Distorted-wave method in reactions with complex particles

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The application of the finite-range distorted-wave method (FRDWM) to the quantitative analysis of the characteristics of direct nuclear reactions with complex particles (including heavy ions) with allowance for reaction mechanisms different from the cluster stripping mechanism, in particular exchange processes, is considered. The exact equations of the distorted-wave method are obtained in the framework of the three-body problem together with general formulas for calculating the differential cross sections of arbitrary binary reactions by means of the FRDWM. Methods of exact and approximate allowance for a finite range of the interaction are discussed. Specific FRDWM programs are analyzed and their applicability is considered.

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INTRODUCTION

In recent years, there has been increased interest in nuclear reactions with complex particles (heavy ions, ^{6}Li, ^{7}Li, ^{9}Be ions, α particles), and experimental information has already been accumulated on the angular distributions of the final particles in these reactions, in some cases in a fairly wide range of angles.

The theoretical analysis of such results entails a generalization of the usual distorted-wave method to the case of reactions with complex particles. The generalization must proceed in two directions: first, to analyze the angular distributions of reactions with complex particles, it is necessary to use the distorted-wave method with allowance for the finite size of the interaction region of the particles; second, it is necessary to take into account the different reaction mechanisms, and not only the cluster stripping mechanism, as is done in the usual distorted-wave method. The systematic treatment of such problems requires solution of the many-body problem. However, in some practically important cases the problem of calculating the cross section of the nuclear reaction $A(x,y)B$ (in what follows, to be specific, we shall assume throughout that $x \geqslant y$) can be simplified and reduced to a three-body problem. This can be done if 1) the incident particle x dissociates in the field of nucleus A into two clusters, $x \to y + c$, and nucleus A remains unchanged; 2) particle x is unchanged, and nucleus A decomposes into two parts, $A \to y + C$, one of which is a final particle. Processes of the first kind associated with the break-up of the incident particle can be called *direct*, and processes of the second kind involving the break-up of the target nucleus can be called *exchange processes*. In the approximation of the three-body problem, the direct and exchange processes are not coupled and there is no interference between them. It is easy to see that the amplitudes of the direct and the exchange processes are equal, apart from the replacement of A by x (x by A) and of the angle θ of emission of the produced particles y in the center-of-mass system by the angle $\pi - \theta$. Therefore, in what follows, we shall give all the formulas basically for direct processes and obtain the expressions for the exchange processes by the substitution $A \rightleftarrows x$ and $\theta \to \pi - \theta$.

The aim of the present review is to analyze the present state of the distorted-wave method as applied to nuclear reactions with complex particles. In Sec. 1, we derive the exact equations of the distorted-wave method in the framework of three-body problems. Analysis of these equations shows that in such an approach there exist four reaction mechanisms (stripping and heavy replacement, which are direct processes, and ordinary replacement and heavy stripping, which are exchange processes). In Sec. 2, we give general formulas for calculating the differential cross sections of nuclear reactions by the finite-range distorted-wave method, and we consider possible simplifications of these formulas for definite types of reaction; we also discuss methods of separation of variables in the radial integral. Section 3 is devoted to methods of approximate allowance for recoil effects and the finite range of the interaction. In Sec. 4, we discuss two main variants of exact allowance for recoil effects in the distorted-wave method, namely, separation of the variables in the wave functions of the relative motion of the particles and potentials of the interaction, and separation of the variables in the distorted waves. In Sec. 5, we analyze the computational programs known in the literature based on the "exact" and the approximate variants of the distorted-wave method, we compare the results of the calculations made with the different programs, and we discuss the regions of applicability of the individual programs.

These questions have not hitherto been considered together in the world literature and the present review should, at least partly, fill this gap.

1. EXACT EQUATIONS OF THE DISTORTED-WAVE METHOD IN THE FRAMEWORK OF THE THREE-BODY PROBLEM

Usual Treatment of the Distorted-Wave Method. We consider the reaction $A(x,y)B$, which proceeds as follows:

$$A + x \to A + (c + y) \to (A + c) + y \to B + y.$$

The matrix element of this reaction can be written in the form

$$M_{fi} = \langle \Phi_f \varphi_B \zeta_y \mid V_f \mid \varphi_x \varphi_A \Psi_i^{(+)} \rangle \equiv \langle \Psi_f^{(-)} \varphi_B \zeta_y \mid V_i \mid \varphi_x \varphi_A \Phi_i \rangle, \qquad (1)$$

where $\Psi_i^{(+)}$ and $\Psi_f^{(-)}$ are the exact solutions of the Schrödinger equation for the ingoing and outgoing waves:

$$\Psi_i^{(+)} = \Phi_i + \frac{1}{E - H_i + i\eta} V_i \Psi_i^{(+)}, \quad H_i = H_0 + V_{yc}; \qquad (2)$$

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$$\Psi_f^{(-)} = \Phi_f + \frac{1}{E - H_f - i\eta} V_f \Psi_f^{(-)}, \quad H_f = H_0 + V_{AC}; \tag{2a}$$

$\mathbf{V_f} = \mathbf{V_{yc}} + \mathbf{V_{yA}}$ and $(\mathbf{V_i} = \mathbf{V_{Ac}} + \mathbf{V_{yA}})$ are the interactions in the final and initial states; Φ_i and Φ_f are the plane waves of the relative motion of the incident and emitted particle in the field of the nucleus: $\varphi_A, \varphi_B, \varphi_x, \varphi_y$ are the internal wave functions of the corresponding particles; $\mathbf{H_0}$ is the Hamiltonian of the free motion of the noninteracting particles A, c, and y with total energy E.

We shall calculate the matrix element (1) using the distorted-wave method.[1] By the "exact" distorted-wave method (FRDWM),[2] which takes into account the finite range of the interaction between the particles, we shall in what follows mean the method of calculating the matrix element (1) when the interactions responsible for the rearrangement of the system from the initial to the final state are taken to be the same as in the three-body problem, and the interactions of the initial and final particles with the corresponding nuclei are described in the approximation of the optical model, i.e., in the approximation of the two-body problem, allowance being made for only elastic scattering by the complex potential. The validity of such treatment is still unclear; moreover, it is known that allowance for the many-body, in particular, the three-body corrections to the optical potential[3] must influence its characteristics. Nevertheless, the optical model is as yet the most developed method for describing the elastic scattering of particles by nuclei.

Let us consider the derivation of the exact equations of the FRDWM.[1,4] We introduce the wave functions $\chi_f^{(-)}$ and $\chi_i^{(+)}$ (distorted waves), which are solutions of the Schrödinger equation in the optical potentials $\mathbf{U_f}$ and $\mathbf{U_i}$:

$$\chi_f^{(-)} = \Phi_f + \frac{1}{E - H_f - i\eta} U_f \chi_f^{(-)}. \tag{3}$$

Similarly,

$$\chi_i^{(+)} = \Phi_i + \frac{1}{E - H_i + i\eta} U_i \chi_i^{(+)}. \tag{3a}$$

From (3) we determine the function Φ_f and substitute it in (1). Then for the matrix element of the reaction, we readily obtain the expression

$$\begin{aligned} M_{if} = {} & \langle \chi_f^{(-)} \varphi_B \varphi_y \mid V_f - U_f \mid \varphi_x \varphi_A \Psi_i^{(+)} \rangle \\ & + \langle \chi_f^{(-)} \varphi_B \varphi_y \mid U_f \mid \varphi_x \varphi_A \Psi_i^{(+)} \rangle \\ & - \langle \chi_f^{(-)} \varphi_B \varphi_y \mid U_f \frac{1}{E - H_f + i\eta} V_f \mid \varphi_x \varphi_A \Psi_i^{(+)} \rangle. \end{aligned} \tag{4}$$

Substituting in the second term of Eq. (4) the expression (2) for $\Psi_i^{(+)}$, we obtain

$$\begin{aligned} M_{if} = {} & \langle \chi_f^{(-)} \varphi_B \varphi_y \mid V_f - U_f \mid \varphi_x \varphi_A \Psi_i^{(+)} \rangle \\ & + \langle \chi_f^{(-)} \varphi_B \varphi_y \mid U_f \mid \varphi_x \varphi_A \Phi_i \rangle. \end{aligned} \tag{5}$$

Using the identity

$$\langle \chi_f^{(-)} \mid U_f \mid \Phi_i \rangle \equiv \langle \Phi_f \mid U_i \mid \chi_i^{(+)} \rangle,$$

we can rewrite the matrix element of the reaction in the more perspicuous form

$$\begin{aligned} M_{if} = {} & \langle \chi_f^{(-)} \varphi_B \varphi_y \mid V_f - U_f \mid \varphi_x \varphi_A \Psi_i^{(+)} \rangle \\ & + \langle \Phi_f \varphi_B \varphi_y \mid U_i \mid \varphi_x \varphi_A \chi_i^{(+)} \rangle. \end{aligned} \tag{6}$$

The expressions (5) and (6) for the reaction matrix element do not contain further approximations beyond those contained in the very definition of the distorted-wave method and, formally, they are exact.

In accordance with Ref. 5, the first term in (6) can be transformed by using the distorted wave (3a) in the initial reaction channel. Then $\Psi_i^{(+)}$ can be written as

$$\left.\begin{aligned} \Psi_i^{(+)} &= \chi_i^{(+)} + \frac{1}{E - H_i + i\eta} (V_i - U_i) \Psi_i^{(+)} \\ &= \chi_i^{(+)} + \frac{1}{E - \tilde{H} + i\eta} (V_i - U_i) \chi_i^{(+)}; \\ \tilde{H} &= H_0 + V_i - U_i. \end{aligned}\right\} \tag{7}$$

Substituting the expression (7) in the first term of the matrix element (5), we obtain[5]

$$\begin{aligned} M_{if} = {} & \langle \chi_f^{(-)} \varphi_B \varphi_y \mid (V_f - U_f) \\ & + (V_f - U_f) \frac{1}{E - \tilde{H} + i\eta} (V_i - U_i) \mid \varphi_x \varphi_A \chi_i^{(+)} \rangle \\ & + \langle \Phi_f \varphi_B \varphi_y \mid U_i \mid \varphi_x \varphi_A \chi_i^{(+)} \rangle. \end{aligned} \tag{8}$$

This form of expression of the matrix element corresponds to the introduction of a T matrix that is formally not related to the distortions of the initial and final particles.

Thus, we have obtained the expressions (6) and (8) for the matrix element in the FRDWM. Although these equations can be solved, for example, by means of the formalism of the integral Faddeev equations,[6] the possibilities of present-day computers are quite inadequate for this. Therefore, for practical calculations it is necessary to introduce various approximations in Eqs. (6) and (8). One of them is associated with the fact that the exact solution in the expression (6) is replaced by a distorted wave, or, which is essentially the same, the terms with the Green's function in the first term of Eq. (8) are omitted (ordinary distorted-wave method[1]). Physically, this approximation means that one uses not the exact T matrices describing the rearrangement process, but local operators—interaction potentials at each vertex; thus, one does not take into account possible many-step reaction mechanisms. It is easy to see that this approximation is not fundamental in nature, and retardation effects can at least be taken into account in the first order. There have by now been developed, in particular, methods for calculating two-step mechanisms in reactions with complex particles[7] that enable one to estimate the influence of retardation effects on the shape of the angular distributions. In what follows, we shall restrict ourselves to considering single-step reaction mechanisms.

However, in the usual calculations in the FRDWM one frequently introduces two further important approximations, which are much harder to justify: first, in the exact expressions (6) and (8) the second term is omitted; second, it is assumed that the interaction $\mathbf{V_{yA}}$ from the total interaction in the final state, $\mathbf{V_f} = \mathbf{V_{yc}} + \mathbf{V_{yA}}$, is compensated by the potential $\mathbf{U_f}$ of the optical model, so that the FRDWM matrix element is determined solely by the interaction $\mathbf{V_{yc}}$.

It is obvious that in reactions with complex particles such approximations are not satisfied. Physically, this is due to the fact that the mechanism of such reactions does not, as a rule, reduce solely to the mechanism of cluster stripping and must include more complicated mechanisms, including exchange mechanisms. Above all, these approximations cannot be justified because

the optical potential is introduced artificially in the method of calculating the matrix element considered above, and therefore the neglect of one of the terms of the total amplitude is in no way justified. In addition, the impossibility of complete compensation of the optical potential U_f and the interaction V_{yA} follows directly from the formal point of view from the fact that U_f and V_{yA} depend on different radial variables.[8] Therefore, compensation will hold only in the cases when the mass of the target nucleus can be assumed to be infinitely large—a condition that is hardly ever satisfied in reactions with complex particles.

In order to understand what mechanisms must be taken into account in the FRDWM, staying within the framework of single-step processes, it is necessary to take into account more systematically the many-body (and, as a first step, the three-body) nature of the problem. Therefore, in the following section we shall consider the derivation of the FRDWM equations in the framework of the three-body problem and show that the two above-mentioned approximations cannot be used in the general case to analyze reactions with complex particles.

Derivation of the Equations of the Distorted-Wave Method in the Framework of the Three-Body Problem. We consider the calculation of the matrix element of the reaction $A(x,y)B$ in the framework of the three-body problem. In this approximation, the total potential contains three terms, two of which determine pairwise the interactions in the initial and final states. For the direct processes,

$$\mathbf{H}=\mathbf{H}_0+\mathbf{V};\ \mathbf{V}=\mathbf{V}_{yc}+\mathbf{V}_{yA}+\mathbf{V}_{cA}, \tag{9}$$

and, as we have said above,

$$\mathbf{V}_i=\mathbf{V}_{yA}+\mathbf{V}_{cA};\quad \mathbf{V}_f=\mathbf{V}_{yc}+\mathbf{V}_{yA}. \tag{10}$$

We introduce the truncated Hamiltonian

$$\mathbf{H}'=\mathbf{H}_0+\mathbf{V}_{cA}+\mathbf{V}_{yc} \tag{11}$$

and shall calculate the matrix element (1) in terms of the eigenfunctions of the truncated Hamiltonian (11), which satisfy the Schrödinger equation

$$(\mathbf{H}_0+\mathbf{V}_{cA}+\mathbf{V}_{yc}-E)\,\zeta=0.$$

This equation can be rewritten in the form of an integral Lippmann–Schwinger equation in two equivalent forms by choosing differently the boundary conditions at infinity along the time axis. If $t\to-\infty$, i.e., the asymptotic behavior of the wave function is expressed in terms of the eigenfunctions of the Hamiltonian $\mathbf{H}_i$, then

$$\zeta_i^{(+)}=\Phi_i+\frac{1}{E-\mathbf{H}_i+i\eta}\mathbf{V}_{cA}\zeta_i^{(+)}. \tag{12}$$

On the other hand, this same solution can be expressed in terms of the eigenfunctions of the Hamiltonian $\mathbf{H}_f$ by using the boundary conditions as $t\to\infty$. Then

$$\zeta_f^{(-)}=\Phi_f+\frac{1}{E-\mathbf{H}_f-i\eta}\mathbf{V}_{yc}\zeta_f^{(-)}. \tag{13}$$

In accordance with (1), the exact matrix element of the reaction $A(x,y)B$ contains a sum of two terms:

$$\mathbf{M}_{if}=\langle\Phi_f\varphi_B\varphi_y\mid\mathbf{V}_{yA}\mid\varphi_x\varphi_A\Psi_i^{(+)}\rangle+\langle\Phi_f\varphi_B\varphi_y\mid\mathbf{V}_{yc}\mid\varphi_x\varphi_A\Psi_i^{(+)}\rangle. \tag{14}$$

We transform the first term in (14). For this, we use Eq. (13) to express the plane wave Φ_f of the relative motion of the final particles:

$$\Phi_f=\zeta_f^{(-)}-\frac{1}{E-\mathbf{H}_f-i\eta}\mathbf{V}_{yc}\zeta_f^{(-)}=\zeta_f^{(-)}-\frac{1}{E-\mathbf{H}'-i\eta}\mathbf{V}_{yc}\Phi_f. \tag{15}$$

Then the first term in (14) takes the form

$$\langle\zeta_f^{(-)}\varphi_B\varphi_y\mid\mathbf{V}_{yA}\mid\varphi_x\varphi_A\Psi_i^{(+)}\rangle-\langle\Phi_f\varphi_B\varphi_y\mid\mathbf{V}_{yc}\frac{1}{E-\mathbf{H}'+i\eta}\mathbf{V}_{yA}\mid\varphi_x\varphi_A\Psi_i^{(+)}\rangle. \tag{16}$$

We express the exact solution of the total Hamiltonian (9) in terms of the function $\zeta_i^{(+)}$. We have

$$\Psi_i^{(+)}=\Phi_i+\frac{1}{E-\mathbf{H}_i+i\eta}\mathbf{V}_i\Psi_i^{(+)}=\zeta_i^{(+)}+\frac{1}{E-\mathbf{H}'+i\eta}\mathbf{V}_{yA}\Psi_i^{(+)}=\zeta_i^{(+)}+\frac{1}{E-\mathbf{H}+i\eta}\mathbf{V}_{yA}\zeta_i^{(+)}. \tag{17}$$

It follows that the action of the operators $(E-\mathbf{H}'+i\eta)^{-1}\mathbf{V}_{yA}$ on the function $\Psi_i^{(+)}$ reduces to the difference $\Psi_i^{(+)}-\zeta_i^{(+)}$. Therefore, the expression (16) can be transformed to

$$\langle\zeta_f^{(-)}\varphi_B\varphi_y\mid\mathbf{V}_{yA}\mid\varphi_x\varphi_A\Psi_i^{(+)}\rangle-\langle\Phi_f\varphi_B\varphi_y\mid\mathbf{V}_{yc}\mid\varphi_x\varphi_A\Psi_i^{(+)}\rangle+\langle\Phi_f\varphi_B\varphi_y\mid\mathbf{V}_{yc}\mid\varphi_x\varphi_A\zeta_i^{(+)}\rangle. \tag{18}$$

Substituting (18) in the expression (14) for the matrix element (1), we obtain

$$\mathbf{M}_{if}=\langle\Phi_f\varphi_B\varphi_y\mid\mathbf{V}_{yc}\mid\varphi_x\varphi_B\zeta_i^{(+)}\rangle+\langle\zeta_f^{(-)}\varphi_B\varphi_y\mid\mathbf{V}_{yA}\mid\varphi_x\varphi_A\Psi_i^{(+)}\rangle. \tag{19}$$

The expression (19) is in fact a generalization of the Gell-Mann–Goldberger theorem[9] to the case of nuclear reactions considered in the approximation of the three-body problem. To elucidate the physical meaning of both terms in (19), we express the exact solution $\Psi_i^{(+)}$ and the plane wave Φ_f in terms of the function ζ, using the relations (15) and (17). Then the expression (19) is rewritten as

$$\mathbf{M}_{if}=\langle\zeta_f^{(-)}\varphi_B\varphi_y\mid\mathbf{V}_{yc}-\mathbf{V}_{yc}\frac{1}{E-\mathbf{H}_f+i\eta}\mathbf{V}_{yc}\mid\varphi_x\varphi_A\zeta_i^{(+)}\rangle+\langle\zeta_f^{(-)}\varphi_B\varphi_y\mid\mathbf{V}_{yA}+\mathbf{V}_{yA}\frac{1}{E-\mathbf{H}+i\eta}\mathbf{V}_{yA}\mid\varphi_x\varphi_A\zeta_i^{(+)}\rangle. \tag{20}$$

Equation (20) is an exact expression for the reaction matrix element for the three-body problem. The first term in this equation corresponds to direct capture of particle c, i.e., it can be associated with the cluster stripping mechanism with complete allowance for the interactions in the final state. The second term describes the scattering of particle y, which is bound in the initial projectile x, by the target nucleus A. This mechanism is usually called heavy replacement. Graphically, the matrix element (20) is shown in Fig. 1a, the pole graph corresponding to the cluster stripping mechanism and the triangle graph to the heavy replacement mechanism.

A similar expression can also be written down for the matrix element of the exchange processes:

$$\mathbf{M}_{if}=\langle\zeta_f^{(-)}\varphi_B\varphi_y\mid\mathbf{V}_{yc}-\mathbf{V}_{yc}\frac{1}{E-\mathbf{H}_f+i\eta}\mathbf{V}_{yc}\mid\varphi_x\varphi_A\zeta_i^{(+)}\rangle+\langle\zeta_f^{(-)}\varphi_B\varphi_y\mid\mathbf{V}_{xy}+\mathbf{V}_{xy}\frac{1}{E-\mathbf{H}+i\eta}\mathbf{V}_{xy}\mid\varphi_x\varphi_A\zeta_i^{(+)}\rangle. \tag{21}$$

The first term in the expression (21) corresponds to the interaction of the final particle y and the intermediate nucleus C in the initial nucleus A, i.e., it corresponds to the stripping of a heavy particle (pole graph, see Fig. 1b). The second term, which describes the scattering of particles x and y in the field of the inter-

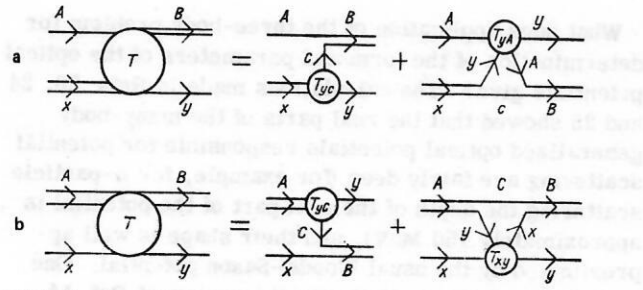

FIG. 1. Graphical representation of the direct (a) and exchange (b) processes in the reaction $A(x, y)B$ in the approximation of the three-body problem.

mediate nucleus C, corresponds to the mechanism of ordinary substitution of clusters (triangle graph, see Fig. 1b).

It follows from the expressions (21) and (22) that in each group of processes the individual mechanisms must add coherently and their interaction must be taken into account. In the framework of the three-body problem, the direct and exchange processes introduce an incoherent contribution to the total reaction cross section (for more details, see Ref. 10).

