The inverse scattering method in the theory of nonlinear evolution equations

V. K. Mel'nikov

Joint Institute for Nuclear Research, Dubna

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An algorithm is proposed for finding nonlinear evolution equations that can be investigated by the inverse scattering method. It is shown that all the equations obtained by means of this algorithm have infinitely many conservation laws. Some well-known examples are given as illustrations.

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INTRODUCTION

During the last decade, mathematical physics has been enriched by a new method of investigating nonlinear evolution equations. Although it applies to the investigation of a comparatively small class of equations, the importance of the obtained results explains the considerable interest that they stimulate. During this period of time it has been shown that a number of equations which are important from the applied point of view belong to this class. Searches for such equations continue, and their number increases with each year. At the present time, there are about 20 known equations that describe nonlinear evolution processes and can be investigated by this method. It has become known as the inverse scattering method, and its essence is as follows.

Consider the nonlinear evolution equation

$$\partial u/\partial t = F(u, \partial u/\partial x, \dots, \partial^n u/\partial x^n). \tag{1}$$

Suppose that it has a solution $u=u(x,t)$, defined for all $x\in(-\infty,\infty)$ and all $t\geqslant 0$. Suppose in addition that as $x\to\pm\infty$ the solution $u(x,t)$ tends fairly rapidly to zero for any fixed $t\geqslant 0$, so that, for example, the condition

$$\int_{-\infty}^{\infty}(1+|x|)\,|u(x,t)|\,dx<\infty \tag{2}$$

is satisfied for all $t\geqslant 0$. We now take the Schrödinger operator $L=-\partial^2/\partial x^2+u$, in which this solution of Eq. (1) occurs as the potential $u=u(x,t)$, and consider for it the standard scattering problem. By virtue of (2), the equation

$$L\varphi=\zeta^2\varphi \tag{3}$$

for any real ζ has two solutions φ_1 and φ_2 satisfying the conditions

$$\varphi_1\sim\exp(-i\zeta x),\quad \varphi_2\sim\exp(i\zeta x),\quad x\to-\infty,$$

and two solutions ψ_1 and ψ_2 satisfying the conditions

$$\psi_1\sim\exp(-i\zeta x),\quad \psi_2\sim\exp(i\zeta x),\quad x\to+\infty.$$

It is easy to see that the Wronskian $W(\varphi_1,\varphi_2)=\varphi_1\varphi_2'-\varphi_1'\varphi_2$ of the pair of solutions φ_1 and φ_2 is $2i\zeta$ and, therefore, $W\neq 0$ for any $\zeta\neq 0$. Hence, for any real $\zeta\neq 0$ the solutions φ_1 and φ_2 form a fundamental system of solutions and, therefore,

$$\psi_\alpha=\sum_{\beta=1}^{2}S_{\alpha\beta}\varphi_\beta,\quad \alpha=1,2, \tag{4}$$

where the elements $S_{\alpha\beta}$ of the matrix S do not depend on x ($\alpha,\beta=1,2$). Further, it is easy to show that the solutions ψ_1 and φ_2 admit analytic continuation with respect to ζ into the lower half-plane. Hence, by virtue of the equation

$$S_{11}=W(\psi_1,\varphi_2)/2i\zeta \tag{5}$$

it follows that the element S_{11} of the matrix S also admits analytic continuation with respect to ζ into the lower half-plane. At the same time, the zeros ζ_m, $m=1,\dots,m_0$, of the function S_{11} lying in the lower half-plane are, by virtue of Eq. (5), points of the discrete spectrum, i.e., for $\zeta=\zeta_m$, Eq. (3) has a solution $\varphi(x,\zeta_m)$ with asymptotic behavior

$$\varphi\sim\begin{cases}\exp(i\zeta_m x), & x\to-\infty;\\ C_m\exp(-i\zeta_m x), & x\to+\infty.\end{cases} \tag{6}$$

Therefore, the solution $\varphi(x,\zeta_m)$ is integrable with respect to x on the complete real axis, i.e.,

$$N_m^2=\int_{-\infty}^{\infty}\varphi^2(x,\zeta_m)\,dx<\infty.$$

Because the potential $u=u(x,t)$ in the definition of the Schrödinger operator L depends on the time t as on a parameter, the scattering data will also be functions of t. A natural question arises: Could it be possible to determine the dependence of the scattering data on the time without solving Eq. (1) itself but only by assuming that the potential $u=u(x,t)$ satisfies Eq. (1) and the condition (2)? If the answer to this question is in the affirmative, we could, instead of directly solving Eq. (1), first find the scattering data for the Schrödinger equation at $t=0$ on the basis of the initial data for Eq. (1), then continue the scattering data to all $t\geqslant 0$, and, finally, having solved the inverse problem for the Schrödinger equation, find the solution to Eq. (1) for all $t\geqslant 0$. And although *a priori* it is not clear whether this route will be shorter or simpler for the actual determination of the solution to Eq. (1), there is no doubt that the availability of a new mathematical formalism may make it possible to find new properties of already known solutions to previously studied equations. This is what has happened! It was found first of all that the equations which can be investigated by the inverse scattering method have very special solutions of the solitary-wave type. These solutions, which have become known as solitons, are characterized above all by the fact that after interaction with one another they reestablish their initial form. Further, it was found that for such equations an arbitrary solution of the solitary-wave type decomposes with increas-

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ing t into a finite number of solitons. Last, but not least, it was found that any solution of such equations satisfies an infinite number of conservation laws.

The idea of applying the inverse scattering method to the investigation of a nonlinear evolution equation is due to a group of American scientists, which at various stages in the investigation included Gardner, Green, Zabusky, Kruskal, and Miura. In 1967, they applied this idea to the investigation of the Korteweg–de Vries equation, which is well known in the theory of nonlinear waves:

$$u_t - 6uu_x + u_{xxx} = 0. \tag{7}$$

The basis of their investigation[1] was a fact that they had noted, namely, that the operator

$$Q = \partial/\partial t + \partial^3/\partial x^3 - 3(u + \zeta^2)\,\partial/\partial x \tag{8}$$

carries any solution of Eq. (3) into an (in general, different) solution to the same equation if the function $u = u(x, t)$ in Eq. (8) and in the definition of the Schrödinger operator L satisfies Eq. (7). Indeed, since the commutator of the operator Q with the Schrödinger operator L has the form

$$[Q, L] = u_t - 6uu_x + u_{xxx} + 3u_x(L - \zeta^2),$$

it follows from Eqs. (3) and (7) that $[Q, L]\varphi = 0$, i.e.,

$$(L - \zeta^2)\,Q\varphi = 0. \tag{9}$$

Equation (9) proved to be very fruitful. Using it, the authors of Ref. 1 found that if a solution to Eq. (7) satisfying the condition (2) is substituted in the Schrödinger operator, then the dependence of the S matrix on the time will be determined by the equation

$$\partial S/\partial t + [\Gamma, S] = 0, \quad \Gamma = \mathrm{diag}\,(-4i\zeta^3, 4i\zeta^3).$$

For by virtue of Eq. (8) and the definition of the solutions φ_1, φ_2 and ψ_1, ψ_2 of Eq. (3)

$$Q\varphi_1 = 4i\zeta^3\varphi_1; \quad Q\varphi_2 = -4i\zeta^3\varphi_2; \quad Q\psi_1 = 4i\zeta^3\psi_1; \quad Q\psi_2 = -4i\zeta^3\psi_2. \tag{10}$$

In addition, in accordance with Eq. (4)

$$Q\psi_\alpha = \sum_{\beta=1}^{2}\left(\frac{\partial S_{\alpha\beta}}{\partial t}\varphi_\beta + S_{\alpha\beta}Q\varphi_\beta\right), \quad \alpha = 1, 2.$$

Using Eqs. (10), we then obtain

$$4i\zeta^3\psi_1 = (\partial S_{11}/\partial t + 4i\zeta^3 S_{11})\,\varphi_1 + (\partial S_{12}/\partial t - 4i\zeta^3 S_{12})\,\varphi_2;$$
$$-4i\zeta^3\psi_2 = (\partial S_{21}/\partial t + 4i\zeta^3 S_{21})\,\varphi_1 + (\partial S_{22}/\partial t - 4i\zeta^3 S_{22})\,\varphi_2,$$

i.e., in accordance with Eq. (4), we have

$$\partial S_{11}/\partial t = 0; \quad \partial S_{12}/\partial t - 8i\zeta^3 S_{12} = 0;$$
$$\partial S_{21}/\partial t + 8i\zeta^3 S_{21} = 0; \quad \partial S_{22}/\partial t = 0.$$

It follows from this in particular that the element S_{11} of the matrix S does not depend on the time. Nor, therefore, do the positions of the zeros of this function. Thus, the points of the discrete spectrum and, therefore, their number do not change with the time if the potential $u = u(x, t)$ in the definition of the Schrödinger operator L satisfies Eq. (7) and the condition (2). Finally, for the eigenfunctions of the discrete spectrum we have in accordance with the equations

$$Q\varphi_2 = -4i\zeta_m^3\varphi_2; \quad Q\psi_1 = 4i\zeta_m^3\psi_1$$

and the condition (6)

$$Q\varphi_2 = (\partial C_m/\partial t)\,\psi_1 + C_m Q\psi_1,$$

i.e.,

$$\partial C_m/\partial t + 8i\zeta_m^3 C_m = 0.$$

Thus, by means of the operator Q we can determine the time evolution of all scattering data for the Schrödinger equation. This enabled the authors of Ref. 1 to apply the inverse scattering method to the solution of the KdV equation. In the series of papers[2-7] which followed Ref. 1, these authors made a comprehensive analysis of the obtained solution, as a result of which they proved the existence of solitons and conservation laws and established a number of other interesting properties of the KdV equation. In particular, it was established in Ref. 5 that the KdV equation is a Hamiltonian system with infinitely many degrees of freedom. Somewhat later, Zakharov and Faddeev proved[8] that the KdV equation is a completely integrable Hamiltonian system, and they found variables of the action–angle type.

Soon after the publication of Ref. 1, Lax[9] found that the operator[1)]

$$A = 4\frac{\partial^3}{\partial x^3} - 3\left(u\frac{\partial}{\partial x} + \frac{\partial}{\partial x}u\right)$$

which is intimately related to L and Q, satisfies by virtue of the KdV equation the relation

$$\partial L/\partial t + [A, L] = 0, \tag{11}$$

i.e., the operator A plays the part of the Hamiltonian in the well-known Heisenberg equation of quantum mechanics. The relation (11) was an effective means for finding equations amenable to investigation by the inverse scattering method. By means of a suitably chosen pair of operators L and A, which came to be known as a Lax pair, several equations that have applied importance were investigated.[10-17] The first paper was by Zakharov and Shabat,[10] who investigated the nonlinear Schrödinger equation

$$iu_t + u_{xx} + \varkappa|u|^2 u = 0, \quad \varkappa > 0. \tag{12}$$

As Lax pair, they introduced the operators

$$L = i\begin{vmatrix} 1+p & 0 \\ 0 & 1-p \end{vmatrix}\frac{\partial}{\partial x} + \begin{vmatrix} 0 & \bar{u} \\ u & 0 \end{vmatrix}, \quad \varkappa = \frac{2}{1-p^2},$$
$$A = -p\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\frac{\partial^2}{\partial x^2} + \begin{vmatrix} |u|^2/(1+p) & i\bar{u}_x \\ -iu_x & -|u|^2/(1-p) \end{vmatrix}.$$

When these operators are substituted in the left-hand side of the equation

$$\partial L/\partial t + i[A, L] = 0.$$

The result is zero if the complex-valued function $u = u(x, t)$ satisfies Eq. (12).

However, in 1973 Ablowitz, Kaup, Newell, and Segur[18] made an analogous investigation of the equation

$$u_{xt} = \sin u, \tag{13}$$

which is encountered in various branches of mathematical physics and is known as the sine–Gordon equation. They were the first to eschew the search for a Lax pair generating Eq. (13), and obtained this equation as the condition for the existence of a fundamental system of solutions $v = |v_{\alpha\beta}|$, $\alpha, \beta = 1, 2$, common to the two linear systems of equations

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1)The relation $Q = \partial/\partial t + A + 3\,\partial/\partial x\,(L - \zeta^2)$ holds.

$$\begin{aligned}\partial v_1/\partial x+i\zeta v_1=q(x,t)v_2,\\ \partial v_2/\partial x-i\zeta v_2=-q(x,t)v_1\end{aligned}\quad (14)$$

and

$$\partial v_1/\partial t=(i/4\zeta)(v_1\cos u+v_2\sin u),$$
$$\partial v_2/\partial t=(i/4\zeta)(v_1\sin u-v_2\cos u),$$

it being necessary to set $q=-u_x/2$ in the system (14). They showed that in this manner one can obtain the nonlinear Schrödinger equation, the KdV equation, and the modified KdV equation.[19]

The approach proposed in Refs. 18–20 was so encouraging that after their publication the Heisenberg equation (11) was also regarded as the condition of compatability of two linear equations:

$$(L-\eta)\varphi=0,\quad \varphi_t+A\varphi=0.$$

However, neither the Heisenberg equation nor the condition of compatability provides the most adequate expression for the idea underlying Ref. 1. As,we have already noted, the key point in the use of the inverse scattering method is the existence of a pair of linear operators L and Q with the following properties.

Suppose the operator L acts on variables which we shall nominally call spatial; suppose, further, the operator L depends on the time t as on a parameter, and suppose, finally, $\varphi=\varphi(x,t,\eta)$ is a solution of the equation

$$(L-\eta)\varphi=0. \quad (15)$$

Let us now consider the conditions under which there exists a linear operator Q such that for any φ satisfying (15) it happens that $\psi=Q\varphi$ also satisfies (15), i.e., the equation

$$(L-\eta)Q\varphi=0 \quad (16)$$

follows from (15). It is easy to see that if such an operator Q exists it is not unique, since the addition to Q of any operator of the form $g(L-\eta)$ with arbitrary operator g preserves the considered property of the operator Q. Further, it follows from (15) and (16) that for any φ satisfying (15)

$$[Q,L-\eta]\varphi=0.$$

This equation means that in the situation we are considering there exists an operator R such that we have the operator relationship

$$[Q,L-\eta]=R(L-\eta). \quad (17)$$

Equation (17) is invariant under the commutative group G_L of transformations generated by the operator L of the form

$$Q\to Q+g(L-\eta),\quad R\to R+[g,L-\eta],$$

where the operator $g\in G_L$. This important property of (17) will be widely used below.

