Microscopic theory of the nucleus in the framework of restricted dynamics

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A new approach to the microscopic theory of the nucleus is developed on the basis of special expansions of the Hamiltonian in the framework of restricted dynamics. A microscopic collective model of the nucleus is formulated and used to elucidate the meaning of the phenomenological rotational–vibrational model. The new zeroth-order Hamiltonian, which is defined as the exactly solvable part of an arbitrary Hamiltonian of a nucleus, is described in its general features.

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INTRODUCTION

The present paper presents a new approach to the microscopic theory of the nucleus based on some simple principles formulated in Ref. 1. The point of departure of the theory is the idea of "restriction of operators." To construct a microscopic theory of the nucleus, we shall use appropriate spaces and operators acting on these spaces, an important circumstance being the fact that the definition of an operator depends on the space on which it acts. The theory is developed in terms of a well-defined many-particle Hilbert space $\mathscr{R}$ and Hamiltonians H acting on $\mathscr{R}$.

For the construction of the theory, the following idea is basic: *The dependence of H on the space $\mathscr{R}$ on which it acts makes it possible to change radically the properties of H if the action of this operator is restricted to a subspace $\mathscr{R}_0$ of $\mathscr{R}$.* This as yet not very perspicuous assertion will in what follows acquire a very definite and constructive content.

The reason why it is necessary to introduce the subspace $\mathscr{R}_0$ of the space $\mathscr{R}$ and restrict the operators is to be found in the extreme complexity of the solution to the many-body problem in nuclear theory. When one is constructing a suitable theory, it is necessary to have at one's disposal a formalism that permits one to simplify the problem to such an extent that it becomes solvable in practice. At the same time, it is desirable that each Hamiltonian H should have a corresponding simplified problem. On the other hand, this procedure must to a certain degree smooth the specific properties of the original Hamiltonian H such as the details of the nucleon-nucleon interaction potential, etc. Without this, one cannot hope to create a sufficiently universal theory that is not too sensitive to the detailed form of the original Hamiltonian of the nucleus.

Since a direct way of separating a model Hamiltonian from H is not known, we use an indirect approach based on the properties of the subspace $\mathscr{R}_0$. A radical change of the properties of the Hamiltonian H, whose action is extended only to $\mathscr{R}_0$, makes it possible to separate from H the model Hamiltonian H_0. Different ways of restricting H lead to different H_0, and therefore the mechanism of restricting H gives a method for constructing different models of the nucleus. The only problem resides in the choice of the subspace $\mathscr{R}_0$. We shall see that a particular choice of $\mathscr{R}_0$ can be suggested by the form of the integrals of the motion of the original Hamiltonian H, by means of which one can construct the simplest, kinematically correct basis in $\mathscr{R}$. In this sense, therefore, the properties of the Hamiltonian H determine the subspaces $\mathscr{R}_0$ and, thus, the Hamiltonian H_0 as well. In other words, by means of definite rules H generates the simplest, smoothed Hamiltonian H_0.

Suppose we are given an arbitrary microscopic nuclear Hamiltonian H which acts on the many-particle Hilbert space $\mathscr{R}$. Then the microscopic theory is constructed in the framework of restricted dynamics in accordance with the following rules (Ref. 1).

1. It is necessary to study all the exact integrals of the motion of the Hamiltonian H.

2. It is necessary to construct in $\mathscr{R}$ the simplest basis that satisfies all the kinematic requirements that follow from the symmetry properties of H.

3. Choosing some characteristics of the simplest basis, one must fix the subspace $\mathscr{R}_0$ in $\mathscr{R}$, and then restricting H to $\mathscr{R}_0$, separate H_0 from H.

4. The Schrödinger equation for H_0 must be solved.

The picture of the structure of the nucleus described by the eigenfunctions Ψ_0 of H_0 will be simplified to the same extent to which H_0 is simplified compared with H. The Hamiltonian H_0 will reflect only some of the features of the Hamiltonian H that generated it, and therefore Ψ_0 will contain only some properties of the true states of the nucleus. We shall therefore say that Ψ_0 provides a model of the nucleus, the model depending on H and $\mathscr{R}_0$. The theory based on such a model is then perfected by taking one further step.

5. With Ψ_0 used as a function of the zeroth approximation, the most important terms of the residual interaction $\overline{H} = H - H_0$ are taken into account.

The first two rules concern the kinematic aspects of the theory, and therefore this part of the program can logically be called the *kinematic basis of the microscopic theory of the nucleus*. These questions are considered in Refs. 2–3.

The present review is devoted to the third and fourth parts of the program, in which we are concerned with the *dynamical foundations of the microscopic theory of*

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the nucleus. In particular, considerable attention is devoted to studying the collective degrees of freedom of the nucleus. Recently, significant progress has been achieved in understanding at the microscopic level the phenomenological theory proposed and developed by Bohr, Mottelson, and Rainwater. The new microscopic approach to the study of collective effects was stimulated by the ideas used in the rotational-vibrational model of the nucleus, although they do not follow directly from them. It will be shown that the dynamical equation describing the collective effects is none other than the Schrödinger equation for a Hamiltonian H_0 separated in a definite manner from H.

The microscopic theory described in the present review has a more general nature and can be presented in a form such that the applicability of its conclusions for describing collective degrees of freedom is revealed only in the final stage. Despite this possibility of axiomatic construction of the theory, a more heuristic approach has been used to simplify the present exposition. From the start we shall use concepts such as collective Hamiltonian, collective variables, and so forth. The reader should not confuse them with the analogous but by no means equivalent concepts used in phenomenological theory. Subsequently, when the connection is established at the appropriate place between the microscopic and the phenomenological approaches, the patient reader will see the different meanings attached to these concepts and the extent to which the phenomenological theory has been absorbed by the microscopic theory.

Such are the general features of the basic ideas of the present theory. We now turn to a more detailed exposition of the material. A demonstration of the computational possibilities of the theory would require a considerable expansion of the review, and we must therefore restrict ourselves to the presentation of concrete results. Additional information can be found in Refs. 1–4, and also in other papers, to which references are given at the appropriate points in the text.

1. PRELIMINARY COMMENTS

The basic ideas of the overwhelming majority of nuclear models are taken from the theory of other many-particle quantum systems. For example, the shell picture is based on the assumption that there exists in the nucleus a single-particle field similar to the field in which atomic electrons move. However, this assumption is not obvious. One can point out the significant difference between the physical conditions in atoms and nuclei, which arises not only because these systems consist of particles whose interaction laws are so different, but also for geometrical reasons. The main part of the single-particle field for atomic electrons is produced by a central heavy particle. Of course, such a situation does not obtain in nuclei, in which the nucleons themselves generate a self-consistent but not necessarily single-particle field. Why must it be of the same type as the self-consistent single-particle field for atomic electrons? Does the shell picture correspond to physical reality or are we only accustomed to it?

In nuclear theory, pairing properties have been taken over from the phenomenon of superconductivity. Now the tendency for paired particles to form in Fermi systems is due, among other factors, to the infinite extension of a system, and in a real nucleus this condition is satisfied only approximately.

The most typical effect deduced from pairing properties is the characteristic gap in the spectrum of low-lying excitations. However, this effect is not observed in all regions of mass numbers, and one cannot therefore assert that in the theory of the nucleus the pairing properties are as important as in the theory of superconductivity.

Collective effects emphasize another aspect of many-sided nuclear theory. The original assumptions of the rotational-vibrational model, which is based on a picture of small surface vibrations in an incompressible liquid, are so classical and contradict the shell ideas that this model loses many quantum features such as the Pauli principle, the possibility of deriving a collective potential energy from a microscopic Hamiltonian of the nucleus, and so forth.

These critical comments concerning well-known models of the nucleus are not given at all to reduce their importance. The remarkable results obtained by means of these models in the interpretation of different properties of nuclei have been worthily treated and presented in many books. It is only important to emphasize that most models are based on ideas taken from theories of other quantum systems, and it must be remembered that they cannot always be transferred uncritically to nuclear theory.

It is necessary to elucidate a common unacceptable feature of these models. Despite the very different initial conceptions (single-particle field, pairing property, or semiclassical collective degrees of freedom), a common negative feature of them is violation of the integrals of the motion. The various types of violation (spurious states, states with approximate value of the total angular momentum of the nucleus or even with approximate number of particles, neglect of the Pauli principle, etc.) have appeared as an undesirable burden from theories of other many-particle quantum systems.

The great importance of the integrals of the motion is now widely recognized in relativistic and nonrelativistic quantum mechanics. In the construction of theories in different fields of physics they are taken into account as a first necessary condition. Why is their significance so frequently ignored in nuclear theory? The opinion is sometimes advanced that it is not worth bothering especially about the integrals of the motion at the start because they can be recovered in the final state of the calculations by means of the projection technique. But this approach, besides technical difficulties, hides a different and more pernicious danger. Allowance for the integrals of the motion can significantly change the physical picture taken initially as the basis of a particular model. A clear example of this situation is encountered in the nuclear shell model, in which the shell

concepts contradict the kinematic requirements that follow from the translational invariance of the nuclear Hamiltonian.[2,3]

In each model, the kinematic requirements are violated in different ways. This may be one of the reasons for their manifest dissonance. The concept of collective motion, for example, strongly contradicts the picture of independent particles. On the one hand, there are too many models of the nucleus for one to be able to believe all of them, but, on the other, one has the feeling that the majority of them have managed to encompass something important. There has long been a need to reconcile these models, and there is hope that the kinematic requirements will be very helpful in implementing this difficult task. Indeed, why should one not take into account the integrals of the motion as the first step in nuclear theory? What physical picture shall we then perceive?

We shall now, in Sec. 2, begin with a survey of the general ideas which can be taken as the basis for creating a kinematically correct microscopic theory of the nucleus constructed in the framework of restricted dynamics. These ideas were put forward in Refs. 1 and 5.

2. GENERAL FORMULATION OF THE METHOD

We have argued above that violation of the symmetry principles must be regarded as a serious shortcoming of a theory. For this reason, we shall from the very beginning construct a kinematically correct theory that at no stage violates the exact integrals of the motion deduced from the symmetry properties of the nuclear Hamiltonian. We shall also eschew analogies between nuclear theory and the theories of other quantum systems. We shall develop a theory adapted to the description of specific properties of the nucleus, which is a compact system of interacting particles well localized in space. Two variants of the theory are possible. The first uses the isospin formalism, and we shall then speak of the nucleus as consisting of nucleons, while the second does not use the isospin formalism, and we shall then speak of a proton-neutron nucleus. In the present review, we shall consider only the first variant of the theory, but all our results can be modified to the associated proton-neutron system.

We now describe the general method of constructing a microscopic theory of the nucleus in the framework of restricted dynamics.[1,5] It is well known that the nuclear Hamiltonian H conserves well the spatial parity π, the total angular momentum and its projection JM_J, the antisymmetry a of the nuclear wave function, and, finally, the number of protons and neutrons, i.e., M_T, the isospin projection, and n, the total number of nucleons. The energy is not included in this list of the set of integrals of the motion. For what follows, it is convenient to denote the set of these integrals of the motion by the single letter $\Lambda \equiv \{\pi J M_J M_T n a\}$.

We begin the construction of the theory with an appropriate realization of the Hilbert space $\mathscr{R}$. Let $\mathscr{R}$ be decomposed into a direct sum of subspaces $\mathscr{R}^{(\Lambda)}$, i.e., suppose

$$\mathscr{R} = \mathscr{R}^{(\Lambda_1)} \dotplus \mathscr{R}^{(\Lambda_2)} \dotplus \ldots, \tag{1}$$

where the indices of Λ denote certain fixed values of the integrals of the motion.

We have resolved not to violate the integrals of the motion. We shall therefore develop the theory only within a subspace $\mathscr{R}^{(\Lambda)}$ with a definite value of Λ; as a rule, this condition is violated in the models discussed in the previous section. The translationally invariant nuclear Hamiltonian H acts within $\mathscr{R}^{(\Lambda)}$, i.e.,

$$\langle \Lambda\Gamma | H | \Lambda'\Gamma' \rangle = \delta(\Lambda\Lambda') \langle \Gamma | H^{(\Lambda)} | \Gamma' \rangle, \tag{2}$$

where Γ denotes a certain basis in $\mathscr{R}^{(\Lambda)}$.[1)] As yet, $H^{(\Lambda)}$ is an arbitrary Hamiltonian, and its action leads to definite superpositions of the randomly chosen basis $\Psi(\Delta\Gamma)$ in $\mathscr{R}^{(\Lambda)}$, and, of course, the best way to fix this basis is to solve the Schrödinger equation for H; in this case, Γ would be the energy. However, it is in practice impossible to solve the Schrödinger equation exactly for the many-particle nucleus, and we must therefore seek other possibilities.

The dimension of the space $\mathscr{R}^{(\Lambda)}$ is very large. It is helpful to decompose $\mathscr{R}^{(\Lambda)}$ into subspaces with fewer dimensions. With this aim, we seek in the set of Hamiltonians H that act within $\mathscr{R}^{(\Lambda)}$ a simpler Hamiltonian (denoted by h) which possesses additional symmetry properties and, therefore, additional integrals of the motion. We use one or several of these integrals of the motion (we denote them by K) for additional fixing of the basis in $\mathscr{R}^{(\Lambda)}$. We have

$$\mathscr{R}^{(\Lambda)} = \mathscr{R}^{(\Lambda K_1)} \dotplus \mathscr{R}^{(\Lambda K_2)} \dotplus \ldots. \tag{3}$$

In the new basis, the matrix of the operator H takes the form

$$\langle \Lambda K\Gamma | H | \Lambda' K'\Gamma' \rangle = \delta(\Lambda\Lambda') \langle K\Gamma | H^{(\Lambda)} | K'\Gamma' \rangle, \tag{4}$$

where Γ now denotes some basis in $\mathscr{R}^{(\Lambda K)}$.

The decomposition (4) gives a many-particle Hilbert space suitable for our purposes. This decomposition is implemented by means of two Hamiltonians: the arbitrary "large" Hamiltonian H and the "little" Hamiltonian h. To be specific, we shall assume that H has the form

$$H = H_{\text{kin}} + H_{\text{Coul}} + H_c + H_v + H_t, \tag{5}$$

where the first two terms are, respectively, the operators of the kinetic energy and the Coulomb energy. The three last terms denote the central, vector, and tensor forces with all exhange and arbitrary potentials of the nucleon-nucleon interaction. As yet, there is no need to discuss the actual form of h.

We now come to the most important part of this section. We separate from H a new Hamiltonian. For this, we write H in the form of two terms:

$$H = H_0 + \bar{H}, \tag{6}$$

where H_0 and $\bar{H}$ conserve the integrals of the motion Λ. We define H_0 as all that part of H for which the condition

$$\langle K\Gamma | H_0^{(\Lambda)} | K'\Gamma' \rangle = \delta(\bar{K}\bar{K}') \langle \Gamma | H_0^{(\Lambda K)} | \Gamma' \rangle \tag{7}$$

[1)] In the present review, the Hamiltonian acts throughout within $R^{(\Lambda)}$, and we shall therefore sometimes omit the superscript Λ of H.

is satisfied for all K in $\mathscr{R}^{(\Lambda)}$. It follows from this definition that $\overline{H}$ has at least one nondiagonal element with respect to K. It is important to emphasize that the condition (7) is not equivalent to the allowance for all K-diagonal matrix elements of H, because H also has nonzero K-diagonal matrix elements.

It is noteworthy that *the operator* $H^{(\Lambda)}$ *and the space* $\mathscr{R}^{(\Lambda K)}$ *determine the new Hamiltonian* $H_0^{(\Lambda K)}$. In what follows, for brevity, we shall say that $H_0^{(\Lambda K)}$ is projected from $H^{(\Lambda)}$ by means of $\mathscr{R}^{(\Lambda K)}$ or, simply, that $H_0^{(\Lambda K)}$ is projected from $H^{(\Lambda)}$.[2)] We now write down the Schrödinger equation for $H_0^{(\Lambda K)}$:

$$H^{(\Lambda K)}\Psi_0^{(\Lambda K)} = \varepsilon_0^{(\Lambda K)}\Psi_0^{(\Lambda K)}. \tag{8}$$

We can now introduce a model of the nucleus.[1,5] We shall say that the physical picture of the nuclear structure described by the wave function $\Psi_0^{(\Lambda K)}$ is the dynamical model of the nucleus generated by the Hamiltonian H restricted to the subspace $\mathscr{R}^{(\Lambda K)}$, or, briefly, *a model of the nucleus with restricted dynamics*.

The definitions given above are very general. It is by no means trivial to ensure fulfillment of the condition (7), and one can hope for success only when the space $\mathscr{R}^{(\Lambda K)}$ is spanned by a basis with exceptionally good properties. Such a basis will be introduced in the following section. The degree of complexity of the Schrödinger equation (8) depends on the choice of the additional quantum numbers K. From the point of view of the practical possibilities of solution of Eq. (8), a particularly favorable situation arises when K is chosen such that the dimension of the space $\mathscr{R}^{(\Lambda K)}$ is finite. In this case, the operator $H_0^{(\Lambda K)}$ is represented in accordance with (7) by a Hermitian matrix of finite dimension, and the Schrödinger equation (8) can be solved exactly.

If the general method formulated in this section is to become viable in practice, it is necessary to study the kinematic and dynamical aspects of the complete problem already mentioned in the Introduction. The kinematic part concerns the detailed description of the spaces $\mathscr{R}^{(\Lambda K)}$, including also the method of constructing bases in these spaces,[2,3] and it is assumed that the reader is familiar with this material. The dynamical part begins with the separation of $H_0^{(\Lambda K)}$ from $H^{(\Lambda)}$ and the study of the general properties of the spectrum and the eigenstates of the Hamiltonian $H_0^{(\Lambda K)}$. This is a very large group of questions, and only some of them can be discussed in the present review.

3. DIFFERENT TYPES OF RESTRICTED HAMILTONIANS

In Refs. 2 and 3, the exact integrals of the motion of the Hamiltonian H are discussed and the simplest, kinematically correct realization of a basis in $\mathscr{R}^{(\Lambda)}$ is described. This realization also specifies the Hamiltonian h. It can be shown (see, for example, Refs. 2 and 3) that this Hamiltonian is the first Casimir operator of the unitary group $U_{3(n-1)}$, where n is the number of nucleons of the nucleus.

In $\mathscr{R}^{(\Lambda)}$, there exist the following two unitarily equivalent bases:

$$\Psi\left(\begin{matrix}\pi E\Omega\beta\,(LS)\,JM_J\\ TM_T\tilde{\alpha}\omega\alpha\,(\lambda\tilde{\lambda})\,a\end{matrix}\middle|\,Q',\,Q\right) \tag{9}$$

and

$$\Psi\left(\begin{matrix}\pi E\gamma\,(LS)\,JM_J\\ TM_T\tilde{\alpha}\delta\omega\alpha\,(\lambda\tilde{\lambda})\,a\end{matrix}\middle|\,Q',\,Q\right). \tag{10}$$

These functions depend on the spin-isospin variables, the set of which is denoted by Q in (9) and (10). They also depend on the $3(n-1)$ orbital variables, denoted together in (9) and (10) by Q'. The set Q' consists of either the Jacobi variables $\rho(i_i^s = 1, 2, \ldots, n-1;\ s = 1, 2, 3)$, or six collective variables ξ and $3(n-3)$ noncollective (so-called internal) variables q; the variables ξ and q are described below.

Before we explain the meaning of the quantum numbers in (9) and (10), we introduce the following notation: The letters U and O denote the unitary and orthogonal groups, and O^+ the orthogonal groups without reflection. We also require three symmetric groups: S_n, $S_n^{(r)}$, and $S_n^{(\sigma\tau)}$. The first of them acts on all the variables, the second only on the orbital variables, and the third on the spin–isospin variables. For both functions (9) and (10), E has a triple meaning and denotes simultaneously the $U_{3(n-1)}$-irreducible representation $[E0\ldots0]$, the U_{n-1}-irreducible representation $[E_1E_2E_30\ldots0]$, and the U_3-irreducible representation $[E_1E_2E_3]$, the relation $E_1+E_2+E_3=E$ holding. The parity π of both functions is positive for even E and negative for odd E. The quantum numbers L, S and T denote the orbital and spin angular momenta and isospin moment of the nucleus; the quantum numbers J, M_J, M_T, and a have already been introduced; ω denotes the O_{n-1}-irreducible representation $(\omega_1\omega_2\omega_30\ldots0)$. The permutational symmetry of the functions (9) and (10) is characterized by the $S_n^{(r)}$-irreducible representations λ (orbital Young tableau), which has the form $[4\ldots43\ldots32\ldots21\ldots1]$, i.e., contains k_1 units, k_2 doublets, k_3 triplets, and k_4 quartets, the relation $k_1+2k_2+3k_3+4k_4=n$ holding; the letter $\tilde{\lambda}$ has a double meaning and denotes either an $S^{\sigma\tau}$-irreducible representation, i.e., the Young tableau conjugate to the tableau λ ($\tilde{\lambda}$ is uniquely determined by λ), or a U_4-irreducible representation. This representation is reduced in accordance with the chain $U_4 \supset U_2 \sim U_2 \times U_2$, and the repetition index of identical S and T in $\tilde{\lambda}$ is denoted by $\tilde{\alpha}$. The index α is the repetition index for the chain $O_{n-1} \supset S_n^r$, i.e., it distinguishes identical λ contained in a given ω. Similarly, δ and γ in (10) denote the repetition indices in the chains $U_{n-1} \supset O_{n-1}$ and $U_3 \supset O_3^+$. The quantum number Ω in (9) denotes the $O_{3(n-1)}$-irreducible representation $(\Omega0\ldots0)$, and β is the repetition index for the chain $O_{3(n-1)} \supset O_3^+ \times O_{n-1}$.

The transformation between the basis (9) and (10) is achieved by means of the orthogonal matrix $M^{(E\omega L)}$ with matrix elements

$$M^{(E\omega L)}_{[L_1E_2E_3]\gamma\delta,\,\Omega\beta}. \tag{11}$$

[2)] We shall also call this procedure restriction of the operator $H^{(\Lambda)}$ to the subspace $\mathscr{R}^{(\Lambda K)}$.

This matrix is diagonal with respect to the $U_{3(n-1)}$-irreducible representation E, and also with respect to ω and L, and does not depend on $S, T, M_T, \bar{\alpha}, \tilde{\alpha}, \lambda, J, M_J$. We shall call the basis functions (9) the functions of the unitary scheme, and the functions (10) the functions of the orthogonal scheme.