Finally, we introduce the optical potentials in the exact matrix elements (20) and (21). For this, we express the solutions $\zeta_i^{(+)}$ and $\zeta_f^{(-)}$ in terms of the optical wave functions $\chi_i^{(+)}$ and $\chi_f^{(-)}$:

$$\left.\begin{aligned}\zeta_f^{(-)} &= \left(1+\frac{1}{E-H'-i\eta}(V_{yc}-U_f)\right)\chi_i^{(-)};\\ \zeta_i^{(+)} &= \left(1+\frac{1}{E-H'-i\eta}(V_{yA}-U_i)\right)\chi_i^{(+)}.\end{aligned}\right\} \tag{22}$$

Substituting the expressions (22) in (20), we obtain an expression for the matrix element (1) in the exact formulation of the FRDWM:

$$M_{if} = \langle\chi_f^{(-)}\varphi_B\varphi_y \mid T_{yc} \mid \varphi_A\varphi_A\chi_i^{(+)}\rangle + \langle\chi_f^{(-)}\varphi_B\varphi_y \mid T_{yA} \mid \varphi_A\varphi_A\chi_i^{(+)}\rangle + \text{terms that take into account multiple rescattering.} \tag{23}$$

Equation (23) is fully identical to Eqs. (5) and (8), though, in contrast to them, the terms that admit a perspicuous physical interpretation are explicitly separated in the expression (23). The FRDWM reaction matrix element (23) contains two terms of the same order of smallness, neither of which is compensated by the optical potential (the corresponding terms, containing $V-U$, contain at the least double scattering). This means that the approximations considered above and frequently introduced in the FRDWM are in fact in the general case not satisfied.

The T matrices in the expression (23) are determined by the equations

$$T_{yc} = V_{yc} - V_{yc}\frac{1}{E-H_0+i\eta}V_{yc} - \ldots; \tag{24}$$

$$T_{yA} = V_{yA} + V_{yA}\frac{1}{E-H_0-V_{yA}+i\eta}V_{yA} + \ldots, \tag{25}$$

which are shown in Fig. 2 in the form of an infinite series of graphs with an ever more complicated topological structure. The replacement of the T matrices by the interaction potentials means that we take into account only the first terms of the corresponding expansions, i.e., the single-step reaction mechanisms. It follows from (23) and the analogous expression for the

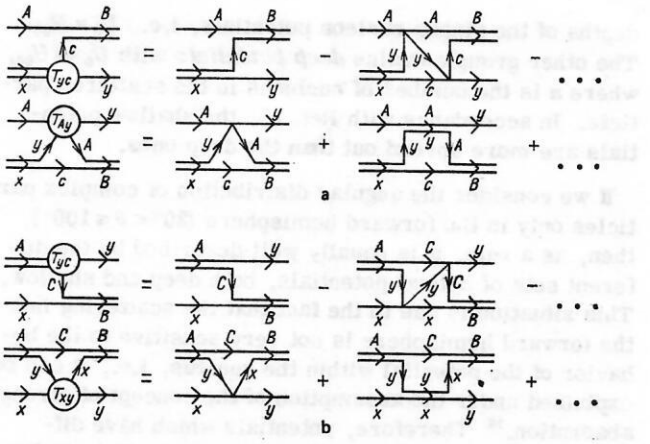

FIG. 2. Graphical representation of the integral equations for T matrices of direct (a) and exchange (b) processes.

exchange processes that there are four such mechanisms in reactions with complex particles: stripping and heavy replacement, which are direct processes, and ordinary replacement and heavy stripping, which are exchange processes, and none of these mechanisms can *a priori* be ignored.

Optical Model for the Scattering of Complex Particles. As we have already said, in the distorted-wave method the interaction of the initial and final particles with the nuclei is described in the framework of the optical model, which takes into account elastic scattering by the complex potential. Since a fair number of detailed reviews has been devoted to the optical model (see, for example, Refs. 11 and 12), we shall not dwell here in detail on a discussion of all its problems. We shall consider only those aspects of the optical model that are associated with the scattering of complex particles, in particular the questions relating to the form and parameters of the optical potentials.

The total number of parameters of optical potentials, including central, spin-orbit nuclear and Coulomb interactions (a detailed discussion of the components of the optical potentials and their radial dependence is given, for example, in Ref. 11), is fairly large (16 when the Coulomb range is included). If the spin–orbit interaction is ignored, the number of parameters is reduced to 10. Usually, one employs an optical model containing only a central potential without surface absorption, although there are programs for calculations on the basis of a many-parameter optical model.[13]

It can now be regarded as established that the scattering of nucleons by nuclei can be described fairly well by an optical model with potentials whose real parts have depth $U_{00} \approx 30-50$ MeV (Ref. 11). At the same time, for the description of the scattering of complex particles there are different disjunct sets of optical potentials. A detailed list of the optical-potential parameters for the scattering of neutrons, protons, deuterons, tritons, ^{3}He, α particles, and heavy ions is given in the review of Ref. 14. Among all the sets of optical potentials for complex particles, we can distinguish two groups of potentials. One of them is the group of so-called *shallow potentials*, which have a depth of the order of the

depths of the single-nucleon potentials, i.e., $U_0 \approx U_{00}$. The other group contains *deep potentials* with $U_0 \approx nU_{00}$, where n is the number of nucleons in the scattered particle. In accordance with Ref. 15, the shallow potentials are more spread out than the deep ones.

If we consider the angular distribution of complex particles only in the forward hemisphere ($20° \leq \theta \leq 100°$), then, as a rule, it is equally well described by the different sets of optical potentials, both deep and shallow. This situation is due to the fact that the scattering into the forward hemisphere is not very sensitive to the behavior of the potential within the nucleus, i.e., it can be explained under the assumption of the concept of strong absorption.[16] Therefore, potentials which have different depths but a similar "tail" describe the angular distribution at small angles equally well and, from this point of view, are indistinguishable. This circumstance leads to the disjunct sets of optical potentials. The angular distributions in the backward hemisphere, where there are anomalously high peaks of the cross section in the case of scattering of complex particles by light nuclei (see the review of Ref. 17 for a detailed discussion of anomalous backward scattering) are described significantly better by the deep potentials.[1)] Nevertheless, the problem of the unique determination of the parameters of the potentials of the optical model solely on the basis of an analysis of the angular distributions of elastic scattering is evidently essentially insoluble, since the elastic scattering of complex particles is not determined by potential scattering alone. Other reaction mechanisms, in particular, exchange processes, can make an important contribution to the angular distribution, especially at large angles.

In this connection, it becomes particularly important to interpret the optical model on the basis of microscopic approaches. One of them consists of the systematic use of the many-body problem to determine the interaction in the initial and final states. In the exact equations (20) and (21) of the three-body problem, one can explicitly separate[3] integral operators that play the part of interaction operators (nonlocal and complex) in the entrance and exit reaction channels. It is natural to call these operators generalized optical potentials for the elastic scattering of complex particles. In Ref. 20, an approximate method is proposed for calculating the real parts of the generalized optical potentials in the three-body problem, the Pauli principle, which plays a fairly important part in the scattering of complex particles,[21] being taken into account directly. As is shown in Ref. 20, the formulas for calculating the real parts of the generalized optical potentials are formally identical to the ones used in the folding model. However, in contrast to the usual folding model,[22,23] the unrenormalized two-body potentials are averaged not over the mean density of the nucleon distribution in the nucleus, but over the density of the distribution of nucleon associations (such a folding model was used in particular in Ref. 24).

What does application of the three-body problem for determination of the form and parameters of the optical potentials give? The calculations made in Refs. 20, 24 and 25 showed that the real parts of the many-body generalized optical potentials responsible for potential scattering are fairly deep (for example, for α-particle scattering the depth of the real part of the potential is approximately 150 MeV), and their shape is well approximated by the usual Woods–Saxon potential. One can, for example, note that in the review of Ref. 14 eight sets of parameters of optical potentials are given for the scattering of α particles with energy from 16 to 27 MeV by ^{12}C and only one of these sets corresponds to three-body generalized optical potentials.[25] Therefore, the microscopic approach makes it possible to reduce considerably the uncertainty in the parameters of the optical potentials and identify the physical optical potentials among the sets obtained by fitting.

The real parts of the generalized optical potentials for the scattering of α particles by ^7Li and ^{12}C calculated in the three-body approximation[20,25] and the angular distributions of 18-MeV α particles scattered elastically by ^7Li and of 26.6-MeV α particles scattered by ^{12}C (experimental data taken from Refs. 26 and 27) as calculated in accordance with the optical model with these potentials are shown in Figs. 3 and 4. It can be seen that the general agreement with the experiment is completely satisfactory, although a detailed description is not achieved, since only potential scattering was taken into account in the calculation.

We also mention two further microscopic approaches for describing the scattering of complex particles. One of them[28] involves the introduction into the optical potentials of a repulsive core, which simulates nonlinear terms in the equations of motion of the particles in the field of the nucleus (the occurrence of nonlinear terms in the Hamiltonian is due to the compressibility of the nucleus in the interaction with the incident particle). Another approach uses two-center optical potentials.[29] Both approaches make it possible to obtain reasonable agreement for the angular distributions of elastic scattering of α particles by light nuclei in the complete range of angles, but the corresponding optical potentials

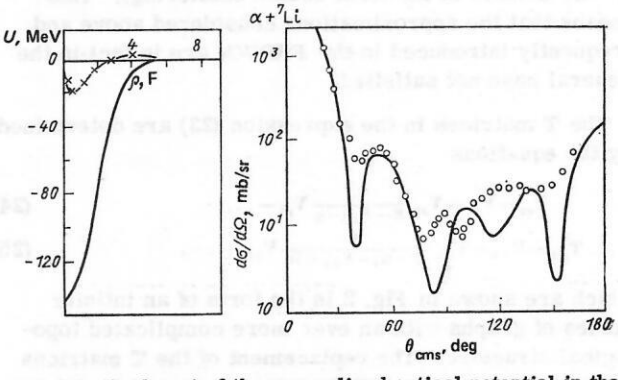

FIG. 3. Real part of the generalized optical potential in the three-body problem for the system $\alpha + {}^7Li$ (Ref. 21) and the anangular distribution of α particles scattered elastically by ^7Li at $E_\alpha = 18$ MeV calculated with this potential. The experimental data are taken from Ref. 27.

[1)]It may be noted that the angular distribution of α particles[18] and ^6Li ions[19] cannot be described by shallow potentials even in the forward hemisphere.

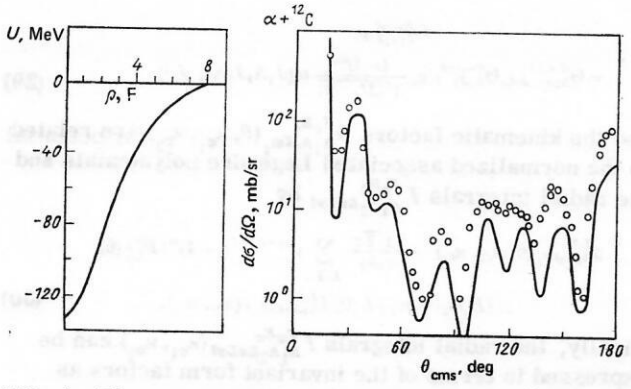

FIG. 4. The same as in Fig. 3 for the system $\alpha + {}^{12}C$ and scattering of α particles by ^{12}C at $E_\alpha = 26.6$ MeV (Ref. 26). The experimental data are taken from Ref. 28.

cannot be used directly in the FRDWM, since they effectively take into account not only potential scattering but also the influence of other mechanisms.

The angular distribution of the elastic scattering of complex particles can be described in agreement with the experiments only when allowance is made for the many-body nature of the scattering and the contribution of more complicated mechanisms. This can be done in the framework of the model of quasimolecular resonances,[30] which can be formed in both the entrance and exit reaction channels. As is shown in Ref. 31, such resonances are equivalent to phenomenological models of the Regge pole type,[32,33] i.e., they must lead to enhancement of backward peaks in the scattering. On the other hand, the occurrence of quasimolecular resonances has been interpreted on a number of occasions[34] as the result of exchange of nucleons between the incident particle and the nucleus, and described phenomenologically by an L-split potential,[35] i.e., by a potential with a correction whose sign depends on the parity of the partial wave. Finally, it was noted in Ref. 36 that the influence of quasimolecular resonances on the elastic scattering of particles can be taken into account directly in optical potentials whose imaginary parts either contain an L dependence[37] or have Woods–Saxon form with an arbitrary exponent.[38] As is shown in Ref. 39, all these phenomenological corrections to potential scattering can be attributed to exchange mechanisms (see Fig. 1b) in the actual process of elastic scattering of complex particles.

Finally, we must speak about the many-body corrections to the optical potentials. As is shown in Ref. 25, the generalized optical potentials calculated in the three-body problem correctly reproduce the main features of the four-body optical potentials. Therefore, the optical Woods–Saxon potentials can, with an appropriate choice of the parameters (equivalent to the parameters of the generalized optical potentials in the three-body problem), evidently be used in the FRDWM to calculate the angular distributions of reactions with complex particles.

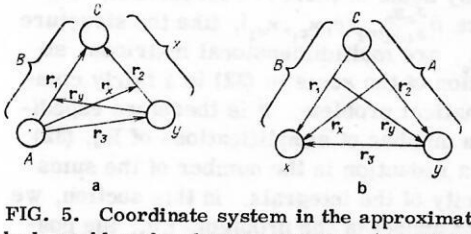

FIG. 5. Coordinate system in the approximation of the three-body problem for the reaction $A(x, y)B$: a) direct processes, b) exchange processes.

2. GENERAL FORMULAS FOR CALCULATING THE DIFFERENTIAL CROSS SECTIONS OF NUCLEAR REACTIONS BY THE FINITE-RANGE DISTORTED-WAVE METHOD

Structure of the FRDWM Formulas for the Reaction Cross Section. It follows from what we have said above that the FRDWM is based on the following approximations:

1) the matrix element of the nuclear reaction $A(x,y)B$ is calculated in the framework of the three-body problem;

2) instead of the exact T matrices, which are the solution of Eqs. (24) and (25), one uses local operators—interaction potentials, i.e., one considers only single-step reaction mechanisms;

3) the interaction of the particles with the nuclei in the initial and final states is described by the optical model;

4) finally, in the FRDWM one introduces one further approximation of a nonfundamental nature—the spin-orbit interaction is ignored. The dependence of the cross sections on the spins of the particles is determined solely by a weight factor.

Remaining in the framework of these approximations, we now turn to the calculation of the differential cross section of the reaction $A(x,y)B$. As before, we shall consider only direct processes, bearing in mind that all relations for the exchange processes can be obtained by the substitution $A \rightleftarrows x$ and $\theta \to \pi - \theta$.

In accordance with Ref. 2, the differential cross section $d\sigma/d\Omega$ in the FRDWM is expressed in terms of invariant form factors which depend on the wave functions of the nuclei participating in the reaction and the corresponding reaction potentials. The problem of finding the invariant form factors reduces to a calculation of the overlap integral

$$J = \langle \Psi_B \Psi_y \mid V \mid \Psi_x \Psi_A \rangle$$

for each of the four mechanisms, the integration being only over the internal variables, and the interaction potential V is specified by a definite choice of the mechanism. Figure 5 shows the coordinate system for the direct and exchange processes in the reaction $A(x,y)B$. In accordance with Fig. 5a, the potential $V(r_2)$ corresponds to the stripping mechanism, and the potential $V(r_3)$ to the heavy-replacement mechanism. The matrix element of the reaction is found by averaging the overlap integral over the distorted waves of the initial and final particles. In what follows, we shall use the partial waves $\chi_{Lx}(k_x r_x)$, $\chi_{Ly}(k_y r_y)$, which are determined by the expression[40]

$$\chi(kr) = \frac{4\pi}{kr} \sum_L \chi_L(kr) i^L Y_{LM}(\mathbf{r}) Y^*_{LM}(\mathbf{k}). \qquad (26)$$

An important aspect of the calculation of the overlap integral is the splitting of the wave function of the nucleus into the wave functions of composite subsystems, which may have their own excited states, and the wave function $\Psi_{\Lambda\mu}(\varkappa r)$ of their relative motion with given number of nodes and orbital angular momentum Λ. To solve this problem, it is necessary to introduce reduced widths (amplitudes of the spectroscopic factors) for the decay of the nucleus into the subsystems. The reduced widths can be introduced in different ways. First, they can be regarded as parameters, though the realization of this variant is not only laborious but also frequently meaningless, since the number of free parameters can become very large when the problem is treated in this way. Second, the reduced widths (at least, up to their sign) can in principle be extracted from experimental spectroscopic factors. However, the systematic implementation of this procedure is in practice impossible, since the experimental spectroscopic factors are known only for decay to the ground (or lowest excited) states of the final nucleus. Finally, one can use the theoretical values of the reduced widths calculated, for example, on the basis of the modern shell model with intermediate coupling,[41] since it is known that this model makes it possible to obtain spectroscopic factors of nucleons, deuterons, and tritons ^{3}He in reasonable agreement with experiment. Since the shell model with intermediate coupling has been developed basically for nuclei of the $1p$ shell, this method is best justified for light nuclei up to ^{16}O.

Formulas for calculating the reduced widths in the shell model are given in Refs. 42 and 43. In accordance with Ref. 43, the corresponding reduced widths depend on the quantum numbers of the states of the nuclei, and also on the orbital angular momentum Λ and the total angular momentum J. In what follows, to standardize the formulas, we introduce the index 1 to denote the angular momenta Λ_1 and J_1 associated with the decay of nucleus B into A and c. Similarly, we shall use the index 2 to denote the angular momenta Λ_2 and J_2 associated with the decay of particle x into y and c. We shall denote the corresponding reduced widths by $\Theta^{B\to A+c}_{\Lambda_1, J_1, J_cE_c}$.

Following Ref. 2, we introduce the invariant form factor by means of the relation

$$F^{J_cE_c}_{\Lambda_1\Lambda_2L_xL_yl}(r_x, r_y) = \sum_{M_x, M_y, \mu_1, \mu_2} \langle L_xM_xL_yL_y \mid lm_l\rangle \langle \Lambda_1\mu_1\Lambda_2-\mu_2 \mid lm_l\rangle \times \int d\Omega_x\, d\Omega_y Y_{L_xM_x}(\Omega_x)\, Y_{L_yM_y}(\Omega_y) \times \Psi^*_{\Lambda_1\mu_1}(\varkappa_{c_1}, r_1)\,(V(r_2)+V(r_3))\,\Psi_{\Lambda_2\mu_2}(\varkappa_{c_2}, r_2). \quad (27)$$

Then the first two terms of the matrix element (23), corresponding to single-step mechanisms, can be written in the form

$$M_{if} = \sum_{\substack{lJ_1J_2\Lambda_1\Lambda_2\\ J_cE_cm_lM_1M_1}} \langle J_AM_AJ_1M_1 \mid J_BM_B\rangle \times \langle J_yM_y J_2M_2 \mid J_xM_x\rangle (-1)^{J+M_2}\sqrt{(2J_1+1)(2J_2+1)} \times \langle J_2-M_2J_1M_1 \mid lm_l\rangle (-1)^{\Lambda_1+\Lambda_2}\Theta^{\Lambda_1\Lambda_2l}_{J_1J_2J_cE_c}\beta^{J_cE_c}_{\Lambda_1\Lambda_2lm_l}(\theta, \varkappa_{c_1}, \varkappa_{c_2}), \quad (28)$$

where the structure factors $\Theta^{\Lambda_1\Lambda_2l}_{J_1J_2J_cE_c}$ are determined by the expression

$$\Theta^{\Lambda_1\Lambda_2l}_{J_1J_2J_cE_c} = \Theta^{B\to A+c}_{\Lambda_1J_1, J_cE_c}\Theta^{x\to y+c}_{\Lambda_2J_2, J_cE_c}\frac{(-1)^{J_c}}{\sqrt{(2J_c+1)}}\, u(J_1\Lambda_1J_2\Lambda_2;\ J_cl), \quad (29)$$

and the kinematic factors $\beta^{J_cE_c}_{\Lambda_1\Lambda_2lm_l}(\theta, \varkappa_{c_1}, \varkappa_{c_2})$ are related to the normalized associated Legendre polynomials and the radial integrals $I^{J_cE_c}_{\Lambda_1\Lambda_2L_xL_yl}$ by

$$\beta^{J_cE_c}_{\Lambda_1\Lambda_2lm_l}(\theta, \varkappa_{c_1}, \varkappa_{c_2}) = \frac{4\sqrt{2\pi}}{k_xk_y}\sum_{L_xL_y} i^{L_x+L_y-l}(-1)^{m_l}\bar{P}^{m_l}_{L_y}(\theta) \times \langle L_ym_l l-m_l \mid L_x0\rangle I^{J_cE_c}_{\Lambda_1\Lambda_2L_xL_yl}(\varkappa_{c_1}, \varkappa_{c_2}). \quad (30)$$

Finally, the radial integrals $I^{J_cE_c}_{\Lambda_1\Lambda_2L_xL_yl}(\varkappa_{c_1}, \varkappa_{c_2})$ can be expressed in terms of the invariant form factors as follows:

$$I^{J_cE_c}_{\Lambda_1\Lambda_2L_xL_yl}(\varkappa_{c_1}, \varkappa_{c_2}) = \int \chi^*_{L_y}(k_yr_y)\, F^{J_cE_c}_{\Lambda_1\Lambda_2L_xL_yl} \times (r_x, r_y)\,\chi_{L_x}(k_xr_x)\, r_x\, dr_x r_y\, dr_y. \quad (31)$$

Substituting the matrix element (28) in the usual expression for the differential cross section and summing over the magnetic quantum numbers, we readily obtain the final expression for the differential cross section of the reaction:

$$\frac{d\sigma}{d\Omega} = \frac{\mu_{xA}\mu_{yB}}{(2\pi\hbar^2)^2}\frac{k_y}{k_x}\frac{2J_B+1}{2J_A+1} \times \sum_{\substack{J_1J_2\\ lml}}\left|\sum_{\substack{\Lambda_1\Lambda_2\\ J_cE_c}}\beta^{J_cE_c}_{\Lambda_1\Lambda_2lml}(\theta, \varkappa_{c_1}, \varkappa_{c_2})\,\Theta^{\Lambda_1\Lambda_2l}_{J_1J_2J_cE_c}\right|^2. \quad (32)$$

It is evident from the structure of this expression that the cross section of the reaction of many-nucleon transfer with complex particles in the FRDWM contains a coherent summation over the orbital angular momenta Λ_1 and Λ_2 of the motion of particle c in nucleus B and particle x. In addition, a coherent summation over the states with total angular momentum J_c and energy E_c of the intermediate particle c is also taken into account. The presence of these sums does not permit one, on the one hand, to factorize the cross section into kinematic and structure factors (the concept of factorization means in the present case that the differential cross section can be represented in the form of incoherent uncoupled sums of products of structure and kinematic factors). On the other hand, the expression (32) shows that the reaction cross section is determined by not only the magnitudes but also the signs of the reduced widths, i.e., it is very sensitive to the structure factors.

