Kinetic equation for a dynamical system interacting with a phonon field

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A preprint of one of the authors [N. N. Bogolyubov, Preprint E17-11822, JINR, Dubna (1978)] is generalized, and methods are formulated for studying an electron–phonon system and eliminating the phonon operators from the corresponding kinetic equations. In particular, for the interaction of an electron with a phonon field a polaron kinetic equation is obtained, this leading in an appropriate approximation to the exact Boltzmann equation for a polaron. Methods are also proposed for calculating the response functions (impedance and admittance); these methods are based on the introduction of an approximating Hamiltonian with linear interaction. The probability density for the particle distribution is calculated.

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1. GENERALIZED KINETIC EQUATION

We consider a dynamical system S interacting with a phonon field Σ. We denote by X_s the arguments of the wave functions for the isolated system S and by $X_\Sigma = (\ldots n_k \ldots)$ the population numbers of the field Σ. Then the dynamical state of the system (S,Σ) can be characterized by wave functions of the type

$$\Psi = \Psi(X_s, X_\Sigma). \tag{1}$$

We shall denote by

$$F(t, S),\ f(S) \tag{2}$$

operators that can depend explicitly on time t and act on $\Psi(X_s, X_\Sigma)$ only as a function of X_s. By

$$G(t, \Sigma),\ g(\Sigma) \tag{3}$$

we shall denote operators that act on Ψ as a function of X_Σ. Operators of this kind are, for example, the Bose amplitudes $\ldots b_k \ldots b_k^+ \ldots$. It is important that $F(t,S)$ and $G(t,\Sigma)$ commute, since they act on different variables in the wave function. In particular, $F(t,S)$ commutes with all b_k and b_k^+. As an example of an operator of the type (3) we can take the Hamiltonian of the phonon field itself:

$$H(\Sigma) = \sum_k \hbar\omega_k b_k^+ b_k,\quad \omega_k > 0. \tag{4}$$

By symbols of the type $\mathfrak{A}(t,S,\Sigma)$ we shall denote operators that act on the variables X_s and the variables X_Σ of the wave functions $\Psi(X_s, X_\Sigma)$.

We emphasize that these operators correspond to the ordinary Schrödinger representation of dynamical variables. We take the case when, in the usual notation, the total Hamiltonian of the system (S,Σ) has the form

$$H_t = H_t(t, S, \Sigma) = \Gamma(t, S) + \sum_{(k)} \{C_k(t, S) b_k + \overset{+}{C}_k(t, S) \overset{+}{b}_k\} + H(\Sigma), \tag{5}$$

where $\Gamma(t,S)$ is the Hamiltonian of the system S; the following term in (5) with the sum over k is the Hamiltonian of the interaction between S and Σ.

We consider two examples of such systems.

1. Polaron Theory. The polaron model describes the motion of an electron in an ionic crystal. The system S consists of one electron in an external electric field $\overline{\mathscr{E}}(t)$:

$$\Gamma(t, S) = p^2/(2m) + \exp(\varepsilon t)\, \overline{E}(t)\, \overline{r};\quad E(t) = -e\overline{\mathscr{E}}(t);$$

$$C_k(t, S) = \frac{\exp(\varepsilon t)}{\sqrt{V}} \mathscr{L}(k) \left(\frac{\hbar}{2\omega_k}\right)^{1/2} \exp(i\overline{k}\overline{r}), \tag{6}$$

where e is the electron charge, $\mathscr{L}^*(k) = \mathscr{L}(k)$, $\overline{r}$ and $\overline{p}$ are the position and momentum of the electron, and $\mathscr{L}(k)$ and ω_k are radially symmetric functions of the wave vector $\overline{k}$.

The summation over $\overline{k}$ is over the usual quasidiscrete spectrum:

$$k = (2\pi n_1/L,\ 2\pi n_2/L,\ 2\pi n_3/L);\quad L^3 = V,$$

where n_1, n_2, n_3 are integers (positive and negative); the factor $\exp(\varepsilon t)$ $(\varepsilon > 0)$ is introduced, as usual, to represent adiabatic switching-on of the interaction.

In this case, the operators of the type $f(S)$ will be functions of the operators $\overline{p}$ and $\overline{r}$; for example,

$$f(\overline{p}),\ \exp(i\overline{k}\overline{r}),\ f(\overline{p})\exp(i\overline{k}\overline{r}),\ \text{etc.}$$

We note finally that in a number of cases the expression $p^2/(2m)$ must be replaced by a more general form $T(p)$ of the electron energy.

Then instead of (6) we have

$$\Gamma(t, S) = T(p) + \exp(\varepsilon t)\, \overline{E}(t)\, \overline{r}. \tag{7}$$

2. Fermion System. The system S is a system of free fermions characterized by the Fermi amplitudes a_f^+ and a_f, and

$$\Gamma(t, S) = \sum_{(f)} \Lambda(f)\, a_f^+ a_f;$$

$$C_k(t, S) = \frac{\exp(\varepsilon t)}{\sqrt{V}} L_k \sum_{(f)} \overset{+}{a}_{f+k} a_f;$$

$$\overset{+}{C}_k(t, S) = \frac{\exp(\varepsilon t)}{\sqrt{V}} L_k^* \sum_{(f)} \overset{+}{a}_f a_{f+k} = \frac{\exp(\varepsilon t)}{\sqrt{V}} L_k^* \sum_{(f)} \overset{+}{a}_{f-k} a_f, \tag{8}$$

where L_k and L_k^* are C-number quantities.

Since fermions have spin, we have here $f=(\mathbf{f},\sigma)$, and the vector $\mathbf{f}$ belongs to the quasidiscrete spectrum (σ is the spin index). The symbol $(f+k)$ is explicitly $f+k=(\mathbf{f}+\mathbf{k},\sigma)$. We can also consider the case of fermions interacting with one another. It is then merely necessary to include in $\Gamma(t,S)$ the terms of the interaction between the fermions, and also their interaction with the external fields.

For dynamical systems of the second type, the operators of the form $f(S)$ will be all combinations of the Fermi amplitudes $\ldots a_f \ldots a_f^+ \ldots$ that do not contain Bose amplitudes; for example, $a_{f_1}^+ a_{f_2}$. We note that dynamical systems of the second type arise in the problem of determining the electrical conductivity of metals, in theory of superconductivity, and elsewhere.

We now return to the Hamiltonian (5) and use the Liouville equation for the statistical operator $\mathcal{D}_t$ of the system (S,Σ),

$$i\hbar\frac{\partial\mathcal{D}}{\partial t}=H(t,S,\Sigma)\mathcal{D}_t-\mathcal{D}_t H(t,S,\Sigma), \tag{9}$$

with the initial condition

$$\mathcal{D}_{t_0}=\rho(S)\mathcal{D}(\Sigma);\quad \mathcal{D}(\Sigma)=Z^{-1}\exp[-\beta H(\Sigma)], \tag{10}$$

where

$$\left.\begin{aligned} &Z=\mathop{\mathrm{Sp}}_{(\Sigma)}\exp[-\beta H(\Sigma)];\\ &\mathop{\mathrm{Sp}}_{(S)}\rho(S)=1;\quad \mathop{\mathrm{Sp}}_{(\Sigma)}\mathcal{D}(\Sigma)=1.\end{aligned}\right\} \tag{11}$$

It can be seen that the initial condition we have adopted corresponds to the situation in which the phonon field Σ is in the state of statistical equilibrium at the time t_0 when its interaction with the dynamical system S characterized by the statistical operator $\rho(S)$ is "switched on."

It follows from (9) that

$$\mathop{\mathrm{Sp}}_{(S,\Sigma)}\mathcal{D}_t=\mathop{\mathrm{Sp}}_{(S,\Sigma)}\mathcal{D}_{t0},$$

$$\mathop{\mathrm{Sp}}_{(S,\Sigma)}\mathcal{D}_t=\mathop{\mathrm{Sp}}_{(S)}\rho(S)\mathop{\mathrm{Sp}}_{(\Sigma)}\mathcal{D}(\Sigma)=1,$$

so that

$$\mathop{\mathrm{Sp}}_{(S,\Sigma)}\mathcal{D}_t=\mathop{\mathrm{Sp}}_{(S,\Sigma)}\mathcal{D}_{t0},$$

and we have the usual normalization for the statistical operator D_t of the dynamical system (S,Σ).

We introduce the operator $U(t,t_0)=U(t,t_0,S,\Sigma)$ which is determined by the equation

$$i\hbar\partial U(t,t_0)/\partial t=H(t,S,\Sigma)U(t,t_0),$$

where $U(t_0,t_0)=1$. Since the Hamiltonian is Hermitian,

$$-i\hbar\partial\overset{+}{U}(t,t_0)/\partial t=\overset{+}{U}(t,t_0)H(t,S,\Sigma),$$

where $\overset{+}{U}(t_0,t_0)=1$. We see that U is unitary, $\overset{+}{U}(t,t_0)=U^{-1}(t,t_0)$. By means of the operators U we deduce $\mathcal{D}_t=U(t,t_0)\mathcal{D}_{t0}U^{-1}(t,t_0)$ from Eq. (9).

We consider some dynamical variable $\mathfrak{A}(t,S,\Sigma)$ in the Schrödinger representation. Its expectation value at the time t is

$$\begin{aligned}\langle\mathfrak{A}\rangle_t&=\mathop{\mathrm{Sp}}_{(S,\Sigma)}\mathfrak{A}(t,S,\Sigma)\mathcal{D}_t\\ &=\mathop{\mathrm{Sp}}_{(S,\Sigma)}\mathfrak{A}(t,S,\Sigma)U(t,t_0)\mathcal{D}_{t0}U^{-1}(t,t_0)\\ &=\mathop{\mathrm{Sp}}_{(S,\Sigma)}\{U^{-1}(t,t_0)\mathfrak{A}(t,S,\Sigma)U(t,t_0)\}\mathcal{D}_{t0}.\end{aligned} \tag{12}$$

It can be seen that the expression

$$U^{-1}(t,t_0)\mathfrak{A}(t,S,\Sigma)U(t,t_0) \tag{13}$$

is the Heisenberg representation for this dynamical variable, the two representations coinciding at $t=t_0$.

We shall denote this Heisenberg representation by the symbol $\mathfrak{A}(t,S_t,\Sigma_t)$:

$$\mathfrak{A}(t,S_t,\Sigma_t)=U^{-1}(t,t_0)\mathfrak{A}(t,S,\Sigma)U(t,t_0). \tag{14}$$

In particular, if we consider the dynamical variable in the Schrödinger representation given by an operator of the type $F(t,S)$, then

$$\begin{aligned}F(t,S_t)&=U^{-1}(t,t_0)F(t,S)U(t,t_0)\\ &=\overset{+}{U}(t,t_0)F(t,S)U(t,t_0).\end{aligned} \tag{15}$$

From (12), we obtain

$$\mathop{\mathrm{Sp}}_{(S,\Sigma)}F(t,S_t)\mathcal{D}_{t0}=\mathop{\mathrm{Sp}}_{(S,\Sigma)}F(t,S)\mathcal{D}_t=\mathop{\mathrm{Sp}}_{(S)}F(t,S)(\mathop{\mathrm{Sp}}_{(\Sigma)}\mathcal{D}_t).$$

We introduce further the reduced statistical operator $\rho_t(S)=\mathop{\mathrm{Sp}}_{(\Sigma)}\mathcal{D}_t$. Then

$$\mathop{\mathrm{Sp}}_{(S,\Sigma)}F(t,S_t)\mathcal{D}_{t0}=\mathop{\mathrm{Sp}}_{(S)}F(t,S)\rho_t(S). \tag{16}$$

We consider the dynamical system with the Hamiltonian (5) and the initial condition (10) for the statistical operator. On the basis of Eq. (14) for the Heisenberg representation, we have

$$\begin{aligned}&[\mathfrak{A}(t,S_t,\Sigma_t);\ \mathfrak{B}(t,S_t,\Sigma_t)]\\ &=U^{-1}(t,t_0)[\mathfrak{A}(t,S,\Sigma);\ \mathfrak{B}(t,S,\Sigma)]U(t,t_0),\end{aligned}$$

where $[\mathfrak{A},\mathfrak{B}]$ denotes the commutator $[\mathfrak{A},\mathfrak{B}]=\mathfrak{A}\mathfrak{B}-\mathfrak{B}\mathfrak{A}$. It can be seen that if the commutator of two dynamical variables in the Schrödinger representation is a number C, then the commutator of these dynamical variables in the Heisenberg representation will have the same value.

