (in particular, to study the dynamics of the coexistence of species). In applications in physics, this system arises in the study of the fine structure of the spectra of plasma (Langmuir) waves and under the boundary conditions $N_{a\,a\to\pm\infty}$ const describes the propagation of a spectral packet of Langmuir oscillations on the background of thermal noise.[28] A detailed review of various applications of chains of infinite length for $N_a(t)$ and other nonlinear differential–difference systems associated with them, including nonlinear induction–capacitance circuits of ladder type, which are used in radio electronic circuits, is contained in the collection of Ref. 8. (Also given in Ref. 8 are special solutions of these equations, in particular of soliton type, obtained both by means of a Bäcklund transformation as well as by computer calculations.) The periodic problem for a one-dimensional Volterra chain (Korteweg–de Vries difference equation) is considered in Ref. 24. In Ref. 11, the inverse scattering technique is used to develop a scheme for its integration for rapidly decreasing initial conditions, and a Lax pair of operators and integrals of the motion are found.

In the review, we shall use the following notation:

G is an arbitrary simple Lie group of rank r;

$\mathscr{H}$ is the Cartan subgroup of G;

$Z^\pm$ are the maximal nilpotent subgroups of G;

$\tilde{G}$, $\tilde{Z}^\pm$, and $\tilde{\mathscr{H}}$ are the complex hulls of the groups $G, Z^\pm$, and $\mathscr{H}$, respectively;

$\mathfrak{g}$ is the Lie algebra of G;

$\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$;

R_+ and R_- are the systems of positive and negative roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$;

k is the Cartan matrix of $\mathfrak{g}$; $\delta_\alpha \equiv 2\sum_\beta k^{-1}_{\alpha\beta}$;

v_α are the elements of the diagonal matrix V which symmetrizes the Cartan matrix: $Vk = k^T V$;

$X_{\pm j}$ are the elements of the root space of root j, $\pm j\in R_\pm$;

h_α are the generators of $\mathfrak{h}$ corresponding to the simple roots α.

For the algebras considered in the review in connection with the problem of integration of nonlinear systems (including infinite-dimensional contragredient algebras) the grading is specified by a Cartan element h of the algebra (or an element that does not belong to the algebra), with respect to which all the generators of the algebra are divided into subsystems with fixed eigenvalue of their elements under the action of h. The embedding of the 3-dimensional subgroup $SU(2)$ in the group G is canonical and is specified by an embedding vector or its Cartan element h,[31] which together with the generators L^+ and L^- satisfies the usual commutation $[h, L^\pm]_- = \pm 2L^\pm$, $[L^+, L^-]_- = h$. The set of generators that correspond to positive and negative roots of $\mathfrak{g}$ and commute with h form together with the elements of $\mathfrak{h}$ a subalgebra $\mathfrak{g}_0\subset\mathfrak{g}$, which is called the invariant subalgebra. We call the corresponding group G_0 the invariance subgroup; it is a gauge group of two-dimensional space[1] and in the case of minimal embedding of $SU(2)$ in G is isomorphic to $\Pi_1^r\otimes U(1)$. We denote by $\mathring{Z}^\pm$ the maximal nilpotent subgroups of G_0; $Z_0^\pm$ is the factor group $Z^\pm/\mathring{Z}^\pm$, and n is the number of positive roots of $\mathfrak{g}/\mathfrak{g}_0$.

[1)]In the one-dimensional case, the Bäcklund transformation for the Toda chain (of infinite length) was obtained, for example, in Refs. 11 and 26 (for detailed bibliographies, see Refs. 8 and 24).

We shall call the eigenvalue λ of an operator-valued quantity A (which takes values in $\mathfrak{g}$) with respect to h, i.e., $[h, A]_- = \lambda A$, the order of A. In accordance with this, we denote by $R_+^{(2)}$ and $R_-^{(2)}$ the subsystems R_+ and R_- that correspond to the generators of $\mathfrak{g}$ having the orders $+2$ and -2, respectively; $\tilde{R}_+$ and $\tilde{R}_-$ are the remaining roots, i.e., $R_\pm = R_\pm / R_\pm^{(2)}$; r^2 is the number of roots $R_+^{(2)}$. It is obvious that in the case of minimal embedding the systems $R_+^{(2)}$ and $R_-^{(2)}$ consist, respectively, of all positive and negative simple roots of $\mathfrak{g}$, and $r_2 = r$.[1]

We represent an arbitrary element g of $\tilde{G}$ in the form of a modified Gauss decomposition:

$$g = M^+ N^- g_0^- = M^- N^+ g_0^+, \tag{3}$$

where $\{M^\pm, N^\pm\} \in \tilde{Z}_0^\pm, g_0^\pm \in G_0$, which is identical with the usual one for $G_0 = \Pi_1^r \otimes U(1)$, when $\tilde{Z}_0^\pm \Rightarrow \tilde{Z}^\pm$, $G_0 \Rightarrow \mathscr{H}$, and $g_0^\pm \Rightarrow \exp H^\pm \in \mathscr{H}$. (In the form given, this decomposition is valid only for finite-dimensional groups.) From (3) there follows the obvious identity

$$(M^-)^{-1} M^+ = N^+ g_0^+ (g_0^-)^{-1} (N^-)^{-1} \tag{4}$$

in accordance with which the group parameter (for example, the Eulerian angles of the three-dimensional subgroups of G) of the elements $N^\pm$, $g_0^+(g_0^-)^{-1}$ are functions of the group parameters of the elements $M^\pm$.

By $A \cap \{R_0\} = 0$ we shall denote the absence of generators corresponding to the roots of a certain subset $R_0 \subset R_\pm$ in the decomposition of the operator-valued quantity A spanned by elements of $\mathfrak{g}$.

All the basic concepts of the theory of Lie algebras and groups used in what follows are contained, for example, in the monographs of Refs. 29 and 30.

1. CYLINDRICALLY SYMMETRIC CONFIGURATIONS OF YANG–MILLS FIELDS FOR ARBITRARY EMBEDDING OF $SU(2)$ IN AN ARBITRARY GAUGE GROUP

1. Equations of motion; the action and the topological charge.

We consider a Yang–Mills field $A_\mu(x)$, $x \in R_4$, $0 \le \mu \le 3$, that is cylindrically symmetric with respect to the total angular momentum $\mathbf{J} = \mathbf{M} + \mathbf{L}$, where $\mathbf{M} = -i\mathbf{x} \times \nabla$ are the operators of spatial rotations, and $\mathbf{L}$ are the generators of some subgroup $SU(2)$ of G, which is, in general, arbitrary (not necessarily semisimple). Then in accordance with Ref. 1, the components of the field $A_\mu(x)$ can be parametrized by four operator structures $W^\mu = W^\mu(r, t)$, $r \equiv \sqrt{\mathbf{x}^2}$, $t \equiv x_0$, which are scalars with respect to the total angular momentum, $[\mathbf{J}, W^\mu]_- = 0$,

$$A_0 = W^0, \quad \mathbf{A} = \mathbf{n} W^1 + \mathbf{M} W^2 + \mathbf{n} \times \mathbf{M} W^3, \tag{5}$$

where $\mathbf{n} \equiv \mathbf{x}/r$. [The quantities W^μ can be represented in the form of the linear combination $W^\mu = \sum_{\{l\}} \varphi_l^\mu(r, t) W^l$ of the operators W^l, which are invariant with respect to the total angular momentum, this last having a spectrum of l eigenvalues which is fixed by the embedding of the $SU(2)$ subalgebra in the simple Lie algebra $\mathfrak{g}$.[1]]

The electric $\mathbf{E}$ and magnetic $\mathbf{H}$ intensities of the gauge field can be calculated in accordance with the usual rules, and in the system $\mathbf{n} = (0, 0, 1)$ can be represented in the form

$$\left.\begin{aligned} &E_0 = (1/2)[D_{z_+}, D_{z_-}]_-; \; H_0 = -(1/2)(z_+ + z_-)^{-2}\{[W^+, W^-]_- + h\}; \\ &E_\pm - H_\pm = -(z_+ + z_-)^{-1}(D_{z_\mp} W^\pm); \; E_\pm + H_\pm = \\ &= (z_+ + z_-)^{-1}(D_{z_\pm} W^\pm). \end{aligned}\right\} \tag{6}$$

Here we have introduced the notation

$$\{E_0, E_\pm\} \equiv \{iE_3, E_1 \pm iE_2\}; \; \{H_0, H_\pm\} \equiv \{iH_3, H_1 \pm iH_2\};$$
$$W_{z_\pm} \equiv -W^1 \mp iW^0, \; W^\pm \equiv \mp (z_+ + z_-)\{[L^\pm, \pm iW^2 - W^3]_- - (z_+ + z_-)^{-1} L^\pm\};$$

$(D_{z_\pm} W) \equiv W_{z_\pm} + [W_{z_\pm}, W]_-$ are covariant derivatives, $\{h, L\} \equiv \{iL_3, L_1 \pm iL_2\}$ are the generators of the $SU(2)$ subgroup of G with h the Cartan element that determines the embedding, $2z_\pm \equiv r \mp it$, and $W_{,z} \equiv \partial W / \partial z$. It follows from (6) that $F_{z_+ z_-} \equiv 2E_0$ is the tensor of the gauge field $W_{z_\pm}$ of the two-dimensional space, $E_\pm$ and $H_\pm$ are the covariant derivatives in the curved space of the sources of the field $W^\pm$, and $V \equiv 2H_0$ plays the part of the interaction of the sources with one another.

In accordance with the expression (6) for the field intensities, the densities of the action $S \equiv -\pi \int dz_+ dz_- s$ and the topological charge $Q \equiv -(1/16\pi) \int dz_+ dz_- q$ have the form

$$s = \mathrm{Sp}\{(1/2)(z_+ + z_-)^2 (F_{z_+ z_-}^2 + V^2) - (D_{z_+} W^-)(D_{z_-} W^+) - (D_{z_+} W^+)(D_{z_-} W^-)\}; \tag{7}$$

$$q = \mathrm{Sp}\{-(z_+ + z_-)^2 F_{z_+ z_-} V + (D_{z_+} W^-)(D_{z_-} W^+) - (D_{z_+} W^+)(D_{z_-} W^-)\} \tag{8}$$

(cf. the corresponding expressions for the minimal embedding in Ref. 1). Thus, the Lagrangian of the system describes the interaction of the charged fields $W^\pm$ and the gauge field $F_{z_+ z_-}$ with self-interaction in the two-dimensional curved space.

The equations of motion can be obtained in terms of these fields by varying the action functional (7):

$$[D_{z_+}, D_{z_-}]_+ W^\pm = \pm [W^\pm, V]_-, \; [D_{z_\pm}, F_{z_+ z_-}]_- = [W^+, (D_{z_\mp} W^-)]_- - [W^-, (D_{z_\pm} W^+)]_-, \tag{9}$$

where $[\ ,\ ]_+$ denotes the anticommutator.

The self-duality equations are a special subclass of this general system:

$$(D_{z_-} W^+) = 0; \; (D_{z_+} W^-) = 0; \; [D_{z_+}, D_{z_-}]_- = [W^-, W^+]_-, \tag{10}$$

and they follow from (6) by equating the components of the electric and magnetic field intensities.[2)] Their expression in the form (10) also requires the additional substitution $W_{z_\pm} \Rightarrow W_{z_\pm} \pm h/2(z_+ + z_-)$, $W^\pm \Rightarrow (z_+ + z_-) W^\pm$, which "straightens" the space and cancels the terms

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[2)]Note that Eqs. (10) are differential equations in two-dimensional space for the functions $u_\alpha^\pm(z_+, z_-)$ and $f_j^\pm(z_+, z_-)$, which occur in the definition of the operators

$$W_{z_\pm} = \sum u_\alpha^\pm \mathfrak{g}_\alpha^0, \quad W^\pm = \sum f_j^\pm X_{\pm j}, \tag{*}$$

where $\mathfrak{g}_\alpha^0$ are the generators of $\mathfrak{g}_0$, $[h, \mathfrak{g}_\alpha^0]_- = 0$, $X_{\pm j}$ are the generators of $\mathfrak{g}$, which have the order ± 2 with respect to h, and $[h, X_{\pm j}]_- = \pm 2 X_{\pm j}$.

$-(1/2)(z_+ + z_-)^{-2h}$ in the expression for H_0. The corresponding equations for anti-self-dual fields are obtained from (10) by the obvious substitution $W^+ \rightleftarrows W^-$.

The system (10) can be rewritten in the form of the relation

$$[\partial/\partial z_+ + W_{z_+} + W^+,\ \partial/\partial z_- + W_{z_-} + W^-]_- = 0, \tag{11}$$

which is a realization of a Lax-type representation[32] for Eqs. (10) (cf. Ref. 33), which describe cylindrically symmetric Yang–Mills fields for arbitrary embedding of $SU(2)$ in the gauge group. [It is obvious that the forms of expression (10) and (11) of the self-duality equations are equivalent, since the terms $[D_{z+}, D_{z-}]_- - [W^-, W^+]_-$, $(D_{z+}W^-)$ and $(D_{z-}W^+)$ on the left-hand side of (11) have different order (± 2 and 0) with respect to the Cartan element h.[2)]]

We emphasize that hitherto we have not made any assumptions about the properties of the gauge group, except for requiring it to have a three-dimensional subgroup $SU(2)$. This circumstance, by virtue of the possibility, as we shall see in what follows, of completely integrating the considered system on the basis of the representation (11), enables us to obtain a large class of completely integrable systems of nonlinear equations in two-dimensional space and to find their solutions explicitly.