Possible Simplifications of the General Formulas for Particular Types of Reaction. It follows from the obtained expressions that in the general case the FRDWM reaction matrix element contains a four-dimensional integral (there is no dependence on the azimuthal angles) and a 12-fold coupled sum. The expression (32) contains 10 sums, and only some of them are incoherent. The kinematic factors $\beta^{J_cE_c}_{\Lambda_1\Lambda_2lm_l}(\theta, \varkappa_{c_1}, \varkappa_{c_2})$, like the structure factors $\Theta^{\Lambda_1\Lambda_2l}_{J_1J_2J_cE_c}$, are multidimensional matrices, so that the calculation of the sums in (32) is a fairly complicated mathematical problem. It is therefore expedient to consider a number of simplifications of Eq. (32) associated with a reduction in the number of the sums and the multiplicity of the integrals. In this section, we consider the first aspect of the problem, i.e., the possibility of reducing the number of sums, both coherent and incoherent.

First of all, in the general formula (32) the number of summations can be reduced by assuming that the wave functions $\Psi_{\Lambda\mu}(\varkappa_c r)$ do not depend on the excitation energy of the intermediate nucleus, i.e., that the form factors (27) do not depend on E_c. (It is clear that this approximation is valid if $E_c^* \ll E - E_c^*$.) We can then calculate the sums over E_c in (32), using the orthogonality of the coefficients of intermediate coupling,[44] and we can also calculate analytically the sums over J_c [the dependence on the states of the intermediate particle c is retained through the orbital angular momentum L_c and the spin S_c (Ref. 43)]. These transformations make it possible to reduce the number of coherent sums in (32) by two. We point out that the cross sections do not contain sums over E_c and J_c if excitation of only one level of particle c in the reaction is possible. This occurs, for example, in reactions with light particles (not heavier than ^{4}He) for direct processes.

A reduction in the number of sums in (27), (29), and (32) is possible for many concrete types of reaction with complex particles. We consider in more detail the most commonly encountered cases.

1. The transferred particle c is not heavier than ^{4}He, i.e., $L_c = 0$ (for exchange processes, such simplifications will occur if the intermediate nucleus C is not heavier than ^{4}He). The formula for the reduced width is then simplified,[43] and in (32) the number of sums remains unchanged. However, if c is an α particle or a nucleon, there are more significant simplifications of the formulas:

a) c is an α particle ($J_c = S_c = T_c = 0$). In (32), there remain the coherent sums over only Λ_1 and Λ_2, and all the remaining summations vanish. Nevertheless, it can be seen that in this case too it is not possible to factorize the cross section into kinematic and structure factors, since Λ_1 and Λ_2 may have several values;

b) the transferred particle is a nucleon ($J_c = S_c = T_c = \frac{1}{2}$). As we have already said above, there is no dependence of the matrix element on E_c in this case. If, in addition, the angular momenta Λ_1 and Λ_2 have a unique value (which, for example, is always the case for nuclei of the $1p$ shell), then in (32) there are no sums over Λ_1 and Λ_2, so that the differential cross section can be represented in the form

$$\frac{d\sigma}{d\Omega} = \frac{\mu_{xA}\mu_{yB}}{(2\pi\hbar^2)^2}\frac{k_y}{k_x}\frac{2J_B+1}{2J_A+1}\sum_{lml} S_l^{\Lambda_1\Lambda_2} |\beta_{\Lambda_1\Lambda_2 lml}(\theta)|^2, \qquad (33)$$

where

$$S_l^{\Lambda_1\Lambda_2} = \sum_{J_1 J_2} S_{\Lambda_1 J_1, S_N}^{B\to A+N} S_{\Lambda_2 J_2, S_N}^{x\to N+y} (2l+1) W^2 (J_1\Lambda_1 J_2\Lambda_2 : S_N l). \qquad (34)$$

It can be seen that even for the partial differential cross section, i.e., the differential cross section with given angular-momentum transfer l, it is not possible to separate the spectroscopic factors of each nucleon as individual factors.

2. The final particle y is not heavier than ^{4}He. In this case, the formula for the reduced width of the decay $x \to c + y$ simplifies somewhat.[43] In the expression (32), the number of sums is not changed. If y is an α particle, then the summation over J_2 in the cross section (32) is absent.

3. The final particle y and the intermediate particle c are not heavier than ^{4}He. In accordance with 1) and 2), the number of sums in the reduced widths[43] is reduced in this case by four, and the expression (32) remains unchanged.

4. The incident particle x is not heavier than ^{4}He. Since $x \geq y$, in this case $y, c \leq {}^4$He. We here have a significant simplification of the computational formulas: first, the formulas for the structure factors (29) simplify; second, for direct processes all the coherent sums vanish, since $\Lambda_2 = 0$ and $\Lambda_1 = l$, i.e., the cross section factorizes rigorously into a kinematic and a structure factor. In addition, for direct processes the expression for the form factor (27) simplifies. The corresponding formulas are given in Sec. 4. For exchange processes, if the dependence of the wave functions on E_c is ignored, there remain coherent sums over Λ_1 and Λ_2, and the form factor (27) itself is not changed.

5. $J_A = 0$ ($J_B = 0$). The reaction takes place either on an even–even nucleus or with the formation of an even–even nucleus. In each of these cases, one sum is absent: either one over J_1 or one over J_2.

Since it is not so much the reaction type as the restrictions on the values of the corresponding angular momenta that are important for the simplification of the general expression (32), the cross sections of all reactions of many-nucleon transfer in which the angular momenta J_c, S_c, Λ_1, and Λ_2 have unique values can be calculated in accordance with the simplified formula (33).

We can now answer the question as to when the cross section in the FRDWM factorizes into kinematic and structure factors. Such factorization occurs rigorously for direct processes in reactions with light nuclei ($x = d, t, {}^3$He, α), but for exchange processes even in reactions with light nuclei there is in general no factorization of the cross section, since coherent sums over Λ_1 and Λ_2 remain in (32). In reactions with heavy ions, the cross section factorizes in processes of single-nucleon transfer in the cases when the nucleon at each of the decay vertices has a unique value of the transferred orbital angular momentum. This requirement is, in particular, satisfied if the transferred nucleon is a $1p$ nucleon. In heavier nuclei, the nucleon wave function contains a coherent sum over the orbital angular momenta, and Λ_1 and Λ_2 cannot have unique values. But if the requirement of uniqueness of the angular momentum transfers at the vertex is satisfied and, in addition, the angular momenta J_1 and J_2 have a unique value, the spectroscopic factors of each nucleon can be separated in the cross section as individual factors. An example of a reaction of this kind is ^{16}O(^{12}C, ^{11}B)^{17}O.

In individual cases, the selection rules permit only one value of Λ_1, Λ_2, J_1, and J_2 in the case of many-nucleon transfer reactions as well, and for these reactions the possibility of factorization is not *a priori* obvious. This means that the number of reactions in which the spectroscopic factors at individual vertices can be separated as factors in the cross section may be some-

what increased.

Problem of Separation of the Variables in the Radial Integral. A basic feature of the integrals that occur in the FRDWM is the circumstance that the interaction potentials and the wave functions of the relative motion, and also the distorted waves, depend on different combinations of relative variables used in the problem, for example, Jacobi coordinates. The problem of separation of variables in the FRDWM is very important, since its solution dictates the possibility of analytic calculation of the integrals over the angular variables and also over the variables that are not related to the interaction of the particles. We note that the problem of separation of the variables can be solved in different ways, depending on the choice of the basis coordinates; in the three-body problem, there must be two ways (for example, $\mathbf{r}_2, \mathbf{r}_3; \mathbf{r}_x, \mathbf{r}_y; \mathbf{r}_1, \mathbf{r}_2$, see Fig. 5).

We consider general methods of solution of the problem of separation of the variables in a wave function that depends on a linear combination of two vectors. The most popular method of separation of variables in functions that depend on a linear combination of two vectors is by means of a Talmi–Moshinsky–Smirnov transformation,[45] the coefficients of which form an orthonormalized matrix for the transition from one set of Jacobi coordinates to the other. However, since this transformation is valid only for the wave functions of a harmonic oscillator, it cannot be used to separate variables in the real wave functions of the FRDWM.

We consider a method of separating the variables that does not require a definite symmetry of the wave function.[46] Suppose we have a function $\varphi_{lm}(\mathbf{R})$ such that

$$\varphi_{lm}(\mathbf{R}) = \varphi_l(R)\, Y_{lm}(\mathbf{R}); \quad \mathbf{R} = \alpha\mathbf{r}_1 + \beta\mathbf{r}_2. \tag{35}$$

It is necessary to represent it as a sum of products of functions of each of the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$. To solve this problem, we use the properties of Dirac δ functions and represent $\varphi_{lm}(\mathbf{R})$ in the form

$$\varphi_{lm}(\mathbf{R}) = \int \varphi_{lm}(\mathbf{r})\, \delta(\mathbf{r}-\mathbf{R})\, d\mathbf{r} = \frac{1}{(2\pi)^3} \int \varphi_{lm}(\mathbf{r}) \exp[i\mathbf{k}(\mathbf{r}-\mathbf{R})]\, d\mathbf{k}\, d\mathbf{r}. \tag{36}$$

We expand the plane waves in a series in Bessel functions and integrate over the angular variables of the vectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}$. Then for the function $\varphi_{lm}(\mathbf{R})$ we obtain the expression

$$\varphi_{lm}(\mathbf{R}) = \frac{4}{\pi} \sum_{l_1 l_2 m_1 m_2} \langle l_1 0 l_2 0 \mid l0\rangle \langle l_1 m_1 l_2 m_2 \mid lm\rangle\, i^{l_1+l_2+l} \times \sqrt{\frac{(2l_1+1)(2l_2+1)}{(2l+1)}}\, A^l_{l_1 l_2}(\alpha r_1, \beta r_2)\, Y_{l_1 m_1}(\mathbf{r}_1)\, Y_{l_2 m_2}(\mathbf{r}_2), \tag{37}$$

where

$$A^l_{l_1 l_2}(\alpha r_1 \beta r_2) = \int r^2 dr \varphi_l(r)\, \Delta_{l_1 l_2 l}(\alpha r_1,\ \beta r_2,\ r); \tag{37a}$$

$$\Delta_{l_1 l_2 l}(\alpha r_1,\ \beta r_2,\ r) = \int k^2\, dk j_{l_1}(\alpha k r_1)\, j_{l_2}(\beta k r_2)\, j_l(kr). \tag{37b}$$

The expressions (37) solve formally the problem we have posed of separating the variables. Indeed, the angular dependence of the right-hand side of (37) is represented as a sum of product of spherical functions of $\mathbf{r}_1$ and $\mathbf{r}_2$. But for the actual separation of the radial variables it is necessary to calculate the two-dimensional integral (37a), which is in itself a difficult problem. This integral can be calculated analytically only when $\varphi_l(R)$ is a spherical Hankel or Bessel function. The corresponding expressions have the form

$$\left.\begin{aligned} &\varphi_l(R) = j_l(kr); \\ &A^l_{l_1 l_2} = \frac{\pi}{2}\, i^{l_1+l_2-l}\, j_{l_1}(\alpha k r_1)\, j_{l_2}(\beta k r_2); \end{aligned}\right\} \tag{38}$$

$$\left.\begin{aligned} &\varphi_l(R) = h^{(1)}_l(i\varkappa r); \\ &A^l_{l_1 l_2} = \frac{\pi}{2}\, i^{l_1+l_2-l}\, h^{(1)}_{l_1}(\alpha i\varkappa r_1)\, j_{l_2}(\beta i\varkappa r_2) \text{ under the condition } \alpha r_1 \geqslant \beta r_2. \end{aligned}\right\} \tag{39}$$

In all the remaining cases, the integral (37a) must be found numerically. The calculations can be simplified if one calculates analytically the integral (37b) of the products of three Bessel functions. In Refs. 47–50, $\Delta_{l_1 l_2 l}$ is calculated analytically in the form of series of different functions: Legendre polynomials, modified harmonic-oscillator functions, associated Legendre polynomials, power-law functions, and Γ functions; however, the convergence of such series is not always rapid.

In Ref. 2, a different method is proposed for separating the variables in the function $\varphi_{lm}(\mathbf{R})$; this uses its angular symmetry, i.e., the fact that the angular dependence of $\varphi_{lm}(\mathbf{R})$ is determined by the spherical functions $Y_{lm}(\mathbf{R})$. For generalized spherical functions of the sum of two vectors the well-known Moshinsky relation holds[51]:

$$R^l Y_{lm}(\mathbf{R}) \sum_{\lambda=0,1,\ldots,l} \sqrt{\frac{4\pi}{2\lambda+1}}\, (\alpha r_1)^{l-\lambda} (\beta r_2)^{\lambda} \binom{2l+1}{\lambda 2}^{1/2} \times \langle l-\lambda m-\mu \lambda\mu \mid lm\rangle\, Y_{l-\lambda\, m-\mu}(\mathbf{r}_1)\, Y_{\lambda\mu}(\mathbf{r}_2). \tag{40}$$

The expression (40) makes it possible to separate the variables in the angular part of the function $\varphi_{lm}(\mathbf{R})$. To separate the variables in the remaining scalar part $\varphi_l(R)/R^l$, it must be expanded in a series in Legendre polynomials in the cosine of the angle between the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$:

$$\frac{1}{R^l}\varphi_l(R) = 2\pi \sum_{KQ} g_K(r_1,\ r_2)\, Y^*_{KQ}(\mathbf{r}_1)\, Y_{KQ}(\mathbf{r}_2), \tag{41}$$

where

$$g_K(r_1,\ r_2) = \int_{-1}^{1} P_K(\mu)\, \frac{1}{R^l}\, \varphi_l(R)\, d\mu; \quad \mu = \cos(\widehat{\mathbf{r}_1 \mathbf{r}_2}). \tag{42}$$

As a result, for the function $\varphi_{lm}(R)$ we obtain finally

$$\varphi_{lm}(\mathbf{R}) = \sqrt{2l+1} \sum_{\lambda=0,1,\ldots,l} \sum_{Kl_1 l_2} \sqrt{\pi}\, (\alpha r_1)^{l-\lambda} (\beta r_2)^{\lambda} \times \binom{2l}{2\lambda}^{1/2} (-1)^K (2K+1)\, g_K(r_1,\ r_2) \times \langle l-\lambda 0 K 0 \mid l_1 0\rangle \langle \lambda 0 K 0 \mid l_2 0\rangle \langle l_1 m_1 l_2 m_2 \mid lm\rangle\, W(l-\lambda l_1 \lambda l_2;\ Kl) \times Y_{l_1 m_1}(\mathbf{r}_1)\, Y_{l_2 m_2}(\mathbf{r}_2). \tag{43}$$

Comparison of the expressions (37) and (43) shows that the use of the angular symmetry of the wave functions makes it possible to simplify significantly the expression for the function $\varphi_{lm}(R)$ when its variables are separated. In particular, whereas (37) contains a sum of double integrals, the expression (43) is determined solely by single integrals, which significantly simplifies the actual calculations. Therefore, many FRDWM programs use the second method of separating the variables, which involves calculating the invariant form fac-

tor in terms of the quantities $g_K(r_x, r_y)$ given by the expression (42).

3. APPROXIMATE ALLOWANCE FOR THE EFFECTS OF RECOIL AND FINITE RANGE OF THE INTERACTION

As follows from the previous section, the concrete realization of the general formalism of the FRDWM is a very laborious computational task, above all because of the presence of the multiple integrals in the expression (32). Various approximate methods have been proposed that make it possible to reduce the multiplicity of the integration in the calculation of the angular distributions. It should be said that the majority of these methods apply to direct processes, and only one mechanism, stripping, is taken into account. The qualitative estimates of Ref. 10 suggest that the influence of heavy replacement is important only for the lightest nuclei, so that the neglect of this mechanism is justified in the majority of cases.

For the stripping mechanism, one can *a priori* expect the presence of certain restrictions on the possible values of the coordinates $\mathbf{r}_x$ and $\mathbf{r}_y$, on which the distorted waves depend. Indeed, in accordance with Fig. 5, these coordinates can be expressed in terms of the Jacobi coordinates $\mathbf{r}_2$ and $\mathbf{r}_3$ as follows (to simplify the expressions, we shall denote the masses by the symbols of the corresponding particles):

$$\mathbf{r}_x = \mathbf{r}_3 + \frac{c}{x}\mathbf{r}_2; \quad \mathbf{r}_y = \frac{A}{B}\mathbf{r}_3 - \frac{c}{B}\mathbf{r}_2; \quad \mathbf{r}_1 = \mathbf{r}_2 + \mathbf{r}_3. \tag{44}$$

The values of the modulus of the coordinate $\mathbf{r}_2$ are bounded by the dimensions of the region of interaction of particles y and c. Therefore, on the $(\mathbf{r}_2, \mathbf{r}_3)$ coordinate plane the values of the variable $\mathbf{r}_x(\mathbf{r}_y)$ are grouped on a strip along the direction of $\mathbf{r}_3(A\mathbf{r}_3/B)$. The width of this strip depends on the size of the interaction region and on the mass ratios c/x and c/B and, as a rule, does not exceed 1-2 F. This enables us to regard the second terms in (44) as corrections to the "basic" coordinate $\mathbf{r}_3$, which is important for different approximate methods of separation of the variables in the distorted waves.

Allowance for Finite-Range and Recoil Effects in the Zeroth Approximation. We now consider approximate variants of the FRDWM associated with the neglect of the second terms in the expressions for $\mathbf{r}_x$ and $\mathbf{r}_y$ in (44). The distorted waves now depend only on a single coordinate, and the variables separate in the easiest manner. Such an approximation is valid if either the coordinate ratio r_2/r_3 or the mass ratios c/x and c/B can be regarded as small. We shall consider each of these cases separately.

Zero-range approximation (ZRA).[40] In this approximation, the dimensions of the region of interaction of particles y and c are assumed to be small compared with the characteristic dimensions of the system, i.e., it is assumed that

$$r_2/r_3 \ll 1.$$

This condition means that particle y is emitted at the same point at which particle c is captured. In this case, the strip of integration on the (r_2, r_3) plane degenerates into a straight line, which coincides with the ordinate axis [on the (r_x, r_y) plane, the ZRA strip of integration is transformed into a straight line with slope A/B]. Mathematically, the zero-range approximation reduces to the replacement of the potential $V(r_2)$ by the Fermi pseudopotential[52]

$$V(r_2)\,\Psi_{\Lambda_2\mu_2}(r_2) = N^{\mathrm{ZRA}}\delta(r_2)\,\delta_{\Lambda_2 0}\delta_{\mu_2 0}/r_2^2.$$

This leads to a significant simplification of the form factor (27):

$$F^{\mathrm{ZRA}}_{L_x L_y l}(r_3) = N^{\mathrm{ZRA}}\sqrt{(2l+1)(2L_x+1)(2L_y+1)} \times \langle L_x 0 L_y 0 \mid l0\rangle\,\Psi_l(r_3)\,\frac{1}{r_3^2}\,\delta(r_2). \tag{45}$$

As a result, the integral (31) becomes one-dimensional:

$$I^{\mathrm{ZRA}}_{L_x L_y l} = N^{\mathrm{ZRA}}\sqrt{(2l+1)(2L_x+1)(2L_y+1)} \times \langle L_x 0 L_y 0 \mid l0\rangle \int \chi^*_{L_y}\left(k_y\frac{A}{B}r\right)\Psi_l(r)\,\chi_{L_x}(k_x r)\,dr. \tag{46}$$

Thus, in the zero-range approximation the cross section (32) is determined by a one-dimensional integral; moreover, the number of sums in (32) is reduced, since the invariant form factor depends only on the angular momenta of the corresponding partial waves and the angular momentum transfer l. Therefore, the calculations of the angular distributions in the zero-range approximation are simplest. However, the use of this approximation is justified only in reactions with deuterons. To calculate the characteristics of reactions with heavy ions, the approximation is basically inapplicable, since the condition $r_2/r_3 \gg 1$ is not satisfied.

Distributed-density approximation (DDA).[53] In reactions with the ions ^{6}Li, ^{7}Li, and ^{9}Be involving transfer to the nucleus of a complex cluster (for example, an α particle) at relatively low energies, it is important to take into account not only the finite range but also the surface nature of the interaction. For this purpose, it was suggested in Ref. 53 that one should introduce a model invariant form factor, this being described by a function $\rho(x)$ with a fairly sharp peak on the boundary of the nucleus, i.e., at $x = r - R$. In this approximation, the radial integral takes the form

$$\left.\begin{aligned} &I^{\mathrm{DDA}}_{L_x L_y l} = \sqrt{(2l+1)(2L_x+1)(2L_y+1)}\,\langle L_x 0 L_y 0 \mid l0\rangle \\ &\times \int dr_x \Psi_l(r_x)\int dr'_y \chi^*_{L_y}\left(k_y\frac{A}{B}r'_y\right)\rho(r-R)\,\chi_{L_x}(k_x r'_y); \\ &\mathbf{r} = \frac{cM}{xB}\mathbf{r}_2 = \mathbf{r}_x - \frac{A}{B}\mathbf{r}_y; \quad r'_y = \frac{y}{x}r_y + \frac{cM}{xB}\mathbf{r}_x; \\ &M \equiv x + A \equiv y + B. \end{aligned}\right\} \tag{47}$$

For the weight function $\rho(r - R)$, a Gaussian distribution was proposed in Ref. 53:

$$\rho(r-R) = \left(\sqrt{\pi}\xi\right)^{-1/2} D_0 \exp\{-[(r-R)/\xi]^2\}.$$

Thus, in the distributed-density approximation the particles y, c, and A are situated on a single straight line, but not at one point, as in the zero-range approximation, and the interaction between them takes place in a region with center at the point R and width of order ξ. It follows from Eq. (47) that in the distributed-density approximation the radial integral reduces to the product of two one-dimensional integrals, but some additional parameters are then introduced (R, ξ, and D_0).

