Master equations in the theory of nuclear reactions

V. E. Bunakov

Institute of Nuclear Physics, USSR Academy of Sciences, Leningrad

Fiz. Elem. Chastits At. Yadra **11**, 1285–1333 (November–December 1980)

The currently existing microscopic, semiphenomenological, and phenomenological models used to describe nuclear reactions are reviewed. Their advantages and shortcomings are emphasized. Particular attention is devoted to analyzing phenomenological models of nuclear reactions based on the use of master equations (nonstationary variants of the exciton models, intranuclear cascade model, etc.). A unified theory of nuclear reactions based on quantum kinetic equations for finite open systems is discussed. Conditions for the transition to these equations for the nonstationary Schrödinger equation in many-body theory are formulated. It is shown that all possible models of nuclear reactions used in practice are approximate solutions to the master equations for finite open systems. The reliability of the description of the experimental data in these models is analyzed from this point of view, and consideration is given to their various modifications, which make it possible to improve the agreement between the results of the various models and give a more accurate description of the data.

PACS numbers: 24.10. – i, 24.90. + d

INTRODUCTION

Several new directions are manifested in the modern development of nuclear physics. One of them is associated with an increasing importance of nuclear reactions in both theoretical and experimental studies. In the past, the most important questions in nuclear structure were "separated" fairly clearly from the mechanisms of the nuclear reactions, which played a subsidiary part in nuclear studies. But nuclear spectroscopy has now gone beyond the traditional framework of weakly excited bound states of nuclei and encompasses the region of high excitation energies, in which the concept of individual states of a system gradually becomes meaningless because the resonances of the compound system overlap to a greater and greater extent. As a consequence, problems relating to nuclear structure and the reaction mechanisms become ever more intimately entwined. A second tendency is associated with the accumulation of experimental information, improvement in its quality, and an increase in its complexity. To discover a new phenomenon or process on the background of ones already known, it is necessary to make a quantitative analysis of the already known processes, which significantly raises the requirements that must be met by modern theory.

In this connection, we require a theory of nuclear reactions capable of giving a fairly complete quantitative description of processes in a wide range of energies of the incident particles and reaction products.

In the present paper, we shall restrict ourselves to analyzing the state of the modern theory of nuclear reactions involving nucleons. We shall begin by considering briefly the main classes of theoretical models used in this theory. These models can be divided into two almost disjunct groups. In the first, we have the theoretically complete microscopic approaches, which demonstrate in principle how the theory of nuclear reactions should be constructed. The second group consists of phenomenological and semiphenomenological models used in practical calculations of nuclear reactions. It is to this group that we shall devote the main attention in our review. Unfortunately, the majority of the models in this class had until recently no connection at all with fundamental theory. Many of them (for example, the so-called pre-equilibrium decay models) were based on the use of phenomenological master equations (or balance equations). Until recently, the question of the connection between these equations and the quantum-mechanical approach in the theory of reactions remained unsolved (see, for example, Feshbach's review in Ref. 1). However, in recent time some research groups, including ours, have achieved successes in this direction.[2–5] Therefore, in describing the existing phenomenological models, we attempt to explain why they "work," using the theory of nuclear reactions based on quantum kinetic equations for finite open systems.

The final section of the review is devoted to considering the accuracy of the existing models and possible modifications that improve them.

Any correct theory of reactions is based on the nonstationary Schrödinger equation for many-body systems:

$$i\hbar\partial\Psi/\partial t = H\Psi \tag{1}$$

with appropriate initial and boundary conditions. To describe even the simplest case of potential scattering, it is necessary to construct wave packets out of solutions to the stationary Schrödinger equation

$$H\psi = E\psi \tag{2}$$

and follow their evolution in space and in time. This is a very complicated mathematical procedure, and it has been properly treated in only a few textbooks (see, for example, Refs. 6 and 7). The Lippmann–Schwinger formalism uses the approximation of very long (in time) packets and makes it possible to reduce the problem to the stationary case with specific boundary conditions. This formalism provides the basis of the R-matrix theory and Feshbach's unified theory of reactions, which led to the creation of a large number of microscopic approaches to nuclear reactions. In all the listed approaches, the total Hamiltonian H is usually divided into two parts:

$$H = H_0 + V. \tag{3}$$

The part H_0 determines a system of basis wave functions in an average field, and the "residual interactions" V determine the intensities of transitions be-

 0090-4759/80/060507-21$02.80

tween the states of the basis. The various methods of choosing the basis correspond to different microscopic models and we have, for example, the shell model in the continuum,[8] the eigenchannel method,[9] and Soloviev's model for low-energy resonance states.[10] However, all these methods are usually too complicated for practical use. First, it is very difficult to find a solution (diagonalize the matrix of the Hamiltonian) for a sufficiently complete basis. Second, even if the basis is sufficiently complete, the accuracy in the calculation of the position of each resonance does not exceed a few hundred keV, whereas the mean distance D between resonances in the majority of real nuclei is considerably less (for medium and heavy nuclei $D \approx 1$ eV). This leads to a paradox; for one can find the position and width of individual resonances both theoretically and experimentally, but one can only compare characteristics averaged over a large group of resonances in an interval ΔE of the order of several MeV. Besides this, all the microscopic approaches based on the use of the Lippmann–Schwinger equation have a further serious shortcoming. The point is that this equation does not have a unique solution in the case of more than two particles in the continuum (for example, incident particle + target nucleus), and therefore these approaches can be used only for energies that do not permit the escape of two nucleons from the nucleus, i.e., in the region of energies $E \lesssim 10$ MeV of the incident particles.

1. PRACTICAL APPROACHES BASED ON PHENOMENOLOGICAL MODELS

Because of these difficulties, a different group of models is used for the practical description of nuclear reactions. They are all purely or semiphenomenological. One of the most popular among them is the optical model, which describes the elastic scattering cross section averaged over the energy by means of a single-particle Schrödinger equation with complex potential. The use of this model can be justified theoretically (see, for example, Refs. 7 and 11–14) in the framework of a microscopic description. The rate of absorption of the incident beam in this model is determined by the imaginary part of the optical potential, the imaginary part being associated with the matrix elements of the residual interactions V averaged over a sufficiently large energy interval ΔE. Despite its simplicity, the optical model makes it possible to establish a connection with the fundamental equations of the microscopic approaches. It is therefore the best model for describing the averaged elastic (or total) cross sections. However, it cannot be used without further modification to describe inelastic processes.

One such modification is the Distorted Wave Born Approximation (DWBA), in which the amplitude T_{if} of an inelastic process is determined by an expression of Born type

$$T_{if} \sim \langle \bar{\psi}_f | V | \bar{\psi}_i \rangle. \tag{4}$$

Here, $\bar{\psi}_i$ and $\bar{\psi}_j$ are wave functions of the elastic and inelastic channels calculated in the potential of the optical model. The DWBA (and its high-energy modifications—the eikonal method and Glauber theory) also belongs to the semiphenomenological class of models, since (4) can be obtained from the microscopic approach (see, for example, Refs. 6 and 7). Such a derivation shows, however, that the expression (4) is suitable only for the description of direct reactions that take place directly after the initial particle collides with the nucleus. To the DWBA cross section it is therefore necessary to add contributions from the compound nucleus and pre-equilibrium processes. It is almost impossible to estimate these contributions, and in practice they are used as free parameters to match theory and experiment.

Because of this incompleteness of the optical model and the DWBA, various purely phenomenological models are used in practice, and these have absolutely no connection with fundamental Schrödinger equations. They are all based on intuitive postulates, and therefore contain a large number of adjustable parameters, whose physical meaning depends on the interpretation of the particular model.

The best known model describing the energy and angular distributions of the secondary particles is the intranuclear cascade model. Many books and reviews (see, for example, Ref. 15) have been devoted to the description of this model. For many years, the model was used as a basis for describing reactions with high-energy (hundreds of MeV) incident particles. Recently, however, some more or less successful attempts have been made to apply the model to reactions with particles of lower energies.[16,17]

It is usually assumed that the intranuclear cascade model makes it possible to simulate in a classical manner the history of the interaction between the nucleon and the nucleus by means of a computer. The algorithm of such an approach[18] is as follows. Using a computer, we "play" a set of cases of collision between the incident particle and the nucleus. In each of these cases, the initial particles generates a cascade of secondary particles, which leave the nucleus with definite energy and momentum. The number of such secondary particles and their momentum distribution vary strongly from case to case. However, if we consider a large number of collisions and average over all the results, we obtain the most probable distribution of the emitted particles. One can follow the classical trajectory of the incident nucleon within the nucleus until the first collision, which leads to the formation of pairs of secondary nucleons and holes in the Fermi surface of the target nucleus. One can then follow the classical trajectory of each secondary nucleon separately until it leaves the nucleus or collides once more. The sum of the energies of the holes produced in the cascade determines the excitation energy of the residual nucleus. The intranuclear cascade model is virtually the only phenomenological model that predicts the angular distributions of the secondary nucleons. Another advantage of the model is that it uses the experimental values of nucleon–nucleon cross sections, and not adjustable parameters like the other models.

The algorithm of the model was proposed by Serber and Goldberger[18] in 1947–1948 and was based on

analogy with the classical algorithm of the theory of particle transport in matter.

Really serious attempts to justify it in the case of nucleon-nucleus interactions on the basis of the quantum theory of multiple scattering were undertaken only ten years later by Watson (see Chap. 11 in the monograph of Ref. 6 and the references there). Watson used the following approximations.

1. The gas approximation (the range of the nucleon-nucleon interaction r_0 is assumed to be much shorter than the mean free path Λ_{fr} of the particle in the nucleus).

2. The de Broglie wavelength of the incident particle is much less than Λ_{fr}.

3. The approximation of quasifree scattering (the energy transferred by the incident particle to a particle of the nucleus, E_{tr}, is much greater than the average kinetic K_{av} and potential U_{av} energies of the particles in the nucleus).

4. The approximation of closure (in Glauber's studies on quasielastic scattering, this approximation is called the completeness approximation). It means that the cross section of the quantum-mechanical process is summed (or averaged) over all final states of the target corresponding to emission of a quasielastic particle. It is assumed that the excitation energy of the target in the final state is in an interval $\Delta W \approx \sqrt{(4/3)E_{tr}K_{av}}$ less than the energy resolution of the detectors or the beam. Because of the influence of the Pauli principle, the gas approximation is satisfied in a nucleus. In fact, it is the Pauli principle that is responsible for the applicability of the optical model, the shell model, and, essentially, the entire formalism of modern nuclear physics. A large mean free path is a second but not too stringent restriction of Watson's theory. The third and fourth restrictions are the most stringent. For because of the large value of Λ_{fr} we can describe the nucleus fairly well as a system consisting of a weakly interacting Fermi gas in the average field U. Then $K_{av} \approx U_{av} \approx E_F/2$ (E_F is the Fermi energy), and the third and fourth conditions entail $E_{tr} > \Delta W > E_F$. Of course, the energy E_i of the incident particle must be greater than E_{tr} and, therefore, greater than K_{av} and U_{av} (this is called the weak-coupling approximation). It is these approximations that made it possible to reduce the problem to the case of the scattering of a classical particle by a gas of unbound and uncorrelated nucleons, i.e., to eliminate the coherence effect inherent in every quantum-mechanical scattering problem. Watson obtained a classical transport equation for stationary flux of incident particles and homogeneous scattering medium that is not changed in the rescattering process (in gas kinetics, this is called the linearization approximation) in the single-energy approximation (i.e., only fast particles with energy $E_i \gg E_{tr}$ were considered). All these aspects together led to a partial justification of the classical-trajectory algorithm of the intranuclear cascade model, but they did not make it possible to describe the dynamics of the excitation of the nucleus during the reaction process or, therefore, the spectra of the secondary slow particles. These shortcomings in the theoretical justification made it necessary to introduce into the algorithm an adjustable parameter—the cutoff energy of the secondary particles (see, for example, Ref. 15)—and led to the appearance of various phenomenological modifications of the intranuclear cascade model (allowance for reflection and refraction of particles, specific allowance for the Pauli principle,[17] and so forth).

Since the cascade model can pretend to a description of only the fast stage of the process, it is in practice always augmented by the evaporation model, which reproduces reasonably the spectra in the case of the interaction of low-energy particles with nuclei. This model is based on the analogy with a heated liquid. It is assumed that the excitation energy which remains in the nucleus after the fast stage is redistributed among the large number of nucleons (compound-nucleus model), so that one case use the concept of a thermal distribution of a Fermi gas of nucleons within the residual nucleus (see, for example, Ref. 19). One can then use the hypothesis of thermodynamic equilibrium between the heated nucleus and the saturated vapor of emitted nucleons[20,21] and obtain the following expression for the probability of emission $I(\varepsilon)$ in unit time of a nucleon with energy ε from nucleus A with the formation of nucleus B with excitation energy $E_B = E_A - \varepsilon - E_{bd}$:

$$I(\varepsilon)\,d\varepsilon = (\rho_B(E_B)/\rho_A(E_A))\,\sigma(E_B,\varepsilon)\,(\bar{g}\mathscr{M}\varepsilon/\pi^2\hbar^3)\,d\varepsilon. \tag{5}$$

Here, $\rho(E)$ is the density of states at the excitation energy E, σ is the cross section for absorption of the nucleon with energy ε by the compound nucleus B (excited to the energy E_B) leading to the formation of the compound system A with excitation energy E_A [this cross section is usually replaced by the cross section $\sigma_{abs}(\varepsilon)$ for the absorption of the nucleon by the unexcited nucleus calculated in the optical model], $\bar{g} = 2s+1$ is the spin factor, and $\mathscr{M}$ is the reduced mass of the system B + nucleon. Using now the connection between the temperature T and the density of states

$$(d\ln\rho/dE)|_{E=E_0} = 1/T(E_0), \tag{6}$$

we can obtain from (5) an approximate expression of Maxwellian type for the evaporation spectrum:

$$I(\varepsilon)\,d\varepsilon = C(\varepsilon,T)\exp(-\varepsilon/T). \tag{7}$$

Here, T denotes either the initial temperature T_A (if $E_A \gg E_{bd}, E_B \gg \varepsilon$) or the final temperature T_B (if $\varepsilon \ll E_A - E_{bd}$), and the factor C is assumed to have a smooth dependence on ε, T, and some other parameters of the systems A and B. Because of these uncertainties, the expression (7) is hardly ever used to calculate the absolute cross sections, and T serves as an adjustable parameter in the description of experimental data by means of Eq. (7). The listed shortcomings of the model arise, as always, because there is no connection between the model and fundamental Schrödinger equations. Such a situation gives rise to a number of questions: Can one use the condition of thermodynamic equilibrium for a system in which there is no saturating vapor? What temperature characterizes the evaporation spectrum (7) if each new particle that leaves the system

leads to a significant decrease in the excitation energy (and this is the usual situation in the final nucleus), etc? However, the evaporation model proved very helpful for describing the low-energy part of the spectrum of emitted particles. This led to the creation of a combined cascade–evaporation model, in which it is assumed that the entire energy of the residual nucleus after the cascade (which exists above all in the form of of hole excitations) goes over into thermal energy and leads to evaporation. In other words, it is assumed that immediately after the cascade stage the nucleus goes over into a state of thermodynamic equilibrium. The cooling of the system during the evaporation stage is taken into account in this approach phenomenologically, and the Monte Carlo method and Eq. (5) for the evaporation model are used to determine the energy and form of the evaporating particle. The energy carried away by the evaporating particle is then subtracted from the excitation energy of the system, and Eq. (6) is used to determine the new temperature of the system. The evaporation formula (5) is then used again. Of course, the theoretical justification of such a procedure, like the possibility of using the concept of thermodynamic equilibrium for a finite open system, remains open. However, even this combined model did not make it possible to reproduce completely the experimental spectra.