We denote the Heisenberg representation for the Bose amplitudes by $\ldots b_k(t),\ldots b_k^+(t)\ldots$. Then by definition (14), $b_k(t_0)=b_k$, $b_k^+(t_0)=b_k^+$. Since b_k^+ and b_k commute with $\Gamma(t,S)$, $C_k(t,S)$, $C_k^+(t,S)$, we see that

$$\left.\begin{aligned}&[b_k(t);\ \Gamma(t,S_t)]=0;\quad [\overset{+}{b}_k(t);\ \Gamma(t,S_t)]=0;\\ &[b_k(t);\ C_k(t,S_t)]=0;\quad [\overset{+}{b}_k(t);\ C_k(t,S_t)]=0;\\ &[b_k(t);\ C_k^+(t,S_t)]=0;\quad [\overset{+}{b}_k(t);\ \overset{+}{C}_k(t,S_t)]=0.\end{aligned}\right\} \tag{17}$$

For the same reason,

$$[H(\Sigma_t);\ f(t,S_t)]=0. \tag{18}$$

It is clear that $b_k^+(t)$ and $b_k(t)$ have the same usual commutation relations as b_k^+ and b_k.

With allowance for (5) and (17), the equations for the Bose amplitudes give

$$i\hbar\partial b_k(t)/\partial t=[b_k(t);\ H(t,S_t,\Sigma_t)];$$

$$i\hbar\partial b_k(t)/\partial t=\hbar\omega_k b_k(t)+\overset{+}{C}_k(t,S_t),$$

i. e.,

$$\partial b_k(t)/\partial t=-i\omega_k b_k(t)-(i/\hbar)\overset{+}{C}_k(t,S_t).$$

The conjugate equation is

$$\partial\overset{+}{b}_k(t)/\partial t=i\omega_k\overset{+}{b}_k(t)+(i/\hbar)C_k(t,S_t).$$

Taking into account the initial conditions, we obtain

$$\left.\begin{aligned}&b_k(t)=\tilde{b}_k(t)-i\mathfrak{B}_k(t),\quad \tilde{b}_k(t)\\ &\qquad=\exp[-i\omega_k(t-t_0)]b_k;\\ &\mathfrak{B}_k(t)=\frac{1}{\hbar}\int_{t_0}^{t}\exp[-i\omega_k(t-\tau)]\overset{+}{C}_k(\tau,S_\tau)\,d\tau;\end{aligned}\right\} \tag{19}$$

$$\left.\begin{aligned}\overset{+}{b}_k(t)&=\tilde{b}_k^+(t)+i\overset{+}{\mathfrak{B}}_k(t);\quad \tilde{b}_k^+(t)=\exp[i\omega_k(t-t_0)]\overset{+}{b}_k;\\ \overset{+}{\mathfrak{B}}_k(t)&=\frac{1}{\hbar}\int_{t_0}^{t}\exp[i\omega_k(t-\tau)]\overset{+}{C}_k(\tau,S_\tau)\,d\tau.\end{aligned}\right\}\quad(20)$$

We consider the dynamical variable that in the Schrödinger representation is represented by an operator of the type $f(S)$ with no explicit dependence on the time.

Using (5) and (17), we can write the equations of motion for $f(S_t)$,

$$i\hbar\partial f(S_t)/\partial t=[f(S_t);\ H(t,S_t,\Sigma_t)],$$

in the form

$$i\hbar\partial f(S_t)/\partial t=[f(S_t);\ \Gamma(t,S_t)]+\sum_{(k)}b_k(t)[f(S_t);\ C_k(t,S_t)]$$
$$+\sum_{(k)}\overset{+}{b}_k(t)[f(S_t);\ \overset{+}{C}_k(t,S_t)].$$

Substituting here (19) and (20) and performing the operation $\underset{(S,\Sigma)}{\mathrm{Sp}}\ldots\mathscr{D}_{t0}$, we obtain

$$\begin{aligned}&i\hbar\frac{\partial}{\partial t}\underset{(S,\Sigma)}{\mathrm{Sp}}f(S_t)\mathscr{D}_{t0}+\underset{(S,\Sigma)}{\mathrm{Sp}}[\Gamma(t,S_t);\ f(S_t)]\mathscr{D}_{t0}\\&=-i\sum_{(k)}\mathrm{Sp}\,\mathfrak{B}_k(t)[f(S_t);\ C_k(t,S_t)]\mathscr{D}_{t0}\\&+i\sum_{(k)}\underset{(S,\Sigma)}{\mathrm{Sp}}\overset{+}{\mathfrak{B}}_k(t)[f(S_t);\ \overset{+}{C}_k(t,S_t)]\mathscr{D}_{t0}\\&+\sum_{(k)}\underset{(S,\Sigma)}{\mathrm{Sp}}\tilde{b}_k(t)[f(S_t);\ C_k(t,S_t)]\mathscr{D}_{t0}\\&+\sum_{(k)}\underset{(S,\Sigma)}{\mathrm{Sp}}\tilde{b}_k^+(t)[f(S_t);\ \overset{+}{C}_k(t,S_t)]\mathscr{D}_{t0}.\end{aligned}\quad(21)$$

To eliminate the Bose amplitudes $\tilde{b}_k$ and $\tilde{b}_k^+$ on the right-hand side of Eq. (21), we formulate a lemma.

The expectation values of the product of the operator $\tilde{b}_k(t)$ and the operator $\mathfrak{A}(S,\Sigma)$ satisfy the relations

$$\begin{aligned}\underset{(S,\Sigma)}{\mathrm{Sp}}\tilde{b}_k(t)\mathfrak{A}(S,\Sigma)\mathscr{D}_{t0}&=(1+N_k)\underset{(S,\Sigma)}{\mathrm{Sp}}\{\tilde{b}_k(t)\mathfrak{A}(S,\Sigma)\\&-\mathfrak{A}(S,\Sigma)\tilde{b}_k(t)\}\mathscr{D}_{t0};\\ \underset{(S,\Sigma)}{\mathrm{Sp}}\mathfrak{A}(S,\Sigma)\tilde{b}_k(t)\mathscr{D}_{t0}&=N_k\underset{(S,\Sigma)}{\mathrm{Sp}}\{\tilde{b}_k(t)\mathfrak{A}(S,\Sigma)-\mathfrak{A}(S,\Sigma)\tilde{b}_k(t)\}\mathscr{D}_{t0},\end{aligned}$$

where

$$N_k=\exp(-\beta\hbar\omega_k)/[1-\exp(-\beta\hbar\omega_k)].$$

The proof is given in Appendix 1.

Taking $\mathfrak{A}(S,\Sigma)=[f(S_t);C_k(t,S_t)]$, we have

$$\left.\begin{aligned}&\underset{(S,\Sigma)}{\mathrm{Sp}}\tilde{b}_k(t)[f(S_t);\ C_k(t,S_t)]\mathscr{D}_{t0}\\&=(1+N_k)\underset{(S,\Sigma)}{\mathrm{Sp}}[\tilde{b}_k(t);\ [f(S_t);\ C_k(t,S_t)]]\mathscr{D}_{t0};\\&\underset{(S,\Sigma)}{\mathrm{Sp}}\tilde{b}^+(t)[f(S_t);\ C_k^+(t,S_t)]\mathscr{D}_{t0}\\&=N_k\underset{(S,\Sigma)}{\mathrm{Sp}}[[f(S_t);\ \overset{+}{C}_k(t,S_t)];\ \tilde{b}_k^+(t)]\mathscr{D}_{t0}.\end{aligned}\right\}\quad(22)$$

Since $\ldots b_k\ldots\overset{+}{b}_k\ldots$ commute with $[f(S);C_k(t,S)]$ and $[f(S);\overset{+}{C}_k(t,S)]$,

$$[b_k(t);\ [f(S_t);\ C_k(t,S_t)]]=0;$$
$$[[f(S_t);\ \overset{+}{C}_k(t,S_t)];\ \overset{+}{b}_k(t)]=0.$$

Substituting here (19) and (20), we find

$$\left.\begin{aligned}&[\tilde{b}_k(t);\ [f(S_t);\ C_k(t,S_t)]]=i[\mathfrak{B}_k(t);\ [f(S_t);\ C_k(t,S_t)]]\\&=i\mathfrak{B}_k(t)[f(S_t);\ C_k(t,S_t)]-i[f(S_t);\ C_k(t,S_t)]\mathfrak{B}_k(t)];\\&[[f(S_t);\ \overset{+}{C}_k(t,S_t)];\ \tilde{b}_k^+]=i[[f(S_t);\ \overset{+}{C}_k(t,S_t)];\ \overset{+}{\mathfrak{B}}_k(t)]\\&=i\mathfrak{B}_k^+(t)[f(S_t);\ \overset{+}{C}_k(t,S_t)]-i[f(S_t);\ \overset{+}{C}_k(t,S_t)]\mathfrak{B}_k^+(t).\end{aligned}\right\}\quad(23)$$

Using (22) and (23), we obtain from (21)

$$\begin{aligned}&i\hbar\frac{\partial}{\partial t}\underset{(S,\Sigma)}{\mathrm{Sp}}f(S_t)\mathscr{D}_{t0}+\underset{(S,\Sigma)}{\mathrm{Sp}}[\Gamma(t,S_t);\ f(S_t)]\mathscr{D}_{t0}\\&=\sum_{(k)}i\{N_k\underset{(S,\Sigma)}{\mathrm{Sp}}\mathfrak{B}_k(t)[f(S_t);\ C_k(t,S_t)]\mathscr{D}_{t0}\\&+(1+N_k)\underset{(S,\Sigma)}{\mathrm{Sp}}[C_k(t,S_t);\ f(S_t)]\mathfrak{B}_k(t)\mathscr{D}_{t0}\}\\&+i\sum_{(k)}\{(1+N_k)\underset{(S,\Sigma)}{\mathrm{Sp}}\overset{+}{\mathfrak{B}}_k(t)[f(S_t);\ \overset{+}{C}_k(t,S_t)]\mathscr{D}_{t0}\\&+N_k\underset{(S,\Sigma)}{\mathrm{Sp}}[\overset{+}{C}_k(t,S_t);\ f(S_t)]\overset{+}{\mathfrak{B}}_k(t)\mathscr{D}_{t0}\}.\end{aligned}$$

Note that with allowance for (16)

$$\underset{(S,\Sigma)}{\mathrm{Sp}}f(S_t)\mathscr{D}_{t0}=\underset{(S)}{\mathrm{Sp}}f(S)\rho_t(S),$$
$$\underset{(S,\Sigma)}{\mathrm{Sp}}[\Gamma(t,S_t);\ f(S_t)]\mathscr{D}_{t0}=\underset{(S)}{\mathrm{Sp}}\{\Gamma(t,S)f(S)-f(S)\Gamma(t,S)\}\rho_t(S);$$

we now replace $\mathfrak{B}_k(t)$ and $\mathfrak{B}_k^+(t)$ by their expressions (19) and (20) and divide both sides by $i\hbar$. We then have

$$\begin{aligned}&\underset{(S)}{\mathrm{Sp}}\left\{f(S)\frac{\partial\rho_t(S)}{\partial t}+\frac{\Gamma(t,S)f(S)-f(S)\Gamma(t,S)}{i\hbar}\rho_t(S)\right\}\\&=\frac{1}{\hbar^2}\sum_{(k)}\int_{t_0}^{t}d\tau\underset{(S,\Sigma)}{\mathrm{Sp}}\exp[-i\omega_k(t-\tau)]\{N_k\overset{+}{C}_k(\tau,S_\tau)[f(S_t);\ C_k(t,S_t)]\\&+(1+N_k)[\overset{+}{C}_k(t,S_t);\ f(S_t)]C_k(\tau,S_\tau)\}\mathscr{D}_{t0}\\&+\frac{1}{\hbar^2}\sum_{(k)}\int_{t_0}^{t}d\tau\underset{(S,\Sigma)}{\mathrm{Sp}}\exp[i\omega_k(t-\tau)]\{(1+N_k)C_k(\tau,S_\tau)\\&\times[f(S_t);\ \overset{+}{C}_k(t,S_t)]+N_k[\overset{+}{C}_k(t,S_t);\ f(S_t)]C_k(\tau,S_\tau)\}\mathscr{D}_{t0}.\end{aligned}\quad(24)$$