No-recoil approximation. This is based on the

assumption that the mass of the transferred particle c is much less than the mass of particle x of nucleus A, i.e., the recoil of the target nucleus due to the transferred particle is ignored and one takes into account only the total recoil of the system, since nucleus A is not assumed to have infinite mass. In this approximation, particle c moves along the straight line joining y and A, although the actual direction of this straight line does not, in contrast to the distributed-density approximation, remain unchanged, since it depends on the direction of emission of particle y. The strip of integration in this case, as in the zero-range approximation, degenerates into a straight line, but the degeneracy is due not to the smallness of r_2/r_3, but to the smallness of the mass ratios. It is clear that this approximation is best justified when one nucleon is transferred in reactions with heavy ions.

The no-recoil approximation also simplifies significantly the expression for the invariant form factor (27), since in it the variables in all the integrands except the wave function $\Psi_{\Lambda_1\mu_1}(\varkappa_{c_1}, \mathbf{r}_1)$ separate together. If the expansion (37) is used, the invariant form factor in this approximation is given by

$$F^{\text{n.o.}}_{\Lambda_1\Lambda_2L_xL_yl}(r_2,\ r_3)=\frac{2}{\pi}\sqrt{(2\Lambda_1+1)(2\Lambda_2+1)(2L_x+1)(2L_y+1)}\,i^{\Lambda_2}\times\langle\Lambda_1 0\Lambda_2 0\mid l0\rangle\langle L_x 0L_y 0\mid l0\rangle A^{*\Lambda_1}_{\Lambda_2 l}(r_2,\ r_3)\,V(r_2)\,\Psi_{\Lambda_2}(\varkappa_{c_2},\ r_2). \quad (48)$$

As a result, the integral (31) takes the form

$$I^{\text{n.o.}}_{\Lambda_1\Lambda_2L_xL_yl}=\int\chi^*_{L_y}\left(k_y\frac{A}{B}r_3\right)F^{\text{n.o.}}_{\Lambda_1\Lambda_2L_xL_yl}\times(r_2,\ r_3)\,\chi_{L_x}(k_xr_3)\,dr_3. \quad (49)$$

Thus, the no-recoil approximation permits analytic calculation of the integral over the angular variables in the form factor (27) without the introduction of additional summations. However, the radial integrals (49) remain three-dimensional, and they can be calculated only numerically. If one uses the expansion of the integrals (37b) in eigenfunctions of the generalized harmonic oscillator,[48] the integral in (49) can be represented as an infinite sum of products of one-dimensional integrals.[54] The integral (49) can be reduced to a product of one-dimensional integrals if[55] the distorted waves are treated in the generator-coordinate method[56] with Gaussian wave packets, and the functions $\Psi_{\Lambda_1\mu_1}(\varkappa_{c_1}, \mathbf{r}_1)$ and $\Psi_{\Lambda_2\mu_2}(\varkappa_{c_2}, \mathbf{r}_2)$ are replaced by the eigenfunctions of a harmonic oscillator. Then the variables in the integral (49) can be separated by means of a Talmi–Moshinsky–Smirnov transformation.[45]

For analytic separation of the variables in the integral (49), it is necessary to use additional assumptions about the form of the wave functions $\Psi_{\Lambda\mu}(\varkappa_c, \mathbf{r})$. For this, it is customary to use the Buttle–Goldfarb approximation (BGA),[57] which consists of replacing the radial function $\Psi_{\Lambda_1}(\varkappa_{c_1}, \mathbf{r}_1)$ by a Hankel function. This approximation should be best satisfied in reactions with heavy ions below the Coulomb barrier. Braun-Münzinger and Harney[58] have shown that because of the presence of strong absorption this approximation is also valid if the energy of the ions exceeds the Coulomb barrier (however, this conclusion cannot be regarded as rigorous, since all the estimates in Ref. 58 are of a numerical nature).

When the Buttle–Goldfarb approximation is used, the radial integrals (49) must be truncated below, since the rate of divergence of the Hankel functions at small r is greater than the rate of decrease of the distorted waves. It is particularly important to introduce a cutoff of the integrals for partial waves with small angular momenta, when there is virtually no centrifugal barrier. We note that, in its physical meaning, this approximation is equivalent to the peripheral model of nuclear reactions,[59] in which the vertex functions are regarded as constants.

In the Buttle–Goldfarb approximation, the wave function of the captured nucleon in the final nucleus has the form

$$\Psi_{\Lambda_1}(\varkappa_{c_1},\ r_1)\equiv\Psi_{\Lambda_1}(\varkappa_{c_1},\ |\mathbf{r}_2+\mathbf{r}_3|)\to N^{\text{BGA}}h_{\Lambda_1}(\varkappa_{c_1},\ |\mathbf{r}_2+\mathbf{r}_3|). \quad (50)$$

Then the matrix $A^{*\Lambda_1}_{\Lambda_2 l}$ can be calculated analytically in accordance with (39), and the integral (49) can be represented as a product of two one-dimensional integrals:

$$I^{\text{n.o. BGA}}_{\Lambda_1\Lambda_2L_xL_yl}=i^{\Lambda_1+\Lambda_2-l}N^{\text{BGA}}\sqrt{(2\Lambda_1+1)(2\Lambda_2+1)}\times\sqrt{(2L_x+1)(2L_y+1)}\langle\Lambda_1 0\Lambda_2 0\mid l0\rangle\langle L_x 0L_y 0\mid l0\rangle\times\int r_2^2 j^*_{\Lambda_2}(i\varkappa_{c_1},\ r_2)\,V(r_2)\,\Psi_{\Lambda_2}(\varkappa_{c_2},\ r_2)\,dr_2\times\int\chi^*_{L_y}\left(k_y\frac{A}{B}r_3\right)h_l^{(1)}(i\varkappa_{c_1},\ r_3)\,\chi_{L_x}(k_xr_3)\,dr_3. \quad (51)$$

The calculation of the integrals (51) does not present especial difficulty. But if a rectangular well is used for $V(r_2)$,[60] then the integral over dr_2 in (51) can be calculated analytically. Thus, in the no-recoil approximation and when the Buttle–Goldfarb approximation is used, the kinematic integral, as in the distributed-density approximation, is replaced by a product of two one-dimensional integrals.

Note that the Buttle–Goldfarb approximation is related to the surface nature of the reaction and not to the neglect of recoil effects. Therefore, it can also be used in the FRDWM variants in which recoil effects are taken into account (see below).

Thus, allowance for the finite-range and recoil effects in the zeroth order makes it possible to simplify considerably the calculations of the kinematic factor (30). However, the actual value of these approximations is greatly reduced by the circumstance that the selection rules for them restrict the possible values of the angular momentum transfers by the condition of normal parity. Physically, this restriction is due to the fact that in the zeroth order the position of all particles in space is unambiguously fixed. Mathematically, the parity selection rules follow from the parity conditions of the sums $\Lambda_1+\Lambda_2+l$ and L_x+L_y+l, which, in their turn, follow from the presence of Clebsch–Gordan coefficients with zero projections in the expressions (45), (47), and (48). Therefore, correct allowance for the finite-range and recoil effects is needed above all to take into account correctly all possible angular momentum transfers l in the cross section (32).

Approximate Allowance for Recoil Effects: Plane-Wave Modulation. We now consider methods of separating the variables in the distorted waves associated with approximate factorization of the distorted waves

themselves. This factorization can be achieved in several ways. Here, we shall consider the simplest, namely, the representation of the distorted wave in the form of the product of a plane wave and a modulating factor.

Formally, the distorted waves $\chi^{(\pm)}(\mathbf{kr})$ can be expressed in the form of an infinite series analogous to (26), each term of the series containing the product of the amplitude function $\mathbf{B}^{(\pm)}(\mathbf{kr})$ and the plane wave $\exp(i\mathbf{kr})$ (expansion of the distorted wave in plane waves). Such a representation is most convenient when the phase of the factor $\mathbf{B}^{(\pm)}(\mathbf{kr})$ varies with k slower than the phase of the plane wave. In this case, a single principal term, in which the phase of the plane wave does not depend on r, is distinguished in the series, and $\mathbf{B}^{(\pm)}(\mathbf{kr})$ can be regarded as a factor that modulates both the amplitude and phase of the plane wave.

If the modulating factor has a sinusoidal nature, this approximate treatment of the distorted waves is called the *eikonal approximation*[61]:

$$\chi^{(\pm)}(\mathbf{kr}) = \exp(-\gamma kR)\exp[\pm(1+\beta\pm i\gamma)\,\mathbf{kr}]. \tag{52}$$

The parameters in the expression (52) have the following physical meaning.[61] First, $(1+\beta)k$ is the modified wave number (it can be expressed in terms of the quasiclassical momentum at the turning point); the parameter γ ensures damping of the distorted wave, i.e., it is associated with the presence of absorption in the optical potential; the parameter R normalizes the wave function (52), and its value is usually taken equal to the radius of the nucleus.

Since the influence of the distortions is taken into account in the eikonal approximation only in the phase of the plane wave, this approximation is in essence very close to the quasiclassical approximation (improved somewhat by the introduction of the parameter γ). Therefore, the condition of applicability of the eikonal approximation is identical with the usual quasiclassical condition

$$1/k = \lambda \ll R. \tag{53}$$

For example, for 25-MeV α particles the wavelength is $\lambda \approx 0.6$ F, and when they interact with nuclei; which have a radius $R \approx 4$–6 F, the condition (53) is fairly well satisfied.

We note one further important circumstance. Since no restrictions at all are imposed on the masses and the coordinates of the particles in the eikonal approximation, both the finite range and the recoil can be taken into account accurately in this approximation. Therefore, the eikonal approximation can be used to calculate the angular distributions of both direct and exchange processes.[62] In this sense, the discussion of the eikonal approximation in the present section is to a large degree formal. The corresponding computational program will be considered in the section devoted to accurate FRDWM programs. The simple form of the distorted wave in the eikonal approximation permits complete separation of the variables in the integral (31), and if the Buttle–Goldfarb approximation[57] is used, analytic expressions for the cross section (32) can be obtained.

If one considers only the stripping mechanism in single-nucleon transfer reactions, i.e., one assumes that the ratios c/x and c/B are small, then one can introduce a different way of separating the variables in the distorted waves, thereby somewhat improving the quasiclassical approximation. In this case, the terms in the modulating factors containing the ratios c/x and c/B are omitted, while in the plane waves the coordinates $\mathbf{r}_x$ and $\mathbf{r}_y$ are taken into account exactly [Dodd-Greider approximation[63] (DGA)]. In this approximation, the distorted waves take the form

$$\chi_x^{(-)}(\mathbf{k}_x\mathbf{r}_x) \approx \mathbf{B}_x^{(+)}(\mathbf{k}_x\mathbf{r}_3)\exp(i\mathbf{k}_x\mathbf{r}_x);$$
$$\chi_y^{(-)}(\mathbf{k}_y\mathbf{r}_y) \approx \mathbf{B}_y^{(-)}\left(\mathbf{k}_y\frac{A}{B}\mathbf{r}_3\right)\exp(i\mathbf{k}_y\mathbf{r}_y). \tag{54}$$

To estimate the region of applicability of such an approximation, we expand $\mathbf{B}_x^{(+)}(\mathbf{k}_x\mathbf{r}_x)$ in a Taylor series in the neighborhood of the point $\mathbf{r}_x = \mathbf{r}_3$ and take into account the terms of first order in c/x:

$$\mathbf{B}_x^{(+)}(\mathbf{k}_x\mathbf{r}_x) = \mathbf{B}_x^{(+)}(\mathbf{k}_x\mathbf{r}_3) + (c/x)\,\mathbf{r}_2\nabla\mathbf{B}_x^{(+)}(\mathbf{k}_x\mathbf{r}_x)|_{r_x=r_3} + \ldots.$$

The second term in this expansion can be ignored if

$$c/x\,\mathbf{r}_2\nabla\mathbf{B}_x^{(+)}(\mathbf{k}_x\mathbf{r}_x)|_{r_x=r_3}/\mathbf{B}_x^{(+)}(k_xr_3) \ll 1,$$

which can be transformed to

$$R|\nabla\mathbf{B}|/B_x^{(+)}(k_xr_3) \ll x/c. \tag{55}$$

If instead of $\mathbf{B}_x^{(+)}(\mathbf{k}_x\mathbf{r}_3)$ we use the functions of the eikonal approximation, the condition (55) takes the form

$$c/x \ll \lambda/R \ll 1. \tag{55a}$$

As one would expect, the quasiclassical inequality (53) in such an approximation is augmented by a further inequality, i.e., the conditions imposed on the ratio λ/R are made more stringent.

The expressions (54) make it possible to simplify the calculation of the kinematic factor (30). If the Buttle–Goldfarb approximation[57] is used, the kinematic factor in the Dodd-Greider approximation takes the form

$$\beta_{\Lambda_1\Lambda_2lml}(\mathbf{k}_x,\ \mathbf{k}_y,\ \varkappa_{c_1},\ \varkappa_{c_2}) = N^{\mathrm{BGA}}\sum_{l_1m_1LM} i^{l_1+\Lambda_1+\Lambda_2+L}$$
$$\times(-1)^{L+M}V\overline{(2\Lambda_1+1)(2l_1+1)}\langle l_1m_1lm_l|L-M\rangle$$
$$\times\Gamma_{\Lambda_1\Lambda_2l}(l_1,\ L)\,Y_{LM}(\mathbf{Q})\,G_{l_1}(\mathbf{k}_x,\ \mathbf{k}_y), \tag{56}$$

where

$$G_{l_1}(\mathbf{k}_x,\ \mathbf{k}_y) = \int d\mathbf{r}_3\mathbf{B}_y^{(-)}\left(\mathbf{k}_y\frac{A}{B}\mathbf{r}_3\right)\exp(i\mathbf{qr}_3)\,h_{l_1}^{(1)*}(i\varkappa_{c_1}r_3)\,Y_{l_1m_1}$$
$$\times(\mathbf{r}_3)\,\mathbf{B}_x^{(+)}(\mathbf{k}_x\mathbf{r}_3); \tag{57}$$
$$\Gamma_{\Lambda_1\Lambda_2l}(l_1,\ L) = \sum_{l_2}V\overline{(2l_2+1)}\langle l_10\Lambda_10|l_20\rangle$$
$$\times\langle l_20L0|\Lambda_20\rangle W(l_1\Lambda_1L\Lambda_2:l_2l)\,A_L(l_2,\ \Lambda_2); \tag{58}$$
$$A_L(l_2,\ \Lambda_2) = i^{l_2}\int r_2^2 j_{l_2}^*(i\varkappa_{c_1}r_2)\,j_L(Qr_2)\,V(r_2)\,\Psi_{\Lambda_2}(\varkappa_{c_2}r_2)\,dr_2. \tag{59}$$

The recoil momentum $\mathbf{Q}$ and the momentum transfer $\mathbf{q}$ are determined in the usual manner:

$$\mathbf{Q} = (c/x)\,\mathbf{k}_x + (c/B)\,\mathbf{k}_y; \tag{60}$$
$$\mathbf{q} = \mathbf{k}_x - (A/B)\,\mathbf{k}_y. \tag{60a}$$

The recoil corrections are included in the integrals (59), since the angular momentum L appears because of the presence of the terms containing c/x and c/B in the distorted waves (54). Since the integrands in (59) are localized in the region in which the nuclear forces act ($r_2 \leq R$), the integrals $A_L(l_2, \Lambda_2)$ will decrease rapidly for $L > QR$ because of the presence of the Bessel func-

tion $j_L(QR)$, and the sum over L is actually finite (for given L, the summation over l_1 and l_2 is restricted by the selection rules). Therefore, when allowance is made for the recoil effects in the first order the kinematic factor (30), as in the no-recoil approximation (the Buttle–Goldfarb approximation being used), contains products of two integrals, but the number of sums in (56)–(59) is increased by three compared with (30) if the radial integral is determined by the expression (51).

In Ref. 63, the modulating functions $B^{(\pm)}(\mathbf{kr})$ are considered in the strong-absorption model,[16] i.e., they vanish within the region of integration and in the shadow region. In this case, the integrals (57) have their simplest form and can be calculated analytically. Such a treatment is frequently called the diffraction approximation.

Approximate Allowance for Recoil Effects: Modulation of the Distorted Wave. We consider a different method of approximate factorization of the distorted waves based on modulation of the distorted wave itself. This method is based on the assumption that the ratios c/x and c/B are small, i.e., it is applicable only for the calculation of the angular distributions of the stripping mechanism in single-nucleon transfer reactions.

The distorted wave $\varkappa^{(\pm)}(\mathbf{kr})$ $(\mathbf{r}=\mathbf{R}+\boldsymbol{\rho})$ in the neighborhood of the point R can be written in the form

$$\chi^{(\pm)}\{\mathbf{k}(\mathbf{R}+\boldsymbol{\rho})\}=\exp(\boldsymbol{\rho}\nabla)\,\chi(\mathbf{kr})|_{\mathbf{r}=\mathbf{R}}. \tag{61}$$

This exact expression can be approximated in different ways. We consider some of them.

Use of perturbation theory. The exponential factor in the expression (61) is expanded in a series around the point $\mathbf{r}_x=\mathbf{r}_3$, $\mathbf{r}_y=(A/B)\mathbf{r}_3$; only the terms of first order in c/x and c/B are taken into account, and the translation operator is treated in the quasiclassical approximation.[64] Then for the distorted waves we have the expressions

$$\chi^{(+)}(\mathbf{k}_x\mathbf{r}_x)=(1+\mathrm{i}\,(c/x)\,\mathbf{r}_3\mathbf{k}_x)\,\chi^{(+)}(\mathbf{k}_x\mathbf{r}_3); \tag{62}$$

$$\chi^{(-)}(\mathbf{k}_y\mathbf{r}_y)=(1-\mathrm{i}\,(c/B)\,\mathbf{r}_2\mathbf{k}_y)\,\chi^{(-)}(\mathbf{k}_y\,(A/B)\,\mathbf{r}_3). \tag{62a}$$

As a result, the kinematic factor (30) contains four terms, one of which is identical to the one obtained earlier in the no-recoil approximation; two others correspond to allowance for recoil effects in the first order order, and the final term determines the recoil effects in the second order (it is proportional to c^2/xB). As the numerical estimates of Ref. 65 show, the contribution of the first-order correction terms to single-nucleon transfer reactions involving heavy ions is appreciable (it is 20–30% of the contribution of the zeroth term in the no-recoil approximation). At the same time, the contribution of the second-order terms is insignificant (~1%) and can be ignored. The same estimates confirm the validity of the expansions (62) and (62a) for the distorted waves, since all the following terms of these expansions are at the least of second order.

Using the Buttle–Goldfarb approximation,[57] one can calculate the first-order corrections to the kinematic factor [in the zeroth order, it is determined by the expressions (30) and (51)]. The corresponding corrections have the form

$$\begin{aligned}
&\Delta\beta_{\Lambda_1\Lambda_2 l m_l}(\theta,\varkappa_{c_1}\varkappa_{c_2})\\
&=-\frac{4\pi}{k_x k_y}N^{\mathrm{BGA}}\sum_{l_1 l_2 L_x L_y M_x M_y}\frac{i^{L_x-L_y}}{\sqrt{2l+1}}\\
&\times\langle L_y M_y l_2 m_l\,|\,L_x M_x\rangle\langle l0 m_l\,|\,l_2 m_l\rangle\\
&\times\langle l_1 0 l_2 0\,|\,\Lambda_1 0\rangle\langle l0\Lambda_2 0\,|\,l_1 0\rangle\langle L_y 0 l_2 0\,|\,L_x 0\rangle\\
&\times A(l_2,\Lambda_2)\,Y_{L_x M_x}(\mathbf{k}_x)\,Y_{L_y M_y}(\mathbf{k}_y)\\
&\times\int\chi^{*}_{l_y}\left(k_y\frac{A}{B}r\right)h_{l_1}^{(1)*}(i\varkappa_{c_1}r)\,\chi_{L_x}(k_x r)\,dr,
\end{aligned} \tag{63}$$

and the integral $A(l_2,\Lambda_2)$ can be obtained from the integral (59) by replacement of the Bessel function $j_L(Qr_2)$ by Qr_2.

Baltz and Kahana[66] proposed a different method of using perturbation theory; in this method, only the distorted wave of the final particle is expanded in a Taylor series around the point $\mathbf{r}_y=\mathbf{r}_x$. Then

$$\mathbf{r}_y=\frac{A}{B}\mathbf{r}_x-\frac{cM}{xB}\mathbf{r}_2,$$

and, if the translation operator is again treated in the quasiclassical approximation, the distorted wave takes the form

$$\chi^{(-)}(\mathbf{k}_y\mathbf{r}_y)=\left(1-\mathrm{i}\,\frac{cM}{xB}\mathbf{r}_2\mathbf{k}_y\right)\chi^{(-)}\left(\mathbf{k}_y\frac{A}{B}\mathbf{r}_x\right). \tag{64}$$

The first term of this expansion gives the same kinematic factor as in the no-recoil approximation, while the second gives first-order corrections analogous to those considered above. However, the influence of the second-order terms in the expansion (64) may be more significant than in the expansions (62) and (62a), since the mass coefficient $cM/(xB)$ is appreciably larger than the coefficients c/x and c/B.