To make the equations obtained by means of (17) differential with respect to t, we take the operators Q and R in the form

$$Q=\sum_{s=0}^{s_0}Q_{s_0-s}\frac{\partial^s}{\partial t^s},\quad R=\sum_{s=0}^{s_0}R_{s_0-s}\frac{\partial^s}{\partial t^s}, \quad (18)$$

where Q_s, R_s, $s=0,1,\ldots,s_0$, act only on the spatial variables and depend on t only as on a parameter. Substituting Eqs. (18) in (17), we obtain the following relations for Q_0, R_0 and Q_1, R_1:

$$[Q_0,L-\eta]=R_0(L-\eta); \quad (19)$$

$$s_0Q_0\,\partial L/\partial t+[Q_1,L-\eta]=s_0R_0\,\partial L/\partial t+R_1(L-\eta). \quad (20)$$

Further, the relations which follow from (17) for Q_s and R_s for $s>1$ (if $s_0>1$) do not depend on (20), and therefore the use of operators Q and R of the form (18) with $s_0>1$ does not give anything new compared with the first-order operators. Therefore, in all that follows, we shall restrict ourselves to the case $s_0=1$. Finally, if Q_0 is the identity operator and R_0 the null operator, the relation (19) is automatically satisfied, and (20) takes the form ($s_0=1$):

$$\partial L/\partial t+[\mathscr{A},L-\eta]=\mathscr{B}(L-\eta). \quad (21)$$

There is an intimate connection between the relations (19), (20), and (21). For if Q_0 and R_0 satisfy (19) and $\mathscr{A}$ and $\mathscr{B}$ satisfy (21), it is readily seen that the operators

$$Q_1=s_0Q_0\mathscr{A};\quad R_1=s_0(Q_0\mathscr{B}+R_0\mathscr{A}-R_0\mathscr{B})$$

satisfy (20). Therefore, in this case, the equation generated by the relation (20) contains all solutions to the equation generated by the relation (21). Moreover, if Q_0 and R_0 satisfy (19) and Q_1 and R_1 satisfy (20) and there exist Q_0^{-1} and $(Q_0-R_0)^{-1}$, then, as is readily seen, the operators

$$\mathscr{A}=(1/s_0)Q_0^{-1}Q_1;\quad \mathscr{B}=(1/s_0)(Q_0-R_0)^{-1}(R_1-R_0Q_0^{-1}Q_1)$$

satisfy (21). It follows that in the case under consideration all the solutions to the equation generated by the relation (20) also satisfy the equation generated by (21). Thus, in this case (20) and (21) generate the same equations. And although in general there exist pairs of operators Q_0 and R_0 which satisfy the relation (19) but for which at least one of the operators Q_0^{-1} or $(Q_0-R_0)^{-1}$ does not exist, no equation is at present known which could be obtained by means of (19) and (20) and not by means of (21). Therefore, in what follows, we shall consider only equations generated by the relation (21). Further, we restrict ourselves to cases in which the spectral parameter η is a scalar quantity and, therefore, (21) can be written in the form

$$\partial L/\partial t+[\mathscr{A},L]=\mathscr{B}(L-\eta). \quad (22)$$

For a large class of operators L, we shall describe all pairs of operators $\mathscr{A}$ and $\mathscr{B}$ that depend rationally on the spectral parameter η and for which (22) is equivalent to a system of differential equations. All the equations obtained in this manner have several infinite series of conservation laws. However, the use of the inverse scattering method for the actual determination of the solutions to these equations by no means always succeeds because the inverse problem itself has so far been solved in comparatively few cases.

1. DERIVATION OF EQUATIONS GENERATED BY A MATRIX OPERATOR OF FIRST ORDER

We now turn to the systematic study of equations generated by different classes of operators L. We begin with the simplest class of first-order differential operators. Namely, let L be an operator of the form

$$L=\Lambda^{-1}(\partial+u), \quad (23)$$

where ∂ is the operator of differentiation with respect to the spatial variable x, Λ is a diagonal matrix with different diagonal elements $\lambda_r \in C$, $r=1, \ldots, r_0$, and $u=u(x, t)$ is a square matrix of order r_0 with vanishing diagonal elements. In addition, we assume that among the diagonal elements of Λ none vanish. In fact, all the obtained results are invariant under the substitution $\Lambda \to \Lambda + \lambda E$, where E is the unit matrix, and λ is a complex parameter, and therefore this assumption is not important. However, to give meaning to some expressions of the type (23), we shall assume that the assumption is satisfied.

We now describe all solutions of the operator relation (22) that are differential operators of the form

$$\mathcal{A}=\sum_{m=0}^{n} \alpha_m \partial^m, \quad \mathcal{B}=\sum_{m=0}^{n} \beta_m \partial^m \tag{24}$$

whose coefficients depend rationally on the spectral parameter η. However, by virtue of the assumptions we have made any operator $\mathcal{A}$ of the form (24) for $n>0$ can be represented in the form

$$\mathcal{A}=\mathcal{A}'+g(L-\eta), \tag{25}$$

where g is a differential operator of order $n-1$, and $\mathcal{A}'$ is an operator of zeroth order, i.e., simply a matrix. Setting further

$$\mathcal{B}=\mathcal{B}'+[g, L], \tag{26}$$

we find that if the pair of operators $\mathcal{A}$, $\mathcal{B}$ satisfies (22), then the pair of operators $\mathcal{A}'$, $\mathcal{B}'$ obtained by means of (25) and (26) also satisfies (22). It is also easy to see that if $\mathcal{A}$, $\mathcal{B}$ were to depend rationally on η, the $\mathcal{A}'$, $\mathcal{B}'$ obtained in accordance with (25) and (26) would also depend rationally on η. Moreover, if $\mathcal{A}$, $\mathcal{B}$ were polynomials of η, then $\mathcal{A}'$, $\mathcal{B}'$ would also be polynomials. Further, it follows from (22) that $\mathcal{B}'$ also has zeroth order, i.e., is a matrix. Using this fact everywhere unless stated otherwise, we shall assume that the operators $\mathcal{A}$ and $\mathcal{B}$ are simply matrices. In this case, using (23) and (22), we can readily prove the equation

$$\mathcal{B} = \mathcal{A} - \Lambda^{-1}\mathcal{A}\Lambda,$$

and then (22) can be expressed by means of it in the form

$$\partial u/\partial t - \partial \mathcal{A}/\partial x - [u, \mathcal{A}] + \eta[\Lambda, \mathcal{A}] = 0. \tag{27}$$

This equation is the condition for the existence of a common fundamental solution of the two linear systems of equations

$$\partial\varphi/\partial x + u\varphi = \eta\Lambda\varphi; \quad \partial\varphi/\partial t + \mathcal{A}\varphi = 0.$$

We now attempt to satisfy (27) by means of a matrix $\mathcal{A}$ that depends polynomially on η, i.e.,

$$\mathcal{A}=\sum_{m=0}^{n} A_m \eta^{n-m}. \tag{28}$$

Substituting (28) in (27) and equating to zero the coefficients of the different powers of η, we obtain the system of equations

$$\left.\begin{array}{l}[\Lambda, A_0]=0; \\ [\Lambda, A_m]-[u, A_{m-1}]-\partial A_{m-1}/\partial x=0, \quad m=1, \ldots, n;\end{array}\right\} \tag{29}$$

$$\partial u/\partial t - \partial A_n/\partial x - [u, A_n] = 0. \tag{30}$$

Thus, if the matrix A_0 is diagonal, and the matrices $A_1, \ldots, A_n$ for $n>0$ are such that the condition (29) is satisfied, the evolution equation that follows from (27) has the form (30). The structure of the obtained equation is entirely determined by the matrix A_n. It is therefore necessary to consider the solvability and nature of the solution of Eqs. (29).

Equations (29) represent a recursion relation that connects the elements of the matrix A_m to the elements of the matrix A_{m-1} for $m>0$. It is therefore natural to attempt to determine successively the matrices $A_1, A_2, \ldots, A_n$. Indeed, it follows from (29) that the elements $A_{m,\mu\nu}$ of the matrix A_m off the main diagonal can be readily determined by means of the equation

$$A_{m,\mu\nu}=\frac{1}{\lambda_\mu-\lambda_\nu}\left\{\frac{\partial A_{m-1,\mu\nu}}{\partial x}+\sum_{r=1}^{r_0}\left(u_{\mu r}A_{m-1,r\nu}-A_{m-1,\mu r}u_{r\nu}\right)\right\}. \tag{31}$$

However, to find the diagonal elements of the matrix A_m we obtain the differential equation

$$\frac{\partial A_{m,\mu\mu}}{\partial x}=\sum_{r=1}^{r_0}\left(A_{m,\mu r}u_{r\mu}-u_{\mu r}A_{m,r\mu}\right). \tag{32}$$

And although the right-hand side of this equation obviously does not contain the element $A_{m,\mu\mu}$, its determination nevertheless requires one quadrature. However, as was shown for the first time in Ref. 21, if the diagonal elements of the matrix A_0 are taken to be independent of x, the right-hand side of (32) for any $m>0$ can be represented in the form

$$\sum_{r=1}^{r_0}\left(A_{m,\mu r}u_{r\mu}-u_{\mu r}A_{m,r\mu}\right)=\frac{d}{dx}p_{m,\mu},$$

where $p_{m,\mu}$ is a polynomial in the elements of the matrix u and its derivatives with respect to x up to order $m-1$. It follows that if

$$A_0=\operatorname{diag}(a_1, \ldots, a_{r_0}),$$

where the elements a_r do not depend on x, then the elements of the matrices A_m for $m>0$ are polynomials in the elements of the matrix u and its derivatives with respect to x up to order $m-1$, and the coefficients of these polynomials do not depend on x and t but depend only on the diagonal elements of the matrices A_0 and Λ, or, more precisely on their pairwise differences $\lambda_\mu-\lambda_\nu$ and $a_\mu-a_\nu$. Further, it follows from Eqs. (31) and (32) that (29) determines the matrix A_m up to a constant diagonal matrix. At the same time, of course, the matrix A_m also depends on how we have used the arbitrariness in the choice of the constants of integration in the determination of the diagonal elements of the matrices $A_{m'}$ with $m'<m$. Among all these solutions, it is necessary to distinguish one special solution, which is obtained under the condition that at all the stages the constants of integration are zero. This solution is characterized by the property that the elements of the matrix A_m in this case are either zero or quasihomogeneous polynomials of rank m in the elements of the matrix u and its derivatives with respect to x up to order $m-1$.[2)] By means of this solution, any other solution to Eqs. (29) can be obtained in accordance with

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[2)]The polynomial Q of the elements of the matrix u and its derivatives with respect to x is said to be quasihomogeneous of rank m if when λ^{k+1} is substituted in Q in place of the elements of the matrix $u^{(k)}=\partial^k u/\partial x^k$ each monomial Q_α takes the form $c_\alpha\lambda^m$, where c_α is a nonvanishing constant.

the equation

$$A'_m = A_m + \sum_{\mu=1}^{m}\sum_{r=1}^{r_0} c_r^{(\mu)} \frac{\partial A_{m-\mu}}{\partial a_r}, \quad m>0,$$

where $c_r^{(\mu)}$ do not depend on x.

It should be noted that the definition of the matrices A_m does not depend on the degree n of the polynomial (28). Therefore, determining by means of (29) the matrices A_m for any $m>0$, we obtain the possibility of taking polynomials of the form (28) with arbitrary $n \geqslant 0$. The equations of the form (30) thus obtained can be written by virtue of (29) in the form

$$u_t = [\Lambda, A_{n+1}], \tag{33}$$

which is in complete agreement with the assumption made earlier concerning the vanishing of the diagonal elements of the matrix u. Since the elements of the matrix A_{n+1} are polynomials in the elements of the matrix u and its derivatives with respect to x up to order n, Eq. (33) is a system of $r_0(r_0-1)\partial$ differential equations of first order in t and n-th order in x. Because of the basic importance of this assertion, we shall give its proof.

2. SOLVABILITY OF EQUATIONS (29)

Consider the equation

$$\partial A/\partial x + [u, A] - \eta[\Lambda, A] = 0, \tag{34}$$

which is obtained from (27) by omitting the term $\partial u/\partial t$. The general solution to Eq. (34) can be written in the form

$$A = \varphi A_0 \varphi^{-1}, \tag{35}$$

where $\varphi = \varphi(x, \eta)$ is a fundamental solution of the system

$$\partial\varphi/\partial x + u\varphi = \eta\Lambda\varphi, \tag{36}$$

and the matrix A_0 does not depend on x. We now assume that the matrix $u = u(x)$ in the system (36) satisfies for any $x \in (-\infty, \infty)$ the condition

$$\int_{-\infty}^{x} \| u(z) \| \, dz < \infty, \tag{37}$$

where $\|u\| = \max_\alpha \sum_\beta |u_{\alpha\beta}|$. We assume further that the diagonal elements of the matrix Λ are purely imaginary. Under these conditions, the system (36) for any real η has a fundamental matrix of solutions which satisfies the condition

$$\lim_{x\to-\infty} \varphi(x, \eta) \exp(-\eta\Lambda x) = \lim_{x\to-\infty} \exp(-\eta\Lambda x)\varphi(x, \eta) = E. \tag{38}$$

Indeed, we set

$$\varphi = \exp(\eta\Lambda x)\Phi. \tag{39}$$

Substituting this expression in (36), we obtain a differential equation for Φ:

$$\Phi_x + \exp(-\eta\Lambda x) u(x) \exp(\eta\Lambda x)\Phi = 0.$$

From the last equation there follows the integral equation

$$\Phi = \Phi_0 - \int_{-\infty}^{x} \exp(-\eta\Lambda z) u(z) \exp(\eta\Lambda z)\Phi(z, \eta)\, dz, \tag{40}$$

which can be solved by successive approximation. Indeed, we set $\Phi_0 = E$, and for $m>0$ we determine Φ_m by means of the recursion relation

$$\Phi_m = E - \int_{-\infty}^{x} \exp(-\eta\Lambda z) u(z) \exp(\eta\Lambda z)\Phi_{m-1}(z, \eta)\, dz.$$

We now show that the series

$$\Phi_0 + \sum_{m=1}^{\infty} (\Phi_m - \Phi_{m-1}) \tag{41}$$

converges uniformly with respect to x on any half-interval $(-\infty, c]$, where c is arbitrary. Indeed, the difference

$$\Psi_m = \Phi_m - \Phi_{m-1}, \quad m>0, \quad \Psi_0 = E,$$

for any $m>0$ satisfies the recursion relation

$$\Psi_m = -\int_{-\infty}^{x} \exp(-\eta\Lambda z) u(z) \exp(\eta\Lambda z)\Psi_{m-1}(z, \eta)\, dz.$$

From this we find that for any $m>0$

$$\| \Psi_m(x, \eta) \| \leqslant \int_{-\infty}^{x} \| u(z) \| \, \| \Psi_{m-1}(z, \eta) \| \, dz.$$

Using this inequality and induction we can readily prove the inequality

$$\| \Psi_m(x, \eta) \| \leqslant \frac{1}{m!} \left(\int_{-\infty}^{x} \| u(z) \| \, dz \right)^m$$

from which our assertion about the convergence of the series (41) follows directly. It then follows that the matrix Φ defined by means of (41) is a solution to the integral equation (40), with $\Phi \to E$ as $x \to -\infty$, and on the basis of (39) we see that (38) is correct.