Here we give a very brief description of the simplest basis in $\mathscr{R}^{(\Lambda)}$. Details, including references to the original papers in which the functions (9) and (10) are introduced, their properties are studied, methods of recursive construction of them are developed, etc., are described in Refs. 1–4.

In the construction of a microscopic theory of the nucleus in the framework of restricted dynamics we shall use bases of the unitary or orthogonal scheme possessing rich sets of quantum numbers. We shall discuss the suitability of these quantum numbers for restricting the Hamiltonian H. By the assertion in the general formulation of the method, both the bases (9) and (10) are characterized by all the exact integrals of the motion $\Lambda \equiv \{\pi J M_J M_T n a\}$. The remaining quantum numbers of the functions (9) and (10) form the set $\{K\Gamma\}$ and therefore $\{K\Gamma\} \equiv \{E\Omega\beta\omega\alpha\tilde{\lambda}\tilde{\alpha}TLS\}$ for (9) and $\{K\Gamma\} \equiv \{E\gamma\delta\omega\alpha\lambda\alpha TLS\}$ for (10).

Each of the sets $\{K\Gamma\}$ consists of indices of three types.

First, there are the repetition indices β, α, $\tilde{\alpha}$ or γ, δ, α, $\tilde{\alpha}$. They are not indices which characterize irreducible representations, and it is therefore not worth attempting to interpret them as additional integrals of the motion.

Second, there are indices of irreducible representations of groups whose rank does not depend on the number of nucleons. For the orthogonal or the unitary scheme, we have the quantum numbers S, T, L, further $\tilde{\lambda}$ (as U_4 representation) and additionally for the unitary scheme $\tilde{E}$ (as U_3 representation).

Third, there are indices of irreducible representations of groups whose rank depends on the number of nucleons, i.e., groups associated with the many-particle properties of the basis. For both schemes, we have E, ω, λ (E as $U_{3(n-1)}$ representation and λ as $S_n^{(r)}$ representation), and also Ω for the orthogonal scheme and $\tilde{E}$ (as U_{n-1} representation) for the unitary scheme.

In accordance with the general propositions of Sec. 2, as an additional integral of the motion we can choose one or several of the quantum numbers of the second or the third set. At the first glance, it would seem that there are too many possibilities, but we shall see that the majority of them must be eliminated for various reasons.

Before we discuss the actual quantum numbers in the two sets $\{K\Gamma\}$, we make a remark. All the additional integrals of the motion that will be discussed in what follows correspond to irreducible representations of continuous or discrete groups, and therefore the realization of the basis in $\mathscr{R}^{(\Lambda)}$ by means of the functions (9) or (10) is exceptionally convenient: If the condition (7) is satisfied, they make it possible to use a powerful algebraic formalism. If the additional integral of the motion K corresponds to an irreducible representation of some group, then in group-theoretical language the projection of $H^{(\Lambda K)}$ from $H^{(\Lambda)}$ amounts to the separation from $H^{(\Lambda)}$ of its scalar part with respect to transformations of this group.

We now discuss the concrete sets $\{K\Gamma\}$. We begin with the quantum numbers for the groups whose rank does not depend on the number of nucleons. The additional integral of the motion $\tilde{\lambda}$ (as U_4 representation) must be eliminated for the following reason. If we project H_0 from H, restricting H to the subspace $\mathscr{R}^{(\Lambda\tilde{\lambda})}$, i.e., we take only the U_4-scalar part of the operator H, then the eigenvalues of the Hamiltonian $H^{(\Lambda\tilde{\lambda})}$, will be degenerate with respect to S and T. However, $\mathbf{S}+\mathbf{L}=\mathbf{J}$, so that the degeneracy with respect with S will entail partial degeneracy with respect to the total angular momentum J of the nucleus. As a consequence of this, the spectrum of the Hamiltonian $H_0^{(\Lambda\tilde{\lambda})}$ will be too simple, and it will therefore not be worthwhile introducing a model of the nucleus based on this Hamiltonian. The two other possibilities based on use of the quantum numbers S and T to restrict H are also unsuitable, since it is fairly simple to take into account the nondiagonality with respect to S and T (for which it is only necessary to know the Clebsch-Gordan coefficients of the group U_4; see Ref. 1), and there is no need to use these quantum numbers as additional integrals of the motion.

From what we have said above, we conclude that the spin-isospin quantum numbers should not be used as additional integrals of the motion. There is, however, a different and deeper reason for this conclusion. The nuclear Hamiltonian depends on the spin-isospin variables through operators of the groups U_4 and $S_n^{(\sigma\tau)}$, in connection with which this dependence has a purely algebraic nature. Therefore, there is no need to restrict H to spin-isospin quantum numbers; the restriction technique is helpful essentially only to simplify the orbital part of the dynamical equations.

We now consider the possibilities associated with the quantum numbers that characterize the irreducible representations of the other groups whose rank does not depend on the number of nucleons. There remain only two groups not yet considered, namely U_3 and O_3^*. From H, we can separate either an O_3^*-scalar or U_3-scalar part. At the first glance, the latter possibility should be eliminated for the same reason as in the case of a U_3-scalar Hamiltonian; restriction to an irreducible space of the group U_3 leads to degeneracy of the energy with respect to the orbital quantum number L and, therefore, to partial degeneracy with respect to J. However, this objection is invalid because, in contrast to the U_4-scalar Hamiltonian, the degeneracy with respect to L can be lifted in this case by means of an additional term in the operator H_0.

We now discuss the quantum numbers associated with the many-particle properties of the bases (9) or (10). We take first E and Ω with E as $U_{3(n-1)}$-irreducible representation. It is obvious that the orbital part of the $U_{3(n-1)}$-scalar Hamiltonian H_0 separated from H is al-

most as simple as the Hamiltonian h and is therefore not suitable for our purposes. Somewhat more complicated, but still too simple, is the Hamiltonian H_0 separated by means of the $O_{3(n-1)}$-irreducible representation Ω. The matrix of the $U_{3(n-1)}$- or $O_{3(n-1)}$-scalar Hamiltonian has a high degree of degeneracy and, in particular, does not depend on L. This degeneracy cannot be lifted by additional terms, so that the quantum numbers of the groups $U_{3(n-1)}$ and $O_{3(n-1)}$ are not suitable for constructing a model with restricted dynamics.

We have still to consider three quantum numbers: λ (as $S_n^{(r)}$ representation), E (as U_{n-1} representation), and ω. The use of all of them as additional integrals of the motion leads to complicated Hamiltonians H_0. Projection by means of the quantum number λ gives the Hamiltonian of the supermultiplet scheme[6,7] (see also Refs. 8 and 4). In the framework of thsi scheme, numerous studies have been made of a number of properties of nuclei such as the probabilities of β and γ transitions, the magnetic moments, the general dependence of the binding energy on the supermultiplet quantum numbers, and so forth.

Nevertheless, this scheme is too general for studying a larger group of questions. As was noted in Ref. 8, it requires further particularization. This can be achieved by introducing the quantum numbers E and ω characterizing the U_{n-1}- and O_{n-1}-irreducible properties of the basis functions (9) and (10). The quantum numbers E and ω are the most important additional integrals of the motion for constructing a microscopic theory of the nucleus in the framework of restricted dynamics.

Let us summarize. Besides the well-known supermultiplet scheme, which in many respects is too general, a constructive and nontrivial Hamiltonian H_0 can be expected only in the case of restriction of H to O_3^+-, U_3-, O_{n-1}-, and U_{n-1}-irreducible spaces. It is obvious that the most complicated Hamiltonian H_0 has the least symmetry, and we therefore obtain the most complicated model for projecting H_0 from H by means of the additional integrals of the motion ω and L. In this case, the Hamiltonian H_0 consists of three terms: H_1', which is an O_3^+ scalar but not an O_{n-1} scalar; H_3', which is an O_{n-1} scalar, but not an O_3^+ scalar; and finally, the term H_2', which is scalar with respect to both groups O_3^+ and O_{n-1}. For this Hamiltonian, we introduce the notation

$$H_0' = H_1' + H_2' + H_3'. \tag{12}$$

A second possibility is associated with the unitary groups U_3 and U_{n-1}. As before, we introduce the Hamiltonian H_0'', which consists of three terms: H_1'', which is a U_{n-1} scalar but not a U_{n-1} scalar; H_3'', which is a U_{n-1} scalar but not a U_3 scalar; and the term H_2'', which is a scalar with respect to both groups U_3 and U_{n-1}. We introduce the notation

$$H_0'' = H_1'' + H_2'' + H_3''. \tag{13}$$

If we need to combine the two terms in (12) or (13), we shall write $H_{12}'' = H_1'' + H_2''$, etc. Note also that the Hamiltonian H can be restricted to irreducible spaces of the groups U_3 and O_{n-1} or the groups O_3^+ and U_{n-1}.

We conclude from what has been said above that in the framework of the method of constructing models discussed in the Introduction we obtain, besides the Hamiltonian of the supermultiplet scheme, two further suitable Hamiltonians H_0' and H_0''; these operators are particularized by the realization of $\mathscr{R}^{\Lambda}$ by means of the kinematically correct bases (9) or (10). Now, in accordance with the fourth part of the program, we turn to a more detailed study of these Hamiltonians and the properties of their eigenstates. For these purposes, we use an appropriate mathematical formalism, to the description of which we now turn.

4. GENERAL FORM OF THE OPERATORS AND REPLACEMENT OF THE VARIABLES

The restricted volume of the present paper does not enable us to discuss in detail the properties of all the terms of the Hamiltonians (12) and (13), and we therefore concentrate our main attention on the operator $H_{23}' = H_2' + H_3'$, which, as we shall see in what follows, has the most direct bearing on the study of the collective degrees of freedom of the nucleus. By definition, this operator is the O_{n-1}-scalar part of the nuclear Hamiltonian H. Our immediate aim is to elucidate the physical meaning of the as yet abstractly defined Hamiltonian H_{23}'. The result is very simple; H_{23}' is the entire collective part of the original Hamiltonian H. In what follows, we shall also obtain an expansion of H (and even a more general expansion of arbitrary operators used in nuclear theory) in a series whose first terms represent the O_{n-1}-scalar (i.e., collective) term, the remaining members of the series giving terms that depend on both the collective and the noncollective (internal) nuclear variables. When the Hilbert space $\mathscr{R}^{(\Lambda)}$ realized on the basis functions (9) or (10) is used, such an expansion is natural for studying collective effects and effects due to the interaction between the collective and internal degrees of freedom of the nucleus.

The problem posed above can be solved in two ways, the first of which is based exclusively on matrix representation of the operators with subsequent expansion of them in an O_{n-1}-irreducible series. This representation is optimal in many respects. However, the matrix expansion is not so perspicuous as another expansion, which is based partly on a matrix and partly on a coordinate representation.

Therefore, we shall consider below only the second method of O_{n-1}-irreducible expansion of the arbitrary operators encountered in nuclear theory. This method was proposed in Ref. 9.

It is not easy to obtain an O_{n-1}-irreducible expansion, and therefore, to make the following exposition more comprehensible, we shall note the main stages.

We first discuss the general properties and structure of the operators to be expanded. We then describe briefly the formulas for the transition to the new, collective and internal, variables, including formulas for the differential operators.

The next step in obtaining the O_{n-1}-irreducible expansion of the operators involves a convenient realization of the Hilbert space with respect to which the transition from the coordinate to the matrix representation is made. We then elucidate the meaning of the O_{n-1}-scalar Hamiltonian H'_{23}, study the matrix of this Hamiltonian, construct a complete basis, which depends on the collective variables, and, finally, separate the collective part from H.

We begin by considering the general form of the operators encountered in nuclear theory. Following Ref. 9, we study translationally invariant operators of the form

$$\hat{\mathcal{O}}^{\varkappa'\varkappa''}_{\mu'\mu''} = \sum_{i=1}^{n} U^{\varkappa'}_{\mu'}(\sigma_i\sigma_j\tau_i\tau_j)\,\hat{W}^{\varkappa''}_{\mu''}(\mathbf{r}_i-\mathbf{r}_j). \tag{14}$$

Here, σ_i and τ_i denote the spin and isospin variables of nucleon i; $\hat{U}^{\varkappa'}_{\mu'}$ is an arbitrary two-particle spin-isospin operator symmetric under transposition of the indices i and j which transforms in accordance with the O_3^+-irreducible representation $\varkappa'$, whose basis we denote by μ'. Similarly, $\hat{W}^{\alpha''}_{\mu''}$ is an arbitrary operator symmetric in i and j that depends on the distance between the nucleons and transforms in accordance with the O_3^+-irreducible representation $\varkappa''$ with basis μ''. Each term of the Hamiltonian (5) can be written in the form (see, for example, (8.8)–(8.12) in Ref. 1)

$$\sum_{\mu}(-1)^{\mu}\hat{\mathcal{O}}^{\varkappa\varkappa}_{\mu-\mu} = \sum_{i>j=1}^{n}\sum_{\mu}(-1)^{\mu}\hat{U}^{\varkappa}_{\mu}(\sigma_i\sigma_j\tau_i\tau_j)\,\hat{W}^{\varkappa}_{-\mu}(\mathbf{r}_i-\mathbf{r}_j). \tag{15}$$

Together with (14), we consider operators of single-particle type. As a rule, they are expressed in terms of the single-nucleon variables $\mathbf{r}_i$ and, therefore, are not in general translationally invariant. To emphasize explicitly this property, we introduce the variables $\dot{\mathbf{r}}_i = \mathbf{r}_i - \mathbf{R}$, which are referred to the center-of-mass vector $\mathbf{R}$, where $n\mathbf{R} = \mathbf{r}_1 + \cdots + \mathbf{r}_n$. In the variables $\dot{\mathbf{r}}_i$, the operators of single-particle type have the general form [see, for example, (8.13) in Ref. 1]

$$\hat{\mathcal{O}}^{\varkappa'\varkappa''}_{\mu'\mu''}(\mathbf{R}) = \sum_{i=1}^{n}\hat{U}^{\varkappa'}_{\mu'}(\sigma_i\tau_i)\,\hat{W}^{\varkappa''}_{\mu''}(\dot{\mathbf{r}}+\mathbf{R}), \tag{16}$$

where the introduced notation is analogous to the notation in (15). We use the circumstance that on a basis of antisymmetric functions

$$\langle\Gamma|\hat{\mathcal{O}}^{\varkappa'\varkappa''}_{\mu'\mu''}|\Gamma'\rangle = n\langle\Gamma|\hat{U}^{\varkappa'}_{\mu'}(\sigma_n\tau_n)\,\hat{W}^{\varkappa''}_{\mu''}(\dot{\mathbf{r}}+\mathbf{R})|\Gamma'\rangle \tag{17}$$

for operators of single-particle type, and,

$$\langle\Gamma|\hat{O}^{\varkappa'\varkappa''}_{\mu'\mu''}|\Gamma'\rangle = [n(n-1)/2\,\langle\Gamma|\hat{U}^{\varkappa'}_{\mu'}(\sigma_{n-1}\tau_{n-1}\sigma_n\tau_n)\,\hat{W}^{\varkappa''}_{\mu''}(\mathbf{r}_{n-1}-\mathbf{r}_n)|\Gamma'\rangle] \tag{18}$$

for operators of two-particle type. The last two formulas show that the problem of O_{n-1}-irreducible expansion of many-particle operators reduces to the expansion of either the single-particle operator $\hat{W}^{\varkappa''}_{\mu''}(\dot{\mathbf{r}}+\mathbf{R})$, or the two-particle operator $\hat{W}^{\varkappa''}_{\mu''}(\mathbf{r}_{n-1}-\mathbf{r}_n)$. We also recall that in the standard definition of the Jacobi vectors ρ_t [see, for example, (16.5) in Ref. 4],

$$\left.\begin{aligned}\dot{\mathbf{r}}_n &= -\sqrt{(n-1)/n}\,\boldsymbol{\rho}_{n-1};\\ \mathbf{r}_{n-1}-\mathbf{r}_n &\equiv \sqrt{2}\,\boldsymbol{\rho}_n = \sqrt{n/(n-1)}\,\boldsymbol{\rho}_{n-1} - \sqrt{(n-2)/(n-1)}\,\boldsymbol{\rho}_{n-2},\end{aligned}\right\} \tag{19}$$

and, hence, both operators $\hat{W}$ have the form

$$\hat{W}^{\varkappa''}_{\mu''}(c_t\boldsymbol{\rho}_t - c'_t\mathbf{R}), \tag{20}$$

where the index t takes the two values $n-1$ and a; $\sqrt{n}c_t = -\sqrt{n-1}$, $c'_t = 1$ for single-particle and $c'_t = \sqrt{2}$, $c' = 0$ for central two-particle operators. For simplicity, we shall simply write $\hat{W}(\rho_t)$ instead of the expression (20) in what follows.

We must now implement the change of variables in the operator (20). For this, we use the explicit expression for the Jacobi variables in terms of the new variables. Instead of the three-dimensional Jacobi vectors, it is convenient to consider the r_0-dimensional vectors ρ_t. This generalization makes it possible to trace the symmetry between the r_0-dimensional space in which the particles move the r-dimensional space whose dimension is associated with the total number of particles. It is also helpful when one is considering methodological problems such as the two-dimensional motion of particles, etc. In the cases when it is necessary to go over from the general expressions to $n-1$ quasiparticles moving in three-dimensional space, we shall without further explanation assume that $r_0 = 3$ and $r = n-1$. We denote the components of the vectors ρ_t^s by ρ_t^s, where $i = 1, 2, \ldots, r$ and $s = 1, 2, \ldots, r_0$. In what follows, we shall also use the special index s_0, which takes the values $s_0 = 1, 2, \ldots, r_0$ for $r_0 \leqslant r$ and $s_0 = 1 + r_0 - r, 2 + r_0 - r, \ldots, r_0$ for $r_0 \geqslant r$.

We introduce the operator $\hat{T}$ of rotation of a-dimensional space. We denote the matrix of the rotation operator in the plane $(p-1)p$ $(p = 2, 3, \ldots, a)$, which depends on the parameter $\vartheta_p^{(t)}$, by $T^t_{(p-1)p}(\vartheta_p^t)$. Besides the rotation operators, we also require the reflection operator $\hat{\sigma}$. We denote the elements of the reflection group of r-dimensional space by σ_r (for details on the operators $\hat{T}$ and $\hat{\sigma}$, see, for example, Ref. 2 and 3). We also introduce the following product of rotation matrices:

$$\begin{aligned}D(a, b) = \coprod_{k=0}^{a-2} T^{(a-b+1)}_{a-b-k,\,a-b+1-k}(\vartheta^{(a-b+1)}_{a-b+1-k})\\ \times T^{(a-b+2)}_{a-b+1-k,\,a-b+2-k}(\vartheta^{(a-b+2)}_{a-b+2-k}) \cdots T^{(a)}_{a-1-k,\,a-k}(\vartheta^{(a)}_{a-k}),\end{aligned} \tag{21}$$

in which it is assumed that the factors with smaller k are on the left and only matrices $T(p-1)p$ with $p > 1$ are taken. The variables ϑ_p take the values $0 \leqslant \vartheta_p < \pi$ for $p > 2$ and $0 \leqslant \vartheta_2 < 2\pi$ for $p = 2$. We define the r-dimensional matrix $D^{(1r)}(q^t)$, where

$$D^{(1r)}(q^+) = \begin{cases} D(r, r_0) & \text{for } r_0 < r;\\ D(r, r-1) & \text{for } r_0 \geqslant r.\end{cases} \tag{22}$$

Similarly, we define the r_0-dimensional matrix $D^{(1r_0)}(G^+)$, which is given by the expression (22) if the indices r_0 and r in it are interchanged. In these expressions, q^+ and G^+ denote, respectively, the sets of $(1/2)r_0(r_0-1)$ and $(1/2)r_0(2r-r_0-1)$ continuous variables ϑ for $r_0 \leqslant r$ and $(1/2)r(2r_0-r-1)$ and $(1/2)r(r-1)$ continuous variables ϑ for $r_0 \geqslant r$. In group theory, q^+ are called the variables defined on the factor space $O^+_{r-\tilde{r}_0}/O^+_r$, where $\tilde{r} = \min(r_0, r)$. The variables G^+ have a similar meaning (for details see, for example, Refs. 2 and 3).

We augment the operation of rotation by the operation of reflection and introduce an r-dimensional matrix with reflection:

$$D^{(1r)}(\sigma q^+) = \sigma_r D^{(1r)}(q^+), \tag{23}$$

where σ_r is either the r-dimensional unit matrix [and then we shall write $\sigma = 0$ in the argument of the matrix

on the left-hand side of the expression (23)], or an r-dimensional reflection matrix, i.e., a matrix that differs from the r-dimensional unit matrix only in the sign of the last diagonal element [and then we shall set $\sigma=1$ on the left-hand side of (23)].

The cases of single-particle and central two-particle operators can be encompassed by introducing the following transformation of the Jacobi vectors [see (16.9) in Ref. 4]:

$$|\rho_1 \dots \rho_{r-2}\rho_s\rho_a| = |\rho_1 \dots \rho_{r-2}\rho_{r-1}\rho_r| \begin{vmatrix} e_{r-2} & 0 & 0 \\ 0 & \sqrt{(r+1)/2r} & -\sqrt{(r-1)/2r} \\ 0 & \sqrt{(r-1)/2r} & \sqrt{(r+1)/2r} \end{vmatrix}, \quad (24)$$

where e_{r-2} is the $(r-2)$-dimensional unit matrix. For $r=n-1$, we obtain from (24) the vector ρ_a. Of course, the transformation (24) belongs to the group O_r. We denote the matrix of this transformation by g_0 and introduce

$$D^{(1r)}(\sigma q^+ g) = D^{(1r)}(\sigma q^+)\, g_0. \quad (25)$$

In what follows, it will be convenient to assume that g on the left-hand side of the expression (25) takes two values, namely, that g is either the matrix in (24) (and we shall then write $g=g_0$) or the r-dimensional unit matrix (and we shall then set $g=e$).