Thus, we have constructed a generalized kinetic equation. We now turn to the polaron model. For this, we substitute the expressions (6) into the right-hand side of this equation, i. e.,

$$\Gamma(t,S)=T(\bar{p})+\exp(\varepsilon t)\bar{E}(t)\bar{r};$$
$$C_k(t,S)=\frac{\exp(\varepsilon t)}{\sqrt{V}}\mathscr{L}(k)\left(\frac{\hbar}{2\omega_k}\right)^{1/2}\exp[i(\bar{k}\bar{r})],$$

and we then find

$$\begin{aligned}&\underset{(S)}{\mathrm{Sp}}\left\{f(S)\frac{\partial\rho_t(S)}{\partial t}\right.\\&\left.+\frac{\exp(\varepsilon t)\bar{E}(t)(\bar{r}f(S)-f(S)\bar{r})+T(p)f(S)-f(S)T(p)}{i\hbar}\rho_t(S)\right\}\\&=\frac{1}{V}\exp(2\varepsilon t)\sum_{(k)}\frac{\mathscr{L}^2(k)}{2\hbar\omega_k}\int_{t_0}^{t}d\tau\exp[-\varepsilon(t-\tau)]\{N_k\exp[-i\omega_k(t-\tau)]\\&+(1+N_k)\exp[i\omega_k(t-\tau)]\}\underset{(S,\Sigma)}{\mathrm{Sp}}\{\exp(-i\bar{k}\bar{r}_\tau)f(S_t)\exp(i\bar{k}\bar{r}_t)\\&-\exp(-i\bar{k}\bar{r}_\tau)\exp(i\bar{k}\bar{r}_t)f(S_t)\}\mathscr{D}_{t0}\\&+\frac{1}{V}\exp(2\varepsilon t)\sum_{(k)}\frac{\mathscr{L}^2(k)}{2\hbar\omega_k}\int_{t_0}^{t}d\tau\exp[-\varepsilon(t-\tau)]\\&\times\{(1+N_k)\exp[-i\omega_k(t-\tau)]+N_k\exp[i\omega_k(t-\tau)]\}\\&\times\underset{(S,\Sigma)}{\mathrm{Sp}}\{\exp(i\bar{k}\bar{r}_t)f(S_t)\exp(-i\bar{k}\bar{r}_\tau)\\&-f(S_t)\exp(i\bar{k}\bar{r}_t)\exp(-i\bar{k}\bar{r}_\tau)\}\mathscr{D}_{t0}.\end{aligned}\quad(25)$$

It is interesting to note that the phonon operators do not occur explicitly in this equation. The right-hand side depends only on the electron trajectory $\bar{r}(\tau),\bar{p}(\tau),t_0\leq\tau<t$.

We emphasize that the electron operators $\bar{r}(\tau)$ and $\bar{p}(\tau)$ depend in a complicated manner on the initial values of $\bar{r},\bar{p},\ldots b_k,b_k^+$. Therefore, to obtain from the last equation an explicit expression, we must restrict ourselves to an appropriate approximation. For example, setting $f(S)=f(\bar{p})$ and replacing the complicated dependence $\bar{r}(\tau)$ in the "zeroth approximation" by uniform motion, $\bar{r}(\tau)=\bar{r}(t)-\bar{p}(t)(t-\tau)/m$, in the framework of the Fröhlich model and using explicitly the small parameter which characterizes the intensity of the electron–phonon interaction, we can construct an explicit Boltzmann equation for the polaron with integral term corresponding to single-phonon absorption and emission[1] of quanta of the phonon field.

We now consider the case of spatial homogeneity, when

$$f(S)=f(\bar{p}),$$

and hence

$$f(S_t)=f(\bar{p}_t).$$

We have

$$\bar{r}f(\bar{p})-f(\bar{p})\,\bar{r}=i\hbar\,\partial f(\bar{p})/\partial\bar{p}.$$

We note the formula

$$\underset{(S)}{\mathrm{Sp}}\; f(\bar{p})\,\rho_t(S)=\int f(\bar{p})\,W_t(\bar{p})\,d\bar{p},$$

where

$$W_t(\bar{p}_0)=\underset{(S)}{\mathrm{Sp}}\;\delta(\bar{p}-\bar{p}_0)\,\rho_t(S). \tag{26}$$

Let $\hat{p}_t$ be the momentum operator in the Schrödinger representation. Then, with allowance for (16),

$$\underset{(S,\Sigma)}{\mathrm{Sp}}\; F(\bar{p}_t)\,\mathscr{D}_{t0}=\int F(\bar{p})\,W_t(\bar{p})\,d\bar{p}.$$

It follows from (11) and (26) that $\int W_t(\bar{p})d\bar{p}=1$. It is clear that $W_t(\bar{p})$ can be interpreted as the possibility density at the time t.

We write the left-hand side of (24) in the form

$$\underset{(S)}{\mathrm{Sp}}\left\{f(\bar{p})\,\frac{\partial\rho_t(S)}{\partial t}+\exp(\varepsilon t)\,\bar{E}(t)\,\frac{\partial f(\bar{p})}{\partial\bar{p}}\,\rho_t(S)\right\}$$

$$=\int d\bar{p}\left\{f(\bar{p})\,\frac{\partial W_t(\bar{p})}{\partial t}+\exp(\varepsilon t)\,\bar{E}(t)\,\frac{\partial f(\bar{p})}{\partial\bar{p}}\,W_t(\bar{p})\right\}.$$

It is easy to see that

$$\begin{aligned}&\exp(i\bar{k}\bar{r})\,f(\bar{p})=f(\bar{p}-\hbar\bar{k})\exp(i\bar{k}\bar{r});\\&f(\bar{p})\exp(i\bar{k}\bar{r})=\exp(i\bar{k}\bar{r})\,f(\bar{p}+\hbar\bar{k}),\end{aligned} \tag{27}$$

and also

$$f(\bar{p}_t)\exp(i\bar{k}\bar{r}_t)=\exp(i\bar{k}\bar{r}_t)\,f(\bar{p}_t+\hbar\bar{k});$$

$$\exp(i\bar{k}\bar{r}_t)\,f(\bar{p}_t)=f(\bar{p}_t-\hbar\bar{k})\exp(i\bar{k}\bar{r}_t).$$

Using the invariance in both terms on the right-hand side of (25) under the replacement of k by $-k$ and the remarks made above, we obtain

$$\begin{aligned}&\int d\bar{p}f(\bar{p})\left\{\frac{\partial W_t(\bar{p})}{\partial t}-\exp(\varepsilon t)\,\bar{E}(t)\,\frac{\partial W_t(\bar{p})}{\partial\bar{p}}\right\}\\&=\int d\bar{p}\left\{f(\bar{p})\,\frac{\partial W_t(\bar{p})}{\partial t}+\exp(\varepsilon t)\,\bar{E}(t)\,\frac{\partial f(\bar{p})}{\partial\bar{p}}\,W_t(\bar{p})\right\}\\&=\exp(2\varepsilon t)\,\frac{1}{V}\sum_{(k)}\frac{\mathscr{L}^2(k)}{2\hbar\omega_k}\int_{t_0}^{t}d\tau\exp[-\varepsilon(t-\tau)]\\&\times\{N_k\exp[-i\omega_k(t-\tau)]+(1+N_k)\exp[i\omega_k(t-\tau)]\}\\&-\underset{(S,\Sigma)}{\mathrm{Sp}}\{\exp(i\bar{k}\bar{r}_\tau)\exp[-i\bar{k}\bar{r}_t]\{f(\bar{p}_t-\hbar\bar{k})-f(\bar{p}_t)\}\,\mathscr{D}_{t0}\}\\&+\exp(2\varepsilon t)\,\frac{1}{V}\sum_{(k)}\frac{\mathscr{L}^2(k)}{2\hbar\omega_k}\int_{t_0}^{t}d\tau\exp[-\varepsilon(t-\tau)]\{(1+N_k)\\&\times\exp[-i\omega_k(t-\tau)]+N_k\exp[i\omega_k(t-\tau)]\}\underset{(S,\Sigma)}{\mathrm{Sp}}(\{f(\bar{p}_t-\hbar\bar{k})\\&-f(\bar{p}_t)\}\exp(i\bar{k}\bar{r}_t)\exp(-i\bar{k}\bar{r}_\tau)\,\mathscr{D}_{t0}),\end{aligned} \tag{28}$$

where $\mathscr{D}_{t0}=\rho(S)\mathscr{D}(\Sigma)$.

This exact relation will be considered in Sec. 2 as a source for obtaining approximate equations.

Finally, we note that the generalized equation (24) can also have other applications. For example, it can also be used to study the motion of electrons in a metal and to derive corresponding kinetic equations. For this it is only necessary to take as the operator function in Eq. (24)

$$f(S)=\overset{+}{a}_f a_f$$

and

$$\Gamma(t,S)=\sum_{(f)}T_f\overset{+}{a}_f a_f.$$

Then

$$\underset{(S)}{\mathrm{Sp}}\,f(S)\,\rho_t(S)=\underset{(S)}{\mathrm{Sp}}\,\overset{+}{a}_f a_f\rho_t(S)=\langle\overset{+}{a}_f(t)\,a_f(t)\rangle_{t0}=n_f(t)$$

and

$$\underset{(S)}{\mathrm{Sp}}\,f(S)\,\frac{\partial\rho_t(S)}{\partial t}=\frac{\partial}{\partial t}\,n_f(t),$$

and as the operator $C_k(t,S)$ we take the expression

$$C_k(t,S)=\frac{\exp(\varepsilon t)}{\sqrt{V}}\,\mathscr{L}_k\sum_{(f)}\overset{+}{a}_{f+k}(t),\,a_f(t).$$

We particularize further the operators $\overset{+}{a}_{f+k}(t), a_f(t)$ in this construction, making an "approximation" by assuming that they satisfy the equation of motion without interaction,

$$i\hbar\,da_f/dt=T(f)\,a_f(t),$$

from which

$$a_f(\tau)=\exp[-i(T_f/\hbar)(\tau-t)]\,a_f(t);\quad \overset{+}{a}_f(\tau)=\exp[i(T_f/\hbar)(\tau-t)]\,\overset{+}{a}_f(t).$$

Then

$$C_k(\tau,S_\tau)=\frac{\exp(\varepsilon\tau)}{\sqrt{V}}\sum_{(f)}\exp\left[-i\,\frac{(T_{f+k}-T_f)}{\hbar}(t-\tau)\right]\overset{+}{a}_{f+k}(t)\,a_f(t).$$

Taking into account these remarks and substituting the "approximate" expression for $C_k(\tau,S_\tau)$ into the generalized kinetic equation (24), and making some simple transformations and the standard passage to the limit $t_0\to-\infty$, $\varepsilon\to 0$, we find the well-known quantum kinetic equation of Bloch, on which the theory of the electrical conductivity and thermal conductivity of metals and semiconductors is based.[5]

2. KINETIC EQUATIONS IN THE FIRST APPROXIMATION FOR WEAK INTERACTIONS

We consider here the case of weak interactions. It is convenient to characterize the coupling constant by a small parameter, which we shall denote by α, under the condition that $\mathscr{L}^2(k)$ is proportional to α.

For example, in the framework of Fröhlich's polaron model the standard dimensionless parameter characterizing the intensity of the electron–phonon interaction is in our notation

$$\alpha=\frac{g^2}{4\pi\hbar\omega^2}\sqrt{\frac{m}{2\hbar\omega}}. \tag{29}$$

We shall also assume that the external force E is formally proportional to the small parameter.