2. Integration of the duality equations for compact groups.

The representation (11) can be regarded as a condition for the vector $A_\pm \equiv W_\pm + W^\pm$ to be a gradient vector, i.e.,

$$A_{z_+} = g^{-1} g_{,z_+};\quad A_{z_-} = g^{-1} g_{,z_-}, \tag{12}$$

where $g \in \tilde{G}$. In the parametrization (3), the expressions (12) take the form

$$A_{z_+} = (g_0^+)^{-1}(N^+)^{-1}(M^-)^{-1} M^-_{,z_+} N^+ g_0^+ + (g_0^+)^{-1}(N^+)^{-1} N^+_{,z_+} g_0^+ + (g_0^+)^{-1} g^+_{0,z_+}; \tag{13}$$

$$A_{z_-} = (g_0^-)^{-1}(N^-)^{-1}(M^+)^{-1} M^+_{,z_-} N^- g_0^- + (g_0^-)^{-1}(N^-)^{-1} N^-_{,z_-} g_0^- + (g_0^-)^{-1} g^-_{0,z_-}. \tag{14}$$

In accordance with the definition, the decompositions of A_{z_+} and A_{z_-} with respect to the generators of $\mathfrak{g}$ contain, besides the elements of $\mathfrak{g}_0$, only the positive and negative roots of $\mathfrak{g}/\mathfrak{g}_0$ of order $+2$ and -2, respectively; in other words,

$$A_{z_+} \cap \{R_-, \tilde{R}_+\} = 0;\quad A_{z_-} \cap \{R_+, \tilde{R}_-\} = 0. \tag{15}$$

With allowance for these conditions, we obtain from (13) and (14)

$$M^-_{,z_+} = 0;\quad M^+_{,z_-} = 0,\ \text{ i.e., } M^+ = M^+(z_+),\ M^- = M^-(z_-); \tag{16}$$

$$(N^-)^{-1} N^-_{,z_-} \cap \{\tilde{R}_-\} = 0; \tag{17}$$

$$(N^+)^{-1} N^+_{,z_+} \cap \{\tilde{R}_+\} = 0. \tag{18}$$

Because the parameters of the elements N^+, N^-, $g_0^+(g_0^-)^{-1}$ can be expressed by virtues of (4) in terms of the parameters of M^+ and M^-, the conditions (17) and (18) must also be reformulated in terms of these elements, thus reducing the problem to their determination. We show that each of the relations (17) and (18) is a system of $n - r_2$ equations that in accordance with the identity (4) connect the first derivatives of the group parameters of the elements M^+ and M^-. To see this, let us, for example, consider (17). Because M^+ depends only on z_+, the realization of the condition (17) in (14) does not depend on the form of the element M^+, which can be set equal to the unit. Then Eq. (4) can be rewritten in the form $M^- = N^- g_0^- (g_0^+)^{-1} (N^+)^{-1}$, as a consequence of which the elements N^+ and $g_0^+(g_0^-)^{-1}$ in it also become the unit, and therefore in (17) we can replace N^- by M^-. Similarly, in (18) the element N^+ can be replaced by M^+. As a result, the system (17)–(18) is equivalent to the conditions

$$(M^+)^{-1} M^+_{,z_+} \cap \{\tilde{R}_+\} = 0;\quad (M^-)^{-1} M^-_{,z_-} \cap \{\tilde{R}_-\} = 0, \tag{19}$$

by virtue of which the n parameters of M^+ and M^- are connected by $n - r_2$ equations, and, therefore, each of the elements M^+ and M^- depends on $n - (n - r_2) \equiv r_2$ arbitrary parameters, which are functions of z_+ and z_-, respectively.

We now turn to the solution of the system (19), which can be rewritten in the form

$$\left.\begin{aligned} M^+_{,z_+} &= M^+ \sum_{\alpha \in R_+^{(2)}} \varphi_{+\alpha}(z_+) X_{+\alpha}; \\ M^-_{,z_-} &= M^- \sum_{\alpha \in R_+^{(2)}} \varphi_{-\alpha}(z_-) X_{-\alpha}; \end{aligned}\right\} \tag{20}$$

where $X_{\pm\alpha}$ are the elements of $\mathfrak{g}$ corresponding to the roots $\pm\alpha \in R_+^{(2)}$; $\varphi_{\pm\alpha}(z_\pm)$ are arbitrary functions of their arguments. The solution of Eqs. (20) (like the S-matrix equation; see, for example, Ref. 34) can be represented in the form of an anti-$Z_\pm$-ordered exponential with Lagrangian $L^\pm \equiv \sum_{\alpha \in R_+^2} \varphi_{\pm\alpha}(z_\pm) X_{\pm\alpha}$:

$$M^\pm = \tilde{\mathcal{Z}}_\pm \exp \int^{z_\pm} \hat{L}^\pm(z_\pm)\, dz_\pm, \tag{21}$$

or, in terms of repeated integrals of retarded commutators (see, for example, Ref. 35),

$$\left.\begin{aligned} M^\pm &= \exp \sum_{m=1} \frac{1}{m!} \int \cdots \int \prod_{i=1}^{m} dz_\pm^{(i)} \theta(z_\pm^{(i-1)} - z_\pm^{(i)}) \\ &\times R_m^\pm(z_\pm^{(1)}, \ldots, z_\pm^{(m)}); \\ R_1^\pm &= L_1^\pm;\ R_2^\pm, = [L_1^\pm, L_2^\pm]_-; \\ R_3^\pm &= [L_1^\pm, [L_2^\pm, L_3^\pm]_-]_- + [L_3^\pm, [L_2^\pm, L_1^\pm]_-]_-; \\ R_4^\pm &= 2\,[L_1^\pm, [L_2^\pm, [L_3^\pm, L_4^\pm]_-]_-]_- + 2[L_3^\pm, [L_2^\pm, [L_4^\pm, L_1^\pm]_-]_-]_- \\ &+ 2[L_4^\pm, [L_1^\pm, [L_3^\pm, L_2^\pm]_-]_-]_- + 2[L_4^\pm, [L_3^\pm, [L_1^\pm, L_2^\pm]_-]_-]_-; \\ R_m^\pm &= \sum_{i_1, \ldots, i_m} [L_{i_1}^\pm, [L_{i_2}^\pm, [L_{i_3}^\pm, [\ldots [L_{i_{m-1}}^\pm, L_{i_m}^\pm]_-]_-]_-]_-]_-, \end{aligned}\right\} \tag{22}$$

where

$$z_\pm^{(0)} \equiv z_\pm;\quad \theta(z) \equiv \begin{cases} 1, & z \geqslant 0, \\ 0, & z < 0 \end{cases}\quad L_i^\pm \equiv \hat{L}^\pm(z_\pm^{(i)}).$$

At the same time, because the groups considered here are of finite dimension, the series in the exponential of the expression (22) contains a finite number of terms.

Thus, Eqs. (21) or (22) solve the problem of integrating the system (10), since the group parameters of the remaining unknown elements $N^\pm$ and $g_0^+(g_0^-)^{-1}$ can be expressed by means of the identity (4) in terms of the parameters $M^\pm$.

3. *Monopole configurations*. Like the special case of

minimal embedding considered in Ref. 1, the above construction of general solutions of the duality equations in R_4 for arbitrary embedding of $SU(2)$ in G solves also the problem of describing spherically symmetric monopoles (or dyons) in Minkowski space (with Higgs field φ in the adjoint representation of G) in the Bogomol'nyi–Prasad–Sommerfield limit.[36] The Hamiltonian density $\mathscr{H}$ for purely magnetic, time-independent solutions is given by the expression

$$\mathscr{H}=(1/2)\,\mathrm{Sp}\,\hat{\mathrm{H}}^2+(1/2)\mathrm{Sp}\,(\mathrm{D}\varphi)^2\equiv(1/2)\,\mathrm{Sp}\,(\hat{\mathrm{H}}\mp\mathrm{D}\varphi)^2\pm\mathrm{Sp}\,(\hat{\mathrm{H}}\cdot\mathrm{D}\varphi), \quad (23)$$

where $\hat{\mathrm{H}}$ is the magnetic intensity of the Yang–Mills field $\hat{A}_\mu=(0,\hat{A})$ parametrized by the structures $\mathbf{W}=\mathbf{W}(\mathfrak{r})$ [see (5)] and $(\mathrm{D}\varphi)\equiv\mathbf{n}\varphi,{}_{r}-[\hat{A},\varphi]_-$.

In the Bogomol'nyi–Prasad–Sommerfield limit, any solution of the differential equations $\hat{\mathrm{H}}=\mathrm{D}\varphi$ $(\hat{\mathrm{H}}=-\mathrm{D}\varphi)$ realizes a minimum of the energy $\mathbf{E}=\int d^3x\mathscr{H}\geqslant\int d^3x$ $\mathrm{Sp}(\hat{\mathrm{H}}\cdot\mathrm{D}\varphi)$, the minimum being determined by the formula

$$\left.\begin{array}{l} 2\pi\int\limits_0^{\mathfrak{r}} d\mathfrak{r}\,\mathrm{Sp}\,\{\mathfrak{r}^2(D_{\mathfrak{r}}\varphi)^2-[\hat{W}^+,\varphi]_-[\hat{W}^-,\varphi]_-\},\\ (D_{\mathfrak{r}}\varphi)\equiv\varphi,{}_{\mathfrak{r}}-[\hat{W}^1,\varphi]_-. \end{array}\right\} \quad (24)$$

Comparing (24) with the integral over $\mathfrak{r}$ of the action density s given by Eq. (7), in which the structures $W_{z_\pm}$ and $W^\pm$ satisfy the self-duality equations and the transition to the static limit is made, we see that they are equal for $W^0=\varphi$ and $\hat{\mathbf{W}}=\mathbf{W}$. Thus, the preimages $\hat{\mathbf{A}}=\mathbf{A}$ and $\varphi=A_0$ of static self-dual fields $(A_0,\mathbf{A})$ in R_4 are monopole solutions in Minkowski space with Higgs field in the adjoint representation of G containing an $SU(2)$ subgroup embedded in an arbitrary manner.

To describe nonsingular magnetic monopoles, it is necessary to go the the static limit in the above expressions for the general solutions of Eqs. (10), which are determined by $2r^{(2)}$ arbitrary functions, and ensure that the energy functional (24) is finite by imposing appropriate boundary conditions. Then the parameters [the constants of integration of the system of ordinary differential equations for the functions $u^\pm_\alpha$ and $f^\pm_i$(*), which depend in the static limit on $z_++z_-\equiv r$] cease to be independent and are connected by algebriac relations (cf. Ref. 1).

The energy M of the monopole system and the matrix of its magnetic charge g_0 are determined by the $\mathfrak{r}\to\infty$ asymptotic behavior of the magnetic field $H_r\equiv\mathbf{n}\cdot\hat{\mathbf{H}}$ and the Higgs field φ in accordance with the formulas[1]

$$M=\lim_{\mathfrak{r}\to\infty}4\pi\mathfrak{r}^2\,\mathrm{Sp}\,(H_{\mathfrak{r}}\varphi)\ \text{and}\ H_{\mathfrak{r}}\underset{\mathfrak{r}\to\infty}{\Rightarrow}g_0/4\alpha\mathfrak{r}^2,$$

where

$$H_{\mathfrak{r}}=-(1/2)\{\sum_{i,j\in R^{(2)}}f_i^+f_j^-[X_i,X_{-j}]-\mathfrak{r}^{-2}h\};$$
$$\varphi=(1/2)\sum_\alpha(u_\alpha^+-u_\alpha^-)\,g_\alpha+(1/2)\,\mathfrak{r}^{-1}h.$$

4. *Topological charge of the dual configuration.* By means of simple algebraic manipulations, the density (8) of the topological charge of a self-dual system can be expressed in terms of the operators $W^\pm$ and $W_{z_\pm}$:

$$q=\mathrm{Sp}\,h\,[W_{z_-,z_+}-W_{z_+,z_-}-(W^+W^-),{}_{z_+z_-}]. \quad (25)$$

The last term in Eq. (25) can, by means of the equation $E_0=H_0$ and the relation $W^+=(1/2)\,[h,W^+]_-$, be reduced to the form

$$-\mathrm{Sp}\,h\,(W^+W^-),{}_{z_+z_-}=(1/2)\,\mathrm{Sp}\,h\,[(z_++z_-)^2\times(W_{z_-,z_+}-W_{z_+,z_-})],{}_{z_+z_-},$$

whence

$$q=\mathrm{Sp}\,h\,\{[(1/2)\,\partial^2/\partial z_+\partial z_-(z_++z_-)^2+1]\,(W_{z_-,z_+}-W_{z_+,z_-})\},$$

or, making in the derivation of Eqs. (10) the substitution $W_{z_\pm}\Rightarrow W_{z_\pm}\pm h/2(z_++z_-)$, we obtain

$$q=\mathrm{Sp}\,h\,\{[(1/2)\,\partial^2/\partial z_+\partial z_-(z_++z_-)^2+1]\times[W_{z_-,z_+}-W_{z_+,z_-}+h(z_++z_-)^{-2}]\}. \quad (26)$$

Because $\mathrm{Sp}\,hW^\pm\equiv0$, in the final formula we can replace $W_{z_\pm}$ identically by $A_{z_\pm}$, which are defined by the expression (12). Then, using the decomposition (13) and introducing the notation $\mathscr{P}_0\equiv g_0^-(g_0^+)^{-1}$, we arrive at the equations

$$\mathrm{Sp}\,h\,(W_{z_-,z_+}-W_{z_+,z_-})=\mathrm{Sp}\,h\,(\mathscr{P}_0^{-1}\mathscr{P}_{0,z_+}),{}_{z_-}=\partial^2/\partial z_+\partial z_-\sum_\alpha\tau_\alpha\,\mathrm{Sp}\,(h_\alpha h), \quad (27)$$

where τ_α are the parameters of the Cartan element $\exp\sum_\alpha h_\alpha\tau_\alpha$ in the Gauss decomposition for $\mathscr{P}_0$, and they can be expressed in terms of the higher vectors $\xi^{\{1\alpha\}}$ of the fundamental representations of G with weight $\{1_\alpha\}\equiv\{\underbrace{0,\ldots,0}_{\alpha-1},1,0,\ldots,0\}$ in accordance with the formula

$$\tau_\alpha=\ln\xi^{\{1_\alpha\}}((M^+)^{-1}M^-). \quad (28)$$

Indeed, taking the matrix element of both sides of the identity (4), $(M^+)^{-1}M^-=N^-\mathscr{P}_0(N^+)^{-1}$, between the states of highest weight $\{l\}\equiv\{l_1,\ldots,l_r\}$, we have

$$\langle l|(M^+)^{-1}M^-|l\rangle\equiv\xi^{\{l\}}((M^+)^{-1}M^-)=\langle l|\exp\sum_\alpha h_\alpha\tau_\alpha|l\rangle=\exp\sum_\alpha l_\alpha\tau_\alpha$$

($\xi^{\{l\}}$ is the highest vector of the irreducible representation of G with highest weight $\{l\}$). From this formula (26) follows. Substituting (27) and (28) in the right-hand side of (26), we obtain the following final expression for the density of the topological charge (or action) of the self-dual configuration:

$$q=-\frac{1}{2}\left[\frac{1}{2}\frac{\partial^2}{\partial z_+\partial z_-}(z_++z_-)^2+1\right]\times\frac{\partial^2}{\partial z_+\partial z_-}\sum_{\alpha,\beta}t_\beta v_\alpha k_{\alpha\beta}\ln\frac{\xi^{\{1_\alpha\}}((M^+)^{-1}M^-)}{(z_++z_-)^{t_\alpha}}, \quad (29)$$

where t_α are the coefficients of the decomposition of the Cartan element h with respect to the generators of $\mathfrak{h}$ (the components of the embedding vector); $\mathrm{Sp}\,(h_\alpha h_\beta)=(1/2)\nu_\alpha k_{\alpha\beta}\equiv(1/2)\nu_\beta k_{\beta\alpha}$. We recall that for minimal embedding $t_\alpha\equiv\delta_\alpha$, since in this case all the positive simple roots have order $+2$ with respect to the Cartan element h of the embedding: $[h,X_\alpha]_-=\sum_\beta t_\beta[h_\beta,X_\alpha]_-=\sum_\beta t_\beta k_{\alpha\beta}X_\alpha=2X_\alpha$. Note that the charge of the magnetic monopole is also determined by the static limit of Eq. (29).