This prompted the development of a new group of so-called pre-equilibrium models, which are based on the following idea. The cascade stage of the process leads to the appearance of a large number of holes in the Fermi distribution of the target nucleus and a certain number of excited particles above the Fermi surface. However, the distribution of these quasiparticles is still very far from the equilibrium thermal distribution. A time is required for such a system to arrive at equilibrium by "mixing" its energy in two-body nucleon–nucleon collisions. During this time, a certain number of particles may be emitted from the system, carrying away some of the excitation energy. The first variant of such models was proposed by Griffin[22] (exciton model). However, we shall consider more complicated nonstationary variants of the pre-equilibrium models, since, as we shall see later, it is easier for them to establish a connection with fundamental Schrödinger equations than it is for the simplified case of closed (integrated over the time) forms of the pre-equilibrium models. The most detailed variant of such nonstationary models is the Harp–Miller–Berne model,[23,16] which uses master equations. The basic idea of this model is very perspicuous. If we believe in the validity of the evaporation model and that the nucleons of a nucleus can "thermalize" like any quantum or classical gas, we must consider what equations describe this process of thermalization in the statistical theory of gases. Such a description is given by the Uehling–Uhlenbeck equations[24] in quantum mechanics or the Boltzmann equations in the classical case. Therefore, in the Harp-Miller-Berne model the nucleus is regarded as a "can" filled with a Fermi gas at zero temperature (step Fermi distribution). To describe the reaction initiated by the nucleon, the initial particle is placed at the initial time t_0 in the "can" in one of the free levels with high energy (Fig. 1a). To simplify the calculations, the complete energy space of the single-particle levels is divided into cells $\Delta\varepsilon$ (Fig. 1b). Each i-th cell contains g_i single-particle states:

$$g_i = \int_{\varepsilon_i-\Delta\varepsilon/2}^{\varepsilon_i+\Delta\varepsilon/2} \rho(\varepsilon)\, d\varepsilon, \tag{8}$$

where the density of the states of the Fermi gas is

$$\rho(\varepsilon)\, d\varepsilon = \bar{g} V_0 d^3p/(2\pi\hbar)^3 = 4\pi V_0 (2m)^{3/2} \varepsilon^{1/2}\, d\varepsilon/(2\pi\hbar)^3. \tag{9}$$

Here, $\bar{g}=2s+1$ is the spin factor, V_0 is the volume of the nucleus, m is the mass of the nucleon, and p and ε are the momentum and energy (measured from the bottom of the nuclear potential).

Assuming that the population number in the cell is n_i, we can calculate the total number of particles in the i-th cell:

$$N_i = n_i g_i. \tag{10}$$

Such a system of particles will remain unchanged in time if we do not allow them to collide with each other, thereby exchanging energy. The transition probability (in unit time) for the collision $i+j \to k+l$ is taken in the form

$$\omega_{ij\to kl} = \sigma_N(\varepsilon_i+\varepsilon_j)\,[2(\varepsilon_i+\varepsilon_j)/m]^{1/2}/V_0 \sum_{k,l} g_k g_l \delta_{\varepsilon_i+\varepsilon_j,\,\varepsilon_k+\varepsilon_l}, \tag{11}$$

where σ_N is the cross section of elastic nulceon–nucleon collisions averaged over the angles (or simply taken for angle 90° between the velocity vectors of the colliding particles). Because of the Kronecker δ symbol, the summation in the denominator is over only the states within the nucleus that are allowed in nucleon–nucleon collisions.

To obtain the master equation (balance equation), it is simply necessary to follow the change in the number of particles N_i in each cell. This change occurs for the following reasons. The collision of particles in cells k and l generates a particle in cell i and a particle in cell j. The resulting change in the number of particles is, with allowance for the Pauli principle, Eq. (11), and the energy conservation law,

$$\frac{dN_i}{dt}(k+l\to i+j) = \sum_{kl,j} \omega_{kl\to ij} n_k n_l (1-n_i)(1-n_j)\,\delta(\varepsilon_i+\varepsilon_j-\varepsilon_k-\varepsilon_l)\, g_i g_k g_l g_j. \tag{12}$$

Collisions of the type $i+j \to k+l$ reduce N_i:

$$\frac{dN_i}{dt}(i+j\to k+l) = -\sum_{kl,j} \omega_{ij\to kl} n_i n_j (1-n_k)(1-n_l)\,\delta(\varepsilon_i+\varepsilon_j-\varepsilon_k-\varepsilon_l)\, g_i g_k g_l g_j. \tag{13}$$

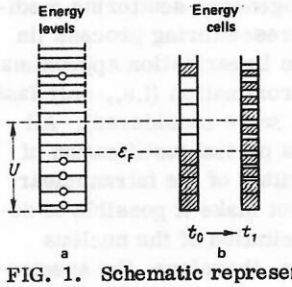

FIG. 1. Schematic representation of the Harp–Miller–Berne model.

Summing now the "arrivals" (12) and "departures" (13), we obtain the well-known standard quantum master equations

$$\frac{dn_i}{dt}=\sum_{klj}\omega_{ijkl}\,\delta(\varepsilon_i+\varepsilon_j-\varepsilon_k-\varepsilon_l)\,[n_k n_l(1-n_i)(1-n_j) - n_i n_j(1-n_k)(1-n_l)]\,g_k g_l g_j. \quad (14)$$

(As usual, we assume that in the elementary collisions $\omega_{ij\to kl}=\omega_{kl\to ij}=\omega_{ijkl}$.) This equation describes the process of thermalization in statistical physics, and we know that its solution for large times leads to a thermal distribution for the Fermi gas:

$$n_i^{th}=\{\exp[(\varepsilon_i-E_F)/T]+1\}^{-1}. \quad (15)$$

This means that the particles in a closed system (even if it has very large dimensions) will exchange energy in binary collisions until the system arrives in thermodynamic equilibrium.

However, the nucleus is not a closed system, since it can emit particles with energy in the continuum ($\varepsilon > E_F + E_{bd}$). Therefore, an additional term was introduced artificially into Eq. (14) in the Harp–Miller–Berne model to take into account the loss of particles from the volume of the nucleus to the "laboratory volume" Ω, in which their energy is $\varepsilon_{ia}=\varepsilon_i - U$ (U is the depth of the nuclear potential):

$$\frac{dN_i}{dt}(i\to ia)=-n_i g_i \omega_{i\to ia} g_{ia}\,\delta(\varepsilon_i-U-\varepsilon_{ia}). \quad (16)$$

The probability $\omega_{i\to ia}$ is assumed (in the spirit of the evaporation model) to be proportional to the cross section $\sigma_{abs}(\varepsilon_{ia})$ of nucleon absorption in the optical model:

$$\omega_{i\to ia}=\sigma_{abs}(\varepsilon_{ia})\,[2\varepsilon_{ia}/m]^{3/2}/g_i\Omega. \quad (17)$$

By analogy with g_i, the number g_{ia} is defined as the number of single-particle states in the external laboratory volume Ω. Thus, in the model it is necessary to solve the system of coupled equations

$$\frac{dn_i}{dt}=\sum_{jhl}\omega_{ijkl}g_k g_j g_l\,\delta(\varepsilon_i+\varepsilon_j-\varepsilon_k-\varepsilon_l)\,[n_k n_l(1-n_i)(1-n_j) - n_i n_j(1-n_k)(1-n_l)] - n_i\omega_{i\to ia}g_{ia}\,\delta(\varepsilon_i-U-\varepsilon_{ia}). \quad (18)$$

A shortcoming of this model is its computational complexity. In addition, the model does not enable one to predict the angular distributions (however, this applies in practice to the majority of pre-equilibrium models).

Despite the rather crude approximations [rectangular well, averaging over the angular dependence in the nucleon-nucleon cross sections σ_N (11)], the model does not contain explicitly adjustable parameters and, in contrast to the intranuclear cascade model, contains very detailed information about the quantum-mechanical properties of the system and its dynamics (the population numbers of the individual single-particle states of the quantum system). This brings it as close as possible to the microscopic description of the nucleus in the shell model and makes it very promising from the point of view of establishing a connection with the microscopic theory of nuclear reactions. It should also be noted that in principle equations of the type (14) can be obtained from the Schrödinger equation when definite conditions are satisfied; many eminent scientists have contributed to the development of different approaches to such a transition (for example, Pauli, Bogolyubov, Landau, Born, Kirkwood, Van Hove, Prigogine). Just the listing of these approaches and the establishment of correspondences between the employed assumptions would provide the subject of a separate monograph and, of course, cannot be made in the present review. The main difficulty in our case is to establish how well these assumptions are satisfied in the case of nuclear reactions. We shall return to these questions in the following section. Another difficulty is that all the derivations of balance equations were made for closed systems, and therefore led to Eqs. (14) without the specific term (16) in the Harp–Miller–Berne equations. If the term (16)–(17) is regarded as an intuitive guess, this form still does not stand up to criticism. First, the division of space into interior and exterior parts is essentially classical in nature. In quantum mechanics, every continuum state is defined in the complete coordinate space ($0 \leqslant r \leqslant \infty$). Therefore, in a microscopic theory of nuclear reactions it is sufficient to transfer a particle by two-body collisions from a discrete state to the continuum [such transitions are already included in the terms (12) and (13)]. No additional interaction of the type (16) is required to carry a continuum particle out of the nucleus. The connection between the probability (and, therefore, cross section) of particle emission and the absorption cross section of the form (17) is taken from the evaporation model, in which it is postulated (see, for example, Ref. 13) that the nucleus completely absorbs the incident beam and forms a completely thermalized compound system, avoiding the direct and pre-equilibrium stages. Therefore, the use of Eq. (17) precisely for calculating the pre-equilibrium process requires further justification and can hardly give a correct description of fast processes such as quasielastic scattering and direct reactions.

As we have already said, the Harp–Miller–Berne model gives a very detailed description of the pre-equilibrium process by means of realistic parameters obtained independently (the range and depth of the average nuclear potential and the free nucleon–nucleon cross section). For this detailed description it is necessary to pay— it is necessary to solve the coupled differential equations (18). To simplify the calculations (and the model), one can give up a detailed description of the population numbers and characterize the state of the nucleus by the excitation energy E and the total number of particles (above the Fermi surface) and holes ($n=p+h$) without worrying how this energy is distributed between the particles and the holes, i.e., assuming that all the ways of distributing energy between them are equally probable. Although it is customary to refer to particles above the Fermi surface and holes as "quasiparticles," historically they were given the name "excitons" in pre-equilibrium models, and the models that characterize the state of the nucleus by the number of excitons n were called exciton models. Instead of the evolution of the population numbers $n_i(t)$, one studies in the exciton model the evolution of the probability $P(n,t)$ for the presence of n excitons in the nucleus (Fig. 2). The initial state in reactions with nucleons is characterized by the exciton number $n_0=1$. Then a binary collision of the incident particle with one of the par-

ticles of the nucleus can change the energy of the incident particle and generate a particle–hole pair, i.e., a system with exciton number $n=3$ (in the exciton model, it is assumed that all possible configurations with $n=3$ arise with equal probability). A subsequent collision between one of the two excited particles and a particle below the Fermi surface or between the two excited particles can lead to the creation of the following states: 1) $n=5$ (production of a new pair); 2) $n=1$ (one of the particles fills a hole); 3) $n=3$ (a new two-particle–hole state is formed). The probabilities λ_+, λ_-, and λ_0 of each of these events are determined in the model by the average matrix element M of the two-body interaction and the density of allowed final states:

$$\lambda_{n\to n'}=(2\pi/\hbar)\,|M|^2\rho_{n'}(E). \tag{19}$$

The number of allowed states ρ_n for each type of transition was calculated by Williams[25] for the case of an equidistant single-particle spectrum with density g:

$$\left.\begin{aligned}&\lambda_{n\to n+2}\equiv\lambda_+=(2\pi/\hbar)\,|M|^2\,g^3E^2/(p+h+1);\\&\lambda_{n\to n-2}\equiv\lambda_-=(2\pi/\hbar)\,|M|^2\,gph\,(p+h-2);\\&\lambda_{n\to n}\equiv\lambda_0=(2\pi/\hbar)\,|M|^2\,[3\,(p+h)-2]\,g^2E/4.\end{aligned}\right\} \tag{20}$$

It is clear that equilibrium will be reached when the rates of increase and decrease, λ_+ and λ_-, respectively, of the exciton number are equal. Hence and from Eqs. (20) one can determine (for the typical case $gE\gg1$) the equilibrium number of excitons: $\bar{n}\approx\sqrt{2gE}$. It can be seen from (20) that for small exciton numbers $n\ll\bar{n}$ the process of increase in the complexity of the state predominates: $\lambda_+\gg\lambda_-$. Therefore, from an initial nonequilibrium state the system prefers to move toward equilibrium, "mixing" the energy among an even greater number of particles. Using now the probabilities $P(n,t)$, we can write down instead of Eq. (14) a different system of balance equations[26]:

$$\begin{aligned}dP(n,t)/dt=&P(n-2,t)\,\lambda_+(n-2)+P(n+2,t)\,\lambda_-(n+2)\\&-P(n,t)\,[\lambda_+(n)+\lambda_-(n)].\end{aligned} \tag{21}$$

For given exciton number, one can find configurations in which at least one particle has excitation energy exceeding its binding energy E_{bd}. This particle can be emitted from the nucleus. The probability of such emission in unit time is defined in the exciton model as follows:

$$I(n,\varepsilon_a)\,d\varepsilon_a=\frac{\rho_{n-1}(E-\varepsilon)\,g(\varepsilon)}{\rho_n(E)}\,\frac{\sigma_{abs}(\varepsilon_a)\,[2\varepsilon_a/m]^{1/2}}{g(\varepsilon)\,\Omega}\,\frac{\Omega\,(2m)^{3/2}\varepsilon_a^{1/2}d\varepsilon_a}{2\pi^2\hbar^3}. \tag{22}$$

Here, $\varepsilon_a=\varepsilon-U$ is the "asymptotic" energy of the emitted particle (in the exterior region), $g(\varepsilon)$ is the density of single-particle states within the nucleus, and the final factor determines the number of single-particle states in the laboratory volume [equal to g_{ia} in Eq. (16) of the Harp–Miller–Berne model]. The first factor in (22) determines the relative weight of configurations with one particle in the continuum among all n-exciton configurations of the nucleus. This factor replaces the probability $n_i(t)$ in the Harp–Miller–Berne model [see (16)], since in the exciton model one makes the simplifying assumption that all the ways of dividing the energy between the excitons are statistically equally probable. The other two factors in (22) determine the probability ε_a of emission of a particle from the excited nucleus [cf. (16) and (17)]. Our doubts regarding the use of the absorption cross section σ_{abs} in (22) were already expressed earlier in connection with the expression (17) in the Harp–Miller–Berne model.

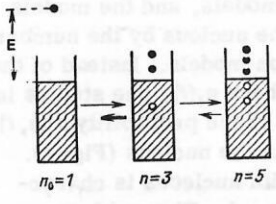

FIG. 2. Schematic representation of the evolution of the system in exciton models.

In more complicated variants of the exciton balance equations,[28,29] allowance is made for the emission of particles [ignored in Eqs. (21)]:

$$\begin{aligned}dP(n,t)/at=&P(n-2,t)\,\lambda_+(n-2)+P(n+2,t)\,\lambda_-(n+2)\\&-P(n,t)\Big[\lambda_+(n)+\lambda_-(n)+\sum_x\int de_a I_x(n,\varepsilon_a)\Big],\end{aligned} \tag{21a}$$

where x denotes the species of the emitted particle, and $I(n,\varepsilon_a)$ is determined by the expression (22). From the computational point of view, it is much simpler to use the system of equations (21a) than Eq. (18) of the Harp–Miller–Berne model. But for this simplicity a high price must be paid, since the matrix element M [see (20)] is now a free parameter of the theory which is very difficult to relate to the experimental nucleon–nucleon cross sections used in the intranuclear cascade model and the Harp–Miller–Berne model (a more detailed investigation of this question can be found, for example, in Refs. 28–31). Since it is impossible to determine the absolute magnitude of the energy loss in the pre-equilibrium process, it is very difficult to determine the temperature of the equilibrium part of the spectrum in the framework of the exciton model. Therefore, the uncertainty in M enables one to change not only the absolute magnitude but also the profile of the spectrum. In the majority of the later variants of exciton models it is customary to take the initial number of excitons to be $n_0=3$ in the case of reactions induced by nucleons. However, the justification for this choice has not yet been provided, nor has it been established whether the exciton model encompasses the stage of the direct reactions. All this has the consequence that the exciton models do not predict the absolute magnitude of the cross sections and are normalized to achieve the best agreement with experiment. Therefore, at the present time the exciton models are effectively used to parametrize experimental data already obtained.