In the "zeroth approximation," when we completely ignore the interaction, we can write down the equation

$$i\hbar\, dr/d\tau = rT(p) - T(p)\, r, \tag{30}$$

from which it follows that

$$\bar{r}_\tau = \exp[iT(p)(\tau-\tau_0)/\hbar]\, \bar{r}_{\tau_0} \exp[-iT(p)(\tau-\tau_0)/\hbar].$$

Suppose $\tau_0 = t$; then

$$\bar{r}_\tau = \exp[iT(p)(\tau-t)/\hbar]\, \bar{r}_t \exp[-iT(p)(\tau-t)/\hbar]$$

and

$$\exp(i\bar{k}\bar{r}_\tau) = \exp\{(i/\hbar)\, T(p_t)(\tau-t)\} \exp\{i\bar{k}\bar{r}_t\} \times \exp\{(-i/\hbar)\, T(p_t)(\tau-t)\}. \tag{31}$$

Using (27) and shifting $\exp(i\bar{k}\bar{r}_t)$ to the right in Eq. (31), we find

$$\exp(i\bar{k}\bar{r}_\tau) = \exp[iT(\bar{p}_t)(\tau-t)/\hbar] \exp[-iT(\bar{p}_t-\hbar\bar{k})(\tau-t)/\hbar] \times \exp(i\bar{k}\bar{r}_t) = \exp[i\{T(\bar{p}_t) - T(\bar{p}_t-\hbar\bar{k})\}(\tau-t)/\hbar] \exp(i\bar{k}\bar{r}_t), \tag{32}$$

and also

$$\exp(i\bar{k}\bar{r}_\tau) = \exp(i\bar{k}\bar{r}_t) \exp[i\{T(\bar{p}_t+\hbar\bar{k}) - T(\bar{p}_t)\}(\tau-t)/\hbar]. \tag{33}$$

Making here the substitution $k \to -k$, we obtain

$$\exp[-i\bar{k}\bar{r}_\tau] = \exp(-i\bar{k}\bar{r}_t) \exp[i\{T(\bar{p}_t-\hbar\bar{k}) - T(\bar{p}_t)\}(\tau-t)/\hbar].$$

This "approximation" will be used in (28) only for the terms proportional to α. Namely, we substitute (32) and (33) into the expression of which the trace is taken and use the zeroth-order approximation as follows:

$$\left.\begin{aligned}
\mathscr{E}_{\text{app}} &= \{\operatorname*{Sp}_{(S,\Sigma)} \exp(i\bar{k}\bar{r}_\tau) \exp(-i\bar{k}\bar{r}_t)\, [f(\bar{p}_t-\hbar\bar{k}) - f(\bar{p}_t)]\, \mathscr{D}_{t0}\}_{\text{app}} = \operatorname*{Sp}_{(S,\Sigma)} \exp[i\{T(\bar{p}_t) \\
&\quad - T(\bar{p}_t-\hbar\bar{k})\}(\tau-t)/\hbar] \times [f(\bar{p}_t-\hbar\bar{k}) - f(\bar{p}_t)]\, \mathscr{D}_{t0};\\
\mathscr{E}^*_{\text{app}} &= \{\operatorname*{Sp}_{(S,\Sigma)} [f(\bar{p}_t-\hbar\bar{k}) - f(\bar{p}_t)] \exp(i\bar{k}\bar{r}_t) \exp(-i\bar{k}\bar{r}_\tau)\}_{\text{app}}\\
&= \operatorname*{Sp}_{(S,\Sigma)} [f(\bar{p}_t-\hbar\bar{k}) - f(\bar{p}_t)] \exp[i\{T(\bar{p}_t-\hbar\bar{k}) - T(\bar{p}_t)\}(\tau-t)/\hbar]\, \mathscr{D}_{t0}.
\end{aligned}\right\} \tag{34}$$

We should point out that these expressions are multiplied by $\mathscr{L}^2(k)$, which is proportional to the small parameter.

Thus, we assume that the terms of first order in α on the right-hand side of (28) are correctly estimated. This is in fact the approximation we wished to obtain. We shall then go to the limit $V \to \infty$, $t_0 \to -\infty$, and in the final result take $\varepsilon \to 0$. But first we expand the expressions given by Eqs. (34) for $\mathscr{E}_{\text{app}}$ and $\mathscr{E}^*_{\text{app}}$.

We turn to the relation

$$\operatorname*{Sp}_{(S,\Sigma)} F(\bar{p}_t)\, \mathscr{D}_{t0} = \int F(\bar{p})\, W_t(\bar{p})\, d\bar{p},$$

which holds for an arbitrary function $F(\bar{p})$ of the momentum. For $F(\bar{p})$, we choose

$$F(\bar{p}) = \exp[i\{T(\bar{p}) - T(\bar{p}-\hbar\bar{k})\}(\tau-t)/\hbar]\, [f(\bar{p}-\hbar\bar{k}) - f(\bar{p})],$$

and then

$$\mathscr{E}_{\text{app}} = \int d\bar{p} \exp[i\{T(\bar{p}) - T(\bar{p}-\hbar\bar{k})\}(\tau-t)/\hbar]\, [f(\bar{p}-\hbar\bar{k}) - f(\bar{p})]\, W_t(\bar{p}) = \int_{(\bar{p}\to\bar{p}+\hbar\bar{k})} d\bar{p} \exp[i\{T(\bar{p}+\hbar\bar{k}) - T(\bar{p})\}(\tau-t)/\hbar] \times f(\bar{p})\, W_t(\bar{p}+\hbar\bar{k}) - \int d\bar{p} \exp[i\{T(\bar{p}) - T(\bar{p}-\hbar\bar{k})\}(\tau-t)/\hbar] \times f(\bar{p})\, W_t(\bar{p}).$$

It is easy to see that $\mathscr{E}^*_{\text{app}}$ is the complex conjugate of $\mathscr{E}_{\text{app}}$:

$$\mathscr{E}^*_{\text{app}} = \int d\bar{p} \exp[i\{T(p) - T(p+\hbar\bar{k})\}(\tau-t)/\hbar]\, f(\bar{p})\, W_t(\bar{p}+\hbar\bar{k}) - \int d\bar{p} \exp[i\{T(\bar{p}-\hbar\bar{k}) - T(\bar{p})\}(\tau-t)/\hbar]\, f(\bar{p})\, W_t(\bar{p}). \tag{35}$$

We now substitute these relations in Eq. (28). Going to the limit $V \to \infty$, we replace the sums $(1/V)\sum_{(k)}(\ldots)$ by appropriate integrals $(2\pi)^3 \int dk(\ldots)$. In what follows, it is convenient also to make the transformation $\bar{k} \to -\bar{k}$ in the integrals containing $W_t(\bar{p})$. We also introduce the new variable of integration $t - \tau = \xi$, so that

$$\int_{t_0}^{t} d\tau(\ldots) = \int_0^{t-t_0} d\xi(\ldots).$$

In the limit $t_0 \to -\infty$, these integrals become $\int_0^\infty d\xi(\ldots)$.

Thus, we can write the equation in the first approximation in the form

$$\int d\bar{p} f(\bar{p}) \left\{\frac{\partial W_t(\bar{p})}{\partial t} - \exp(\varepsilon t)\, \bar{E}(t) \frac{\partial W_t(\bar{p})}{\partial \bar{p}}\right\} = \frac{\exp(2\varepsilon t)}{(2\pi)^3} \int d\bar{p} f(\bar{p}) \int d\bar{k} \frac{\mathscr{L}^2(k)}{2\hbar\omega_k} A_\varepsilon(\bar{p}, \bar{k}),$$

where

$$A_\varepsilon(\bar{p}, \bar{k}) = \int_0^\infty d\xi \exp(-\varepsilon\xi)\, ((1+N_k)\exp(i\omega_k\xi) + N_k \exp(-i\omega_k\xi))\, \{\exp(-i\xi\Delta_{p,k})\, W_t(\bar{p}+\hbar\bar{k}) - \exp(i\xi\Delta_{p,k})\, W_t(p)\} + \int_0^\infty d\xi \exp(-\varepsilon\xi)\, ((1+N_k)\exp(-i\omega_k\xi) + N_k \exp(i\omega_k\xi))\, \{\exp(i\xi\Delta_{p,k})\, W_t(\bar{p}+\hbar\bar{k}) - \exp(-i\xi\Delta_{p,k})\, W_t(\bar{p})\};$$

$$\Delta_{p,k} = [T(\bar{p}+\hbar\bar{k}) - T(\bar{p})]/\hbar.$$

Becuase $f(\bar{p})$ is an arbitrary function of the momentum $\bar{p}$, this equation reduces to

$$\frac{\partial W_t(p)}{\partial t} - \exp(\varepsilon t)\, \bar{E}(t) \frac{\partial W_t(\bar{p})}{\partial \bar{p}} = \frac{\exp(2\varepsilon t)}{(2\pi)^3} \int d\bar{k} \frac{\mathscr{L}^2(k)}{2\hbar\omega_k} A_\varepsilon(\bar{p}, \bar{k}). \tag{36}$$

Collecting the terms in the expression $A_\varepsilon(\bar{p}, \bar{k})$, we find

$$A_\varepsilon(\bar{p}, \bar{k}) = \{(1+N_k)\, W_t(\bar{p}+\hbar\bar{k}) - N_k W_t(\bar{p})\} \times \left\{\int_0^\infty \exp(-\varepsilon\xi) \exp[-i\xi(\Delta_{p,k} - \omega_k)]\, d\xi + \int_0^\infty \exp(-\varepsilon\xi) \exp[i\xi(\Delta_{p,k} - \omega_k)]\, d\xi\right\} + \{N_k W_t(\bar{p}+\hbar\bar{k}) - (1+N_k)\, W_t(\bar{p})\} \times \left\{\int_0^\infty \exp(-\varepsilon\xi) \exp[-i\xi(\Delta_{p,k} + \omega_k)]\, d\xi + \int_0^\infty \exp(-\varepsilon\xi) \exp[i\xi(\Delta_{p,k} + \omega_k)]\, d\xi\right\},$$

in which $N_k = \exp(-\beta\hbar\omega_k)/[1 - \exp(-\beta\hbar\omega_k)]$, or, in abbreviated form,

$$A_\varepsilon(\bar{p}, \bar{k}) = \frac{W_t(\bar{p}+\hbar\bar{k}) - \exp(-\beta\hbar\omega_k)\, W_t(\bar{p})}{1 - \exp(-\beta\hbar\omega_k)}\, \mathscr{D}_\varepsilon(\Delta_{p,k} - \omega_k) + \frac{W_t(\bar{p}+\hbar\bar{k}) \exp(-\beta\hbar\omega_k) - W_t(\bar{p})}{1 - \exp(-\beta\hbar\omega_k)}\, \mathscr{D}_\varepsilon(\Delta_{p,k} + \omega_k),$$

where

$$\mathscr{D}_\varepsilon(Z) = \int_{-\infty}^{+\infty} \exp(-\varepsilon|\xi|) \exp(i\xi Z)\, d\xi.$$

Note also that

$$\lim_{\varepsilon\to 0}\mathscr{D}_\varepsilon(\Delta_{p,k}\mp\omega_k)=2\pi\delta(\Delta_{p,k}\mp\omega_k)$$

$$=2\pi\delta([\hbar\Delta_{p,k}\mp\hbar\omega_k]/\hbar)=2\pi\hbar\delta(\hbar\Delta_{p,k}\mp\hbar\omega_k).$$

Therefore

$$\lim_{\varepsilon\to 0}A_\varepsilon(\bar p,\bar k)=\frac{2\pi\hbar}{1-\exp(-\beta\hbar\omega_k)}\{W_t(\bar p+\hbar\bar k)$$
$$-\exp(-\beta\hbar\omega_k)W_t(\bar p)\}\delta(T(\bar p+\hbar\bar k)-T(p)-\hbar\omega_k)$$
$$+\frac{2\pi\hbar}{1-\exp(-\beta\hbar\omega_k)}\{W_t(\bar p+\hbar\bar k)\exp(-\beta\hbar\omega_k)$$
$$-W_t(\bar p)\}\delta(T(\bar p+\hbar\bar k)-T(\bar p)+\hbar\omega_k).$$

We now make the final step, going to the limit $\varepsilon\to 0$ in Eq. (36). As a result, we obtain the equation of the first approximation in the final form

$$\frac{\partial W_t(\bar p)}{\partial t}-\bar E(t)\frac{\partial W_t(\bar p)}{\partial\bar p}=$$
$$=\frac{1}{(2\pi)^2}\int d\bar k\frac{\mathscr{L}^2(k)}{2\omega_k[1-\exp(-\beta\hbar\omega_k)]}\{W_t(\bar p+\hbar\bar k)$$
$$-\exp(-\beta\hbar\omega_k)W_t(\bar p)\}\delta(T(\bar p+\hbar\bar k)-T(\bar p)-\hbar\omega_k)$$
$$+\frac{1}{(2\pi)^2}\int d\bar k\frac{\mathscr{L}^2(k)}{2\omega_k[1-\exp(-\beta\hbar\omega_k)]}\{W_t(\bar p+\hbar\bar k)\exp(-\beta\hbar\omega_k)$$
$$-W_t(\bar p)\}\delta(T(\bar p+\hbar\bar k)-T(\bar p)+\hbar\omega_k).\tag{37}$$

Thus, we have obtained generalized Boltzmann equations.