The fundamental matrix of solutions of the system (36) obtained above has one further remarkable property, namely,

$$\lim_{\eta\to\pm\infty} \varphi(x, \eta) \exp(-\eta\Lambda x) = \lim_{\eta\to\pm\infty} \exp(-\eta\Lambda x)\varphi(x, \eta) = E, \tag{42}$$

the transition to the limit being uniform with respect to x on any half-interval $(-\infty, c]$, where c is arbitrary. Indeed, setting in (40)

$$\Phi = \Phi_0 + \Psi, \quad \Phi_0 = E, \tag{43}$$

we obtain an equation for Ψ:

$$\Psi = f(x, \eta) - \int_{-\infty}^{x} \exp(-\eta\Lambda z) u(z) \exp(\eta\Lambda z)\Psi(z, \eta)\, dz, \tag{44}$$

where

$$f = -\int_{-\infty}^{x} \exp(-\eta\Lambda z) u(z) \exp(\eta\Lambda z)\, dz.$$

Remembering our earlier assumption about the vanishing of the diagonal elements of the matrix u, we have

$$f_{\mu\nu} = -\int_{-\infty}^{x} u_{\mu\nu}(z) \exp[\eta(\lambda_\nu - \lambda_\mu) z]\, dz, \quad \mu \neq \nu, \quad f_{\mu\mu} = 0.$$

In accordance with the Riemann–Lebesque theorem and by virtue of the inequality (37), $|f_{\mu\nu}| \to 0$ as $\eta \to \pm\infty$, and, as is readily seen, the passage to the limit is uniform with respect to x on any half-interval $(-\infty, c]$, where c is arbitrary. It follows that the solution of Eq. (44) satisfies the condition $\|\Psi\| \to 0$ as $\eta \to \pm\infty$ uniformly with respect to x on any half-interval $(-\infty, c]$. Hence, it follows from (39) and (43) that Eq. (42) holds.

Now suppose that the matrix A_0 in (35) is diagonal with different diagonal elements a_r, $r=1, \ldots, r_0$. Suppose that a_r does not depend on x or the parameter η. By (38) and (42),

$$\lim_{x\to-\infty} A = \lim_{\eta\to\pm 0} A = A_0. \tag{45}$$

Now suppose that the elements of the matrix u have continuous derivatives with respect to x of any order that for all $x \in (-\infty, \infty)$ satisfy the conditions

$$\int_{-\infty}^{x} \| u^{(k)}(z) \| \, dz < \infty, \tag{46}$$

where $u^{(k)} = \partial^k u/\partial x^k$, $k > 0$. Under these conditions, the solution (35) has an asymptotic expansion as $\eta \to \pm\infty$ of the form

$$A \sim \sum_{m=0}^{\infty} A_m \eta^{-m}, \tag{47}$$

where the matrix A_0 is by virtue of (45) the same as in Eq. (35), and A_m for $m > 0$ satisfy (29) and a certain additional condition which will be given shortly.

We begin with the construction of the asymptotic series (47). We determine the matrices A_m for $m > 0$ successively in accordance with Eq. (29), determining the elements off the principal diagonal by means of Eq. (31) and the elements on it by means of the equation

$$A_{m,\mu\mu} = \sum_{r=1}^{r_0} \int_{-\infty}^{x} (A_{m,\mu r}(z)\, u_{r\mu}(z) - u_{\mu r}(z)\, A_{m,r\mu}(z))\, dz. \tag{48}$$

Using (37) and (46), we can readily show that the matrices A_m defined in this manner for all $m > 0$, $k > 0$ and any $x \in (-\infty, \infty)$ satisfy the inequalities

$$\int_{-\infty}^{x} \| [\Lambda, A_m(z)] \| \, dz < \infty; \quad \int_{-\infty}^{x} \| A_m^{(k)}(z) \| \, dz < \infty, \tag{49}$$

where $A_m^{(k)} = \partial^k A_m/\partial x^k$, $k > 0$. It follows from this in particular that Eq. (48) is correct.

We show that the series (47) defined thus is an asymptotic expansion of the solution (35); for suppose

$$K_n = A - \sum_{m=0}^{n} A_m \eta^{-m}.$$

Then by (29) and (34)

$$\partial K_n/\partial x + [u, K_n] - \eta [\Lambda, K_n] = -[\Lambda, A_{n+1}]\, \eta^{-n}. \tag{50}$$

Setting now

$$K_n = \varphi \mathscr{K}_n \varphi^{-1}, \tag{51}$$

where $\varphi = \varphi(x, \eta)$ is a fundamental solution of the system (36) satisfying the conditions (38) and (42), and using (36) and (50), we obtain an equation for $\mathscr{K}_n$:

$$\partial \mathscr{K}_n/\partial x = -\varphi^{-1} [\Lambda, A_{n+1}]\, \varphi \eta^{-n}.$$

Using (36), we can transform this equation to

$$\partial \mathscr{K}_n/\partial x = \{\partial(\varphi^{-1} A_{n+1} \varphi)/\partial x - \varphi^{-1}(\partial A_{n+1}/\partial x + [u, A_{n+1}])\, \varphi\}\, \eta^{-(n+1)},$$

i.e., in accordance with (29) we have

$$\partial \mathscr{K}_n/\partial x = \{\partial(\varphi^{-1} A_{n+1} \varphi)/\partial x - \varphi^{-1} [\Lambda, A_{n+2}]\, \varphi\}\, \eta^{-(n+1)}.$$

Hence, using (51), we obtain

$$K_n = \{A_{n+1} - \varphi(x, \eta) \int_{-\infty}^{x} \varphi^{-1}(z, \eta)\, [\Lambda, A_{n+2}(z)]\, \varphi(z, \eta)\, dz\, \varphi^{-1}(x, \eta)\}\, \eta^{-(n+1)}.$$

It follows from this equation in accordance with (42) and (49) that $\eta^n K_n \to 0$ as $\eta \to \pm\infty$ uniformly with respect to x on any half-interval $(-\infty, c]$. Therefore, the series (47) really is an asymptotic (as $\eta \to \pm\infty$) expansion of the solution (35).

In the polynomial $p(z) = \det|A_0 - zE|$ we now substitute the solution (35) in place of z. We obtain the matrix equation $p(A) = 0$. In this equation, we substitute the series (47) in place of A. Grouping the terms with the same powers of η and equating to zero the obtained sum, we find equations of the form

$$\sum_{r=1}^{r_0} p_r \sum_{k=0}^{r-1} A_0^{r-k-1} A_m A_0^k = P_m(A_0, A_1, \ldots, A_{m-1}), \tag{52}$$

where p_r is the coefficient of z^r in the polynomial $p(z)$, and P_m is a polynomial in the matrices $A_0, A_1 \ldots, A_{m-1}$, with $P_1 \equiv 0$. It is easy to see that on the left-hand side of Eq. (52) we have a diagonal matrix with elements $p'(a_\mu) A_{m,\mu\mu}$ on the principal diagonal. It follows that $A_{1,\mu\mu} = 0$ for $\mu = 1, \ldots, r_0$, and for $m > 1$ the diagonal elements of the matrices A_m can be expressed polynomially in terms of the elements of the matrices $A_1, \ldots, A_{m-1}$. Using these facts, we can readily show by induction that the elements of the matrices A_m are polynomials in the elements of the matrix u and its derivatives with respect to x up to order $m-1$. Further, it is easy to show that the elements of the matrix A_m either vanish or are quasihomogeneous polynomials of rank m in the elements of the matrix u and its derivatives with respect to x of corresponding order.

It follows from (31) and (48) that the elements of the matrices A_m for $m > 0$ are homogeneous polynomials of degree m in $\lambda_{\mu\nu} = (\lambda_\mu - \lambda_\nu)^{-1}$ with $\mu \neq \nu$. It is clear from this that Eq. (29) has a solution A_m which depends polynomially on the elements of the matrix u and its derivatives with respect to x not only for purely imaginary λ_r, as was assumed earlier, but also for all complex values that at least satisfy the condition $\lambda_\mu \neq \lambda_\nu$ for $\mu \neq \nu$. Finally, it must be noted that the conditions (37) and (46) are also not necessary for the solvability of Eq. (29). To see this, we multiply the elements of the matrix u by an infinitely differentiable function $w(x)$ which vanishes for $x \leq x_0$ and is equal to unity for $x \geq x_1 > x_0$, and we then reduce the case of an arbitrary infinitely differentiable matrix u to the case considered here, since for $x \geq x_1$ Eqs. (29) with the matrix u and the matrix wu are identical.

We now drop the requirement that the diagonal elements of the matrix u vanish. It is easy to see that in the investigation of the solvability of (29) this requirement was used just once, namely, in the proof of Eq. (42). If we drop this requirement, Eq. (42) is violated, but we do have the equation

$$\lim_{\eta\to\pm\infty} \varphi(x, \eta) \exp(-\eta \Lambda x) = \lim_{\eta\to\pm\infty} \exp(-\eta \Lambda x)\, \varphi(x, \eta) = \sigma(x), \tag{53}$$

where

$$\sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_{r_0}), \quad \sigma_r = \exp\left(-\int_{-\infty}^{x} u_{rr}(z)\, dz\right), \quad r = 1, \ldots, r_0.$$

Indeed, if $\varphi = \varphi(x, \eta)$ is a solution of the system (36) satisfying the condition (38), then $\psi=\sigma^{-1}\varphi$ is a solution to the following system satisfying the condition (38):

$$\partial\psi/\partial x+v\psi=\eta\Lambda\psi, \tag{54}$$

and the diagonal elements of the matrix v vanish. Therefore, Eq. (42) holds for the solution of the system (54). Hence, for the solution of the system (36), Eq. (53) holds. Further, it is easy to see that the replacement of (42) by (53) does not lead to any difficulties in either the proof of (45) or the proof of the asymptotic properties of the series (47). Thus, if the conditions (37) and (46) are satisfied, and the diagonal elements of the matrix Λ are purely imaginary, then the solution (35) of Eq. (34) has in the limits $\eta\to\pm\infty$ an asymptotic expansion of the form (47) irrespective of whether or not the diagonal elements of the matrix u vanish. We shall make essential use of this remark in the following section.

3. CONSERVATION LAWS

Equation (33) has a number of remarkable properties. First of all, it has r_0 infinite series of conservation laws, i. e., there exist r_0 infinite sequences of quantities T_{mr}, $m>0$, $r=1,\ldots,r_0$, which are polynomials in the elements of the matrix u and its derivatives with respect to x of corresponding order such that when any solution of Eq. (33) is substituted in T_{mr}

$$\partial T_{mr}/\partial t=\partial X_{mnr}/\partial x, \tag{55}$$

where X_{mnr} are also polynomials in the elements of the matrix u and its derivatives with respect to x of sufficiently high order. At the same time, the polynomials T_{mr} do not depend explicitly on the number n of Eq. (33), while the polynomials X_{mnr} do depend explicitly on n. The quantities T_{mr} can be determined in terms of the trace of the matrix ΛA_{m+1} by means of the equation

$$T_{mr}=\partial\,\mathrm{Sp}\,(\Lambda A_{m+1})/\partial a_r. \tag{56}$$

To prove (55), we require the important equation

$$\delta T_{mr}/\delta\tilde{u}=m\,\partial A_m/\partial a_r, \tag{57}$$

where the tilde over the matrix u denotes the transpose, and the variational derivative $\delta T_{mr}/\delta\tilde{u}$ is defined by the equation

$$\frac{\delta T_{mr}}{\delta\tilde{u}}=\sum_{k=0}^{\infty}(-1)^k\frac{d^k}{dx^k}\frac{\partial T_{mr}}{\partial\tilde{u}^{(k)}},\quad u^{(k)}=\frac{\partial^k u}{\partial x^k}. \tag{58}$$

Equation (57) was proved for the first time in Ref. 21. We shall prove (57) first for the case when the elements of the matrix u vanish identically for $x\leqslant x_0$. We shall assume that the elements of the matrix u have continuous derivatives with respect to x of any order and, therefore, they all vanish identically for $x\leqslant x_0$. In addition, we shall assume for the time being that the diagonal elements of the matrix Λ are purely imaginary.

We consider the functional

$$H=\int_{x_0}^{x_0'}\mathrm{Sp}\,(\Lambda A)\,dx, \tag{59}$$

where A is the solution (35) of Eq. (34), and $x_0'>x_0$ is arbitrary. We now find the variational derivative $\delta H/\delta\tilde{u}$ for variation of the elements of the matrix u at the point $x\in(x_0,x_0')$. By Eq. (35),

$$\delta H=\int_{x_0}^{x_0'}\mathrm{Sp}\,(\Lambda\delta\varphi A_0\varphi^{-1}+\Lambda\varphi A_0\delta\varphi^{-1})\,dx, \tag{60}$$

where $\delta\varphi$ is a variation of a solution $\varphi=\varphi(x,\eta)$ to Eq. (36) that satisfies the condition $\varphi=\exp(\eta\Lambda x)$ for $x\leqslant x_0$, and the variation $\delta\varphi^{-1}$ of the matrix φ^{-1} is obviously equal to

$$\delta\varphi^{-1}=-\varphi^{-1}(\delta\varphi)\,\varphi^{-1}. \tag{61}$$

The variation $\delta\varphi$ satisfies the equation

$$\partial(\delta\varphi)/\partial x+u\delta\varphi-\eta\Lambda\delta\varphi=-(\delta u)\,\varphi$$

and the condition $\delta\varphi=0$ for $x\leqslant x_0$. Setting

$$\delta\varphi=\varphi\Phi, \tag{62}$$

we readily find

$$\Phi_x=-\varphi^{-1}(\delta u)\,\varphi,$$

i. e., for $x>x_0$ we have

$$\Phi=-\int_{x_0}^{x}\varphi^{-1}(z,\eta)\,\delta u(z)\,\varphi(z,\eta)\,dz. \tag{63}$$

Substituting (61) and (62) in (60), we obtain

$$\delta H=\int_{x_0}^{x_0'}\mathrm{Sp}\,(\Lambda\varphi\Phi A_0\varphi^{-1}-\Lambda\varphi A_0\Phi\varphi^{-1})\,dx. \tag{64}$$

Using (63), we integrate by parts in (64), obtaining

$$\delta H=\int_{x_0}^{x_0'}\mathrm{Sp}\left\{\varphi(x,\eta)\int_{x}^{x_0'}[\varphi^{-1}(z,\eta)\,\Lambda\varphi(z,\eta),A_0]\,dz\times\varphi^{-1}(x,\eta)\,\delta u(x)\right\}dx.$$

We then obtain directly

$$\frac{\delta H}{\delta\tilde{u}}=\varphi(x,\eta)\int_{x}^{x_0'}[\varphi^{-1}(z,\eta)\,\Lambda\varphi(z,\eta),A_0]\,dz\,\varphi^{-1}(x,\eta). \tag{65}$$

We now find the matrix $\partial A/\partial\eta$. By virtue of (34), we have an equation for $B=\partial A/\partial\eta$:

$$\partial B/\partial x+[u,B]-\eta[\Lambda,B]=[\Lambda,A].$$

Suppose $B=\varphi\mathscr{B}\varphi^{-1}$. In accordance with (35) and (36), there now follows an equation for $\mathscr{B}$:

$$\partial\mathscr{B}/\partial x=[\varphi^{-1}\Lambda\varphi,A_0].$$

Integrating this equation and using the equation $\partial A/\partial\eta=0$ for $x\leqslant x_0$, we find that for $x>x_0$

$$\frac{\partial A}{\partial\eta}=\varphi(x,\eta)\int_{x_0}^{x}[\varphi^{-1}(z,\eta)\,\Lambda\varphi(z,\eta),A_0]\,dz\,\varphi^{-1}(x,\eta). \tag{66}$$

Note that in deriving (65) and (66) we have not used the assumption that the diagonal elements of the matrix u vanish. Moreover, to obtain the diagonal elements of the matrix $\delta H/\delta\tilde{u}$, we need to vary the diagonal elements of the matrix u, i. e., to consider the functional (59) for the case when the diagonal elements of the matrix u are nonzero.