5. *Concluding remarks.* The main result of the present section is the construction of general solutions of the self-duality equations (10), these guaranteeing, when appropriate boundary conditions are imposed, a description of cylindrically symmetric instantons and non-

singular monopoles for arbitrary embedding of $SU(2)$ in the compact gauge group G. The proposed invariant method of integrating Eqs. (10) is based on explicit realization of a Lax-type representation by the pair of operators $A_{z_\pm}$ (12), which take values in the algebra. All the constructions of subsection 2 can be completely transferred to the more general case when the operator-valued structures $W^\pm$ in the representation (11) contain contributions of the generators of the algebra up to a certain order $|s| \geqslant 2$, these leading to the following modification of Eqs. (20):

$$M^\pm_{,z_\pm} = M^\pm \sum_{p=2}^{|s|} \sum_{\alpha \in R_+^{(p)}} \varphi_{\pm\alpha}(z_\pm)\, X_{\pm\alpha}.$$

Since we are unaware of any physical applications for generalizations of this kind, we shall content ourselves with noting the possibility of integrating such systems.

In the following sections, particularizing the form of the algebras and the choice of the grading in them, we consider a number of nonlinear systems of mathematical physics and, in particular, the two-dimensional generalized Toda chain (both finite and with periodic boundary conditions); these systems admit complete integration on the basis of general formulas of the type (11) and (21). We note that if the "kinetic" part of the equations of the resulting system associated with particular algebraic structures is to be the two-dimensional Laplacian, the corresponding invariance subgroup must be Abelian.

2. INTEGRATION OF TWO-DIMENSIONAL NONLINEAR EQUATIONS ASSOCIATED WITH FINITE-DIMENSIONAL ALGEBRAS

1. Generalized two-dimensional (finite, nonperiodic) Toda chain. An important example of the concrete realization of the general results of Sec. 1, whose use is not restricted to the theory of gauge fields, is the case of minimal embedding of $SU(2)$ in a simple compact Lie algebra $\mathfrak{g}$. For this embedding, the Cartan element h satisfies the commutation relations $[h, X_{\pm\alpha}] = \pm 2X_{\pm\alpha}$ with the generators $X_{\pm\alpha}$ of $\mathfrak{g}$ corresponding to the simple roots, the invariance subgroup is $G_0 = \Pi_1^r \otimes U(1)$ (Ref. 1), and the formulas (*) in footnote 2 are rewritten in the form $W_{z\pm} = \sum_\alpha u^\pm_\alpha h_\alpha$, $W^\pm = \sum_\alpha f^\pm_\alpha X_{\pm\alpha}$. Then the Lax-type representation (11) leads in conjunction with the relations

$$[h_\alpha, h_\beta]_- = 0;\ [h_\alpha, X_{\pm\beta}]_- = \pm k_{\beta\alpha} X_{\pm\beta},\quad [X_\alpha, X_{-\beta}]_- = \delta_{\alpha\beta} h_\alpha \tag{30}$$

to the following system of equations for $u^\pm_\alpha$ and $f^\pm_\alpha$:

$$\left.\begin{array}{l} (\ln f^+_\alpha)_{,z_-} = -(ku^-)_\alpha;\ (\ln f^-_\alpha)_{,z_+} = (ku^+)_\alpha; \\ u^+_{\alpha,z_-} - u^-_{\alpha,z_+} = f^+_\alpha f^-_\alpha, \end{array}\right\} \tag{31}$$

whence, introducing the functions $\rho_\alpha \equiv (\ln f^\pm_\alpha) f^-_\alpha$, which are invariant under G_0, we have

$$\rho_{\alpha, z_+z_-} = \sum_{\beta=1}^r k_{\alpha\beta} \exp \rho_\beta,\quad 1 \leqslant \alpha \leqslant r. \tag{32}$$

This last system describes the generalized Toda chain with fixed ends ($\rho_0 = \rho_{r+1} = -\infty$) and goes over into the system (1) as a result of the substitution $\rho_\alpha = \sum_{\beta=1}^r k_{\alpha\beta} x_\beta$ because of the nondegeneracy of the Cartan matrices k of simple Lie algebras.

In accordance with the general scheme, the solution of the system (32) is determined by the element

$$K \equiv (M^+)^{-1} M^- = N^- \exp(H)\, (N^+)^{-1}, \tag{33}$$

in which the elements $M^\pm$ satisfy the equations

$$\left.\begin{array}{l} M^+_{,z_+} = M^+ \sum\limits_\alpha \varphi_{+\alpha}(z_+)\, X_{+\alpha} \equiv M^+ \hat{L}^+; \\ M^-_{,z_-} = M^- \sum\limits_\alpha \varphi_{-\alpha}(z_-)\, X_{-\alpha} \equiv M^- \hat{L}^-,\ M^\pm \in \tilde{Z}^\pm \equiv \tilde{Z}^\pm_0 \end{array}\right\} \tag{34}$$

and are expressed by Eqs. (21).

Here, $\exp H \equiv g_0^-(g_0^+)^{-1} = \exp \sum_\alpha H_\alpha h_\alpha$, $g_0^\pm \equiv \exp H^\pm \in \mathscr{H}$.

The Gauss decomposition (33) makes it possible to determine explicitly $N^\pm$ and $\exp H$ and thus to find the functions $f^\pm_\alpha$, in terms of which the solutions to the system (32) are expressed. The calculation of the explicit expressions for $N^\pm$ is a fairly laborious problem. However, we shall show that the solution to the system (1) is completely determined by the functions H_α in (33). To this end, we substitute the decomposition (33) in the expression for

$$K^{-1} K_{,z_-} \equiv \hat{L}^-,\ K_{,z_+} K^{-1} \equiv -\hat{L}^+, \tag{35}$$

obtaining

$$\exp(-H)\,(N^-)^{-1} N^-_{,z_-} \exp H + H_{,z_-} - (N^+)^{-1} N^+_{,z_-} = (N^+)^{-1} \hat{L}^- N^+,$$
$$-\exp H\, (N^+)^{-1} N^+_{,z_+} \exp(-H) + H_{,z_+} + (N^-)^{-1} N^-_{,z_+} = -(N^-)^{-1} \hat{L}^+ N^-.$$

From this we obtain the relations

$$\left.\begin{array}{l} (N^-)^{-1} N^-_{,z_-} = \exp H \hat{L}^- \exp(-H);\ H_{,z_-} - (N^+)^{-1} N^+_{,z_-} \\ = (N^+)^{-1} \hat{L}^- N^+ - \hat{L}^-; \\ (N^+)^{-1} N^+_{,z_+} = \exp(-H)\, \hat{L}^+ \exp H;\ H_{,z_+} + (N^-)^{-1} N^-_{,z_+} \\ = -(N^-)^{-1} \hat{L}^+ N^- + \hat{L}^+ \end{array}\right\} \tag{36}$$

Then, introducing the current-like operators $J_{z+} \equiv \exp(-H)\hat{L} + \exp H$ and $J_{z-} \equiv H_{z-} + \hat{L}^- - (N^+)^{-1}\hat{L} - N^+$, we find from the condition of their compatibility in the form $[\partial/\partial z_+ + J_{z+}, \partial/\partial z_- + J_{z-}]_- = 0$

$$H_{,z_+z_-} = [\hat{L}^-, \exp(-H)\hat{L}^+ \exp H]_-. \tag{37}$$

Using (34) for the operators $\hat{L}^\pm$ and the commutation relations (30), and decomposing the element H with respect to the generators h_α of $\mathfrak{h}$, we obtain from (37)

$$\begin{aligned} H_{,z_+z_-} &= \sum_{\alpha,\beta} \varphi_{-\alpha}, \varphi_{+\beta} [X_{-\alpha}, \exp(-\sum_\gamma H_\gamma h_\gamma)\, X_{+\beta} \exp(\sum_\theta H_\theta h_\theta)] \\ &= \sum_{\alpha,\beta} \varphi_{-\alpha}\varphi_{+\beta} \exp[-(kH)_\beta]\, [X_{-\alpha}, X_{+\beta}]_- \\ &= -\sum_\alpha \varphi_{+\alpha}\varphi_{-\alpha} \exp[-(kH)_\alpha]\, h_\alpha, \end{aligned}$$

i.e.,

$$H_{\alpha, z_+z_-} = -\varphi_{+\alpha}(z_+)\, \varphi_{-\alpha}(z_-) \exp[-(kH)_\alpha], \tag{38}$$

or, introducing the functions

$$\left.\begin{array}{l} x_\alpha \equiv -H_\alpha + \sum\limits_{\beta=1}^r k^{-1}_{\alpha\beta} \ln[\varphi_{+\beta}\varphi_{-\beta}]; \\ \exp(-x_\alpha) = \exp\{-\sum\limits_{\beta=1}^r k^{-1}_{\alpha\beta} \ln[\varphi_{+\beta}(z_+)\, \varphi_{-\beta}(z_-)]\} \\ \times\, \xi^{\{1_\alpha\}}((M^+)^{-1} M^-), \end{array}\right\} \tag{39}$$

we see that x_α satisfy the system (1).

The density of the topological charge of a self-dual configuration for minimal embedding of $su(2)$ in $\mathfrak{g}$ is a direct consequence of the general formula (29) for

$t_\alpha \equiv \delta_\alpha$, which, after summation over the indices α and β with allowance for the equation

$$\sum_\alpha v_\alpha \ln \xi^{\{1_\alpha\}} = -(vx)_{, z_+z_-} = \partial^2/\partial z_+ \partial z_- \ln \xi^{\{v\}},$$

can be represented in terms of the highest vectors $\xi^{\{v\}}$ of the irreducible representation of G with weight $\{v\}$:

$$q = -\left[(1/2)\frac{\partial^2}{\partial z_+ \partial z_-}(z_+ + z_-)^2 + 1\right]\frac{\partial^2}{\partial z_+ \partial z_-}\ln\frac{\xi^{\{v\}}((M^+)^{-1}M^-)}{(z_+ + z_-)^{(v\delta)}}. \quad (40)$$

For the solutions of the system (1), we give one further derivation of Eq. (39), this derivation generalizing in an invariant manner the solutions (1) for the series A_r.[1] We denote by $\hat{X}_i$ and $\hat{\hat{X}}_i$, respectively, an element of the eigenspace of the i-th root of the left and right regular representation of G, and by $\hat{h}_\alpha$ and $\hat{\hat{h}}_\alpha$ the generators of the corresponding regular representations of the Cartan subgroup $\mathscr{H}$. We have $[\hat{Y}, \hat{\hat{Y}}]_- \equiv 0\ \forall\ \hat{Y} \equiv \{\hat{X}_i, \hat{h}_\alpha\}$, $\hat{\hat{Y}} \equiv \{\hat{\hat{X}}_i, \hat{\hat{h}}_\alpha\}$. In accordance with the definition $\xi^{\{l\}}(K) \equiv \langle l|K|l\rangle$ of the highest vector of the irreducible representation of G with weight $\{l\}$,

$$\left.\begin{array}{l}\hat{X}_{-i}\xi^{\{l\}} = \hat{\hat{X}}_{+i}\xi^{\{l\}} = 0\ \forall\ i \in R_+;\\ \hat{h}_\alpha \xi^{\{l\}} = -\hat{\hat{h}}_\alpha \xi^{\{l\}} = l_\alpha \xi^{\{l\}}.\end{array}\right. \quad (41)$$