Local Momentum Approximation (LMA). We consider one further approximation, in which one does not use perturbation theory for the modulating factor of the distorted wave but assumes that the interaction of the particles takes place on the surface of the nucleus. In this case, it can be assumed that for each r in a small neighborhood of $\mathbf{r}_3$ the function $\chi(\mathbf{kr})$ can be represented in the approximate form

$$\chi(\mathbf{kr})\approx A(\mathbf{r})\exp\{\mathrm{i}\mathbf{q}(\mathbf{r})\mathbf{r}\},$$

and, in contrast to the eikonal approximation, the phase of the distorted wave may vary as a function of r. Then the expression (61) takes the form

$$\chi^{(+)}(\mathbf{k}_x\mathbf{r}_x)=\exp\left[\mathrm{i}\,\frac{c}{x}\,\mathbf{r}_2\mathbf{q}_x(\mathbf{r}_3)\right]\chi^{(+)}(\mathbf{k}_x\mathbf{r}_3); \tag{65}$$

$$\chi^{(-)}(\mathbf{k}_y\mathbf{r}_y)=\exp\left[-\mathrm{i}\,\frac{c}{B}\,\mathbf{r}_2\mathbf{q}_y(\mathbf{r}_3)\right]\chi^{(-)}\left(\mathbf{k}_y\frac{A}{B}\mathbf{r}_3\right). \tag{65a}$$

The quantities $\mathbf{q}_x(\mathbf{r})$ and $\mathbf{q}_y(\mathbf{r})$ are called *local momenta*. Such an approximate factorization of the distorted waves is the essence of the local momentum approximation.[58] When is such an approximation justified? To obtain qualitative estimates, we expand the potential of the optical model in a series in the neighborhood of the point $\mathbf{r}=\mathbf{r}_3$:

$$\mathbf{U}(\mathbf{r})=\mathbf{U}(\mathbf{r}_3)+\mathbf{x}\nabla\mathbf{U}(\mathbf{r})|_{\mathbf{r}=\mathbf{r}_3}+\ldots.$$

If the function (65) and (65a) are to be solutions of the Schrödinger equation with a potential containing only the first term of this expansion, the following conditions must be satisfied:

$$|\mathbf{q}(\mathbf{r})|\approx(2m/\hbar^2)\sqrt{E-\mathbf{U}(\mathbf{r}_3)}; \tag{66}$$

$$|\mathbf{x}|\ll[E-\mathbf{U}(\mathbf{r}_3)]/|\nabla\mathbf{U}|. \tag{66a}$$

In other words, if the local momentum approximation is to be used it is necessary that the local momenta have the order of the quasiclassical wave numbers at the turning point and, in addition, the region of localization of the interaction of the particles must be much smaller than the reciprocal relative gradient of the potential. If the reaction takes place on the surface of the nucleus, as in reactions with strongly absorbing particles, the relative gradients of the optical potentials are not large, and, as is shown in Ref. 58, their reciprocal values reach ~40 F. Then the local momenta can be set equal to $q(R)$, where R is the radius of the nucleus. The condition (66) determines the absolute value of q, and not its direction, and therefore the cross section in the local momentum approximation should not depend on the direction of the local momenta. According to Ref. 67, the local momentum approximation is best satisfied for quasiclassical particles with a definite trajectory of the motion.

Using the expressions (65) and (65a), one can readily show that in the local momentum approximation the kinematic factor, as in the Dodd–Greider approximation, is determined by the expression (56). The difference from the latter approximation resides, first, in the fact that the integral (57) in the local momentum approximation has a somewhat different form:

$$G_{l_1}(\mathbf{k}_x, \mathbf{k}_y) = \int \chi^{(-)*}\left(\mathbf{k}_y \frac{A}{B}\mathbf{r}_3\right) h_{l_1}^{(1)*}(ix_{c_1}r_3) \times Y^*_{l_1 m_1}(\mathbf{r}_3)\,\chi^{(+)}(\mathbf{k}_x\mathbf{r}_3)\,d\mathbf{r}_3, \quad (67)$$

where the distorted waves are solutions of the Schrödinger equation, and not model functions, as in the Dodd–Greider approximation. Second, in the local momentum approximation the recoil momentum Q is determined by the expression

$$\mathbf{Q} = (c/x)\,\mathbf{q}_x + (c/B)\,\mathbf{q}_y, \quad (68)$$

and, in contrast to the Dodd–Greider approximation, does not have a definite direction.

Approximation of large momentum transfers.[109] Finally, we consider one further variant of approximate factorization of the distorted waves that, like the local momentum approximation, does not involve the use of perturbation theory. In this variant, it is assumed that the momentum transferred in the collision of the incident particle with the target nucleus appreciably exceeds the momentum of the relative motion in the colliding nuclei. This assumption enables one to ignore the gradient of the distorted wave with respect to the coordinate $\mathbf{r}_2$ in the action of the translation operator in (61), as a result of which the distorted waves in this approximation can be represented in the form [cf. the expressions (54) and (65)]

$$\chi^{(+)}(\mathbf{k}_x\mathbf{r}_x) = \exp\{i(\mathbf{k}_x\mathbf{r}_x - \mathbf{k}_{0x}\mathbf{r}_3)\}\,\chi^{(+)}(\mathbf{k}_x\mathbf{r}_3); \quad (69)$$

$$\chi^{(-)}(\mathbf{k}_y\mathbf{r}_y) = \exp\left\{i\left(\mathbf{k}_y\mathbf{r}_y - \mathbf{k}_{0y}\frac{A}{B}\mathbf{r}_3\right)\right\}\chi^{(-)}\left(\mathbf{k}_y\frac{A}{B}\mathbf{r}_3\right). \quad (69a)$$

In these expressions, the momenta $\mathbf{k}_{0x}$ and $\mathbf{k}_{0y}$ are given by

$$\mathbf{k}_{0x} = \frac{yM}{x(A+y)}\mathbf{k}_x; \quad \mathbf{k}_{0y} = \frac{AM}{(A+y)B}\mathbf{k}_y.$$

Formally, the factorization (69) for the distorted wave does not require the assumption that the mass of the transferred particle is small compared with the mass of the heavy ion. Nevertheless, this assumption is actually used in the approximation, since for small ratios c/x and c/A the approximation of large momentum transfers is in fact best satisfied, i.e., one can ignore the gradient of the distorted wave with respect to the coordinate $\mathbf{r}_2$.

It is clear that this approximation, like the local momentum approximation, is best justified at high energies of the incident particles. However, in the approximation of large momentum transfers the restrictions on the energy are more stringent. Indeed, if we consider single-nucleon transfer reactions with heavy ions, the momenta $\mathbf{k}_{0x}$ and $\mathbf{k}_{0y}$ are close to $\mathbf{k}_x$ and $\mathbf{k}_y$, respectively, and the factorization (69) of the distorted wave in this approximation reduces to the factorization (65) in the local momentum approximation if the quasiclassical momentum on the boundary of the nucleus is equal to the momentum $\mathbf{k}_x$ and $\mathbf{k}_y$, respectively. To ensure such equality, the energy of the incident particles must, in accordance with (66), satisfy the condition

$$E \gg |U(\mathbf{r}_3)|,$$

or, as is shown in Ref. 109,

$$E/A > 10 \text{ MeV/nucleon}.$$

For given ion energy, this condition obviously imposes an upper bound on the ion mass. Thus, the approximation of large momentum transfers is best justified in single-nucleon transfer reactions with relatively light ions (^{12}C, ^{16}O) of high energies (several hundred MeV).

The rather stringent restrictions on the energy and mass of the ions in this approximation cast doubt on the validity of using the Buttle–Goldfarb approximation. Therefore, the kinematic factor in the approximation of large momentum transfers is determined not by Eq. (56) but by the more general expression

$$\beta_{lm_l}^{\text{ALMT}} = \sum_{\mu_1,\mu_2} (-1)^{\Lambda_1+\Lambda_2-\mu_2}\frac{1}{\sqrt{2l+1}}\langle\Lambda_1 -\mu_1\Lambda_2\mu_2 \mid lm_l\rangle \times \int G_{\Lambda_1\mu_1}(\mathbf{k}_x, \mathbf{k}_y, \mathbf{r}_2)\exp(i\mathbf{Q}\mathbf{r}_2)\,V(\mathbf{r}_2)\,\Psi_{\Lambda_2\mu_2}(\mathbf{r}_2)\,d\mathbf{r}_2, \quad (70)$$

where

$$G_{\Lambda_1\mu_1}(\mathbf{k}_x, \mathbf{k}_y, \mathbf{r}_2) = \int \chi^{(-)*}\left(\mathbf{k}_y\frac{A}{B}\mathbf{r}_3\right) \times \exp(i\mathbf{q}\mathbf{r}_3)\,\Psi_{\Lambda_1\mu_1}(\mathbf{r}_2+\mathbf{r}_3)\,\chi^{(+)}(\mathbf{k}_x\mathbf{r}_3)\,d\mathbf{r}_3. \quad (70a)$$

In (70) and (70a), the momentum transfer $\mathbf{Q}$ (recoil momentum) is determined in the same way as in the Dodd–Greider approximation [see the expression (60)], and the momentum $\mathbf{q}$ in the approximation of large momentum transfers is not equal to the corresponding momentum in the Dodd–Greider approximation and the local momentum approximation but is given by

$$\mathbf{q} = \frac{cA}{x(A+y)}\left(\mathbf{k}_x - \frac{xy}{B^2}\mathbf{k}_y\right).$$

As follows directly from (70) and (70a), the kinematic factor in the approximation of large momentum transfers is determined by a fourfold integral. To reduce the multiplicity of integration in this approximation, one uses the quasiclassical approach, which makes it possible to estimate the integral (70a) analytically. Let us consider this approach in somewhat more detail.

The kinematic factor (70) can be written in the momentum representation in the form

$$\beta_{lm_l}^{\mathrm{ALMT}} = \sum_{\mu_1\mu_2} (-1)^{\Lambda_1+\Lambda_2-\mu_2} \frac{1}{\sqrt{2l+1}} \langle \Lambda_1 -\mu_1, \Lambda_2 -\mu_2 \mid lm_l \rangle \qquad (71)$$
$$\times \int \Psi^*_{\Lambda_1\mu_1}(\tilde{\mathbf{q}})\, S(\tilde{\mathbf{q}}-\mathbf{q})\, \xi_{\Lambda_2\mu_2}(\tilde{\mathbf{q}}-\mathbf{Q})\, d\tilde{\mathbf{q}},$$

where

$$\xi_{\Lambda_2\mu_2}(\tilde{\mathbf{q}}-\mathbf{Q}) = 1/(2\pi)^{3/2} \int V(\mathbf{r}_2)\, \Psi_{\Lambda_2\mu_2}(\mathbf{r}_2) \exp[i(\tilde{\mathbf{q}}-\mathbf{Q})\mathbf{r}_2]\, d\mathbf{r}_2,$$

and the factor $S(\tilde{\mathbf{q}}-\mathbf{q})$, which depends only on the dynamics of the colliding particles, is determined by the expression

$$S(\tilde{\mathbf{q}}-\mathbf{q}) = \int \exp[i(\tilde{\mathbf{q}}-\mathbf{q})\mathbf{r}]\, \chi^{(-)*}\left(\mathbf{k}_y \frac{A}{B}\mathbf{r}\right) \chi^{(+)}(\mathbf{k}_x\mathbf{r})\, d\mathbf{r}. \qquad (72)$$

In the quasiclassical approximation, it can be expressed in terms of the phase shifts of elastic scattering in the entrance and exit channels[110]:

$$S(\tilde{\mathbf{q}}-\mathbf{q}) = \frac{2\pi^2}{k_0 Q}\, \delta\{(\tilde{\mathbf{q}}-\mathbf{q})\hat{\mathbf{k}}\} \sum_{L=0}^{\infty} (2L+1)$$
$$\times \exp(2i\delta_L) P_L(1-|\tilde{\mathbf{q}}-\mathbf{q}|^2/2k_0^2), \qquad (72a)$$

where

$$\mathbf{k} = \tfrac{1}{2}(\mathbf{k}_x + \mathbf{k}_y); \quad \mathbf{k}_0 = \tfrac{1}{2}(\mathbf{k}_{0x} + \mathbf{k}_{0y});$$
$$\delta_L = \tfrac{1}{2}(\delta_{L_x} + \delta_{L_y}).$$

The δ function in the expression (72a) determines the direction of the momentum $\tilde{\mathbf{q}}-\mathbf{q}$ (it must be parallel to the half-sum of the momenta of the incident and emitted ions). Such an approximation of the factor (72) is valid provided[109]

$$|(\tilde{\mathbf{q}}-\mathbf{q})/Q| \left|1-\frac{k_0}{Q}\cos\frac{\theta}{2}\right| \ll \frac{\pi}{L_0},$$

where $kL_0 \sim R$ corresponds to the boundary of the nucleus. This condition restricts the region of applicability of the approximation of large momentum transfers to not only high energies (large values of k_0 and Q) but also small angles of emission of the final particles.

It is readily seen from the expression (59) that the transition to the no-recoil approximation in the Dodd–Greider approximation, in the local momentum approximation, and in the approximation of large momentum transfers corresponds to vanishing recoil momentum $\mathbf{Q}$. At the same time, $L=0$, $l_1=l$, and the kinematic factor (56) can be transformed to the form (30), the corresponding integrals being determined by the expression (51).

Even approximate allowance for recoil effects leads to a smoothing of the angular distributions in reactions with complex particles compared with the distributions calculated in the no-recoil approximation. This is due in the first place to the circumstance that the recoil effects enable one to obtain selection rules with respect to the angular momentum transfer that are not restricted by the condition of normal parity (the particles c, y, and A need not be on one straight line). In addition, a smoothing of the angular distribution is due to the presence of the recoil factor (59) in the kinematic factor (56). If the recoil momentum (60) and (68) is such that it has the order of magnitude of the region of localization of the function $\Psi_{\Lambda_2}(\varkappa_{c_2}, r_2)$, the interference between the different terms in the sum (58) may appreciably smooth the factors $\Gamma_{\Lambda_1 \Lambda' 1}(l_1, L)$, and this leads to a general smoothing of the angular distribution.[63]

It should be emphasized that allowance for recoil effects necessarily requires use of a potential $V(r_2)$ with only a finite range of the interaction. Indeed, if one introduces a potential with zero range of the interaction, the integral (59) becomes equal to unity and the effects of smoothing of the angular distribution (including those due to transitions with "non-normal" parity) are lost.

4. EXACT ALLOWANCE FOR RECOIL EFFECTS

In the variants of the FRDWM that take into account accurately the recoil it is necessary to solve the problem of separating the variables either in the distorted waves, if the Jacobi coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$ are taken as basis coordinates, or in the wave functions of the relative motion and the interaction potentials, if the coordinates $\mathbf{r}_x$ and $\mathbf{r}_y$ are taken as basis coordinates. In neither case can one separate the variables analytically in either the distorted waves or the wave functions $\Psi_{\Lambda_1\mu_1}(\varkappa_{c_1}\mathbf{r}_1)$ and $\Psi_{\Lambda_2\mu_2}(\varkappa_{c_2}\mathbf{r}_2)$. The last circumstance is due to the fact that when correct allowance is made for recoil effects one cannot completely ignore the inner region of the nucleus, i.e., one cannot use the Buttle–Goldfarb approximation[57] to approximate the wave functions of the relative motion of the particles. In many FRDWM variants, these functions are found by numerical solution of the Schrödinger equation with total energy equal to the binding energy of the particles (to solve the equation it is necessary to specify not only the binding energy but also the parameters of the optical potential of the interaction of the corresponding particles and the number of nodes of the required wave function). If reactions with light particles (d, t, α, ^{3}He, ^{6}Li) are considered, the wave function of the relative motion of the particles can be specified in an analytic form, analogous to the Hulthén function. Finally, for the wave functions of the relative motion one can use harmonic-oscillator functions fitted to Hankel functions. None of these methods of determining the radial wave function of the relative motion permits analytic separation of its variables. Therefore, in exact FRDWM variants a decrease in the multiplicity of the kinematic integral can be achieved only by integration over the angular variables. This increases the number of sums in the cross section (32), and the additional one-dimensional integrals (37a) and (42) appear.

After these preliminary comments, we turn to the analysis of the concrete methods of exact allowance for recoil effects in the FRDWM. We consider two of them: separation of the variables in the wave functions of the relative motion of the particles and the interaction potentials, and separation of the variables in the distorted waves.

Separation of the Variables in the Wave Functions of the Relative Motion of the Particles. In this FRDWM variant, the coordinates $\mathbf{r}_x$ and $\mathbf{r}_y$ are taken as basis coordinates of the three-body problem (see Fig. 5). Since the invariant form factor (27) depends on the Jacobi coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$ in the calculation of the

overlap integrals in the three-body problem, the transition to the coordinates $\mathbf{r}_x$ and $\mathbf{r}_y$ requires the introduction of the Jacobian J_0 connecting the volume elements for these sets of coordinates. The coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$ can be expressed in terms of $\mathbf{r}_x$ and $\mathbf{r}_y$ by the relations (see Fig. 5)

$$\mathbf{r}_1=(\alpha(\mathbf{r}_x-\gamma\mathbf{r}_y); \quad \alpha=xB/cM;$$
$$\mathbf{r}_2=\alpha(\delta\mathbf{r}_x-\mathbf{r}_y); \quad \gamma=y/x;$$
$$\mathbf{r}_3=\frac{1}{M}(x\mathbf{r}_x+B\mathbf{r}_y); \quad \delta=A/B; \quad M\equiv A+x\equiv B+y$$

(we point out once more that for exchange processes analogous relations between the coordinates can be obtained by making the substitution $A\rightleftarrows x$ and replacing the angle θ by $\pi-\theta$). As a result, the Jacobian J_0 for the transition to the coordinates $\mathbf{r}_x$ and $\mathbf{r}_y$ has the form

$$J_0^{\text{d.tr.}}=\alpha^3\equiv\left(\frac{xB}{cM}\right)^3; \quad J_0^{\text{i.tr.}}=\left(\frac{AB}{CM}\right)^3.$$

For the separation of the variables in the wave functions $\Psi_{\Lambda_1\mu_1}(\varkappa_{c_1}, \mathbf{r}_1)$ and $\Psi_{\Lambda_2\mu_2}(\varkappa_{c_2}, \mathbf{r}_2)$ and the interaction potentials $\mathbf{V}(\mathbf{r}_2)\{\mathbf{V}(\mathbf{r}_3)\}$, we introduce the quantities $g_{K\Lambda_1\Lambda_2}(r_x, r_y)$ by means of the relations [see (42)]

$$g^{\text{str}}_{K\Lambda_1\Lambda_2}(r_x, r_y)$$
$$=\int_{-1}^{+1}\frac{d\mu}{r_1^{\Lambda_1}r_2^{\Lambda_2}}\Psi^*_{\Lambda_1}(\varkappa_{c_1}, r_1)V(r_2)\Psi_{\Lambda_2}(\varkappa_{c_2}, r_2)P_K(\mu); \tag{73}$$

$$g^{\text{h.rep}}_{K\Lambda_1\Lambda_2}(r_x, r_y)$$
$$=\int_{-1}^{+1}\frac{d\mu}{r_1^{\Lambda_1}r_2^{\Lambda_2}}\Psi^*_{\Lambda_1}(\varkappa_{c_1}, r_1)V(r_3)\Psi_{\Lambda_2}(\varkappa_{c_2}, r_2)P_K(\mu). \tag{74}$$

Then, using the expansion (43), for the invariant form factor (27) we can obtain (Refs. 2, 43, 69, and 70)

$$F^{J_cE_c}_{\Lambda_1\Lambda_2L_xL_yl}(r_x, r_y)$$
$$=\frac{1}{2}\alpha^{\Lambda_1+\Lambda_2}r_x^{\Lambda_1}(-r_y)^{\Lambda_2}(2\Lambda_1+1)(2\Lambda_2+1)$$
$$\times\sum_K(-1)^K(2K+1)\{g^{\text{str}}_{K\Lambda_1\Lambda_2}(r_x, r_y)+g^{\text{h.rep.}}_{K\Lambda_1\Lambda_2}(r_x, r_y)\}$$
$$\times\sum_{\lambda_1\lambda_2}(r_x)^{-\lambda_1}(\delta r_x)^{\lambda_2}(-r_y)^{-\lambda_2}(-\gamma r_y)^{\lambda_1}$$
$$\times\binom{2\Lambda_1}{2\lambda_1}^{1/2}\binom{2\Lambda_2}{2\lambda_2}^{1/2}$$
$$\times\sum_{l_1l_2}\sqrt{(2l_1+1)(2l_2+1)}\langle\Lambda_1-\lambda_1\lambda_20\mid l_10\rangle$$
$$\times\langle l_10K0\mid L_x0\rangle\langle\lambda_10\Lambda_2-\lambda_20\mid l_20\rangle$$
$$\times\langle l_20K0\mid L_y0\rangle W(l_2L_yl_1L_x:Kl)$$
$$\times\begin{Bmatrix}\Lambda_1-\lambda_1 & \lambda_1 & \Lambda_1\\ \lambda_2 & \Lambda_2-\lambda_2 & \Lambda_2\\ l_1 & l_2 & l\end{Bmatrix}. \tag{75}$$

The expression (75) in conjunction with (31) completely solves the problem of calculating the kinematic factor (30) when exact allowance is made for recoil effects. It can be seen that in this case the kinematic factor contains the sums of integrals (31), the additional one-dimensional integrals (73) and (74), and five additional sums (over $K, \lambda_1, \lambda_2, l_1, l_2$). The sum rules in (75) require parities of the sum $\Lambda_1+\Lambda_2+L_x+L_y$, but the values of the angular momentum transfer l are not restricted by parity.

The number of terms in the sums over λ_1 and λ_2 in the form factor (75) is bounded above by the values of the angular momenta Λ_1 and Λ_2. With regard to the number of essential terms in the sum over K, it depends on the rate at which the quantities $g_{K\Lambda_1\Lambda_2}(r_x, r_y)$, which are

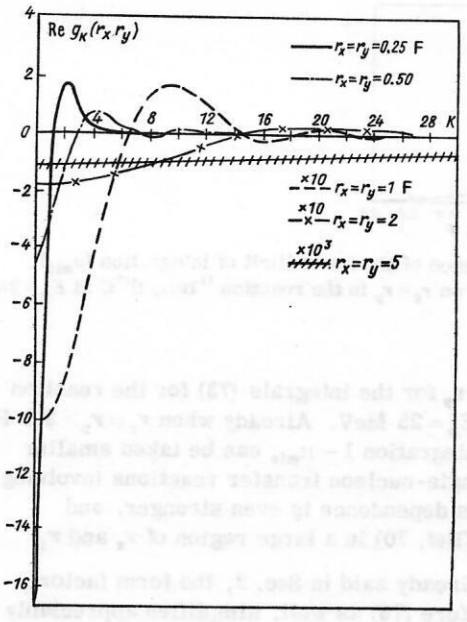

FIG. 6. Dependence on K of the real part of $g_{K\Lambda_1\Lambda_2}(r_x, r_y)$ for the reaction $^{11}\text{B}(\alpha, t)^{12}\text{C}$ at $E_\alpha=25$ MeV ($r_x=r_y$, $\Lambda_1=1$, $\Lambda_2=0$).

determined by the integrals (73) and (74), decrease with increasing K. Since for the stripping mechanism r_2 is not very large, while for the heavy-replacement mechanism r_3 has the order of the radius of the nucleus, the integrals (73) and (74) have an appreciable value only within a comparatively narrow strip along the straight line $r_y=(A/B)r_x$, which corresponds to the zero-range approximation. For small r_x and r_y (smaller than the characteristic range r_0 of the interaction), the dependence of the quantities $g_{K\Lambda_1\Lambda_2}(r_x, r_y)$ on K is fairly strong, and they are oscillating functions of K (the period of the oscillations increases with increasing r_x and r_y) and they decrease rapidly with increasing K. At large values of r_x and r_y ($\geqslant r_0$), the integrals $g_{K\Lambda_1\Lambda_2}(r_x, r_y)$ decrease rapidly, and their dependence on K virtually disappears. Figure 6 shows the real parts of $g_{K\Lambda_1\Lambda_2}(r_x, r_y)$ for $r_x=r_y$, i.e., near the straight line $r_y=(A/B)r_x$, for the reaction $^{11}\text{B}(\alpha, t)^{12}\text{C}$ at $E_\alpha=25$ MeV. It can be seen that already at $r_x=r_y=5$ F for all K from 0 to 30 the integrals $g_{K\Lambda_2\Lambda_2}(r_x, r_y)$ are effectively constants and smaller than the corresponding values at small radii by more than four orders of magnitude. Thus, in the sum (75) we can restrict ourselves to the maximal value $K_{\max}$, at which $g_{K\Lambda_1\Lambda_2}(r_x, r_y)$ are fairly small and are virtually independent of K. For the chosen $K_{\max}$, the selection rules restrict the number of terms in the sums over L_x and L_y.