From (65) and (66) there follows the equation

$$\delta H/\delta\tilde{u}+\partial A/\partial\eta=\varphi(x,\eta)\,J(\eta)\,\varphi^{-1}(x,\eta), \tag{67}$$

where

$$J=\int_{x_0}^{x_0'}[\varphi^{-1}(z,\eta)\,\Lambda\varphi(z,\eta),A_0]\,dz.$$

The quantity J determined in this manner tends to zero as $\eta\to\pm\infty$ faster than any negative power of η, i.e., for any $n>0$,

$$\lim_{\eta\to\pm\infty}\eta^n J(\eta)=0. \tag{68}$$

Indeed, by virtue of Eq. (67) the quantity $J(\eta)$ is equal to the value of the matrix $\exp(-\eta\Lambda x)(\delta H/\delta\tilde{u})\exp(\eta\Lambda x)$ at the point $x=x_0$. In accordance with (59), the matrix $\delta H/\delta\tilde{u}$ has in the limits $\eta\to\pm\infty$ an asymptotic expansion of the form

$$\frac{\delta H}{\delta\tilde{u}}\sim\sum_{m=0}^{\infty}\frac{\delta H_m}{\delta\tilde{u}}\eta^{-m}, \tag{69}$$

where

$$H_m=\int_{x_0}^{x_0'}\operatorname{Sp}(\Lambda A_m)\,dx. \tag{70}$$

Since A_0 does not depend on u, nor does H_0 depend on u, and therefore, $\delta H_0/\delta\tilde{u}=0$. Further, $H_1=0$, since the diagonal elements of A_1 vanish, as we have already noted. Finally, for $m>1$ we have $\delta H_m/\delta\tilde{u}=0$ at the point $x=x_0$; for since the elements of the matrix A_m are quasihomogeneous polynomials of rank $m>1$, the elements of the matrix $\delta H_m/\delta\tilde{u}$ are quasihomogeneous polynomials of rank $m-1>0$, i.e., do not contain free terms. Since the elements of the matrix u and all its derivatives with respect to x vanish at the point $x=x_0$, it follows that $\delta H_m/\delta\tilde{u}=0$ at the point $x=x_0$ for $m>1$. Thus, we have proved Eq. (68). By virtue of this equation and on the basis of (67) we find that the asymptotic expansions as $\eta\to\pm\infty$ of $\delta H/\delta\tilde{u}$ and $\partial A/\partial\eta$ differ only in sign. Further, it is easy to show that the asymptotic expansion as $\eta\to\pm\infty$ of $\partial A/\partial\eta$ has the form

$$\frac{\partial A}{\partial\eta}\sim-\sum_{m=1}^{\infty}mA_m\eta^{-(m+1)}. \tag{71}$$

It follows that the asymptotic series (69) and (71) differ only in sign. Hence, for any $m>0$,

$$\delta H_{m+1}/\delta\tilde{u}=mA_m. \tag{72}$$

From the last equation and (56), (58), and (70) it follows that (57) holds for the case $u\equiv0$ for $x\leqslant x_0$. However, multiplying the elements of the matrix u by an infinitely differentiable function $w(x)$ which vanishes for $x\leqslant x_0$ and is equal to unity for $x\geqslant x_1>x_0$, we reduce the case of an arbitrary infinitely differentiable matrix u to the case considered here. Further, taking into account the nature of the dependence of the right- and left-hand sides of Eq. (57) on the diagonal elements of the matrix Λ, we can readily show that this equation holds for all complex λ_r satisfying the condition $\lambda_\mu\neq\lambda_\nu$ for $\mu\neq\nu$.

We now show how the quantity T_{mr} defined by means of Eq. (56) varies with the time if any solution of Eq. (33) is substituted in it. We have

$$\frac{\partial}{\partial t}T_{mr}=\operatorname{Sp}\left(\sum_{s=0}^{\infty}\frac{\partial T_{mr}}{\partial\tilde{u}^{(s)}}\frac{\partial u^{(s)}}{\partial t}\right)=\operatorname{Sp}\left(\sum_{s=0}^{\infty}\frac{\partial T_{mr}}{\partial\tilde{u}^{(s)}}\left[\Lambda,\frac{\partial^s A_{n+1}}{\partial x^s}\right]\right).$$

We set

$$Z_{mnr}=\operatorname{Sp}\left(\sum_{s=1}^{\infty}\sum_{\sigma=0}^{s-1}(-1)^\sigma\left[\Lambda,\frac{\partial^{s-\sigma-1}A_{n+1}}{\partial x^{s-\sigma-1}}\right]\frac{\partial^\sigma}{\partial x^\sigma}\frac{\partial T_{mr}}{\partial\tilde{u}^{(s)}}\right).$$

Then

$$\frac{\partial}{\partial t}T_{mr}=\operatorname{Sp}\left(\frac{\delta T_{mr}}{\delta\tilde{u}}[\Lambda,A_{n+1}]\right)+\frac{\partial}{\partial x}Z_{mnr}.$$

Then by virtue of (57)

$$\frac{\partial}{\partial t}T_{mr}=m\Theta_{mnr}+\frac{\partial}{\partial x}Z_{mnr},$$

where

$$\Theta_{mnr}=\operatorname{Sp}\left\{\frac{\partial A_m}{\partial a_r}[\Lambda,A_{n+1}]\right\}.$$

We now multiply the equation

$$\left[\Lambda,\frac{\partial A_{m+1}}{\partial a_r}\right]-\left[u,\frac{\partial A_m}{\partial a_r}\right]-\frac{\partial}{\partial x}\frac{\partial A_m}{\partial a_r}=0$$

from the right by A_n and add it to the equation

$$[\Lambda,A_{n+1}]-[u,A_n]-\frac{\partial A_n}{\partial x}=0$$

multiplied from the left by $\partial A_m/\partial a_r$. As a result, we obtain the equation

$$\frac{\partial A_m}{\partial a_r}[\Lambda,A_{n+1}]-\frac{\partial A_{m+1}}{\partial a_r}[\Lambda,A_n]+\left[\Lambda,\frac{\partial A_{m+1}}{\partial a_r}A_n\right]=\left[u,\frac{\partial A_m}{\partial a_r}A_n\right]+\frac{\partial}{\partial x}\left(\frac{\partial A_m}{\partial a_r}A_n\right),$$

from which there follows

$$\Theta_{mnr}=\Theta_{m+1,\,n-1,\,r}+\frac{\partial}{\partial x}\operatorname{Sp}\left(\frac{\partial A_m}{\partial a_r}A_n\right).$$

Solving this recursion relation, we readily obtain

$$\Theta_{mnr}=\frac{\partial}{\partial x}\operatorname{Sp}\left(\sum_{h=0}^{n}\frac{\partial A_{m+h}}{\partial a_r}A_{n-h}\right).$$

We now set

$$X_{mnr}=Y_{mnr}+Z_{mnr},$$

where

$$Y_{mnr}=m\operatorname{Sp}\left(\sum_{h=0}^{n}\frac{\partial A_{m+h}}{\partial a_r}A_{n-h}\right).$$

It is readily seen that the quantities X_{mnr} defined in this manner satisfy Eq. (55).

Note that by virtue of Eq. (72), Eq. (33) can be written in the Hamiltonian form

$$u_t=[\Lambda,\delta\hat{H}_n/\delta\tilde{u}],$$

where

$$\hat{H}_n=\frac{1}{n+1}\int_{x_0}^{x_0'}\operatorname{Sp}(\Lambda,A_{n+2})\,dx.$$

Further, it is readily seen that by virtue of (33) the following Heisenberg equation holds:

$$\partial L/\partial t+[P,L]=0,$$

where the operator P has the form

$$P=\sum_{m=0}^{n}A_mL^{n-m}. \tag{73}$$

The analogy between Eqs. (28) and (73) also holds in a more general case. It is based on the equation

$$A=P+g(L-\eta), \tag{74}$$

where

$$g=-\sum_{m=0}^{n-1}\sum_{h=0}^{n-m-1}A_mL^{n-m-h-1}\eta^h.$$

Indeed, substituting (74) in (22), we obtain

$$\partial L/\partial t+[P, L]=Q(L-\eta),$$

where the operator $Q=\mathscr{B}-[g, L]$, depends polynomially on the parameter η. Since the left-hand side of this equation does not depend on η, we have $Q=0$.

4. THE CASE OF RATIONAL DEPENDENCE OF THE OPERATOR $\mathscr{A}$ ON THE SPECTRAL PARAMETER

We now consider the case when the operator $\mathscr{A}$ depends rationally on the spectral parameter η. Bearing in mind what we have said above, we shall assume that the operator $\mathscr{A}$ has zeroth order, i.e., is simply a matrix. On the basis of this, we set

$$\mathscr{A}=\sum_{m=0}^{n} A_m\eta^{n-m}+\sum_{\mu=1}^{\mu_0}\sum_{p=1}^{p_\mu}\frac{\alpha_{\mu p}}{(\eta-\eta_\mu)^p}, \tag{75}$$

where the matrices A_m and $\alpha_{\mu p}$ do not depend on η. Then by virtue of (27), $[\Lambda, A_0]=0$; the matrices A_m for $m=1, \ldots, n$, if $n>0$, satisfy (29); the matrices $\alpha_{\mu p}$ satisfy the conditions

$$\partial\alpha_{\mu p}/\partial x+[u, \alpha_{\mu p}]-\eta_\mu[\Lambda, \alpha_{\mu p}]-[\Lambda, \alpha_{\mu, p+1}]=0, \tag{76}$$

where $\alpha_{\mu p}=0$ for $p>p_\mu$, and, finally, the evolution equation that follows from (27) has the form

$$\frac{\partial u}{\partial t}-\frac{\partial A_n}{\partial x}-[u, A_n]+\sum_{\mu=1}^{\mu_0}[\Lambda, \alpha_{\mu 1}]=0. \tag{77}$$

In the general case, Eq. (76) for the matrices $\alpha_{\mu p}$ does not have a solution whose elements are polynomials of the elements of the matrix u and its derivatives with respect to x. Therefore, in the general case Eq. (76) must be integrated simultaneously with Eq. (77). To obtain a unique solution in this case we must specify not only the value of the matrix u at $t=0$ but also the values of the matrices $\alpha_{\mu p}$ at some $x=x_0$. However, in some cases the system of equations (76) and (77) can be reduced to a single evolution equation. We shall consider this in more detail somewhat later, and we shall now show that the system of equations (76) and (77) has r_0 infinite series of conservation laws of the form

$$\partial T_{mr}/\partial t=\partial\hat{X}_{mnr}/\partial x, \tag{78}$$

where T_{mr} is defined by means of Eq. (56), $\hat{X}_{mnr}=X_{mnr}-\check{X}_{mr}$, and X_{mnr} are determined by the manner described in Sec. 3 and do not depend explicitly on the matrices $\alpha_{\mu p}$, while the quantities $\check{X}_{mr}$ are polynomials in the elements of the matrix u and its derivatives with respect to x and depend linearly on the elements of the matrices $\alpha_{\mu p}$ and their derivatives with respect to x.

Indeed, suppose

$$\sum_{\mu=1}^{\mu_0}\sum_{p=1}^{p_\mu}\frac{\alpha_{\mu p}}{(\eta-\eta_\mu)^p}=\sum_{k=1}^{\infty}\alpha_k\eta^{-k},$$

where

$$\alpha_k=\sum_{\mu=1}^{\mu_0}\sum_{p=1}^{p_{\mu k}}\frac{(k-1)!}{(k-p)!\,(p-1)!}\eta_\mu^{k-p}\alpha_{\mu p}, \quad p_{\mu k}=\min(p_\mu, k).$$

Then from (76) there follows an equation for α_k:

$$\partial\alpha_k/\partial x+[u, \alpha_k]-[\Lambda, \alpha_{k+1}]=0, \quad k>0, \tag{79}$$

and the evolution equation (77) has the form

$$\partial u/\partial t-\partial A_n/\partial x-[u, A_n]+[\Lambda, \alpha_1]=0.$$

Using (29), we can write this equation in the form

$$u_t=[\Lambda, A_{n+1}]-[\Lambda, \alpha_1]. \tag{80}$$

Let us now consider how the quantity T_{mr} defined by means of (56) varies with the time if we substitute in it any solution of Eq. (80). In accordance with the results of Sec. 3, we have

$$\frac{\partial}{\partial t}T_{mr}=\frac{\partial}{\partial x}X_{mnr}-\mathrm{Sp}\left(\sum_{s=0}^{\infty}\frac{\partial T_{mr}}{\partial\widetilde{u}^{(s)}}\left[\Lambda, \frac{\partial^s\alpha_1}{\partial x^s}\right]\right).$$

We now set

$$\check{Z}_{mr}=\mathrm{Sp}\left(\sum_{s=1}^{\infty}\sum_{\sigma=0}^{s-1}(-1)^\sigma\left[\Lambda, \frac{\partial^{s-\sigma-1}\alpha_1}{\partial x^{s-\sigma-1}}\right]\frac{\partial^\sigma}{\partial x^\sigma}\frac{\partial T_{mr}}{\partial\widetilde{u}^{(s)}}\right).$$

Then

$$\frac{\partial}{\partial t}T_{mr}=\frac{\partial}{\partial x}X_{mnr}-\mathrm{Sp}\left(\frac{\delta T_{mr}}{\delta\widetilde{u}}[\Lambda, \alpha_1]\right)-\frac{\partial}{\partial x}\check{Z}_{mr}. \tag{81}$$

Suppose

$$\check{\Theta}_{mrk}=\mathrm{Sp}\left(\frac{\partial A_m}{\partial a_r}[\Lambda, \alpha_k]\right)-\mathrm{Sp}\left(\alpha_k\left[\Lambda, \frac{\partial A_m}{\partial a_r}\right)\right]. \tag{82}$$

We multiply the equation

$$\left[\Lambda, \frac{\partial A_m}{\partial a_r}\right]-\left[u, \frac{\partial A_{m-1}}{\partial a_r}\right]-\frac{\partial}{\partial x}\frac{\partial A_{m-1}}{\partial a_r}=0$$

from the left by α_k and subtract from it (79) multiplied by $\partial A_{m-1}/\partial a_r$. We obtain

$$\alpha_k\left[\Lambda, \frac{\partial A_m}{\partial a_r}\right]-\alpha_{k+1}\left[\Lambda, \frac{\partial A_{m-1}}{\partial a_r}\right]+\left[\Lambda, \alpha_{k+1}\frac{\partial A_{m-1}}{\partial a_r}\right]$$
$$=\left[u, \alpha_k\frac{\partial A_{m-1}}{\partial a_r}\right]+\frac{\partial}{\partial x}\left(\alpha_k\frac{\partial A_{m-1}}{\partial a_r}\right).$$

Hence, in accordance with (82),

$$\check{\Theta}_{mrk}=\check{\Theta}_{m-1, r, k+1}-\frac{\partial}{\partial x}\mathrm{Sp}\left(\alpha_k\frac{\partial A_{m-1}}{\partial a_r}\right).$$

Solving this recusion relation, we obtain

$$\check{\Theta}_{mrk}=-\frac{\partial}{\partial x}\mathrm{Sp}\left(\sum_{s=0}^{m-1}\alpha_{k+s}\frac{\partial A_{m-s-1}}{\partial a_r}\right).$$

Then in accordance with (82) we obtain

$$\mathrm{Sp}\left(\alpha_1\left[\Lambda, \frac{\partial A_m}{\partial a_r}\right]\right)=\frac{\partial}{\partial x}\mathrm{Sp}\left(\sum_{h=1}^{m}\alpha_h\frac{\partial A_{m-h}}{\partial a_r}\right).$$

We now set

$$\check{Y}_{mr}=m\,\mathrm{Sp}\left(\sum_{k=1}^{m}\alpha_k\frac{\partial A_{m-k}}{\partial a_r}\right).$$

Suppose, further, $\check{X}_{mr}=-\check{Y}_{mr}+\check{Z}_{mr}$ and $\hat{X}_{mnr}=X_{mnr}-\check{X}_{mr}$. Then, using (81), we see that Eq. (78) holds.