It is easy to show by directly verifying the fulfillment of relations (41) that the determinant of second order

$$\det\begin{pmatrix}\xi^{\{l\}} & \hat{X}_\alpha \xi^{\{l\}}\\ \hat{\hat{X}}_{-\alpha}\xi^{\{l\}} & \hat{X}_\alpha \hat{\hat{X}}_{-\alpha}\xi^{\{l\}}\end{pmatrix}$$

is none other than the highest vector $\xi^{\{L(\alpha)\}}$ of the irreducible representation of G with weight $\{L_{(\alpha)}\} = \{2l_\beta - k_{\alpha\beta},\ 1 \leqslant \beta \leqslant r\}$ with proportionality factor l_α, which is determined with allowance for the equation $\xi^{\{l\}}(K)| = 1$. We shall also require a formula that expresses $\xi^{\{l\}}$ in terms of the highest vectors of the fundamental representations of G (see, for example, Ref. 29):

$$\xi^{\{l\}} = \prod_{\alpha=1}^{r}[\xi^{\{1_\alpha\}}]^{l_\alpha}, \quad (42)$$

by virtue of which

$$\det\begin{pmatrix}\xi^{\{l\}} & \hat{X}_\alpha \xi^{\{l\}}\\ \hat{\hat{X}}_{-\alpha}\xi^{\{l\}} & \hat{X}_\alpha \hat{\hat{X}}_{-\alpha}\xi^{\{l\}}\end{pmatrix} \equiv l_\alpha \xi^{\{2l_\beta - k_{\alpha\beta}\}} = l_\alpha [\xi^{\{l\}}]^2 \prod_\beta [\xi^{\{1_\beta\}}]^{-k_{\alpha\beta}}. \quad (43)$$

Taking as K in $\tilde{G}$ the element $(M^+)^{-1}M^-$ in which $M^\pm = M^\pm(z_\pm)$ satisfy Eqs. (34), we obtain

$$\left.\begin{array}{l}\frac{d}{dz_+}\xi^{\{l\}} = -\sum_\alpha \varphi_{+\alpha}\langle l|X_{+\alpha}(M^+)^{-1}M^-|l\rangle = -\sum_\alpha \varphi_{+\alpha}(z_+)\hat{X}_\alpha \xi^{\{l\}};\\ \frac{d}{dz_-}\xi^{\{l\}} = \sum_\alpha \varphi_{-\alpha}(z_-)\hat{\hat{X}}_{-\alpha}\xi^{\{l\}}.\end{array}\right\} \quad (44)$$

Taking $\xi^{\{l\}}$ in (44) to be the highest vector $\xi^{\{1_\alpha\}}$ of the fundamental representation and noting that

$$\hat{\hat{X}}_{-\beta}\xi^{\{1_\alpha\}} = \hat{X}_\beta \xi^{\{1_\alpha\}} \equiv 0\ \forall \cdot \beta \neq \alpha,$$

we obtain

$$\frac{d}{dz_+}\xi^{\{1_\alpha\}} = -\varphi_{+\alpha}(z_+)\hat{X}_\alpha \xi^{\{1_\alpha\}}; \quad \frac{d}{dz_-}\xi^{\{1_\alpha\}} = \varphi_{-\alpha}(z_-)\hat{\hat{X}}_{-\alpha}\xi^{\{1_\alpha\}}. \quad (45)$$

and, using (43), we find

$$\det\begin{pmatrix}\xi^{\{1_\alpha\}} & \xi^{\{1_\alpha\}}_{, z_+}\\ \xi^{\{1_\alpha\}}_{, z_-} & \xi^{\{1_\alpha\}}_{, z_+z_-}\end{pmatrix} = -\varphi_\alpha \varphi_{-\alpha}\begin{pmatrix}\xi^{\{1_\alpha\}} & \hat{X}_\alpha \xi^{\{1_\alpha\}}\\ \hat{\hat{X}}_{-\alpha}\xi^{\{1_\alpha\}} & \hat{X}_\alpha \hat{\hat{X}}_{-\alpha}\xi^{\{1_\alpha\}}\end{pmatrix} = -\varphi_{+\alpha}\varphi_{-\alpha}\xi^{\{2(1_\alpha) - k_{\alpha\beta}\}} = -\varphi_{+\alpha}\varphi_{-\alpha}\prod_{\beta \neq \alpha}[\xi^{\{1_\beta\}}]^{-k_{\alpha\beta}}. \quad (46)$$

Comparing (46) with the system (1), written down for the functions

$$\left.\begin{array}{l}X_\alpha \equiv \exp(-x_\alpha),\ 1 \leqslant \alpha \leqslant r;\\ \det\begin{pmatrix}X_\alpha & X_{\alpha, z_+}\\ X_{\alpha, z_-} & X_{\alpha, z_+z_-}\end{pmatrix} = -\prod_{\beta \neq \alpha} X_\beta^{-k_{\alpha\beta}},\end{array}\right\} \quad (47)$$

we arrive at the expression (39) for the solutions of the system (1).

2. Examples of completely integrable self-duality equations for nonsemisimple Lie algebras. We here consider the nonlinear equations at which we arrive in accordance with the general scheme developed in Sec. 1 by taking as gauge group a connected Lie group $\hat{G}$ represented in accordance with the Levi–Mal'tsev theorem (see, for example, Ref. 29) as the semidirect product of its solvable and semisimple subgroups. In what follows, we shall restrict ourselves to connected Lie groups possessing Abelian invariance subgroups and, therefore, nonlinear systems of the form

$$x_{a, z_+z_-} = \Phi_a(x). \quad (48)$$

It should be noted that the problem of distinguishing such groups among all connected groups is fairly complicated and there is as yet no general criterion for their selection.

Since the Lie algebra $\hat{\mathfrak{g}}$ of the group $\hat{G}$ can be decomposed into the direct sum of its radical $\mathfrak{R}$ and semisimple subalgebra $\mathfrak{g}$, with

$$[\mathfrak{g}, \mathfrak{g}]_- \subset \mathfrak{g};\ [\mathfrak{R}, \mathfrak{R}]_- \subset \mathfrak{R};\ [\mathfrak{R}, \mathfrak{g}]_- \subset \mathfrak{R}, \quad (49)$$

it is natural that the structure of the system of resulting nonlinear equations (48) will have the same nature, i.e., will contain as subsystem equations of the type (1) corresponding to the subgroup G.

a) We consider as $\hat{G}$ an arbitrary simple Lie group G with commutative multiplet $\mathfrak{R}^{\{1\}}$ lying in its adjoint representation. Then the operators $A_{z_\pm}$, which realize the Lax-type representation for the system (10), can be represented in the form

$$A_{z_\pm} = \sum_\alpha [u_\alpha^\pm h_\alpha + f_\alpha^\pm X_{\pm\alpha} + U_\alpha^\pm H_\alpha + F_\alpha^\pm Y_{\pm\alpha}], \quad (50)$$

where $u_\alpha^\pm, f_\alpha^\pm, U_\alpha^\pm, F_\alpha^\pm$ are functions of z_+ and z_-; $\{X_{\pm\alpha}, h_\alpha; 1 \leqslant \alpha \leqslant r\}$ are the generators of $\mathfrak{g}$ corresponding to the simple roots; and $\{Y_{\pm\alpha}, H_\alpha, 1 \leqslant \alpha \leqslant r\}$ are the generators of $\mathfrak{R}^{\{1\}}$, which satisfy in accordance with (49) the commutation relations

$$\left.\begin{array}{l}[H_\alpha, R_\beta]_- = 0;\ [H_\alpha, X_{\pm\beta}]_- = \pm k_{\beta\alpha}Y_{\pm\beta};\\ [h_\alpha, Y_{\pm\beta}]_- = \pm k_{\beta\alpha}Y_{\pm\beta},\ [X_\alpha, Y_{-\beta}]_- = \delta_{\alpha\beta}H_\alpha.\end{array}\right\} \quad (51)$$

With allowance for (50) and (51), the representation (11) in the considered case leads to Eqs. (31) for $u_\alpha^\pm$ and $f_\alpha^\pm$ and to the system

$$\begin{array}{l}(F_\alpha^-/f_\alpha^-)_{, z_+} = (kU^+)_\alpha;\ (F_\alpha^+/f_\alpha^+)_{, z_-} = -(kU^-)_\alpha;\\ U^+_{\alpha, z_-} - U^-_{\alpha, z_+} = (F_\alpha^- f_\alpha^+ + F_\alpha^+ f_\alpha^-),\end{array}$$

whence, introducing the functions $\rho_\alpha \equiv \ln f_\alpha^+ f_\alpha^-$ and $\sigma_\alpha \equiv F_\alpha^-/f_\alpha^- + F_\alpha^+/f_\alpha^+$, we finally obtain

$$\rho_{\alpha, z_+z_-} = \sum_\beta k_{\alpha\beta}\exp\rho_\beta;\quad \sigma_{\alpha, z_+z_-} = \sum_\beta k_{\alpha\beta}\sigma_\beta \exp\rho_\beta. \quad (52)$$

We shall not give here the results of integrating this system because they can be obtained by taking as gauge group the direct product $G \otimes G$ of two simple Lie groups with subsequent contraction operation.

b) As a second example, we consider the semidirect product of the group $SU(2)$ and the commutative multiplet $\mathfrak{R}^{\{l\}}$ of dimension l with respect to the angular momentum of the group $SU(2)$, the generators of the multiplet satisfying the following commutation relations with the generators h and $X_\pm$ of the subalgebra $su(2)$:

$$[h,\ R_m]_- = 2mR_m;\ [X_\pm,\ R_m]_- = \sqrt{(l\mp m)(l\pm m+1)}\,R_{m\pm 1}. \tag{53}$$

In accordance with the general scheme, we set

$$A_{z_\pm} = u^\pm h + f^\pm X_\pm + f_0^\pm R_0 + f_1^\pm R_{\pm 1}. \tag{54}$$

Then the representation (11) leads to equations of the form (31) for $u^\pm$ and $f^\pm$ and

$$f^+_{0,\,z_-} - f^-_{0,\,z_+} = \sqrt{l(l+1)}\,(f^+ f_1^- - f^- f_1^+);$$
$$(f_1^+/f^+)_{,\,z_-} = \sqrt{l(l+1)}\,f_0^-;\ (f_1^-/f^-)_{,\,z_+} = \sqrt{l(l+1)}\,f_0^+;$$

whence for the functions $\rho \equiv \ln(f^+ f^-)$ and $\sigma \equiv f_1^+/f^+ - f_1^-/f^-$ we obtain

$$\rho_{,\,z_+z_-} = 2\exp\rho;\ \sigma_{,\,z_+z_-} = l(l+1)\,\sigma\exp\rho. \tag{55}$$

It is easy to show that the system (55) has the general solutions

$$\left.\begin{aligned} &\rho = \ln[\Phi_{+,\,z_+}\Phi_{-,\,z_-}(\Phi_+ + \Phi_-)^{-2}];\\ &\sigma = (\Phi_+ + \Phi_-)^{l+1}\left[\frac{d^l}{d\Phi_-^l}\frac{\varphi_-(\Phi_-)}{(\Phi_+ + \Phi_-)^{l+1}} + \frac{d^l}{d\Phi_+^l}\frac{\varphi_+(\Phi_+)}{(\Phi_+ + \Phi_-)^{l+1}}\right]\end{aligned}\right\} \tag{56}$$

for integral l. Here, $\varphi_\pm(\Phi_\pm)$ and $\Phi_\pm(z_\pm)$ are arbitrary functions of their arguments.

3\. *Two-dimensionalized system of Volterra equations (difference Korteweg–de Vries equations) as Bäcklund transformation of the Toda chain and their complete integration.* Here, we integrate the finite system (2) of Volterra-type equations. The proposed method of constructing general solutions of this system, which depend on $2r+1$ arbitrary functions, is based on the circumstance that this system can be regarded as the Bäcklund transformation for the two-dimensionalized (finite, nonperiodic) Toda chain.

It follows directly from the system (2) that the functions

$$\rho_\alpha \equiv \ln(N_{2\alpha-1}N_{2\alpha});\ \rho'_\alpha \equiv \ln(N_{2\alpha}N_{2\alpha+1}),\ 1\leqslant\alpha\leqslant r,$$

satisfy the equations

$$\left.\begin{aligned} &\rho_{\alpha,\,z_+z_-} = 2\exp\rho_\alpha - \exp\rho_{\alpha+1} - \exp\rho_{\alpha-1},\ \rho_0 = \rho_{r+1} = -\infty;\\ &\rho'_{\alpha,\,z_+z_-} = 2\exp\rho'_\alpha - \exp\rho'_{\alpha+1} - \exp\rho'_{\alpha-1},\ \rho'_0 = \rho'_{r+1} = -\infty,\\ &1\leqslant\alpha\leqslant r,\end{aligned}\right\} \tag{57}$$

which describe the two-dimensionalized (finite, nonperiodic) Toda chain associated with the series A_r (32). In other words, the system (2) can be regarded as the realization of a Bäcklund transformation for Eqs. (57). Since the general solutions of the system (57) contain $2r$ arbitrary functions, whereas the general solutions of (2) are determined by $2r+1$ arbitrary functions, the Bäcklund transformation (2) for the system (57) possesses a functional arbitrariness in the form of one arbitrary function. The scheme for integrating the system (2) consists of establishing a connection between the $2r$ arbitrary functions $\Phi_{+\alpha}(z_+)$ and $\Phi_{-\alpha}(z_-)$, $1\leqslant\alpha\leqslant r$, which determine in accordance with the results of subsection 2 the general solutions of the system (57) for ρ_α, $1\leqslant\alpha\leqslant r$, and the $2r$ functions $\Phi'_{+\alpha}(z_+)$ and $\Phi'_{-\alpha}(z_-)$, which determine the solutions of (57) for ρ'_α. The solutions of the system (2) are completely determined by the relations

$$N_{2\alpha-1}N_{2\alpha} = \exp\rho_\alpha;\ N_{2\alpha}N_{2\alpha+1} = \exp\rho'_\alpha,\quad 1\leqslant\alpha\leqslant r, \tag{58}$$

and the equation

$$N_{1,\,z_+} = -\exp\rho_1, \tag{59}$$

by means of which the missing arbitrary function is introduced.