We now consider the important special case

$$T(p)=p^2/(2m),$$

when the argument of the δ functions contains the expressions $\delta[(\bar p+\hbar\bar k)^2/(2m)-\bar p^2/(2m)\mp\hbar\omega_k]$.

It is then obvious that Eq. (37) will be the ordinary Boltzmann equation, the integral terms on the right-hand side corresponding to single-phonon emission and absorption. Such a Boltzmann equation has been intensively studied in the investigation of transport properties. To study the stationary state, when the electric field does not depend on the time, we have from (37)

$$-\bar E\frac{\partial W(\bar p)}{\partial\bar p}=\frac{1}{(2\pi)^2}\int d\bar k\frac{\mathscr{L}^2(k)}{2\omega_k[1-\exp(-\beta\hbar\omega_k)]}\{W(\bar p+\hbar\bar k)$$
$$-\exp(-\beta\hbar\omega_k)W(\bar p)\}\delta\left(\frac{(\bar p+\hbar\bar k)^2}{2m}-\frac{\bar p^2}{2m}-\hbar\omega_k\right)$$
$$+\frac{1}{(2\pi)^2}\int d\bar k\frac{\mathscr{L}^2(k)}{2\omega_k[1-\exp(-\beta\hbar\omega_k)]}\{W(\bar p+\hbar\bar k)\exp(-\beta\hbar\omega_k)$$
$$-W(\bar p)\}\delta\left(\frac{(\bar p+\hbar\bar k)^2}{2m}-\frac{\bar p^2}{2m}+\hbar\omega_k\right).\tag{38}$$

For the special case of low temperatures, the factor $\exp(-\beta\hbar\omega_k)$ in (38) can be ignored.

The equation obtained in this manner was considered in Ref. 2 by Devreese and Evard for the Fröhlich polaron model. They found a very complicated behavior of the distribution function $W(p)$, this evidently indicating an essential singularity at $E=0$.

In conclusion, we say some words about a simplified method of approximation to the determination of the dependence between the applied electric field and the mean equilibrium velocity $\bar v$ of the electron.

We multiply both sides of Eq. (38) by $\bar p$ and integrate over the complete momentum space. After simple transformations, we find

$$-\bar E=\frac{1}{(2\pi)^2}\int d\bar k\frac{\mathscr{L}^2(\bar k)\hbar\bar k}{2\omega_k[1-\exp(-\beta\hbar\omega_k)]}\int d\bar pW(\bar p)\delta\left(-\frac{(\hbar k)^2}{2m}\right.$$
$$\left.+\hbar\frac{\bar k\bar p}{m}-\hbar\omega_k\right)-\frac{1}{(2\pi)^2}\int d\bar k\frac{\mathscr{L}^2(k)\hbar\bar k}{2\omega_k[\exp(\beta\hbar\omega_k)-1]}$$
$$\times\int d\bar pW(\bar p)\delta\left(\frac{(\hbar k)^2}{2m}+\hbar\frac{\bar k\bar p}{m}-\hbar\omega_k\right).\tag{39}$$

Here, in accordance with the notation of Sec. 1,

$$\bar E=-e_c\bar{\mathscr{E}},\tag{40}$$

where $\mathscr{E}$ is the external electric field.

Thus, this is an exact consequence of the Boltzmann equation. We make a "rough approximation," choosing as a trial function for $W(\bar p)$ a "shifted" Maxwellian distribution function with mean velocity $\bar v$:

$$\bar W(\bar p)=\rho_M(\bar p-m\bar v);\quad\rho_M(\bar p)=\left(\frac{\beta}{2m\pi}\right)^{3/2}\exp(-\beta p^2/2m),$$

and we substitute this distribution in Eq. (39). This leads to the approximate equation

$$e_c\bar{\mathscr{E}}=\frac{1}{(2\pi)^2}\int d\bar k\frac{\mathscr{L}^2(k)\hbar\bar k}{2\omega_k[1-\exp(-\beta\hbar\omega_k)]}$$
$$\times\int d\bar p\rho_M(\bar p)\delta\left(-\frac{(\hbar k)^2}{2m}+\frac{\hbar\bar k\bar p}{m}-\hbar(\omega_k-\bar k\bar v)\right)$$
$$-\frac{1}{(2\pi)^2}\int d\bar k\frac{\mathscr{L}^2(k)\hbar\bar k}{2\omega_k[\exp(\beta\hbar\omega_k)-1]}$$
$$\times\int d\bar p\rho_M(\bar p)\delta\left(\frac{(\hbar k)^2}{2m}+\hbar\frac{\bar k\bar p}{m}-\hbar(\omega_k-\bar k\bar v)\right).\tag{41}$$

Note that

$$\delta\left(-\frac{(\hbar\bar k)^2}{2m}+\frac{\hbar\bar k\bar p}{m}-\hbar(\omega_k-\bar k\bar v)\right)$$
$$=\frac{1}{(2\pi)}\int_{-\infty}^{+\infty}\exp\left[i\left(\frac{(\hbar k)^2}{2m}-\frac{\hbar\bar k\bar p}{m}+\hbar(\omega_k-\bar k\bar v)\right)\xi\right]d\xi;$$

$$\delta\left(\frac{(\hbar k)^2}{2m}+\frac{\hbar\bar k\bar p}{m}-\hbar(\omega_k-\bar k\bar v)\right)$$
$$=\frac{1}{(2\pi)}\int_{-\infty}^{+\infty}\exp\left[i\left(\frac{(\hbar k)^2}{2m}+\frac{\hbar\bar k\bar p}{m}-\hbar(\omega_k-\bar k\bar v)\right)\xi\right]d\xi$$

and

$$\int\rho_M(\bar p)\exp(-i\xi\hbar\bar k\bar p/m)\,d\bar p=\exp[-(\hbar k)^2\xi^2/(2m\beta)];$$
$$\int\rho_M(\bar p)\exp(i\xi\hbar\bar k\bar p/m)\,d\bar p=\exp[-(\hbar k)^2\xi^2/(2m\beta)].$$

Therefore, from (41) we find

$$e_c\bar{\mathscr{E}}=\int_{-\infty}^{+\infty}d\xi\frac{1}{(2\pi)^3}\int\frac{\mathscr{L}^2(k)\hbar\bar k}{2\omega_k}\left\{\frac{\exp[i\hbar(\omega_k-\bar k\bar v)\xi]}{1-\exp(-\beta\hbar\omega_k)}\right.$$
$$\left.-\frac{\exp[-i\hbar(\omega_k-\bar k\bar v)\xi]}{\exp(\beta\hbar\omega_k)-1}\right\}\exp\left[-\frac{(\hbar k)^2}{2m}\left(\frac{\xi^2}{\beta}-i\xi\right)\right].\tag{42}$$

This approximation equation was obtained by Thornber and Feynman[3] for weak interactions.[1)] They found that the mobility obtained from (42) in the weak-coupling limit does not agree with the mobility obtained from the

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[1)] In their notation and in the system of units

$$\hbar=1,\quad C_k=(1/\sqrt{V})/(2\omega_k)^{1/2}\,\mathscr{L}(k);\ \bar E=e_c\bar{\mathscr{E}}$$

Eq. (42) has the form

$$\bar E=\int_{-\infty}^{+\infty}d\xi\sum_{(k)}|C_k|^2\bar k\left\{\frac{\exp[i(\omega_k-\bar k\bar v)\xi]}{1-\exp(-\beta\omega_k)}-\frac{\exp[-i(\omega_k-\bar k\bar v)\xi]}{\exp(\beta\omega_k)-1}\right\}$$
$$\times\exp\left[-\frac{k^2}{2m}\left(\frac{\xi^2}{\beta}-i\xi\right)\right].$$

This corresponds to Eq. (17) in Ref. 3.

standard treatment of the Boltzmann equation.

We see here that this discrepancy is due to the use of an inadequate approximation—the use as a trial momentum distribution function of the Maxwell distribution around the mean velocity $\overline{v}$ in Eq. (39), which is an exact consequence of the Boltzmann equation. The connection between Eq. (42) and the use of the Maxwell distribution as a trial function of the equilibrium distribution was also noted by Devreese (private communication).

3. LINEAR POLARON MODEL

We wish to show here that the results of Ref. 2, and also Ref. 3, relating to the calculation of the inpedance in the polaron model can be obtained directly without the use of functional integration.

We begin by considering the exact equation (38), in which as the arbitrary function $f(\overline{p})$ of the momenta we choose

$$f(\overline{p})=\overline{p}. \tag{43}$$

We denote the expectation value of the electron momentum by

$$\langle\overline{p}\rangle_t=\int\overline{p}W_t(\overline{p})\,d\overline{p}.$$

We introduce the notation

$$\underset{(S,\,\Sigma)}{\mathrm{Sp}}\exp[\mathrm{i}\overline{k}\overline{r}(\tau)]\exp[-\mathrm{i}\overline{k}\overline{r}(t)]\mathscr{D}_{t0}=\Phi_k(t,\tau,t_0), \tag{44}$$

and then

$$\underset{(S,\,\Sigma)}{\mathrm{Sp}}\exp[\mathrm{i}\overline{k}\overline{r}(t)]\exp[-\mathrm{i}\overline{k}\overline{r}(\tau)]\mathscr{D}_{t0}$$
$$=\mathrm{Sp}\{\exp[\mathrm{i}\overline{k}\overline{r}(\tau)]\exp[-\mathrm{i}\overline{k}\overline{r}(t)]\}^{+}\mathscr{D}_{t0}=\Phi_k^{*}(t,\tau,t_0).$$

It follows from Eq. (38), using (43) and (44), that

$$d\langle\overline{p}\rangle_t/dt+\exp(\varepsilon t)\overline{E}(t)$$
$$=-\frac{1}{V}\exp(2\varepsilon t)\sum_{k}\frac{\mathscr{L}^2(\overline{k})\,\overline{k}}{2\omega_k[1-\exp(-\beta\hbar\omega_k)]}\int_{t_0}^{t}d\tau\{\exp[\mathrm{i}\omega_k(t-\tau)]$$
$$+\exp[-\mathrm{i}\omega_k(t-\tau)]\exp[-\beta\hbar\omega_k]\}\exp[-\varepsilon(t-\tau)]\Phi_k(t,\tau,t_0)$$
$$-\frac{1}{V}\exp(2\varepsilon t)\sum_{(k)}\frac{\mathscr{L}^2(\overline{k})\,\overline{k}}{2\omega_k[1-\exp(-\beta\hbar\omega_k)]}$$
$$\times\int_{t_0}^{t}d\tau\exp[-\varepsilon(t-\tau)]\{\exp[-\mathrm{i}\omega_k(t-\tau)]$$
$$+\exp[\mathrm{i}\omega_k(t-\tau)]\exp[-\beta\hbar\omega_k]\}\Phi_k^{*}(t,\tau,t_0). \tag{45}$$

This is an exact relation. To obtain from it some approximate *equation*, we must find a corresponding approximation of the expressions $\Phi_k(t,\tau,t_0)$ explicitly. For this, it is expedient to introduce a model Hamiltonian leading to exactly solvable equations of motion. To obtain a suitable approximation, this model Hamiltonian must be chosen such that the behavior of $\overline{r}(t)$ resembles in some sense the behavior of $\overline{r}(t)$ corresponding to the exact Hamiltonian (5).