We note finally that by virtue of (76) and (77) the Heisenberg equations holds. Indeed, we set

$$P=\sum_{m=0}^{n}A_mL^{n-m}+\sum_{\mu=1}^{\mu_0}\sum_{p=1}^{p_\mu}\alpha_{\mu p}(L-\eta_\mu)^{-p}.$$

Then by (75)

$$\mathscr{A}=P+g(L-\eta),$$

where

$$g=-\sum_{m=0}^{n-1}\sum_{h=0}^{n-m-1}A_mL^{n-m-h-1}\eta^h$$
$$+\sum_{\mu=1}^{\mu_0}\sum_{p=1}^{p_\mu}\alpha_{\mu p}\sum_{h=1}^{p}(\eta-\eta_\mu)^{-p+h-1}(L-\eta_\mu)^{-h}.$$

Hence, in accordance with (22),

$$\partial L/\partial t+[P, L]=Q(L-\eta), \tag{83}$$

where the operator $Q=\mathscr{B}-[g, L]$ depends rationally on the spectral parameter η. Since the left-hand side of (83) does not depend on η, it follows that $Q=0$, i.e., the left-hand side of the equation vanishes by virtue of Eqs. (76) and (77). Note that in the above arguments we have not used the explicit form of the operator L but only the fact that it does not depend on the parameter η. Therefore, the argument also goes through in the more general case. The connection between the relation (22) and the Heisenberg equation was established for the first time in Refs. 22 and 23.

5. DIRECT AND INVERSE SCATTERING PROBLEMS

To find the solution of Eqs. (33) and (77), we use the inverse scattering method. However, the possibilities of this method at the present time by no means correspond to the arising needs. The case of second-order matrices ($r_0=2$) has been most fully investigated. In this case, we shall use the approach originally applied by Gel'fand, Levitan, and Marchenko to the solution of the inverse problem for a second-order equation (the Sturm–Liouville equation or the Schrödinger equation).[24]

Thus, we consider the system

$$\partial\varphi/\partial x+U\varphi=i\zeta\Lambda\varphi, \tag{84}$$

where

$$U=\begin{vmatrix}0 & u\\ v & 0\end{vmatrix},\quad \Lambda=\begin{vmatrix}1 & 0\\ 0 & -1\end{vmatrix},$$

where the functions $u=u(x)$ and $v=v(x)$ satisfy the condition

$$\int_{-\infty}^{\infty}(|u(x)|+|v(x)|)\,dx<\infty. \tag{85}$$

In accordance with the results of Sec. 2, the system (84) for any real ζ has a fundamental solution $\varphi^-=\varphi^-(x,\zeta)$ satisfying the condition

$$\lim_{x\to-\infty}\varphi^-(x,\zeta)\exp(-i\zeta\Lambda x)=\lim_{x\to-\infty}\exp(-i\zeta\Lambda x)\varphi^-(x,\zeta)=E,$$

and a fundamental solution $\varphi^+=\varphi^+(x,\zeta)$ satisfying the condition

$$\lim_{x\to\infty}\varphi^+(x,\zeta)\exp(-i\zeta\Lambda x)=\lim_{x\to\infty}\exp(-i\zeta\Lambda x)\varphi^+(x,\zeta)=E.$$

These two solutions are related by

$$\varphi^+=\varphi^-S, \tag{86}$$

where the matrix $S=S(\zeta)$ does not depend on x. Further, by virtue of the inequality (85) the first column of the matrix φ^- and the last column of the matrix φ^+ admit analytic continuation with respect to ζ into the lower half-plane, and the last column of the matrix φ^- and the first column of φ^+ admit analytic continuation with respect to ζ into the upper half-plane. Indeed, let us set

$$\varphi_{11}^-=\exp(i\zeta x)\Phi_1;\quad \varphi_{21}^-=\exp(-i\zeta x)\Phi_2. \tag{87}$$

Then in accordance with (40) the vector $\Phi=(\Phi_1,\Phi_2)$ satisfies the system of integral equations

$$\left.\begin{aligned}\Phi_1&=1-\int_{-\infty}^{x}u(z)\exp(-2i\zeta z)\Phi_2(z,\zeta)\,dz;\\ \Phi_2&=-\int_{-\infty}^{x}v(z)\exp(2i\zeta z)\Phi_1(z,\zeta)\,dz.\end{aligned}\right\} \tag{88}$$

We now set

$$\Phi_1=\Psi_1;\quad \Phi_2=\exp(2i\zeta x)\Psi_2. \tag{89}$$

Then the system (88) takes the form

$$\left.\begin{aligned}\Psi_1&=1-\int_{-\infty}^{x}u(z)\Psi_2(z,\zeta)\,dz;\\ \Psi_2&=-\exp(-2i\zeta x)\int_{-\infty}^{x}v(z)\exp(2i\zeta z)\Psi_1(z,\zeta)\,dz.\end{aligned}\right\} \tag{90}$$

To solve the system (90), we use for $\operatorname{Im}\zeta\leqslant 0$ the method of successive approximation. The obtained solution satisfies the condition

$$|\Psi_1-1|+|\Psi_2|\to 0,\quad |\zeta|\to\infty,$$

uniformly with respect to $\arg\zeta$ for $-\pi\leqslant\arg\zeta\leqslant 0$. It then follows by (87) and (90) that the first column of the matrix φ^- satisfies the condition

$$|\varphi_{11}^-(x,\zeta)\exp(-i\zeta x)-1|+|\varphi_{21}^-(x,\zeta)\exp(-i\zeta x)|\to 0,\quad |\zeta|\to\infty, \tag{91}$$

uniformly with respect to $\arg\zeta$ for $-\pi\leqslant\arg\zeta\leqslant 0$. Similarly, we can prove that the last column of the matrix φ^- satisfies the condition

$$|\varphi_{12}^-(x,\zeta)\exp(i\zeta x)|+|\varphi_{22}^-(x,\zeta)\exp(i\zeta x)-1|\to 0,\quad |\zeta|\to\infty, \tag{92}$$

uniformly with respect to $\arg\zeta$ for $0\leqslant\arg\zeta\leqslant\pi$.

We consider now the integrals

$$\left.\begin{aligned}K_{11}&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\{\varphi_{11}^-(x,\zeta)-\exp(i\zeta x)\}\exp(-i\zeta y)\,d\zeta;\\ K_{21}&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi_{21}^-(x,\zeta)\exp(-i\zeta y)\,d\zeta.\end{aligned}\right\} \tag{93}$$

We show that for $y>x$

$$K_{11}=K_{21}=0. \tag{94}$$

To do this, we take in the ζ plane a contour C with clockwise orientation bounded by the interval $[-R, R]$ of the real axis and the lower half of the circle $|\zeta|=R$, and consider the integral

$$I_C=\frac{1}{2\pi}\oint_C f(x,\zeta)\exp(i\zeta(x-y))\,d\zeta,$$

where

$$f=\varphi_{11}^-(x,\zeta)\exp(-i\zeta x)-1.$$

By the analyticity of the integrand, $I_C=0$. Further, suppose

$$I_C=I_R+I_R^*,$$

where

$$I_R=\frac{1}{2\pi}\int_{-R}^{R}f(x,\zeta)\exp(i\zeta(x-y))\,d\zeta.$$

Using Jordan's lemma and also (91) we find that for $y>x$

$$\lim_{R\to\infty}I_R^*=0.$$

It follows that $\lim_{R\to\infty}I_R=0$, i.e., $K_{11}=0$ for $y>x$. Similarly, $K_{21}=0$ for $y>x$. In (93) we now make an inverse Fourier transformation. Using (94), we obtain

$$\left.\begin{aligned}\varphi_{11}^-&=\exp(i\zeta x)+\int_{-\infty}^{x}K_{11}(x,y)\exp(i\zeta y)\,dy;\\ \varphi_{21}^-&=\int_{-\infty}^{x}K_{21}(x,y)\exp(i\zeta y)\,dy,\end{aligned}\right\} \tag{95}$$

Eqs. (95) holding in the half-plane $\mathrm{Im}\zeta \le 0$.

Similarly, using (92) we show that the last column of the matrix φ^- can be represented in the form

$$\left.\begin{array}{l}\varphi_{12}^- = \int_{-\infty}^{x} K_{12}(x, y)\exp(-i\zeta y)\,dy;\\ \varphi_{22}^- = \exp(-i\zeta x) + \int_{-\infty}^{x} K_{22}(x, y)\exp(-i\zeta y)\,dy,\end{array}\right\} \quad (96)$$

Eqs. (96) holding in the half-plane $\mathrm{Im}\zeta \ge 0$.

Equations (95) and (96) can be combined in the single matrix equation

$$\varphi^- = \exp(i\zeta\Lambda x) + \int_{-\infty}^{x} K(x, y)\exp(i\zeta\Lambda y)\,dy. \quad (97)$$

Substituting (97) in (84) and making some simple calculations, we find that the kernel $K(x, y)$ for $y \le x$ satisfies the equation

$$\frac{\partial}{\partial x}K(x, y) + \Lambda\frac{\partial}{\partial y}K(x, y)\Lambda^{-1} + U(x)K(x, y) = 0 \quad (98)$$

and on the straight line $y = x$ the condition

$$\Lambda K(x, x)\Lambda^{-1} - K(x, x) = U(x),$$

i. e.,

$$u(x) = -2K_{12}(x, x); \quad v(x) = -2K_{21}(x, x). \quad (99)$$

Further, in accordance with (86),

$$S_{11} = \varphi_{11}^+\varphi_{22}^- - \varphi_{12}^-\varphi_{21}^+; \quad S_{12} = \varphi_{12}^+\varphi_{22}^- - \varphi_{12}^-\varphi_{22}^+;$$
$$S_{21} = \varphi_{11}^-\varphi_{21}^+ - \varphi_{11}^+\varphi_{21}^-; \quad S_{22} = \varphi_{11}^-\varphi_{22}^+ - \varphi_{12}^+\varphi_{21}^-.$$

It follows from these equations that S_{11} is an analytic function of ζ in the upper half-plane with $S_{11} \to 1$ as $|\zeta| \to \infty$, and S_{22} is an analytic function of ζ in the lower half-plane and $S_{22} \to 1$ as $|\zeta| \to \infty$. To the zeros of the function S_{11} in the upper half-plane there correspond points of the discrete spectrum, i. e., if $S_{11} = 0$ for $\zeta = \zeta_m$, then there exists a constant C_m such that

$$\varphi_{11}^+(x, \zeta_m) = C_m\varphi_{12}^-(x, \zeta_m); \quad \varphi_{21}^+(x, \zeta_m) = C_m\varphi_{22}^-(x, \zeta_m). \quad (100)$$

It follows from (100) that for $\zeta = \zeta_m$ the system (84) has a solution which decreases exponentially as $x \to \pm\infty$. Similarly, to the zeros of the function S_{22} in the lower half-plane there also correspond points of the discrete spectrum, i. e., if $S_{22} = 0$ for $\zeta = \zeta_n^*$, there exists a constant C_n^* such that

$$\varphi_{12}^+(x, \zeta_n^*) = C_n^*\varphi_{11}^-(x, \zeta_n^*); \quad \varphi_{22}^+(x, \zeta_n^*) = C_n^*\varphi_{21}^-(x, \zeta_n^*). \quad (101)$$

Therefore, the system (84) for $\zeta = \zeta_n^*$ has a solution which decreases exponentially as $x \to \pm\infty$.

The set of the S matrix, the zeros $\zeta_1, \ldots, \zeta_{m_0}$ of the function S_{11} in the upper half-plane, the zeros $\zeta_1^*, \ldots, \zeta_{n_0}^*$ of the function S_{22} in the lower half-plane, and also the constants C_m and C_n^* in (100) and (101) are called scattering data. The finding of the scattering data from a given matrix U is called the direct scattering problem. The inverse scattering problem consists of determining the matrix U from the scattering data. This is achieved as follows.

Suppose S_{11} does not vanish on the real axis, and the zeros of S_{11} in the upper half-plane are all simple, i. e., if $S_{11} = 0$ for $\zeta = \zeta_m$, then $S_{11}'(\zeta_m) \ne 0$. Further, we consider an equation that follows from (86):

$$\varphi_{11}^+/S_{11} = \varphi_{11}^- + (S_{21}/S_{11})\varphi_{12}^-.$$

It follows from this equation that

$$I_1 = I_2 + I_3, \quad (102)$$

where

$$I_1 = \int_{-\infty}^{\infty}\left[\frac{\varphi_{11}^+(x, \zeta)}{S_{11}(\zeta)} - \exp(i\zeta x)\right]\exp(-i\zeta y)\,d\zeta = 2\pi i\sum_{m=1}^{m_0}\frac{\varphi_{11}^+(x, \zeta_m)}{S_{11}'(\zeta_m)}\exp(-i\zeta_m y);$$

$$I_2 = \int_{-\infty}^{\infty}[\varphi_{11}^-(x, \zeta) - \exp(i\zeta x)]\exp(-i\zeta y)\,d\zeta = 2\pi K_{11}(x, y);$$

$$I_3 = \int_{-\infty}^{\infty}\frac{S_{21}(\zeta)}{S_{11}(\zeta)}\varphi_{12}^-(x, \zeta)\exp(-i\zeta y)\,d\zeta = \int_{-\infty}^{x}K_{12}(x, z)\int_{-\infty}^{\infty}\frac{S_{21}(\zeta)}{S_{11}(\zeta)}\exp[-i\zeta(z+y)]\,d\zeta\,dz.$$

Setting

$$f_1(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{S_{21}(\zeta)}{S_{11}(\zeta)}\exp(-i\zeta w)\,d\zeta;$$

$$g_{11} = i\sum_{m=1}^{m_0}\frac{\varphi_{11}^+(x, \zeta_m)}{S_{11}'(\zeta_m)}\exp(-i\zeta_m y),$$

we write (102) in the form

$$K_{11}(x, y) + \int_{-\infty}^{x}K_{12}(x, z)f_1(z+y)\,dz = g_{11}(x, y). \quad (103)$$

Consider the equation

$$\varphi_{21}^+/S_{11} = \varphi_{21}^- + (S_{21}/S_{11})\varphi_{22}^-.$$

It follows from it that

$$J_1 = J_2 + J_3, \quad (104)$$

where

$$J_1 = \int_{-\infty}^{\infty}\frac{\varphi_{21}^+(x, \zeta)}{S_{11}(\zeta)}\exp(-i\zeta y)\,d\zeta = 2\pi i\sum_{m=1}^{m_0}\frac{\varphi_{21}^+(x, \zeta_m)}{S_{11}'(\zeta_m)}\exp(-i\zeta_m y);$$

$$J_2 = \int_{-\infty}^{\infty}\varphi_{21}^-(x, \zeta)\exp(-i\zeta y)\,d\zeta = 2\pi K_{21}(x, y);$$

$$J_3 = \int_{-\infty}^{\infty}\frac{S_{21}(\zeta)}{S_{11}(\zeta)}\varphi_{22}^-(x, \zeta)\exp(-i\zeta y)\,d\zeta = \int_{-\infty}^{\infty}\frac{S_{21}(\zeta)}{S_{11}(\zeta)}\exp[-i\zeta(x+y)]\,d\zeta + \int_{-\infty}^{x}K_{22}(x, z)\int_{-\infty}^{\infty}\frac{S_{21}(\zeta)}{S_{11}(\zeta)}\exp[-i\zeta(z+y)]\,d\zeta\,dz.$$