We illustrate this construction for the example of the simplest case of the group $SU(2)$ $(r=1)$, for which the system (2) has the form

$$N_{1,\,z_+} = -N_1N_2;\ N_{2,\,z_-} = N_2(N_3 - N_1);\ N_{3,\,z_+} = N_2N_3. \tag{60}$$

Substituting the well-known general solution $\exp\rho(=N_1N_2) = \Phi_{+,z_+}\Phi_{-,z_-}/(1-\Phi_+\Phi_-)^2$ of Liouville's equation into the first of the equations (60) and integrating them successively, we obtain

$$\left.\begin{aligned} &N_1 = u^{-1}[1-\Phi_+(\Phi_- + u\Phi_{-,\,z_-})]/(1-\Phi_+\Phi_-);\\ &N_2 = u\Phi_{+,\,z_+}\Phi_{-,\,z_-}/(1-\Phi_+\Phi_-)\,[1-\Phi_+(\Phi_- + u\Phi_{-,\,z_-})];\\ &N_3 = u^{-1}\frac{(\Phi_- + u\Phi_{-,\,z_-})_{,\,z_-}}{\Phi_{-,\,z_-}}\frac{1-\Phi_+\Phi_-}{[1-\Phi_+(\Phi_- + u\Phi_{-,\,z_-})]},\end{aligned}\right\} \tag{61}$$

where $\Phi_+(z_+)$, $\Phi_-(z_-)$, and $u(z_-)$ are arbitrary functions of their arguments. Thus, the Bäcklund transformation which is realized by the system (60) and relates the solutions ρ and ρ' of Liouville's equation leads to the replacement of the arbitrary function $\Phi_-(z_-)$ in $\exp\rho$ by the arbitrary function $\Phi_-(z_-)+u(z_-)\Phi_{-,z_-}$ in the solution $\exp\rho'$ $(=N_2N_3)$.

We require explicit solutions of Eqs. (57) in polynomial form; in accordance with the reduction scheme of Ref. 1, they can be written conveniently for the series A_r in terms of the functions $\exp(-x_\alpha)\equiv X_\alpha$,

$$X_\alpha \equiv \exp\left(-\sum_{\beta=1}^{r} k^{-1}_{\alpha\beta}\rho_\beta\right),\quad 1\leqslant\alpha\leqslant r, \tag{62}$$

which satisfy the system [cf. (47)]

$$X_{\alpha,\,z_+z_-}X_\alpha - X_{\alpha,\,z_+}X_{\alpha,\,z_-} = -X_{\alpha-1}X_{\alpha+1},\quad 1\leqslant\alpha\leqslant r,\quad X_0 = X_{r+1} = 1. \tag{63}$$

The solutions of the system (63), which depend on the $2r$ arbitrary functions $\Phi_{+\alpha}(z_+)$ and $\Phi_-(z_-)$, are given by

$$X_\alpha = (-1)^{\alpha(\alpha-1)/2}\Delta_\alpha(X),\quad 2\leqslant\alpha\leqslant r; \tag{64}$$

$$X_1 \equiv X = [\Delta_r(\Phi_+)\,\Delta_r(\Phi_-)]^{-1/(r+1)}\left[1+\sum_{\alpha=1}^{r}(-1)^\alpha\Phi_{+\alpha}\Phi_{-\alpha}\right], \tag{65}$$

where $\Delta_\alpha(X)$ are the principal minors of the matrix

$$X_{\alpha\beta} \equiv X_{,\underbrace{z_+\ldots z_+}_{\alpha-1}\underbrace{z_-\ldots z_-}_{\beta-1}},$$

and $\Delta_r(\Phi_+)$ and $\Delta_r(\Phi_-)$ are the determinants of the matrices

$$\Phi_{\beta,\,+\alpha} \equiv \Phi_{+\alpha,\underbrace{z_+\ldots z_+}_{\beta}}$$

and

$$\Phi_{\beta,\,-\alpha} \equiv \Phi_{-\alpha,\underbrace{z_-\ldots z_-}_{\beta}}$$

respectively. Therefore, in accordance with (64) for X_2, we arrive at the following expression for the function $\exp\rho_1 \equiv X_2 X_1^{-2} \equiv -(\ln X)_{,z_+z_-}$:

$$\exp\rho_1 = [-\sum_\alpha (-1)^\alpha \Phi_{+\alpha,z_+}\Phi_{-\alpha,z_-} + \sum_{\alpha<\beta} (\Phi_{+\alpha}\Phi_{+\beta,z_+} - \Phi_{+\alpha,z_+}\Phi_{+\beta})(\Phi_{-\alpha}\Phi_{-\beta,z_-} - \Phi_{-\alpha,z_-}\Phi_{-\beta})] \times [1+\sum_\alpha (-1)^\alpha \Phi_{+\alpha}\Phi_{-\alpha}]^{-2}, \tag{66}$$

and this is needed for the solution of Eq. (59). Equation (59) in conjunction with the equation $N_1N_2 = \exp\rho_1 = -(\ln X)_{,z_+z_-}$ enables us to find an expression for N_1, $N_1 = (\ln X)_{,z_-} + g(z_-)$, where $g(z_-)$ is the arbitrary function which realizes the functional arbitrariness (of one function) of the Bäcklund transformation noted above. Then for the function N_2, we obtain $N_2 = \exp\rho_1[(\ln X)_{,z_-} + g(z_-)]^{-1}$, whence, using the second of the equations (2), $N_{2,z_-} = N_2(N_3 - N_1) = \exp\rho_1' - \exp\rho_1$, we have

$$\begin{aligned}\exp\rho_1' = {}& \exp\rho_1 - Y^{-2}(Y')^{-2}\{[Y^2(uZ)_{,z_-} - Y^2_{,z_-}(uZ)] + (1/2)u^2[Y^2_{,z_-}Z_{,z_-} - Y^2_{,z_-z_-}Z]\} = \\ & = (Y')^{-2}\{-\sum_\alpha (-1)^\alpha \Phi_{+\alpha,z_+}(\Phi_{-\alpha} + u\Phi_{-\alpha,z_-})_{,z_-} + \sum_{\alpha<\beta}(\Phi_{+\alpha}\Phi_{+\beta,z_+} - \Phi_{+\alpha,z_+}\Phi_{+\beta})[(\Phi_{-\alpha}+u\Phi_{-\alpha,z_-}) \\ & \times (\Phi_{-\beta}+u\Phi_{-\beta,z_-})_{,z_-} - (\Phi_{-\alpha}+u\Phi_{-\alpha,z_-})_{,z_-}(\Phi_{-\beta}+u\Phi_{-\beta,z_-})]\},\end{aligned} \tag{67}$$

where

$$Z \equiv (1/2)Y_{,z_+z_-} - 2Y_{,z_+}Y_{,z_-}; \quad Y \equiv 1 + \sum_\alpha (-1)^\alpha \Phi_{+\alpha}\Phi_{-\alpha};$$
$$Y' \equiv 1 + \sum_\alpha (-1)^\alpha \Phi_{+\alpha}(\Phi_{-\alpha} + u\Phi_{-\alpha,z_-}),$$

and the functions u and g, which depend on the single variable z_-, are related by

$$g - [1/(r+1)][\ln\Delta_r(\Phi_-)]_{,z_-} = u^{-1}. \tag{68}$$

The expression (67) makes it possible to find explicitly the transformation connecting the functions $\Phi_{+\alpha}$ and $\Phi_{-\alpha}$, which determine the solution of the Toda chain (57) for ρ_α, $1 \leqslant \alpha \leqslant r$, to the functions $\Phi'_{+\alpha}$ and $\Phi'_{-\alpha}$, which determine the solutions ρ'_α, $1 \leqslant \alpha \leqslant r$, of the same system. It is simplest to arrive at (67) by noting that the denominator of the expression on the right-hand side of (67) contains the product $Y^2(Y')^2$ of the squares of two polynomials, one of which is equal to the denominator of Eq. (66) for $\exp\rho_1$, whereas the other, Y', corresponds to the solution $\exp\rho_1'$. Thus, the result of integrating the Bäcklund transformation for the system (57) can be represented in the form[3)]

$$X_\alpha = X_\alpha(\Phi_+, \Phi_-), \quad X'_\alpha = X_\alpha(\Phi_+, \Phi_- + u\Phi_{-,z_-}). \tag{69}$$

This enables us by means of the relations (58),

$$N_{2\alpha-1}N_{2\alpha} = \exp\rho_\alpha(\Phi_+, \Phi_-);$$
$$N_{2\alpha}N_{2\alpha+1} = \exp\rho_\alpha(\Phi_+, \Phi_- + u\Phi_{-,z_-}),$$

and knowing the general solutions X_α of the system (63) and Eq. (67), to reconstruct the general solutions of the system (2), which depend on the $2r+1$ arbitrary functions $\Phi_{+\alpha}$, $\Phi_{-\alpha}$, and $u(z_-)$, $1 \leqslant \alpha \leqslant r$. The final expressions for the functions $N_{2\alpha-1}$ and $N_{2\alpha}$ have the form

$$\left.\begin{aligned} N_{2\alpha-1} &= u^{-1}\left[\frac{\Delta_r(\Phi_- + u\Phi_{-,z_-})}{\Delta_r(\Phi_-)}\right]^{\frac{1}{r+1}} X_{\alpha-1}X'_\alpha(X'_{\alpha-1}X_\alpha)^{-1}; \\ N_{2\alpha} &= u\left[\frac{\Delta_r(\Phi_-)}{\Delta_r(\Phi_- + u\Phi_{-,z_-})}\right]^{\frac{1}{r+1}} X'_{\alpha-1}X_{\alpha+1}(X_\alpha X'_\alpha)^{-1}. \end{aligned}\right\} \tag{70}$$

Note that Eqs. (2) in the case of an even number of functions N_a can also be regarded as a Bäcklund transformation connecting the solutions of the system (57) for the algebra A_r to the solutions for A_{r-1}, which actually corresponds to replacement of the boundary conditions $N_0 = N_{2r+2} = 0$ by $N_0 = N_{2r+1} = 0$. If the solutions (70) obtained above are used, this procedure reduces to trivial operations, the condition $N_{2r+1} = 0$ leading to the ordinary differential equation $\Delta_r(\Phi_- + u\Phi_{-,z_-}) = 0$, which connects the $2r+1$ functions $\Phi_{+\alpha}$, $\Phi_{-\alpha}$, and u, namely, $u^{-1} = -[\ln(a_0 + \sum_\alpha a_\alpha \Phi_{-\alpha})]_{,z_-}$, where a_0 and a_α are arbitrary constants. The solutions of the system (2) then obtained for an even number of functions N_a depend on $2r$ arbitrary functions. [We emphasize that the vanishing of $\Delta_r(\Phi'_{-\alpha})$, which ensures fulfillment of the boundary condition N_{2r+1}, by no means entails the vanishing of all the remaining functions, since the presence in (70) of the common factor $\Delta_r(\Phi'_-)$ is compensated for N_a, $1 \leqslant a \leqslant 2r$, by the corresponding dependence on it of the functions X'_α.]

The solutions of the system (2) in the one-dimensional case can be obtained from the constructed general solutions (70) of the two-dimensionalized Volterra equations by the substitution $\Phi_{+\alpha} = c_{+\alpha}\exp m_\alpha z_+$, $\Phi_{-\alpha} = c_{-\alpha}\exp(-m_\alpha z_-)$, $u(z_-) \Rightarrow u_0 = \text{const}$. The functions X and X', which determine in accordance with (65) the solutions X_α and X'_α, take the form

$$\left.\begin{aligned} X &= [\prod_1^r d_\alpha m_\alpha^2 W^2 \exp(t\sum_{\beta=1}^r m_\beta)]^{-1/(r+1)} \times [1 + \sum_{\gamma=1}^r (-1)^\gamma d_\gamma \exp(tm_\gamma)]; \\ X' &= [\prod_1^r d_\alpha m_\alpha^2(1+u_0m_\alpha)W^2\exp(t\sum_{\beta=1}^r m_\beta]^{-1/(r+1)} \times [1+\sum_{\gamma=1}^r (-1)^\gamma d_\gamma(1+u_0m_\gamma)\exp(tm_\gamma)], \end{aligned}\right\} \tag{71}$$

where $d_\alpha \equiv c_{+\alpha}c_{-\alpha}$, $t \equiv z_+ - z_-$, $W \equiv W(m_1, \ldots, m_r)$ is the Vandermonde determinant (cf. Ref. 1).

Thus, the solutions of the one-dimensional Volterra equations are characterized by $2r+1$ arbitrary parameters d_α, m_α, and u_0 and can be written in the form

$$\left.\begin{aligned} N_{2\alpha-1} &= u_0^{-1}\left[\prod_{\beta=1}^r (1+u_0m_\beta)\right]^{1/(r+1)} \frac{\Delta_\alpha(X')\Delta_{\alpha-1}(X)}{\Delta_\alpha(X)\Delta_{\alpha-1}(X')}; \\ N_{2\alpha} &= u_0\left[\prod_{\beta=1}^r (1+u_0m_\beta)\right]^{-1/(r+1)} \frac{\Delta_{\alpha+1}(X)\Delta_{\alpha-1}(X')}{\Delta_\alpha(X)\Delta_\alpha(X')}, \end{aligned}\right\} \tag{72}$$

where $\Delta_\alpha(X)$ and $\Delta_\alpha(X')$ are the principal minors of the matrices $X_{\alpha\beta} \equiv d^{\alpha+\beta-2}X/dt^{\alpha+\beta-2}$ and $X'_{\alpha\beta} \equiv d^{\alpha+\beta-2}X'/dt^{\alpha+\beta-2}$. In the case of an even number of functions N_a, the parameters d_α, m_α, and u_0 cease to be independent.