At large values of r_x and r_y, the integrands in the expressions (73) and (74) have the form of functions with a sharp peak at the point $\mu=1$. On the one hand, this explains why the integrals $g_{K\Lambda_2\Lambda_2}(r_x, r_y)$ in this region of r_x and r_y values do not depend on K, and, on the other hand, it enables one to choose the lower limit of integration $\mu_{\min}$ close to unity, which significantly shortens the necessary computing time. The degree of proximity to unity depends on the narrowness of the strip, i.e., the smallness of $[cM/(\varkappa B)]r_2$. Figure 7 shows $\mu_{\min}$ as a

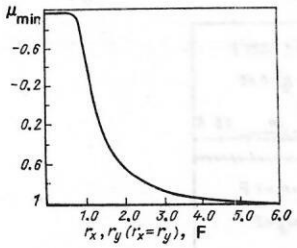

FIG. 7. Dependence of the lower limit of integration (μ_{min}) with respect to μ on $r_x = r_y$ in the reaction $^{11}B(\alpha, t)^{12}C$ at $E_\alpha = 25$ MeV.

function of $r_x = r_y$ for the integrals (73) for the reaction $^{11}B(\alpha, t)^{12}C$ at $E_\alpha = 25$ MeV. Already when $r_x = r_y > 3.5$ F the region of integration $1 - \mu_{min}$ can be taken smaller than 0.1. In single-nucleon transfer reactions involving heavy ions, this dependence is even stronger, and $1 - \mu_{min} < 0.01$ (Ref. 70) in a large region of r_x and r_y.

As we have already said in Sec. 2, the form factor (27), and therefore (75) as well, simplifies appreciably if the incident particle x is not heavier than the α particle. In this case, $\Lambda_2 = 0$ and $\Lambda_1 = l$, and in (75) the summation over Λ_2, l_1, l_2 disappears. As a result, the invariant form factor (27) for direct processes in reactions with α particles has the form

$$F_{L_x L_y l}(r_x, r_y) = \frac{1}{2}\sqrt{2l+1}\sum_K \alpha^l r_x^l (-1)^K (2K+1) \times \{g_K^{str}(r_x, r_y) + g_K^{h.rep.}(r_x, r_y)\} \times \sum_\lambda \binom{2l}{2\lambda}^{1/2} \left(-\frac{\gamma r_y}{r_x}\right)^\lambda \langle \lambda 0 K \mid L_x 0\rangle \times \langle l - \lambda 0 K \mid L_y 0\rangle W(L_x \lambda L_y l - \lambda : Kl). \quad (76)$$

It follows from this expression that for reactions with α particles the selection rules restrict the angular momentum transfers l by the condition of normal parity even if an interaction of finite range is used.

Finally, we consider the transition from the form factors (75) and (76) to the form factor (45) in the zero-range approximation. In this case $\mu = 1$ and $P_K(\mu) = 1$ irrespective of the value of K. As a result, the integral (73) is also independent of K and takes the form

$$g_K^{ZRA}(r_x, r_y) = (1/r_3^l)\,\Psi_l(\varkappa_{c_1}, r_3)\, N^{ZRA}(\delta(r_2)/r_2^2)\,\delta_{\Lambda_2 0}. \quad (77)$$

Then the sums over K and λ in the expression (76) can be readily calculated analytically by means of the well-known relations.[72] As a result, the form factor (76) is transformed to (45).

Thus, the separation of the variables in the wave functions of the relative motion and in the interaction potential makes it possible to calculate analytically the integrals over the angular variables in terms of known algebraic factors and the one-dimensional integrals $g_K(r_x, r_y)$. Although the number of additional summations is appreciable in this method of calculating the kinematic factor, the actual number of terms in the sums is reasonable (it is limited by the selection rules and the region of localization of the interaction of the particles). In this variant of the FRDWM, both direct and exchange processes can be treated symmetrically, since additional restrictions on the masses of the particles are not introduced.

Separation of the Variables in the Distorted Waves. In the method of exact allowance for recoil effects considered above, the radial integral (31) remains two-dimensional. The separation of the variables in the distorted waves may lead to a decrease in the multiplicity of integration.

If in the partial waves of the initial and the final particle $\mathbf{r}_x$ and $\mathbf{r}_y$ are expressed in terms of $\mathbf{r}_1$ and $\mathbf{r}_2$ by means of (44), and then the variables r_1 and r_2 are separated by means of the matrices (37a), the radial integral (31) for the stripping mechanism takes the form

$$I_{\Lambda_1\Lambda_2 L_x L_y l} = \frac{4}{\pi^3}\sum_{l_1 l_2 l_1' l_2'} i^{l_1+l_2-(l_1'+l_2')+L_x-L_y} \begin{Bmatrix} l_1 & l_2 & L_x \\ l_1' & l_2' & L_y \\ \Lambda_1 & \Lambda_2 & l \end{Bmatrix} \times (2l_1+1)(2l_2+1)(2l_1'+1)(2l_2'+1) \times \langle l_1 0 l_2 0 \mid L_x 0\rangle \langle l_1' 0 l_2' 0 \mid L_y 0\rangle \times \int A_{l_1' l_2'}^{*L_y}\left(\frac{A}{B} r_1, -r_2\right) \Psi_{\Lambda_1}^*(\varkappa_{c_1}, r_1)\, V(r_2) \times \Psi_{\Lambda_2}(\varkappa_{c_2}, r_2)\, A_{l_1 l_2}^{L_x}\left(r_1, -\frac{y}{x} r_1\right) r_1 dr_1 r_2 dr_2. \quad (78)$$

To calculate this integral, McMahan and Tobocman[73] used an expansion of the integrals (37b) with respect to the eigenfunctions of a generalized harmonic oscillator[48] and found that the integral (78) can be reduced to a form containing infinite sums of products of four one-dimensional integrals. However, the number of additional sums for this method of separating the variables is 15. But the most important shortcoming of this method is that the number of terms in the sums cannot be bounded above by any physical considerations, and the convergence of the corresponding series can be verified only numerically. To improve the convergence in Ref. 73, the authors used the method of Ref. 74, in which the functions $\Psi_{\Lambda_1}(\varkappa_{c_1}, r_1)$ and $\Psi_{\Lambda_2}(\varkappa_{c_2}, r_2)$ are expanded in harmonic-oscillator eigenfunctions. In this case, one can achieve a certain reduction of the sums through the orthogonality of the coefficients of the different expansions (if this orthogonality holds). But in fact the use of such a method of separating the variables in the distorted waves involves great computational difficulties, despite the decrease in the multiplicity of the integration.

A less laborious method of separating the variables in the distorted waves was proposed by Charlton.[75] In this method, the partial distorted waves are expanded in a series in Bessel functions (expansion of the distorted wave in plane waves), but, in contrast to the approximate methods considered in Sec. 3, an appreciable number of terms is taken into account in the expansion:

$$\chi_L^{(\pm)}(kr) = \frac{kr}{4\pi}\sum_{n=1}^{N(L)} \mathcal{D}_{nL}^{(\pm)} j_L(k_n r). \quad (79)$$

The coefficients $\mathcal{A}_{nL}^{(\pm)}$ are expressed in the usual manner

$$\mathcal{A}_{nL}^{(\pm)} = \sum_{n'=1}^{N(L)} N_{nn'}^L b_{n'L}^{(\pm)} \quad (80)$$

in terms of the overlap integrals $b_{n'L}^{(\pm)}$, which are determined by the expression

$$b_{nL}^{(\pm)} = \frac{4\pi}{k} \int_0^{R_{max}} r\,dr\,\chi_L^{(\pm)}(kr)\,j_L(k_n r), \tag{80a}$$

and the normalizing matrix

$$(N_{nn'}^{L})^{-1} = \int_0^{R_{max}} r^2\,dr\,j_L(k_n r)\,j_L(k_{n'} r). \tag{80b}$$

The quantities k_n must be chosen to make the functions $j_L(k_n r)$ form an orthonormal system. The correctness of this method of expansion is discussed in Ref. 76, where, in particular, the restrictions imposed on the k_n are considered.

When (79) and (38) are used, the integral (31) takes the form

$$\begin{aligned} I_{\Lambda_1\Lambda_2 L_x L_y l} &= \frac{k_x k_y}{(4\pi)^2} \\ &\times \frac{1}{2} \sum_{n_x n_x' n_y n_y'}^{N_x(L_x) N_y(L_y)} N_{n_x n_x'}^{L_x} N_{n_y n_y'}^{L_y} \\ &\times b_{n_y L_y}^{(-)} b_{n_x L_x}^{(+)} (k_{ny})^{\Lambda_1} (k_{nx})^{\lambda_2} (2\Lambda_1+1)(2\Lambda_2+1) \\ &\times \sum_K (-1)^K (2K+1)\, g_{K\Lambda_1\Lambda_2}(k_{nx}, k_{ny}) \\ &\times \sum_{\lambda_1,\lambda_2} \binom{2\Lambda_1}{2\lambda_1}^{1/2} \binom{2\Lambda_2}{2\lambda_2}^{1/2} \left(-\frac{A}{B}\frac{k_{ny}}{k_{nx}}\right)^{\lambda_1} \left(-\frac{y}{x}\frac{k_{nx}}{k_{ny}}\right)^{\lambda_2} \\ &\times \sum_{l_1, l_2} \sqrt{(2l_1+1)(2l+1)}\, W(L_x l_1 L_y l_2 : Kl) \\ &\times \langle \Lambda_1 - \lambda_1 0 \lambda_2 0 \mid l_1 0\rangle \langle l_1 0 K 0 \mid L_x 0\rangle \\ &\times \langle \lambda_1 0 \Lambda_2 - \lambda_2 0 \mid l_2 0\rangle \langle l_2 0 K 0 \mid L_y 0\rangle \begin{Bmatrix} \Lambda_1-\lambda_1 & \lambda_1 & \Lambda_1 \\ \lambda_2 & \Lambda_2-\lambda_2 & \Lambda_2 \\ l_1 & l_2 & l \end{Bmatrix}, \end{aligned} \tag{81}$$

where

$$\left.\begin{aligned} &g_{K\Lambda_1\Lambda_2}(k_{nx}, k_{ny}) \\ &= \int_{-1}^{+1} F_{\Lambda_1}^*(\varkappa_{c_1}, q_1)\, F_{\Lambda_2}(\varkappa_{c_2}, q_2)\, P_K(\mu)\, d\mu; \\ &\mathbf{q}_1 = -\frac{A}{B}\mathbf{k}_{ny} + \mathbf{k}_{nx};\ \mathbf{q}_2 = \mathbf{k}_{ny} - \frac{y}{x}\mathbf{k}_{nx}; \end{aligned}\right\} \tag{82}$$

$$F_{\Lambda_1}^*(\varkappa_{c_1}, q_1) = q_1^{-\Lambda_1} \int \Psi_{\Lambda_1}^*(\varkappa_{c_1}, r_1)\, j_{\Lambda_1}(q_1 r_1)\, r_1^2\, dr_1; \tag{83}$$

$$F_{\Lambda_2}(\varkappa_{c_2}, q_2) = q_2^{-\Lambda_2} \int \Psi_{\Lambda_2}(\varkappa_{c_2}, r_2)\, j_{\Lambda_2}(q_2 r_2)\, r_2^2\, dr_2. \tag{84}$$

Thus, if the distorted waves are expanded in a series in plane waves, the integration over the angular variables can be performed fairly readily—the integral (31) contains the product of three one-dimensional integrals, and the number of additional sums is equal to nine, i.e., four more than in the methods associated with separation of the variables in the wave functions and the interaction potentials. This method is more economic than other methods of calculating the integrals in the FRDWM if the series (79) includes a comparatively small number of terms ($N(L) \leq 15$–20). The expansion (79) is best justified in reactions with light particles at not very high energies. In reactions with strongly absorbing particles, it is necessary to take into account a larger number of terms in the expansion (79), and the advantage of this approach is lost.

Finally, we note that in Ref. 49 a method was proposed for calculating the kinematic factor (30) by separating the variables in the distorted wave of the final particle and in the wave function $\Psi_{\Lambda 1}^*(\varkappa_{c_1}, r_1)$ when the coordinates $\mathbf{r}_2$ and $\mathbf{r}_x$ are used as basis coordinates. In this case, the integral (31) has the form

$$\begin{aligned} &I_{\Lambda_1\Lambda_2 L_x L_y l} \\ &= \frac{4}{\pi^3} \sum_{l_1 l_2 l_1' l_2' \mathscr{L}} i^{l_1+l_2+l_1'+l_2'-\Lambda_1-L_y} \begin{Bmatrix} l_1 & l_2 & \Lambda_1 \\ l_1' & l_2' & L_y \\ L_x & \Lambda_2 & \mathscr{L} \end{Bmatrix} \\ &\times u(\Lambda_1 \Lambda_2 L_y L_x : l\mathscr{L}) \frac{1}{\sqrt{2L_y+1}} \\ &\times \int r_2^2\, dr_2\, r_x\, dr_x\, A_{l_1 l_2}^{*L_y}\left(\frac{1}{\alpha} r_2, \frac{A}{B} r_x\right) \\ &\times A_{l_1 l_2}^{*\Lambda_1}\left(\frac{y}{x} r_2, r_x\right) V(r_2)\, \Psi_{\Lambda_2}(\varkappa_{c_2}, r_2)\, \chi_{L_x}(k_x r_x). \end{aligned}$$

However, it is not clear whether this method of separating the variables gives a gain over the previous method, since the matrices A are here obtained numerically, the integrals (37b) being determined solely in terms of Γ functions.

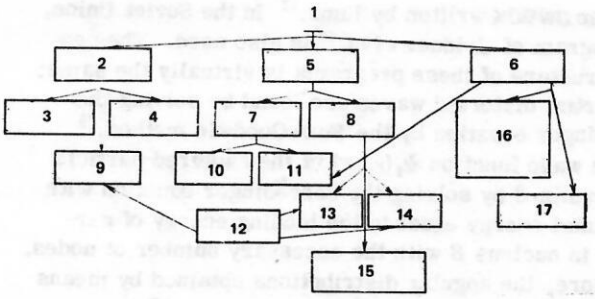

FIG. 8. Scheme of the basic variants (approximate and exact) for taking into account recoil effects in the FRDWM. 1—FRDWM; 2—Allowance for recoil effects in zeroth order; 3—Zero-range approximation; 4—Distributed-density approximation; 5—Allowance for recoil effects in first order; 6—Exact allowance for recoil effects; 7—Amplitude modulation of plane wave; 8—Modulation of the distorted wave; 9—No-recoil approximation; 10—Diffraction approximation; 11—Eikonal approximation; 12—Buttle–Goldfarb approximation; 13—Expansion in a Taylor series; 14—Local momentum approximation; 15—Approximation of large momentum transfers; 16—Separation of variables in wave functions of the relative motion and interaction potentials; 17—Separation of variables in distorted waves.

However, it is not clear whether this method of separating the variables gives a gain over the previous method, since the matrices A are here obtained numerically, the integrals (37b) being determined solely in terms of Γ functions.

To conclude this section, Fig. 8 shows the scheme of the various FRDWM variants—from the most approximate to the most systematic as considered in Secs. 3 and 4.

5. BRIEF DESCRIPTION OF ACTUAL FRDWM PROGRAMS AND ANALYSIS OF THE POSSIBILITIES OF USING THEM FOR THE SOLUTION OF PHYSICAL PROBLEMS

We consider now the best known FRDWM programs, beginning with the simplest used in the approximation of a zero-range interaction. We shall not go into a detailed discussion of the construction of each program and its mathematical algorithms, and we refer the interested reader to the appropriate descriptions. We shall mention only those features of a program that are related to the physical aspects of the corresponding problem. All the programs considered below are written in FORTRAN and in this sense are universal.

Programs that Take into Account Approximately Recoil Effects for the Stripping Mechanism. Programs for the zero-range approximation. Best known here is the

program DWUCK written by Kunz.[77] In the Soviet Union, the program of Gridnev *et al.*[78] is also used. The general structure of these programs is virtually the same: The partial distorted waves are found by solving the Schrödinger equation by the Fox–Goodwin method,[79] and the wave function $\Psi_l(\varkappa_c, r)$ of the captured particle is determined by solving the Schrödinger equation with given total energy equal to the binding energy of particle c in nucleus B with the necessary number of nodes. Therefore, the angular distributions obtained by means of these programs are virtually the same. The computing time for one angular distribution using these programs on a BÉSM-6 computer is 20–30 sec.

Programs for the no-recoil approximation. There are two such programs in which the Buttle–Goldfarb approximation is not used.[57] In one of them, RDRC (Ref. 54), the kinematic integral is calculated by expanding the integrals (37b) in eigenfunctions of a generalized harmonic oscillator. The other program, SETILL (Ref. 80), uses a representation of the distorted waves by means of the generator-coordinate method in the form of modified wave packets of Gaussian type.[55] Because of the use of the oscillator basis, the computing time for an angular distribution in both programs is fairly short (1–2 min). Figure 9 shows the angular distributions for three reactions calculated with the RDRC program together with the experimental data (Refs. 81–83). It can be seen that the theoretical angular distribution of the deuterons from the reaction $^{12}C(^6Li, d)^{16}O^*$ (see Fig. 9a) describes only the general nature of the experimental curve. This can be attributed to the fact that the condition for the use of the no-recoil approximation ($c/x, c/B \ll 1$) is not satisfied in this reaction. The angular distribution of the ^{13}C ions from the reaction $^{13}C(^{12}C, ^{13}C)^{12}C$ (see Fig. 9b) also disagrees with the experiment, though in this case the mass ratios c/x and c/B are fairly small. But for this reaction one can have two possible values of the angular momentum transfers l (0 and 1), and in the no-recoil approximation the selection rules (see Sec. 3) limit the possible l by the condition of normal parity, i.e., $l = 0$. As a result, in this approximation the theoretical angular distribution oscillates strongly, which does not correspond to the experimental results.[82] Finally, the theoretical angular distribution of the ^{15}N ions from the reaction $^{11}B(^{16}O, ^{15}N)^{12}C$ agrees qualitatively with the experiment,

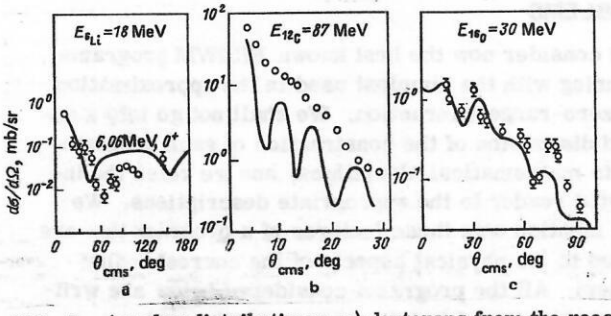

FIG. 9. Angular distributions: a) deuterons from the reaction $^{12}C(^6Li, d)^{16}O^*$ (6.06 MeV) (Ref. 81); b) ^{13}C ions from the reaction $^{13}C(^{12}C, ^{13}C)^{12}C$ (Ref. 82); c) ^{15}N ions from the reaction $^{11}B(^{16}O, ^{15}N)^{12}C$ (Ref. 83). The curves were calculated with the program RDRC.[54]

since in this reaction the mass ratios c/x and c/B are small and the selection rules permit only one l value. Note that it is impossible to reconcile any of these curves with the absolute values given by the experiment (the spectroscopic factors, which are determined from the normalization of the theoretical curves at the first maximum, are, as a rule, several times larger than the values given by shell calculations[44]).

Thus, the use of the no-recoil approximation does not permit one to obtain the absolute value of the cross section even in reactions with heavy ions. The theoretical form of the angular distributions agrees with the experiments only when the selection rules permit one value of the transferred angular momentum l and the conditions c/x, $c/B \ll 1$ are satisfied.

The program BRUNHILD (Ref. 84) for the local momentum approximation.[58] In this program, the kinematic factor is calculated in accordance with (56), (58), (59), and (67), i.e., using the Buttle–Goldfarb approximation, while the angular distribution is calculated in accordance with (33). The partial distorted waves are calculated in the same way as in the program explained in Ref. 77.