Although it is easier to solve Eqs. (21a) numerically than Eqs. (18), the solution nevertheless requires a certain computer time. To avoid this difficulty, one can use one of the many closed exciton models. Since the calculation of

$$\sigma(\varepsilon_a)\,de_a\sim\sum_n de_a I(n,\varepsilon_a)\int dt\,P(n,t) \tag{22a}$$

requires some knowledge of $P(n,t)$, all these closed forms of models use different additional assumptions about $\int P(n,t)dt$. (We shall not consider these models, since they are described in detail in Refs. 29–33.) On

the one hand, in view of the descriptive nature of the exciton models mentioned above, the closed forms are not much inferior to the nonstationary exciton models when applied to experiments. On the other hand, they are much easier for calculation and satisfactory parametrization of experimental data.

To summarize our brief survey of the existing position in the theory of nuclear reactions, we present the scheme in Fig. 3. [We have not included the statistical theory in this scheme or in its study, despite the fact that it is a good example of a practical semiphenomenological theory; this is because in the exposition that follows we use a variant of the statistical approach to nuclear reactions which is a good bridge from the Schrödinger equations (1) to the phenomenological models and which partly overlaps[4] with more standard statistical approaches.]

From the foregoing discussion we have seen that the main shortcoming of the microscopic models is their inapplicability (or excessive complexity) for practical calculations. The semiphenomenological models are very practical but, unfortunately, do not cover the complete process of the reactions. The phenomenological models are also very practical but do not contain fundamental derivations linking them to the microscopic approach. This is bad not only from the theoretical point of view but also because it makes it impossible to determine the regions of applicability of the models and leaves space for parameters that are very obscurely determined (or are simply free) and for further modifications dictated by the inclinations of the authors rather than physical significance. Despite all this, they all give a reasonable description of the experimental data. The question of why these models agree reasonably with the experiments is of direct interest for theoreticians.

2. CONNECTION BETWEEN PHENOMENOLOGICAL MODELS AND QUANTUM MECHANICS

To find the connection between the most general nonstationary Schrödinger equation and the phenomenological models, we investigate them in more detail. As we have said in the Introduction, the microscopic models generally use almost monochromatic ($\Delta E \to 0$) wave packets and go over to the stationary equation (2). The solution ψ of this equation is obtained by diagonalizing the matrix of the Hamiltonian using a basis of the eigenfunctions φ_λ of H_0:

$$\psi = \mathrm{S}\, C_\lambda \varphi_\lambda. \tag{23}$$

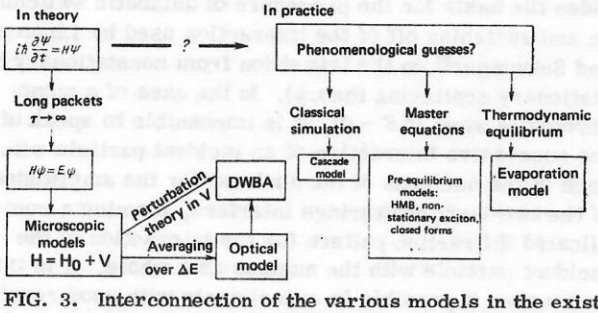

FIG. 3. Interconnection of the various models in the existing theory of nuclear reactions.

As usual, the symbol S denotes the sum over the bound states and the integral over the continuum. The Lippmann–Schwinger formalism makes it possible to relate the coefficients C_λ to the S matrix and the reaction cross sections. As H_0, one usually takes (see, for example, Ref. 8) the shell Hamiltonian of the model of noninteracting particles. Then φ_λ are antisymmetrized products of single-particle functions of the shell model corresponding to the configuration λ. When at least one particle in the given configuration is in the continuum, the index λ includes the momentum p of this particle as well. As the wave functions of particles in the continuum one can use the sets $u_{\mathbf{p}}^{(+)}(\mathbf{r})$ or $u_{\mathbf{p}}^{(-)}(\mathbf{r})$. In our problem, it is appropriate to use the function $u_{\mathbf{p}}^{(+)}$ for the initial configuration, since it describes an incident particle with momentum p. For the other configurations, it is more convenient to use the solutions $u_{\mathbf{q}}^{(-)}$, since by means of them it is simpler to describe the angular distribution of the secondary particles emitted from the nucleus with momentum q. When we considered the phenomenological approaches, we pointed out that the balance equations of the Harp–Miller–Berne model make it very close to the quantum-mechanical approach. Considering the term (16), we also mentioned that the wave function of a continuum state in quantum mechanics is defined in the complete coordinate space. A good example of this kind is the plane wave

$$\varphi_E(r, t) = \exp[ikr - (\mathrm{i}/\hbar) Et], \tag{24}$$

which shows that a particle in a given state can be found everywhere ($-\infty \leqslant r \leqslant \infty$) with equal probability. However, it must not be forgotten that a particle must move in space and time. To see this motion in quantum mechanics, it is necessary to construct a wave packet out of the plane waves (24):

$$\tilde{\varphi}_E(r, t) = \int A(E, E')\, \varphi_{E'}(r, t)\, dE'. \tag{25}$$

One can show (see, for example, Ref. 6) that the physical results depend only on the mean energy E and the characteristic width ΔE of the distribution A but not on the specific form of this distribution (in particular, step, Lorentzian, or Gaussian form). If we take for simplicity a step function A with width ΔE,

$$A(E, E') = \begin{cases} 1/2\Delta E, & E - \Delta E \leqslant E' \leqslant E + \Delta E; \\ 0, & \text{otherwise,} \end{cases}$$

then from (25) we obtain (remembering that $E = k^2\hbar^2/2M$)

$$\tilde{\varphi}_E(r, t) = \frac{1}{2\Delta E} \int_{E-\Delta E}^{E+\Delta E} dE' \exp[ik'r - (\mathrm{i}/\hbar) E't]$$
$$\approx \exp[ikr - (\mathrm{i}/\hbar) Et] \frac{\sin(r - vt)\Delta k}{(r - vt)\Delta k} \equiv \bar{\varphi}_E(r, t)\, G(r, t). \tag{26}$$

Here, $v = k\hbar/M$ is the velocity of the particle, $\bar{\varphi}_E$ is a plane wave with energy E equal to the average energy of the packet, and $G(r, t) = \sin(r - vt)\Delta k \times 1/(r - vt)\Delta k$ is a function which describes the evolution of the packet as a whole and shows that at a given time the particle of the packet can be localized in space with accuracy $\Delta r \sim 1/\Delta k$ and that it is displaced in time along the classical trajectory $r = vt$. If we now place this packet within a volume of radius $R \gg \Delta r$ (i.e., $R\Delta k \gg 1$), it will propagate over the volume as a classical particle and

spend the time $t=R/v$ in it. However, if the volume is small compared with the spatial extension of the packet, $R<\Delta r$ (i.e., $R\Delta k<1$), the packet will leave the volume in a time $t\approx\Delta r/v=\hbar/\Delta E=\tau$ (τ is the length of the packet in time). This last case ($\tau\to\infty$) is the one usually considered in the stationary description of reactions. When there is a scattering potential within the volume, the emergence of the packet from the volume is delayed (see, for example, Refs. 35 and 6) by the time $t_{ret}\approx -i\hbar\partial\ln f(E)/\partial E$, where $f(E)$ is the amplitude for scattering of the particle by the given potential. Therefore, the total time taken for the packet to leave the volume is $\tau+t_{ret}$. In the case when narrow potential resonances exist, $t_{ret}\approx\hbar/\Gamma$ (Γ is the width of the resonance). However, as we shall see later, such a situation hardly arises in the real Woods–Saxon potential. All this shows that if we wish to observe the emission from the interior region of the nucleus of particles that populate the continuum states we must use the technique of wave packets. We also see that the rate of departure of the packet from the interior region of the potential is determined by the time $\tau=\hbar/\Delta E$ (or more precisely, the time $\tau+t_{ret}$). Therefore, in our approach we shall use in place of the stationary functions the wave packets $\tilde{\varphi}_\lambda(t)$:

$$\tilde{\varphi}_\lambda(t)=\int A(E_\lambda, E'_\lambda)\,\varphi_\lambda(E'_\lambda, t)\,dE'_\lambda=\bar{\varphi}_\lambda(t)\,G_\lambda(t). \tag{27}$$

Here $\varphi_\lambda(t)=\varphi_\lambda\exp[-(i/\hbar)E_\lambda t]$, and $\tilde{\varphi}$ and G have the same meaning as in (26). Instead of the usual expansion (23), we now use the nonstationary expansion

$$\Psi(t)=\sum_\lambda \bar{C}_\lambda(t)\,\tilde{\varphi}_\lambda(t)=\sum_\lambda \tilde{C}_\lambda(t)\,\bar{\varphi}_\lambda(t), \tag{28}$$

which gives a solution to the nonstationary equation (1). Here, $\bar{C}_\lambda$ are averaged over an energy interval ΔE, and

$$\tilde{C}_\lambda(t)\equiv\bar{C}_\lambda(t)\,G_\lambda(t) \tag{29}$$

describes the evolution of $\bar{C}_\lambda(t)$ of the given component $\bar{\varphi}_\lambda$ of the wave function as well as the evolution $G_\lambda(t)$ of the packets as a whole.

By their nature, the balance equations are statistical. Therefore, in all usual derivations of such equations from the Schrödinger equation, it is necessary to operate with quantities averaged over a large statistical ensemble. In the case of nuclear reactions, we consider an ensemble consisting of a large number of pairs formed by "one target nucleus + one particle in the incident beam." This is a very natural ensemble formed in any experiment that investigates nuclear reactions. Using the usual techniques (see the more detailed derivation in Ref. 2), we can now go over from the Schrödinger equation (1) to balance equations for the coefficients (29):

$$\partial\langle|\tilde{C}_\lambda|^2\rangle/\partial t=(2\pi/\hbar)\sum_{\lambda'}|\bar{V}_{\lambda\lambda'}|^2\,\delta^{\Delta E}(E_\lambda-E_{\lambda'})\,[\langle|\tilde{C}_{\lambda'}|^2\rangle-\langle|\tilde{C}_\lambda|^2\rangle]. \tag{30}$$

Here, $V_{\lambda\lambda'}=\langle\bar{\varphi}_\lambda|V|\bar{\varphi}_{\lambda'}\rangle$ are the matrix elements of the residual interactions, $\delta^{\Delta E}(E)=(1/\pi E)\sin(E/\Delta E)$ denotes the "final" δ function (nonvanishing in a neighborhood ΔE of $E=0$), and the angular brackets denote averaging over the statistical ensemble and will be omitted in what follows.

Equations (30), like all balance equations, appear very perspicuous from the classical point of view: the change in the probability is due to the balance between the inflow and the outflow associated with the residual two-body interactions. However, such an interpretation is in general invalid in quantum mechanics, since successive collisions will interfere with one another, and the simple probabilities $|V_{\lambda\lambda'}|^2$ in Eq. (30) will be augmented by interference cross terms. Therefore, Eqs. (30) can be derived from the Schrödinger equation (1) only if the following two conditions are satisfied.[2]

1) Wave packets are used which have a large spread in energy (short packets in the time) such that $\Delta E>\Gamma_{spr}$, where Γ_{spr} characterizes the change of C_λ in the time $\tau\approx\hbar/\Delta E$. In the first stages of the reaction, Γ_{spr} is approximately equal to the imaginary part W_{opt} of the optical potential. Then this quantity gradually decreases during the process of thermalization to the value equal to the reciprocal lifetime of the compound state (to Γ_{comp} in the case of isolated compound resonances).

2) The "representatives" of the statistical ensemble are completely uncorrelated, i.e., there is no correlation between the particles of the incident beam.

Let us consider in somewhat more detail the conditions of derivation of Eq. (30) and their connection with the derivations used in statistical physics to which we have already referred. As is well known, the main physical condition needed for the derivation of a kinetic equation in statistical physics is the gas approximation (the mean free path λ_{fr} exceeds the range r_0 of the two-body interaction). This condition is satisfied in real nuclei by virtue of the Pauli principle. For classical particles with mean velocity v, the gas approximation makes it possible to introduce a time scale, the collision time $\tau_0=r_0/v$, and the mean free time $t_{fr}=\lambda_{fr}/v$. Roughly speaking, it is the condition $\tau_0<t_{fr}$ which enables one in statistical physics to regard the motion of particles in a rarefied gas as independent displacement along free trajectories (left-hand side of the Boltzmann equation) that are only occasionally perturbed by two-body collisions. However, in quantum mechanics a continuum particle can be localized only by introducing a wave packet with spread Δr and duration $\tau=\hbar/\Delta E$. A more accurate localization of the particle within a given packet Δr is prohibited by the uncertainty principle. Therefore, the collision time for a sufficiently monochromatic particle ($\Delta r>r_0$) is now determined by the packet duration τ (it is this that provides the basis for the procedure of adiabatic switching on and switching off of the interaction used by Lippmann and Schwinger[36] on the transition from nonstationary to stationary scattering theory). In the case of a monochromatic wave ($\Delta E\to 0$), it is impossible to speak of the successive interaction of an incident particle with each of the nucleons of the nucleus; for the amplitudes of the two-body scatterings interfere, forming a complicated diffraction pattern for the interaction of the incident particle with the nucleus as a whole. It is this that makes it possible in experiments with good resolution (using, for example, Glauber's method) to reconstruct the density distribution or the transition density

of nucleons in nuclei from the obtained diffraction pattern in the elastic and inelastic channels. Therefore, the meaning of the first condition becomes clear if we recall that the mean free time in quantum mechanics is $t_{opt} \approx \hbar/W_{opt} = \hbar/\Gamma_{spr}$. The condition $\Delta E > \Gamma_{spr}$ can be written in the form $\tau < t_{fr}$. It replaces in the quantum-mechanical case ($\tau_0 < \tau$) the condition $\tau_0 < t_{fr}$ of the gas approximation. For if the packet duration is shorter than the interval of time between successive collisions, the amplitudes of these collisions will not interfere with one another (since nonoverlapping packets do not interfere). An argument of this kind was used by Friedman and Weisskopf[14] in the time interpretation of the optical model (see also Ref. 12). They pointed out that for averaging over a sufficiently large energy interval ΔE the packets of incident particles spend in the nucleus a time $\tau \approx \hbar/\Delta E$, which is sufficient for only one or two collisions. On the one hand, this makes it possible to treat the residual two-body interactions in the spirit of nonstationary perturbation theory. On the other hand, the packets of particles trapped by the system and forming a compound nucleus can re-enter the elastic channel much later, and therefore will not overlap or interfere with the initial fast particles. This last made it possible to divide the process into fast and slow stages whose cross sections do not interfere with one another. Our approach is a generalization of the idea of the optical model to inelastic channels.