Setting

$$g_{21} = i\sum_{m=1}^{m_0}\frac{\varphi_{21}^+(x, \zeta_m)}{S_{11}'(\zeta_m)}\exp(-i\zeta_m y),$$

we write Eq. (104) in the form

$$K_{21}(x, y) + f_1(x+y) + \int_{-\infty}^{x}K_{22}(x, z)f_1(z+y)\,dz = g_{21}(x, y). \quad (105)$$

We now assume that S_{22} vanishes nowhere on the real axis and that its zeros in the lower half-plane are all simple, i. e., if $S_{22} = 0$ for $\zeta = \zeta_n^*$, then $S_{22}'(\zeta^*) \ne 0$. Similarly (see above), transforming the equation

$$\varphi_{12}^+/S_{22} = (S_{12}/S_{22})\varphi_{11}^- + \varphi_{12}^-; \quad \varphi_{22}^+/S_{22} = (S_{12}/S_{22})\varphi_{21}^- + \varphi_{22}^-,$$

we obtain

$$K_{12}(x, y)+f_2(x+y)+\int_{-\infty}^{x} K_{11}(x, z) f_2(z+y)\,dz=g_{12}(x, y);$$
$$K_{22}(x, y)+\int_{-\infty}^{x} K_{21}(x, z) f_2(z+y)\,dz=g_{22}(x, y), \qquad (106)$$

where

$$f_2(w)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{S_{12}(\zeta)}{S_{22}(\zeta)}\exp(i\zeta w)\,d\zeta;$$

$$g_{12}=-i\sum_{n=1}^{n_0}\frac{\varphi_{12}^{+}(x, \zeta_n^*)}{S_{22}'(\zeta_n^*)}\exp(i\zeta_n^* y);$$

$$g_{22}=-i\sum_{n=1}^{n_0}\frac{\varphi_{22}^{+}(x, \zeta_n^*)}{S_{22}'(\zeta_n^*)}\exp(i\zeta_n^* y).$$

In the expressions for $g_{\alpha\beta}$ we now replace the solution $\varphi_{\alpha\beta}^{+}$ by means of Eqs. (100) and (101) by the solution $\varphi_{\alpha\beta}^{-}$, $\alpha, \beta=1, 2$. We also use the representation (97) for the solution φ^{-}. We obtain the equations

$$g_{11}=\int_{-\infty}^{x} K_{12}(x, z)\, g_1(z+y)\,dz;$$

$$g_{12}=g_2(x+y)+\int_{-\infty}^{x} K_{11}(x, z)\, g_2(z+y)\,dz;$$

$$g_{21}=g_1(x+y)+\int_{-\infty}^{x} K_{22}(x, z)\, g_1(z+y)\,dz;$$

$$g_{22}=\int_{-\infty}^{x} K_{21}(x, z)\, g_2(z+y)\,dz,$$

where

$$g_1(w)=i\sum_{m=1}^{m_0}\frac{C_m}{S_{11}'(\zeta_m)}\exp(-i\zeta_m w);$$

$$g_2(w)=-i\sum_{n=1}^{n_0}\frac{C_n^*}{S_{22}'(\zeta_n^*)}\exp(i\zeta_n^* w).$$

We now set

$$F_1=f_1-g_1;\quad F_2=f_2-g_2.$$

As a result, Eqs. (103), (105), and (106) take the form

$$K_{11}(x, y)+\int_{-\infty}^{x} K_{12}(x, z) F_1(z+y)\,dz=0;$$

$$K_{12}(x, y)+F_2(x+y)+\int_{-\infty}^{x} K_{11}(x, z) F_2(z+y)\,dz=0;$$

$$K_{21}(x, y)+F_1(x+y)+\int_{-\infty}^{x} K_{22}(x, z) F_1(z+y)\,dz=0;$$

$$K_{22}(x, y)+\int_{-\infty}^{x} K_{21}(x, z) F_2(z+y)\,dz=0.$$

These equations can be written in the form of the single matrix equation

$$K(x, y)+F(x+y)+\int_{-\infty}^{x} K(x, z) F(z+y)\,dz=0, \qquad (107)$$

where

$$F=\begin{vmatrix} 0 & F_2 \\ F_1 & 0 \end{vmatrix}.$$

Thus, using the scattering data we determine the elements of the matrix F. Further, we solve the integral equation (107) and find $K(x, y)$. Finally, by means of Eqs. (99) we find the functions u and v, i.e., the matrix U in Eq. (84). Such is the scheme for the solution of the inverse problem.

In general, the solution of the integral equation (107) is not a simple problem. However, in one case a solution can be obtained explicitly. This is the case of the so-called reflectionless potentials, when the S matrix has diagonal form. Then the kernel F of the integral equation (107) is degenerate, and, therefore, the problem reduces to the solution of a system of linear algebraic equations. It is precisely the reflectionless potentials that correspond to soliton solutions of nonlinear evolution equations.

6. REFLECTIONLESS POTENTIALS

Suppose the S matrix has a diagonal form, i.e., $S_{12}=S_{21}=0$. Then on the real axis we have $S_{11}S_{22}=1$, and, therefore, the function S_{11} admits continuation with respect to ζ into the lower half-plane as a meromorphic function with simple poles at the points $\zeta_1^*, \ldots, \zeta_{n_0}^*$. Similarly, the function S_{22} admits continuation with respect to ζ into the upper half-plane as a meromorphic function with simple poles at the points $\zeta_1, \ldots, \zeta_{m_0}$. It follows that the functions

$$S_1=\frac{\prod_{n=1}^{n_0}(\zeta-\zeta_n^*)}{\prod_{m=1}^{m_0}(\zeta-\zeta_m)}S_{11}(\zeta),\quad S_2=\frac{\prod_{m=1}^{m_0}(\zeta-\zeta_m)}{\prod_{n=1}^{n_0}(\zeta-\zeta_n^*)}S_{22}(\zeta)$$

are defined in the complete complex plane of ζ and do not have singularities in it. Further, it is readily seen that

$$\lim_{|\zeta|\to\infty}\zeta^{m_0-n_0}S_1(\zeta)=\lim_{|\zeta|\to\infty}\zeta^{n_0-m_0}S_2(\zeta)=1.$$

It follows directly from this that at least either $S_1(m_0 \geqslant n_0)$ or $S_2(n_0 \geqslant m_0)$ is bounded in the neighborhood of the point at infinity. Hence, by Liouville's theorem we find that either S_1 or S_2 is a constant, whence by $S_1S_2=1$ it follows that both the functions S_1 and S_2 are constants, and this is possible only if $m_0=n_0$. From this last fact and the circumstance that $S_{11}\to 1$ as $|\zeta|\to\infty$ and $S_{22}\to 1$ as $|\zeta|\to\infty$ it follows that

$$S_{11}=\prod_{m=1}^{m_0}\left(\frac{\zeta-\zeta_m}{\zeta-\zeta_m^*}\right);\quad S_{22}=\prod_{m=1}^{m_0}\left(\frac{\zeta-\zeta_m^*}{\zeta-\zeta_m}\right). \qquad (108)$$

Thus, in the considered case specification of the $2m_0$ points $\zeta_1, \ldots, \zeta_m, \zeta_1^*, \ldots, \zeta_{m_0}^*$ completely determines the S matrix. We also specify the $2m_0$ constants $C_1, \ldots, C_{m_0}, C_1^*, \ldots, C_{m_0}^*$. By means of these data, the kernel F of the integral equation (107) is completely determined. It has the form

$$F_1=-i\sum_{m=1}^{m_0}\frac{C_m}{S_{11}'(\zeta_m)}\exp(-i\zeta_m w);$$

$$F_2=i\sum_{m=1}^{m_0}\frac{C_m^*}{S_{22}'(\zeta_m^*)}\exp(i\zeta_m^* w),$$

where $S_{11}'(\zeta_m)$ and $S_{22}'(\zeta_m^*)$ are determined by Eqs. (108).

We now set

$$N_m=\left(2\int_{-\infty}^{\infty}\varphi_{12}^{-}(x, \zeta_m)\,\varphi_{22}^{-}(x, \zeta_m)\,dx\right)^{-1/2};$$

$$N_m^*=\left(-2\int_{-\infty}^{\infty}\varphi_{11}^{-}(x, \zeta_m^*)\,\varphi_{21}^{-}(x, \zeta_m^*)\,dx\right)^{-1/2}.$$

Then

$$N_m = (-\mathrm{i}C_m/S'_{11}(\zeta_m))^{1/2}; \quad N_m^* = (-\mathrm{i}C_m^*/S'_{22}(\zeta_m^*))^{1/2}.$$

For suppose $\varphi = (\varphi_1, \varphi_2)$ is a solution of the system (84), and $\psi = (\psi_1, \psi_2)$ a solution of the system (84) for $\zeta = \zeta'$. Using (84), we can readily prove that

$$d(\varphi_1\psi_2 - \varphi_2\psi_1)/dx = \mathrm{i}(\zeta - \zeta')(\varphi_1\psi_2 + \varphi_2\psi_1). \tag{109}$$

Differentiating this equation with respect to ζ and then setting $\zeta' = \zeta$, we obtain

$$\frac{d}{dx}\left(\frac{\partial\varphi_1}{\partial\zeta}\psi_2 - \frac{\partial\varphi_2}{\partial\zeta}\psi_1\right) = \mathrm{i}(\varphi_1\psi_2 + \varphi_2\psi_1). \tag{110}$$

Similarly, differentiating (109) with respect to ζ' and then setting $\zeta' = \zeta$, we obtain

$$\frac{d}{dx}\left(\varphi_1\frac{\partial\psi_2}{\partial\zeta} - \varphi_2\frac{\partial\psi_1}{\partial\zeta}\right) = -\mathrm{i}(\varphi_1\psi_2 + \varphi_2\psi_1). \tag{111}$$

In Eqs. (110) and (111) we now set

$$\varphi_1 = \varphi_{11}^+(x, \zeta_m); \quad \varphi_2 = \varphi_{21}^+(x, \zeta_m);$$
$$\psi_1 = \varphi_{12}^-(x, \zeta_m); \quad \psi_2 = \varphi_{22}^-(x, \zeta_m).$$

Further, we integrate (110) with respect to the spatial variable from x to ∞ and (111) from $-\infty$ to x and subtract the one result from the other. Then, using (100), we obtain

$$S'_{11}(\zeta_m) = -2\mathrm{i}C_m \int_{-\infty}^{\infty} \varphi_{12}^-(x, \zeta_m)\,\varphi_{22}^-(x, \zeta_m)\,dx. \tag{112}$$

Setting in Eqs. (110) and (111)

$$\varphi_1 = \varphi_{11}^-(x, \zeta_m^*); \quad \varphi_2 = \varphi_{21}^-(x, \zeta_m^*);$$
$$\psi_1 = \varphi_{12}^+(x, \zeta_m^*); \quad \psi_2 = \varphi_{22}^+(x, \zeta_m^*),$$

we obtain similarly

$$S'_{22}(\zeta_m^*) = 2\mathrm{i}C_m^* \int_{-\infty}^{\infty} \varphi_{11}^-(x, \zeta_m^*)\,\varphi_{21}^-(x, \zeta_m^*)\,dx. \tag{113}$$

We now note that if $v = -\bar{u}$ in Eq. (84) then the points $\zeta_1, \ldots, \zeta_{m_0}, \zeta_1^*, \ldots, \zeta_{m_0}^*$ can be numbered such that $\zeta_m^* = \bar{\zeta}_m$, $m = 1, \ldots, m_0$. It then follows in accordance with (108), (112), and (113) that $C_m^* = -\bar{C}_m$, $m = 1, \ldots, m_0$. Therefore, the quantities $N_1, \ldots, N_{m_0}$, $N_1^*, \ldots, N_{m_0}^*$ defined above satisfy in this case the condition $N_m^* = \bar{N}_m$, $m = 1, \ldots, m_0$.

Now suppose

$$\left.\begin{aligned} K_{11} &= \sum_{m=1}^{m_0} N_m p_{11,m}(x)\exp(-\mathrm{i}\zeta_m y); \\ K_{12} &= \sum_{m=1}^{m_0} N_m^* p_{12,m}(x)\exp(\mathrm{i}\zeta_m^* y). \end{aligned}\right\} \tag{114}$$

Substituting (114) in (107), we obtain the system of linear algebraic equations

$$\left.\begin{aligned} &p_{11,m} + \sum_{\mu=1}^{m_0} N_m N_\mu^* \frac{\exp(\mathrm{i}(\zeta_\mu^* - \zeta_m)x)}{\mathrm{i}(\zeta_\mu^* - \zeta_m)} p_{12,\mu} = 0; \\ &-\sum_{\mu=1}^{m_0} N_\mu N_m^* \frac{\exp(\mathrm{i}(\zeta_m^* - \zeta_\mu)x)}{\mathrm{i}(\zeta_m^* - \zeta_\mu)} p_{11,\mu} + p_{12,m} = N_m^*\exp(\mathrm{i}\zeta_m^* x). \end{aligned}\right\}$$

The matrix R of this system has the form

$$R = \begin{vmatrix} E & \tilde{M} \\ -M & E \end{vmatrix}.$$

Suppose the matrix R_m is obtained from R by replacing its m-th column by the column

$$(0, \ldots, 0, N_1^*\exp(\mathrm{i}\zeta_1^* x), \ldots, N_{m_0}^*\exp(\mathrm{i}\zeta_{m_0}^* x)).$$

Suppose further

$$\Delta = \det R = \det|E + M\tilde{M}|; \quad \Delta_m = \det R_m.$$

Then by Cramer's rule

$$p_{11,m} = \Delta_m/\Delta; \quad p_{12,m} = \Delta_{m_0+m}/\Delta. \tag{115}$$

We also set

$$\left.\begin{aligned} K_{21} &= \sum_{m=1}^{m_0} N_m p_{21,m}(x)\exp(-\mathrm{i}\zeta_m y); \\ K_{22} &= \sum_{m=1}^{m_0} N_m^* p_{22,m}(x)\exp(\mathrm{i}\zeta_m^* y). \end{aligned}\right\} \tag{116}$$

Substituting (116) in (107), we obtain the system of linear algebraic equations

$$p_{21,m} + \sum_{\mu=1}^{m_0} N_m N_\mu^* \frac{\exp(\mathrm{i}(\zeta_\mu^* - \zeta_m)x)}{\mathrm{i}(\zeta_\mu^* - \zeta_m)} p_{22,\mu} + N_m\exp(-\mathrm{i}\zeta_m x) = 0;$$
$$-\sum_{\mu=1}^{m_0} N_\mu N_m^* \frac{\exp(\mathrm{i}(\zeta_m^* - \zeta_\mu)x)}{\mathrm{i}(\zeta_m^* - \zeta_\mu)} p_{21,\mu} + p_{22,m} = 0.$$