The matrix elements of the matrices $D^{(1r_0)}(G^+)$ and $D^{(1r)}(\sigma q^+ g)$ will be used as functions by means of which we implement the change of variables. The expression for ρ_i^s in terms of the new variables has the form

$$\rho_i^s = \sum_{s_0} \rho^{(s_0)} D^{1r_0}_{s_0 s}(G^+)\, D^{(1r)}_{r-r_0+s_0,\, i}(\sigma^+ q^+ g), \quad (26)$$

where $\rho^{(s_0)}$ are variables of radial type taking values in the interval $0 \leqslant \rho^{(s_0)} < +\infty$. For $r_0=3$, $\sigma=0$, and $g=e$ this expression, written in a somewhat different form, was proposed in Refs. 10 and 11 (see also Ref. 12). The variables $\rho^{(s_0)}$, G^+, and q^+ were used in implicit form in Ref. 13. The discrete variables specifying reflection were introduced in Ref. 14.

In general, the operator (20) also depends on the derivatives with respect to ρ_i^s. For the replacement of the variables in the differential operators, it is necessary to know the expression for the derivatives with respect to ρ_i^s in the new variables. For $r_0=3$, it was obtained in Ref. 15. Generalized to the case of arbitrary r_0, this expression has the form[1]

$$\begin{aligned}\frac{\partial}{\partial\rho_i^s} = {} & \sum_{s_0'} D^{(1r_0)}_{s_0 s} D^{(1r)}_{r-r_0+s_0',\, i} \frac{\partial}{\partial\rho^{(s_0')}} - \sum_{s'}\sum_{i'} D^{(1r_0)}_{s's} D^{(1r_0)}_{s's} D^{(1r)}_{i'r}\, \hat{\mathcal{R}}(s'i') \\ & - \sum_{s_0'>s_0''} \Big[\frac{\rho^{(s_0')}}{(\rho^{(s_0')})^2-(\rho^{(s_0'')})^2} D^{(1r_0)}_{s_0' s} D^{(1r)}_{r-r_0+s_0',\, i} \\ & + \frac{\rho^{(s_0')}}{(\rho^{(s_0')})^2-(\rho^{(s_0'')})^2} D^{(1r_0)}_{s_0'' s} D^{(1r)}_{r-r_0+s_0',\, i} \Big] \hat{\mathcal{J}}_{r-r_0+s_0',\, r-r_0+s} \\ & - \sum_{s_0'>s_0''} \Big[\frac{\rho^{(s_0')}}{(\rho^{(s_0)})^2-(\rho^{(s_0'')})^2} D^{(1r_0)}_{s_0' s} D^{(1r)}_{r-r_0+s_0',\, i} \\ & + \frac{\rho^{(s_0'')}}{(\rho^{(s_0')})^2-(\rho^{(s_0'')})^2} D^{(1r_0)}_{s_0'' s} D^{(1r)}_{r-r_0+s_0'',\, i} \Big] \hat{\mathcal{L}}_{s_0' s_0''}, \end{aligned} \quad (27)$$

where s_0' and s_0'' take the same values as s_0 in (26). In (27), the matrix elements of $D^{(1r_0)}$ and $D^{(1r)}$ depend, respectively, on G^+ and $\sigma q^+ g$. If $r>r_0$, then in the second term of the expression (27) the summation indices take the values $s'=1,2,\dots,r_0$ and $i'=1,2,\dots,r-r_0$, and

$$\hat{\mathcal{R}}(s'i') = (1/\rho^{(s')})\, \hat{\mathcal{J}}_{r-r_0+s',\, i}. \quad (28)$$

If $r>r_0$, then $s'=1,2,\dots,r_0-r, i'=1,2,\dots,r$ and

$$\hat{\mathcal{R}}(s'i') = (1/\rho^{(r_0-r+i')})\, \hat{\mathcal{L}}_{r_0-r+i',\, s'}. \quad (29)$$

For $r=r_0$,

$$\hat{\mathcal{R}}(s'i') \equiv 0. \quad (30)$$

In (27), $\hat{\mathcal{J}}$ and $\hat{\mathcal{L}}$ are the operators of left shift for the groups $O^+_{r_0}$ and O^+_r (Refs. 2 and 3). To give their definition, it is convenient to modify somewhat the notation for the angular variables ϑ. We relabel the set of variables in the order in which they occur in the product (21) as follows:

$$\begin{aligned} & \vartheta_n^{(a-b+1)}\vartheta_a^{(a-b+2)} \dots \vartheta_a^{(a)},\ \vartheta_{a-1}^{(a-b)}\vartheta_{a-1}^{(a-b+1)} \\ & \dots \vartheta_{a-1}^{(a-1)}, \dots, \vartheta_4^{(2)}\vartheta_4^{(3)}\vartheta_4^{(4)},\ \vartheta_3^{(2)}\vartheta_3^{(3)},\ \vartheta_2^{(2)}, \end{aligned} \quad (31)$$

or, briefly, $\vartheta_t^{(\varkappa)}$, where $t=a, a-1, \dots, 2$ and $\varkappa=t-b+1, t-b+2, \dots, t (\varkappa>1)$. Let $q_\varkappa^+$ be all the variables in (31) to the left of $\vartheta_t^{(\varkappa)}$, and $\bar{q}_t^+$ denote the variable $\vartheta_t^{(\varkappa)}$ and all the variables to the right of $\vartheta_t^{(\varkappa)}$. Then, by definition, the operators $\hat{\mathcal{J}}_{jj'}$ (and, similarly, the operators $\hat{\mathcal{L}}_{jj'}$) can be found from the system of equations

$$\frac{\partial}{\partial\vartheta_t^{(\varkappa)}} = \sum_{j>j'} \hat{\mathcal{J}}_{jj'} D^{(11)}_{jj',\, \varkappa,\, \varkappa-1}(q_{t\varkappa}^+) \quad (32)$$

for all t and $\varkappa$. In (32), we have introduced the matrix elements

$$\begin{aligned} D^{(11)}_{j_1j_2,\, j_3j_4}(q_{t\varkappa}^+) = {} & D^{(1r)}_{j_1j_3}(q_{t\varkappa}^+)\, D^{(1r)}_{j_2j_4}(q_{t\varkappa}^+) \\ & - D^{(1r)}_{j_2j_4}(q_{t\varkappa}^+) - D^{(1r)}_{j_1j_4}(q_{t\varkappa}^+)\, D^{(1r)}_{j_2j_3}(q_{t\varkappa}^+). \end{aligned} \quad (33)$$

The matrix $D^{(11)}$ is the matrix of the O_r-irreducible representation $(11)\equiv(110\cdots0)$, which follows from the definition (33) and the explicit form of the corresponding Clebsch-Gordan coefficients for the group O_r [see (3.17) in Ref. 16]; for $r>3$, the infinitesimal operators $\hat{\mathcal{J}}_{jj'}$ themselves also transform in accordance with the representation (11). It can be shown that $\hat{\mathcal{J}}_{ii'}$ satisfy the commutation relations

$$[\hat{\mathcal{J}}_{i_1i_2},\, \hat{\mathcal{J}}_{i_3i_4}] = \hat{\mathcal{J}}_{i_1i_3}\delta(i_2i_4) + \hat{\mathcal{J}}_{i_2i_4}\delta(i_2i_3) - \hat{\mathcal{J}}_{i_2i_3}\delta(i_1i_4) - \hat{\mathcal{J}}_{i_1i_4}\delta(i_1i_3). \quad (34)$$

In calculations with the expressions (27), it is convenient to use the algebra of tensor operators for the group O_r; an example of such calculations is given in Ref. 1. To calculate the submatrices, it is helpful to know the explicit expression of one of the operators $\hat{\mathcal{J}}$. The operator

$$\hat{\mathcal{J}}_{a-b+1,\, a-b} = \partial/\partial\vartheta_{n-1}^{(n-3)}, \quad (35)$$

has the simplest form. In (35) we have used the notation employed in (27). It is readily verified that the operator (35) is an O_{r-4} scalar which transforms in accordance with the O_{r-3}-irreducible representation (1), the O_{r-2}-irreducible representation (11), and the O_{r-1}-irreducible representation (11); these properties of the transformation must be known when the Wigner-Eckart theorem for operators of the group O_r is used.

5. O_{n-1}-IRREDUCIBLE EXPANSION OF THE OPERATORS $\hat{W}(\rho_t)$

We use Eqs. (26) and (27) for two sets of Jacobi variables, namely $\rho_1, \dots, \rho_{r-2}, \rho_{r-1}, \rho_r$ or

$\rho_1, \ldots, \rho_{r-2}, \rho_s, \rho_a$, which are related by the transformation (24). In (26), we take $g = e$ and consider this expression written down for the two sets of angular variables q^+ and q'^+ corresponding to each set of Jacobi coordinates above. The connection between the angles made by the sets q^+ and q'^+ can be found by substituting in (24) the expressions (26) for both sets of Jacobi coordinates. Using the orthogonality of the matrices $D^{(1r_0)}$, we obtain the following equation for the matrix elements of $D^{(1r)}$:

$$D^{(1r)}_{r-r_0+s_0,\,i}(\sigma q'^+) = D^{(1r)}_{r-r_0+s,\,i}(\sigma q^+ g_0). \tag{36}$$

Substituting the expressions (21) for the matrix elements of $D^{(1r)}$ and canceling the identical operators T, and also σ, we obtain for $r_0 = 3$

$$T^{(r-2)}_{r-3\,r-2}(\vartheta^{(r-2)}_{r-2})\,T^{(r-1)}_{r-2\,r-1}(\vartheta^{(r-1)}_{r-1})\,T^{(r)}_{r-1\,r}(\vartheta^{(r)}_{r})\,T^{(r-2)}_{r-4\,r-3}(\vartheta^{(r-2)}_{r-3})\,T^{(r-1)}_{r-3\,r-1}$$
$$\times(\vartheta^{(r-1)}_{r-2})\,T^{(r)}_{r-2\,r-1}(\vartheta^{r}_{r-1}) = T^{(r-2)}_{r-3\,r-2}(\vartheta^{r-2}_{a})\,T^{(r-1)}_{r-2\,r-1}(\vartheta^{r-1}_{a})\,T^{(r)}_{r-1\,r}(\vartheta^{r}_{a})$$
$$\times T^{(r-2)}_{r-4\,r-3}(\vartheta^{r-2}_{s})\,T^{(r-1)}_{r-3\,r-2}(\vartheta^{(r-1)}_{s})\,T^{(r)}_{r-2\,r-1}(\vartheta^{(r)}_{s})\,g_0^{-1}. \tag{37}$$

On the right-hand side of this last matrix relation the introduced angular variables $\vartheta^{r-2}_a, \vartheta^{(r-1)}_a, \vartheta^{(r)}_a, \vartheta^{(r-2)}_s, \vartheta^{(r-1)}_s, \vartheta^{(r)}_s$ range over the same intervals of values as the corresponding variables $\vartheta^{(r-2)}_{r-2}, \vartheta^{(r-1)}_{r-1}, \vartheta^{(r)}_r, \vartheta^{(r-2)}_{r-3}, \vartheta^{(r-1)}_{r-2}, \vartheta^{(r)}_{r-1}$. We recall that for small r there remain in (37) only the operators $T_{p-1\,p}$ with $p > 1$ and, therefore, there are less than six parameters ϑ. In what follows, it will be convenient to denote both these sets of angles in a unified manner; we shall write $\vartheta_3, \vartheta_2, \vartheta_1, \vartheta'_3, \vartheta'_2, \vartheta'_1$, understanding by this the angles on the left-hand side of (37) if we are concerned with the variables ρ_r, ρ_{r-1}, or the angles on the right-hand side of (37) if we are concerned with the variables ρ_a, ρ_s.

Multiplying the matrices T in (37), we readily see[1] that the penultimate column of the matrices $D^{(1r)}(q^+)$ or $D^{(1r)}(q'^+)$ depends on the six angles $\vartheta_3, \vartheta_2, \vartheta_1, \vartheta'_3, \vartheta'_2, \vartheta'_1$. The last column of this matrix depends only on the three angles $\vartheta_3, \vartheta_2, \vartheta_1$, the set of which we denote by $\overline{q}^+$. Hence, we conclude that for $i = r-1, r$ the right- and left-hand sides of the relations (36) depend only on six continuous variables, and Eqs. (37) express the functional connection between them. The explicit form of the matrix elements in (37) is given in Ref. 1 [see (8.21)]. Below, we give their expressions only for the final column. For $r > 3$ they are all equal to zero except for the following four matrix elements:

$$\left.\begin{aligned} &D^{(1r)}_{r-3,\,r}(\overline{q}^+) = \sin\vartheta_3 \sin\vartheta_2 \sin\vartheta_1;\\ &D^{(1r)}_{r-2,\,r}(\overline{q}^+) = \cos\vartheta_3 \sin\vartheta_2 \sin\vartheta_1;\\ &D^{(1r)}_{r-1,\,r}(\overline{q}^+) = \cos\vartheta_2 \sin\vartheta_1;\\ &D^{(1r)}_{r,\,r}(\overline{q}^+) = \cos\vartheta_1. \end{aligned}\right\} \tag{38}$$

Writing out explicitly the components of the three-dimensional vectors $\rho_t (t = n-1$ or $t = a)$, we obtain

$$\rho^s_t = \sum_{s_0} \rho^{(s_0)} D^{(1_3)}_{s_0 s}(G^+)\, D^{(1\,n-1)}_{n-4+s_0,\,r}(\sigma\overline{q}^+) \tag{39}$$

and, therefore,

$$|\rho_t| = \Big(\sum_{s_0} (\rho^{(s_0)})^2\, D^{(1\,n-1)}_{n-4+s_0,\,n-1}(\sigma\overline{q}^+)\Big)^{1/2}, \tag{40}$$

where the matrix elements of $D^{(1\,n-1)}(\overline{q}^+)$ have the expressions (38), and $D^{(1_3)}_{s_0 s}(G^+)$ are the matrix elements of the matrix of three-dimensional rotations in the Cartesian basis and are defined by means of (22). The matrix elements (38) also occur in the expressions for the derivative with respect to ρ^s_t. Because $D^{(1\,n-1)}_{i'\,n-1}(\overline{q}^+)$ with $i' < n-4$ vanishes, the expression (27) for the variable ρ^s_t for $n > 4$ contains only one term with respect to i', which contains the operator $\hat{\mathcal{J}}_{n-4+s',\,n-4}$. To summarize, we conclude that the derivative with respect to ρ^s_t depends only on the continuous variables $\rho^{(s_0)}$, G^+, and $\overline{q}^+$, and also on σ and the infinitesimal operators $\hat{\mathcal{J}}_{s_0 s'_0}$ and $\hat{\mathcal{L}}_{s_0 s_0}$.

Equations (39), (27), and (38) give explicit expressions for ρ^s_t and the derivatives with respect to ρ^s_t in terms of the new variables $\rho^{(s_0)}$, G^+, and $\sigma\overline{q}^+$. Substituting these expressions in (20), we obtain the operator $\hat{W}(\rho_t)$:

$$\hat{W}^{(\varkappa r)}_{\mu''} = W^{\varkappa r}_{\mu''}(\rho^{(s_0)},\ \partial/\partial\rho^{(s_0)},\ G^+,\ \hat{\mathcal{L}}_{s_0 s'_0},\ \sigma\overline{q}^+,\ \hat{\mathcal{J}}_{s_0 s'_0}). \tag{41}$$

This expression, together with Eqs. (17) or (18), completes the first stage in the O_{n-1}-irreducible expansion of arbitrary operators of the form (14) or (16).

We implement the second stage of this expansion in two ways. First, we average (41) with respect to the variables $\sigma\overline{q}^+$, i.e., with respect to these variables we go over to a matrix representation. We then take definite linear combinations of the averaged operator and thus complete the O_{n-1}-irreducible expansion of the operator $\hat{W}(\rho_t)$.

Before we begin to describe a suitable class of functions on which the averaging of the operator (41) is to be implemented, we introduce some more notation. We denote the set of variables that augment $\overline{q}^+$ to q^+ or to q'^+ by q^+_1, so that $q^+ \equiv \overline{q}^+ q^+_1$ or $q'^+ \equiv \overline{q}'^+ q^+_1$, depending on whether we are considering the Jacobi coordinates ρ_{n-1}, ρ_{n-2} or ρ_a, ρ_s. Now let Q in (9) or (10) denote the set of variables $\rho^{(s_0)}$, G^+, and $\sigma\overline{q}^+ q^+_1$. These functions are characterized by the orbital quantum number L, and therefore their dependence on the angles G^+ can be separated in the form

$$D^L_{KM}(G^+). \tag{42}$$

Here, D^L are matrices of an irreducible representation of O^+_3. Similarly, the dependence of the functions (9) or (10) on the variables $\sigma\overline{q}^+ q^+_1$ can be separated in the form of the functions[14]

$$D^{\omega}_{\nu\theta,\ \alpha\lambda\mu}(\sigma\overline{q}^+ q^+_1), \tag{43}$$

where D^ω are the matrices of the O_{n-1}-irreducible representation ω, and μ is the basis of the $S^{(r)}_n$-irreducible representation λ. The functions (42) and (43) are similar in many respects, and this similarity is due to the circumstance that (42) and (43) are matrix elements of matrices of irreducible representations of orthogonal groups. However, there are some differences between them. First, the columns of the matrix D^L are labeled by the projections M, the presence of which makes it possible to construct the functions (9) or (10) with total angular momentum J. Similarly, the columns in (43) are labeled by the indices $\alpha\lambda\mu$, the presence of which makes it possible to construct antisymmetric basis functions. Because of the last requirement, we use irreducible matrices D^ω of the orthogonal group with reflection, depending on the variable σ as well as the continuous variables. The presence of the discrete

variable is important in the construction of the functions (43); it is impossible to fix the correct basis characterized by a definite Young tableau for the functions $D^{\omega}_{\nu^0,\alpha\lambda\mu}$ defined only on the set of continuous variables $\overline{q}^{+}q_1^{+}$.[3)]

A second difference between (42) and (43) is reflected in the designation of the rows; for $n>3$, all the rows of the matrix D^{L} are present in the expansions of (9) and (10) in terms of (42), whereas in the expansion with respect to the matrix elements (43) only some of the rows of the matrix D^{ω} are used[1-3,14]; this circumstance is indicated in (43) by the superscript 0 of ν. Several different chains can be used to label ν^0, and one of them is associated with the group of three-dimensional rotations with reflection, which is embedded in O_{n-1} in accordance with $O_{n-1}\supset O_3\dot{+}O_{n-4}$. In this case, ν^0 denotes the set of indices $\beta_\nu l_\nu m_\nu$, where $l_\nu m_\nu$ is the quasi angular momentum and its projection, which are associated with the abstract three-dimensional space spanned by the Jacobi vectors $\rho_{n-3},\rho_{n-2},\rho_{n-1}$; β_ν is the repetition index corresponding to $O_{n-1}\supset O_3+O_{n-4}$. These quantum numbers are introduced in Ref. 14.

Because of the O_r-irreducible properties, (43) is an optimal basis for expanding arbitrary functions that depend on the variables $\sigma\overline{q}^{+}q_1^{+}$. This set gives precisely the class of functions on which the operator (41) should be averaged. It is first convenient to separate the dependence of the functions (43) on the variables $\overline{q}^{+}$. This is achieved by means of a certain numerical matrix A introduced in Ref. 18, whose matrix elements are called coefficients of fractional parentage. To obtain the corresponding expansion of the functions (43), we use the identity $\sigma\overline{q}^{+}q_1^{+}\equiv\sigma\overline{q}^{+}\sigma_1^{-1}\sigma^{-1}\sigma_0^{-1}\sigma_0\sigma_1 q_1^{+}$, in which σ_1 and σ_0 are discrete variables that augment the elements of the groups O^{+}_{n-2} and O^{+}_{1}, which are embedded in O^{+}_{n-1} in accordance with $O^{+}_{n-1}\supset O^{+}_{n-2}\dot{+}O^{+}_{1}$, to the groups O_{n-2} and O_1, which are embedded in O_{n-1} in accordance with $O_{n-1}\supset O_{n-2}\dot{+}O_1$.[4)] Further, we decompose the matrix elements (43) into a sum of products of elements that depend, respectively, on $\sigma\overline{q}^{+}\sigma_1^{-1}\sigma_0^{-1}$, σ_0, and $\sigma_1 q_1^{+}$, simultaneously forming linear combinations of them by means of the matrix elements of A. The final expressions have the form

$$D^{\omega}_{\nu^0,\,\alpha\lambda\bar{\lambda}\bar{\mu}}(\sigma\overline{q}^{+}q_1^{+})=\sum_{\bar{\omega}\alpha\nu_0}D^{\omega}_{\nu^0,\,\bar{\omega}\bar{\nu}^0}(\sigma\overline{q}^{+}\sigma_1^{-1})\,D^{\bar{\omega}}_{\nu_0,\,\alpha\bar{\lambda}\bar{\mu}}(\sigma_1 q_1^{+})\,A^{(\omega\bar{\lambda})}_{\bar{\omega}\alpha,\lambda} \tag{44}$$

for the single-particle operators and

$$D^{\omega}_{\nu^0,\,\alpha\lambda\lambda_{12}\bar{\bar{\lambda}}\bar{\bar{\mu}}}(\sigma\overline{q}^{+}q_1^{+})=\sum_{\bar{\omega}\bar{\bar{\omega}}\bar{\bar{\alpha}}\bar{\nu}_0}D^{\omega}_{\nu^0,\,\bar{\omega}\bar{\nu}^0}(\sigma\overline{q}^{+}\sigma_1^{-1})\,D^{\bar{\omega}}_{\nu_0,\,\bar{\bar{\omega}}\bar{\bar{\alpha}}\bar{\bar{\lambda}}\bar{\bar{\mu}}}(\sigma_1 q_1^{+})\,A^{(\omega\lambda_{12}\bar{\bar{\lambda}})}_{\bar{\bar{\omega}}\bar{\bar{\alpha}},\alpha\lambda} \tag{45}$$

for the central two-particle operators. In the two last formulas, $\bar{\omega},\bar{\bar{\omega}},\bar{\bar{\bar{\omega}}},\bar{\lambda},\bar{\bar{\lambda}}$ and λ_{12} denote, respectively, irreducible representations for the groups $O_{n-2},O_{n-3},O_{n-4},S_{n-1},S_{n-2}$ and S_2; $\bar{\alpha}$ and $\bar{\bar{\alpha}}$ are the repetition indices for the chains $O_{n-2}\supset S_{n-1}$ and $O_{n-3}\supset S_{n-2}$, and $\bar{\mu}$ and $\bar{\bar{\mu}}$ are bases for the representations $\bar{\lambda}$ and $\bar{\bar{\lambda}}$. The properties of the matrices of the reflection operators in (44) and (45) are considered in Ref. 20.