The meaning of the second condition is somewhat deeper, since it actually makes the first condition more precise. The point is that when we spoke of short packets we did not say how they are obtained physically. In a real experiment, as was noted by Austern,[7,34] each of the physical packets of the accelerator has a very small energy spread (i.e., $\tau = \hbar/\delta E$ is very large compared with even the lifetime of the compound nucleus). An experiment with poor resolution recognizes only that the accelerator beam consists of an incoherent (uncorrelated with respect to the initial phase) mixture of these long-wavelength packets, whose energy "centers of gravity" E_i are spread over the interval ΔE. Therefore, each of the packets separately gives a scattering amplitude f containing all interference effects. It can be roughly divided into a part f_1, which depends smoothly on the energy (and corresponds to the fast direct processes), and a part f_2, which oscillates rapidly over the interval ΔE and corresponds to the slower stages of the reaction: $f = f_1 + f_2$. The cross section $\sigma = |f|^2 = |f_1|^2 + |f_2|^2 + f_1^* f_2 + f_1 f_2^*$ contains information about the interference between the fast and slow stages. However, for uncorrelated packets the averaging over the energy eliminates all interference terms, and the mean cross section will again consist of the sum of the cross sections of the fast and the slow stages averaged over the energy, as in the case with short packets. Thus, Austern demonstrated the equivalence of using in the scattering problem short packets or an incoherent mixture of long packets (i.e., a poor energy resolution). This equivalence can also be demonstrated in density-matrix language. Indeed, let us consider the density matrix for a macroscopic ensemble whose parts are subsystems consisting of one beam particle + one nucleus in the material of the target. In the representation by means of the functions $\varphi_\lambda(t, t_0) = \varphi_\lambda \exp[-(i/\hbar)E_\lambda(t - t_0)]$ (φ_λ are the wave functions of the shell model), the density matrix of the system takes the form

$$\rho_{\lambda_1\lambda_2} = \mathrm{S}_i\, \omega_i C_{\lambda_1}^{i*} C_{\lambda_2}^{i}, \tag{31}$$

where C_λ are the coefficients in Eq. (23). To find the correlation function and the dependence $\rho(t)$, it is necessary to know the dependence $C_\lambda^i(t)$ for long, almost monochromatic packets in the accelerator. For the transition probability, we can use the coefficients $C_\lambda^i(t) = \langle \varphi_\lambda | \Psi_i(t) \rangle$ obtained in nonstationary scattering theory with long packets, i.e., with the adiabatic switching on of the interaction at a time t_0 in the distant past as proposed by Lippmann and Schwinger (see Ref. 36 or Chaps. 3 and 5 in Ref. 6):

$$C_\lambda^i(t, t_0) = \int_{-\infty}^{t} \langle \varphi_\lambda | V | \psi_i^{(+)} \rangle \exp[(i/\hbar)(E_i - E_\lambda)(t' - t_0)]\, dt'. \tag{32}$$

Here, $\psi_i^{(+)}$ is the eigenfunction (23) of Eqs. (2) and (3) corresponding to the energy eigenvalue E_i, and $\langle \varphi_\lambda | V | \psi_i \rangle = T_{\lambda i}(E_i)$ is the amplitude for transition to the state φ_λ used in the usual stationary scattering theory. Since the beam has an energy spread ΔE, averaging over the ensemble means in the first place that

$$\mathrm{S}_i\, \omega_i \to \frac{1}{2\pi} \int \frac{\Delta E}{(\bar{E}_i - E_i)^2 + (\Delta E/2)^2}\, dE_i. \tag{33}$$

(To simplify the calculations, we have chosen the Lorentzian form of the energy distribution in the incident beam.) Using Eqs. (31)–(33), we can obtain for the correlation function the expression

$$\rho_{\lambda_1\lambda_2}(t_1, t_2) = \frac{1}{2\pi} \int \frac{\Delta E\, dE_i}{(\bar{E}_i - E_i)^2 + (\Delta E/2)^2} T_{\lambda_1 i}^*(E_i)\, T_{\lambda_2 i}(E_i) \times \int_{-\infty}^{t_1} \exp[-(i/\hbar)(E_{\lambda_1} - E_i)(t' - t_0)]\, dt' \times \int_{-\infty}^{t_2} \exp[(i/\hbar)(E_i - E_{\lambda_2})(t'' - t_0)]\, dt''. \tag{34}$$

Closing the contour of integration with respect to E_i in the upper or lower half-plane, depending on the sign of $t_1 - t_2$, and noting that (see, for example, Ref. 12) a shift of the amplitude $T_\lambda(E_i)$ into the complex plane by the amount ΔE is equivalent to averaging of it over an energy ΔE, $T_\lambda(\bar{E}_i + i\Delta E) = \bar{T}_\lambda(\bar{E}_i)$, we obtain

$$\rho_{\lambda_1\lambda_2}(t_1, t_2) = \bar{C}_{\lambda_1}^*(t_1)\, \bar{C}_{\lambda_2}(t_2) \times \exp[-|t_1 - t_2|/\tau] \exp[(i/\hbar)(E_{\lambda_2} - E_{\lambda_1})\, t_0], \tag{34a}$$

where we have the following coefficients averaged over the energy:

$$\bar{C}_\lambda(t) = \bar{T}_{\lambda i}(\bar{E}_i)\, \delta^{\Delta E}(E_\lambda - E_i) \exp[(i/\hbar)(E_i - E_\lambda)\, t]; \quad \tau = \hbar/\Delta E. \tag{35}$$

For simplicity, we have considered the case of a smooth energy dependence of the amplitudes T_λ. When the contribution from the formation of a compound nucleus is important in the amplitude, it is necessary to include in (33) an averaging over intervals ΔE of the final energies E_{λ_1} and E_{λ_2} to ensure that in the expression for $\rho_{\lambda_1\lambda_2}(t_1, t_2)$ the correlation-weakening factor $\exp(-|t_1 - t_2|/\tau)$ again appears:

$$\rho_{\lambda_1\lambda_2}(t_1, t_2) = \exp(-|t_1 - t_2|/\tau)$$
$$\times \frac{1}{2\pi} \int \frac{\Delta E_i dE_i}{(\bar{E}_i - E_i)^2 + (\Delta E_i/2)^2} T^*_{\lambda_1 i}(E_i)\, T_{\lambda_2 i}(E_i)\, \delta^{\Delta E_i}(\bar{E}_{\lambda_1} - E_i)$$
$$\times \delta^{\Delta E}(\bar{E}_{\lambda_2} - E_i) \exp[(i/\hbar)(\bar{E}_{\lambda_2} - \bar{E}_{\lambda_1}) t_0]$$
$$\times \exp\{(i/\hbar)[(\bar{E}_{\lambda_1} - E_i) t_1 + (E_i - \bar{E}_{\lambda_2}) t_2]\}.$$

It can be seen that when such averaging is performed, the requirements on the interval ΔE_i of the initial states are made less stringent. As will be shown below, this interval must be sufficiently large to include, at given excitation energy, the levels of all the discrete configurations λ, i.e., many resonances of the system.

Equation (34a) shows that averaging over the ensemble of initial energies in the interval ΔE leads to the disappearance of the correlations for amplitudes separated by a time interval $|t_1 - t_2| > \tau = \hbar/\Delta E$, and this is the result that is obtained with short wave packets. Thus, we have shown that averaging over the initial energy (and, thus, by virtue of the conservation laws, over the final energy as well) is indeed equivalent to using short wave packets. Simultaneously, we see that ΔE is the uncertainty of the energy of the compound system as a whole and that it does not increase with the time. Since the coherence time τ in (34a) also remains constant, we can, if we use short packets, ignore their "spreading" with the time, as was done in (26).

However, to derive the master equations we must also show that the nondiagonal elements of the density matrix are small compared with the diagonal ones: $\rho_{\lambda\lambda} > \rho_{\lambda\lambda_1}$. For this, there are several possibilities. It is simplest, as in statistical theories of nuclear reactions, to assume random phases of the matrix elements $\langle \varphi_\lambda | V | \psi_i \rangle$, which is done in Refs. 4 and 5 by Weidenmüller *et al*. However, one can use the absence of correlation of the particles in the incident beam and average (31) over the time t_0 of switching on of the interaction. Then the oscillating factor in (34a) leads to the appearance in the density matrix of the additional factor

$$\frac{1}{T} \int_{T_0 - T/2}^{T_0 + T/2} \exp[(i/\hbar)(E_{\lambda_1} - E_{\lambda_2}) t_0]\, dt_0$$
$$= \exp[(i/\hbar)(E_{\lambda_1} - E_{\lambda_2}) T_0] \frac{\hbar \sin[(E_{\lambda_1} - E_{\lambda_2}) T/\hbar]}{T(E_{\lambda_1} - E_{\lambda_2})}. \quad (35a)$$

Since this factor is nonzero only in the interval $|E_{\lambda_1} - E_{\lambda_2}| \leq \hbar/T$, it leads in conjunction with the final δ function of the expression (35) in the case $T > \tau$ to an additional smallness $\rho_{\lambda\lambda_1}/\rho_{\lambda\lambda} \sim \tau/T$ when we integrate or sum (for configurations of the discrete spectrum) over the states λ (such summation is always necessary in the derivation of master equations). Note that since we have invariance under a shift of the time we could dispense altogether with the concept of the initial time t_0 and perform the averaging (34) with respect to the time t and obtain the same result because of the presence in the expression of the factor $\exp[(E_{\lambda_1} - E_{\lambda_2})t/\hbar]$. This "coarse-graining" of the time description was used by Kirkwood[37] in deriving master equations for classical particles and is equivalent[38] to the procedure of separating the slow and fast processes in the evolution of correlation functions proposed by Bogolyubov.[39] This fact in conjunction with the exponential correlation-weakening factor in (34a) shows that our averaging over the ensemble is physically equivalent to the well-known condition of correlation weakening used by Bogolyubov in the derivation of kinetic equations. It is somewhat more complicated to establish the physical connection between our averaging procedure and the random phase approximation, since the conditions for the validity of this approximation are not known. It is however clear that the approximation must be satisfied better, the greater is the number of degrees of freedom of the system that can participate in the process (i.e., for sufficiently large A and, probably, excitation energies E). In the derivation, we use the concepts of the mean free path or the optical potential, which are also valid only for systems with a large number of degrees of freedom. In estimating the smallness of the nondiagonal elements $\rho_{\lambda\lambda_1}$ for discrete states, it is necessary to use the concept of the density of these states, which is meaningful (see, for example, Chap. 2 in Ref. 13) only for $AE/E_F \gg 1$, and this is the condition of applicability of the usual statistical approaches in nuclear physics.

Having shown that the procedure of averaging over the spread of energies of the system is equivalent to the procedure of using short wave packets, we can[2] use the states (27) to construct the density matrix with coefficients $\langle \bar{C}_\lambda(t) \rangle$. Using nonstationary perturbation theory in the interaction V up to terms of second order (and the condition $\Delta E > W_{opt}$ and the applicability of the gas approximation to nuclei enable us to do this), and ignoring the nondiagonal terms in the density matrix, we obtain Eq. (30). As was shown by Bogolyubov and Gurov[40] (see also Ref. 41), the use of higher approximations in the interaction V leads to an additional Hartree–Fock self-consistency condition for the average field in which the particles of the nucleus move. Since, however, the parameters of the average field of the nucleus obtained in Hartree–Fock calculations differ little from the parameters of the "frozen" field of the shell model (especially when allowance is made for averaging over ΔE), we shall for the moment use only Eq. (30) without the additional self-consistency condition. Therefore, the use of the non-self-consistent basis of the shell model for constructing the packets makes it possible to go beyond the framework of the second order of perturbation theory. Of course, one then loses the possibility of taking into account the zero-sound excitation branch. We shall return to this question somewhat later (and we shall also see that in the construction of the evaporation model it is necessary to take into account the change in the parameters of the average field associated with the emission of particles). Allowance for the distorting influence of the average field on the wave functions of the basis automatically enables us to describe as well in rough details coherent effects such as quasielastic scattering. Indeed, in the theory of multiple scattering of Goldberger and Watson (see Chap. 11 in Ref. 6) it is noted that in the case when an incident particle in a collision with a gas particle transfers momentum $p < \hbar/\lambda_{fr}$ to the latter the scattered waves can interfere with one another. This interference leads to the appearance of a refractive index or a "pseudopotential," which distorts

the motion of the fast particles in the gas. Noting the connection between the mean free path λ_{fr} and the imaginary part of the optical potential, $\hbar/\lambda_{fr} = (2/v)W_{opt}$, we see that coherent effects are important for energy transfers $E_{tr} < W_{opt}$. The introduction of the averaging interval $\Delta E > W_{opt}$ in our approach automatically divides collisions into two classes. For $E_{tr} \leq \Delta E$, we assume that the system remains in the initial state $\bar{\varphi}_{\lambda_0}$ and the continuum particle is only elastically scattered in the average field of the nucleus. Elastic (or rather, quasi-elastic because of the averaging over the interval) scattering is automatically included in the problem, since the continuum wave functions in the Slater determinants φ_λ are solutions to the scattering problem in the average field of the nucleus and have asymptotically the form $u_k^{(\pm)} = \exp(ikz) + f(\theta)\exp(\pm ikr)/r$. For $E_{tr} > \Delta E$, the system goes over into a new state $\bar{\varphi}_\lambda$, and the inelastic incoherent scattering is described by the system of equations (30).

Thus, we see that Eqs. (30) are suitable for describing experiments with poor resolution (with sufficiently large ΔE). Therefore, the cross sections obtained by means of the coefficients $\bar{C}_\lambda$ (see below) will describe only the averaged behavior of the experimental cross sections. This is the price that we pay for replacing the complicated Schrödinger equations (1) by the much simpler ones (30).

Equation (30) differs in one important respect from the master equations of the type (14) that are usually obtained. The symbol $\partial/\partial t$ in it stands for $|G_\lambda|^2 \partial |C_\lambda|^2/\partial t$. In the terminology of fluid dynamics, this is the "substantial" derivative, which describes the change of $|\bar{C}_\lambda|^2$ within a moving wave packet. We are situated within a moving packet and follow the change in the number of particles in the packet due to binary collisions. To determine the total change taking place within a fixed volume in space (for example, within the nuclear volume V_0), it is necessary to go over to the "local" derivative $d|\tilde{C}|^2/dt$, which is related to the substantial derivative $\partial|\tilde{C}|^2/\partial t$ by

$$\partial |\tilde{C}_\lambda|^2/\partial t = d|\tilde{C}_\lambda|^2/dt - |\bar{C}_\lambda|^2 \partial |G_\lambda|^2/\partial t. \tag{36}$$

Thus, if we wish to describe by means of Eqs. (30) what occurs within the nuclear volume [i.e., we wish to use d/dt in (30)], we must add to the usual term describing collisions [the right-hand side of (30)] the term $|C_\lambda|^2 \partial |G_\lambda|^2/dt$, which shows that even in the absence of collisions a continuum particle leaves the nuclear volume in a time determined by the packet duration τ. This is a specific property of an open finite system, and distinguishes it from closed systems, for which all derivations of kinetic equations have been made. If ΔE is sufficiently small, so that $R\Delta k < 1$ [see the discussion of Eq. (26)], where R is radius of the nucleus, the wave packet $G_\lambda(t)$ of continuum configurations (i.e., configurations containing at least one particle in the continuum) can be approximated within the volume $r \leq R$ by a step function:

$$G_\lambda(t_i, t) = \begin{cases} 1, & t_i \leqslant t \leqslant t_i + \tau; \\ 0, & \text{otherwise.} \end{cases} \tag{37}$$

For the interval τ we can now replace $\bar{C}_\lambda$ in Eq. (30) by $\bar{C}_\lambda$. The presence of the extra term leads to special initial conditions for the solution of (30), namely, the integration of (30) for each successive interval τ must be made with null initial conditions for $\bar{C}_\lambda$ belonging to the continuum. This means that at the end of each interval τ wave packets of a continuum configuration leave the nucleus and their population in the next stage begins from zero. Each wave packet $\tilde{\varphi}_\lambda$ which leaves the interior region of the nucleus in the time $t_0 \leq t \leq t_0 + \tau$ can be registered in the region $r \to \infty$, making a contribution to the cross section for transition from the initial state λ_0 to the final state λ formed during the first stage of the process. By definition[6] this cross section is equal to the transition probability $|\bar{C}_\lambda(t_0, t_0 + \tau)|^2$ referred to unit incident flux and multiplied by the number of allowed final states $d\rho_\lambda$:

$$d\sigma_{\lambda_0 \to \lambda}^{t_0}/d\rho_\lambda = |\bar{C}_\lambda(t_0, t_0 + \tau)|^2/F. \tag{38}$$

Here, F is the incident flux:

$$F = v_0 |\bar{C}_{\lambda_0}(t_0)|^2. \tag{39}$$

For narrow proton resonances under the Coulomb barrier and for bound states, the outflow takes place, as we have already noted, at the rate $1/t_{ret} \approx \Gamma/\hbar \approx 0$. This fact can be expressed differently by saying that all bound-state configurations are localized the entire time within the nucleus, and therefore they have $G_\lambda(t) = \text{const}$ for all values of t and the corresponding $\bar{C}_\lambda(t)$ vary monotonically (there are no outflow terms).