For numerical verification of the validity of the local momentum approximation, it is necessary to establish, first, the absence of a dependence of the angular distribution on the direction of the recoil momentum (68) and, second, the convergence of the series in L in (56) through rapid decrease of the integral (59) for $L > QR$. The angular distribution of ^{13}N ions from the reaction $^{12}C(^{14}N, ^{13}N)^{13}C$ ($E^{14}N = 79$ MeV) (Ref. 85) calculated in accordance with the BRUNHILD program[86] for different values of the angle θ_Q is shown in Fig. 10a. It can be seen that for $\theta \leq 40°$ the angular distribution really is insensitive to the direction of the recoil momentum. Figure 10b shows the angular distribution of ions from the reaction $^{144}Sm(^{16}O, ^{14}C)^{146}Gd$ ($E^{16}O = 104$ MeV) calculated[86] for different numbers of terms in the sum over L in the expression (56). The results of the calculations show that whereas the zeroth order (no-recoil approxi-

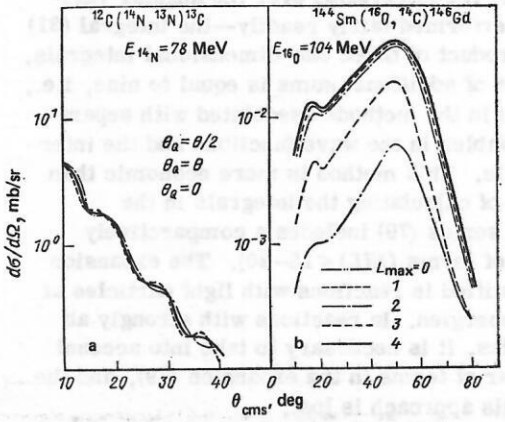

FIG. 10. Verification of the local momentum approximation in reactions with heavy ions. The calculated curves, which were calculated[86] by the program BRUNHILD[84] show (a) the weak sensitivity of $d\sigma/d\Omega$ to the direction of the recoil momentum (68); b) the dependence of $d\sigma/d\Omega$ on the number of terms L_{max} in the sum over L in the kinematic factor (56).

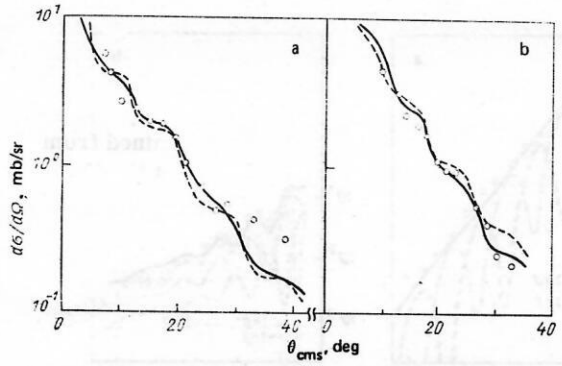

FIG. 11. Angular distribution of ^{13}N ions from the reaction $^{12}C(^{14}N, ^{13}N)^{13}C$ (Refs. 85 and 87): a) $E_{14N} = 78$ MeV; b) $E_{14N} = 100$ MeV. The continuous curve was calculated by the program LOLA[88] and the dashed curve by the program BRUNHILD[84] (the parameters of the optical potentials are the same).

mation) disagrees strongly with the first, and the first also differs strongly from the second, the subsequent terms of the series in L make an almost insignificant contribution to the kinematic factor (56). Finally, for additional verification of the validity of the local momentum approximation, the angular distribution for the reaction $^{12}C(^{14}N, ^{13}N)^{13}C$ calculated by BRUNHILD was compared with the experimental distribution[85, 87] and with the angular distribution calculated by the program LOLA[88] (see below), in which the recoil is taken into account exactly. The corresponding angular distributions are shown in Fig. 11. It can be seen that the calculated angular distributions agree with each other (and with the experiment) in the region of angles $\theta \leq 30°$. So do the absolute values of the cross sections–the structure factor (34) in this reaction is found to be 0.53 by LOLA and 0.47 by BRUNHILD.

The volume of the program BRUNHILD is about 30 0008 words. A typical angular distribution with one value of the angular momentum transfer and with the maximal number of partial waves ($L_{x,y}^{max} \approx 100$) is calculated on the CDC-3300 in about 6 min if the terms of zeroth, first, and second order are taken into account in the sum over L in (56).

We note one further circumstance. Since the program of Ref. 84 uses not only the local momentum approximation but also the Buttle–Goldfarb approximation,[57] it is necessary to test the sensitivity of the angular distributions to the choice of the wave functions $\Psi_{\Lambda_1}(\varkappa_{c_1}, r_1)$ of the captured particle. As is shown in Ref. 89, the Buttle–Goldfarb approximation is apparently not satisfied in reactions with light particles. It is therefore more expedient to use BRUNHILD to calculate the angular distributions and determine the structure factors in single-nucleon transfer reactions involving heavy ions in a narrow interval of forward angles (up to 40°).

The program WOMBLE for the approximation of large momentum transfers.[111] In this program, the kinematic factor is calculated in accordance with (71) using the quasiclassical estimate (72a) for the factor $S(\tilde{q} - q)$.

The validity of the approximation of large momentum

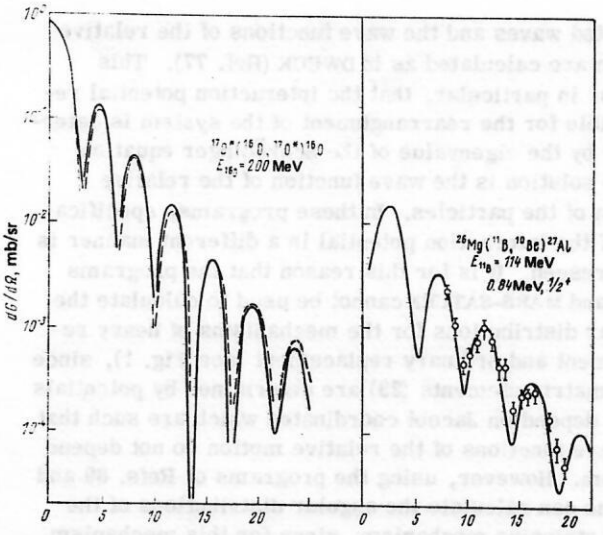

FIG. 12. Angular distributions of $^{17}O^*$ and ^{10}Be ions from the reactions $^{17}O^*(^{16}O, ^{17}O^*)^{16}O$ ($E_{16O} = 200$ MeV) (a) and $^{26}Mg(^{11}B, ^{10}Be)^{27}Al^*$ ($E_{11B} = 114$ MeV) (b) calculated by the program WOMBLE[109] (continuous curves). The experimental results are taken from Ref. 112. The broken curve shows the angular distribution calculated by LOLA.[112]

transfers was verified numerically by comparing the results of calculation of $d\sigma/d\Omega$ by WOMBLE with calculations.[88] Figure 12a shows the angular distribution of the hypothetical reaction $^{17}O^*(^{16}O, ^{17}O^*)^{16}O$ calculated in Refs. 109 and 112 by these programs. In the region of small angles [where the estimate (72a) is valid], the two calculated curves virtually coincide. Figure 12b shows the angular distribution of the ^{10}Be ions from the reaction $^{26}Mg(^{11}B, ^{10}Be)^{27}Al$ calculated by WOMBLE together with the experimental data[112] at angles less than 25°. It can be seen that the theoretical curve describes the experiment quite well. It is important to emphasize that the WOMBLE calculation makes it possible to reconcile the experiment with the absolute value of the calculated cross section (the spectroscopic factor of the neutron in ^{27}Al extracted from the normalization of the theoretical curve is fairly close to the results of other authors).

We point out that the computational advantages of WOMBLE are evidently most fully realized in the calculation of the angular distributions of the products of single-nucleon transfer reactions with relatively light ions of high energies in the case of scattering by comparatively light target nuclei (up to ^{40}Ca).

Programs that Take into Account Exactly Recoil Effects for the Stripping Mechanism in Reactions with Heavy Ions. We consider here the two most widely used FRDWM programs: LOLA (Ref. 88) and MARS-SATURN(Ref. 90), in which the differential cross section is calculated in accordance with (33) with the structure factor (34). These programs can be used to calculate the angular distributions of reactions with complex particles if each of the orbital angular momenta Λ_1 and Λ_2 has one possible value. The invariant form factor is calculated in accordance with (75), i.e., the recoil is taken into account exactly through the quantities $g_{K\Lambda_1\Lambda_2}(r_x, r_y)$ determined by the expressions (73). The

distorted waves and the wave functions of the relative motion are calculated as in DWUCK (Ref. 77). This means, in particular, that the interaction potential responsible for the rearrangement of the system is determined by the eigenvalue of the Schrödinger equation whose solution is the wave function of the relative motion of the particles. In these programs, specification of the interaction potential in a different manner is not foreseen. It is for this reason that the programs LOLA and MARS-SATURN cannot be used to calculate the angular distributions for the mechanisms of heavy replacement and ordinary replacement (see Fig. 1), since their matrix elements (23) are determined by potentials which depend on Jacobi coordinates which are such that the wave functions of the relative motion do not depend on them. However, using the programs of Refs. 89 and 90, one can calculate the angular distributions of the heavy stripping mechanism, since for this mechanism the interaction potential and the wave function of the relative motion of particles y and C in nucleus A depend on the same coordinate.

We now consider some computational features of the programs of Refs. 88 and 90. In both programs, an input parameter is the width Δ of the strip of integration, i.e., the maximally possible deviation of the coordinate r_y from $(A/B)r_x$. Specification of Δ significantly reduces the size of the integration mesh in the double integral (31). In LOLA, the value of Δ can vary in a very wide range. In MARS-SATURN, Δ is bounded by the value 2 F, and, therefore, this program can be used if $[cM/(xB)]r_2 < 2$ F. The program of Ref. 90 uses an integration mesh with step h_0 within the strip with respect to the variable $\delta_{xy} = r_y - (A/B)r_x$ and with larger step $h = nh_0$ $(n > 1)$ with respect to the variable r_y (at the $n-1$ intermediate points, the values of $F_{\Lambda_1\Lambda_2 L_x L_y l}(r_x, r_y)$ are obtained by interpolation). Because of the weak dependence of the form factor on r_y, this does not result in any effective loss of accuracy, and the computing rate is increased by n times. Therefore, the MARS-SATURN program is currently to be regarded as the fastest FRDWM program, although all its computational advantages appear only in calculations of the angular distributions of single-nucleon transfer reactions involving heavy ions on sufficiently heavy nuclei, when Δ is very small and n may reach 20.

These programs are about equally long and contain $300\,000_8$ words. The maximal L_x^{max} and L_y^{max} of a partial wave in these programs is 100, and the actual number of partial waves taken into account, i.e., $L_{x,y}^{max} - L_{x,y}^{min}$, is about 50. If $L_{x,y}^{min} > 0$, a cutoff of the integrals (31) at the lower limit is automatically introduced, the value of the cutoff radius being determined by $kR_{min} = L_{x,y}^{min}$. The program MARS-SATURN foresees the possibility of introducing three values of the angular momentum transfer; LOLA calculates the angular distribution for each angular momentum transfer l independently. During the calculations, these programs use magnetic tape to store intermediate results. The computing time for one angular distribution is about 15 min.

We now analyze the physical results obtained by means of these programs.

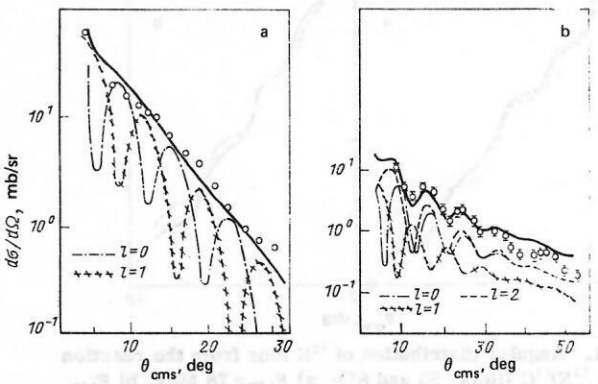

FIG. 13. Angular distributions of ^{13}C and ^{11}B ions from the reactions $^{13}C(^{12}C, ^{13}C)^{12}C$ (a) and $^{11}B(^{12}C, ^{11}B)^{12}C$ (b) (Ref. 82); the curves were calculated[69] by the program LOLA; the thin curves show the contribution of the individual angular momentum transfers and the thick curves the total cross section.

We begin with LOLA. The angular distributions for the reactions $^{13}C(^{12}C, ^{13}C)^{12}C$ (a) and $^{11}B(^{12}C, ^{11}B)^{12}C$ (b) for 87-MeV incident ^{12}C ions[82] are shown in Fig. 13. The first of these angular distributions does not have a clearly expressed diffraction structure, whereas the second exhibits perceptible maxima and minima. The calculation of the angular distributions of these reactions for the stripping mechanism by LOLA (continuous curves) shows that the qualitative difference between the behaviors of the angular distributions of these reactions is due not to the difference between their mechanisms, but to the difference between the selection rules for the angular momentum transfer l. Two angular momenta, $l = 0$ and 1, contribute to the cross section of the reaction in Fig. 13a, which leads, in contrast to the no-recoil approximation (see Fig. 9b), to a pronounced smoothing of the angular distribution. Three angular momenta, $l = 0$, 1, and 2, contribute to the cross section of the reaction in Fig. 13b, and the angular distributions of the components with $l = 0$ and 2 are similar, so that the total angular distribution has a pronounced diffraction structure. Note that the calculation gives the correct absolute value of the cross section: The structure factor (34) extracted from the normalization of the theoretical cross section is 0.43 for the first reaction (see Fig. 13a), whereas a shell-model calculation[44] gives 0.61. For the second reaction (see Fig. 13b), the corresponding values of the structure factors are found to be 3.1 and 2.85 (Ref. 44).

We now discuss the numerical results obtained by means of MARS-SATURN.[90] In Ref. 70, the sensitivity of the angular distributions to the choice of the parameters n and Δ was investigated for the example of the reaction $^{88}Sr(^{16}O, ^{15}N)^{89}Y$ at $E^{16}O = 59$ MeV, for which the angular distribution was also calculated in Ref. 91 by LOLA. It was found[70] that up to 120° the angular distributions are virtually insensitive to n if it varies from 3 to 15 and the strip width Δ is 0.05 and 0.1 F. The absolute value of the theoretical angular distribution at the maximum ($\theta \approx 70°$) ($d\sigma/d\Omega = 0.353$ mb/sr for $\Delta = 0.05$ F, $n \approx 15$) agrees very well with the same value calculated by LOLA ($d\sigma/d\Omega = 0.358$ mb/sr).

In Fig. 14a, we show the experimental[92] angular dis-

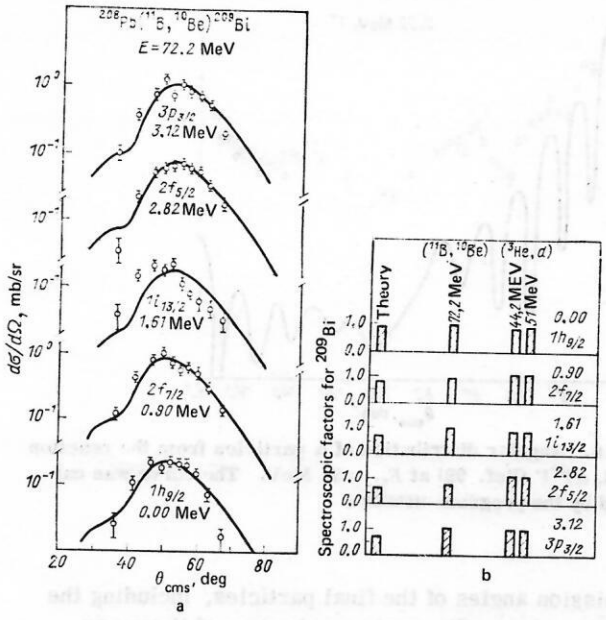

FIG. 14. Angular distributions of ^{10}Be ions from the reaction ^{208}Pb(^{11}B, ^{10}Be)^{209}Bi (Ref. 92) (a) and spectroscopic factors of the proton in the nucleus ^{209}Bi for different single-particle configurations corresponding to the lowest states of this nucleus (b). The curves were calculated[70] by the program MARS–SATURN. The spectroscopic factors for the (^{3}He, d) reaction are taken from Ref. 93 and the theoretical spectroscopic factors from Ref. 94.

tributions of ^{10}Be ions from the reaction ^{208}Pb(^{11}B, ^{10}Be)-^{209}Bi with excitation of several low-lying states of ^{209}Bi and compare them with the distributions (continuous curves) calculated in Ref. 70 by MARS-SATURN. In Fig. 14b, we show the single-particle spectroscopic factors for these states obtained in this calculation, the ones found earlier from analysis of the (^{3}He, d) reaction,[93] and also the results of the theoretical calculations of Ref. 94. It can be seen that the proton spectroscopic factors in the nucleus ^{209}Bi determined by all methods agree well.

Thus, the programs of Refs. 88 and 90 make it possible to calculate the angular distributions for single-nucleon transfer reactions with heavy ions in quantitative agreement with the experiments. On the one hand, this confirms that the approximations on which these programs are based are justified for these reactions and, on the other, it shows that the determining mechanism of these reactions is stripping of a nucleon, at least in the forward hemisphere. Note also that LOLA and MARS-SATURN can be used to calculate the angular distributions of reactions with transfer of not only a nucleon but also more complex particles if the angular momenta Λ_1 and Λ_2 have unique values. These programs can also be used to calculate heavy stripping if this condition is satisfied. We point out only that to calculate the angular distributions of reactions with light ions (^{6}Li, ^{7}Li) on light nuclei it is better to use LOLA, since the computational advantages of MARS-SATURN are hardly realized in the calculation of such reactions.

Progams that Take into Account Exactly Recoil Effects for the Stripping Mechanism in Reactions with Light Ions. The program HI-DRC (Ref. 73). This program is based on exact allowance for recoil effects in the separation of the variables in the distorted waves by an expansion of them in series in eigenfunctions of a generalized harmonic oscillator. As we have already said, such a method of separating the variables in the three-body problem leads to a large number (15) of additional sums, and it is therefore actually implementable only for calculating the angular distributions of the stripping mechanism in reactions with light ions at not high energies on relatively light nuclei. This is connected with the circumstance that the wave functions of the lowest states of nuclei of the 1p shell overlap strongly with the harmonic-oscillator wave functions, and the number of distorted waves that must be taken into account is not too large. As a result, the series in the eigenfunctions of the generalized harmonic oscillator converge fairly rapidly, and the additional sums contain a small number of terms.

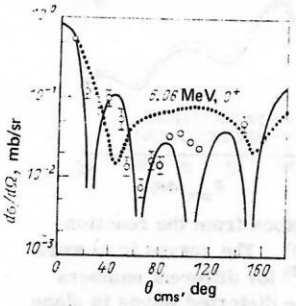

FIG. 15. Angular distribution of deuterons from the reaction ^{12}C(^{6}Li, d)^{16}O* (6.06 MeV) (Ref. 81). The continuous curve was calculated[73] by the program HI-DRC; the dotted curve, by the program RDRC (Ref. 54).

Figure 15 shows the experimental angular distribution of deuterons from the reaction ^{12}C(^{6}Li, d)^{16}O* ($E^*=6.06$ MeV) at $E_{^6\mathrm{Li}}=18$ MeV (Ref. 81), and also the distributions calculated[73] by the program HI-DRC (continuous curve) and in the no-recoil approximation by means of the program RDRC (Ref. 54) (dotted curve) (see also Fig. 9a). The necessary number of partial waves for calculating this reaction is $L_{\max}=9$, the number of terms of the series in the expansions of each distorted wave in the eigenfunctions of the generalized harmonic oscillator is about 20, and the number of intermediate sums is 12. The time required to calculate an angular distribution in the program HI-DRC is about 10 min. As the obtained results show, allowance for the recoil effects makes it possible to obtain the angular distribution of the deuterons in qualitative agreement with the experiment. Neglect of recoil effects does not permit one to obtain agreement with the experiment even at forward angles ($\theta \leq 60°$).

The program MERCURY.[95] In this program, the radial integral $I_{\Lambda_1\Lambda_2 L_x L_y l}$ is calculated by (81), i.e., it has a relatively simple form. Since the number of terms in the expansion (79) for partial waves in reactions with light ions is comparatively small, the additional sums do not contain a too large number of terms. The program MERCURY foresees $L_{x,y}^{\max}=50$ and $N_{\max}=20$. The time required to calculate one angular distribution on a CDC-6500 is about 5 min.

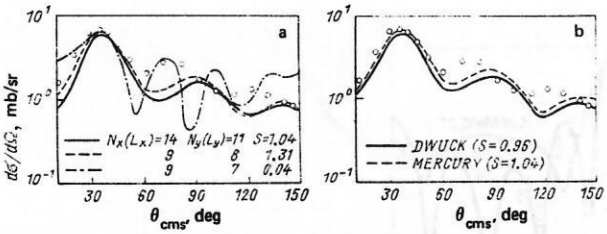

FIG. 16. Angular distribution of protons from the reaction $^{40}Ca(d, p)^{41}Ca$ at $E_d = 11$ MeV (Ref. 96). The curves in a) were calculated[75] by the program MERCURY[95] for different numbers of terms $N(L)$ in the expansions of the distorted waves in plane waves; b) is a comparison of the results of the calculation by the program MERCURY (continuous curve) for $N_x(L_x) = 14$, $N_y(L_y) = 11$ and by the program DWUCK[77] (broken curve).

The program was tested by calculating the angular distribution of protons from the reaction $^{40}Ca(d,p)^{41}Ca$. The corresponding angular distributions (experimental data taken from Ref. 96) calculated by MERCURY for different numbers of terms $N(L)$ in the expansion (79) in the entrance and exit channels of the reaction are shown in Fig. 16a. The results of the MERCURY calculations were compared with the DWUCK results[77] (see Fig. 16b), since for this reaction one can expect the zero-range approximation to be well satisfied. It can be seen that the dependence of the angular distribution on $N(L)$ is fairly strong (although it affects basically the degree of the oscillations rather than the general nature of the decrease of the angular distribution at large angles), and it is only for $N(L) > 10$ that the angular distribution agrees reasonably with the DWUCK distribution both in profile and absolute magnitude.