In (44) and (45) we have now separated $\sigma\overline{q}^{+}$ and can average the operator (41) over these variables. The volume element for q^{+} is (see, for example, Ref. 22)

$$d\tau_{n-1}(q^{+})=(V_0|(\tau_{n-1}))^{-1}\prod_{p=n-1}^{2}(\sin\vartheta_p^{(n-1)})^{p-2}\,d\vartheta_p^{(n-1)}$$
$$\times\prod_{p=n-2}^{2}(\sin\vartheta_p^{(n-2)})^{p-2}\,d\vartheta_p^{(n-2)}\prod_{p=n-3}^{2}(\sin\vartheta_p^{(n-3)})^{p-2}\,d\vartheta_p^{(n-3)}. \tag{46}$$

For the volume element augmented by the discrete variable, we have

$$d\tau_{n-1}(\sigma q^{+})=d\tau_{n-1}(q^{+})/2. \tag{47}$$

In (46), we have introduced the notation

$$V_0|(\tau_{n-1})=\int d\tau_{n-1}(q^{+}). \tag{48}$$

In Ref. 14, the orthogonality property

$$d_\omega\sum_{\delta}\int d\tau_{n-1}(\sigma q^{+})\,D^{*\omega}_{\nu^0\mu}(\sigma q^{+})\,D^{\omega'}_{\nu'^0\mu'}(\sigma q^{+})=\delta(\omega\omega')\,\delta(\nu^0\nu'^0)\,\delta(\mu\mu'), \tag{49}$$

where d_ω is the dimension of the representation ω, is proved. It is obvious that (47) can be rewritten as

$$d\tau_{n-1}(\sigma q^{+})=d\tau(\sigma\overline{q}^{+})\,d\tau_{n-2}(\sigma_1 q_1^{+}), \tag{50}$$

where $d\tau_{n-2}(q_1^{+})$ has the form (46) and

$$d\tau(\sigma\overline{q}^{+})=\frac{1}{2}\frac{V_0|(\tau_{n-2})}{V_0|(\tau_{n-1})}(\sin\vartheta_1)^{n-3}(\sin\vartheta_2)^{n-4}(\sin\vartheta_3)^{n-5}\,d\vartheta_1\,d\vartheta_2\,d\vartheta_3. \tag{51}$$

Standard calculations using (49) lead to the following expression for the matrix elements of the single-particle operator (41) on the basis of the functions (44) (Ref. 20):

$$\langle\nu^0\omega\alpha\lambda\bar{\lambda}\bar{\mu}|\hat{W}|\nu'^0\omega'\alpha'\lambda'\bar{\lambda}'\bar{\mu}'\rangle=\delta(\bar{\lambda}\bar{\lambda}')\,\delta(\bar{\mu}\bar{\mu}')\sum_{\bar{\omega}}\hat{I}^{(\nu^0\nu'^0)}_{\omega\bar{\omega},\,\omega'\bar{\omega}}Q_{\omega\bar{\omega},\,\omega'\bar{\omega}}. \tag{52}$$

Similarly, for the matrix elements of the central two-particle operator on the basis of the functions (45), we have

$$\langle\nu^0\omega\alpha\lambda\lambda_{12}\bar{\bar{\lambda}}\bar{\bar{\mu}}|\hat{W}|\nu'^0\omega'\alpha'\lambda'\lambda'_{12}\bar{\bar{\lambda}}'\bar{\bar{\mu}}'\rangle$$
$$=\delta(\lambda_{12}\lambda'_{12})\,\delta(\bar{\bar{\lambda}}\bar{\bar{\lambda}}')\,\delta(\bar{\bar{\mu}}\bar{\bar{\mu}}')\sum_{\bar{\bar{\omega}}}\hat{I}^{(\nu^0\nu'^0)}_{\bar{\bar{\omega}}\bar{\bar{\omega}},\,\omega'\bar{\bar{\omega}}}Q_{\bar{\bar{\omega}}\bar{\bar{\omega}},\,\omega'\bar{\bar{\omega}}}. \tag{53}$$

In the last two formulas, we have introduced the notation

$$Q_{\omega\bar{\omega},\,\omega'\bar{\omega}}=\sum_{\bar{\alpha}}A^{(\omega\bar{\lambda})}_{\omega\bar{\alpha},\,\alpha\lambda}A^{(\omega'\bar{\lambda})}_{\omega'\bar{\alpha},\,\alpha'\lambda'} \tag{54}$$

for (52) and

$$Q_{\bar{\bar{\omega}}\bar{\bar{\omega}},\,\omega'\bar{\bar{\omega}}}=\sum_{\bar{\bar{\omega}}\bar{\bar{\alpha}}}A^{(\omega\lambda_{12}\bar{\bar{\lambda}})}_{\bar{\bar{\omega}}\bar{\bar{\omega}}\bar{\bar{\alpha}},\,\alpha\lambda}A^{(\omega'\lambda_{12}\bar{\bar{\lambda}})}_{\bar{\bar{\omega}}\bar{\bar{\omega}}\bar{\bar{\alpha}},\,\alpha'\lambda'} \tag{55}$$

for (53). The quantity Q is called the generalized density matrix.

The integrals $\hat{I}$ in (52) and (53) have the form

$$\hat{I}^{(\nu^0\nu'^0)}_{\omega\bar{\omega},\,\omega'\bar{\omega}}\left(\rho^{(s_0)},\frac{\partial}{\partial\rho^{(s_0)}},G^{+}\mathcal{L}_{s_0s'_0}\right)=\sqrt{\frac{d_{\omega'}}{d_\omega}}\sum_{\chi\beta}d_\chi C^{\chi\omega'\beta\omega}_{0\,0\,\bar{\omega}\,\bar{\omega}}$$
$$\times\sum_{\mu\nu''^0}\left(\frac{1}{2}\sum_{\sigma}\int d\tau(\overline{q}^{+})\,D^{\chi}_{0\,u}((\sigma\overline{q}^{+})^{-1})\,\langle\omega'\nu'^0|\hat{W}(\sigma\overline{q}^{+})|\omega'\nu''^0\rangle\right)C^{\chi\omega'\beta\omega}_{\mu\nu''^0\nu^0}, \tag{56}$$

[3)] This circumstance was not noted in Ref. 17, in which results taken from Ref. 14 were rewritten in a simplified form.

[4)] This identity becomes more perspicuous when one considers the general case of fractional-parentage separation for a system consisting of r quasiparticles into subsystems consisting of r_1 and r_2 quasiparticles ($r_1+r_2=r$). We then have the chain $O^{+}_{r}\supset O^{+}_{r_1}\dot{+}O^{+}_{r_2}$ (its parameters are $q_0^{+}q_1^{+}q_2^{+}$),[19] and also $O_r\supset O_{r_1}\dot{+}O_{r_2}$ and the identity $\sigma_r q_0^{+}q_{r_2}^{+}q_{r_1}^{+}\equiv\sigma_r q_0^{+}\sigma_{r_1}^{-1}\sigma_{r_2}^{-1}(\sigma_{r_2}q_{r_2}^{+})(\sigma_{r_1}q_{r_1}^{+})$ adapted to it. Here, we consider only the special case $r_1=n-2$ and $r_2=1$, when $O^{+}_{r_2}$ degenerates to the trivial group O^{+}_{1}, which consits of a single element.

where C are Clebsch-Gordan coefficients for the group O_{n-1}, χ are all possible O_{n-1}-irreducible representations in the direct product $\chi \times \omega' \to \beta\omega$, where β is a repetition index, and $\bar{0}$ denotes the O_{n-2}-scalar representation. The matrix elements of the operator $\hat{W}$ in (56) arise because of the dependence of $\hat{W}$ on the infinitesimal operators of left shift $\hat{\jmath}$. These operators act on the rows of the matrix D^{ω}, and therefore $\hat{W}$, being a function of $\hat{\jmath}$, acts on D^{ω} as follows:

$$\hat{W}D^{\omega}_{\nu^0\nu} = \sum_{\nu'^0} \langle \omega\nu^0|\hat{W}|\omega\nu'^0\rangle D^{\omega}_{\nu'^0\nu}. \tag{57}$$

When $\hat{W}$ does not depend on $\hat{\jmath}$, the matrix element of the operator $\hat{W}$ has the form

$$\langle \omega'\nu'^0|\hat{W}|\omega'\nu''^0\rangle = \delta(\nu'^0\nu''^0)\hat{W}, \tag{58}$$

and then the integral (56) simplifies accordingly.

We now introduce the O_{n-1}-irreducible integrals

$$\begin{aligned}&\hat{I}^{(\nu^0\nu'^0)}_{\omega\omega'(\chi)\beta}(\rho^{(s_0)},\ \partial/\partial\rho^{(s_0)},\ G^{+},\ \hat{\mathcal{L}}_{s_0s'_0})\\ &= d_{\chi}\sqrt{\frac{d_{\omega'}}{d_{\omega}}}\sum_{\mu\nu''^0}\Big(\frac{1}{2}\sum_{\sigma}\int d\tau\,(\bar{q}^{+})\,D^{\chi}_{0,\mu}((\sigma\bar{q}^{+})^{-1})\\ &\quad\times\langle\omega'\nu'^0|\hat{W}(\sigma\bar{q}^{+})|\omega'\nu''^0\rangle\Big)C^{\chi\omega'\beta\omega}_{\mu\nu''^0\nu^0},\end{aligned} \tag{59}$$

and also the O_{n-1}-irreducible components of the generalized density matrix

$$Q_{\omega\omega'(\chi)\beta} = \sum_{\bar{\omega}} Q_{\omega\bar{\omega},\,\bar{\omega}'\omega}C^{\chi\omega'\beta\omega}_{0\bar{\omega}\bar{\omega}}. \tag{60}$$

It is readily verified that both expressions (52) and (53) can be written in the form

$$\langle\nu^0\omega\alpha\lambda\mu|\hat{W}|\nu'^0\omega'\alpha'\lambda'\mu'\rangle = \delta(\mu\mu')\sum_{\chi\beta}\hat{I}^{(\nu^0\nu'^0)}_{\omega\omega'(\chi)\beta}Q_{\omega\omega'(\chi)\beta}, \tag{61}$$

where μ denotes $\bar{\lambda}\bar{\mu}$ for the single-particle and $\lambda_{12}\overline{\overline{\lambda\mu}}$ for the central two-particle operators.

In making actual calculations, it must be borne in mind that the matrix elements of (52) and (53) are given in different bases of the $S_n^{(r)}$-irreducible representation λ and that different variables $\bar{q}^{+}$ for D^{ω} were used in their calculation. It is shown in Ref. 22 that these two types of matrix element are correlated if the two-particle coefficients of fractional parentage are calculated in accordance with the formula

$$A^{(\omega\lambda_{12}\bar{\bar{\lambda}})}_{\bar{\bar{\omega}}\bar{\bar{\omega}}\bar{\bar{\alpha}},\,\alpha\lambda} = \sum_{\bar{\bar{\omega}}'} D^{(\bar{\bar{\omega}}\bar{\bar{\omega}})}_{\bar{\bar{\omega}},\,\bar{\bar{\omega}}'}(g_0^{-1})\sum_{\bar{\lambda}}\Big(\sum_{\bar{\bar{\alpha}}}A^{(\omega\bar{\bar{\lambda}})}_{\bar{\bar{\omega}}'\bar{\bar{\alpha}},\,\alpha\lambda}A^{(\bar{\bar{\omega}}\bar{\bar{\lambda}})}_{\bar{\bar{\omega}}\bar{\bar{\alpha}},\,\bar{\alpha}\bar{\lambda}}\Big)M^{(\lambda\lambda'')}_{\bar{\lambda}\lambda_{12}}, \tag{62}$$

where $D^{\omega}(g_0^{-1})$ is the matrix of the O_{n-1}-irreducible representation ω for the group element g_0^{-1} in (24), and M is the known matrix of the simple transformation between the two bases of the representation λ used to calculate the matrix elements (52) and (53).

6. COLLECTIVE PART OF AN OPERATOR

Equation (61) gives the O_{n-1}-irreducible expansion of arbitrary operators $\hat{W}$ having the form (20). This expansion has not yet been generalized to the many-particle operators (14) or (16) formed from central two-particle or single-particle components. To realize such a generalization, one should substitute (61) in (17) or (18) and, giving a definite meaning to the sets of quantum numbers Γ, obtain an O_{n-1}-irreducible expansion of the many-particle operator $\hat{O}$.

There exists a well-developed technique of such expansion, the basic stages of which have been set forth in Refs. 4, 23, and 24. However, the description of these questions in a general form goes beyond the scope of the present review, and we therefore restrict ourselves to studying only the first term in the expansion of the operators in an O_{n-1}-irreducible series. It will soon become clear that this term has a perspicuous physical meaning. But first we recall an important assertion relating to the variables ρ^{s_0}, G^{+}, and σq^{+}.

These variables have a noteworthy property that makes it possible to sort them in accordance with a single criterion, which is important for understanding the nature of the collective degrees of freedom of the nucleus. We return to the case of the $r_0 r$ variables ρ_i^s with arbitrary r_0 and introduce the following convenient notation: we denote the set of variables $\rho^{(s_0)}$, G^{+} by ξ and the set of variables σq^{+} by q. Suppose the number of variables ξ is $N(\xi)$, the number of variables q^{+} is $N(q^{+})$, and the number of discrete variables σ is $N(\sigma)$. We have noted above that

$$N(\xi) = \begin{cases} (1/2)\,r_0(r_0+1), & r_0 \leqslant r; \\ r_0 r - (1/2)\,r(r-1), & r_0 \geqslant r. \end{cases} \tag{63}$$

We find the number $N(q^{+})$ from the relation

$$N(\xi) + N(q^{+}) = r_0 r, \tag{64}$$

which reflects the law of conservation of the number and species of the particles. Obviously, $N(\sigma) = 1$ for $r \geqslant 0$ and $N(\sigma) = 0$ for $r = 0$; the latter case is encountered for $r = n-1$ and $n = 1$.

The variables ξ differ significantly from the variables q in their permutational properties: All the $N(\xi)$ variables of the set ξ are scalars of the group $S_n^{(r)}$ acting on the orbital indices of the particles, whereas none of the variables of the set q are of this kind, since they transform in accordance with the $S_n^{(r)}$-irreducible representation $[n-1, 1]$. It is shown in Ref. 2, that preserving the total number of variables $r_0 r$, it is impossible to separate from the set q any $S_n^{(r)}$-scalar variables without thereby losing the possibility of fulfilling the Pauli principle.

This property of the sets of variables ξ and q enables us to regard ξ as collective variables and q are noncollective (internal) variables of the nucleus, the number of the two types of variables being strictly fixed because of the relation (64). This definition, which was proposed in Ref. 2, agrees with the intuitive physical interpretation of the collective degrees of freedom of a finite Fermi system; namely, by collective variables it is natural to understand the variables that are averaged to such an extent with respect to the indices of the individual particles that they "do not feel" the particles. The variables of the set ξ are of this kind. It also follows from this definition that variables that are nonscalar with respect to transformations of the group $S_n^{(r)}$ must be eliminated from the list of candidates for the status of collective variables, since they are not sufficiently averaged with respect to the indices of the individual particles and, therefore, "feel" their presence.

What we have said above applies to the sets ξ and q used in a theory constructed by means of the isospin

formalism, i.e., if the nucleus is regarded as a system consisting of particles of one species. When we are concerned with a theory in which the nucleus is regarded as a two-component quantum system consisting of two Fermi subsystems (proton and neutron), the total number of collective variables of such a system is made up of the number of collective variables of the two subsystems plus the number of variables that specify the relative motion of the centers of mass of the two subsystems.

To encompass both cases, it is convenient to speak of a system consisting of k Fermi subsystems, each of which consists of r_i+1 particles ($i=1,2,\ldots,k$) moving in an r_0-dimensional space.[5)] The total number r of particles is $r=r_1+r_2+\cdots+r_k+k$. Each subsystem is described by $N_i(\xi)$, $N_i(q^+)$, and $N_i(\sigma)$ variables of the type ξ, q^+, and σ. We introduce the numbers

$$\left.\begin{aligned} N_\xi &= r_0(k-1)+\sum_{i=1}^{k} N_i(\xi);\\ N_{q^+} &= \sum_{i=1}^{k} N_i(q^+);\\ N_\sigma &= \sum_{i=1}^{k} N_i(\sigma), \end{aligned}\right\} \tag{65}$$

where $r_0(k-1)$ gives the number of variables describing the relative motion of all the subsystems. It is obvious that $N_\xi+N_{q^+}=r_0 r$.

Summarizing, we state the following theorem: *without violating the requirements that follow from the kinematic properties of the Hamiltonian of the k-component system introduced above, one can introduce N_ξ collective variables. The remaining N_{q^+} continuous and N_σ discrete variables are essentially noncollective (internal) variables.* For $r_0=1,2$ this theorem was stated and proved in Ref. 2 (see also Ref. 3). It follows from the theorem in accordance with (63) that when $r_i \geq r_0$ the number of collective variables of subsystem i does not depend on the number of particles and is determined solely by the dimension r_0 of the space. Therefore, in the theory of the nucleus developed using the isospin formalism one can introduce only six collective variables ξ. For what follows, this is an important property of the variables $\rho^{(s_0)}$, G^+.

It is interesting to see how the collective variables are "absorbed" when two subsystems of equivalent fermions are combined into a single system. Suppose we have two nuclei consisting of n and n'' particles ($n'>3$ and $n''>3$). We denote for them the sets of collective variables by ξ' and ξ''. We introduce also three variables η, which describe the relative motion of the two subsystems; altogether, we have 15 collective variables ξ', ξ'', and η. Until the subsystems are combined, the Pauli principle is imposed on the n' and n'' particles separately. When the subsystems are combined into a single system, the requirement of the Pauli principle is extended to all $n'+n''$ particles. To satisfy this requirement, we replace ξ', ξ'', and η by other variables, which have the following properties: We introduce nine nonscalar variables q^0 which "feel" permutations of the particles between the two subsystems, and six scalar, i.e., collective, variables ξ for the combined system. We conclude from what we have said that it is the Pauli principle that is the reason why the total number of collective variables is always reduced to six in the combined system. The explicit connection between the sets of variables ξ', ξ'', η and q^0, ξ was obtained in Ref. 19.

Having now elucidated the meaning of the variables ξ, we return to our study of the expansion (61). We take its first term with $\chi=(0)$. In this case, the index β is not important and

$$C^{0\omega'\omega}_{\bar\nu^{\prime 0}\nu^0}=\delta(\omega'\omega)\,\delta(\nu^{\prime 0}\nu^0). \tag{66}$$

Substituting this expression in (59), we obtain[1]

$$\begin{aligned}&\hat I^{(\nu^0\nu^{\prime 0})}_{\omega\omega'(0)}(\rho^{(s_0)}\partial/\partial\rho^{(s_0)},\ G^+,\ \hat{\mathcal{L}}_{s_0s_0'}\\ &=\delta(\omega\omega')\frac{1}{2}\sum_\sigma\int d\tau(\bar q^+)\langle\omega\nu^{\prime 0}|\hat W(\bar q^0\sigma)|\omega\nu^0\rangle,\end{aligned} \tag{67}$$

where we recall that in the general case $\hat W$ depends on the derivatives with respect to $\rho^{(s_0)}$ and the infinitesimal operators $\hat{\mathcal{L}}_{s_0s_0'}$ and $\hat J_{s_0s_0'}$ of left shift. In particular, if $\hat W$ has the form (58), then, using (58), we find from (67) that[9,25]

$$\hat I^{(\nu^0\nu^{\prime 0})}_{\omega\omega'(0)}(\rho^{(s_0)},\ G^+)=\delta(\omega\omega')\,\delta(\nu^0\nu^{\prime 0})\frac{1}{2}\sum_\sigma\int d\tau(\bar q^+)\hat W(\sigma\bar q^*). \tag{68}$$

The final results (67) and (68) are very simple: To obtain the O_{n-1}-scalar part of an arbitrary single-particle or central two-particle operator acting on a Hilbert space spanned by O_{n-1}-irreducible spaces, it is necessary to average the matrix element of the operator $\hat W$ with respect to the variables $\bar q^+$ and sum over the two values of the variable σ. In accordance with (51), the volume element $d\tau$ has the expression

$$d\tau(\bar q^+)=N_\tau(\sin\vartheta_1)^{n-3}(\sin\vartheta_2)^{n-4}(\sin\vartheta_3)^{n-5}\,d\vartheta_1\,d\vartheta_2\,d\vartheta_3, \tag{69}$$

where

$$N_\tau=\Big(\int\int\int(\sin\vartheta_1)^{n-3}(\sin\vartheta_2)^{n-4}(\sin\vartheta_3)^{n-5}\,d\vartheta_1\,d\vartheta_2\,d\vartheta_3\Big)^{-1}. \tag{70}$$

We recall that ν_1, ν_2, ν_3 stand for $\vartheta^{(n-1)}_{n-1}$, $\vartheta^{(n-2)}_{n-2}$, $\vartheta^{(n-3)}_{n-3}$ for the single-particle operators and $\vartheta^{(n-1)}_a$, $\vartheta^{(n-2)}_a$, $\vartheta^{(n-3)}_a$ for the central two-particle operators. For $n=4$ or $n=3$, the integral over the variables $\bar q^+$ becomes two- or one-dimensional. For $n=5$, the variables $\bar q^+$ have limits that are nonstandard. If $\hat W$ does not depend on σ, then in (67) or (68) the factor 1/2 and the sum over σ can be omitted.

The expression (67) depends only on the variables ξ and the derivatives with respect to them. As we have already said, these are collective variables of the nucleus. Therefore, the O_{n-1}-scalar term of an arbitrary operator $\hat W$ is its collective part; this gives us the physical meaning of this term. To emphasize this circumstance, we introduce the special notation

$$\hat I^{(\nu^0\nu^{\prime 0})}_{\omega\omega'0}\equiv\delta(\omega\omega')\Big[\hat W^{(\omega)}_{\rm coll}\Big(\rho^{(s_0)},\ \frac{\partial}{\partial\rho^{(s_0)}},\ G^+,\ \hat{\mathcal{L}}_{s_0s_0'}\Big)\Big]_{\nu^0\nu^{\prime 0}}. \tag{71}$$

It can be seen from (71) that $\hat W^{(\omega)}_{\rm coll}$ is the matrix operator of left shift acting on the O_{n-1}-irreducible space ω. In particular, when $\hat W$ does not depend on the deriva-

[5)] The requirement that the i-th subsystem consist of fermions is important only when $r_1>0$; for $r_1=0$, there is only one particle in the subsystem, and the Pauli principle does not play any part, so that this particle need not necessarily be a Fermi particle.

tives with respect to the variables q^+, $\hat{W}^{(\omega)}_{\text{coll}}$ is a diagonal matrix whose elements do not depend on the basis indices ν^0. In other words, in this case $\hat{W}^{(\omega)}_{\text{coll}}$ is also an O_{n-1}-scalar with respect to left shift.