During the following time interval $t_0 + \tau \leq t \leq t_0 + 2\tau$, the continuum configurations will again be populated by transitions from the bound configurations excited during the preceding first stage (note that because V represents two-particle interactions these are basically "doorway" configurations of 3-quasiparticle type). At the end of the new interval, the continuum packets again leave the nucleus, making an additional contribution to the cross section (38) of the preceding stage. Thus, the cross section formed at time $t_0 + 2\tau$ is the sum of the two components

$$\sigma_{\lambda_0 \to \lambda}^{t_0} + \sigma_{\lambda_0 \to \lambda}^{t_0 + \tau}, \tag{38a}$$

which can be called the contributions from the direct reactions and from the "doorway" states. Proceeding similarly for the subsequent intervals t_i, we obtain contributions to the cross section from the excitation of configurations which are more and more complicated:

$$\sigma_{\lambda_0 \to \lambda} = \sum_i \sigma_{\lambda_0 \to \lambda}^{t_i}, \tag{38b}$$

or, going over from summation to integration,

$$\frac{d\sigma_{\lambda_0 \to \lambda}}{d\rho_\lambda} = \frac{1}{\tau} \int \frac{d\sigma_{\lambda_0 \to \lambda}(t)}{d\rho_\lambda} dt = \frac{\int |\bar{C}_\lambda(t)|^2 dt}{\tau F}. \tag{40}$$

It should be noted that in the limiting case $V = 0$ (absence of residual interactions between the nucleons) we obtain from (38) and (39) a formula for the cross section of scattering by the average potential $U(r)$. For in this case Eq. (30) gives simply the flux conservation law $|\bar{C}_{\lambda_0}(t)|^2 = |\bar{C}_{\lambda_0}(t_0)|^2$. As we have already said, the configuration φ_{λ_0} is constructed using the single-particle functions $u_p^{(+)}(r)$. To determine the angular distribution, it is necessary to go over to the functions $u_k^{(-)}(r)$, i.e., to go over from $C_{\lambda_0(p)}^{(+)}$ to $C_{\lambda(k)}^{(-)}$, using the following form-

ula[6]: $C^{(-)}_{\lambda(\mathbf{k})} = \sum_{\lambda_0(\mathbf{p})} S^{\text{pot}}_{\mathbf{kp}} C^{(+)}_{\lambda_0(\mathbf{p})}$, where $S^{\text{pot}}_{\mathbf{kp}}$ is the S matrix of potential scattering of a particle in the state with momentum $\mathbf{p}$ into a state with momentum $\mathbf{k}$. Using the initial condition $|\bar{C}^{(+)}_{\lambda}(t_0)|^2 = \delta_{\lambda\lambda_0} |C_{\lambda_0}(t_0)|^2$, we obtain the relation $\bar{C}^{(-)}_{\lambda(\mathbf{k})} = |S^{\text{pot}}_{\mathbf{kp}}|^2 |\bar{C}^{(+)}_{\lambda_0}|^2$ and the scattering cross section [see (38) and (39)]

$$d\sigma_{\mathbf{p}\to\mathbf{k}}/d\rho_{\mathbf{k}} = |S^{\text{pot}}_{\mathbf{kp}}|^2/v_p. \tag{41}$$

The same expression can be obtained from Eq. (40) by noting that the entire process of the particle's leaving the interaction region of the potential occupies a time τ (the integration over t is up to the time $t_0 + \tau$). It is easy (see, for example, Refs. 6 and 42) to write Eq. (41) in the usual form in terms of the amplitude $f(\theta)$ of potential scattering:

$$d\sigma_{\text{el}}/d\Omega_{\mathbf{k}} = |\bar{f}(\mathbf{q} = \mathbf{p} - \mathbf{k})|^2. \tag{42}$$

Of course, the averaging over the energy ΔE leads to a spreading of the diffraction pattern with respect to the momentum transfer within $\Delta q \sim 1/\lambda_{\text{fr}}$ (and, strictly speaking, the scattering itself is quasielastic). If $1/\Delta k \sim \lambda_{\text{fr}} > R_{\text{nuc}}$, the diffraction pattern (minimum at $q \sim 1/R_{\text{nuc}}$) is not completely spread out. But when $\lambda_{\text{fr}} < R_{\text{nuc}}$ (we shall consider this case later), the situation tends to the scattering picture obtained for a classical particle scattered by a potential of range R_{nuc}. In particular, the spreading of the diffraction pattern with respect to Δq enables one to understand the so-called extinction paradox (see, for example, Ref. 43), which is as follows. At high energies ($E \to \infty$), the cross section for scattering by a potential of range a in optics and in quantum mechanics is twice the corresponding cross section πa^2 in classical mechanics due to Fraunhofer diffraction, which has a sharp maximum in the region of small scattering angles $\theta \lesssim 1/ka$. The averaging over Δq means that it is impossible to distinguish experimentally particles scattered at such a small angle from incident particles that are not scattered at all. Of course, the Markov-type equations (30) do not by themselves describe elastic scattering. However, in contrast to all other equations of kinetic theory, they take into account the fact that the quantum-mechanical system is "open" by means of the basis states $u^{(+)}_{\mathbf{p}}$ and $u^{(-)}_{\mathbf{k}}$ of the scattering problem. This, in turn, makes it possible to use correctly formulas for the S matrix of the form (38)–(42), which determine the scattering cross section. The averaged S matrices corresponding to (38) are related to the coefficients $\bar{C}_{\lambda}$ in the usual (see, for example, Ref. 6) manner:

$$|\bar{S}^{t_i}_{\lambda\lambda_0}|^2 = |\bar{C}_{\lambda}(t_i, t_i + \tau)|^2.$$

We now show[2,3] that our master equations in the various approximations will give the results obtained in the microscopic and semiphenomenological models. Let us consider the solutions of Eqs. (30) for the fastest stage of the reaction ($t_0 \leq t \leq t_0 + \tau$) in the first order of perturbation theory in V. We again use the initial condition

$$|\bar{C}_{\lambda}(t_0)|^2 = \delta_{\lambda\lambda_0} |\bar{C}_{\lambda_0}(t_0)|^2.$$

In the elastic entrance channel we can ignore the "reverse" transitions from the inelastic channels λ:

$$\frac{d|\bar{C}_{\lambda_0}(t_0, t)|^2}{dt} = -\frac{2\pi}{\hbar} |\bar{C}_{\lambda_0}(t_0, t)|^2 \sum_{\lambda} |\bar{V}_{\lambda\lambda_0}|^2 \delta^{\Delta E}(E_{\lambda_0} - E_{\lambda}). \tag{43}$$

Introducing the matrix element $\bar{V}$, which is averaged over the configurations λ, and the density of states ρ_{λ} in the inelastic channels, we obtain the law of attenuation for a packet passing through the nucleus:

$$|\bar{C}_{\lambda_0}(t_0, t)|^2 = |\bar{C}_{\lambda_0}(t_0)|^2 \exp[-(W/\hbar)(t - t_0)]. \tag{44}$$

The rate of absorption W in this expression is determined in the same way as the imaginary part of the optical potential W_{opt} (see, for example, Ref. 7) when the latter is derived from microscopic theory:

$$W(E_{\lambda_0}) = 2\pi |\bar{V}|^2 \rho_{\lambda}(E_{\lambda_0}), \tag{45}$$

where ρ_{λ} is the total density of the configurations of the "doorway" states λ. To find the cross section of the inelastic process $\lambda_0 \to \lambda$ in the same approximation, we can ignore in (30) all transitions except $\lambda_0 \to \lambda$:

$$d|\bar{C}_{\lambda}(t_0, t)|^2/dt = (2\pi/\hbar) |\bar{C}_{\lambda}(t_0, t)|^2 |\langle\varphi^{(-)}_{\lambda}|V|\varphi^{(+)}_{\lambda_0}\rangle|^2 \delta^{\Delta E}(E_{\lambda_0} - E_{\lambda}). \tag{46}$$

If we assume that the absorption in the entrance channel is small, $[C_{\lambda_0}(t_0, t) \approx C_{\lambda_0}(t_0)]$, then from (38)–(40) we can arrive at the following expression for the cross section of the fast direct process:

$$(d\sigma_{\lambda\lambda_0}/\partial\rho_{\lambda}) = \frac{2\pi\rho/l}{\hbar^2 k_{\lambda_0}} |\langle\bar{\varphi}^{(-)}_{\lambda}|V|\bar{\varphi}^{(+)}_{\lambda_0}\rangle|^2. \tag{47}$$

It can be seen that this is the Born expression with allowance for the distortion of the initial and final states in the average field of H_0. Using (43) and (44), we can introduce absorption in the channels λ_0 and λ. Then (47) becomes exactly equal to the usual formula of the DWBA [see (4)].

We shall now show that the purely phenomenological models can also be derived as special cases of approximate solutions to Eqs. (30) under different conditions. For this, it is more convenient to go over from the representation of the occupation probabilities $|C_{\lambda}(t)|^2$ of the configurations to the single-particle occupation numbers $\bar{n}_k(t)$ (this transition is described, for example, in Ref. 44). Then instead of (30) we obtain

$$\frac{\partial\langle\tilde{n}_k\rangle}{\partial t} = \frac{\pi}{\hbar} \sum_{ijl} |\bar{V}_{ijkl}|^2 \delta^{\Delta\varepsilon}(\varepsilon_i + \varepsilon_j - \varepsilon_k - \varepsilon_l) \times [\langle\tilde{n}_i\rangle\langle\tilde{n}_j\rangle(1 - \langle\tilde{n}_k\rangle)(1 - \langle\tilde{n}_l\rangle) - \langle\tilde{n}_k\rangle\langle\tilde{n}_l\rangle(1 - \langle\tilde{n}_i\rangle)(1 - \langle\tilde{n}_j\rangle)]. \tag{30a}$$

Here, the angular brackets again denote averaging over the statistical ensemble and will in what follows be omitted; $\tilde{n}_k(t) = \bar{n}_k(t) |G_k(t)|^2$ by analogy with the expression (29); and $G_k(t)$ describes the evolution of the single-particle wave packet with quantum numbers k. The "substantial" derivative $\partial/\partial t$ on the left-hand side of Eq. (30a) is related to the local derivative d/dt by an equation analogous to (31):

$$\partial\tilde{n}_k/\partial t \equiv |G_k|^2 \partial\bar{n}_k/\partial t = d\tilde{n}_k/dt - \bar{n}_k \partial|G_k|^2/\partial t. \tag{31a}$$

Let us consider the connection between the "single-particle" uncertainty of the energy $\Delta\varepsilon$ in (30a) and the energy uncertainty ΔE of the complete system in (30). At the inital time, the entire uncertainty ΔE is associated with the uncertainty in the energy of the incident particle: $\Delta\varepsilon = \Delta E$. However, in each subsequent two-body collision in the system additional quasiparticles are formed with their own energy uncertainty. The total uncertainty $\Delta E(t)$ in the energy of the system

is made up of the uncertainties $\Delta\varepsilon$ of the quasiparticles $N_{qu}(t)$ formed up to the time t: $\Delta E = \Delta\varepsilon N_{qu}(t)$. In the ordinary nonstationary Schrödinger equation, $\Delta\varepsilon = \Delta E$ during the whole of the reaction. However, we have derived equations for quantities averaged over the statistical ensemble of nuclei by means of Eqs. (34) and (35) and have seen that the value of ΔE does not change with the time. This means that in the average over the ensemble of compound nuclei $\Delta\varepsilon(t) = \Delta E/N_{qu}(t)$.

Thus, we have obtained the master equation of Uehling–Uhlenbeck type (14) with the extra term $\bar{n}_k \partial |G_k|^2/dt$ which we attempted to find when discussing the equations of the Harp–Miller–Berne model. The detailed form of this term depends on the detailed choice of the form of the packet $A(E_k, E_k')$ [see Eqs. (25) and (26)], but it is clear that it differs from the corresponding term (16)–(17) of the Harp–Miller–Berne model. Let us consider first the case satisfying the condition $R\Delta k < 1$. As we have already seen, the rate of emission of particles is determined by the packet duration τ. One can, for example, choose a Lorentzian form $A(E, E')$. This has the consequence that

$$|G_k(t)|^2 = \exp(-t/\tau). \tag{32a}$$

Then the extra term takes the form

$$\bar{n}\partial|G|^2/\partial t = -\tilde{n}/\tau, \tag{48}$$

and we can write down Eq. (30a) in the interior region of the nucleus:

$$\frac{d\tilde{n}_k}{dt} = \sum_{ijl} \omega_{ijkl}\delta(\varepsilon_i + \varepsilon_j - \varepsilon_k - \varepsilon_l)\, g_i g_j g_l \times [\tilde{n}_i\tilde{n}_j(1-\tilde{n}_k)(1-\tilde{n}_l) - \tilde{n}_k\tilde{n}_l(1-\tilde{n}_i)(1-\tilde{n}_j)] - \tilde{n}_k/\tau. \tag{30b}$$

As we have already said several times, the most general form for the rate of outflow, with allowance for the presence of the average field, must be $\tilde{n}/(\tau + t_{ret})$. The most important contribution from $t_{ret} = -i\hbar\partial \ln f(\varepsilon)/\partial\varepsilon$ arises in the case of a narrow potential resonance. Then the outflow term takes the form $\tilde{n}\Gamma\Delta\varepsilon/\hbar(\Delta\varepsilon + \Gamma)$, and for $\Gamma < \Delta\varepsilon - \tilde{n}\Gamma/\hbar$. In a rough description of Eq. (30b), we shall ignore the existence of these resonances for neutrons, and we shall assume that proton resonances below the Coulomb barrier, as also bound states, are infinitely narrow: $\Gamma \to 0$.

In the expression (30b), we introduced the transition probabilities $\omega_{ijkl} = (\pi/\hbar)|\bar{V}_{ijkl}|^2 = (\pi/\hbar)|\langle\bar{u}_i\bar{u}_j|V|\bar{u}_k\bar{u}_l\rangle|^2$, which are related to the matrix elements of the two-body interactions for the single-particle states u_i of the average field H_0 averaged over the directions of the momenta $\mathbf{p}_i$ (for the continuum) and the magnetic quantum numbers (for bound states). We also replaced the integrals over the continuum by sums, which led to the appearance of the numbers g of single-particle states and Kronecker deltas instead of the final δ functions. Comparing (30b) with the Harp–Miller–Berne equations (18), we see that they differ in two important respects.

1) In our case, the form of the extra term is determined by the value of τ and reflects the simple fact that the packet of continuum states leaves the fixed volume in a time $\tau = \hbar/\Delta\varepsilon$.

2) The densities ρ of the single-particle states needed in the calculation of g [see Eq. (8)] for the continuum states are very different from the ones used in the Harp–Miller–Berne model. In principle, they are determined by the single-particle potential phase shifts $\partial\delta/\partial k$ (see, for example, Ref. 45), but if we ignore these quantities ($\partial\delta/\partial k < R$, which means essentially the absence of narrow single-particle potential resonances), we can use[45] the expression for a Fermi gas, which is analogous to Eq. (9),

$$\rho(\varepsilon_a)\, d\varepsilon = (2s+1)\, V_0 d^3 p_a/(2\pi\hbar)^3 = 4\pi V_0 (2m)^{3/2} \varepsilon_a^{1/2}\, d\varepsilon/(2\pi\hbar^3), \tag{9a}$$

but, in contrast to (9), contains the asymptotic values of the energy, ε_a, and the momentum, p_a, and the ε_a are much smaller than the "local" energies ε used in the Harp–Miller–Berne equations:

$$\varepsilon_a = \varepsilon - U \tag{49}$$

(U is the depth of the nuclear potential).

Both the differences between Eqs. (30b) and (18) are due to the more correct treatment of the boundary conditions for open systems made in the derivation of Eq. (30b). We shall show later what important numerical differences result from this.

The finite extent and open nature of the system also lead to specific relaxation times of the system. During the fastest stage $t \approx \tau$, the incident particle and the continuum particles produced in the direct reactions leave the system, as we have shown, and there remain in the system $N_{qu}(t_0)$ quasiparticles in bound orbitals and a corresponding excitation energy $E_{exc}(t_0)$. Let us consider the most complicated and typical case when the system can still emit secondary particles [$E_{exc}(t_0) > E_{bd}$) and when the mean excitation energy per quasiparticle is large:

$$\varepsilon_{exc}(t_0) = E_{exc}/N_{qu} > E_{bd}. \tag{50}$$

In this case, when the excited particles collide with the other particles of the nucleus (for $N_{qu} \ll A$, the collisions of the quasiparticles with one another can be ignored) the probabilities for secondary particles to appear in the discrete states and in the continuum are comparable. This means that there are comparable probabilities for the process leading to a further spreading of the Fermi distribution and an increase in $N_{qu}(t)$ and the probability of emission from the nucleus of secondary particles, which reduces $E_{exc}(t)$. During this stage, which has a characteristic duration of the order of a few mean free times t_{fr}, the pre-equilibrium contribution to the cross section is basically formed. The regime of the system changes abruptly when $\bar{\varepsilon}_{exc}(t)$ becomes less than the binding energy:

$$E_{exc}(t)/N_{qu}(t) < E_{bd}. \tag{50a}$$

When this stage has been reached, the collision of an excited particle with a particle of the nucleus can no longer (of course, on the average) "eject" a secondary particle into the continuum. Therefore, the system becomes "quasiclosed" and a thermal distribution is established in it over a time of order t_{fr}, as in an ordinary closed system. Of course, this state too is not in perfect equilibrium. As long as $E_{exc}(t) > E_{bd}$, particles can be emitted from the system. However, such emission must occur only through the collisions of excited particles from the most distant (and therefore small) ends of the thermal distribution. This greatly reduces

the probability of particle emission from the nucleus and increases the lifetime of the system in the new, evaporation stage. Of course, evaporation leads to a further decrease in $E_{exc}(t)$. However, the thermal distribution of the system will now adjust adiabatically to the new $E_{exc}(t)$. We shall consider these questions in more detail in the following section, and for the moment we shall merely emphasize once more the basic features of open systems: 1) the possibility of two processes that differ strongly in time and their nature, the transition between them being determined by the conditions (50) and (50a); 2) the existence of a quasiequilibrium regime with slowly varying temperature $T(t)$ (evaporation model with cooling). Since (see, for example, Ref. 13) $T \approx E_{exc}/N_{qu}$, one can speak of evaporation only for temperatures $T \lesssim E_{bd}$. There is also a limitation on the accuracy of the thermal description. It is clear that one can consider a distribution only for $\Delta\varepsilon < T$. Bearing in mind the connection we have already discussed between $\Delta\varepsilon$ and the total energy uncertainty ΔE, this last condition means that $\Delta E < E_{exc}$.