The one-dimensional Volterra equations of the type considered above are intimately related to the nonlinear

3)One can see the validity of Eq. (67), which actually already contains the result of the transformation $\Phi'_{+\alpha} = \Phi_{+\alpha}$ and $\Phi'_{-\alpha} = \Phi_{-\alpha} + u\Phi_{-\alpha,z_-}$, directly by substituting the expressions $\exp\rho_1 = -(\ln Y)_{,z_+z_-}$, $\exp\rho_1' = -(\ln Y')_{,z_+z_-}$ and $N_1 \equiv Y'/uY$ into the equation $\exp\rho_1' - \exp\rho_1 = N_{2,z_-} = [N_1^{-1}\exp\rho_1]_{,z_-}$, which has to be verified; one then obtains the chain of relations $\{(\ln Y'/Y)_{,z_+} - (\ln Y)_{,z_+z_-} uY/Y'\}_{,z_-} = \{Y^{-1}Y'^{-1}[YY'_{,z_+} - Y_{,z_+} - Y_{,z_+}Y' - u(YY_{,z_+z_-} - Y_{,z_+}Y_{,z_-})]\}_{,z_-} = 0$, which completes the verification.

differential–difference system

$$M_{\alpha,t}=(1+M_\alpha^2)(M_{\alpha+1}-M_{\alpha-1}), \tag{73}$$

which can be made finite by imposing the boundary conditions $M_{-1}=-M_{N+1}\equiv i$:

$$R_{\alpha,t}=(1-R_\alpha^2)(R_{\alpha+1}-R_{\alpha-1}), \quad 0\leqslant\alpha\leqslant N, \tag{74}$$

where $R_\alpha\equiv -iM_\alpha$, $R_{-1}=-R_{N+1}\equiv 1$.

The connection between the solutions of the Volterra equations and the system (74) is given by the relations (see, for example, Ref. 8)

$$N_\alpha=(1+R_\alpha)(1-R_{\alpha-1}), \quad 1\leqslant\alpha\leqslant N, \tag{75}$$

which are a finite discrete variant of the Miura transformation.[37] Knowing the solutions of the Volterra equations, we can readily construct the exact solutions of the system (74) in accordance with Eqs. (75) and one of Eqs. (74). Indeed, the first equation in (74), $R_{0,t}=(1-R_0^2)(R_1-1)$, and the relation $N_1=(1+R_1)(1-R_0)$ enable us to eliminate R_1 and reduce the problem of finding the function $R_0\equiv f^{-1}-1$ to integration of the equation $f_{,t}+(N_1-4)f+2=0$, whose solution has the form

$$f=-2\left[1+\sum_\alpha(-1)^\alpha d_\alpha \exp(m_\alpha t)\right]^{-1}\left\{\lambda^{-1}+c_0\exp(-\lambda t)+\sum_\alpha(-1)^\alpha\frac{d_\alpha}{m_\alpha+\lambda}\exp(m_\alpha t)\right\}, \quad \lambda\equiv u_0^{-1}-4,\ c_0=\text{const}. \tag{76}$$

After this, the remaining unknown functions R_α, $1\leqslant\alpha\leqslant N$, can be expressed algebraically in accordance with (75) in terms of the known functions N_α and R_0. The final expression for the solutions can be represented in the form of the continued fractions

$$1+R_\alpha=\cfrac{N_\alpha}{2-\cfrac{N_{\alpha-1}}{2-\cfrac{N_{\alpha-2}}{2-\cfrac{N_{\alpha-3}}{\ddots\cfrac{}{2-\cfrac{N_3}{2-\cfrac{N_2}{2-\cfrac{N_1}{1-R_0}}}}}}}}, \quad 1\leqslant\alpha\leqslant N \tag{77}$$

In the case of an odd number of functions R_α, the general formulas (77) are not changed except for the dependence noted above that arises between the parameters d_α, m_α, and u_0.

As we have seen, the decisive factor in the construction of the general solutions of the Volterra equations (and their two-dimensional generalizations) was knowledge of the general solutions of the corresponding Toda chain, for which the former serve as the Bäcklund transformation. In this connection, it would be interesting to construct an analog of Eqs. (2), which would play the part of the Bäcklund transformations for the generalized Toda chain (32) corresponding to an arbitrary simple Lie algebra.

Note that for the equations of the two-dimensional Toda chain with periodic boundary conditions the Bäcklund transformation in the form (2) obviously leads to the conditions $N_0=N_{2r}$, $N_{2r+1}=N_1$, which, generally speaking, make it possible to integrate (2) in this case on the basis of the solutions for the periodic Toda chain (see the following section of the review).

3. INTEGRATION OF TWO-DIMENSIONAL NONLINEAR EQUATIONS ASSOCIATED WITH INFINITE-DIMENSIONAL GRADED ALGEBRAS

In this section, we consider systems of equations with exponential nonlinearities of the form

$$\rho_{\alpha,z_+z_-}=(k\exp\rho)_\alpha; \tag{78a}$$

$$x_{\alpha,z_+z_-}=\exp(kx)_\alpha \tag{78b}$$

with arbitrary numerical matrix k; for degenerate k, the connection between these systems does not have the nature of an identical substitution of functions $(\rho_\alpha=(kx)_\alpha)$. The representation (11) of Lax type in the case of compact algebras suggests a method for constructing a pair of operators for the systems (78).

To this end, we introduce a system of $3r$ operators $X_{\pm\alpha}$, h_α, $1\leqslant\alpha\leqslant r$, making them satisfy the commutation relations

$$\left[h_\alpha,\ \begin{matrix}h_\beta\\ X_{\pm\beta}\end{matrix}\right]_-=\begin{matrix}0\\ \pm k_{\beta\alpha}X_{\pm\beta}\end{matrix}; \quad [X_{+\alpha},X_{-\beta}]_-=\delta_{\alpha\beta}h_\alpha, \tag{79}$$

where k is the matrix which occurs in the system (78).

The relations (79) were studied in a number of papers (see, for example, Ref. 21), in which they were classified under certain restrictions on the matrix k. If (79) is to be accorded the nature of an algebra (finite- or infinite-dimensional), it is necessary to investigate the properties of the m-fold commutators $[X_{\pm\alpha_1}[X_{\pm\alpha_2}[\ldots[X_{\pm\alpha_{m-1}},X_{\pm\alpha_m}]]]]\equiv X_{\pm\alpha_1\ldots\alpha_m}$, which belong to the invariant subspace $C_{\pm m}$ with grading $\pm m$. The requirement that there be an invariant bilinear form in the representation space makes it possible to determine the norm of the element $X_{\pm\alpha_1\ldots\alpha_m}$ in the form $N_{\alpha_1,\ldots,\alpha_m}=(X_{-\alpha_m,\ldots,\alpha_1},X_{+\alpha_1,\ldots,\alpha_m})$ $((X_{+\alpha},X_{-\beta})=\delta_{\alpha\beta})$ and calculated by means of the Jacobi identity, the relations (79), and the definition of $X_{\pm\alpha_1,\ldots,\alpha_m}$ (as multiple commutator).

The algebras that arise in this manner are classified by calculating the dimension $D(m)$ of the invariant subspace with grading m (i.e., the number of linearly independent elements with nonvanishing norm). As quantitative criterion, we choose the ratio limit

$$\lim_{m\to\infty}\ln\left(\sum_{s=1}^m D(s)\right)/\ln m\equiv d. \tag{80}$$

These are the following three possibilities:

1) $d=0$, which corresponds to finite-dimensional simple Lie algebras;

2) $0<d=\text{const}<\infty$; the algebras which arise in this case are infinite-dimensional and are called algebras of finite growth;

3) $d=\infty$, which corresponds to infinite-dimensional, unbounded algebras.

In the first two cases, one can make a complete classification of the matrices k (the so-called generalized Cartan matrices), i.e., one can list them all and associate generalized Dynkin schemes with them. Naturally, semisimple Lie algebras arise as a special case of algebras of finite growth for which the determinants of the matrices k are nonvanishing.

By means of the generators (79), we construct a pair of operators $A_{z\pm}$ in the form [cf. formulas (*) in footnote 2]

$$A_{z_+} = \sum_\alpha (u_\alpha^+ h_\alpha + f_\alpha^+ X_{+\alpha}); \quad A_{z_-} = \sum_\alpha (u_\alpha^- h_\alpha + f_\alpha^- X_{-\alpha}). \tag{81}$$

Then the Lax-type representation $[\partial/\partial z_+ + A_{z+},\ \partial/\partial z_- + A_{z_-}] = 0$ leads to the system (78a) for $\rho_\alpha \equiv \ln f_\alpha^+ f_\alpha^-$.

To find the solutions of the system (78b), we introduce, as before, the operators $M^\pm$, defining them as the solutions of the S-matrix-type equations

$$M^\pm_{,z_\pm} = M^\pm \hat{L}^\pm; \quad \hat{L}^\pm = \sum_\alpha \varphi_{\pm\alpha}(z_\pm) X_{\pm\alpha}, \tag{82}$$

where $\varphi_{\pm\alpha}(z_\pm)$ are arbitrary functions of their arguments expressed in the form of anti-$\mathcal{L}_\pm$-ordered exponentials (21). We emphasize that in the general case the series in the arguments of the exponentials (22) are not bounded and contain an infinite number of terms (semisimple Lie algebras are an exception). For infinite-dimensional algebras, the concept of a group element becomes meaningless (at the same time, nilpotent infinite-dimensional subalgebras can be exponentiated, and the elements $M^\pm$ exist), and therefore the element $K \equiv (M^+)^{-1} M^-$ cannot, in general, be given a rigorously defined meaning. For this reason, we shall consider the solutions of (82) in the form of a series of successive approximations, the operators $\hat{L}^\pm$ or, ultimately, the functions $\varphi_{+\alpha}\varphi_{-\alpha}$ playing the part of a "small parameter." Then, substituting in K the expressions for $M^\pm$ up to some p-th order and determining to the same accuracy the elements of $N^\pm$ and $\exp H$ from the relation

$$K \approx N^- \exp(H)(N^+)^{-1}, \tag{83}$$

we conclude (repeating literally the arguments of Sec. 2.1) that $H \equiv \sum_\alpha h_\alpha H_\alpha$ satisfies the system of equations

$$\left.\begin{aligned} &H_{,z_+z_-} \approx [\hat{L}^-,\ \exp(-H)\,\hat{L}^+ \exp H];\\ &H_{\alpha,\,z_+z_-} \approx -\varphi_{+\alpha}\varphi_{-\alpha} \exp[-(kH)_\alpha]. \end{aligned}\right\} \tag{84}$$

Here, the symbol $\approx$ signifies fulfillment of the corresponding equations up to terms of p-th order in powers of $\varphi_{+\alpha_1} \ldots \varphi_{+\alpha_p} \varphi_{-\alpha_1} \ldots \varphi_{-\alpha_p}$. To make the transition from (84) to (78b), it must be noted that the functions $\varphi_{+\alpha}$ and $\varphi_{-\alpha}$ are related by

$$\sum_\alpha \ln \varphi_{+\alpha} \lambda_\alpha^{(s)} = 0, \quad \sum_\alpha \ln \varphi_{-\alpha} \lambda_\alpha^{(s)} = 0,$$

where $\lambda_\alpha^{(s)}$ is the set of left eigenvectors of the matrix k with vanishing eigenvalues: $(\lambda^{(s)} k)_\alpha = 0$. In the case of nondegenerate matrices k, all the functions $\varphi_{\pm\alpha}$ are functionally independent. Thus, the problem of solving the system (78b) for infinite-dimensional algebras is now concentrated around the problem of investigating the conditions of convergence of perturbation-theory series.

To obtain closed explicit expressions for the functions $\exp(-x_\alpha)$, which give the solutions of the system (78b), we introduce "generators of the highest vectors" R_α^+ and R_α^- of the fundamental representations of the algebra defined by Eqs. (79), making them satisfy the following commutation relations and orthonormally condition:

$$\left.\begin{aligned} &[X_{\pm\alpha},\ R_\beta^\pm]_- = 0; \quad [h_\alpha,\ R_\beta^\pm]_- = \pm\delta_{\alpha\beta} R_\beta^\pm;\\ &(R_\beta^-,\ R_\alpha^+) = \delta_{\alpha\beta}. \end{aligned}\right\} \tag{85}$$

By means of these generators, we find from the expansion (83) in an obvious manner

$$\begin{aligned} \exp H_\alpha &= (M^+ R_\alpha^- (M^+)^{-1},\ \ M^- R_\alpha^+ (M^-)^{-1}\\ &= \sum_n (-1)^n (\alpha_1, \ldots, \alpha_n)_+ (\alpha_1\beta_2, \ldots, \beta_n)_-\\ &\times \langle \alpha_1 | \alpha_1\beta_2 \ldots, \beta_n, \alpha_n \ldots \alpha_1 | \alpha_1 \rangle, \alpha_1 \equiv \alpha \end{aligned} \tag{86}$$

[the last equation is obtained using (85)]. Here, we have adopted the notation

$$(\alpha_1, \ldots, \alpha_n)_\pm \equiv \int^{z_\pm} \varphi_{\pm\alpha_1} dz_\pm^1 \int^{z_\pm^1} \varphi_{\pm\alpha_2} dz_\pm^2 \ldots \int^{z_\pm^{n-1}} \varphi_{\pm\alpha_n} dz_\pm^n,$$

which is a multiple integral of the indicated sequence of functions; the scalar product in (86) for the vectors $\alpha_n \ldots \alpha_1 | \alpha_1 \rangle \equiv [X_{-\alpha_n \cdots \alpha_1},\ R_{\alpha_1}^+]$, $\langle \alpha_1 | \alpha_1\beta_2 \ldots \beta_n \equiv [X_{+\beta_n \cdots \beta_2\alpha_1},\ R_{\alpha_1}^-]$ can be calculated by means of the Jacobi identities and Eqs. (79) and (85). Note that the expression (86) for $\exp H_\alpha$ is purely algebraic, and it determines this quantity as the scalar product of two vectors $M^\pm R_\alpha^\mp (M^\pm)^{-1}$.