In the case $\chi=(0)$, we obtain from (60)

$$Q_{\omega\omega'(0)}=\delta(\omega\omega')\sum_{\bar{\omega}}Q_{\omega\bar{\omega},\omega\bar{\omega}}. \tag{72}$$

Substituting (54) or (55) in (72) and using the orthogonality of the matrix A, we can show that the sum in (72) is equal to $\delta(\alpha\alpha')\delta(\lambda\lambda')$. Therefore, the O_{n-1}-scalar component of the density matrix has the form

$$Q_{\omega\omega'(0)}=\delta(\omega\omega')\,\delta(\alpha\alpha')\,\delta(\lambda\lambda'). \tag{73}$$

Therefore, as must be the case, the O_{n-1}-scalar term in (61) is diagonal with respect to $\omega\alpha\lambda$ and equal to $[\hat{W}^{(\omega)}_{\text{coll}}]_{\nu^0\omega'_0}$.

7. MICROSCOPIC COLLECTIVE HAMILTONIAN OF THE NUCLEUS

In the preceding section, we have separated the collective part W_{coll} of an arbitrary single-particle or central two-particle operator $\hat{W}$. The next step in studying the collective properties of the many-particle operators (14) and (16) involves their factorization into orbital and spin-isospin parts. The operators (14) and (16) are obviously scalars with respect to the group S_n, which permutes simultaneously the orbital and spin-isospin indices. However, (14) and (16) are not scalars with respect to the groups $S_n^{(r)}$ and $S_n^{(\sigma\tau)}$, which act separately on the orbital and spin–isospin indices; they are reducible with respect to transformations of these groups. It is shown in Ref. 26 (see also Ref. 4) that the operator (16) contains two $S_n^{(r)}\times S_n^{(\sigma\tau)}$-irreducible components, namely $[n]$ and $[n-1,1]$, and that the operator (14) contains the three components $[n]$, $[n-1,1]$, and $[n-2,2]$. In what follows, we shall need to separate explicitly only the $S_n^{(r)}\times S_n^{(\sigma\tau)}$-scalar component. In accordance with (8.44) and (8.30) in Ref. 4, we have

$$\hat{O}^{\varkappa'\varkappa''}_{\mu'\mu''}(\mathbf{R})=\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\hat{U}^{\varkappa'}_{\mu'}(\sigma_i\tau_i)\Big)\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\hat{W}^{\varkappa''}_{\mu''}(\mathbf{r}_i+\mathbf{R})\Big)+\hat{\bar{\mathcal{O}}}^{\varkappa'\varkappa''}_{\mu'\mu''}(\mathbf{R}) \tag{74}$$

for the operator (16), and

$$\hat{O}^{\varkappa'\varkappa''}_{\mu'\mu''}=\Big(\sqrt{\frac{2}{n(n-1)}}\sum_{i>j=1}^{n}\hat{U}^{\varkappa'}_{\mu'}(\sigma_i\sigma_j\tau_i\tau_j)\Big)\times\Big(\sqrt{\frac{2}{n(n-1)}}\sum_{i<j=1}^{n}\hat{W}^{\varkappa''}_{\mu''}(\mathbf{r}_i-\mathbf{r}_j)\Big)+\hat{\bar{\mathcal{O}}}^{\varkappa'\varkappa''}_{\mu'\mu''} \tag{75}$$

for the operator (14). In these expressions, $\hat{\bar{O}}$ denotes one irreducible term $[n-1,1]$ in the case (74) and the two irreducible terms $[n-1,1]$ and $[n-2,2]$ in the case (75). The basis functions (9) or (10) contain irreducible characteristics of the groups $S_n^{(r)}$ and $S_n^{(\sigma\tau)}$, and, therefore, using the expansions (74) or (75) and the usual technique for calculating the matrix elements of irreducible operators, adapted to the symmetric group (for details, see Ref. 4), we obtain

$$\left\langle\begin{matrix}\pi\Gamma_0(LS)JM_J\\ TM_T\widetilde{\alpha}(\lambda\widetilde{\lambda})a\end{matrix}\middle|\,\hat{O}^{(\varkappa'\varkappa'')\varkappa}_{q}\,\middle|\begin{matrix}\pi'\Gamma'_0(L'S')J'M'_J\\ T'M'_T\widetilde{\alpha}'(\lambda'\widetilde{\lambda}')a\end{matrix}\right\rangle$$
$$=N(n)\,\delta(\lambda\lambda')\,\delta(\widetilde{\lambda}\widetilde{\lambda}')\,((2L+1)(2S+1)(2\varkappa+1)(2J'+1))^{1/2}$$
$$\times\begin{Bmatrix}L&S&J\\ L'&S'&J'\\ \varkappa'&\varkappa''&\varkappa\end{Bmatrix}C^{\varkappa J'J}_{\mu M'_J M_J}\langle\widetilde{\lambda}\widetilde{\alpha}STM_T\|\hat{U}^{\varkappa'}\|\widetilde{\lambda}\widetilde{\alpha}S'T'M'_T\rangle$$
$$\times\langle\pi\Gamma_0L\lambda\|\hat{W}^{\varkappa''}\|\pi'\Gamma'_0L'\lambda\rangle+\langle|\hat{\bar{O}}|\rangle, \tag{76}$$

where Γ_0 is $E\Omega\beta\omega\alpha$ for the basis (9) or $E\gamma\delta\omega\alpha$ for the basis (10). In (76), the curly brackets denote the standard $9j$ coefficient for the group of three-dimensional rotations, $\langle|\hat{\bar{O}}|\rangle$ is the abbreviated notation for the matrix element of the operator $\hat{\bar{O}}$, and

$$\bar{\mathcal{O}}^{(\varkappa'\varkappa'')\varkappa}_{\mu}=\sum_{\mu'\mu''}\bar{\mathcal{O}}^{\varkappa'\varkappa''}_{\mu'\mu''}C^{\varkappa'\varkappa''\varkappa}_{\mu'\mu''\mu}. \tag{77}$$

In (76), $\hat{U}$ and $\hat{W}$ depend on the same variables as in (17) or (18); $N(n)=n$ for the single-particle and $2N(n)=n(n-1)$ for the two-particle operators. The submatrices in (76) are defined for the group O_3^+ when the Wigner-Eckart theorem is used in the form (9.3) of Ref. 4.

It is easy to see that for an O_{n-1}-scalar operator the matrix element $\langle|\hat{\bar{O}}|\rangle$ is zero. Indeed, the group $S_n^{(r)}$ is embedded in O_{n-1} in accordance with $O_{n-1}\supset S_n^{(r)}$, and therefore an arbitrary O_{n-1}-scalar operator is always an $S_n^{(r)}$-scalar as well and, therefore, does not contain $\hat{\bar{O}}$ components.

In (76), we substitute the operator $\hat{W}_{\text{coll}}$ in place of $\hat{W}$. Then by what we have said above, the term $\langle|\hat{\bar{O}}|\rangle$ vanishes. The matrix elements of the spin–isospin operators $U^{\varkappa'}$ are completely determined by the algebra of the groups U_4 and $S_n^{(\sigma\tau)}$; the technique for calculating them is described in Ref. 4. Therefore, in the presence of known submatrices of the operator $\hat{W}_{\text{coll}}$ the matrix of the collective part of the many-particle operator (14) or (16) also becomes known. In other words, the single-particle or translationally invariant two-particle operator W_{coll} acting on the O_{n-1}-irreducible space ω uniquely determines the many-particle operator $O^{(\varkappa\varkappa'')\varkappa}_{\text{coll}\mu}$. In turn, we have shown above that $\hat{W}_{\text{coll}}$ is uniquely determined by the operator $\hat{W}$. We therefore conclude that the separation of the collective part from an arbitrary operator is unique.

This assertion was proved by means of an explicit construction using a partly matrix representation and a partly coordinate representation. However, from the proof of the existence of an operator acting on a separable Hilbert space in a definite representation there follows the proof of its existence in an arbitrary representation.[27] We therefore conclude that $\hat{\mathcal{O}}_{\text{coll}}$ is uniquely determined in any representation and that for it there exists the expansion[9,25]

$$\hat{\Theta}=\hat{\Theta}_{\text{coll}}+\sum_{\chi\neq(0),\,\beta}\hat{\Theta}^{(\varkappa)\beta}, \tag{78}$$

where χ and β have the same meaning as in (61). In particular, in the coordinate representation

$$\hat{\Theta}(\rho^{(s_0)},\,\partial/\partial\rho^{(s_0)},\,G^+,\,\hat{\mathcal{L}}_{s_0s'_0},\,q,\,\hat{\mathcal{Y}}_{s_0s'_0};\,Q)$$
$$=\hat{\Theta}_{\text{coll}}\Big(\rho^{(s_0)},\,\frac{\partial}{\partial\rho^{(s_0)}},\,G^+,\,\hat{\mathcal{L}}_{s_0s'_0},\,\mathcal{Y}_{s_0s'};\,Q\Big)$$
$$+\sum_{\chi\neq(0),\,\beta}\Theta^{(\chi)\beta}\Big(\rho^{(s_0)},\,\frac{\partial}{\partial\rho^{(s_0)}},\,G^+,\,\hat{\mathcal{L}}_{s_0s'},\,\hat{\mathcal{Y}}_{s_0s'_0};\,Q\Big). \tag{79}$$

The first term in (79) depends on $\hat{\mathcal{Y}}_{s_0s'_0}$ in the form of O_{n-1}-scalar operators, and therefore $\hat{\bar{\mathcal{O}}}_{\text{coll}}$ does not depend on the internal variables q, whereas the remaining terms do in general depend on them. The irreducible properties of the expansion (79) guarantee unique separation of $\hat{\bar{\mathcal{O}}}_{\text{coll}}$ from $\hat{\bar{\mathcal{O}}}$: The operator $\hat{\bar{\mathcal{O}}}-\hat{\bar{\mathcal{O}}}_{\text{coll}}$ no longer contains an O_{n-1}-scalar, i.e., purely

collective, component.

In the general formulation of the method in Sec. 2, we said that the additional integrals of the motion permit separation from H of the simplest Hamiltonian H_0. This assertion can now be demonstrated explicitly if ω is used as an additional integral of the motion. We rewrite (78) for the Hamiltonian H:

$$H = H_{\text{coll}} + H_{\text{coll. intern}}, \tag{80}$$

where $H_{\text{coll. intern}}$ denotes all the terms that are not scalar with respect to transformations of O_{n-1}. In accordance with the notation introduced in Sec. 3,

$$H_{\text{coll}} \equiv H' = H'_2 + H'_3. \tag{81}$$

The Hamiltonian of the nucleus conserves J, and in (76) we therefore set $\varkappa = 0$ and use the circumstance that the last term in (76) vanishes. Then for the basis functions (10) we obtain the following general expression for the matrix elements of the matrix of the collective Hamiltonian of the nucleus[9]:

$$\left\langle \begin{matrix} E\gamma\,(LS)\,JM_J \\ TM_T\tilde{\alpha}\delta\omega\alpha\,(\lambda\tilde{\lambda})\,a \end{matrix} \middle| H_{\text{coll}}^{(\varkappa\kappa)0} \middle| \begin{matrix} E'\gamma'\,(L'S')\,JM_J \\ T'M_T\tilde{\alpha}'\delta'\omega'\alpha'\,(\lambda'\tilde{\lambda}')\,a \end{matrix} \right\rangle$$
$$= \frac{1}{2} n(n-1)\,\delta(\omega\omega')\,\delta(\alpha\alpha')\,\delta(\lambda\lambda')\,\delta(TT')\,(-1)^{L+S'+J}$$
$$\times \left((2L+1)(2S+1)\right)^{1/2} \begin{Bmatrix} L & S & J \\ S' & L' & \varkappa \end{Bmatrix}$$
$$\times \langle \tilde{\lambda}\tilde{\alpha} S T M_T \| \hat{U}^{\varkappa} \| \tilde{\lambda}\tilde{\alpha}' S' T M_T \rangle \langle E\gamma L\delta\omega \| \hat{W}_{\text{coll}}^{\varkappa} \| E'\gamma' L'\delta'\omega \rangle. \tag{82}$$

The reason why (82) is diagonal with respect to the quantum numbers ω, α, and λ is obvious. The diagonality with respect to T is less obvious. Nevertheless, it is established in a more detailed study of the matrix of the operator H_{coll} if allowance is made for the explicit dependence of all the terms of the Hamiltonian (5) on the spin–isospin operators.[28] To summarize, we conclude that H_{coll}, which is obtained by restricting H to the subspace $\mathscr{R}^{(\Lambda\omega)}$, acts, because of the above diagonality, within $\mathscr{R}^{(\Lambda\omega\alpha\lambda T)}$; it is assumed that the basis functions (9) and (10) are orthogonal with respect to the index α.

The method for calculating the spin-isospin submatrices in (82) has by now been developed to such an extent that they are known for many practically interesting Young tableaux (for details, see Refs. 4 and 28). It therefore remains to consider only the submatrices of the operator $\hat{W}_{\text{coll}}$. Besides the technical aspect, which relates to the method of calculation, this question is intimately related to understanding the microscopic meaning of the phenomenological rotational-vibrational model of the nucleus. The following section is devoted to the discussion of this question.

8. HILBERT SPACE FOR COLLECTIVE HAMILTONIAN OF THE NUCLEUS

We now construct a basis in the Hilbert space on which which the operator $\hat{W}_{\text{coll}}$ acts. When we are concerned only with orbital variables, we shall denote the corresponding Hilbert space by $\tilde{R}^{(\Lambda\omega)}$. We span $\tilde{R}^{(\Lambda\omega)}$ by a complete system of functions that depend on the variables ξ, i.e., the variables $\rho^{(s_0)}, G^+$. A complete basis for G^+ is provided by the functions (42), and it therefore remains only to find a complete system of functions that depend on the variables $\rho^{(s_0)}$ which is suited to the specific problem. These functions can be constructed with respect to the quantum numbers of either the orthogonal or the unitary scheme. In both cases, they are the matrix elements of matrices of irreducible representations with specially chosen bases. For the orthogonal scheme, we have[2,3]

$$\Theta\left(\begin{matrix} E\Omega\beta LK \\ \omega\nu^0 \end{matrix} \middle| \rho\tilde{\vartheta}_1\tilde{\vartheta}_2\right) = R_{E\Omega}(\rho)\,(d\Omega/d_L\,d_\omega)^{1/2}\,D^{\Omega}_{\bar{0},\,\beta LK\omega\nu^0}(\tilde{\vartheta}_1\tilde{\vartheta}_2), \tag{83}$$

where we have used the variables $\rho, \tilde{\vartheta}_1, \tilde{\vartheta}_2$, which are related to the variables ρ^{s_0} by

$$\left.\begin{aligned} \rho^x &= \rho \sin\tilde{\vartheta}_1 \sin\tilde{\vartheta}_2; \\ \rho^y &= -\rho \sin\tilde{\vartheta}_1 \cos\tilde{\vartheta}_2; \\ \rho^{(z)} &= \rho\cos\tilde{\vartheta}_1, \end{aligned}\right\} \tag{84}$$

and $0 \leqslant \tilde{\vartheta}_1, \tilde{\vartheta}_2 \leqslant -\pi/2$. In (83), D^{Ω} is the matrix of the $O_{3(n-1)}$-irreducible representation Ω, whose columns are labeled by the quantum numbers already encountered in the functions (9), and $\bar{0}$ denotes an $O_{3(n-3)-1}$-scalar row. The functions $R_{E\Omega}$ in (83) are the radial functions of a $3(n-1)$-dimensional oscillator and depend on the global radius ρ.[6)]

The basis functions for the space $\tilde{\mathscr{R}}^{(\Lambda\omega)}$ in the case of a unitary scheme can be represented in the form[1]

$$\Theta\left(\begin{matrix} E\,[E_1E_2E_3]\,\gamma LK \\ \delta\omega\nu^0 \end{matrix} \middle| \eta^{(s_0)}\right) | \text{micr}\rangle$$
$$= \left(\frac{d_E}{d_L d_\omega}\right)^{1/2} D^{E}_{\bar{0}[E_1E_2E_3]\gamma LK\delta\omega\nu^0}(\eta^{(s_0)}) \,|\, \text{micr}\rangle, \tag{85}$$

where D^E is the matrix of the $U_{3(n-1)}$-irreducible representation E, $\bar{0}$ denotes an $U_{3(n-1)-1}$-scalar row, $\eta^{(s_0)}$ are the creation operators of oscillator quanta for the variables $\rho^{(s_0)}$, and $|\,\text{micr}\rangle$ is the vacuum state. The columns of the matrix D^E are labeled by the quantum numbers encountered in the functions (10); for clarity, we have written out explicitly in (85) the U_3- and U_{n-1}-irreducible representations $[E_1E_2E_3]$. The functions (83) and (85) are related by the transformation (11).

The basis functions (85) can be constructed explicitly in the coordinate representation. In Ref. 22, the following expression is obtained:

$$\Theta\left(\begin{matrix} E\gamma LK \\ \alpha\omega\bar{\bar{\omega}}\bar{\bar{\bar{0}}} \end{matrix} \middle| \rho^{(x)},\ \rho^{(y)},\ \rho^{(z)}\right)$$
$$= \sum_{\substack{E_x\bar{\bar{E}}_y E_z \\ \omega_x\omega_y\omega_z}} R_{Ex\omega_x}(\rho^{(x)})\,R_{E_y\omega_y}(\rho^{(y)})\,R_{E_z\omega_z}(\rho^{z})\,B_{E_x\omega_xE_y\omega_yE_z\omega_z}, \tag{86}$$

where

$$B = \sum_{E_{xy}\omega_{xy}} (d_{\omega_x} d_{\omega_y} d_{\omega_z})^{1/2}\, D^{(\omega_x\bar{\bar{0}})}_{\bar{0}\,\omega_x}\, D^{(\bar{\bar{\omega}}_x\bar{\bar{\bar{0}}})}_{\bar{\bar{0}}\,\bar{\bar{\omega}}}\, D^{(\omega_y\bar{\bar{0}})}_{\bar{0}\,\omega_y}$$
$$\times C^{E_x\ E_y\ E_{xy}}_{\omega_x\bar{\bar{\omega}}\ \omega_y\bar{\bar{0}}\ \alpha_{xy}\omega_{xy}}\, C^{E_{xy}\ E_z\ E}_{\alpha_{xy}\omega_{xy}\bar{\bar{\omega}}\ \omega_z\bar{\bar{0}}\ \alpha\omega\bar{\bar{\omega}}}\, M^{(E)}_{E_{xy}E_x,\,\gamma LK}. \tag{87}$$

In (86), R are the radial functions of an $(n-1)$-dimensional oscillator, $\bar{\omega}$ and $\bar{\bar{\omega}}$ denote irreducible representations of the groups O_{n-2} and O_{n-3}, and $\bar{0}$ is the O_{n-4}-scalar representation; in (87), D are simple factors whose explicit form is given in Ref. 22 [see the expression (2.14)]. The matrix $M^{(E)}$ transforms the

[6)] Replacing the functions $R_{E\Omega}(\rho)$ in (83) by a complete set of functions belonging to the continuous and discrete spectrum, one can extend the Hilbert space $\tilde{\mathscr{R}}^{(\Lambda\omega)}$ and thus the Hilbert space $\mathscr{R}$ to an equipped Hilbert space. Thus, the theory developed here can be generalized to a larger group of problems associated with nonstationary processes. However, we shall not consider these questions in the present paper.

canonical basis $U_3 \supset U_2 \supset U_1$ into the basis $U_3 \supset O_3^+ \supset O_2^+$ (Refs. 29 and 30). The coefficients C in (87) are isoscalar factors of the Clebsch-Gordan coefficients for the chain $U_{n-1} \supset O_{n-1} \supset O_{n-2} \supset O_{n-3}$; α_{xy} is the repetition index of the O_{n-1}-representation ω_{xy} contained in the U_{n-1}-representation E_{xy}. The functions (86) are normalized by the condition[22]

$$\frac{1}{d_L d_\omega} \sum_{K\bar{\omega}\bar{\bar{\omega}}} \int d\tau \left(\rho^{(s_0)} \Theta^* \begin{pmatrix} E\gamma LK \\ \alpha\omega\bar{\omega}\bar{\bar{\omega}}\bar{\bar{0}} \end{pmatrix} \middle| \rho^{(s_0)}\right) \times \Theta \left(\begin{matrix} E'\gamma' LK \\ \alpha\omega\bar{\omega}\bar{\bar{\omega}}\bar{\bar{0}} \end{matrix} \middle| \rho^{(s_0)}\right) = \delta(EE')\,\delta(\gamma\gamma')\,\delta(\alpha\alpha'), \tag{88}$$

where for $n > 3$

$$d\tau(\rho^{(s_0)}) = \frac{2^{n-4}(\Gamma((n-1)/2))^2}{3\Gamma(n-3)} \mid (\rho^{(x)})^2 - (\rho^{(y)})^2 \mid\mid (\rho^{(y)})^2 - (\rho^{(z)})^2 \mid\mid (\rho^{(z)})^2 - (\rho^{(x)})^2 \mid (\rho^{(x)}\rho^{(y)}\rho^{(z)})^{n-4}\, d\rho^{(x)}\, d\rho^{(y)}\, d\rho^{(z)}. \tag{89}$$

The non-normalized volume element (89) is obtained in Ref. 10. The normalization is obtained in Ref. 22, in which the details of the construction of the basis (86) are also described.