The angular distributions of α particles from the reaction $^{12}C(^6Li, \alpha)^{14}N$ at $E_{^6Li} = 33$ MeV (Ref. 97), deuterons from the reaction $^{12}C)^6Li, d)^{16}O^*$ at $E_{^6Li} = 18$ MeV (Ref. 81), and ^8Be ions from the reaction $^{40}Ca(^{12}C, ^8Be)$-^{44}Ti at $E_{^{12}C} = 45$ MeV (Ref. 98) are shown in Fig. 17. These curves were obtained by the MERCURY program in the quoted papers for the cluster stripping reaction. The broken curve in Fig. 17b corresponds to the HI-DRC calculation (see Fig. 15). The differential cross sections calculated by these programs give a good description of the experimental data in the complete range

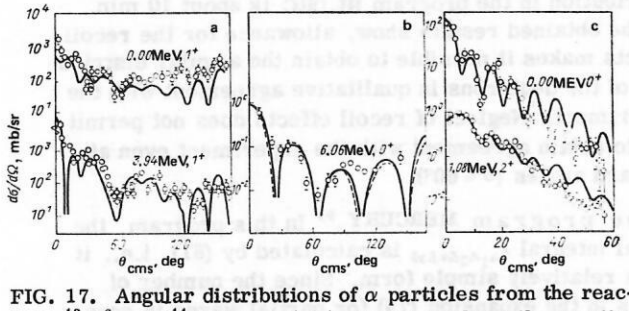

FIG. 17. Angular distributions of α particles from the reactions $^{12}C(^6Li, \alpha)^{14}N$ (Ref. 97) (a), deuterons from the reaction $^{12}C(^6Li, d)^{16}O$ (Ref. 81) (b), and ^8Be ions from the reaction $^{40}Ca(^{12}C, ^8Be)^{44}Ti$ (Ref. 98) (c). The calculations were made for the cluster stripping mechanism by the programs MERCURY (continuous curves) and HI-DRC (broken curve).

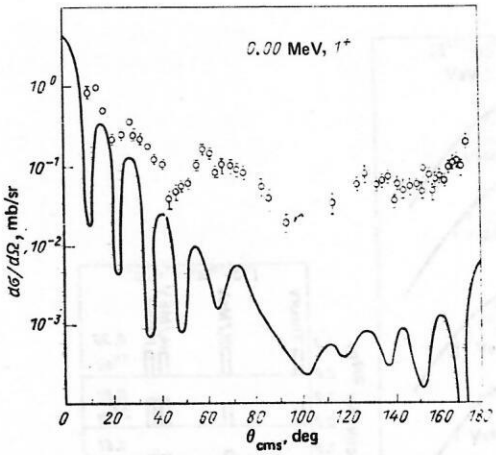

FIG. 18. Angular distribution of α particles from the reaction $^{16}O(^6Li, \alpha)^{18}F$ (Ref. 99) at $E_{^6Li} = 34$ MeV. The curve was calculated by the program MERCURY.

of emission angles of the final particles, including the absolute values. The main mechanism of these particular reactions is evidently the stripping mechanism. However, this is not always the case. In some reactions for which calculations have been made for the stripping mechanism with exact allowance for recoil effects the calculated angular distributions cannot be reconciled with the experimental distributions in a wide range of angles. For example, Fig. 18 shows the angular distribution of α particles from the reaction $^{16}O(^6Li, \alpha)^{18}F$ (Ref. 99) and the theoretical curve calculated by MERCURY. It can be seen that the curve does not even qualitatively describe the experiment. The authors of Ref. 99 attribute this to the circumstance that the mechanism of this reaction does not reduce to the stripping mechanism, and for quantitative description of the experiment it is necessary to take into account exchange processes associated with the break-up of the target nucleus.

Programs that Take into Account Exactly Recoil Effects for Direct and Exchange Processes in Reactions with Light Particles. As follows from the previous sections, the mechanism of single-nucleon transfer reactions with heavy ions is determined by the stripping mechanism. But if a complex cluster is transferred to the nucleus, it is necessary to take into account exchange mechanisms associated with the break-up of the target nucleus. The calculation of the angular distributions of exchange processes for an arbitrary reaction $A(x,y)B$ is very complicated (see Sec. 2). However, for reactions with α particles (and lighter particles), when all the nucleons in particle x are in the s state of relative motion, the calculation of the cross sections (32) simplifies appreciably, since in this case one can introduce products of the amplitudes of the reduced widths $\Theta^{B\to C+x}_{J_1\Lambda_1, J_c}$ and $\Theta^{A\to C+y}_{J_2\Lambda_2, J_c}$ summed over J_1, J_2, and J_c in accordance with

$$\Theta_{\Lambda_1\Lambda_2 lsj} = \sum_{J_1 J_2 J_c} \Theta^{B\to C+x}_{J_1\Lambda_1, J_c}\Theta^{A\to C+y}_{J_2\Lambda_2, J_c} \times (-1)^{J_c}\sqrt{\frac{2J_A+1}{2J_c+1}}\, u(J_1 G_x J_2 s_y : J_c S) \times u(\Lambda_2 J_A J_1 s : J_2 j)\, u(j\Lambda_2 J_B \Lambda_1 : J_1 l). \quad (85)$$

As a result, if the dependence of the wave functions on E_c is ignored, there remain in the cross section (32) coherent sums over only Λ_1 and Λ_2. At the same time, the variables are separated in the wave functions of the relative motion and in the interaction potentials, so that the invariant form factor must be calculated in accordance with (75). For direct processes, the structure factor in (32) for the reaction with α particles (and lighter particles) is determined solely by the width $\Theta^{B\to A+C}_{lsj}$ (where j is the spin of the channel),[42] so that the cross section (32) always factorizes into a spectroscopic factor and a kinematic part. In this case, the invariant form factor can be calculated by (76).

The need to introduce a finite range of the interaction to calculate exchange processes is more or less obvious, but to calculate direct processes in reactions with light particles it would seem that a zero-range interaction is sufficient. However, this is not so, since even in single-nucleon transfer reactions with light particles on light nuclei the recoil of the complete system is not small. But if one considers transfer reactions involving several nucleons, the effects associated with recoil become even more important. From this point of view, the use of a zero-range interaction is justified best in (d,p) and (d,n) reactions, since here the recoil is least significant.

After these preliminary comments, we now analyze the actual programs.

The program AICINT (Ref. 100). This is designed to calculate the cross sections of exchange processes if the distorted waves of the initial and final particle are treated in the eikonal approximation,[61,62] which requires for its use only the fulfillment of the condition (53). The angular distributions are calculated in accordance with (32) and the structure factors in accordance with (85), while the amplitudes of the reduced widths are calculated in the shell model with intermediate coupling. The dimensions of the structure matrices correspond to three possible values for both Λ_1 and Λ_2. The length of AICINT is 20 6008 words, and about 1 min is required to calculate one angular distribution on a BESM-6.

The experimental angular distributions of tritons from (α,t) reactions on ^{7}Li and ^{11}B nuclei,[101] and of deuterons from (α,d) reactions on ^{13}C nuclei[102] at $E_\alpha = 25$ MeV, and the AICINT distributions calculated in Ref. 62 for the mechanism of stripping of a heavy particle in the region of large angles are shown in Fig. 19. It can be seen that the theoretical curves agree quite well with the experimental curves in both their profiles and their absolute magnitudes (the normalization constant N for all reactions is near unity). It is important to note that the parameters β and γ of the eikonal approximation depend weakly on the nucleus and on the type of reaction. This indicates that both the reaction mechanism and the method used to take into account the distortions have been chosen correctly.

The programs FUNLY-2 (Ref. 103) and OLYMP (Ref. 104). These are designed to calculate the angular distributions of direct exchange

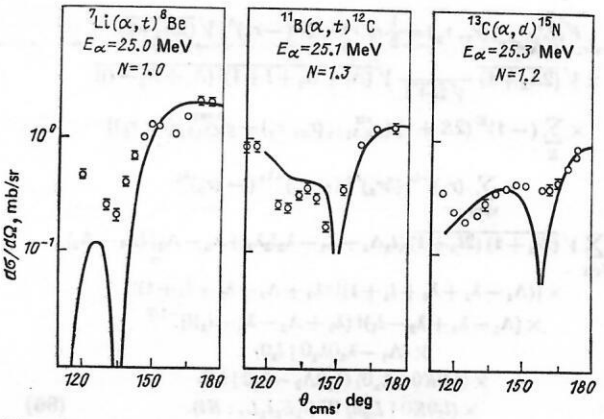

FIG. 19. Angular distributions of tritons from (α, t) reactions on ^{7}Li and ^{11}B nuclei[101] and deuterons from (α, d) reactions on ^{13}C nuclei[102] at $E_\alpha = 25$ MeV in the region of backward angles. The curves were calculated by the program AICINT.[62]

processes in reactions with light particles (not heavier than α particles), above all for nuclei of the $1p$ shell. The two programs have much in common in their structure. The wave functions of the relative motion of the particles in these programs are either found by solving the Schrödinger equation or are specified analytically (or numerically); the interaction potentials of the particles can be introduced into the program independently of the method of determining the wave functions. These programs can be used to calculate the angular distributions of all four mechanisms. The distorted waves are calculated in FUNLY-2 in the same way as in DWUCK,[77] and in OLYMP in the same way as in the program described in Ref. 78. The maximal number of partial waves $L^{max}_{x,y}$ for FUNLY-2 is 50, and for OLYMP it is 30. The maximal number of points of the integration mesh for the calculation of the integral (31) in FUNLY-2 is 4×10^4 and in OLYMP it is 10^4. However, in FUNLY-2 the integral (31) is calculated on the complete coordinate plane, whereas in OLYMP the strip width Δ is introduced, this being specified as an input parameter, which makes it cimpossible to increase the value of R_{max} keeping the same number of mesh points.

On the other hand, FUNLY-2 and MARS-SATURN differ both in the manner of calculating the invariant form factor and in the method of organizing the operation of the program itself. In FUNLY-2, the invariant form factor for both direct and exchange processes is calculated in accordance with Eq. (76) (Ref. 68), which is possible only for reactions in which one of the two angular momenta Λ vanishes for both the direct and exchange processes. In fact, this condition (it is satisfied in the reactions in which the intermediate nucleus C for the exchange processes is an α particle) greatly reduces the number of reactions with light particles that can be analyzed by means of FUNLY-2. In OLYMP, only the form factor of direct processes is calculated in accordance with (76); to calculate the form factor for exchange processes, one uses the general formula (75) simplified by contraction of the 9-symbol with two triangles of vector addition that degenerate into a straight line[108]:

$$F^{J_c E_c}_{\Lambda_1\Lambda_2 L_x L_y l}(r_x, r_y) = \frac{1}{2}\alpha'^{\,l+\Lambda_2} r_x^{\Lambda_1} (-r_y)^{\Lambda_2} \sqrt{(2\Lambda_1+1)}$$
$$\times \sqrt{(2\Lambda_2+1)}\,\frac{1}{\sqrt{2l+1}}\sqrt{(\Lambda_1+\Lambda_2+l+1)!\,(\Lambda_1+\Lambda_2-l)!}$$
$$\times \sum_K (-1)^K (2K+1)\{g^{\text{str}}_{K\Lambda_1\Lambda_2}(r_x, r_y) + g^{\text{h.rep.}}_{K\Lambda_1\Lambda_2}(r_x, r_y)\}$$
$$\times \sum_{\lambda_1\lambda_2} (r_x)^{-\lambda_1} (\delta r_x)^{\lambda_2} (-r_y)^{-\lambda_2} (-\gamma r_y)^{\lambda_1}$$
$$\times \sum_{l_1 l_2} \sqrt{(2l_1+1)(2l_2+1)}\,\langle l_1\Lambda_1 - \lambda_1 - \lambda_2 l_2\lambda_1 + \lambda_2 - \Lambda_2 \mid l\Lambda_1 - \Lambda_2\rangle$$
$$\times [(\Lambda_1 - \lambda_1 + \lambda_2 + l_1 + 1)!\,(\lambda_1 + \Lambda_2 - \lambda_2 + l_2 + 1)!$$
$$\times (\Lambda_1 - \lambda_1 + \lambda_2 - l_1)!\,(\lambda_1 + \Lambda_2 - \lambda_2 - l_1)!]^{-1/2}$$
$$\times \langle \Lambda_1 - \lambda_1 0 \lambda_2 0 \mid l_1 0\rangle$$
$$\times \langle l_1 0 K 0 \mid L_x 0\rangle \langle \lambda_1 0 \Lambda_2 - \lambda_2 0 \mid l_2 0\rangle$$
$$\times \langle l_2 0 K 0 \mid L_y 0\rangle W(l_1 L_y l_1 L_x : K l). \qquad (86)$$

The structure factors (85) are introduced into the program as input data. If the shell model with intermediate coupling is used for their calculation, this also restricts the choice of the possible reactions ($A, B \leqslant {}^{16}$O), but not so strongly as in FUNLY-2.

Further, FUNLY-2 (like LOLA and MARS-SATURN) uses three magnetic tapes during calculation to store intermediate results, since the distorted waves, the wave functions, and the quantities $g_{K\Lambda_1\Lambda_2}(r_x, r_y)$ are calculated independently. The program OLYMP foresees a different method of calculation by a displacement of the signs of the sums in such a way that to the right of the sign of the corresponding sum there are no expressions that do not depend on the summation index. When the calculation is organized in this way, it is necessary to introduce into the memory only the distorted waves and algebraic coefficients. This makes it possible to calculate the angular distributions without using magnetic tapes and with fewer repeated calculations, which greatly shortens the computing time and the requirements on the memory.

The length of FUNLY-2 is $300\,000_8$ words, and that of OLYMP is $100\,00_8$ words. The time required to calculate one angular distributions by FUNLY-2 on a CDC-6500 is ~5 min, and OLYMP requires about 10 min to calculate one for direct processes on a BÉSM-6 and ~10–15 min for exchange processes.

The angular distributions of the particles from the (d,p) and (α,t) single-nucleon transfer reactions on ^{12}C and ^{27}Al nuclei[101,105] are given in Fig. 20. The curves shown the results of calculations for the stripping mechanism by OLYMP and LOLA and by the program described in Ref. 78. It can be seen that whereas allowance for the finite range of the interaction in the (d,p) reaction has virtually no influence on either the profile or the absolute value of the cross section (the cross sections coincide in these calculations without an additional normalization), the situation is quite different for the (α,t) reaction. The effects of recoil and the finite range have a strong influence on the profile of the angular distribution, especially in the (α,t) reaction on ^{11}B. However, for (α,t) reactions, even allowance for the finite range does not make it possible to obtain an angular distribution in agreement with the experiment if only the stripping mechanism is considered. This circumstance indicates once more the importance of exchange mechanisms in reactions with α particles.

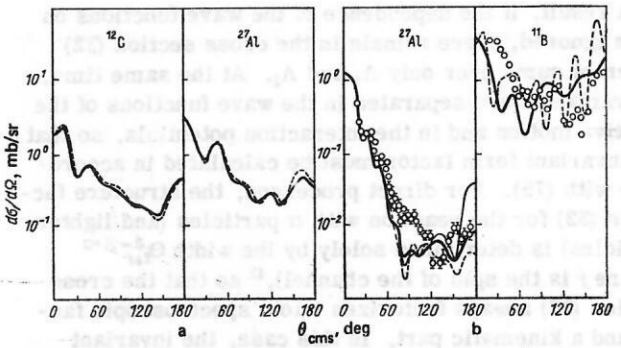

FIG. 20. Angular distributions of particles from (d, p) reactions on ^{12}C and ^{27}Al nuclei (a) and from (α, t) reactions on ^{11}B and ^{27}Al nuclei (b).[101,105] The curves were calculated by the programs OLYMP (heavy curves) and LOLA (thin curve) and by the program described in Ref. 78 (broken curves).

The experimental angular distribution of tritons from the reaction ^{7}Li$(d,t)^6$Li at $E_d = 12$ MeV (Ref. 106) and the distributions calculated[107] by FUNLY-2 for direct and exchange processes, and also the total angular distribution defined as the incoherent sum of the direct and exchange processes, are shown in Fig. 21. It should be emphasized that the angular distributions for both the direct and the exchange processes were obtained by coherent addition of the amplitudes of the individual mechanisms. The results in Fig. 20 clearly show that in the reaction ^{7}Li$(d,t)^6$Li the contribution of each of the four mechanisms to the reaction cross section is not small in the entire range of angles, and only allowance for all four mechanisms makes it possible to obtain a theoretical angular distribution in agreement with the experimental one. (Note that all the curves in Fig. 21 are given without any normalizing factors.)

Finally, we give Table I, which contains the most widely used programs and a description of their computational features.

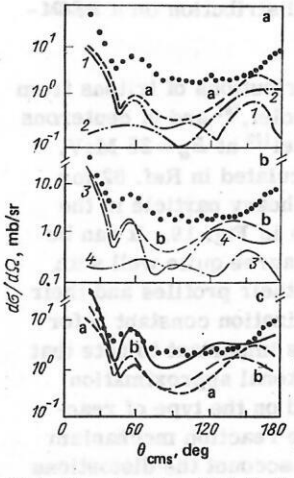

FIG. 21. Angular distribution of tritons from the reaction ^{7}Li$(d,t)^6$Li (Ref. 106). The curves were calculated by the program FUNLY-2 a) direct processes; 1) stripping; 2) heavy replacement; b) exchange processes; 3) ordinary replacement; 4) heavy stripping; c) angular distribution that is the incoherent sum of the direct (a) and exchange (b) processes.

TABLE I.

Program	Type of reaction	Possible reaction mechanisms	Restrictions on the angular momentum transfer	Number of integrals: one-dimensional	Number of integrals: two-dimensional	Number of summations: coherent	Number of summations: incoherent	Approximate computing time on a BÉSM-6 computer, min
DWUCK Gridnev	Reactionswith light particles on relatively heavy nuclei	Stripping	l of only one parity	1	–	2	2	0.3
RCDC	Single-nucleon transfer reactions with heavy ions, $c/x, c/A \gg 1$	»	The same	3	–	6	2	1
BRUNHILD	Single-nucleon transfer reactions with high-energy heavy ions, $c/x, c/A \ll 1$, $\vartheta < 40°$	»	No restrictions	2	–	8	2	6
WOMBLE	Single-nucleon transfer reactions with relatively light high-energy ions on comparatively light targets ($E > 10$ MeV/nucleon, $\theta < 30°$)	Stripping	No restrictions	2	–	1	2	5
LOLA	Transfer reactions under the condition that J_C, S_C, T_C, Λ_1, and Λ_2 have one value	Stripping, heavy stripping	The same	1	1	7	1	15
MARS – SATURN	Single-nucleon transfer reactions with heavy ions on heavy nuclei	Stripping		1	1	7	2	5
FUNLY-2	Reactions on the lightest nuclei when $C = \alpha$	Direct and exchange processes	l of only one parity	1	1	4	1	5
OLYMP-1	$x \leqslant 4$; $A, B \leqslant 16$		The same	1	1	4	2	10
OLYMP-2	$x \leqslant 4$; $A, B \leqslant 16$	Direct processes	No restrictions	1	1	9	2	15
MERCURY	$x \leqslant 6$; $A \leqslant 40$; $E \lesssim 30$–50 MeV	Stripping	The same	3	–	11	1	10
AICINT	$x \leqslant 4$; $A, B \leqslant 16$	Exchange processes	» »	2	–	–	7	1
HI – DRC	$x \leqslant 6$; $A \leqslant 40$; $E \lesssim 30$–50 MeV	Stripping	» »	4	–	17	1	10–30

CONCLUSIONS

From the point of view of its theoretical justification, the most vulnerable point of the distorted-wave method is to be found in the circumstance that by its very definition this method is based on the use of the optical model. Therefore, strictly speaking, it must be classed as a semiempirical method. On the other hand, in the most consistent FRDWM variants the rearrangement of the particles participating in the reaction is described exactly in the framework of the three-body problem (other approximations such as, for example, allowance for only single-step mechanisms are not of a fundamental nature). In this sense one can say that the edifice of the FRDWM has a fairly sound foundation, and this is confirmed by the results presented in the present review.

The results given here refer only to the analysis of the angular distributions formed in particle reactions. However, the FRDWM formalism can also be readily extended to calculate the angular dependences of reaction characteristics that are associated with nondiagonal elements of the density matrix (for example, the polarizations of the final particles, the angular correlations of these particles with the decay products of the final nucleus when it is formed in an excited state). In particular, to calculate the angular dependence of the polarization of the emitted particles it is necessary to introduce a spin–orbit interaction in the optical potentials and in the potentials responsible for the rearrangement of the particles. The angular correlations can be expressed directly in terms of the quantities $\beta_{\Lambda_1\Lambda_2 l m_l}(\vartheta, \varkappa_{c_2}, \varkappa_{c_1})$ determined by the expression (30), and they can therefore be calculated on the basis of the existing FRDWM programs.

It follows from the results given in the review that hitherto the FRDWM has been used only to describe and analyze experimental results that were already obtained and has not been used to predict qualitative features of the angular distributions of concrete reactions. The extent to which the FRDWM has predictive power is a very moot point. It concerns, in the first place, the sensitivity of the results of the calculation to the parameters of the method and the reliability of their determination. The parameters needed for analysis are introduced into the FRDWM programs independently (the parameters of the optical potentials from fitting of the angular distributions of elastic scattering and from microscopic approaches, the spectroscopic factors from theoretical model calculations). All the calculations made in the FRDWM of the angular distributions of the stripping mechanism, and the majority of the calculations are of this kind, show that the theoretical curves vary most strongly when the parameters of the optical potentials are changed. Microscopic calculations of the many-body optical potentials make it possible to reduce considerably the arbitrariness in the choice of the parameters and thus raise the reliability of the predicted calculated angular distributions.

With regard to the structure factors, the angular distributions of the different mechanisms are sensitive to them to different degrees. For direct processes in the cases when the cross section factorizes into a kinematic and a structure factor (see Sec. 2), the profile of the angular distribution for one angular momentum transfer l does not depend on the spectroscopic factor, since this factor only normalizes the absolute value of the cross section. If several l values can participate in the reaction, the profile of the angular distribution already depends strongly on the ratios of the corresponding spectroscopic factors even for direct processes. For exchange processes, the sensitivity of the angular distributions to the structure factors is much stronger, since even for one value of l the structure factors have a strong influence on the profile of the angular distributions.

However, special investigations into the sensitivity of the FRDWM to the parameters of the method have not been made. Nor have the relations of the cross sections between transitions for different reactions having the same entrance or exit channel been investigated. Therefore, there are as yet no weighty arguments for drawing definite conclusions about the FRDWM in its capacity to make sufficiently accurate predictions of the absolute values of the reaction cross sections and their angular characteristics.

It is significantly harder to justify the validity of using the FRDWM to analyze the energy dependences of the cross sections (excitation functions). On the one hand, the experimental excitation functions frequently behave

extremely irregularly, and one observes in them resonances of different profiles and amplitudes; on the other hand, the optical model, like the distorted-wave method itself, can pretend only to a description of the smooth differential cross section averaged over a certain energy interval. Therefore, the analysis of the excitation functions requires a further improvement in the method itself.

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