Let us now consider the case of ΔE so large that $R\Delta k > 1$. It is then possible to localize the wave packet of a particle within the nucleus with accuracy $\Delta r \sim 1/\Delta k$. Once Δr becomes smaller than the characteristic length (the mean free path in the case of a homogeneous distribution of nuclear matter or the spread of the surface in the Fermi-distribution model), one can use the approximation of a local momentum, ascribing the particle the momentum $\mathbf{p}(\mathbf{r})$ in the intervals between successive collisions. Then the part of the quantum number k in Eq. (30a) will be played by the momentum $\mathbf{p}_k$, $G_k(r, t)$ takes the form (26), and the population numbers $\tilde{n}_k(t)$ are transformed, by definition, into the ordinary classical distribution function $\rho(\mathbf{p}_k, r, t)$:

$$\tilde{n}_k(t) = \bar{n}_{\mathbf{p}_k}(t) |G_k(r, t)|^2 = \rho(\mathbf{p}_k, r, t). \tag{51}$$

Instead of the master equations (30a), we obtain equations of Boltzmann type:

$$\frac{\partial \rho(\mathbf{r}, \mathbf{p}_k, t)}{\partial t} = \int d\mathbf{p}_i\, d\mathbf{p}_j\, d\mathbf{p}_l\, P_{ijkl} \{\rho(\mathbf{r}, \mathbf{p}_i, t)\, \rho(\mathbf{r}, \mathbf{p}_j, t)\, [1 - \rho(\mathbf{r}, \mathbf{p}_k, t)]\, [1 - \rho(\mathbf{r}, \mathbf{p}_l, t)] - \rho(\mathbf{r}, \mathbf{p}_k, t)\, \rho(\mathbf{r}, \mathbf{p}_l, t)\, [1 - \rho(\mathbf{r}, \mathbf{p}_i, t)]\, [1 - \rho(\mathbf{r}, \mathbf{p}_j, t)]\}. \tag{52}$$

The collision term $P_{ijkl} = (\pi/\hbar) |\langle u_{\mathbf{p}_i} u_{\mathbf{p}_j} | V | u_{\mathbf{p}_l} u_{\mathbf{p}_k} \rangle|^2 \times \delta^{\Delta\varepsilon}(\varepsilon_i + \varepsilon_j - \varepsilon_k - \varepsilon_l)$ contains in the matrix element an integration over the volume $\Delta V \sim 1/(\Delta k)^3$. The derivative $\partial\rho/\partial t$ is the ordinary substantial derivative, which is related to the local derivative by the usual relation

$$\partial\rho/\partial t = d\rho/dt + (\mathbf{p}/m)\, \nabla\rho + (\partial U/\partial \mathbf{r})\, \partial\rho/\partial\mathbf{p}. \tag{53}$$

The second and third terms in (53) are obtained from the extra term $\bar{n}\partial |G|^2/\partial t$ in Eq. (31a), it being necessary to bear in mind that now [see, for example, (26)] the probability G for existence of a packet is nonzero only in a neighborhood Δr of the point $\mathbf{r} = [\mathbf{p}(r)/m]t$: $|G(\mathbf{r}, \mathbf{p}, t)|^2 = |G\{\mathbf{r} - [\mathbf{p}(r)/m]t\}|^2$. Therefore

$$-\partial |G|^2/\partial t = (\mathbf{p}/m)\, \nabla |G|^2 + (p/m)\, (\partial |G|^2/\partial \mathbf{p})\, \partial\mathbf{p}/\partial\mathbf{r}. \tag{53a}$$

A local momentum $\mathbf{p}(\mathbf{r})$ can be introduced only in the quasiclassical limit, when $\mathbf{p}^2(\mathbf{r})/2m = \mathbf{p}_a^2/2m + U(\mathbf{r})$, where $U(r)$ is the average potential of the nucleus; $\mathbf{p}_a$ is the asymptotic momentum outside this potential. Therefore $(\mathbf{p}/m)\partial\mathbf{p}(\mathbf{r})/\partial\mathbf{r} = \partial U(\mathbf{r})/\partial\mathbf{r}$. This last relation in conjunction with (53a) and (51) leads to the classical equation (53). Equations (52) and (53) differ from Boltzmann equations only by the allowance for the Pauli principle. One can now use the well-developed computational methods of gas dynamics and transport phenomena based on Boltzmann equations. One of the methods of solution of these equations in gas dynamics (see, for example, Ref. 46) is based on the method of random sampling and is very close to the algorithm of the intranuclear cascade model. In transport theory (see, for example, Ref. 47), it is very common to simplify the solution by using the linearization approximation, which means that the density of the medium in the phase space changes very little when a fast particle moves through it. When the dimensions of the system are comparable with the mean free path, a good approximation is the method of successive collisions. A fairly accurate solution is then obtained by means of the method of random sampling, which is identical with Goldberger's algorithm for the intranuclear cascade model (which was in fact taken by Goldberger from the transport theory of Ulam and von Neumann). In the special case of a stationary incident flux and the single-energy approximation (the scattered particles do not lose energy) Watson's transport equation discussed above is obtained from the method of successive collisions. Thus, the algorithm of the intranuclear cascade model is a method of approximate solution (in the case of linearization) of Eqs. (52) and (53). It should be noted that in the case of very good localization of the packet, $\Delta r < r_0$ (r_0 is the distance between the nucleons), we have $\Delta E > E_F$, and we arrive at the case considered by Glauber and by Goldberger and Watson (see above).

Completing our derivation of the phenomenological models, we note that to obtain the exciton balance equations (21) or (21a) we can also use Eq. (30), recalling that $|C_\lambda(t)|^2 \equiv P(\lambda, t)$ is the probability of population of configuration λ in the nucleus. Averaging Eq. (30) over the different configurations containing the same number of excitons, we obtain an equation of the form (21a), where again it is necessary to modify the density of the continuum states that occur in the $\lambda_\pm$ [see (19) by analogy with Eq. (9a)]. With regard to the rate of outflow (22), we can by analogy with Eqs. (32) and (32a) assert that it is inversely proportional to τ.

Thus, we have shown that the considered master equations for finite open systems overlap strongly with the semiphenomenological models and constitute a good "bridge" between the existing phenomenological models and the fundamental Schrödinger equations. Therefore, instead of the scheme in Fig. 3 we can propose the scheme in Fig. 4. Again, we have not included statistical theory in the scheme. As we have seen earlier, the discussed master equations have a statistical nature, since the nonstationary Schrödinger equations are averaged over the statistical ensemble of colliding pairs of the "incident nucleon + target nucleus" type. This led to the nonstationary form of the master equations (30). As we have already noted above, the group of Agassi, Weidenmüller, and Mantzouranis[4,5] obtained

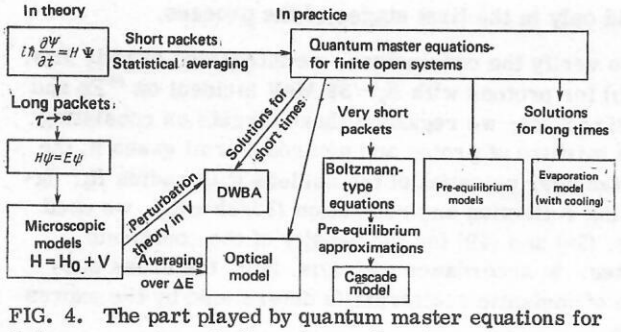

FIG. 4. The part played by quantum master equations for finite open systems in the scheme of modern models of nuclear reactions.

results close to ours. They began with the stationary Lippmann–Schwinger approach to the microscopic theory of reactions and obtained equations identical with the exciton balance equations (21a) integrated over the time: $\int_0^\infty dt$. In deriving their stationary equations,[4,5] they used virtually the same conditions as we did: averaging over an energy $\Delta E > \Gamma_{spr}$ and the random phase approximation for the matrix elements, which, as was shown above, leads to the same results (vanishing of the nondiagonal terms of the density matrix) as our ensemble averaging (35a), as time averaging in Kirkwood's derivation, and as Bogolyubov's method of correlation weakening. The need for integration over the time in their approach is undoubtedly due to the circumstance that they began with Lippmann–Schwinger equations, which are obtained from the nonstationary Schrödinger equation (1) by integration over the time in the case of an infinitely long wave packet ($\tau \to \infty$). The use of the Lippmann–Schwinger equations led to additional restrictions in their theory—the impossibility of considering the case of several nucleons in the continuum (i.e., the case when the energy of the incident particle exceeds ≈10 MeV). Since we do not use a Lippmann–Schwinger equation, our approach is in principle free of these limitations and enables us to understand the intranuclear cascade model for high energies. However, the stationary approach of Refs. 4 and 5 is closer to the microscopic theory on which the ordinary statistical models are based. This makes it possible to obtain many results of ordinary statistical theory (for example, the Hauser–Feshbach formula) in more general cases than usual.

Let us discuss briefly the choice of the basis of H_0 in our approach. In deriving Eqs. (30), we used the condition $\Delta E > \Gamma_{spr}$. It is natural to choose a basis such that ΔE is smaller than the characteristic distance between the states of the basis. For the basis of the shell model, this characteristic distance is of order $2\hbar\omega \approx 20$ MeV, whereas Γ_{spr} for the single-particle states is equal [see (37)–(39)] to the imaginary part of the optical potential W_{opt}. It can be seen from this that this basis can be used to describe reactions with nucleons of low and intermediate (hundreds of MeV) energies, for which meson production is not yet very important. We have already pointed out that in principle Eq. (30) must be solved simultaneously with the self-consistency conditions, and not in the "frozen" field of the shell model. It is readily seen (see, for example, Ref. 48) that the addition to our kinetic equations of such a self-consistency condition leads to the possible appearance of solutions of a new type of zero-sound vibrations, i.e., collective excitations of the type of giant multipole resonances). In practice, such an addition would make it necessary to carry out time-dependent Hartree–Fock-type calculations simultaneously with the solution of the master equations. It is clear that such a "brute-force" solution is too cumbersome from the computational point of view and needs to be done in an approximate manner. For the moment such effects can be ignored, since the existing theoretical[49] and experimental[50] estimates show that the contribution of the zero-sound excitations to the cross section of reactions with nucleons is very small. One may also choose a larger interval ΔE and go over to a quasiclassical basis (in the approximation of a local momentum), as was done in the derivation of the Boltzmann-type equations (52). As we shall see later, one can also attempt to combine features of different bases.

The inclusion of meson production in the master equations does not present particular difficulties. However, the extremely short mean free paths of π mesons [especially in the region of the (3, 3) resonance] may make it necessary to use very large averaging intervals ΔE comparable with the π-meson mass in the theory. This question is currently under discussion (see also Ref. 51).

We have restricted ourselves to the description of reactions with emission of nucleons. Formally, it is easy to include in the theory emission of clusters as well, provided the microscopic description has a corresponding basis set. Unfortunately, such a basis does not yet exist. This is also the main difficulty in attempts to apply the kinetic approach to reactions with heavy ions.[52] It was suggested in Ref. 53 that the collision of two heavy ions should be described by a system of Boltzmann equations together with time-dependent Hartree–Fock equations. The proposal appears obvious, but, as we have already said, even in the case of collision of a nucleon with a nucleus the direct solution of such a system of equations is not yet possible.

Thus, we have shown that the existing phenomenological models can be related to the fundamental equations of quantum mechanics. We now consider the practical validity of the approximations used in their derivation and some possibilities of their improvement.

3. HOW RELIABLE ARE THE RESULTS OF PHENOMENOLOGICAL MODELS?

In the present section, we consider only models that have predictive power, i.e., that enable one to calculate the absolute values of the cross sections and the profiles of the spectra using parameters found from independent experiments. The Harp–Miller–Berne model and the intranuclear cascade model belong to this class. A detailed comparison of the results obtained in these models with experiments can be found in Refs. 30 and 16. Typical cases of such comparison, which make it possible to draw conclusions about the accuracy of the models, are given in Figs. 5 and 6. The Harp–

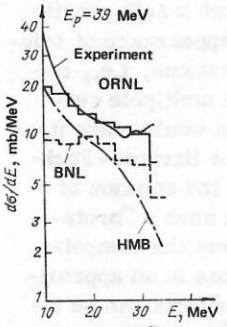

FIG. 5. Comparison of the spectra of secondary protons obtained in different variants of pre-equilibrium models for the reaction $^{56}Fe(p,p')$.

Miller–Berne model usually gives cross sections that are 2–3 times smaller than the experimental ones. Even smaller values of the cross sections are obtained in the BNL (Brookhaven National Laboratory) cascade model. A better description of the experimental spectra is given by the ORNL (Oak Ridge National Laboratory) cascade model. The main difference between these cascade models is that the latter uses a three-step approximation for the nuclear density $\rho(r)$ and does not take into account refraction and reflection of particles on the boundary of the potential, while the former uses a seven-step approximation for the nuclear density and the potential and takes into account refraction and reflection. One must ask why it is that the more detailed (and, apparently, more realistic) BNL model gives worse results. It is also puzzling that the classical ORNL cascade description should be better than the Harp–Miller–Berne solution of the quantum-mechanical equations.

Answers to these problems can be obtained using the general approach set forth in Sec. 2.[54] As we have seen, both the cascade model and the Harp–Miller–Berne method are obtained by solving the master equations (30) for different averaging intervals (of course, in the Harp–Miller–Berne model it is necessary to use the correct values for the outflow and density of the continuum states). Another differences is that in the Harp–Miller–Berne model the master equations are solved exactly (numerically), while the intranuclear cascade model provides only an approximate solution of the Boltzmann-type equations (52). The main approximation here is that in the calculation of the right-hand side of (52) in the intranuclear cascade model the spreading of the Fermi surface due to preceding collisions is ignored [this linearization makes it possible to "decouple" the system of equations (52) and solve each of them separately]. It is clear that this approximation can be valid only in the first stages of the process.

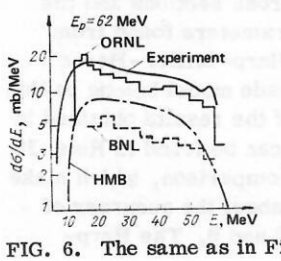

FIG. 6. The same as in Fig. 5 for the reaction $^{209}Bi(p,p')$.