For the solutions of the system (78b), we obtain by means of Eqs. (83), (84), and (86) the following formal final expression, whose domain of convergence requires an additional investigation:

$$\begin{aligned} \exp(-x_\alpha) = \exp\Big(-\sum_{s=1}^n \mu_\alpha^s \ln f_{+s} f_{-s}\Big) \exp\Big(-\sum_{t=n+1}^r \lambda_\alpha^t \ln f_{+t} f_{-t}\Big)\\ \times (M^+ R_\alpha^- (M^+)^{-1},\ M^- R_\alpha^+ (M^-)^{-1}), \end{aligned} \tag{87}$$

where $(k\mu^s)_\alpha = 0.1 \leqslant s \leqslant n$; $(k\lambda^t)_\alpha = E_t \lambda_\alpha^t$, $n+1 \leqslant t \leqslant r (E_t \neq 0)$, and the functions $\varphi_{\pm\alpha}$, which occur in the definition of $\hat{L}^\pm$ in (82), are related to $f_{\pm\alpha} (n+1 \leqslant s \leqslant r)$ by

$$\varphi_{+\alpha}\varphi_{-\alpha} = \prod_{s=n+1}^r (f_{+s} f_{-s})^{\lambda_\alpha^s}. \tag{88}$$

From (87), we can obtain a formal solution of the Goursat problem for the system (78). To see this, we make the solutions of (82) satisfy the initial conditions $M^+ = 1$ for $z_+ = a_+$ and $M^- = 1$ for $z_- = a_-$. Then as follows from (86) and (87),

$$\exp(-x_\alpha)|_{z_- = a_-} = c \exp\Big[-\sum_{s=1}^n \mu_\alpha^s \ln f_{+s}(z_+) - \sum_{t=n+1} \lambda_\alpha^t \ln f_{+t}(z_+)\Big],$$

i.e., the functions f_{+s}, $1 \leqslant s \leqslant n$, are determined by the values of $\exp(-x_\alpha)$ for fixed $z_- = a_-$. Similarly, the functions f_{-s}, $1 \leqslant s \leqslant n$, can be found from the values of $\exp(-x_\alpha)$ for $z_+ = a_+$. This gives the solution of the Goursat problem for the system (78b).

In the case of algebras of finite growth, when k is identical to their generalized Cartan matrices, one can make a general estimate for the terms of the p-th approximation in (86) by showing that for functions $\varphi_{\pm\alpha}$ that are bounded on the interval $a_+ \leqslant z_+ \leqslant b_+$, $a_- \leqslant z_- \leqslant b_-$ but otherwise arbitrary the series in (86) is absolutely convergent. To prove this assertion, we require the following two facts from the theory of representations of graded algebras of finite growth, which we give without proof (see Ref. 38). The generalized Cartan matrix of such an algebra is negative definite and has exactly one vanishing eigenvalue. The mutual scalar products of the basis vectors of the fundamental representations, containing equal numbers of raising and lowering operators $R^+_{\alpha_1\alpha_1\ldots\alpha_n} \equiv [X_{-\alpha_n,\ldots,\alpha_1}, R^+_{\alpha_1}]_-$ and $R^-_{\alpha_1\alpha_1,\ldots,\alpha_n} \equiv [X_{+\alpha_n,\ldots,\alpha_1}, R^-_{\alpha_1}]$, have the same sign, being positive for even n and negative for odd n, $(-1)^n (R^-_{\alpha_1,\ldots}, R^+_{\alpha_1,\ldots}) > 0$. We denote by μ^0_α the unique eigenvector of the matrix k with vanishing eigenvalue, $(k\mu^0)_\alpha = 0$. For all algebras of finite growth $\mu^0_\alpha > 0$,[21] and we normalize this vector by the condition $\min \mu^0_\alpha = 1$. The system (84) certainly has a solution of the form $H_\alpha = \mu^0_\alpha H_0$ under the condition $-\mu^0_\alpha(H_0)_{,z_+z_-} = \varphi_{+\alpha}\varphi_{-\alpha}$ for all α, i.e., $\varphi_{+\alpha}\varphi_{-\alpha} \equiv \mu^0_\alpha \varphi_+\varphi_-$ or $\varphi_{+\alpha} = \mu^0_\alpha\varphi_+$, $\varphi_{-\alpha} = \varphi_-$. At the same time, $H_0 = -\int \varphi_+ dz_+ \int \varphi_- dz_-$, and $\hat{L}^+ = \sum_\alpha \mu^0_\alpha X_{+\alpha}\varphi_+$, $\hat{L}^- = \sum_\alpha X_{-\alpha}\varphi_-$. Then for $M^\pm$, we obtain the final expressions

$$\left.\begin{aligned} M^+ &= \exp \sum_\alpha \mu^0_\alpha X_{+\alpha} \int dz_+ \varphi_+ \equiv \exp \hat{X}_+(\varphi_+); \\ M^- &= \exp \sum_\alpha X_{-\alpha} \int dz_- \varphi_- \equiv \exp \hat{X}_-(\varphi_-). \end{aligned}\right\} \tag{89}$$

To determine $\hat{X}_\pm$, we have

$$[\hat{X}_+, \hat{X}_-]_- = \sum_\alpha \mu^0_\alpha h_\alpha \equiv \hat{h}; \ [\hat{h}, X_{\pm\beta}] \equiv \pm \sum_\alpha \mu^0_\alpha k_{\beta\alpha} X_{\pm\beta}.$$

Substituting (89) in (86) and using the following property of the multiple integral,

$$\underbrace{\int \varphi \, dz_1 \int^{z_1} \varphi \, dz_2 \ldots \int^{z_{n-1}} \varphi \, dz_n}_{n} = \left(\int \varphi \, dz\right)^n / n!,$$

we find

$$\begin{aligned} \exp H_\alpha &= (M^+ R^-_\alpha (M^+)^{-1}, M^- R^+_\alpha (M^-)^{-1}) \\ &= \exp\left(-\mu^0_\alpha \int dz_+ \varphi_+ \int dz_- \varphi_-\right) \\ &= \sum_n (-1)^n \frac{(\int \varphi_+ dz_+)^n}{n!} \frac{(\int \varphi_- dz_-)^n}{n!} \\ &\times \sum \mu^0_\alpha \mu^0_\beta \ldots \mu^0_\gamma (R^-_{\alpha\beta,\ldots,\gamma}, R^+_{\alpha\eta\ldots\varkappa}), \end{aligned}$$

i.e., $\sum \mu^0_\alpha \mu^0_\beta \ldots \mu^0_\nu (R^-_{\alpha\beta\ldots\nu}, R^+_{\alpha\eta\ldots\varkappa}) \equiv (\mu^0_\alpha)^n n!$

To estimate the terms of the n-th approximation in (86), we consider the functions $\varphi_{+\alpha}$ and $\varphi_{-\alpha}$, which are bounded on the intervals (a_+, z_+) and (a_-, z_-) and for which $|\varphi_{-\alpha}| \leqslant M_-$, $|\varphi_{+\alpha}| \leqslant M_+ \leqslant \mu^0_\alpha M^+$ (the last inequality follows from the normalization adopted for the vector $\mu^0_\alpha \geqslant 1$). For the n-fold integrals in (86) we have the obvious estimates

$$\begin{aligned} |(\alpha_1\beta_2 \ldots \beta_n)_-| &\leqslant M^n_- (z_- - a_-)^n / n!; \\ |(\alpha_1 \ldots \alpha_n)_+| &\leqslant M^n_+ \mu^0_{\alpha_1} \ldots \mu^0_{\alpha n} (z_+ - a_+)^n / n! \end{aligned} \tag{90}$$

Substituting (90) in (86) and using the above sum rule and the positive definiteness of the scalar products of the basis vectors, we obtain the following inequality for the n-th term of the series S_n:

$$|S_n| \leqslant M^n_+ M^n_- (z_+ - a_+)^n (z_- - a_-)^n / n!$$

Thus, the series which determines the expressions for the solutions of the system (78b) in the case when k coincides with the Cartan matrices of graded algebras of finite growth is absolutely convergent and gives the solutions of the Goursat problem for the system (78b), the solutions depending on the necessary number of arbitrary functions. We note that formally the series (86) also gives the solution of the system (78b) for arbitrary matrices k, but the problem of its domain of convergence requires an additional investigation.

Among the equations of the system (78b) corresponding to infinite-dimensional contragredient algebras of finite growth, the best known are the sine–Gordon equations

$$\varphi^{(1)}_{,z_+z_-} = 2\exp\varphi^{(1)} - 2\exp(-\varphi^{(1)}); \tag{91}$$

$$\varphi^{(2)}_{,z_+z_-} = \exp 2\varphi^{(2)} - 2\exp(-\varphi^{(2)}). \tag{92}$$

These equations follow from (78b) for matrices k of the form $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$, with $\varphi^{(1)} \equiv 2x_1 - 2x_2$ and $\varphi^{(2)} \equiv x_1 - 2x_2$, respectively. Solutions of these equations, which depend on two arbitrary functions, are constructed in Ref. 6 and are a direct consequence of the general expressions of the present section:

$$\exp\varphi^{(1)}/2 = \varphi^{1/2}_{+1}(z_+)\varphi^{1/2}_{-1}(z_-) X^{(1)}_2 (X^{(1)}_1)^{-1}; \tag{93}$$

$$\exp(-\varphi^{(2)}) = \varphi_{+1}(z_+)\varphi_{-1}(z_-) X^{(2)}_2 (X^{(2)}_1)^{-2}; \tag{94}$$

$$X^{(1)}_\alpha = X_\alpha |_{\varphi_{\pm2} = \varphi^{-1}_{\pm1}}; \ X^{(2)}_\alpha = X_\alpha |_{\varphi_{\pm2} = \varphi^{-2}_{\pm1}};$$

$$X_\alpha = (M^+ R^-_\alpha (M^+)^{-1}, M^- R^+_\alpha (M^-)^{-1}), \quad \alpha = 1, 2; \tag{95}$$

$$M^\pm(z_\pm) = \mathfrak{Z}_\pm \exp \int^{z_\pm} dz'_\pm [\varphi_{\pm1}(z'_\pm) X_{\pm1} + \varphi_{\pm2}(z'_\pm) X_{\pm2}].$$

The infinite series that arise when the expressions for X_α are written out in the case under consideration are, as was shown in the general case for algebras of finite growth, absolutely convergent. As an illustration, we write down the first eight terms in the expansion of X_1 in powers of $\varphi_{+\alpha}\varphi_{-\alpha}$ (X_2 is obtained from X_1 by the obvious substitution $\varphi_{\pm1} \rightleftarrows \varphi_{\pm2}$):

$$\begin{aligned} X_1 = \sum_\beta [(-1)^\beta \sum_\alpha c_\alpha |X^\beta_\alpha|^2] &= 1 - |X^1_1|^2 + 2|X^2_1|^2 \\ &- 4|X^3_1|^2 - 2|X^3_2|^2 + 4|X^4_1|^2 + 8|X^4_2|^2 \\ &- 8|X^5_1|^2 - 8|X^5_2|^2 - 16|X^5_3|^2 + 8|X^6_1|^2 + 16|X^6_2|^2 \\ &+ 16|X^6_3|^2 + 32|X^6_4|^2 - 16|X^7_1|^2 - 16|X^7_2|^2 \\ &- 16|X^7_3|^2 - 32|X^7_4|^2 - 64|X^7_5|^2 + 32|X^8_1|^2 \\ &+ 32|X^8_2|^2 + 64|X^8_3|^2 + 64|X^8_4|^2 + 64|X^8_5|^2 + 128|X^8_6|^2 \quad \ldots. \end{aligned}$$

Here, $|X^\beta_\alpha|^2 \equiv X^\beta_{+\alpha} X^\beta_{-\alpha}$; the superscript is the number of the approximation, and the subscript is the serial number of the term of the approximation:

$$\begin{aligned} &X^1_{\pm1} = (1)_\pm \ X^2_{\pm1}; = (12)_\pm; \ X^3_{\pm1} = (122)_\pm; \ X^3_{\pm2} = (121)_\pm; \\ &X^4_{\pm1} = (1212)_\pm + (1221)_\pm; \ X^4_{\pm2} = (1221)_\pm; \ X^5_{\pm1} = (12122)_\pm \\ &+ (12212)_\pm; \ X^5_{\pm2} = (12121)_\pm + 2(12211)_\pm; \ X^5_{\pm3} = (12211)_\pm; \\ &X^6_{\pm1} = (122121)_\pm + (121221)_\pm; \ X^6_{\pm2} = (121211)_\pm + 3(122111)_\pm; \\ &X^6_{\pm3} = 2(122112)_\pm + (121212)_\pm \\ &+ (122121)_\pm + (121221)_\pm; \ X^6_{\pm4} = (122112)_\pm. \end{aligned}$$

To conclude this section, we indicate one further possible way of solving the system (78b), which we write in the form

$$x_{\alpha,z_+z_-} = \varphi_{+\alpha}\varphi_{-\alpha}\exp(kx)_\alpha; \ \hat{x}_{,z_+z_-} = [\exp\hat{x}\hat{L}^+\exp(-\hat{x}), \hat{L}^-]_-, \tag{96}$$

where $\hat{x} \equiv \sum_\alpha h_\alpha x_\alpha$, and the operators $\hat{L}^\pm$ are determined by (82). We shall regard the functions $\varphi_{+\alpha}\varphi_{-\alpha}$ in (96) as small quantities, with respect to which an iteration procedure can be performed. Expanding the solutions x_α of the system (96) in quantities of the corresponding smallness order,

$$x_\alpha = \sum_{s=1}^\infty x^{(s)}_\alpha; \ \hat{x} = \sum_{s=1}^\infty \hat{x}_{(s)},$$

we obtain for the terms of the n-th approximation from (96)

$$\hat{x}_{(n),z_+z_-} = \sum_{l_1,\ldots,l_n} \frac{1}{l_1!,\ldots,l_n!} [[\hat{x}^{l_{n-1}}_{(n-1)} \ldots \hat{x}^{l_1}_{(1)}, \hat{L}^+], \hat{L}^-], \tag{97}$$

where $l_1+2l_2+\ldots+(n-1)l_{n-1}=n-1$, and $[\hat{x}_{(n-1)}^{l_{n-1}}\ldots\hat{x}_{(1)}^{l_1}, \hat{L}^+]_-$ are the sequence of l_1-, ..., l_{n-1}-fold commutators of the mutually commuting operators $\hat{x}_{(1)}, \ldots, \hat{x}_{(n-1)}$ with $\hat{L}^+$. Then, using the standard technique and making some fairly lengthy calculations, we arrive at the formal perturbation series

$$x_\alpha=\sum_n(-1)^n(\alpha\alpha_2, \ldots, \alpha_n)_+(\alpha\beta_2, \ldots, \beta_n)_- \times(R_\alpha^- X_{+\alpha_2}, \ldots, X_{+\alpha_n}, X_{-\beta_n}, \ldots, X_{-\beta_2}R_\alpha^+). \quad (98)$$

Going over from the functions x_α to the exponentials $\exp(-x_\alpha)$, we arrive at the previously obtained expressions for the solutions of the system (78b). Note that from the point of view of physics this transition corresponds to allowance for disconnected diagrams in the calculation of the vacuum expectation value of the S matrix.