Suppose the matrix R_m^* is obtained from R by replacing its m-th column by the column

$$(N_1\exp(-\mathrm{i}\zeta_1 x), \ldots, N_{m_0}\exp(-\mathrm{i}\zeta_{m_0} x), 0, \ldots, 0).$$

Suppose further

$$\Delta_m^* = \det R_m^*.$$

Then in accordance with Cramer's rule,

$$p_{21,m} = -\Delta_m^*/\Delta; \quad p_{22,m} = -\Delta_{m_0+m}^*/\Delta. \tag{117}$$

Using (99) and in accordance with (114) and (115), we have

$$\left.\begin{aligned} u(x) &= -\frac{2}{\Delta}\sum_{m=1}^{m_0} N_m^*\Delta_{m_0+m}\exp(\mathrm{i}\zeta_m^* x); \\ v(x) &= \frac{2}{\Delta}\sum_{m=1}^{m_0} N_m\Delta_m^*\exp(-\mathrm{i}\zeta_m x). \end{aligned}\right\} \tag{118}$$

Formulas (118) give expressions for the required potentials. Further, using the definition of the matrices R_m and R_m^*, we can readily prove the equation

$$\frac{\partial}{\partial x}\ln\Delta = \sum_{m=1}^{m_0}\left[-N_m\frac{\Delta_m}{\Delta}\exp(-\mathrm{i}\zeta_m x) + N_m^*\frac{\Delta_{m_0+m}^*}{\Delta}\exp(\mathrm{i}\zeta_m^* x)\right].$$

Hence, by virtue of Eqs. (114)–(117),

$$K_{11}(x, x) + K_{22}(x, x) = -\partial\ln\Delta/\partial x.$$

In addition, from Eq. (98)

$$dK_{11}(x, x)/dx + u(x)K_{21}(x, x) = 0;$$
$$dK_{22}(x, x)/dx + v(x)K_{12}(x, x) = 0.$$

Using (99), we now find that

$$d[K_{11}(x, x) + K_{22}(x, x)]/dx = u(x)v(x).$$

Thus,

$$\partial^2\ln\Delta/\partial x^2 = -u(x)v(x).$$

7. EXAMPLES

Let us now consider what equations can actually be obtained by the method described above in the case $r_0 = 2$. Setting

$$\Lambda = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}; \quad U = \begin{vmatrix} 0 & u \\ v & 0 \end{vmatrix}; \quad A_m = \begin{vmatrix} \alpha_m & \beta_m \\ \gamma_m & \delta_m \end{vmatrix},$$

in accordance with Eq. (29) (the matrix u is replaced in it by U)

$$[\Lambda, A_{m+1}] - [U, A_m] - \partial A_m/\partial x = 0,$$

we obtain the following recursion relations for the elements of the matrix A_m:

$$\left.\begin{array}{l}\partial\alpha_m/\partial x - v\beta_m + u\gamma_m = 0;\\ 2\beta_{m+1} + (\alpha_m - \delta_m)\,u - \partial\beta_m/\partial x = 0;\\ 2\gamma_{m+1} + (\alpha_m - \delta_m)\,v + \partial\gamma_m/\partial x = 0;\\ \partial\delta_m/\partial x + v\beta_m - u\gamma_m = 0.\end{array}\right\} \quad (119)$$

From the first and last equations of this system it follows that $\partial(\alpha_m + \delta_m)/\partial x = 0$, i.e., $\alpha_m + \delta_m = c_m$, where c_m does not depend on x. Taking $c_m = 0$, we obtain $\delta_m = -\alpha_m$. On the basis of this remark, the system (119) takes the form

$$\left.\begin{array}{l}\partial\alpha_m/\partial x = v\beta_m - u\gamma_m;\\ \beta_{m+1} = -u\alpha_m + (1/2)\,\partial\beta_m/\partial x;\\ \gamma_{m+1} = -v\alpha_m - (1/2)\,\partial\gamma_m/\partial x.\end{array}\right\} \quad (120)$$

We take the matrix A_0 in the form $A_0 = \mathrm{diag}(c, -c)$, i.e., $\alpha_0 = c, \beta_0 = \gamma_0 = 0$. Then, using the system (120), we find

$$\left.\begin{array}{l}\beta_1 = -cu;\quad \gamma_1 = -cv;\quad \alpha_1 = 0;\\ \beta_2 = -cu_x/2;\quad \gamma_2 = cv_x/2;\quad \alpha_2 = -cuv/2;\\ \beta_3 = cu^2v/2 - cu_{xx}/4;\quad \gamma_3 = cuv^2/2 - cv_{xx}/4;\\ \alpha_3 = c\,(uv_x - u_xv)/4;\\ \beta_4 = 3cuu_xv/4 - cu_{xxx}/8;\quad \gamma_4 = -3cuvv_x/4 + cv_{xxx}/8.\end{array}\right| \quad (121)$$

Further, the equations defined in accordance with (33) have in our case the form

$$u_t = 2\beta_{n+1};\quad v_t = -2\gamma_{n+1}.$$

For $n = 2$, we obtain in accordance with (121) the system

$$u_t = cu^2v - cu_{xx}/2,\quad v_t = -cuv^2 + cv_{xx}/2.$$

For $c = -2i$ and $v = \pm\bar{u}$, this system is equivalent to a single equation for the complex-valued function $u = u(x, t)$, which is known as the nonlinear Schrödinger equation:

$$iu_t + u_{xx} = \pm 2\,|u|^2u. \quad (122)$$

For $n = 3$, we have in accordance with (121)

$$u_t + 3cuu_xv/2 - cu_{xxx}/4;\quad v_t = 3cuvv_x/2 - cv_{xxx}/4.$$

For $c = 4$ and $v = u\pm$, the obtained system reduces to the single equation

$$u_t + u_{xxx} = \pm 6u^2u_x, \quad (123)$$

which is known as the modified KdV equation.

Each of the equations (122) and (123) has an infinite sequence of equations, which by analogy with the higher KdV equations could be said to be higher for the nonlinear Schrödinger equation or for the modified KdV equation. In the case of the nonlinear Schrödinger equation, the higher equations have the form

$$u_t = 2g_{k+1},\quad k > 0, \quad (124)$$

where the functions g_m and f_m for $v = \bar{u}$ are determined by the recursion relations

$$\partial f_m/\partial x = \bar{u}g_m - (-1)^m u\bar{g}_m;\ g_{m+1} = -uf_m + (1/2)\,\partial g_m/\partial x,$$

and for $v = -u$ by the relations

$$\partial f_m/\partial x = -\bar{u}g_m + (-1)^m u\bar{g}_m;\quad g_{m+1} = -uf_m + (1/2)\,\partial g_m/\partial x.$$

Equation (122) is itself obtained from (124) for $k = 1$. Similarly, in the case of the modified KdV equation the higher equations have the form

$$u_t = 2g_{2k+2},\quad k > 0, \quad (125)$$

where the functions g_m and f_m for $v = u$ are determined by the recursion relations

$$\partial f_m/\partial x = [1 + (-1)^m]\,ug_m;\quad g_{m+1} = -uf_m + (1/2)\,\partial g_m/\partial x,$$

and for $v = -u$ by

$$\partial f_m/\partial x = -[1 + (-1)^m]\,ug_m;\quad g_{m+1} = -uf_m + (1/2)\,\partial g_m/\partial x.$$

Equation (123) itself is obtained from (125) for $k = 1$.

We now consider what equations can be obtained in the case $r_0 = 2$ which we are considering by means of a matrix $\mathscr{A}$ that depends rationally on the spectral parameter. We consider the simplest case when the matrix $\mathscr{A}$ of the form (75) has a unique pole of first order at the point $\eta = 0$. In accordance with (77), the evolution equation has the form

$$U_t + [\Lambda, \alpha] = [\Lambda, A_{n+1}], \quad (126)$$

where the matrix α satisfies in accordance with (76) the condition

$$\partial\alpha/\partial x + [U, \alpha] = 0, \quad (127)$$

and the matrix A_{n+1} is determined by means of (29). We take the matrix U in the form

$$U = \begin{vmatrix} 0 & a\theta_x \\ b\theta_x & 0 \end{vmatrix}.$$

Then it follows from (127) that

$$\alpha = \begin{vmatrix} f & ag \\ -bg & -f \end{vmatrix}, \quad (128)$$

where $f = f(\theta)$ and $g = g(\theta)$ is an arbitrary solution of the system

$$f' - 2abg = 0;\quad g' - 2f = 0.$$

For $4ab = -1$, the general solution of this system has the form

$$f = \mu\cos\theta + \nu\sin\theta;\quad g = 2\mu\sin\theta - 2\nu\cos\theta, \quad (129)$$

and for $4ab = 1$ the general solution has the form

$$f = \mu\exp\theta + \nu\exp(-\theta);\quad g = 2\mu\exp\theta - 2\nu\exp(-\theta). \quad (130)$$

Using the system (119), we find

$$A_{n+1} = \begin{vmatrix} f_{n+1} & ag_{n+1} \\ (-1)^n bg_{n+1} & -f_{n+1} \end{vmatrix},$$

where the functions f_{n+1} and g_{n+1} are determined by the recursion relations

$$\partial f_n/\partial x = [1 + (-1)^n]\,abg_n\theta_x;\quad g_{n+1} = -f_n\theta_x + (1/2)\,\partial g_n/\partial x.$$

It follows that (126) determines a noncontradictory system only for $n = 2k + 1, k \geqslant 0$. In accordance with (128), this system consists of two identical equations of the form

$$\theta_{xt} + 2g = 2g_{2k+2},\quad k \geqslant 0. \quad (131)$$

In addition, to this family we must add one further equation

$$\theta_{xt} + 2g = 0, \quad (131')$$

which is obtained if in the expression for the matrix $\mathscr{A}$ we retain only the pole term. Formally, Eq. (131′) can be obtained from (131) for $k = -1$. Then Eqs. (131) with $k \geqslant 0$ are higher equations corresponding to (131′). If in (129) we take $\mu = -1/4$ and $\nu = 0$, then (131′) is transformed into the sine–Gordon equation

$$\theta_{xt} = \sin\theta.$$

But if in (130) we take $\mu=-1/4$ and $\nu=0$, then (131′) is transformed into the Liouville equation

$$\theta_{xt}=\exp\theta.$$

To investigate all the equations obtained above for definite boundary conditions, we use the inverse scattering method. To use this method, we must find the time dependence of the scattering data for the operator L if the coefficients of this operator satisfy one of the above equations.

We begin by considering the case when the matrix $\mathcal{A}$ has the form (28), i.e., it depends polynomially on the spectral parameter. Suppose that the functions $u=u(x,t)$ and $v=v(x,t)$ in the matrix U tend for any fixed $t\geqslant 0$ sufficiently rapidly to zero as $x\to\pm\infty$.

By the definition of the matrix $\mathcal{A}$, the operator $\partial/\partial t+\mathcal{A}$ carries any solution of the equation $(L-\eta)\varphi=0$ into a solution of the same equation. Indeed, in accordance with (22),

$$(\partial L/\partial t)\,\varphi-(L-\eta)\,\mathcal{A}\varphi=0.$$

In addition, differentiating the equation $(L-\eta)\varphi=0$ with respect to t, we obtain

$$(\partial L/\partial t)\,\varphi+(L-\eta)\,\varphi_t=0.$$

It follows from these equations that

$$(L-\eta)\,(\varphi_t+\mathcal{A}\varphi)=0,$$

i.e., $\psi=\varphi_t+\mathcal{A}\varphi$ satisfies the equation $(L-\eta)\psi=0$ if φ satisfies this equation. Therefore, in the case under consideration there exist matrices Γ_+ and Γ_- which do not depend on x and are such that

$$\partial\varphi^+/\partial t+\mathcal{A}\varphi^+=\varphi^+\Gamma_+;\quad \partial\varphi^-/\partial t+\mathcal{A}\varphi^-=\varphi^-\Gamma_-. \tag{132}$$

Going to the limit $x\to\infty$ in the first of Eqs. (132) and to the limit $x\to-\infty$ in the second, we obtain

$$\Gamma_+=\Gamma_-=\Gamma=A_0\eta^n=\operatorname{diag}(\gamma_1,\gamma_2)=\operatorname{diag}(c\eta^n,-c\eta^n), \tag{133}$$

where n is the degree of the polynomial (28). Further, by virtue of the equation $\varphi^+=\varphi^-S$ we obtain from the first equation (132)

$$(\partial\varphi^-/\partial t+\mathcal{A}\varphi^-)\,S+\varphi^-\partial S/\partial t=\varphi^-S\Gamma.$$

Using the second equation (132), we obtain from this

$$\partial S/\partial t+[\Gamma,S]=0, \tag{134}$$

i.e.,

$$S=\exp(-\Gamma t)\,S_0\exp(\Gamma t),$$

where S_0 does not depend on t. It follows from the last equation that the diagonal elements of the matrix S do not depend on the time. This means that the positions and numbers of points of the discrete spectrum also do not change with the time.

We now consider how the constants C_m and C_n^* in Eqs. (100) and (101) change with the time. Suppose

$$\varphi_1^+=(\varphi_{11}^+(x,\zeta_m),\varphi_{21}^+(x,\zeta_m));\quad \varphi_2^-=(\varphi_{12}^-(x,\zeta_m),\varphi_{22}^-(x,\zeta_m)).$$

Then from the consideration of the asymptotic behavior as $x\to\infty$ we obtain

$$(\partial/\partial t+\mathcal{A})\,\varphi_1^+=\gamma_1\varphi_1^+, \tag{135}$$

and by considering the asymptotic behavior as $x\to-\infty$ we obtain

$$(\partial/\partial t+\mathcal{A})\,\varphi_2^-=\gamma_2\varphi_2^-. \tag{136}$$

At the same time, the diagonal elements γ_1 and γ_2 of Γ are taken in accordance with (133) for $\eta=\eta_m=i\zeta_m$. Further, using the equation $\varphi_1^+=C_m\varphi_2^-$, we obtain from (135)

$$(\partial C_m/\partial t)\,\varphi_2^-+C_m(\partial/\partial t+\mathcal{A})\,\varphi_2^-=\gamma_1C_m\varphi_2^-.$$

Hence, on the basis of (136)

$$\partial C_m/\partial t=(\gamma_1-\gamma_2)\,C_m. \tag{137}$$

An equation for C_n^* is obtained similarly, and it has the form

$$\partial C_n^*/\partial t+(\gamma_1-\gamma_2)\,C_n^*=0, \tag{138}$$

where the diagonal elements γ_1 and γ_2 of Γ are taken in accordance with (133) for $\eta=\eta_n^*=i\zeta_n^*$. At the same time, for Eq. (124) the matrix Γ has the form

$$\Gamma=\operatorname{diag}(c\eta^{2k},-c\,\eta^{2k})=\operatorname{diag}((-1)^{k+1}2i\zeta^{2k},(-1)^k2i\zeta^{2k}).$$

In particular, for $k=1$ (the nonlinear Schrödinger equation)

$$\Gamma=\operatorname{diag}(2i\zeta^2,-2i\zeta^2). \tag{139}$$

With regard to the discrete spectrum, it does not exist at all for $v=\bar{u}$, while for $v=-\bar{u}$ the points of it form pairs ζ_m, $\zeta_m^*=\bar{\zeta}_m$. For Eq. (125), the matrix Γ has the form

$$\Gamma=\operatorname{diag}(c\eta^{2k+1},-c\eta^{2k+1})=\operatorname{diag}((-1)^k4i\zeta^{2k+1},(-1)^{k+1}4i\zeta^{2k+1}).$$

In particular, for $k=1$ (modified KdV equation)

$$\Gamma=\operatorname{diag}(-4i\zeta^3,4i\zeta^3). \tag{140}$$

For $v=u$ there is no discrete spectrum at all, and for $v=-u$ the points of the discrete spectrum form quartets ζ_m, $\bar{\zeta}_m$, $-\zeta_m$, $-\bar{\zeta}_m$ if $\zeta_m\neq-\bar{\zeta}_m$ and duets ζ_m, $\bar{\zeta}_m=-\zeta_m$ otherwise.