We consider the first case, when the external field is absent,

$$\overline{E}=0. \tag{46}$$

We take the Hamiltonian

$$H_L=\frac{p^2}{2m^*}+\frac{c^2r^2}{2}+\sum_{(k)}\hbar\nu(k)\,b_k^{+}b_k$$
$$+\frac{\mathrm{i}}{\sqrt{V}}\sum_{(k)}\left(\frac{\hbar}{2\nu(k)}\right)^{1/2}L(\overline{k})(\overline{k}\overline{r})(b_k+b_{-k}^{+}), \tag{47}$$

where $L(\overline{k})$ is a spherically symmetric function of $\overline{k}$; $\nu(\overline{k})$ is a spherically symmetric function and is also essentially positive: $\nu(\overline{k})>0$. Until we go to the limit $V\to\infty$, we assume that the volume V is finite, and that the number of terms $\mathfrak{N}_V$ in the sums over k is also finite. Then the corresponding Heisenberg equations of motion form a finite linear system of ordinary differential equations with constant coefficients and, thus, is exactly solvable, i. e.,

$$\left.\begin{aligned}
&d\overline{r}(t)/dt=\overline{p}(t)/m^{*};\\
&\frac{d\overline{p}(t)}{dt}=-c^2\overline{r}(t)-\frac{\mathrm{i}}{\sqrt{V}}\sum_{(k)}\left(\frac{\hbar}{2\nu(k)}\right)^{1/2}L(k)\,\overline{k}\{b_k(t)+b_{-k}^{+}(t)\};\\
&\frac{db_k(t)}{dt}=-\mathrm{i}\nu(k)\,b_k(t)-\frac{1}{\sqrt{V}}\left(\frac{1}{2\hbar\nu(k)}\right)^{1/2}L(k)(\overline{k}\overline{r}(t));\\
&\frac{db_{-k}^{+}(t)}{dt}=\mathrm{i}\nu(k)\,b_{-k}(t)+\frac{1}{\sqrt{V}}\left(\frac{1}{2\hbar\nu(k)}\right)^{1/2}L(k)(\overline{k}\overline{r}(t));\\
&\overline{r}(t_0)=\overline{r};\quad\overline{p}(t_0)=\overline{p};\quad b_k(t_0)=b_k;\quad b_{-k}^{+}(t_0)=b_{-k}^{+}.
\end{aligned}\right\} \tag{48}$$

We now show that for an appropriate choice of the constant c^2 the Hamiltonian (47) becomes translationally invariant. We begin with the identity

$$\sum_{(k)}\hbar\nu(k)\left\{b_k^{+}+\frac{\mathrm{i}(\overline{k}\overline{r})}{\sqrt{V}}\frac{L(k)}{\nu(k)\sqrt{2\hbar\nu(k)}}\right\}\left\{b_k-\frac{\mathrm{i}(\overline{k}\overline{r})}{\sqrt{V}}\frac{L(k)}{\nu(k)\sqrt{2\hbar\nu(k)}}\right\}$$
$$=\sum_{(k)}\hbar\nu(k)\,b_k^{+}b_k+\frac{\mathrm{i}}{\sqrt{V}}\sum_{k}\left(\frac{\hbar}{2\nu(k)}\right)^{1/2}L(k)(\overline{k}\overline{r})\,b_k$$
$$-\frac{\mathrm{i}}{\sqrt{V}}\sum_{(k)}\left(\frac{\hbar}{2\nu(k)}\right)^{1/2}L(k)(\overline{k}\overline{r})\,b_k^{+}+\frac{1}{V}\sum_{k}\frac{L^2(k)}{2\nu^2(k)}(\overline{k}\overline{r})^2$$

and note that by the spherical symmetry of the functions $L(k)$ and $\nu(k)$

$$\frac{1}{V}\sum_{(k)}\frac{L^2(k)}{\nu^2(k)}(\overline{k}\overline{r})^2=r^2\frac{1}{V}\sum_{(k)}\frac{L^2(k)}{3\nu^2(k)}k^2.$$

For this reason,

$$H_L=\frac{p^2}{2m^*}+\left(c^2-\frac{1}{V}\sum_{(k)}\frac{L^2(k)}{3\nu^2(k)}k^2\right)\frac{r^2}{2}$$
$$+\sum_{(k)}\hbar\nu(k)\left\{b_k^{+}+\frac{\mathrm{i}(\overline{k}\overline{r})L(k)}{\sqrt{V}\nu(k)\sqrt{2\hbar\nu(k)}}\right\}\left\{b_k-\frac{\mathrm{i}(\overline{k}\overline{r})L(k)}{\sqrt{V}\nu(k)\sqrt{2\hbar\nu(k)}}\right\}.$$

Therefore, if we take

$$c^2=\frac{1}{V}\sum_{(k)}\frac{L^2(k)k^2}{3\nu^2(k)}, \tag{49}$$

the Hamiltonian H_L becomes invariant under the translation group

$$\overline{r}\to\overline{r}+\overline{R};\quad b_k\to b_k+\frac{\mathrm{i}(\overline{k}\overline{R})L(k)}{\sqrt{V}\nu(k)(2\hbar\nu(k))^{1/2}}. \tag{50}$$

This invariance must lead to the existence of a conserved vector $\overline{\mathscr{P}}$:

$$d\overline{\mathscr{P}}/dt=0. \tag{51}$$

which can be regarded as a form of "total momentum."

To find the expression for $\overline{\mathscr{P}}$, we note that (48) implies

$$\frac{d(b_k(t)-b_{-k}^{+}(t))}{dt}=-\mathrm{i}\nu(k)(b_k(t)+b_{-k}^{+}(t))$$
$$-\frac{2}{\sqrt{V}}\left(\frac{1}{2\hbar\nu(k)}\right)^{1/2}L(k)(\overline{k}\overline{r}(t))$$

and hence

$$\frac{d}{dt}\frac{1}{\sqrt{V}}\sum_{(k)}L(k)\left(\frac{\hbar}{2\nu(k)}\right)^{1/2}\frac{b_k(t)-b_{-k}^{+}(t)}{\nu(k)}\overline{k}$$
$$=-\mathrm{i}\frac{1}{\sqrt{V}}\sum_{(k)}\left(\frac{\hbar}{2\nu(k)}\right)^{1/2}L(k)\,\overline{k}(b_k(t)+b_{-k}^{+}(t))$$
$$-\frac{1}{V}\sum_{(k)}\frac{L^2(k)}{\nu^2(k)}(\overline{k}\overline{r}(t))\,\overline{k}=\frac{d\overline{p}(t)}{dt}+c^2\overline{r}(t)-\frac{1}{V}\sum_{(k)}\frac{L^2(k)}{\nu^2(k)}(\overline{k}\overline{r}(t))\,\overline{k}.$$

But here, by virtue of (49),

$$\frac{1}{V}\sum_{(k)}\frac{L^2(k)}{\nu^2(k)}(\bar{k}\bar{r}(t))\,\bar{k}=\bar{r}(t)\,\frac{1}{V}\sum_{(k)}\frac{L^2(k)\,k^2}{3\nu^2(k)}=c^2\bar{r}(t),$$

and therefore

$$\frac{d}{dt}\left\{\bar{p}(t)-\frac{1}{\sqrt{V}}\sum_{(k)}\frac{\bar{k}L(k)}{\nu(k)}\left(\frac{\hbar}{2\nu(k)}\right)^{1/2}(b_k(t)-b^+_{-k}(t))\right\}=0.$$

It follows from the last expression that the conserved "total momentum" vector has the form

$$\bar{\mathscr{P}}=\bar{p}-\frac{1}{\sqrt{V}}\sum_{(k)}\frac{\bar{k}L(k)}{\nu(k)}\left(\frac{\hbar}{2\nu(k)}\right)^{1/2}(b_k-b^+_{-k}). \tag{52}$$

We now introduce an external electric field, replacing the Hamiltonian H_L by

$$\tilde{H}_L=H_L+\bar{E}(r)\,\bar{r}\,. \tag{53}$$

Since H_L commutes with $\mathscr{P}$ and since

$$[\mathscr{P}_\beta,\,r_\gamma]=[p_\beta,\,r_\gamma]=-i\hbar\delta_{\beta\gamma};\quad \beta,\,\gamma=1,\,2,\,3,$$

we see that

$$d\bar{\mathscr{P}}(t)/dt=-\bar{E}(t). \tag{54}$$

It is clear that for the Hamiltonian (5) under the condition (36) the translation group is determined by the transformation

$$\bar{r}\to\bar{r}+\bar{R};\quad b_k\to b_k\exp(-i\bar{k}\bar{R}). \tag{55}$$

In this situation, the "total momentum" is given by the expression

$$\bar{\mathscr{P}}=\bar{p}+\sum_{(k)}\hbar\bar{k}b^+_k b_k. \tag{56}$$

When the external field is switched on, $\mathscr{P}$ also satisfies Eq. (54).

We consider the Hamiltonian (53), the corresponding Heisenberg equations, and the initial conditions for the statistical operator $\mathscr{D}_t$. We use the same form of these conditions as in (10):

$$\mathscr{D}_{t0}=\rho(S)\,\mathscr{D}_L(\Sigma).$$

Only now, naturally, do we take the operator for the statistical-equilibrium model (Σ) system:

$$\mathscr{D}_L(\Sigma)=\text{const}\exp\left[-\beta\sum_{(k)}\hbar\nu(k)\,b^+_k b_k\right].$$

The equations of motion for the model ($S+\Sigma$) system are

$$m^*\,d\bar{r}(t)/dt=\bar{p}(t);$$

$$\frac{d\bar{p}(t)}{dt}=-c^2\bar{r}(t)-\frac{i}{\sqrt{V}}\sum_{(k)}\left(\frac{\hbar}{2\nu(k)}\right)^{1/2}L(k)\,\bar{k}\,(b_k(t)+b^+_{-k}(t))-\bar{E}(t); \tag{57}$$

$$\frac{db_k(t)}{dt}=-i\nu(k)\,b_k(t)-\frac{1}{\sqrt{V}}\left(\frac{1}{2\hbar\nu(k)}\right)^{1/2}L(k)\,(\bar{k}\bar{r}(t));$$

$$\frac{db^+_{-k}(t)}{dt}=i\nu(k)\,b^+_{-k}(t)+\frac{1}{\sqrt{V}}\left(\frac{1}{2\hbar\nu(k)}\right)^{1/2}L(k)\,(\bar{k}\bar{r}(t)); \tag{58}$$

$$\bar{r}(t_0)=\bar{r};\quad \bar{p}(t_0)=\bar{p};\quad b_k(t_0)=b_k;\quad b^+_{-k}(t_0)=b^+_{-k},$$

from which it follows that

$$b_k(t)=b_k\exp[-i\nu(k)(t-t_0)]-\frac{1}{\sqrt{V}}\left(\frac{1}{2\hbar\nu(k)}\right)^{1/2}L(k)\int_{t_0}^{t}\exp[-i\nu(k)(t-\tau)]\,(\bar{k}\bar{r}(\tau))\,d\tau;$$

$$b^+_{-k}(t)=b^+_{-k}\exp[i\nu(k)(t-t_0)]+\frac{1}{\sqrt{V}}\left(\frac{1}{2\hbar\nu(k)}\right)^{1/2}L(k)\int_{t_0}^{t}\exp[i\nu(k)(t-\tau)]\,(\bar{k}\bar{r}(\tau))\,d\tau.$$

We substitute these expressions in Eq. (57) and obtain

$$\frac{d\bar{p}(t)}{dt}+c^2\bar{r}(t)+\frac{i}{\sqrt{V}}\sum_{k}\frac{L^2(k)\,\bar{k}}{2\nu(k)}\int_{t_0}^{t}d\tau\,(\bar{k}\bar{r}(\tau))\{\exp[i\nu(k)(t-\tau)]-\exp[-i\nu(k)(t-\tau)]\}$$
$$=-\frac{i}{\sqrt{V}}\sum_{(k)}\left(\frac{\hbar}{2\nu(k)}\right)^{1/2}L(k)\,\bar{k}\,(b_k\exp[-i\nu(t-t_0)]+b^+_{-k}\exp[i\nu(k)(t-t_0)])-\bar{E}(t).$$