To verify the conclusions, we integrated exactly Eqs. (30b) for protons with $E_p = 39$ MeV incident on ^{56}Fe and ^{209}Bi targets; we regarded these targets as consisting of a mixture of proton and neutron Fermi gases in the rectangular potential of the nucleus with radius R_0. Ignoring reflection and refraction ($\partial\delta/\partial k < R_0$), we used Eqs. (9a) and (49) for the density of the continuum states. In accordance with Eqs. (40), the cross section of inelastic scattering is determined by the expression

$$d\sigma_\varepsilon/d\varepsilon = \int \bar{n}_\varepsilon(t)\,dt \cdot \rho(\varepsilon)/\tau F. \qquad (54)$$

Figure 7 illustrates the formation in time of the lowest energy part ($\varepsilon \approx 2$ MeV) of the inelastic neutron spectrum $[\sigma_\varepsilon(t) \sim \int_0^\infty n_\varepsilon(t')dt']$ in the case of the ^{56}Fe target. It can be seen that the process is divided into two clearly distinguished stages. The fast pre-equilibrium stage lasts about $(5–10)t_{fr}$ (the mean time between successive collisions is $t_{fr} \approx 10^{-22}$ sec). At the end of this period, the population numbers of the bound states are well described by an equilibrium thermal distribution, whereas the population numbers of the continuum states decrease with the energy much more abruptly. The second stage, corresponding to the plateau in Fig. 7, is characterized by much longer relaxation times (the integration of our equations up to a time 10^{-19} sec introduces a contribution of the order of a percent relative to the cross section of the fast stage formed up to the time 3×10^{-21} sec, although the total contribution to the cross section from the slow stage in this energy region exceeds the pre-equilibrium cross section by an order of magnitude). During this stage, the rate of population of the continuum states decreases strongly because of the exponential smallness of the population numbers n of the bound states above the Fermi energy E_F and $1-n$ of the vacancies below E_F, which make the main contribution to the population of the continuum states. Since this rate of population becomes appreciably less than the outflow rate, it is this that determines the long relaxation times (10^{-15}–10^{-16} sec in our case) in this stage. The possibility of unambiguous separation of the two stages on the basis of $\sigma(t)$ reaching the plateau is extremely important, since it makes it possible to investigate the solutions in the fast and slow stages separately.

As we have already said, one can attempt to solve the system of kinetic equations in the first stage using the approximate method of an intranuclear cascade. Knowledge of the kinetic equations for whose solution this method is used makes it possible to modify the standard algorithm of the intranuclear cascade model. First, since Eq. (52) is symmetric with respect to the replacement of n particles by $1-n$ holes, the intranuclear cas-

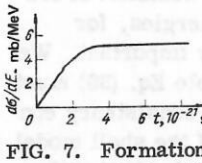

FIG. 7. Formation in time of the cross sections of secondary particles obtained by solving Eqs. (30b).

cade model can also be used to calculate the de-excitation of hole excitations. Therefore, in the calculation we fixed all holes excited in the usual cascade process, and then used the cascade algorithm to follow the de-excitation of holes excited to energy $E_F - E_{bd}$. (The de-excitation of holes in cells with larger energies leads only to an additional spreading of the Fermi surface and not to the emission of particles.) We introduced at the start one further change into the intranuclear cascade model, this being solely in order to compare the cascade solution of the master equations with the exact solutions in the identical model approximations. Calculating the density of the continuum states (9a) needed for the exact solution of Eqs. (30b), we ignore the narrow potential resonances below the centrifugal barrier that arise for $l > k_a R_a$ (l is the angular momentum of the particle and k_a is the asymptotic wave number). Therefore, in the cascade calculations we determined the value of l for the secondary particles after each collision and eliminated collisions that generate particles with $l > k_a R_0$. We used the intranuclear cascade model modified in this way to calculate the spectrum of the secondary nucleons on the same targets ^{56}Fe and ^{209}Bi in the Fermi gas model and compared the results with the spectra obtained in the fast stage by the exact solution of (30b). The results of the comparison for the case of the secondary neutrons are shown in Fig. 8 (the agreement for the spectrum of secondary protons is effectively the same). It can be seen that the correct allowance for the continuum in the open systems modifies the Harp–Miller–Berne and intranuclear cascade models in such a way that the agreement between them becomes considerably better than in the purely phenomenological approaches (see Figs. 5 and 6). In other words, it can be seen that the modified intranuclear cascade model is a good approximation to the exact solution in the pre-equilibrium stage even in the case of fairly low energies of the incident particles (accuracy 20–30%). Another very unexpected result of our calculations is the relatively small contribution to the pre-equilibrium spectra from the de-excitation of the holes. As can be seen in Fig. 8, this process is important only in the low-energy part of the spectrum, which is completely determined by the evaporation process (the contribution of evaporation is not shown in Fig. 8, but it usually exceeds by an order of magnitude the pre-equilibrium contribution in the region of low energies).

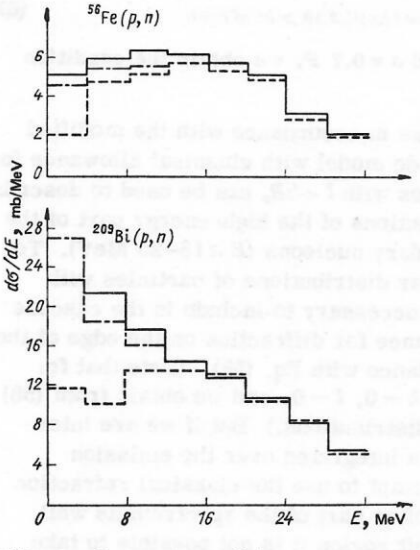

FIG. 8. Comparison of the spectra of secondary neutrons obtained by the exact solution of Eqs. (30b) (solid line) and the modified intranuclear cascade model with the condition $1 < kR_0$ without allowance for de-excitation of holes (broken line) and with allowance for it (chain line).

Hitherto, our main aim was to compare two methods of solution of the master equations in the framework of the simple model of a Fermi gas without refraction. We have seen that the intranuclear cascade model is a fairly simple and reliable method for solving the equations. It would be very appealing to use it as well to describe the angular distributions of the secondary particles with allowance for refraction and diffraction on the edges of the real nuclear potential. For this, we return to the original system (30a) and consider what determines the angular distributions of the secondary particles. Suppose that in the interval τ around the time t_1 and in the neighborhood of the point $\mathbf{r}_1$ within the nucleus a particle in the continuum state u_i interacts with particles u_j of the nuclear matter. This collision during the interval τ generates a new particle in the continuum state u_k with probability [see the solutions of (30a) with the initial conditions $\tilde{n}_i(t_1) = \bar{n}_i(t_1) |G_i(\mathbf{r}_1, t_1)|^2$ and all the remaining states above the Fermi surface unoccupied]

$$\bar{n}_k(t_1) \approx \tau \frac{\pi}{\hbar} n_i(t_1) |G_i(\mathbf{r}_1, t_1)|^2 \times \sum_j \tilde{n}_j |\bar{V}_{ijkl}|^2 \delta^{\Delta\varepsilon}(\varepsilon_i + \varepsilon_j - \varepsilon_k - \varepsilon_l). \quad (55)$$

Using nonstationary perturbation theory, we can take $\tilde{n}_k \ll 1$ and $n_l \ll 1$ and ignore the second term in (30a), which takes into account the absorption of secondary particles. Using the expression (40), we can integrate (55) over all times t_1 for which the particle n_i is within the nucleus. Here, we use the quasiclassical nature of the motion of the particle within the nucleus, i.e., the fact that $dt_1 = dr_1/v_i$ and

$$\int |G(r_1, t_1)|^2 dt_1 = \int_{L(\mathbf{r})_1} \frac{dl(\mathbf{r})}{v_l},$$

where the integration is over the paths of the motion of particle i within the nucleus. Ignoring the absorption of the initial particle, $n_i(t) \approx n_i(t_1)$, we obtain from (40) the cross section for the emission of secondary particles with asymptotic momentum $\hbar\mathbf{k}_k$:

$$\frac{d\sigma}{d\rho_k} = n_i(t_1) \int_{L(\mathbf{r}_1)} \frac{dl(\mathbf{r})}{v_l} \left\{ \frac{\pi}{\hbar v_0} \sum_j \tilde{n}_j(\mathbf{r}) |\bar{V}_{ijkl}|^2 d\rho_l \right\}. \quad (56)$$

Comparing (56) with (47), we see that in the curly brackets we have the cross section for knockout of particles k and l by the incident particle from the nuclear matter at the point $\mathbf{r}$, the cross section being calculated in the DWBA:

$$d\sigma/d\rho_k = n_i(t_1) \int_{L(\mathbf{r})_1} \frac{dl(\mathbf{r})}{v_l} \frac{d\sigma^{\mathrm{DWBA}}_{i\to k,\,l}(\mathbf{r})}{d\rho k}. \quad (56a)$$

Thus, the quasiclassical nature of the nuclear potential has the consequence that particle i moves through the nucleus along the trajectory $L(\mathbf{r}_1)$, knocking out second-

ary particles successively. The same quasiclassical property makes it possible[55] to simplify even further the calculation of the matrix elements $\bar{V}_{ijkl} = \int d\mathbf{r}_1 d\mathbf{r}_2 \bar{u}_k^* \times (\mathbf{r}_1)\bar{u}_l^*(\mathbf{r}_2)V(\mathbf{r}_1-\mathbf{r}_2)\bar{u}_i(\mathbf{r}_1)u_j(\mathbf{r}_2)$, since the integral here is taken in a neighborhood $\Delta r \sim 1/\Delta k$ of the point r. Because of the quasiclassical property, in the interior region of the nucleus $u(\mathbf{r}+\mathbf{x}) = \exp[(i/\hbar)\mathbf{p}(\mathbf{r})]u(\mathbf{r})$. Then

$$|V_{ijkl}|^2 \sim \sigma_N(ij \to kl)|\bar{u}_i(\mathbf{r})|^2|\bar{u}_j(\mathbf{r})|^2|\bar{u}_k(\mathbf{r})|^2|\bar{u}_l(\mathbf{r})|^2, \quad (57)$$

where σ_N is the cross section for scattering of free nucleons with local momenta $\mathbf{p}(\mathbf{r})$. To describe the angular distributions of the secondary particles produced in the state u_k at the point r, it is only necessary to find the connection between the vector r and the asymptotic momentum $\hbar\mathbf{k}$ of the particle by expanding $u_k(r)$ in partial waves:

$$|u_k(\mathbf{r})|^2 = \left|\sum_l \frac{i^l(2l+1)}{4\pi kr}\exp(i\delta_l)\chi_l(kr)P_l\left(\frac{\mathbf{kr}}{kr}\right)\right|^2. \quad (58)$$

Thus, the quasiclassical property of the nuclear potential makes it possible to simplify the finding of the angular distributions by using the algorithm of the intranuclear cascade model in accordance with (56)–(58). As usual, the probability of collision of particle i with the particles of the nucleus is determined by the cross sections σ_N [see (57)], which depend on the local momenta $\mathbf{p}(\mathbf{r})$, and by the density of the nuclear matter $\tilde{n}_j(\mathbf{r}) \sim \rho(\mathbf{p}_j, \mathbf{r})$ [see (56)]. The local momentum of the secondary particle $p_k(\mathbf{r})$ is related to its asymptotic momentum by $\hbar k = \sqrt{p_k^2(r) + 2\mathcal{M}U(r)}$, and the direction of emission $\mathbf{k}/k$ is related to the vector r of the point of collision by Eq. (58). The sum over l in (58) is determined by the properties of the potential through $\chi_l(kr)$. In a rectangular well, because of the sharp reflection on the boundary, there are many narrow potential resonances below the centrifugal barrier, and therefore the summation must be extended to values $l_{\max} = p_k R_0/\hbar$. In the Woods–Saxon potential with diffuse edge, there are virtually no such resonances (see, for example, the numerical calculations of Ref. 56), the wave functions $\chi_l(kr)$ for $l > kR_0$ (below the centrifugal barrier) are exponentially small, and the sum effectively terminates at $l_{\max} = kR_0$. Therefore, the criterion $l < kR_0$ introduced in the modified intranuclear cascade model is a simple but effective means of taking into account the diffuse edge of the nuclear potential. This criterion has a very strong influence on the profile of the spectrum of secondary nucleons (Fig. 9). In the ordinary cascade model with refraction on the abrupt discontinuity of the potential there is total internal reflection of particles with $l > kR_0$, which is the classical analog of a potential resonance below the centrifugal barrier. Being reflected by the edge, the particle returns to the interior of the potential until the next two-body collision decelerates it and enables it to leave the nucleus, augmenting the low-energy part of the spectrum. When the energy of the particle is so high that its motion is also quasiclassical outside the nucleus, it is possible to use the quasiclassical representation for the Legendre polynomials and the method of stationary phase in (58) to show that the expression (58) is nonzero only for motion along a classical path from the point r to infinity, i.e., one can justify the classical refraction algorithm in the intranuclear cascade model. Moreover, such an investigation shows that description of the angular distribution with accuracy $\Delta\theta \approx 20°$ makes it possible to approximate the Woods–Saxon potential by one step. To a greater accuracy we cannot pretend, since we have wave packets with a momentum spread $\Delta p \approx \Delta\varepsilon\sqrt{\mathcal{M}/2\varepsilon}$. An estimate of the validity of the path algorithm of the intranuclear cascade model for the description of the angular distributions of the secondary nucleons can be obtained from the following considerations. To calculate the angle of emission of a nucleon with accuracy $\Delta\theta$, it is necessary that the uncertainty in the momentum Δp of the particle outside the nucleus be sufficiently small:

$$\Delta p/p_a \approx \Delta\theta. \quad (59)$$

However, in the path algorithm of the intranuclear cascade model we use Eq. (53), which is based on the fact that the local momentum $\mathbf{p}(\mathbf{r})$ of the packet is related to the central potential $U(\mathbf{r})$ (refraction of a classical particle). We have already pointed out that for the validity of the expression (53) the packet dimensions must be less than the characteristic diffuseness dimension of the potential, $l \approx 4.4a$ (a is the diffuseness parameter of the Woods–Saxon potential), i.e.,

$$\Delta p > \hbar/l. \quad (60)$$

Comparing (59) and (60), we see that the condition (60) is more stringent than the requirement of quasiclassical behavior. We can now find a condition for the energy of secondary particles for which the path description of the angular distributions is valid:

$$\varepsilon > \Delta\varepsilon = p_a\Delta p/\mathcal{M} \approx (\Delta p)^2/\mathcal{M}\Delta\theta > \hbar^2/\mathcal{M}l^2\Delta\theta. \quad (61)$$

Taking $\Delta\theta \approx 20°$ and $a \approx 0.7$ F, we obtain the condition $\varepsilon > 12$ MeV.

Thus, calculations in accordance with the modified intranuclear cascade model with classical allowance for refraction for states with $l < kR_0$ can be used to describe the angular distributions of the high-energy part of the spectrum of secondary nucleons ($E \gtrsim 15$–20 MeV). To describe the angular distributions of particles with lower energy it is necessary to include in the cascade calculations allowance for diffraction on the edge of the potential in accordance with Eq. (58). (Note that for very low energies $k \to 0$, $l \to 0$, and we obtain from (58) isotropic angular distributions.) But if we are interested in the spectra integrated over the emission angles, we can attempt to use the classical refraction algorithm in the softer part of the spectrum as well. However, in the soft region it is not possible to take into account only the fast processes, since the contribution from the slow stage to the spectrum is here dominant. One could calculate the slow stage by direct integration of Eqs. (30b). (Examination of the two-body

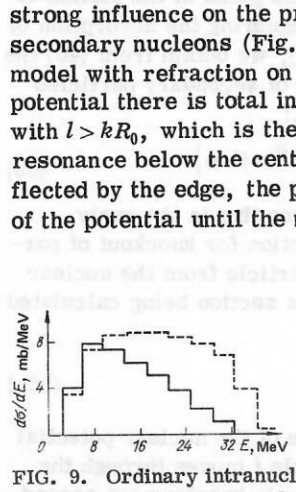

FIG. 9. Ordinary intranuclear cascade model (solid line) and the cascade model with the condition $1 < kR_0$ (broken line).

matrix elements V_{ijkl} of the wave functions in the Woods–Saxon potential shows that if the total excitation energy E_{exc} of the system significantly exceeds the rotational energy E_{rot} of the nucleus as a whole, then the angular distributions for the process of de-excitation of the hole and more complicated states are isotropic.) One can, however, significantly simplify the problem[57] by using the results of the numerical solution of Eqs. (30b). The smallness of the continuum population numbers makes it possible to ignore the second term on the right-hand sides of Eqs. (30b) for $\tilde{n}_k$ corresponding to the continuum:

$$\frac{d\tilde{n}_k}{dt} \approx \sum_{ijl} \omega_{ijkl}\delta(\varepsilon_i+\varepsilon_j-\varepsilon_k-\varepsilon_l) \times g_i g_j g_l \tilde{n}_i \tilde{n}_j (1-\tilde{n}_l) - \frac{\tilde{n}_k}{\tau}. \quad (62)$$

Further, one can use the circumstance that for the slow stage the population numbers of the bound states are well described by the thermal distribution (15). The outflow of particles (and associated outflow of energy) from the system merely has the consequence that the temperature of the system becomes a slowly varying function $T(t)$ of the time. Using this form for the population numbers of the discrete states, we can write the solution (62) in the typical case $T < E_{bd} < E_F$ in the form

$$\bar{n}_k(t) \approx W_{ijkl} \exp[-(E_{bd}+\varepsilon_k^a)/T(t)], \quad (63)$$

where $\varepsilon_k^a = \varepsilon_k - E_F - E_{bd}$ is the kinetic energy of the emitted particle outside the nucleus, and

$$W_{ijkl} = \tau \sum_{\varepsilon_i=E_F}^{E_F+E_{bd}} g_i \sum_{\varepsilon_j=E_F}^{2E_F-E_{bd}-\varepsilon_i} g_j g_l (\varepsilon_l = \varepsilon_i+\varepsilon_j-\varepsilon_k)\,\omega_{ijkl}$$

is the probability of emission of a particle with energy ε_k from the heated nucleus. Substituting (63) in (54), we obtain a formula for the evaporation cross section:

$$\frac{d\sigma(\varepsilon_k^a)}{d\varepsilon_k} = \frac{W_{ijkl}}{F}\frac{d\rho(\varepsilon_k^a)}{\tau} \times \int_{t_1}^{t_{max}} \exp[-(E_{bd}+\varepsilon_k^a)/T(t)]\,dt. \quad (64)$$

The first factor on the right-hand side here is the cross section for emission of a particle with energy ε_k^a by the excited system. It can be seen that in contrast to the absorption cross section in the usual evaporation model, it is explicitly related to the matrix elements of the two-body interactions ω_{ijkl}. Examining the limits of summation in $W_{ijkl}(\varepsilon_l \leq E_F - \varepsilon_k^a, E_F \leq \varepsilon_i, \varepsilon_j \leq E_F + E_{bd})$, we readily see that this cross section tends to $\sigma_{abs}(\varepsilon_k^a)$ in the case $T \ll E_{bd}$ even when the contribution of direct processes of inelastic scattering to σ_{abs} can be ignored. Since the temperature of our system falls during the evaporation process, the temperature factor $\exp(-\varepsilon/T)$ customary in the evaporation model is under the integral sign. Using the thermal distributions for the population numbers of the bound states, we can derive from Eqs. (30b) an energy-balance equation for our system and find in analytic form[57] the time dependence $T(t)$ of the temperature needed to calculate the expression (64). We see that our evaporation model differs in the first place from the usual one by the presence of cooling. There exists a phenomenological evaporation model[58] in which the cooling of the system is taken into account by the method of random sampling. We compared the results of our calculations in accordance with Eq. (64) with the results of this model and showed that the latter reproduces with small error (≈ 10%) the results of (64).