Thus, specifying $u=u_0(x)$ for $t=0$, we can, by solving the direct scattering problem, find the scattering data for $t=0$. We then continue the scattering data for $t>0$ by means of Eqs. (134), (137), and (138). Finally, solving the inverse scattering problem, we obtain the solution $u=u(x,t)$ for $t>0$. At large t, the main part in the kernel $F(w)$ of the integral equation (107) is played by the soliton terms, since in accordance with (139) and (140) the elements S_{12} and S_{21} of the matrix S oscillate with the time, and the quantities C_m and C_n^* defined in accordance with (137) and (138) increase exponentially with the time in general either as $t\to\infty$ or $t\to-\infty$.

For the sine–Gordon equation, the inverse scattering method was used to find solutions $\theta=\theta(x,t)$ satisfying the boundary conditions

$$\theta\to 2k_+\pi,\quad x\to\infty;\quad \theta\to 2k_-\pi,\quad x\to-\infty, \tag{141}$$

where k_+ and k_- are arbitrary integers. For the given boundary conditions, the solution can be found by the method described above with the only difference that in this case one uses the inverse scattering method to find not the solution θ itself but the derivative θ_x, and the obtained result must be integrated to find the solution. It is found that if the condition (141) is satisfied at $t=0$, then it will also be satisfied for any $t>0$, i.e., the quantity $k=k_+-k_-$ does not change with the time. We note only that in this case the matrix Γ has the form

$$\Gamma=\operatorname{diag}(i/4\zeta,-i/4\zeta),$$

and the discrete spectrum, as in the case of the modified KdV equation, forms quartets $\zeta_m, \bar{\zeta}_m, -\zeta_m, -\bar{\zeta}_m$ if $\zeta_m \neq -\bar{\zeta}_m$, and duets $\zeta_m, \bar{\zeta}_m = -\zeta_m$ otherwise.

The inverse problem for the Liouville equation is strongly degenerate. We therefore give a different approach to the solution of the Cauchy and Goursat problems for this equation. To solve the Goursat problem, we consider the Liouville equation in the form

$$\theta_{xt} = 2\exp(-\theta). \tag{142}$$

We find a solution $\theta = \theta(x, t)$ satisfying the conditions

$$\left.\begin{array}{ll}\theta(x, 0) = f(x), & x \in (-\infty, \infty);\\ \theta(0, t) = g(t), & t \in (-\infty, \infty),\end{array}\right\} \tag{143}$$

with $f(0) = g(0) = c$. The required solution has the form

$$\theta = \ln[\varphi_1(x)\psi_1(t) + \varphi_2(x)\psi_2(t)]^2, \tag{144}$$

where

$$\varphi_1 = \exp\left[\frac{1}{2}f(x) - \frac{c}{4}\right]; \quad \varphi_2 = \varphi_1(x)\int_0^x \frac{dz}{\varphi_1^2(z)}; \tag{145}$$

$$\psi_1 = \exp\left[\frac{1}{2}g(t) - \frac{c}{4}\right]; \quad \psi_2 = \psi_1(t)\int_0^t \frac{d\tau}{\psi_1^2(\tau)}. \tag{146}$$

Indeed, it can be verified by direct substitution that the solution (144) satisfies Eq. (142) and the conditions (143). This solution is unique.

The uniqueness is proved as follows. It is readily seen that

$$I = \theta_{xx} + \theta_x^2/2 \tag{147}$$

does not depend on t, and

$$J = \theta_{tt} + \theta_t^2/2 \tag{148}$$

does not depend on x if $\theta = \theta(x, t)$ satisfies Eq. (142). We now set

$$\theta = \ln \Phi^2. \tag{149}$$

Substituting (149) in (147), we obtain

$$\Phi_{xx} - I\Phi/2 = 0, \quad I = f_{xx} + f_x^2/2. \tag{150}$$

Hence

$$\Phi = c_1\varphi_1(x) + c_2\varphi_2(x), \tag{151}$$

where φ_1 and φ_2 are a fundamental system of solutions of Eq. (150) determined by Eqs. (145), and c_1 and c_2 do not depend on x. Further, substituting (149) in (148), we obtain

$$\Phi_{tt} - J\Phi/2 = 0, \quad J = g_{tt} + g_t^2/2. \tag{152}$$

It follows then in accordance with (151) that

$$c_1 = c_{11}\psi_1(t) + c_{12}\psi_2(t); \quad c_2 = c_{21}\psi_1(t) + c_{22}\psi_2(t),$$

where ψ_1 and ψ_2 are a fundamental system of solutions of Eq. (152) determined by Eqs. (146), and $c_{\alpha\beta}$, $\alpha, \beta = 1, 2$, do not depend on x or t. It is readily seen that the solution obtained in this manner satisfies the conditions (143) only for $c_{11} = c_{22} = 1$ and $c_{12} = c_{21} = 0$.

To solve the Cauchy problem, we take the Liouville equation in the form

$$\theta_{xx} - \theta_{tt} = 8\exp(-\theta).$$

We find a solution $\theta = \theta(x, t)$ satisfying the conditions

$$\theta(x, 0) = f(x); \quad \theta_t(x, 0) = g(x); \quad x \in (-\infty, \infty).$$

The required solution is

$$\theta = \ln[\varphi_1(x-t)\psi_1(x+t) + \varphi_2(x-t)\psi_2(x+t)]^2,$$

where $\varphi_1 = \varphi_1(\zeta)$ and $\varphi_2 = \varphi_2(\zeta)$ are a fundamental system of solutions of the equation

$$\varphi_{\zeta\zeta} - M\varphi/8 = 0, \tag{153}$$

and $\psi_1 = \psi_1(\eta)$ and $\psi_2 = \psi_2(\eta)$ are a fundamental system of solutions of the equation

$$\psi_{\eta\eta} - N\psi/8 = 0. \tag{154}$$

We also have

$$M = -8\exp[-f(\zeta)] + 2f''(\zeta) - 2g'(\zeta) + [f'(\zeta) - g(\zeta)]^2/2;$$
$$N = -8\exp[-f(\eta)] + 2f''(\eta) + 2g'(\eta) + [f'(\eta) + g(\eta)]^2/2.$$

In addition, the solutions φ_1 and φ_2 can be chosen arbitrarily but subject to the condition $\varphi_1\varphi_2' - \varphi_1'\varphi_2 = 1$, and the solutions ψ_1 and ψ_2 are chosen subject to the condition $\psi_1\psi_2' - \psi_1'\psi_2 = 1$, so that

$$\left.\begin{array}{l}\varphi_1(x)\psi_1(x) + \varphi_2(x)\psi_2(x) = \exp[f(x)/2];\\ -\varphi_1'(x)\psi_1(x) + \varphi_1(x)\psi_1'(x) - \varphi_2'(x)\psi_2(x)\\ \quad + \varphi_2(x)\psi_2'(x) = g(x)\exp[f(x)/2]/2.\end{array}\right\} \tag{155}$$

We show that this is possible. Suppose

$$F_0 = \exp[f(x)/2]; \quad F_+ = [f'(x) + g(x)]F_0(x)/4;$$
$$F_- = [f'(x) - g(x)]F_0(x)/4.$$

Suppose further

$$\psi_1 = F_0(x)\varphi_2'(x) - F_-(x)\varphi_2(x);$$

$$\psi_2 = -F_0(x)\varphi_1'(x) + F_-(x)\varphi_1(x).$$

Then it can be directly verified that ψ_1 and ψ_2 defined in this manner satisfy Eq. (154) if φ_1 and φ_2 satisfy (153). In addition, $\psi_1\psi_2' - \psi_1'\psi_2 = 1$ if the analogous equation holds for φ_1 and φ_2. Finally, by means of the equations

$$\psi_1' = F_+(x)\varphi_2'(x) - \{[F_+(x)F_-(x) + 1]/F_0(x)\}\varphi_2(x);$$

$$\psi_2' = -F_+(x)\varphi_1'(x) + \{[F_+(x)F_-(x) + 1]/F_0(x)\}\varphi_1(x)$$

we can readily show that (155) holds. But from this the uniqueness of the solution to the Cauchy problem for the Liouville equation follows.

Finally, it should be noted that a similar approach was used in Ref. 25 for the construction of a global solution to the Liouville equation. In Ref. 26, the Liouville equation is investigated by means of the standard inverse scattering method. In recent years, the interest in the Liouville equation has to a large extent been stimulated, on the one hand, by the application of the Liouville equation to the theory of a relativistic string,[27,28] and, on the other, by the connection discovered in Ref. 29 between the Liouville equation and the Born–Infeld equation for a scalar massless field in two-dimensional space-time.[30] The connection between the Born–Infeld equation and a relativistic string is considered in Ref. 31.

8. THE CASE WHEN THE OPERATOR L DEPENDS ON THE SPECTRAL PARAMETER

Hitherto, we have considered the case when the operator L of the form (23) does not depend on the spectral parameter. However, the method used above can also be applied when L depends rationally on the spectral parameter. Let us consider now the case when the matrix u in (23) has the form

$$u = \sum_{k=0}^{k_0} u_k(x, t)\zeta^{k_0-k} + \sum_{\mu=1}^{\mu_0}\sum_{p=1}^{p_\mu}\frac{u_{\mu p}(x, t)}{(\zeta-\zeta_\mu)^p},$$

where the integer k_0 satisfies $k_0 \geq 0$. Further, we take the matrix $\mathscr{A}$ in the form

$$\mathcal{A}=\sum_{m=0}^{n} A_m\zeta^{n-m}+v,$$

where

$$v=\sum_{\nu=1}^{\nu_0}\sum_{q=1}^{q_\nu}\frac{v_{\nu q}(x,t)}{(\zeta-\zeta_\nu')^q}.$$

At the same time, the points $\zeta_1, \ldots, \zeta_{u_0}, \zeta_1', \ldots, \zeta_{\nu_0}'$ lie in the complex ζ plane and are all different.

We now set

$$u=\sum_{k=0}^{\infty} u_k(x, t)\zeta^{k_0-k}, \quad v=\sum_{k=1}^{\infty} v_k(x, t)\zeta^{-k}.$$

Suppose the matrices A_m, $m \geq 0$ are such that for $m=0$

$$[\Lambda, A_0]=0; \tag{156}$$

for $1 \leq m \leq k_0$, if $k_0>0$,

$$[\Lambda, A_m]-\sum_{k=0}^{m-1}[u_k, A_{m-k-1}]=0, \tag{157}$$

and for $m>k_0$

$$[\Lambda, A_m]-\sum_{k=0}^{m-1}[u_k, A_{m-k-1}]-\frac{\partial}{\partial x}A_{m-k_0-1}=0. \tag{158}$$

Consider the equation

$$\partial u/\partial t-\partial\mathcal{A}/\partial x-[u, \mathcal{A}]+\zeta^{k_0+1}[\Lambda, \mathcal{A}]=0, \tag{159}$$

which is obtained from (27) for $\eta=\zeta^{k_0+1}$. In this situation, Eq. (159) is equivalent to the system of equations

$$\left.\begin{array}{l}\partial u_0/dt=[\Lambda, A_{n+1}]-[\Lambda, v_1];\\ \partial u_k/\partial t=[\Lambda, A_{n+k+1}]-[\Lambda, v_{k+1}]\\ \quad-\sum_{\varkappa=0}^{k-1}[u_\varkappa, A_{n+k-\varkappa}]+\sum_{\varkappa=0}^{k-1}[u_\varkappa, v_{k-\varkappa}],\ 1\leqslant k\leqslant k_0;\\ \partial u_{\mu p}/\partial t+\sum_{s=0}^{p_\mu-p}[V_{\mu s}, u_{\mu, p+s}]=0;\\ \partial v_{\nu q}/\partial x+\sum_{s=0}^{q_\nu-q}[U_{\nu s}, v_{\nu, q+s}]=0,\end{array}\right\} \tag{160}$$

where $V_{\nu s}$ is equal to the value of the matrix

$$\frac{1}{s!}\frac{\partial^s}{\partial\zeta^s}(u-\zeta^{k_0+1}\Lambda)$$

at the point $\zeta=\zeta_\nu'$, and $V_{\mu s}$ is equal to the value of the matrix

$$\frac{1}{s!}\frac{\partial^s}{\partial\zeta^s}\left(\sum_{m=0}^{n}A_m\eta^{n-m}+v\right)$$

at the point $\zeta=\zeta_\mu$.

The system (160) has a number of remarkable properties. First of all, the matrices A_m defined by (156)–(158) are such that their elements are polynomials of the elements of the matrices u_k and their derivatives with respect to x of corresponding order. Thus, the system (160) is a system of differential equations. Further, the system (160) has r_0 infinite series of conservation laws of the form (55). Finally, the approach considered in this section can be extended to the case of an arbitrary number of spatial variables. The proofs of all these assertions are contained in Ref. 32.

Finally, we note that the method used here to find the equations generated by the operator relation (22) is as follows. For the case considered here, we transformed the relation (22) to the form (27), and then considered the stationary equation (34) obtained from (27) after neglect of the term $\partial u/\partial t$, and found the exact solution A of Eq. (34). Under definite additional restrictions imposed on the operator L, we then constructed the asymptotic expansion

$$A \sim \sum_{m=0}^{\infty} A_m\eta^{-m} \tag{*}$$

of the previously obtained exact solution A of Eq. (34). Without additional restrictions on the operator L, the series (*) is a formal solution of Eq. (34). However, all the remarkable properties of the terms of the series (*) remain after the additional restrictions have been dropped. Finally, the partial sums of the form

$$\sum_{m=0}^{n} A_m\eta^{n-m}=\eta^n\sum_{m=0}^{n}A_m\eta^{-m}$$

of the series (*) give all (up to equivalence) operators $\mathscr{A}$ and $\mathscr{B}=\Lambda^{-1}[\Lambda, \mathscr{A}]$ that depend polynomially on the parameter η and for which the relation (22) is transformed into a nonlinear evolution equation of the form (33). It can be shown that this program can be realized not only in the case of first-order operators but also in the case of operators of arbitrary order. The central point here is the study of the stationary relation

$$[A, L]=B(L-\eta),$$

obtained from (22) after the term $\partial L/\partial t$ has been dropped and the construction of a formal solution of the form

$$A \sim \sum_{m=0}^{\infty} A_m\eta^{-m}; \quad B \sim \sum_{m=0}^{\infty} B_m\eta^{-m}$$

for this relation. Then the operators

$$\mathcal{A}=\sum_{m=0}^{n} A_m\eta^{n-m}; \quad \mathscr{B}=\sum_{m=0}^{n} B_m\eta^{n-m}$$

give all (up to equivalence) pairs of operators that depend polynomially on η and for which (22) is transformed into a system of nonlinear evolution equations. This program is fully implemented in Refs. 22 and 23. The same program can also be implemented in some other cases.

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