We integrate by parts:

$$i\int_{t_0}^{t}d\tau\,(\bar{k}\bar{r}(\tau))\{\exp[i\nu(k)(t-\tau)]-\exp[-i\nu(k)(t-\tau)]\}$$
$$=-\frac{1}{\nu(k)}\int_{t_0}^{t}(\bar{k}\bar{r}(\tau))\frac{d}{d\tau}\{\exp[i\nu(k)(t-\tau)]+\exp[-i\nu(k)(t-\tau)]\}\,d\tau$$
$$=-2\,\frac{(\bar{k}\bar{r}(t))}{\nu(k)}+\frac{2\,(\bar{k}\bar{r})}{\nu(k)}\cos\nu(k)(t-t_0)+\frac{2}{\nu(k)}\int_{t_0}^{t}d\tau\left(\bar{k}\,\frac{d\bar{r}(\tau)}{d\tau}\right)\cos\nu(k)(t-\tau)$$

and recall that

$$-\frac{1}{V}\sum_{(k)}\frac{L^2(k)\,\bar{k}}{\nu^2(k)}(\bar{k}\bar{r}(t))=-\frac{1}{V}\sum_{(k)}\frac{L^2(k)\,k^2}{3\nu^2(k)}\bar{r}(t)=-c^2\bar{r}(t);$$

$$\frac{1}{V}\sum_{(k)}\frac{L^2(k)\,\bar{k}}{\nu^2(k)}\left(\bar{k}\,\frac{d\bar{r}(\tau)}{d\tau}\right)\cos\nu(k)(t-\tau)=\frac{1}{V}\sum_{(k)}\frac{L^2(k)\,k^2}{3\nu^2(k)}\cos\nu(k)(t-\tau)\,\frac{d\bar{r}(\tau)}{d\tau}.$$

We then obtain

$$\frac{d\bar{p}(t)}{dt}+\frac{1}{m^*}\int_{t_0}^{t}d\tau K(t-\tau)\,\bar{p}(\tau)=-\bar{r}K(t-t_0)$$
$$-\frac{i}{\sqrt{V}}\sum_{(k)}\left(\frac{\hbar}{2\nu(k)}\right)^{1/2}L(k)\,\bar{k}\,(b_k\exp[-i\nu(k)(t-t_0)]+b^+_{-k}\exp[i\nu(k)(t-t_0)])-\bar{E}(t), \tag{59}$$

where

$$K(t-\tau)=\frac{1}{V}\sum_{(k)}\frac{L^2(k)\,k^2}{3\nu^2(k)}\cos\nu(k)(t-\tau). \tag{60}$$

We consider the averaging of this equation with respect to the initial statistical operator

$$\mathscr{D}_{t0}=\rho(S)\,\mathscr{D}(\Sigma) \tag{61}$$

and denote

$$m^*\langle\bar{v}(t)\rangle=\langle\bar{p}(t)\rangle=\underset{(S,\,\Sigma)}{\mathrm{Sp}}\,\bar{p}(t)\,\mathscr{D}_{t0};$$

$$\langle\bar{r}\rangle=\underset{(S,\,\Sigma)}{\mathrm{Sp}}\,\bar{r}\mathscr{D}_{t0}=\underset{(S)}{\mathrm{Sp}}\,\bar{r}\rho(s).$$

Since $\langle b_k\rangle=0$, it follows that $\langle b^+_{-k}\rangle=0$. Equation (59) leads to

$$m^*\frac{d\langle\bar{v}(t)\rangle}{dt}+\int_{t_0}^{t}d\tau\,\langle\bar{v}(\tau)\rangle\,K(t-\tau)=-\langle\bar{r}\rangle\,K(t-t_0)-\bar{E}(t). \tag{62}$$

Here, $\langle\bar{v}(t)\rangle$ is the mean particle velocity.

We now consider the situation when $E(t)$ is a periodic function of t multiplied by the factor $\exp(\varepsilon t)$ ($\varepsilon>0$), which corresponds to the switching-on of the external electric field in the limit $t\to-\infty$. We shall seek stationary solutions of (62), i. e., solutions represented by the product of the factor $\exp(\varepsilon t)$ and a periodic function.

Since Eq. (62) is a linear equation, we can restrict the treatment to the simplest expression

$$\bar{E}(t)=\bar{E}_\omega\exp[(-i\omega+\varepsilon)\,t]. \tag{63}$$

Indeed, if $E(t)$ were a sum of such terms with different frequencies ω, the resulting stable solution of Eq. (62) would have to be a sum of solutions of the form (63).

Thus, we consider the equation

$$m^* \frac{d\langle \overline{v}(t)\rangle}{dt} + \int_{-\infty}^{t} d\tau \langle \overline{v}(\tau)\rangle K(t-\tau) = -\overline{E}_\omega \exp[(-i\omega+\varepsilon)t].$$

Substituting $\langle \overline{v}(t)\rangle = \overline{v}_\omega \exp[(-i\omega+\varepsilon)t]$, we obtain

$$\left\{m^*(-i\omega+\varepsilon) + \int_0^\infty K(t)\exp[(i\omega-\varepsilon)t]\,dt\right\}\overline{v}_\omega = -\overline{E}_\omega.$$

The definition (60) leads to

$$\int_0^\infty K(t)\exp[(i\omega-\varepsilon)t]\,dt$$

$$= \frac{1}{V}\sum_{(k)} \frac{L^2(k)k^2}{6v^2(k)}\left\{\frac{1}{\varepsilon - i(\omega+v(k))} + \frac{1}{\varepsilon - i(\omega - v(k))}\right\}.$$

We denote

$$\frac{1}{V}\sum_{(k)} \frac{L^2(k)k^2}{6v^2(k)}\{\delta(v(k)-\Omega) + \delta(v(k)+\Omega)\} = I(\Omega), \quad (64)$$

and then $I(-\Omega) = I(\Omega)$, $I(\Omega) \geqslant 0$, and

$$\int_0^\infty K(t)\exp[(i\omega-\varepsilon)t]\,dt = i\int_{-\infty}^{+\infty} I(\Omega)\frac{d\Omega}{\omega + i\varepsilon - \Omega}. \quad (65)$$

Therefore

$$\left\{m^*(-i\omega+\varepsilon) + i\int_{-\infty}^{+\infty} I(\Omega)\frac{d\Omega}{\omega+i\varepsilon-\Omega}\right\}\langle\overline{v}(t)\rangle$$

$$= -\overline{E}_\omega \exp[(-i\omega+\varepsilon)t].$$

By virtue of (52), $\overline{E}_\omega = -e_c\overline{\mathscr{E}}_\omega$, and by the definition of the current

$$j_\omega(t) = -e_c\langle\overline{v}(t)\rangle.$$

Hence

$$\left\{m^*(-i\omega+\varepsilon) + i\int_{-\infty}^{+\infty} I(\Omega)\frac{d\Omega}{\omega+i\varepsilon-\Omega}\right\} j_\omega(t)$$

$$= e_c^2\mathscr{E}_\omega \exp[(-i\omega+\varepsilon)t]. \quad (66)$$

We now go to the limit $V \to \infty$, assuming that for any real ω and positive ε

$$\int_{-\infty}^{+\infty} I(\Omega)\frac{d\Omega}{\omega+i\varepsilon-\Omega} \to \int_{-\infty}^{+\infty} J(\Omega)\frac{d\Omega}{\omega+i\varepsilon-\Omega}. \quad (67)$$

After going to the indicated limit, we take $\varepsilon \to 0$ in Eq. (66), and we obtain

$$j_\omega(t) = \frac{1}{Z_+(\omega)} e_c^2\mathscr{E}_\omega \exp(-i\omega t),$$

where

$$Z_+(\omega) = -m^* i\omega + i\int_{-\infty}^{+\infty} J(\Omega)\frac{d\Omega}{\omega - \Omega + i0}. \quad (68)$$

Choosing here the electron charge e_c as the unit, we see that the expression (68) is the impedance corresponding to the frequency $-\omega$.

As we shall see later *in connection with the process of the passage to the limit*, all the expressions *that we shall use*, *including (44)*, depend only on *the function $J(\Omega)$ and not on the particular choice of the functions $v(k)$ and $L(k)$*.

Therefore, we first choose an appropriate expression for $J(\Omega)$. We choose $J(\Omega)$ to satisfy the following conditions:

1) $J(\Omega)$ is an analytic function of its complex variable, regular in the strip $|\mathrm{Im}\,\Omega| \leqslant \eta_0$;

2) $J(\Omega) = J(-\Omega)$;

3) $|J(\Omega)| \leqslant C|\Omega|^2$ for $|\Omega| \geqslant \omega_0$, where ω_0 and C are constants;

4) for real Ω, $J(\Omega) > 0$. (69)

Further, we choose the expressions for $L)k)$ and $v(k)$ such that[2] $v(k) > 0$ and

$$\frac{1}{V}\sum_{v(k)\geqslant\omega} \frac{L^2(k)k^2}{6v^2(k)} < \frac{C_1}{\omega} \ (C_1 \text{ is a } V\text{-independent constant}); \quad (70)$$

$$\frac{1}{V}\sum_{v(k)<\omega} \frac{L^2(k)k^2}{6v^2(k)} \to \int_0^\omega J(\Omega)\,d\Omega, \quad 0<\omega<\infty. \quad (71)$$

In the considered situation, it is clear that the relation (67) holds for any fixed $\varepsilon > 0$ and that the convergence here is uniform with respect to ω (in the interval $-\infty < \omega < +\infty$).

We introduce a function of the complex variable W:

$$\Delta(W) = i\int_{-\infty}^{+\infty} J(\Omega)\,d\Omega/(W-\Omega). \quad (72)$$

We see that it is regular for $|\mathrm{Im}\,W| > 0$. Using the properties (69)–(71), we readily see that

$$\Delta(W) = \lim_{(V\to\infty)} i\int_{-\infty}^{+\infty} I(\Omega)\,d\Omega/(W-\Omega), \quad \mathrm{Im}\,W \neq 0. \quad (73)$$

Here, by virtue of (64),

$$i\int_{-\infty}^{+\infty} J(\Omega)\frac{d\Omega}{(W-\Omega)} = \frac{i}{V}\sum_{(k)}\frac{L^2(k)k^2}{6v^2(k)}\left(\frac{1}{W-v(k)} + \frac{1}{W+v(k)}\right),$$

and therefore this function is analytic in the entire complex plane and has poles on the real axis at $W = \pm v(k)$ as singularities. However, the limit function has a cut along the entire real axis: $\Delta(\omega+i0) - \Delta(\omega - i0) = 2\pi J(\omega) > 0$.

Thus, we have two analytic functions

$$\Delta_+(W) = i\int_{-\infty}^{+\infty} J(\Omega)\,d\Omega/(W-\Omega) \quad \text{for} \quad \mathrm{Im}\,W \geqslant 0;$$

$$\Delta_-(W) = i\int_{-\infty}^{+\infty} J(\Omega)\,d\Omega/(W-\Omega) \quad \text{for} \quad \mathrm{Im}\,W \leqslant 0. \quad (74)$$

By virtue of the condition 4 in (69), these functions are related in a simple manner to each other:

$$\Delta_-(W) = -\Delta_+(-W) \quad \text{for} \quad \mathrm{Im}\,W < 0. \quad (75)$$

Therefore, we need to investigate only one of them, for example, $\Delta_+(W)$. We denote

$$\mathrm{Re}\,W = \omega, \quad \mathrm{Im}\,W = y > 0. \quad (76)$$

Then for any fixed $\omega_1 > 0$

[2]One of the possibilities for finding such expressions for the functions $L(k)$ and $v(k)$ is as follows. We choose $\overline{k} = (2\pi n_1/L,\ 2\pi n_2/L,\ 2\pi n_3/L$, where $L^3 = V$ (n_1, n_2, n_3) are positive and negative integers and it is assumed that $n_1^2+n_2^2+n_3^2 \neq 0$. This eliminates the appearance of a zero value for k in the sums over $\overline{k}$. We then set $v(\overline{k}) = S|\overline{k}|$ and $L^2(\overline{k}) = [2\pi^2(S^3)/|k|^2] \times J(S|\overline{k}|)$, where S is a positive constant that does not depend on V.