Using the dependence of the basis functions on the Eulerian angles G^+, we obtain

$$\Theta\left(\begin{matrix} E\gamma LM \\ \delta\omega\nu^0 \end{matrix} \middle| \rho^{(s_0)},\ G^+\right) = \sum_K \Theta\left(\begin{matrix} E\gamma LK \\ \delta\omega\nu_0 \end{matrix} \middle| \rho^{(s_0)}\right) D^L_{KM}(G^+) \tag{90}$$

for the unitary scheme and

$$\Theta\left(\begin{matrix} E\Omega\rho LM \\ \omega\nu^0 \end{matrix} \middle| \rho^{(s_0)},\ G^+\right) = \sum_K \Theta\left(\begin{matrix} E\Omega\beta LK \\ \omega\nu^0 \end{matrix} \middle| \rho^{(s_0)}\right) D^L_{KM}(G^+) \tag{91}$$

for the orthogonal scheme. The functions (90) or (91) form a complete system of functions that span the Hilbert space $\tilde{\mathscr{R}}^{(\Lambda\omega)}$. The basis functions in the space $\tilde{\mathscr{R}}^{(\Lambda)}$ have the form

$$\Psi\left(\begin{matrix} E\gamma LM \\ \delta\omega\alpha\lambda\mu \end{matrix} \middle| \rho^{(s_0)},\ G^+,\ \sigma q^+\right) = \sum_{\nu^0} \Theta\left(\begin{matrix} E\gamma LK \\ \delta\omega\nu^0 \end{matrix} \middle| \rho^{(s_0)}\right) D^L_{KM}(G^+)\, D^{\omega}_{\nu^0,\,\alpha\lambda\mu}(\sigma q^+). \tag{92}$$

An analogous expansion holds for the orthogonal scheme. The expansion (92) was obtained in Ref. 14.

Using the basis constructed in explicit form in the space $\tilde{\mathscr{R}}^{(\Lambda\omega)}$, we can calculate the matrix elements

$$\langle E\gamma LM\delta\omega \mid \hat{W}_{\text{coll}} \mid E'\gamma' L'M'\delta'\omega\rangle = (d_\omega)^{-1} \sum_{\nu^0\nu'^0} \int d\tau(G^+)\, d\tau(\rho^{(s_0)})\, \Theta^*\left(\begin{matrix} E\gamma LM \\ \delta\omega\nu^0 \end{matrix} \middle| \rho^{(s_0)},\ G^+\right) \times [\hat{W}^{(\omega)}_{\text{coll}}]_{\nu^0\nu'^0} \Theta\left(\begin{matrix} E'\gamma' L'M' \\ \delta'\omega\nu'^0 \end{matrix} \middle| \rho^{(s_0)},\ G^+\right), \tag{93}$$

and then find the submatrices of the operator $\hat{W}_{\text{coll}}$ and substitute them in (82). This completes the explicit construction of the collective Hamiltonian in the matrix representation. Diagonalization of its matrix gives the eigenvalues and eigenfunctions, i.e., the solution to the Schrödinger equation

$$H_{\text{coll}}\Psi_{\text{coll}} = E_{\text{coll}}\Psi_{\text{coll}}. \tag{94}$$

Below, we consider some properties of the eigenstates of the Hamiltonian H_{coll}, and we also discuss the possibilities of practical solution of Eq. (94).

The irreducible properties of the expansion (80) ensure unique separation of all the terms of the Hamiltonian H that depend only on the collective variables. Each additional term will necessarily take into account the interaction between the collective and internal degrees of freedom of the nucleus. Therefore, the microscopic theory of the collective motion in nuclei inherent in Eq. (94) is the most general kinematically correct theory of the collective degrees of freedom of the nucleus when the isospin formalism is used.[7)] *Every kinematically correct theory that studies purely collective phenomena is either equivalent to the theory presented here or is a special case of it.*

We introduce the following definition. We call the physical picture described by the solutions of Eq. (94) *the kinematically correct microscopic collective model of the nucleus*. This model was proposed in Ref. 9.

9. VARIABLES OF THE ROTATIONAL–VIBRATIONAL MODEL OF THE NUCLEUS

The theory of rotational and vibrational forms of motion in nuclei, which was initiated in the well-known work of Bohr, Mottelson, and Rainwater,[32-34] is phenomenological. In this theory, the existence of the collective modes of motion is postulated and not derived from the Schrödinger equation for the many-particle nuclear Hamiltonian. The kinematically correct microscopic collective model of the nucleus presented in the present review is formulated rigorously in the framework of quantum mechanics on the basis of the Hamiltonian H. It is interesting to see whether one can set up a bridge between these two theories and thereby elucidate the meaning of the phenomenological model. In accordance with our assertion at the end of the preceding section, this can certainly be done if the rotational-vibrational model is a kinematically correct model. However, as we have already noted in Sec. 1, it is actually hard to imagine how this can be tested in the form given to the model in the original papers.

The first question that must be discussed concerns the interpretation of the variables β and γ employed in the phenomenological theory. Are they quantum-mechanical variables or certain parameters that describe the deformation of the nucleus? If the former, they are not observables in states with definite energy, and, hence, one can speak only of their mean values. Of course, the expectation values of the operators of other physical quantities, in particular, the transition operators, do not depend on β and γ. In such an interpretation of β and γ, one can attempt to relate them to microscopic dynamical variables. In the latter case, when β and γ are treated as parameters, the expectation values of the operators depend on them, and one can attempt to relate β and γ only to certain parameters (but not variables) of the microscopic Hamiltonian H.

In the extensive literature devoted to questions related to collective excitation of nuclei, among which we mention the monographs of Refs. 35–37 and the reviews of Refs. 38 and 39, this question has not always been clearly posed and elucidated. In the present paper,

[7)] The general features of the theory of collective motion in a two-component system of fermions (i.e., when the isospin formalism is not used) are considered in Ref. 31.

we shall adhere to the first of the two points of view considered above and shall regard β and γ as dynamical variables.

We begin our study of the microscopic meaning of the phenomenological theory by discussing the connection between the variables employed in the two approaches. We introduce six new variables $q^{ss'}$ ($s \leq s' = 1, 2, 3$), which are functions of the collective variables ξ. We choose the dependence of $q^{ss'}$ on ξ such that the kinetic-energy operator for the variables $q^{ss'}$ is identical to the operator used in the Bohr-Mottelson theory. The idea that makes it possible to realize the choice of $q^{ss'}$ is suggested by the transformation to principal axes of the bilinear form $x_1^s x_1^{s'} + x_2^s x_2^{s'} + \cdots + x_n^s x_n^{s'}$, where x_i^s are single-particle Cartesian variables. Such a transformation was used in Ref. 40 to introduce collective variables describing quadrupole deformations of nuclei. This bilinear form is not translationally invariant and, therefore, does not satisfy the kinematic requirements. This violation of the symmetry principles can be readily avoided by replacing x_i^s by the Jacobi variables $\rho_i^{(s)}$. Therefore, we shall proceed from the bilinear form of the variables ρ_i^s with a normalization suitably chosen for what follows and assume that

$$q^{ss'} = \left(\frac{2}{1+\delta(ss')}\right)^{1/2} \sum_{i=1}^{m-1} \rho_i^s \rho_i^{s'}. \tag{95}$$

Substituting in (95) the expression (26) for ρ_i^s, we obtain[2]

$$q^{ss'} = \left(\frac{2}{1+\delta(ss')}\right)^{1/2} \sum_{s_0} (\rho^{(s_0)})^2 D^{(1_3)}_{s_0 s}(G^+) D^{(1_3)}_{s_0 s'}(G^+). \tag{96}$$

It can be seen from this expression that $q^{ss'}$ really does depend only on the collective variables ξ, i.e., that $q^{ss'} = q^{ss'}(\xi)$.

To trace the connection between the variables $q^{ss'}$ and the variables used in the rotational-vibrational model of the nucleus, we use the expression for the Laplace operator in the variables $q^{ss'}$. For brevity, we introduce the notation $(\rho^{(x)})^2 \equiv X$, $(\rho^{(y)})^2 \equiv Y$ and $(\rho^{(z)})^2 \equiv Z$. The calculations lead to the expression[41]

$$\begin{aligned}\Delta_{\text{phen}} = \sum_{s' \geq s=1}^{3} \frac{\partial^2}{\partial (q^{ss'})^2} = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2} \\ + 2\left(\frac{1}{X-Y} + \frac{1}{X-Z}\right)\frac{\partial}{\partial X} + 2\left(\frac{1}{Y-Z} + \frac{1}{Y-X}\right)\frac{\partial}{\partial Y} \\ + 2\left(\frac{1}{Z-X} + \frac{1}{Z-Y}\right)\frac{\partial}{\partial Z} + \frac{2}{(X-Y)^2}\hat{\mathcal{L}}^2_{12} \\ + \frac{2}{(X-Z)^2}\hat{\mathcal{L}}^2_{13} + \frac{2}{(Y-Z)^2}\hat{\mathcal{L}}^2_{23}.\end{aligned} \tag{97}$$

To cast the operator (97) into a more customary form, we rewrite (96) in O_3^+-irreducible form. We have

$$q^{\varkappa}_{\mu} = \sum_{s \geq s'=1}^{3} G^{1\ 1\ \varkappa}_{s\ s'\ \mu} q^{ss'} = \sum_{\mu'} p^{\varkappa}_{\mu'} D^{\varkappa}_{\mu' \mu}(G^+), \tag{98}$$

where $\varkappa = 0$ and 2 and

$$p^{\varkappa}_{\mu} = \sum_{s_0} (\rho^{(s_0)})^2 C^{1\ 1\ \varkappa}_{s_0\ s_0\ \mu}. \tag{99}$$

In particular, for $\varkappa = 0$ we find that

$$\sqrt{3}\, p^0_0 = X + Y + Z = \rho^2 \equiv p_0.$$

For $\varkappa = 2$, using the Clebsch-Gordan coefficients in a Cartesian basis [see, for example, Eq. (7.12) in Ref. 1], we readily find that p^2_{μ} does not vanish only for two values of μ. We denote these values by 11 and 22 and write down explicitly the expressions for the non-vanishing variables p^2_{11} and p^2_{22}:

$$\left.\begin{aligned} p^2_{11} &= (1/\sqrt{6})(X + Y - 2Z) \equiv (1/\sqrt{3})\, p_1; \\ p^2_{22} &= (1/\sqrt{2})(X - Y) = p_2. \end{aligned}\right\} \tag{100}$$

In the variables p_0, p_1, p_2, the operator (97) has the form

$$\begin{aligned}\Delta_{\text{phen}} = 3\frac{\partial^2}{\partial (p_0)^2} + 4\Big[3\frac{\partial^2}{\partial (p_1)^2} + \frac{\partial^2}{\partial (p_2)^2} \\ + 6\frac{p_1}{p_1^2 - p_2^2}\frac{\partial}{\partial p_1} + \frac{p_1^2 - 3p_2^2}{p_2 (p_1^2 - p_2^2)}\frac{\partial}{\partial p_2} \\ + \frac{1}{4(p_2)}\hat{\mathcal{L}}^2_{12} + \frac{1}{(p_1 + p_2)^2}\hat{\mathcal{L}}^2_{13} + \frac{1}{(p_1 + p_2)^2}\hat{\mathcal{L}}^2_{23}\Big].\end{aligned} \tag{101}$$

If in (101) we take $p_1 = \sqrt{3}\, a_0$ and $p_2 = \sqrt{2}\, a_2$, we can readily verify that (101) contains the expressions (6.15) and (6.17) given in Ref. 37. As in Ref. 37, we take

$$\left.\begin{aligned} (1/\sqrt{6})(X + Y - 2Z) &= \beta \cos\gamma; \\ (1/12)(X - Y) &= \beta \sin\gamma \end{aligned}\right\} \tag{102}$$

and introduce the moments of inertia[37]

$$\left.\begin{aligned} \mathcal{J}_{12} &= 4p_2^2 = 4\beta^2 \sin^2\gamma; \\ \mathcal{J}_{13} &= (p_1 + p_2)^2 = 4\beta^2 \sin^2(\gamma - 2\pi/3); \\ \mathcal{J}_{23} &= (p_1 - p_2)^2 = 4\beta^2 \sin^2(\gamma - 4\pi/3). \end{aligned}\right\} \tag{103}$$

In the new variables, (101) takes the form

$$\begin{aligned}\Delta_{\text{phen}} = 3\frac{\partial^2}{\partial (p_0)^2} + 4\Big(\sum_{s>s'=1}^{3} \frac{\hat{\mathcal{L}}^2_{s's}(G^+)}{\mathcal{J}_{s's}(\beta\gamma)} \\ + \frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2} + \frac{1}{\beta^2}\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma}\Big).\end{aligned} \tag{104}$$

The second term of this expression is identical to the well-known expression obtained in Ref. 32 for the Laplace operator of the phenomenological rotational-vibrational model, which confirms the correctness of the choice of the variables $q^{ss'}$ in the form (95). Note that in (102) we could, instead of $\beta\sin\gamma$ and $\beta\cos\gamma$, take $\rho^2\beta\sin\gamma$ and $\rho^2\beta\cos\gamma$; this connection between the variables was used in Ref. 11. In this case, however, in the second term of the expression (104) we should have the factor p_0^{-2}, i.e., the variables p_0 and β, γ in the operator (104) would not separate so fully.

Although the second term in (104) is identical with the kinetic-energy operator used in the Bohr-Mottelson theory, this operator now acquires a different, microscopic meaning: in its derivation, we have been able to avoid the traditional approach and have not required the phenomenological concept of vibration of the surface of the nucleus near the spherically symmetric equilibrium form. The derivation of the operator (104) just demonstrated indicates that in the rotational-vibrational model in its different, quantum-mechanical interpretation one uses quadratic forms in the microscopic variables $\rho^{(s_0)}$; the importance of this circumstance was pointed out in Ref. 25. Note also that in the rotational-vibrational model the variable p_0 is not taken into account for well-known reasons. It is, however, preferable to use all six variables $(\rho^{(s_0)})^2 G^+$; they are needed for simpler establishment of the connection between the phenomenological and microscopic theories—in the microscopic approach, the kinematic principles do not permit us to ignore any variable. In addition, there is one further and more practical rea-

son: Frequently, it is more perspicuous to use the Cartesian variables $(\rho^{(s_0)})^2$ [cf., for example, (97) and (104)]. It can also be shown that the use of the additional degree of freedom described by the variable p_0 is important on the transition from the rotational to the vibrational limit in the phenomenological model.[41]

10. HILBERT SPACE FOR THE HAMILTONIAN OF THE ROTATIONAL-VIBRATIONAL MODEL OF THE NUCLEUS

Let $\tilde{\mathscr{R}}_{\text{phen}}$ be a Hilbert space spanned by a complete set of functions that depend on the six variables $(\rho^{(s_0)})^2$, G^+ or the six equivalent variables $q^{ss'}$. Following our traditional approach, we introduce a basis in $\tilde{\mathscr{R}}_{\text{phen}}$. This is readily done by spanning the space $\tilde{\mathscr{R}}_{\text{phen}}$ by the functions of the following Hamiltonian of a six-dimensional oscillator:

$$h_{\text{phen}} = -\tfrac{1}{2}\Delta_{\text{phen}} + \tfrac{1}{2}\sum_{s_0 \geqslant s_0' = 1}^{3} (q^{s_0 s_0'})^2. \tag{105}$$

In group-theoretical language, this means that the basis introduced in $\tilde{\mathscr{R}}_{\text{phen}}$ transforms in accordance with the irreducible representations $\varepsilon \equiv [\varepsilon 00000]$ of the unitary group U_6, where $\varepsilon = 0, 1, 2, \ldots$. To label the basis functions with definite ε, one can use different chains of subgroups of U_6, for example,

$$U_6 \supset U_5 \supset P_5 \supset O_3^+; \tag{106}$$
$$U_6 \supset O_6 \supset O_5 \supset O_3^+; \tag{107}$$
$$U_6 \supset U_3 \supset O_3^+. \tag{108}$$

These three forms of basis were introduced in Ref. 41. We consider first the final chain, in which it is assumed that the group U_3 is embedded in U_6 in the sense of a plethysm. The general theory of plethysm is given in Ref. 42 (see also Refs. 4 and 43). In this special case, the meaning of plethysm is as follows. Let $[1]_3$ be the space on which the group U_3 acts. By means of the direct product $[1]_3 \times [1]_3$ we construct a nine-dimensional tensor space. Forming linear combinations of the components of this tensor, we separate a three-dimensional antisymmetric subspace $[11]_3$ and six-dimensional symmetric space $[2]_3$. Suppose the symmetric subspace $[2]_3$ is a space on which the group U_6 acts; to emphasize this circumstance, we redenote it by $[1]_6$. It is important that the carrying space $[1]_6$ for the group U_6 is the U_3-irreducible space $[2]_3$, and this property of it defines a specific embedding of U_3 in U_6. This embedding is embedding in the plethysm sense.

The plethysm serves as a very useful tool for studying reduction rules for different, at times intricately embedded, chains of subgroups, for reducing direct products, and so forth.[4,42,43] In particular, the rules of reduction in accordance with the chain (108) are very simple[41]: if ε is an irreducible representation of U_6, all U_3-irreducible representations contained in ε are listed by partitioning the number 2ε into all possible triplets of terms E_1, E_2, E_3 such that $E_1 \geqslant E_2 \geqslant E_3$ and all E_1, E_2, and E_3 are even. These numbers label the U_3-irreducible representations $[E_1E_2E_3]$ contained in ε. Using this rule, we find, for example, that the U_6-representation $[3]_6$ contains the U_3-representations [600], [420], and [222].

We denote the basis functions labeled by means of the quantum numbers of the chain (108) by

$$\Theta(\in [E_1E_2E_3]\ \gamma LM \mid q^{ss'}) = \sum_K \Theta(\in [E_1E_2E_3]\ \gamma LM \mid (\rho^{(s_0)})^2)\, D^L_{KM}(G^+) \tag{109}$$

and consider whether we can find the connection between the basis (109) and the basis (90). The dependence of the two sets of functions on G^+ is the same, and it therefore only remains to compare the functions

$$\Theta\left(\begin{matrix} E\,[E_1F_2E_3]\,\gamma LK \\ \delta\omega\gamma_0 \end{matrix}\middle|\, \rho^{(s_0)}\right) \tag{110}$$

with the functions

$$\Theta(\in [E_1E_2F_3]\ \gamma LK \mid (\rho^{(s)})^2). \tag{111}$$

In (110), the notation of the $U_3 \times U_{n-1}$-irreducible representation is written out explicitly. The basis functions (111) were introduced in Ref. 41.

We can immediately note some differences between (110) and (111). First, as could be seen from (86), the asymptotic behavior of (110) at large $\rho^{(s_0)}$ (vacuum state) is

$$|\,\text{micr}> \approx \exp\left\{-\tfrac{1}{2}\sum_{s_0}(\rho^{(s_0)})^2\right\}, \tag{112}$$

whereas (111) has the behavior

$$|\,\text{phen}> \approx \exp\left\{-\tfrac{1}{2}\sum_{s_0 \geqslant s_0'}(q^{s_0 s_0'})^2\right\} = \exp\left\{-\tfrac{1}{2}\sum_{s_0}(\rho^{(s_0)})^4\right\}. \tag{113}$$

Second, (110) depends on $\rho^{(s_0)}$ in the form $P^{(\omega)}_{\text{micr}}(\rho^{(s_0)})|_{\text{micr}}\rangle$, where $P_{\text{micr}}(\rho^{(s_0)})$ are certain polynomials, whereas (111) always depends on the square of the variables $\rho^{(s_0)}$, i.e., has the form $P_{\text{phen}}((\rho^{(s_0)})^2|_{\text{phen}}\rangle$. A third difference consists in the sets of quantum numbers: The set in (110) is much larger than the set in (111). Therefore, the space $\tilde{\mathscr{R}}^{(\Lambda\omega)}$ spanned by (110) is also larger than $\tilde{\mathscr{R}}_{\text{phen}}$, which is spanned by (111); thus, one can expand (111) in terms of (110) but not vice versa.

We could advance immediately in elucidating the meaning of the phenomenological rotational-vibrational model of the nucleus if we could find a subset of functions in (110) isomorphic to the set (111). To this end, we consider the transformation properties of the variables $q^{ss'}$ with respect to the action of the left-shift operators $\hat{T}_L$ of the group O_{n-1}. If G_{n-1} is an arbitrary element of this group, then by the definition of the left-shift operators (see, for example, Refs. 2, 3), we have

$$\hat{T}_L(G_{n-1})\,\rho^s_i = \sum_{s_0}\rho^{(s_0)} D^{(1s)}_{s_0 s}(G^+)\, D^{(1^{n-1})}_{n-4+s_0,\,i}(G^{-1}_{n-1}q). \tag{114}$$

Recalling the definition (95) of the variables $q^{ss'}$, we readily obtain

$$\hat{T}_L q^{ss'} = q^{ss'}, \tag{115}$$

i.e., $q^{ss'}$ are scalars with respect to the action of the operators of left shift (and also, of course, of right shift) of the group O_{n-1}. This means that $q^{ss'}$ is completely "in sensitive" to the existence of the group O_{n-1} to the same extent as, for example, the radial variable,

being a function of a quadratic form of the Cartesian variables, is "insensitive" to the action of the group O_3^*. In other words, in the space $\tilde{\mathcal{R}}_{\text{phen}}$ any action of the group O_{n-1} is completely frozen; hence, it is also necessary to find a frozen subspace in $\tilde{\mathcal{R}}^{(\Lambda\omega)}$. One can freeze the action of all operators of the group O_{n-1} $\tilde{\mathcal{R}}^{(\Lambda\omega)}$ by taking $\omega=(0)$ in (110), and it is therefore plausible that $\mathcal{R}_{\text{phen}}$ should be isomorphic to the subspace $\tilde{\mathcal{R}}^{(\Lambda,\omega=0)} \equiv \tilde{\mathcal{R}}^{(0)}$.

This conjecture can be readily proved by verifying that there is an isomorphism between the corresponding sets of quantum numbers. The rules for reduction in accordance with $U_{n-1} \supset O_{n-1}$ are well known (see, for example, Ref. 4), and we shall use here only one fact: the U_{n-1}-irreducible representation $[E_1E_2E_3]$ contains an O_{n-1}-scalar representation if and only if all E_1, E_2, E_3 are even. In this case, the repetition index δ takes only one value and, therefore, is unimportant.