CONCLUSIONS

The investigation of the self-duality equations for cylindrically symmetric Yang–Mills fields made in this review completes the problem of constructing the general solutions of these equations for arbitrary embedding of the subgroup $SU(2)$ in an arbitrary simple gauge Lie group and ensures a unified description of instanton and monopole configurations. The method developed for integrating the nonlinear systems (10) that arise is based on the theory of group representations and explicit realization of the Lax-type representation (11) by the operators (12), which take values in the algebra of the corresponding group (which is not in general compact or simple). It is found that the proposed general scheme for integrating the self-duality equations encompasses a large class of nonlinear systems of the type (1), (2), (52), (55), (74), and (78), which are associated with finite-dimensional Lie algebras (both simple and nonsemisimple) and infinite-dimensional contragredient algebras of finite growth, which are encountered in numerous physical applications. In particular, the equations that describe the generalized two-dimensional Toda chain with fixed ends and periodic boundary conditions are included among those listed above. It is important that all the above equations, which at the first glance are in no way related, are presented in the framework of this method as concrete realizations of a single algebraic construction, and the possibility of their complete integration is due to this algebraic basis of the equations.

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APPENDIX 1

The methods developed above for integrating nonlinear dynamical systems associated with grading algebras can be generalized to the supersymmetric case. The odd elements of the corresponding superalgebras are associated with anticommuting (spinor) fields which take values in the Grassmann algebra. Here, we consider in detail the supersymmetric generalization of Liouville's equation associated with a superalgebra of the type $B(0, 1)$ (O Sp(2, 1)), the aim being to follow in detail the differences between the integration of supersymmetric equations and ordinary equations.

The supersymmetric Liouville equation corresponding to the action $dz_+dz_-d\theta_+d\theta_-[-1/2\hat{\Phi}\hat{D}_-\hat{D}_+\hat{\Phi}+\exp\Phi]$, has the form

$$\hat{D}_-\hat{D}_\oplus\hat{\Phi}=\exp\hat{\Phi}, \quad (A.1)$$

where

$$\hat{\Phi}=\hat{\Phi}(z_\pm, \theta_\pm)=\rho(z_\pm)-:\bar{\theta}\omega(z_\pm)-:1/2\bar{\theta}\theta F(z_\pm)$$

is a superscalar field consisting of the two scalar fields ρ and F and the Majorana spinor $\omega^\pm$, which are functions with anticommuting values; the superscalar field depends on the coordinates $z_\pm$ of the two-dimensional space and the Grassmann variables $\theta\equiv\binom{\theta_+}{\theta_-}$;and $\bar{\theta}\equiv(-:\ \theta_-, \theta_+)$. By $\hat{D}_\pm$ we denote the operators of superdifferentiation, $\hat{D}_\pm=\mp\partial/\partial\theta_\pm+\theta_\pm\partial/\partial z_\pm$; $\hat{D}_\pm^2=\mp\partial/\partial z_\pm$, $\hat{D}_+\hat{D}_-=-\hat{D}_-\hat{D}_+$. In the components of the superfield $\hat{\Phi}(F=\exp\rho)$, Eq. (A.1) has the form

$$\begin{aligned}\rho_{,z_+z_-}&=\exp 2\rho+\exp\rho\omega^+\omega^-,\\ \omega^\pm_{,z_\mp}&=\exp\rho\omega^\mp,\end{aligned} \quad (A.2)$$

and in the case $\omega^\pm=0$ it goes over naturally into the ordinary Liouville equation and is identical to the one obtained earlier in Ref. 40.

The superalgebra $B(0, 1)$ (see, for example, Ref. 41) consists of the five elements h, $X_\pm$, and $Y_\pm$, which satisfy the commutation relations

$$\left.\begin{aligned}&[h, X_\pm]_-=\pm 2X_\pm;\ [h, Y_\pm]_-=\pm Y_\pm;\ [X_+, X_-]_-=[Y_+, Y_-]_+=h;\\ &[X_\pm, Y_\pm]_-=0;\ [X_\pm, Y_\mp]_-=Y_\pm;\ [Y_\pm, Y_\pm]_+=\mp 2X_\pm.\end{aligned}\right\} \quad (A.3)$$

We introduce the following operators $A_\pm$ which take values in the algebra $B(0, 1)$:

$$A_\pm=u^\pm h+\varphi^\pm X_\pm+\psi^\pm Y_\pm, \quad (A.4)$$

where $u^\pm(z_+, z_-)$ and $\varphi^\pm(z_+, z_-)$ are ordinary functions, and $\psi^\pm(z_+, z_-)$ are anticommuting functions, $(\psi^\pm)^2=\psi^+\psi^-+\psi^-\psi^+=0$. Then the representation of "zero curvature" for the operators $A_\pm$,[4)]

$$[\partial/\partial z_++A_+, \partial/\partial z_-+A_-]_-=0, \quad (A.5)$$

leads to the system

$$\left.\begin{aligned}&u^-_{,z_+}-u^+_{,z_-}+\varphi^+\varphi^-+\psi^+\psi^-=0;\\ &\varphi^\mp_{,z_\pm}=\pm 2u^\pm\varphi^\mp,\quad \psi^\mp_{,z_\pm}\mp u^\pm\psi^\mp=\varphi^\mp\psi^\pm,\end{aligned}\right\} \quad (A.6)$$

and after the obvious change of variables $\psi^+\psi^-\equiv\exp 2\rho$, $\psi^\pm\equiv\omega^\pm(\varphi^\pm)^{1/2}$ this reduces to Eqs. (A.2).

The representation (A.5) is the condition for the operators $A_\pm$ to be gradients, i.e.,

$$A_\pm=g^{-1}g_{,z_\pm}, \quad (A.7)$$

where g is an element of the complex hull of the supergroup G (Ref. 42) with generators (A.3), and it can be represented in the form of the Gauss decomposition

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4)The matrix realization of the Lax representation for (A.1) contained in Ref. 40 corresponds to the special representation $B(0, 1)$ in the framework of our approach.

$$g = M^+N^- \exp H = M^-N^+ \exp H', \tag{A.8}$$

in which $M^{\pm}$ and $N^{\pm}$ are elements of the complex hulls of the maximal nilpotent subgroups of G spanned by $X_{\pm}$ and $Y_{\pm}$, and H and H' belong to the Cartan subalgebra of G. In what follows, for simplicity we take the gauge $H' = 0$, in which $u^+ = 0$ and $\varphi^-{}_{,z_+} = 0$. It follows from Eqs. (A.4), (A.7), and (A.8) that the elements $M^{\pm}$ can be represented in the form $M^{\pm} = \exp(m^{\pm}X_{\pm} + \varepsilon^{\pm}Y)$, where $m^+(z_+)$, $m^-(z_-)$ and $\varepsilon^+(z_+)$, $\varepsilon^-(z_-)$ are, respectively, ordinary and anticommuting functions of their arguments (cf. (21)). The identity $(M^+)^{-1}M^- = N^- \exp H(N^+)^{-1}$, which follows from (A.8), makes it possible to determine the group parameters of the elements $N^{\pm} \equiv \exp(\tilde{m}^{\pm}X_{\pm} + \tilde{\varepsilon}_{\pm}Y_{\pm})$ and $\exp H = \exp(rh)$ in terms of the arbitrary functions $m^{\pm}$ and $\varepsilon^{\pm}$, which parametrize $M^{\pm}$; namely,

$$\left.\begin{array}{l} \exp(-r) = 1 - m^+m^- - \varepsilon^+\varepsilon^-, \quad \tilde{\varepsilon}^{\pm} = (\varepsilon^{\pm} + m^{\pm}\varepsilon^{\mp}) \exp r, \\ \tilde{m}^{\pm} = m^{\pm} \exp r. \end{array}\right\} \tag{A.9}$$

Substituting (A.8) with the elements $M^{\pm}$, $N^{\pm}$, and $\exp H$ [which are known in accordance with (A.9)] in (A.7) and comparing the result with (A.4), we find the following final expression for the general solutions of the supersymmetric Liouville equation:

$$\left.\begin{array}{l} \varphi^{\pm} = (m^{\pm}_{,z_{\pm}} \pm \varepsilon^{\pm}\varepsilon^{\pm}_{,z_{\pm}}) \exp[(1 \pm 1) r]; \\ u = -(\varepsilon^+\varepsilon^-_{,z_-} + m^+m^-_{,z_-}) \exp r; \\ \psi^+ = \lambda^{-1}\varepsilon^+_{,z_+} + \lambda^{-2}m^+_{,z_+}(\varepsilon^- + \varepsilon^+m^-); \\ \lambda \equiv 1 - m^+m^-; \\ \psi^- = \varepsilon^-_{,z_-}(1 - \lambda^{-1}\varepsilon^+\varepsilon^-) + \lambda^{-1}m^-_{,z_-}(\varepsilon^+ + \varepsilon^-m^+); \\ \exp 2\rho = (m^+_{,z_+} + \varepsilon^+\varepsilon^+_{,z_+})(m^-_{,z_-} - \varepsilon^-\varepsilon^-_{,z_-}) \exp 2r. \end{array}\right\} \tag{A.10}$$

The above method of integrating the supersymmetric Liouville equation (A.1) can be generalized in a natural manner to arbitrary graded superalgebras. The core of the problem is in the construction of the elements $N^{\pm}$ and $\exp H$ in (A.8) from known $M^{\pm}$, which, as in the case of "ordinary" graded algebras, satisfy equations of S-matrix type. To solve the nonlinear equations associated with the graded algebras characterized by a Cartan matrix (in general, generalized),[21] it is necessary to know the value of the element $\exp H$ in (A.8), whose parameters can be determined by calculating the matrix elements of the known operator $(M^+)^{-1}M^-$ between the highest and lowest vectors of the basis. However, the simplest example of the supersymmetric Liouville equation already shows that the calculation of the highest vector of the element $(M^+)^{-1}M^-$, which is equal to $1 - m^+m^- - \varepsilon^+\varepsilon^-$, is not sufficient for the description of the complete solution (A.10) of the system (A.1).

Note that the supersymmetric generalizations of the sine-Gordon equations (see, for example, Refs. 40 and 43) $\tilde{\rho}_{,z+z-} = 2\exp\tilde{\rho} - 2\exp(-\tilde{\rho})$ and $\tilde{\rho}_{,z+z-} = 2\exp\tilde{\rho} - \exp(-2\tilde{\rho})$, which have a nontrivial group of internal symmetries in ordinary space, are apparently related to the superalgebras of finite growth with Dynkin schemes

 and 

and may be integrated like them. All these questions require further investigations.

TABLE I. Table of highest roots of the simple Lie algebras.*

A_r, $r \geqslant 1$	$\pi_1, \pi_1+\pi_2, \ldots, \pi \equiv \sum_{j=1}^{r} \pi_j$	B_r, $r > 1$	$\pi, \pi+\pi_r, \pi+\pi_{r-1}+\pi_r, \ldots, 2\pi-\pi_1$
C_r, $r > 1$	$\pi, \pi+\pi_{r-1}, \pi+\pi_{r-1}+\pi_r, \ldots, 2\pi-\pi_r$	D_r, $r > 2$	$\pi-\pi_{r-1}, \pi-\pi_r, \pi, \pi+\pi_{r-2}, \pi+\pi_{r-3}+\pi_{r-2}, \ldots, 2\pi-\pi_{r-1}-\pi_r-\pi_1$
G_2	(13), (23)	F_4	(1232), (1242), (1243), (2243)
E_6	(101212), (112211), (111212), (112212), (112312), (112322)	E_7	(1112322), (1212322), (1212313), (1212323), (1212423), (1213423), (1223423)
E_8	(12324524), (12323534), (12324534), (12324634), (12324635), (12424635), (13424635), (23424635)		

*The highest roots are arranged in order of ascending height. To simplify the expression of the highest roots of the exceptional Cartan algebras, we have written out only the coefficients of the expansion in the simple roots π_α, $1 \leqslant \alpha \leqslant r$.

APPENDIX 2

To facilitate the use of the obtained expressions for the solutions (39) of the system (1) in the case of simple Lie algebras, we give here explicit formulas for the highest vectors $\xi^{\{l\}}$ of the irreducible representations of the corresponding groups. As is shown in Ref. 39, they are completely determined by the system of highest roots of the group, one of the possible forms of expression of these being given in Table I. Then the highest vectors are expressed by the formula $\xi^{\{l\}}(K) = \Pi_1^r [D_j(a)]^{\mu_j}$, where D_j are the principle minors of the matrix $a_{ab} \equiv \mathrm{Sp}(X_{-a}KX_bK^{-1})$ of the adjoint representation, and they are reckoned from the maximal root s of the system of highest roots: $\mu_j = \sum_1^r l_i(\lambda^{-1})_{ji}$, $\lambda_{ij} \equiv \sum_{a=s-j+1}^{s} a_i$. In the case in which we are interested, $K = (M^+)^{-1}M^-$, so that

$$a_{ab} \equiv \mathrm{Sp}\{[M^+X_{-a}(M^+)^{-1}][M^-X_b(M^-)^{-1}]\}.$$

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