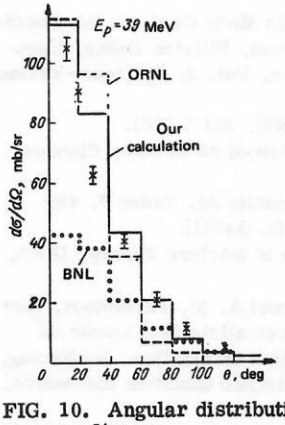

FIG. 10. Angular distribution of secondary protons in the reaction $^{54}Fe+p$.

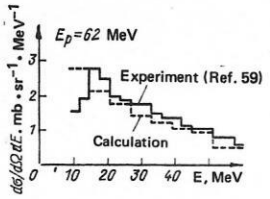

FIG. 11. Spectrum of secondary protons emitted at 60° in the reaction $^{209}Bi+p$.

Thus, we have obtained a method of describing the spectra and angular distributions of secondary nucleons which is as simple as the existing phenomenological models but does not contain adjustable parameters. To describe the fast pre-equilibrium stage, we use the modified intranuclear cascade model, and for the system formed at the end of this stage we use the expressions of the modified evaporation model. The calculations made in this way have been compared with the available experimental data on reactions with protons (E=39–62 MeV) on the nuclei ^{54}Fe, ^{89}Y, ^{208}Pb, and ^{209}Bi. In Fig. 10, we compare our calculated angular distributions of secondary protons with $E \geq 20$ MeV in the reaction $^{54}Fe+p$ with the experimental data of Ref. 59. We also plot the results of calculations in accordance with the ORNL and BNL cascade models. It is now easy to see what is the origin of the shortcomings of both the cascade models. In the BNL model, allowance is made for refraction on a step potential. The absence

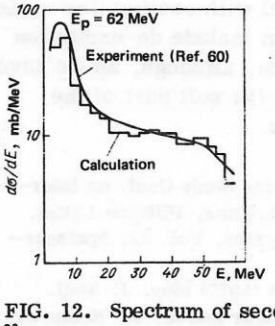

FIG. 12. Spectrum of secondary protons from the reaction $^{89}Y+p$.

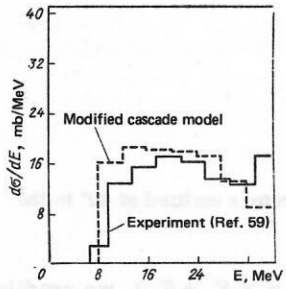

FIG. 13. Comparison of the spectra of secondary protons from the reaction ^{209}Bi(p,p'). The experimental data are averaged over an energy interval $\Delta E = 4$ MeV.

of the criterion $l < kR_0$ leads to a large number of total internal reflections of a particle on the edge of the potential, which, as we have seen, very strongly distorts the spectrum. At small angles it is found to be better to ignore refraction altogether, which is done in the ORNL model. However, the latter gives underestimated yields at large angles, since without allowance for refraction large emission angles of the secondary nucleons are improbable. Our modified intranuclear cascade model, which takes into account refraction and diffuseness of the nuclear potential, agrees better with the experiments at both small and large angles. In Fig. 11, we give an example of the calculation of the energy spectrum of the secondary protons emitted at angle $\theta = 60°$ in the reaction ^{209}Bi$+p$. In Figs. 12 and 13 we show the energy spectra of the secondary protons from the reactions ^{89}Y$+p$ and ^{209}Bi$+p$. The pronounced spread in the hardest part of the spectrum in Fig. 13 is due to the excitation of low-lying collective states of the target nucleus ^{209}Bi. As we have already said, in a model in which the density and the potential are not self-consistent it is impossible to take into account such states. It can be seen that even if we make simplifying assumptions such as that of a Fermi gas, step penetrability of the Coulomb barrier, and classical refraction (but with the condition $l < kR_0$) the calculation yields reasonable agreement (30–40%) with the experiments.

Thus, the master-equation method not only makes it possible to unify all the various phenomenological models of nuclear reactions but also to understand which of them are the most reliable in practical applications. At the present time, we believe that the most promising and practical method of calculation is provided by the intranuclear cascade model (it is sufficient to take into account in it classical refraction on a single-step potential, but the condition $l < kR_0$ must be included) augmented by the evaporation model with cooling (the model of Ref. 58 can be used). One can include de-excitation of holes in the cascade algorithm, although, as we have seen, this only slightly changes the soft part of the spectrum of secondary nucleons.

[1]H. Feshbach, in: Proc. Europhysics Study Conf. on Intermediate Processes in Nuclear Reactions, Plitvice Lakes, Yugoslavia. Lecture Notes in Physics, Vol. 22, Springer–Verlag (1973).

[2]V. E. Bunakov, Yad. Fiz. 25, 505 (1977) [Sov. J. Nucl. Phys. 25, 271 (1977)]; V. E. Bunakov and M. M. Nesterov, in: Materialy X zimneĭ shkoly LIYaF po fizike yadra i élementarnykh chastits (Proc. Tenth Winter School of the Leningrad Institute of Nuclear Physics on Nuclear Physics and Elementary Particles), Part 1, Leningrad (1975).

[3]V. E. Bunakov and M. M. Nestorov, Phys. Lett. B60, 417 (1976).

[4]D. Agassi and H. Weidenmüller, Phys. Lett. B56, 305 (1975).

[5]D. Agassi, H. Weidenmüller, and G. Mantzouranis, Phys. Rep. C22, 147 (1975).

[6]M. L. Goldberger and K. M. Watson, Collision Theory, Wiley, New York (1964) [Russian translation published by Mir, Moscow (1967)].

[7]N. Austern, Direct Nuclear Reaction Theories, Wiley, New York (1970).

[8]C. Mahaux and H. Weidenmüller, Shell-Model Approach to Nuclear Reactions, Amsterdam (1969).

[9]R. F. Barrett *et al.*, Rev. Mod. Phys. 45, 44 (1973).

[10]V. G. Soloviev, in: Proc. Intern. School in Nuclear Structure, Alushta (1972); V. G. Soloviev and L. A. Malov, Nucl. Phys. A196, 433 (1972).

[11]H. Feshbach, Ann. Phys. (N.Y.) 5, 357 (1958).

[12]G. E. Brown, Rev. Mod. Phys. 31, 893 (1959).

[13]A. Bohr and B. R. Mottelson, Nuclear Structure, Vol. 1, Benjamin, New York (1969) [Russian translation published in Moscow (1971)].

[14]F. Friedman and V. Weisskopf, in: Niels Bohr and the Development of Physics, London (1955).

[15]V. S. Barashenkov *et al.*, in: Proc. Intern. JINR–CERN School on High Energy Physics, E2-5813 (1971); V. S. Barashenkov and V. D. Toneev, Vzaimodeĭstvie vysokoénergeticheskikh chastits i atomnykh yader s yadrami (Interaction of High Energy Particles and Nuclei with Nuclei), Moscow (1972); V. S. Barashenkov *et al.*, Nucl. Phys. A187, 631 (1972).

[16]G. D. Harp and J. M. Miller, Phys. Rev. C 3, 1847 (1971).

[17]H. V. Bertini, G. D. Harp, and F. E. Bertrand, Phys. Rev. C 10, 2472 (1974).

[18]M. L. Goldberger, Phys. Rev. 74, 1268 (1948).

[19]L. D. Landau, Phys. Z. Sowjetunion 11, 556 (1937).

[20]J. Frenkel, Phys. Z. Sowjetunion 9, 533 (1936).

[21]V. Weisskopf, Phys. Rev. 52, 295 (1937).

[22]J. J. Griffin, Phys. Rev. Lett. 17, 478 (1966).

[23]G. D. Harp, J. M. Miller, and B. J. Berne, Phys. Rev. 165, 1166 (1968).

[24]E. A. Uehling and G. E. Uhlenbeck, Phys. Rev. 43, 552 (1933).

[25]F. S. Williams, Phys. Lett. B31, 184 (1970).

[26]C. K. Cline and M. Blann, Nucl. Phys. A172, 225 (1971).

[27]C. K. Cline, Nucl. Phys. A193, 417 (1972).

[28]K. K. Gudima, G. A. Ososkov, and V. D. Toneev, Yad. Fiz. 21, 260 (1975) [Sov. J. Nucl. Phys. 21, 138 (1975)].

[29]E. Gadioli and L. Milazzo-Colli, in: Proc. Europhysics Study Conf. on Intermediate Processes in Nuclear Reactions, Plitvice Lakes, Yugoslavia. Lecture Notes in Physics, Vol. 2, Springer–Verlag (1973).

[30]M. Blann, in: Proc. Europhysics Study Conf. on Intermediate Processes in Nuclear Reactions, Plitvice Lakes, Yugoslavia. Lecture Notes in Physics, Vol. 2, Springer–Verlag (1973).

[31]D. Gadioli *et al.*, Phys. Lett. B65, 311 (1976).

[32]C. Kalbach, in: Proc. Intern. School on Neutron Physics, Alushta (1974), Dubna (1974).

[33]K. Seidel *et al.*, Fiz. Elem. Chastits At. Yadra 7, 499 (1976) [Sov. J. Part. Nucl. 7, 192 (1976)].

[34]N. Austern, in: Selected Topics in Nuclear Theory, IAEA, Vienna (1963).

[35]A. I. Baz', Ya. B. Zel'dovich, and A. M. Perelomov, Rasseyanie, reaktsii i raspady v nerelyativistskoĭ kvantovoĭ mekhanike, Moscow (1965); English translation: Scattering, Reactions and Decay in Nonrelativistic Quantum Mechanics, Jerusalem (1969).

[36]J. Schwinger and B. Lippmann, Phys. Rev. 79, 469 (1950).

[37]J. Kirkwood, J. Chem. Phys. 15, 72 (1946); J. Irving and J. Kirkwood, J. Chem. Phys. 18, 817 (1950).
[38]G. Sandri, Ann. Phys. (N.Y.) 24, 332 (1963); G. E. Uhlenbeck and G. W. Ford, Lectures in Statistical Physics, AMS, Providence, R.I. (1963) [Russian translation published in Moscow (1965)].
[39]N. N. Bogolyubov, Zh. Eksp. Teor. Fiz. 16, 8 (1946); Problemy dinamicheskoĭ teorii v statisticheskoĭ fizike, Moscow (1946); English translation published in: "Problems of a dynamical theory in statistical physics," in: Studies in Statistical Mechanics, Vol. 1 (ed. J. de Boer and G. E. Uhlenbeck), North–Holland, Amsterdam (1962).
[40]N. N. Bogolyubov and K. P. Gurov, Zh. Eksp. Teor. Fiz. 17, 614 (1947).
[41]B. A. Rumyantsev and S. A. Kheĭfets, Yad. Fiz. 21, 510 (1975) [Sov. J. Nucl. Phys. 21, 267 (1975)].
[42]A. S. Davydov, Kvantovaya mekhanika, Moscow (1963); English translation: Quantum Mechanics, Pergamon, London (1965).
[43]R. G. Newton, Scattering Theory of Waves and Particles, McGraw–Hill, New York (1966) [Russian translation published in Moscow (1969)].
[44]P. M. Matthews, I. Shapiro, and D. Falkoff, Phys. Rev. 120, 1 (1960); N. Dresden, Rev. Mod. Phys. 33, 265 (1961); M. Leontovich, Zh. Eksp. Teor. Fiz. 5, 211 (1935).
[45]L. D. Landau and V. M. Lifshits, Statisticheskaya fizika, Moscow–Leningrad (1951); English translation: Statistical Physics, Pergamon, London (1958).
[46]J. K. Haviland, Advances in Research and Application 4, 109 (1965); G. A. Bird, J. Fluid Mech. 30, 479 (1967).
[47]Metod Monte-Karlo v probleme perenosa izlucheniya (The Monte Carlo Method in the Problem of Radiative Transfer), Moscow (1975); S. M. Ermakov, Metod Monte-Karlo i smezhnye voprosy (The Monte Carlo Method and Related Questions), Moscow (1975); O. I. Leĭpunskiĭ, B. V. Novozhilov, and V. N. Sakharov, Rasprostranenie gamma-kvantov v veshchestve (Propagation of Gamma Rays in Matter), Moscow (1960).
[48]D. J. Thouless, The Quantum Mechanics of Many-Body Systems, Academic Press, New York (1961) [Russian translation published in Moscow (1975)].
[49]G. Bertsch and S. F. Tsai, Phys. Rev. C 11, 1634 (1975); G. Satchler *et al.*, Nucl. Phys. A245, 189 (1975).
[50]E. F. Kuchnir, Phys. Rev. 161, 1236 (1967).
[51]J. N. Ginocchio, Phys. Rev. C 17, 195 (1978).
[52]S. Ayik and W. Nörenberg, Preprint GSI-VT-I-78, Darmstadt.
[53]H. Orland and R. Schaeffer, Preprint DPh-T/78/41, Saclay.
[54]V. E. Bunakov, M. M. Nesterov, and N. A. Tarasov, Izv. Akad. Nauk SSSR, Ser. Fiz. 41, 2187 (1977); V. E. Bunakov, M. M. Nesterov, and N. A. Tarasov, Phys. Lett. B73, 267 (1978).
[55]V. E. Bunakov, M. M. Nesterov, and N. A. Tarasov, Izv. Akad. Nauk SSSR, Ser. Fiz. 42, 2351 (1979).
[56]V. E. Kolesnikov and V. A. Korotkikh, Izv. Akad. Nauk SSSR, Ser. Fiz. 27, 900 (1963).
[57]V. E. Bunakov, M. M. Nesterov, and N. A. Tarasov, Izv. Akad. Nauk SSSR, Ser. Fiz. 42, 1893 (1978).
[58]I. Dostrovsky *et al.*, Phys. Rev. 111, 1659 (1958); 118, 791 (1960).
[59]F. E. Bertrand and R. W. Peele, Phys. Rev. C 8, 1045 (1973).
[60]F. E. Bertrand and R. W. Peele, Report ORNL-4450, Oak Ridge (1969).

Translated by Julian B. Barbour