Thus, we have found that the functions (110) with $\omega=(0)$, namely the functions

$$\Theta\left(\begin{matrix} E\,[E_1E_2E_3]\,\gamma LM \\ \omega=(0) \end{matrix}\middle|\,\rho^{(s_0)}\right), \tag{116}$$

are characterized by a set of quantum numbers isomorphic to the set in (111). Taking into account the explicit form of the functions $R_{E\Omega}(\rho)$, we can see from the general expression (86) that for $\omega=0$ the polynomials $P^{(0)}_{\text{micr}}$, multiplied by $(\rho^{(x)}\rho^{(y)}\rho^{(z)})^{1/2(n-2)}$, depend only on $(\rho^{(s_0)})$, i.e., there exists a connection between P_{phen} and $P^{(0)}_{\text{micr}}$. In both cases, the asymptotic conditions at large $\rho^{(s)}$ ensure rapid decrease of the basis functions, so that convergence problems do not arise. We summarize this by the following theorem.

THEOREM. The Hilbert space $\tilde{\mathcal{R}}_{\text{phen}}$ is isomorphic to the space $\tilde{\mathcal{R}}^{(0)}$ (Ref. 1).

This theorem is based on algebraic properties. When comparing the bases (110) and (111), we must draw attention to the intervals of values of the variables $q^{ss'}$: $0 \leqslant q^{ss} < +\infty$; these limits must be taken into account in the solution of the differential equation for the operator (105). However, from the algebraic point of view these details are not very important. They can be taken into account by means of normalizing factors. An example of this kind can be found in Ref. 44, where, in particular, it is explicitly demonstrated that the algebraic properties of some coefficients of the expansion in the case of the functions of a one-dimensional oscillator (when $-\infty < x < +\infty$) can be calculated by means of the same formula as the coefficients for oscillator radial functions (i.e., for $0 \leqslant r < +\infty$).

From the practical point of view, the presence of this isomorphism between the two differently constructed spaces $\tilde{\mathcal{R}}_{\text{phen}}$ and $\tilde{\mathcal{R}}^{(0)}$ means that the basis (110) can be used instead of the basis (111) in calculations. For arbitrary ω, the basis (110) depends on the number of particles, whereas the other basis does not depend on it. Therefore, in the realization of the space isomorphic to the space $\tilde{\mathcal{R}}_{\text{phen}}$ one can fix the number of particles and, of course, it is simplest to take the smallest n. The degenerate case with $n=3$, when the set $\rho^{(s_0)}$ consists of only two variables, is not suitable, and therefore one can take $n=4$. From (87) we then obtain the following expression for the coefficients B:

$$B = \sum_{E_{xy}\omega_x} (d_{\omega_x} d_{\omega_y})^{1/2} D^{(\omega_x \bar{\bar{\omega}})}_{\bar{0}\bar{\omega}_x} D^{(\bar{\omega}_x \bar{\bar{0}})}_{\bar{0}\bar{0}} D^{(\omega_y \bar{\bar{0}})}_{\bar{0}\bar{\omega}_x} C^{E_x \quad E_y \quad E_{xy}}_{\omega_x \bar{\omega}_x 0} C^{E_{xy} \quad E_y \quad E_x}_{\omega_x \bar{\omega}_y \omega_x} C^{E_{xy} \quad E_z \quad E}_{\omega_z \quad \omega_z \quad 0} M^{(E)}_{E_{xy}E_z,\,\gamma LK}, \tag{117}$$

where $\bar{\omega}_x$ takes not more than two values, and C are the Clebsch-Gordan coefficients for the group U_3. The explicit form of C and M is known and, therefore, a basis in $\tilde{\mathcal{R}}_{\text{phen}}$ can be constructed explicitly.

We consider the remaining two chains (106) and (107) used in constructing a basis in the space $\tilde{\mathcal{R}}_{\text{phen}}$. In Ref. 44, a matrix is found explicitly that connects these two bases, and therefore it is sufficient to consider one of them, say (107). The basis functions for it obviously have the form

$$\sum_K \Theta\,(\in \Lambda_6\Lambda_5 \varkappa LK \mid \rho_0,\ \beta,\ \gamma)\, D^L_{KM}(G^+)$$
$$= R_{\in\Lambda_6}(\rho_0) \sum_K Y\,(\Lambda_6\Lambda_5 \varkappa LK \mid \beta,\ \gamma)\, D^L_{KM}(G^+), \tag{118}$$

where Λ_6 and Λ_5 denote O_6- and O_5-irreducible representations, and $\varkappa$ is the repetition index corresponding to $O_5 \subset O_3^*$. By means of the reduction rules for the corresponding chains, we can directly establish an isomorphism between the set of basis functions (118) and the set of functions (91) with $\omega=(0)$. However, there is no need to do this because the basis functions (118) are related to the functions (109) by a unitary transformation [see (5.5) in Ref. 41], and the functions (91) are related to the functions (90) by the unitary transformation specified by the matrix with the matrix elements (11). Both these transformations are specified by square matrices, and this fact is sufficient to establish the isomorphism of the spaces spanned by the basis (118) and the basis (91) with $\omega=0$. In particular, it is clear that functions with given Ω can be related to functions with $2\Lambda_6=\Omega$, and therefore the set of quantum numbers Λ_5, $\varkappa$ in (118) is isomorphic to the repetition index δ in (110). In Refs. 45 and 46, basis functions for the chain $U_5 \supset O_5 \supset O_3^*$ are constructed explicitly. Augmenting this construction to the chain (106), we can use the results of the quoted papers to realize a basis in $\tilde{\mathcal{R}}_{\text{phen}}$.

11. MICROSCOPIC ROTATIONAL–VIBRATIONAL MODEL OF THE NUCLEUS

The theorem of the previous section gives the key to understanding the microscopic meaning of the rotational-vibrational model. Let us explain this more clearly. When speaking of the rotational model of the nucleus, we have in mind the theory based on the Hamiltonian

$$H_{\text{phen}}((\rho^{(s_0)})^2,\ G^+) = -(1/2)\,\Delta_{\text{phen}} + V_{\text{phen}}((\rho^{(s_0)})^2,\ G^+) \tag{119}$$

with kinetic-energy operator of the form (97) and arbitrary potential energy V_{phen}, which depends, in general, on $(\rho^{(s_0)})^2$ and G^+; note that a dependence of the potential V_{phen} on the variables G^+ is not ruled out, since the nucleon-nucleon interaction conserves only the total angular momentum J. The usually understood model is encompassed by this definition of the phenomenological rotational-vibrational model. Indeed, if we note the connection (102) between $(\rho^{(s_0)})^2$ and the

variables β and γ and assume that (119) does not depend on the variable p_0, and also assume that V_{phen} does not depend on G^+, then (119) is transformed into the usually employed Hamiltonian of the phenomenological rotational-vibrational model (see, for example, (6.38) in Ref. 37).

In the preceding section, we have shown that the operator (119) acts on the space $\tilde{\mathscr{R}}_{\text{phen}}$. For given V_{phen}, we could in principle calculate the matrix of the operator H_{phen} in the basis (111) or in the basis (118) (one of these bases is suited to the so-called rotational limit, and the other to the vibrational limit[41,47]) and, diagonalizing it, find the spectrum and eigenvalues of the operator (119). In the phenomenological theory, V_{phen} is chosen empirically. In addition, we introduce a dependence on the spin quantum numbers, augmenting H_{phen} with the operators of the total angular momentum. The Hilbert space is accordingly enlarged by the replacement of D^L by D^J, where J is the total angular momentum of the nucleus. One can also take into account the well-known technical details associated with ensuring the requirements that follow from the additional symmetry under the group of discrete transformations, etc.

The isomorphism between the spaces $\tilde{\mathscr{R}}_{\text{phen}}$ and $\tilde{\mathscr{R}}^{(0)}$ proved in Ref. 1 gives a constructive possibility for generalizing the phenomenological theory using the advantages of the microscopic theory. This is done as follows[1]: to obtain a microscopic justification of the phenomenological rotational-vibrational model, it is necessary to develop the theory in the space $\mathscr{R}^{(0)}$, where, we recall, the absence of the tilde on $\mathscr{R}$ means that the spin-isospin degrees of freedom of the nucleus have already been taken into account. However, such a theory has already been presented in Secs. 4–8, so that it remains only to adapt the general formulas to this particular case. For the matrix of the Hamiltonian, setting $\omega=(0)$ in (82), we obtain

$$\left\langle \begin{matrix} E\gamma\,(LS)\,JM_J \\ TM_T\tilde{\alpha}\,(0)\,\tilde{\lambda} \end{matrix} \middle| H^{(\varkappa\varkappa_0)}_{\text{coll}} \middle| \begin{matrix} E'\gamma'\,(L'S')\,JM_J \\ TM_T\tilde{\alpha}'\,(0)\,\tilde{\lambda} \end{matrix} \right\rangle$$
$$=(-1)^{L+S'+J}\,((2L+1)\,(2S+1))^{1/2}\begin{Bmatrix} L & S & J \\ S' & L' & \varkappa \end{Bmatrix}$$
$$\langle \tilde{\lambda}\,\tilde{\alpha}STM_T \| \sum_{i>j=1}^{n} \hat{U}^{\varkappa}(\sigma_i\sigma_j\tau_i\tau_j) \| \tilde{\lambda}\,\tilde{\alpha}'S'TM_T \rangle$$
$$\times (E\gamma L\,(0) \| \hat{V}^{\varkappa}_{\text{coll}} \| E'\gamma'L'\,(0)). \qquad (120)$$

In (120), we have made a small modification, namely, we have omitted the factor $(1/2)n(n-1)$ and replaced $\hat{U}(\rho_a)$ by the sum $\sum\hat{U}(ij)$. The submatrix in (120) is calculated by means of (93) with $\omega=(0)$.

We denote the Hamiltonian obtained by restricting H to the subspace $\mathscr{R}^{(0)}$ by $H_{R\text{-}V}$. The matrix representation of the operator $H_{R\text{-}V}$ is given by Eq. (120), and therefore this operator is defined in any representation. Solving the Schrödinger equation

$$H_{R\text{-}V}\Psi_{R\text{-}V}=E_{R\text{-}V}\Psi_{R\text{-}V}, \qquad (121)$$

we find the spectrum and wave functions $\Psi_{R\text{-}V}$. By definition, we shall call the physical picture described by the functions $\Psi_{R\text{-}V}$ the *microscopic generalized rotational-vibrational model of the nucleus* (generalized because the Hamiltonian $H_{R\text{-}V}$ also takes into account the "breathing" degree of freedom of the nucleus described by the variable ρ). This model was introduced in Ref. 1.

As we have just shown, the Hamiltonian of the microscopic generalized model is H restricted to the O_{n-1}-scalar subspace $\mathscr{R}^{(0)}$. It is important to bear in mind that this restriction violates the Pauli principle. Indeed, if $\omega=(0)$, then $\lambda=[n]$, and hence $\tilde{\lambda}$ must be equal to $[1^n]$. But this is impossible: when $n>4$, spin-isospin functions do not exist for $[1^n]$. There remains nothing else for us to do but assume that the equation $\tilde{\lambda}=[1^n]$ does not necessarily follow from the condition $\lambda=[n]$, i.e., that the $S_n^{(\tau)}$- and $S_n^{(\sigma\tau)}$-irreducible representations λ and $\tilde{\lambda}$ are not coupled into an antisymmetric representation. This, of course, is an additional assumption that violates the kinematic requirements of quantum mechanics. We make this assumption and shall assume that $\tilde{\lambda}$ in (120) takes the values permitted in the physically allowed U_{n-1} states $[E_1E_2E_3]$ with all even E_1, E_2, E_3.[8)]

Summarizing, we conclude that the microscopic generalized rotational-vibrational model of the nucleus is not a kinematically correct model. This property of the theory was lost because of the decision to take $\omega=(0)$. This last decision was, in its turn, due to the properties of the phenomenological Hamiltonian (119), which depends on $q^{ss'}$, i.e., on $(\rho^{(s_0)})^2$ and G^+, and its eigenfunctions are scalars of the group O_{n-1}. Thus, the microscopic model we have defined, being in this respect close to the phenomenological model, gives a natural generalization of the latter.

The principal advantage of the microscopic model over the phenomenological model is that its Hamiltonian $H_{R\text{-}V}$ is not empirical, but is derived from the Hamiltonian H. This circumstance opens up possibilities for studying the dependence of $H_{R\text{-}V}$ on the details of the nucleon–nucleon interaction. In this direction, much work of both analytic and numerical nature is still to be done. Without going into the details, we shall discuss only the question of the dependence of $H_{R\text{-}V}$ on the angular momentum operators.

In this respect, the most interesting terms arise from the vector and tensor forces, which depend on the angular momentum operators. Using (26) and (27) for the variables ρ_a^s and noting the circumstance that in the O_{n-1}-scalar spaces all the operators $\hat{\mathcal{J}}$ vanish, we obtain

$$L_{ss'}(\rho_a^0)=\rho_a^s\partial/\partial\rho_a^{s'}-\rho_a^{s'}\partial/\partial\rho_a^s$$
$$=\sum_{s_0}\rho^{(s_0)}D^{(1_{n-1})}_{n-4+s_0,a}\sum_{s_0'>s_0''}\left(\frac{\rho^{(s_0')}}{(\rho^{(s_0')})^2-(\rho^{(s_0'')})^2}D^{(11)}_{s_0s_0''s's}\right.$$
$$\left.\times D^{(1_{n-1})}_{n-4+s_0',a}+\frac{\rho^{(s_0'')}}{(\rho^{(s_0')})^2-(\rho^{(s_0'')})^2}D^{(11)}_{s_0s_0ss'}D^{(1_{n-1})}_{n-4+s_0'',a}\right)\mathscr{L}_{s's_0''}. \qquad (122)$$

[8)] This in fact means that in this model in such a form a "memory" of the Pauli principle remains, i.e., the Pauli principle is violated only for the orbital part of the nuclear wave function. Note also that the expressions (76) and (82) were obtained for antisymmetric wave functions and one could object that (120) is not valid when we ignore the antisymmetry condition. This would be true for $\langle|\hat{\mathcal{P}}|\rangle\neq 0$. For $\langle|\hat{\mathcal{P}}|\rangle=0$ this formula is suitable for arbitrary $\tilde{\lambda}$ after the modification of (120).

To complete the separation of the operator $\hat{W}_{\text{coll}}$ for the vector forces, it remains to multiply (122) by the potential of the vector interaction and, using (68), to find the O_{n-1}-scalar part of the resulting expression, It is clear that the final result will depend on the variables $\rho^{(s_0)}$, G^+ and the angular momentum $(i)^{-1}\mathcal{L}'_{s_0 s_0}$. The more complicated bilinear dependence of H_{R-V} on $\mathcal{L}'_{s_0 s_0}$ is due to the velocity-dependent tensor forces.

These simple arguments show that H_{R-V} depends on the angular momentum not only through the kinetic-energy operator but also through the vector and tensor forces; additional terms of this kind are sometimes introduced empirically in an attempt at microscopic justification of the phenomenological rotational–vibrational model. In the framework of the theory described above, their origin can be readily understood and one can study their dependence on the details of the potential of the nucleon–nucleon interaction.

The microscopic generalized rotational–vibrational model of the nucleus is defined in the framework of quantum mechanics without recourse to the conjecture of a rotating nucleus and references to a semiclassical picture of nuclear structure. Formally, this model is a special case of a kinematically correct theory. In essence, however, this is not so. The microscopic collective model of the nucleus is constructed in the Hilbert space spanned by the O_{n-1}-irreducible subspaces $\mathcal{R}^{\Lambda\omega}$ with $\omega \geq \omega_{\min}$, where $\omega_{\min}$ in the case $n>4$ cannot take O_{n-1}-scalar values (for details, see Refs. 1 and 4). The generalized microscopic rotational–vibrational model is defined in the space $\mathcal{R}^{(0)}$ for the same reason that the kinematically correct model based on the Hamiltonian H_{coll} was "spoiled" by the use of the "unnatural" variables $(\rho^{(s_0)})^2$. This defect of the theory, introduced in the definition of the Hamiltonian (119), cannot be corrected, so that for $n>4$ the microscopic generalized rotational-vibrational model of the nucleus does not, strictly speaking, have a right of existence in the framework of quantum mechanics; it is defined on the unphysical subspace $\mathcal{R}^{(0)}$ of the many-particle Hilbert space. As was noted in Ref. 25, the solutions of the Schrödinger equation (121) in the space spanned by functions that depend on the variables $(\rho^{(s_0)})^2$ do not give the collective states obtained in a truly quantum approach.

With regard to the phenomenological rotational–vibrational model of the nucleus, its fate depends on the meaning attached to the variables β and γ. If it is assumed that the microscopic meaning of these variables is determined by the relations (102), then all that we have said about the microscopic generalized rotational–vibrational model is transferred to the phenomenological model as well, and then the latter must be deleted from the list of kinematically correct theories.

12. OTHER TYPES OF RESTRICTED HAMILTONIANS

We now return to questions of the general theory based on the idea of restricted dynamics. Section 8 was completed by the construction of the Hamiltonian H_{coll}, which contains all information about the collective degrees of freedom of the nucleus. This Hamiltonian acts on the infinite space $\mathcal{R}^{(\Lambda\omega)}$, and therefore a convergence problem arises in the solution of the Schrödinger equation for H_{coll}. One of the methods of approximate solution of this equation can be based on diagonalization of the matrix (82). With present-day computer facilities it is not possible to carry out such diagonalization for a wide range of mass numbers and guarantee good convergence. Nevertheless, in a number of special cases, when the values of the quantum number ω are favorable from the computational point of view and the requirements on the convergence of the expansions with respect to the basis functions are moderate, such calculations can be made comparatively easily. The difficulty of solving the Schrödinger equation for the collective Hamiltonian H_{coll} also depends, of course, on the complexity of the original Hamiltonian H.

We have spoken above of numerical methods of solution of Eq. (94). As a rule, these are neither so general nor so perspicuous as analytic methods, and it is therefore desirable to make progress in the study of the general properties of H_{coll}. One step can be taken in this direction by expanding H_{coll} in an operator series, the first term then giving a Hamiltonian simpler than H_{coll} but still describing approximately the properties of the collective excitations. We shall see here that such an expansion can be sought in the framework of the general method of projecting operators with restricted dynamics as described in Sec. 2.

We recall that in our discussion in Sec. 2 of different types of restricted Hamiltonians we introduced not only $H_{\text{coll}} \equiv H'_{23}$ but also three further operators, namely H'_1, H''_1, and H''_{23}. We shall discuss the first two of these below and now concentrate our attention on H''_{23}. By definition, this operator is the U_{n-1}-scalar part of the original Hamiltonian H, and knowledge of this circumstance is sufficient to suggest that H''_{23} is contained as a term in the Hamiltonian H_{coll}. Indeed, O_{n-1} is embedded in U_{n-1} in accordance with $U_{n-1} \supset O_{n-1}$, from which it follows that every U_{n-1}-scalar operator is necessarily also a O_{n-1} scalar. Therefore,

$$H_{\text{coll}} = H_{0\,\text{coll}} + \overline{H}_{\text{coll}}, \tag{123}$$

where $H_{0\,\text{coll}} \equiv H''_{23}$, and $\overline{H}_{\text{coll}}$ consists of all terms of H_{coll} that are not U_{n-1} scalars.

The decomposition (123) can be achieved in at least two different ways, the first of which, like (80), is based on the coordinate representation of the collective variables. In practice, it is implemented by the replacement in (67) of the collective variables and their derivatives by creation and annihilation operators of oscillator quanta with subsequent separation from them of the U_{n-1}-scalar terms. The matrix of the operator $H_{0\,\text{coll}}$ is calculated in accordance with (82) with the only difference that the orbital submatrix in it becomes additionally diagonal with respect to $E\delta$ and independent of $\delta\omega$. The orbital submatrices are calculated in accordance with Eq. (93) with allowance for this extra diagonality.

We shall also say a few words about the second

method of deriving the expansions (80) and (123). In the construction of the operators H_{coll} and $H_{0\,\text{coll}}$ in the matrix representation, their explicit dependence on the variables $\rho^{(s_0)}, G^+$ appears only in the intermediate stage, averaging over these variables then being performed in the calculation of (96). It is pertinent to consider whether this intermediate coordinate representation is really necessary, since it is very inconvenient in practical calculations. Indeed, in the case of nontrivial W it is by no means simple to integrate in (67) over $\overline{q}^+$, and one can hope to obtain answers in an analytic form only for simple potentials. Then there arises the problem of calculating in the general case the sixfold integrals (93). Clearly, this problem is also not simple.

The coordinate representation is convenient for understanding the meaning of the collective and internal degrees of freedom of the nucleus, for perspicuous realization of the bases in the space $\mathscr{R}$, and for elucidating the exceptional part played by the groups O_{n-1} and U_{n-1} in the study of the collective degrees of freedom of the nucleus. However, it is inconvenient from the practical point of view. It is well known (see, for example, Ref. 27) that every "good" operator acting on a separate Hilbert space can be represented in matrix form. It is therefore clear that every method developed in the coordinate representation can also be formulated in matrix language. This can be achieved on a basis that is complete with respect to the physical operators that act on $\mathscr{R}$. A description of this basis, including references to original papers, can be found in Refs. 1–4.

The matrix of the operator $H_{0\,\text{coll}}$ is diagonal with respect to the U_{n-1}-irreducible representation $[E_1E_2E_3]$. The dimension of this representation is finite, so that the operator $H_{0\,\text{coll}}$ is also given by a finite-dimensional Hermitian matrix. This means that the Schrödinger equation

$$H_{0\,\text{coll}}\Psi_{0\,\text{coll}} = E_{0\,\text{coll}}\Psi_{0\,\text{coll}} \tag{124}$$

can be solved exactly. In other words, the expansion (123) has, from the point of view of practical application, the important property that its first term gives the exactly solvable part of the Hamiltonian H_{coll}. Therefore, all convergence problems disappear, and Eq. (124) can be solved for a wide range of mass numbers, including examples from the region of medium and heavy nuclei.

One can say something about the spectra of E_{coll} and $E_{0\,\text{coll}}$ by considering the degree of their degeneracy. In Sec. 7, we have already noted that H_{coll} acts on $\mathscr{R}^{(\Lambda\omega\alpha\lambda T)}$, and therefore the Hamiltonian H_{coll} conserves not only Λ but also the quantum numbers $\omega\alpha\lambda T$. Despite the circumstance that the orbital submatrix in (82) for the operator H_{coll} is degenerate with respect to the quantum numbers $\alpha\lambda$, the dependence of the matrix (82) on λ can be restored by means of the spin–isospin operators, whose matrix elements depend on $\tilde{\lambda}$, which, in its turn, stands in a one-to-one correspondence with λ. As a result, we find that $E_{\text{coll}} = E_{\text{coll}}(\pi J, \omega\lambda T)$, and because of the infinite dimensionality of the space $\mathscr{R}^{(\Lambda\omega)}$ this spectrum for given ω, λ, T consists of an infinite number of levels.

The orbital submatrix for the operator $H_{0\,\text{coll}}$ is degenerate with respect to the quantum numbers $\delta\omega\alpha\lambda$, but the dependence of its spectrum on λ can be restored for the same reason as for H_{coll}, and therefore $E_{0\,\text{coll}} = E_{0\,\text{coll}}(\pi J, E\lambda T)$. In contrast to the case of E_{coll}, this spectrum for given E, λ, T consists of a finite number of levels. The residual interaction $H_{\text{coll}} - \overline{H}_{\text{coll}}$ lifts the degeneracy with respect to $\delta\omega$, forming thereby from the finite bands of the spectrum $E_{0\,\text{coll}}$ the complete spectrum E_{coll} of the collective Hamiltonian H_{coll}.

It is interesting to see the extent to which the spectrum E_{coll} is degenerate as compared with the spectrum E of the total Hamiltonian H. The degree of degeneracy of E_{coll} can be estimated on the basis of the number of values of the repetition index α. It is known (see, for example, the tables in Appendix 12 in Ref. 4) that, as a rule, this index is unimportant for low-lying states, and this means that in many cases of practical importance the noncollective part $H - H_{\text{coll}}$ of the Hamiltonian H shifts but does not split the energy levels given by the microscopic collective model. This property of the spectrum E_{coll} (which was noted in Ref. 25) shows that the collective degrees of freedom also make their contribution to the formation of the overwhelming majority of the low-lying levels. It is important to establish the magnitude of the contribution; this problem requires a careful study.

With increasing energy of the collective excitations, i.e., for less symmetric Young tableaux λ, and also for less energetically advantageous values of the quantum number ω, the index α can take an ever larger number of values. Examples are known when this number reaches hundreds and even thousands. In these cases, because of the degeneracy with respect to α, the level density of the spectrum of the collective excitations differs more and more strongly from the level density of the spectrum of the total Hamiltonian H, and, provided the levels are sufficiently stationary, there arises an interesting effect, which was noted in Ref. 1, namely, multiple splitting of the levels of the collective Hamiltonian H_{coll} by the residual interaction $H - H_{\text{coll}}$. Without concrete calculations, it is difficult to establish the excitation energies at which this occurs.

Hitherto, we have studied the chain of spaces $\mathscr{R}^{(\Lambda)} \supset \mathscr{R}^{(\Lambda\omega)} \supset \mathscr{R}^{(\Lambda E)}$ and the corresponding chain of Hamiltonians $H \supset H_{\text{coll}} \supset H_{0\,\text{coll}}$ acting on these spaces. As we "descend" these chains, we increase the symmetry of the model Hamiltonians, simplifying thereby the problem of finding their eigenfunctions. We have seen in Sec. 7 that the collective term H_{coll} (or $H_{0\,\text{coll}}$) is contained only in the $S_n^{(\tau)} \times S_n^{(\sigma\tau)}$-scalar component $H^{[n]}$ of the Hamiltonian H. It follows from this that if we take into account only collective effects we completely ignore the nonsymmetric components $H^{[n-1,1]}$ and $H^{[n-2,2]}$. The magnitude of the matrix elements of these components was estimated in Ref. 48 (see also Ref. 4); it was found that these components are not small, which indicates that the terms $H^{[n-1,1]} + H^{[n-2,2]}$ are important. Therefore, in the restricted Hamiltonian H_0 one must take into account, besides

H_{coll} (or $H_{0\,\text{coll}}$), the terms that have nonvanishing symmetry components $[n-1,1]$ and $[n-2,2]$.

In the theory constructed in the framework of restricted dynamics we have already encountered such terms, the two operators H_1' and H_1'', which are obtained by restricting H to the O_3^+- and U_3-irreducible spaces, respectively. In the framework of restricted dynamics, they take into account the interaction between the collective and internal degrees of freedom of the nucleus, and also, in a certain sense, effects that are the opposite of collective. Let us briefly consider the meaning and importance of these operators.

Both the Hamiltonians H_1' and H_1'' have the symmetry terms $[n]$, $[n-1,1]$, and $[n-2,2]$, and, therefore, the matrix of the operator $\hat{\mathcal{O}}$ in (76) does not vanish for them. The orbital submatrices of these Hamiltonians depend on λ. In other words, in contrast to H_{coll} and $H_{0\,\text{coll}}$, the operators H_1' and H_1'' directly "feel" the orbital Young tableau. It is obvious that if special potentials of the nucleon–nucleon interaction are chosen this sensitivity can be raised to a desired amount. Then H_1' and H_1'' will begin to resemble pairing Hamiltonians, which, as is well known, have similar properties. We conclude that there is a certain similarity between the operators H_1' or H_1'' and short-range forces in the sense that both the former and the latter are sensitive to effects due to the Pauli principle.

Qualitatively and quantitatively, these effects can be most readily discussed for the example of the operator H_1'', which has an incomparable simpler structure than the operator H_1'. At the present time, the spectrum of the operator H_1'' is known for many states in the region of mass numbers $A \leqslant 40$. The levels in these spectra are arranged in such a manner that the states with the most symmetric Young tableaux are fairly clearly separated on the energy scale from the states with less symmetric λ (for details, see Refs. 4 and 1). We can therefore conclude that the operator H_1'' in (13) is the prototype of pairing forces. The Hamiltonian (13) also contains the term H_3'', which is a generalization of the operator of the quadrupole interaction considered in Ref. 49 acting in the framework of one SU_3 configuration. To summarize, we conclude that by restricting H to an irreducible space of the group $U_3 \times U_{n-1}$ we obtain effects similar to the effects responsible for long-range and short-range forces.

The structure of the operator H_1'' can be enriched if one "descends" down the chain of spaces $\mathcal{R}^{(\lambda E)} \supset \mathcal{R}^{(\lambda L)}$ defined by the chain of groups $U_3 \supset O_3^+$ and, accordingly, "ascends" the chain of Hamiltonians $H_1'' \subset H_1'$. Unfortunately, the operator H_1' in (12) is too complicated, which greatly hinders study of the properties of its eigenstates. In fact, H_1' consists of a sum of the kinetic-energy operator and a central interaction (and also, if necessary, a Coulomb term), only the O_{n-1}-scalar part being eliminated from the sum. In order to have a tractable Hamiltonian, we must strongly simplify H_1'. It appears attractive to use H_1' for a short-range interaction of the form

$$V(|\mathbf{r}_i-\mathbf{r}_j|)=V_0\delta(|\mathbf{r}_i-\mathbf{r}_j|), \tag{125}$$

where V_0 is some constant. We introduce the Hamiltonian

$$H_{\text{coll}+\text{pair}}=H_{\text{coll}}+H_{\text{pair}}. \tag{126}$$

where H_{pair} is H_1' with potentials of a central internucleon interaction of the form (125). The operator $H_{\text{coll}+\text{pair}}$ acts on some subspace $\bar{\mathcal{R}}^{(\Lambda)}$ of $\mathcal{R}^{(\Lambda)}$, but the specific characteristics of $\bar{\mathcal{R}}^{(\Lambda)}$ are not known. The operator H_{pair} violates the O_{n-1}-scalar property in (126) that the Hamiltonian H_{coll} possesses. The operator (126) generalizes the Hamiltonian used in nuclear theory that contains a quadrupole interaction and pairing forces. The problem of studying the eigenfunctions and spectrum of the operator $H_{\text{coll}+\text{pair}}$ is complicated and can be solved comparatively easily only in the case of few-nucleon systems.

CONCLUSIONS

Let us recall the main premises of the theory presented here of finite, spatially well-localized Fermi systems consisting of identical particles and discuss some of its aspects. As was emphasized in the Introduction, the mathematical idea used in the construction of the theory enables one to project out of complicated operators simpler operators used for developing the constructive theory of the nucleus. The physical premises that precede this mathematical idea are associated with the extremely specific conditions that exist in finite Fermi systems[3]: From the point of view of "our" three-dimensional space, in which the nucleons move, there are no distinguished force centers in the nucleus, so that there are no weighty grounds for ascribing three-dimensional space a particular role compared with the other, $(n-1)$-dimensional space whose dimension is determined by the number of quasiparticles. It is necessary to construct a theory that takes into account in equal measure the effects associated with the presence of both spaces.

For this reason, it is convenient to regard from the very beginning the orbital part of the wave function as a function that depends on the components of the $r_0 r$-dimensional vector of the Euclidean space $\mathcal{P}_{r_0 r}$, where $r_0=3$ and $r=n-1$. This space encompasses all orbital degrees of freedom of the nucleus. From $\mathcal{P}_{r_0 r}$, one then projects the subspace $\mathcal{P}_{r_0}\times\mathcal{P}_r$, the direct product of the two equally important spaces $\mathcal{P}_{r_0}$ and $\mathcal{P}_r$, whose introduction reflects the two types of indices of the orbital variables of the nucleus. Then, roughly speaking, we obtain a description of the collective effects by averaging the dynamical equations over the indices of the space $\mathcal{P}_r$ and a description of the effects that are the opposite of collective by averaging over the indices of the space $\mathcal{P}_{r_0}$. The algebraic formalism of either the orthogonal groups O_r and O_{r_0} or the unitary groups U_r and U_{r_0} provides the technique of such averaging. In particular, analysis of the space on which the Hamiltonian of the microscopic generalized rotational-vibrational model of the nucleus acts has shown that this model corresponds to a picture that can be said to be deeply frozen in $\mathcal{R}_r$—when the operators of the group O_r acting on this space give zero, i.e., when the motion

in $\mathscr{P}_r$ is completely ignored in the theory.

Although in the present review we have given more prominence to the effects resulting from averaging over the indices of the space $\mathscr{P}_r$, there is nevertheless a complete symmetry in the initial stage of the construction of the theory with respect to the spaces $\mathscr{P}_r$ and $\mathscr{P}_{r_0}$. The difference becomes noticeable on the transition to the exact integrals of the motion. Apart from the parity π, none of them is due solely to the orbital degrees of freedom: To obtain JM_J, it is necessary to have the quantum numbers LM and SM_S, and similarly, to construct an antisymmetric function, it is necessary to have $\lambda\mu$ and $\tilde{\lambda}\tilde{\mu}$.

On the one hand, the experimental data are directly related to the value of the quantum number J, and, therefore, indirectly to L and S when they are combined into J. It has been established in the general theory that J can be introduced even when the motion in $\mathscr{P}_r$ is frozen. On the other hand, the experimental data are indirectly related to the antisymmetry, and hence, to λ and $\tilde{\lambda}$ when they are combined into a. It is known from the general theory that a can be introduced even when the motion in $\mathscr{P}_{r_0}$ is frozen. However, the quantum number a is fixed, whereas J takes numerous values; this is the reason why the spectra of the operators H_{coll} and $H_{0\,\text{coll}}$ are rich and characterized by many values of the observable quantum numbers, while the spectra of the operators H_1' and H_1'' (for details, see Ref. 1) do not give them.

This also enables us to understand why the first stage in the development of the phenomenological rotational–vibrational model of the nucleus, in which the single-particle degrees of freedom are not yet taken into account, so emphasizes collective effects—they are associated with observable quantum numbers. We have also shown that because of the inadmissibly deep freezing in $\mathscr{P}_r$ (which arose because of the use of bilinear forms instead of $\rho^{(s_0)}$) the eigenvectors of the Hamiltonian, which depends on $(\rho^{(s_0)})^2$, lie in the unphysical subspace $\mathscr{R}^{(\omega)}$ of $\mathscr{R}^{(\omega)}$. As a consequence, in such an approach many important features of the spectrum of the real collective excitations are lost.

The further development of the phenomenological theory was directed toward allowance for the coupling between the collective and the single-particle motion. In a kinematically correct theory there is, strictly speaking, no place for such coupling, and this is not only because the kinematics prohibits the occurrence in a nucleus of a single-particle field, which can be replaced by a quasiparticle field (for details, see Refs. 1–3). The principal reason is that the use of collective variables rules out the possibility of introducing single-quasiparticle variables. This follows from a simple count: The total number of variables is $3(n-1)$, and six of them are collective, so that there remain $3(n-3)$ variables, and this number is already insufficient to ascribe a triplet of variables to each of the $n-1$ quasiparticles. We therefore use the set of internal variables and speak of the interaction between the collective and internal degrees of freedom of the nucleus.

The theory expounded in the present paper is based on Eq. (7), which guarantees the condition for restricted dynamics. The condition appears simple only on first acquaintance. In reality, it is difficult to satisfy. How can one isolate some box, cutting off all the nondiagonal elements that connect it to the remaining part of the matrix of the energy operator? How can one find a suitable mechanism for cutting off the nondiagonal elements capable of yielding a restricted Hamiltonian H_0 that is neither too complicated nor too trivial? From the technical point of view, this can be done only through a very specific realization of the Hilbert space based on purely algebraic properties of the constructed basis. What is the reason for this? We recall the following detail, which relates to the reduction rules in accordance with the chain $U_{3(n-1)} \supset U_3 \times U_{n-1}$; namely, the $U_{3(n-1)}$-irreducible representations $[E0\ldots0]$ in this chain decompose into the representations $[E_1E_2E_3]$, which simultaneously characterize the irreducible spaces for both groups U_3 and U_{n-1}. In other words, the U_3-irreducible representation uniquely determines the U_{n-1}-irreducible representation, and vice versa. This is the good property of the basis that ensures the possibility of projecting operators with restricted dynamics; for when one makes the restriction to the irreducible space of the group U_{n-1} (freezing the motion in $\mathscr{P}_{n-1}$), all the diagonal elements are cut off with respect to not only the quantum numbers of the group U_{n-1} but also those of U_3. Despite this, the motion in the three-dimensional space $\mathscr{P}_3$ is not frozen and one obtains a fairly complicated but nevertheless solvable Hamiltonian $H_{0\,\text{coll}}$. The parts played by the spaces $\mathscr{P}_{n-1}$ and $\mathscr{P}_3$ are interchanged when the restriction is made on the irreducible space of the group U_3.

From the mathematical point of view, the unitary group $U_{3(n-1)}$ gives a large set of spaces, which are labeled as $[E_1E_2\ldots E_{3(n-1)}]$. These spaces are spanned by functions that, in general, depend on a large number of variables, this number being much greater than the number of orbital degrees of freedom of the nucleus (see, for example, Ref. 50). If these representations are reduced in accordance with the chain $U_3 \times U_{n-1}$, it is found that, besides the case of representations of the class $[E0\ldots0]$, the irreducible representations of one of the groups U_3 or U_{n-1} do not uniquely determine an irreducible representation of the other group and, therefore, there is no longer place for the freezing property noted above. Why is it that in nuclear theory one uses such a negligible fraction of these spaces, namely, the space $[E0\ldots0]$, which is so suitable for our purposes? The point is that the many-particle problem is studied in nuclear theory in the framework of nonrelativistic quantum mechanics. Then the number of particles and their species are conserved and, as a consequence of this, the Hilbert space on which the nuclear Hamiltonian acts can be spanned by the $U_{3(n-1)}$-irreducible spaces $[E0\ldots0]$.

We note also that the realization of the Hilbert space by means of functions defined on the corresponding factor space of the compact group $U_{3(n-1)}$ reflects the properties of compactness of the physical system,

i.e., the good localization in space of the nucleus. The possibility of extending the Hilbert space to an equipped Hilbert space was noted in Sec. 8.

One cannot hope to elucidate the structure of the nucleus without simplification to a reasonable level of the dynamical many-body problem. In the theory presented in this review, the many-particle problem is simplified by separating H_0' or H_0'' from H, this revealing certain smoothed properties of the original Hamiltonian. The obtained picture of nuclear structure is hidden in the Hamiltonians H_0' or H_0'' and much work must still be done to elucidate all its details. The potentials of the nucleon–nucleon interaction are not known, which dictated the adaptation of the presented theory to an arbitrary Hamiltonian H, i.e., to the set of Hamiltonians that act on the Hilbert space $\mathscr{R}^{(A)}$. This feature of the theory, which is very valuable on account of its generality, raises a number of questions, the answers to which must be found in the future. One of them concerns the sensitivity of the results to the details of the nucleon–nucleon interaction. How does a change in the form or intensity of this interaction affect the expectation values of the various operators calculated on the eigenstates of the Hamiltonians H_0' or H_0''? In particular, how sensitive are the observable quantities to the form of the potentials of the central, vector, and tensor forces?

This question is too complicated for one to be able to answer it in a general form. The sensitivity of the results obtained with approximate wave functions is different for different observables such as, for example, the spectrum, the matrix elements of the operators describing the shape of the nucleus, the transition probabilities, and the other static and dynamic characteristics of nuclei. Of course, the most important of them is the energy spectrum, since one can speak of the properties of the given state only once one has established its energy and other quantum numbers.

With regard to the questions posed above, the Hamiltonian H_0'' has the technically valuable property that the Schrödinger equation for it can be solved exactly. In a certain sense, H_0'' is the largest part of the Hamiltonian H that can still be exactly solved; it is to be expected that as long as H is an arbitrary Hamiltonian each term of it additional to H_0'' will lead to an operator for which the Schrödinger equation is not amenable to analytic methods of solution.

At the first glance, the assertion that every H contains an exactly solvable part H_0'' appears very unexpected. The opinion has become too firmly established that in the quantum theory of many-particle systems there are only a few exactly solvable Hamiltonians such as the harmonic-oscillator Hamiltonian and some other simple Hamiltonians. This opinion arose because of the tradition of dealing with operators in the coordinate representation. However, this is not the only way of specifying them in a separable Hilbert space.[27] Frequently, it is more convenient to use the more flexible matrix representation. It is precisely this flexibility that made it possible to project from the arbitrary H its exactly solvable part H_0''. The advantage of the matrix representation over the coordinate representation can be seen by attempting to construct H_0'' in the purely coordinate representation. This would lead to intractable expansions, some impression of which can be obtained from the complete operator basis adapted to such expansions described in Ref. 1.

In the quantum theory of many-particle systems, exactly solvable Hamiltonians are usually called zeroth-order Hamiltonians. Following this tradition, we call H_0'' the *zeroth-order Hamiltonian with restricted dynamics*. This Hamiltonian defines a certain model of the nucleus, which was proposed in Ref. 5. It reflects, albeit in a very simplified form, the dynamics inherent in the Schrödinger equation for the original Hamiltonian H. As we have emphasized at the corresponding places in this review, H_0'' is generated by the pair of operators H and h, where h is the "little" kinematic Hamiltonian, which even in a simplified form is not capable of reflecting the properties of the original "large" Hamiltonian H.

The picture of nuclear structure described by the eigenfunctions of h is very simple. In accordance with this picture, the nucleus consists of a system of noninteracting quasiparticles, the ensemble of which reflects only the simplest kinematic properties of H (for details, see Refs. 2 and 3). When the pair H and h generate the Hamiltonian H_0'', a dynamical picture of nuclear structure is introduced. The original Hamiltonian H, which is enclosed in a double E box (double, because E appears as U_3- and U_{n-1}-irreducible representation), is transformed into the model Hamiltonian H_0'', which consists of the two terms H_1'' and $H_{0\,\mathrm{coll}}$, which have directly opposite physical meanings. One of them, namely $H_{0\mathrm{coll}}$, is the collective term in the space $\mathscr{P}_3$. The other, H_1'', is also a "collective" term, not in $\mathscr{P}_3$, but in the space $\mathscr{P}_{n-1}$. Indeed, in the space in which the coordinate axes are labeled by the indices of the quasiparticles, "collective" means an averaging over the Cartesian indices. From the point of view of "our" space $\mathscr{P}_3$, this type of freezing generates effects that may be called anticollective.[1]

These two forms of interaction give rise to very definite features of the spectrum, some properties of which are discussed in Ref. 1. In particular, it has been shown that, in general, the Hamiltonians H_1'' and $H_{0\,\mathrm{coll}}$ do not commute, although one does sometimes encounter $U_3 \times U_{n-1}$ boxes of E in which they do commute. Therefore, the diagonalization of the matrix of H_0'' leads to a certain mixing of the collective and "anticollective" effects. The vector and tensor terms of the operator $H_{0\,\mathrm{coll}}$ give rise to a specific splitting of the spectrum of the operator generated by the central interaction. The nature of this splitting depends, among other factors, on the characteristics of the given E box in which the Hamiltonian H is confined. In some boxes, the vector and tensor terms do not play any part, and one can expect a spectrum that in the case of even nuclei is similar to the spectrum of the phenomenological theory. But in other boxes, only the tensor term vanishes, and all the splittings with respect to the quantum number J are due to vector forces

alone. In the phenomenological theory, purely collective effects arise in even nuclei; in the microscopic theory, they are equally important for both even and odd n. The eigenfunctions of the Hamiltonian H_0'' are fairly complicated and varied and not readily amenable to perspicuous interpretation.

The structure of the zeroth-order Hamiltonian H_0'' is incomparably richer than the structure of the Hamiltonian h. Nevertheless, H_0'' should be regarded as the zeroth-order Hamiltonian, with all the consequences that follow from this interpretation. It must not be expected that its eigenfunctions are capable of giving a sufficiently good description of the majority of properties of low-lying levels. Nuclei are too complicated objects for one to be able to hope to explain their properties with the wave functions of the zeroth-order nuclear Hamiltonian. Some important features of nuclear structure are certainly hidden in the residual interaction $H - H_0''$. It is possible that one could identify a fairly simple term in H responsible for the strong interaction between the orbital and spin degrees of freedom of the nucleus (in the language employed in the shell model, it is usually called the strong spin–orbit interaction). In the construction of nuclear theory in the framework of restricted dynamics, allowance is made for collective and "anticollective" effects, which can be studied in either the exactly solvable (Hamiltonian H_0'') or approximately solvable (Hamiltonian H_0') variants of the theory. In both cases, allowance is also made for the coupling between the orbital and spin degrees of freedom, which can be seen from the structure of the matrix (82), though this interaction is inadequately taken into account in this form. Heuristic ideas indicating how this can be done can be obtained from the properties of the expansion (59). The mathematical formalism needed for such investigations has been developed fairly fully and some details are also known. However, these questions go beyond the scope of the present review.

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