$$\Delta_+(\omega+iy)=i\int\limits_{|\Omega-\omega|>\omega_1} J(\Omega)\frac{d\Omega}{\omega+iy-\Omega}$$

$$+i\int\limits_{\omega-\omega_1}^{\omega+\omega_1} J(\Omega)\frac{\omega-\Omega-iy}{(\omega-\Omega)^2+y^2}\,d\Omega,$$

but

$$\int\limits_{\omega-\omega_1}^{\omega+\omega_1}\frac{\omega-\Omega}{(\omega-\Omega)^2+y^2}\,d\Omega=-\int\limits_{-\omega_1}^{\omega_1}\frac{\Omega}{\Omega^2+y^2}\,d\Omega=0$$

and thus

$$\Delta_+(\omega+iy)=i\int\limits_{|\Omega-\omega|>\omega_1} J(\Omega)\frac{d\Omega}{\omega+iy-\Omega}$$

$$+i\int\limits_{\omega-\omega_1}^{\omega+\omega_1}\frac{J(\Omega)-J(\omega)}{(\omega-\Omega)^2+y^2}(\omega-\Omega)\,d\Omega+\int\limits_{\omega-\omega_1}^{\omega+\omega_1} J(\Omega)\frac{y}{(\omega-\Omega)^2+y^2}\,d\Omega. \quad (77)$$

It follows that

$$\Delta_+(\omega)=\lim_{y\to 0}\Delta_+(\omega+iy)=i\int\limits_{|\Omega-\omega|>\omega_1} J(\Omega)\frac{d\Omega}{\omega-\Omega}$$

$$+i\int\limits_{\omega-\omega_1}^{\omega+\omega_1}\frac{J(\Omega)-J(\omega)}{\omega-\Omega}\,d\Omega+\pi J(\omega). \quad (78)$$

Thus, $\Delta_+(\omega)$ is also an analytic function on the real axis. Using (77), we can readily show that

$$|\Delta_+(\omega)|\leqslant \text{const}/|W|,\quad |W|\to\infty. \quad (79)$$

Further

$$\Delta_+(\omega)=\Delta_-(\omega)+2\pi J(\omega)=-\Delta_+(-\omega)+2\pi J(\omega).$$

By virtue of the condition 1 in (69), the function $-\Delta_+(-W)+2\pi J(\omega)$ is analytic in the region

$$0\geqslant \operatorname{Im} W\geqslant -\eta_0. \quad (80)$$

Since it is equal to $\Delta_+(W)$ on the real axis, we see that $\Delta_+(W)$, which was previously defined for $\operatorname{Im} W>0$, can be analytically continued into the region (80).

Thus, we can write

$$\Delta_+(W)=-\Delta_+(W)+2\pi J(W)\quad\text{for}\quad 0\geqslant \operatorname{Im} W\geqslant -\eta_0. \quad (81)$$

We can show that the inequality (76) holds everywhere for

$$\operatorname{Im} W\geqslant -\eta_0. \quad (82)$$

We now consider the impedance function $Z_+(W) = -im^*W+\Delta_+(W)$ in the region (82) and note that in the upper half-plane and on the real axis it has no zeros.

Taking into account (77), we have $\operatorname{Re} Z_+(W)>0$ for $\operatorname{Im} W\geqslant 0$. Therefore, the zeros of the last function in the considered region (82) must, if they exist, all be contained in the region (80). But $\Delta_+(W)\to 0$ as $|W|\to\infty$, and for this reason the zeros of the function $Z_+(W)$ can found only in the restricted region

$$|\operatorname{Re} W|\leqslant \text{const},\quad 0\geqslant \operatorname{Im} W\geqslant -\eta_0. \quad (83)$$

As is well known, an analytic function can have only a finite number of zeros in such a bounded region. If there really are zeroes in (74), we choose $\eta>0$ such that $-\eta$ is larger than the imaginary parts of these zeros. If, on the other hand, the region (83) does not contain zeros of the function $Z_+(W)$ at all, we choose $\eta=N_0$. In either case, we see that, by choosing $\eta>0$ appropriately, we can always ensure that the region

$$\operatorname{Im} W\geqslant -\eta \quad (84)$$

contains no zeros of the impedance function $Z_+(W)$. Therefore, the admittance function $1/Z_+(W)$ is a regular analytic function in the region (84). Its behavior at infinity is determined by the relation

$$\frac{1}{Z_+(W)}=\frac{1}{-im^*W+\Delta_+(W)}=-\frac{1}{imW}$$

$$+\frac{\Delta_+(W)}{im^*W(-im^*W+\Delta_+(W))}=-\frac{1}{im^*W}+O\left(\frac{1}{W^3}\right),\quad |W|\to\infty. \quad (85)$$

We conclude by giving an example We take

$$\Delta_+(W)=i\frac{k_0^2}{2}\left\{\frac{1}{W-\nu_0+i\gamma}+\frac{1}{W+\nu_0+i\gamma}\right\},\quad \gamma>0,\quad \operatorname{Im} W>-\gamma;$$

$$\Delta_-(W)=-\Delta_+(-W)=i\frac{k_0^2}{2}\left\{\frac{1}{W+\nu_0-i\gamma}+\frac{1}{W-\nu_0-i\gamma}\right\},\quad \operatorname{Im} W<\gamma.$$

Then

$$J(\omega)=\frac{1}{(2\pi)}\{\Delta_+(\omega)-\Delta_-(\omega)\}$$

$$=\frac{k_0^2}{2\pi}\left\{\frac{\gamma}{(\omega-\nu_0)^2+\gamma^2}+\frac{\gamma}{(\omega+\nu_0)^2+\gamma^2}\right\}. \quad (86)$$

In this example, all our conditions are satisfied. A similar result would also be obtained if we considered not just the single term (86), but a finite sum of such terms.

After these rather lengthy considerations about the analyticity of the impedance and admittance functions, we return to our fundamental equation (59), in which we take

$$\overline{E}(t)=\sum_{\omega}\overline{E}_\omega\exp(-i\omega t). \quad (87)$$

We solve it by means of a Laplace transformation. Thus, we multiply both sides of (59) by the factor

$$\exp(iWt),\quad W=\Omega+i\delta \quad (88)$$

and integrate over t:

$$\int\limits_{t_0}^{\infty} dt\exp(iWt)\frac{d\overline{p}(t)}{dt}+\frac{1}{m^*}\int\limits_{t_0}^{\infty} dt\exp(iWt)\int\limits_{t_0}^{t} d\tau K(t-\tau)\overline{p}(\tau)$$

$$=-\overline{r}\int\limits_{t_0}^{\infty} dt\exp(iWt)K(t-t_0)-\sum_{(\omega)}\overline{E}_\omega\int\limits_{0}^{\infty}\exp[i(W-\omega)t]\,dt$$

$$-\frac{i}{\sqrt{\overline{V}}}\sum_{(k)}\left(\frac{\hbar}{2\nu(k)}\right)^{1/2}L(k)\overline{k}\left\{b_k\int\limits_{t_0}^{\infty} dt\exp[i(W-\nu(k))t]\right. \quad (89)$$

$$\left.\times\exp[i\nu(k)t_0]+b^+_{-k}\int\limits_{t_0}^{\infty} dt\exp[i(W+\nu(k))t]\exp[-i\nu(k)t_0]\right\}.$$

But

$$\int\limits_{t_0}^{\infty} dt\exp(iWt)\frac{d\overline{p}(t)}{dt}=-\overline{p}\exp(iWt_0)-iw\int\limits_{t_0}^{\infty} dt\exp(iWt)\overline{p}(t);$$

$$\int\limits_{t_0}^{\infty} dt\exp(iWt)\int\limits_{t_0}^{t} d\tau K(t-\tau)\overline{p}(\tau)=\int\limits_{0}^{\infty} K(t)\exp(iWt)dt$$

$$\times\int\limits_{t_0}^{\infty}\exp(iWt)\overline{p}(t)\,dt,$$

and therefore

$$\frac{1}{m^*}\left\{-im^*W+\int\limits_{0}^{\infty} K(t)\exp(iWt)\,dt\right\}\int\limits_{t_0}^{\infty}\overline{p}(t)\exp(iWt)\,dt$$

$$=\overline{p}\exp(iWt_0)-\overline{r}\exp(iWt_0)\int\limits_{0}^{\infty} K(t)\exp(iWt)\,dt$$

$$+\sum_{(\omega)}\overline{E}_\omega\frac{\exp[i(W-\omega)t_0]}{i(W-\omega)}+\frac{1}{\sqrt{\overline{V}}}\sum_{(k)}\left(\frac{\hbar}{2\nu(k)}\right)^{1/2}L(k)\overline{k}$$

$$\times\left\{\frac{b_k\exp(iWt_0)}{W-\nu(k)}+\frac{b^+_{-k}\exp(iWt_0)}{W+\nu(k)}\right\}.$$

Hence, by virtue of (65),

$$\int_0^\infty K(t)\exp(iWt)dt = i\int_{-\infty}^{+\infty} I(\nu)\frac{d\nu}{W-\nu}.$$

We denote

$$\left.\begin{aligned} &-im^*W + i\int_{-\infty}^{+\infty} I(\nu)\frac{d\nu}{W-\nu} = Z^{(V)}(W);\\ &i\int_{-\infty}^{+\infty} I(\nu)\frac{d\nu}{W-\nu} = \Delta^{(V)}(W), \end{aligned}\right\} \tag{90}$$

and we then obtain

$$\int_{t_0}^{\infty}\overline{p}(t)\exp(iWt)\,dt = \frac{m^*\overline{p}\exp(iWt_0)}{Z^{(V)}(W)} - m^* r\frac{\Delta^{(V)}(W)}{Z^{(V)}(W)}\exp(iWt_0)$$

$$-i\sum_{(\omega)} m^*\overline{E}_\omega\frac{\exp(iWt_0)\exp(-iWt_0)}{(W-\omega)Z^{(V)}(W)}$$

$$+\frac{1}{\sqrt{V}}\sum_{(k)} m^*\left(\frac{\hbar}{2\nu(k)}\right)^{1/2}\frac{\mathcal{L}(k)\,\overline{k}\exp(iWt_0)}{Z^{(V)}(W)}\left\{\frac{b_k}{W-\nu(k)}+\frac{b^+_{-k}}{W+\nu(k)}\right\}. \tag{91}$$

Since

$$f(t) = \frac{1}{(2\pi)}\int_{-\infty}^{+\infty}\exp[(\delta - i\Omega)t]\left\{\int_{t_0}^{\infty} f(t)\exp[(i\Omega-\delta)t]\,dt\right\}d\Omega.$$

$$t > t_0.$$

Therefore, using the notation

$$\left.\begin{aligned} &\frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{\exp[(\delta-i\Omega)(t-t_0)]}{(\Omega+i\delta-\nu)Z^{(V)}(\Omega+i\delta)}d\Omega = f(\nu,\ \delta,\ t-t_0);\\ &\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\exp[(\delta-i\Omega)(t-t_0)]}{Z^{(V)}(\Omega+i\delta)}d\Omega\\ &= \frac{1}{2\pi m^*}\int_{-\infty}^{+\infty}\left\{-\frac{1}{i\Omega-\delta}+\frac{\Delta^{(V)}(\Omega+i\delta)}{(i\Omega-\delta)Z^{(V)}(\Omega+i\delta)}\right\}\\ &\times\exp[(\delta-i\Omega)(t-t_0)]\,d\Omega = g_0(\delta,\ t-t_0);\\ &\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\Delta^{(V)}(\Omega+i\delta)}{Z^{(V)}(\Omega+i\delta)}\exp[(\delta-i\Omega)(t-t_0)]\,d\Omega\\ &= g_1(\delta,\ t-t_0), \end{aligned}\right\} \tag{92}$$

we obtain from (91)

$$\left.\begin{aligned} &\overline{p}(t) = \overline{p}^{(S)}(t)+\overline{p}^{(E)}(t)+\overline{p}^{(\Sigma)}(t);\\ &\overline{p}^{(S)}(t) = m^*\overline{p}g_0(\delta,\ t-t_0) - m^*\overline{r}g_1(\delta,\ t-t_0);\\ &\overline{p}^{(E)}(t) = -m^*\sum_{(\omega)}\overline{E}_\omega f(\omega,\ \delta,\ t-t_0)\exp(-i\omega t_0);\\ &\overline{p}^{(\Sigma)}(t) = \frac{-im^*}{\sqrt{V}}\sum_{(k)}\left(\frac{\hbar}{2\nu(k)}\right)^{1/2} L(k)\,\overline{k}\{b_k f(\nu(k),\ \delta,\ t-t_0)\\ &\qquad + b^+_{-k}f(-\nu(k),\ \delta,\ t-t_0)\}. \end{aligned}\right. \tag{93}$$

It should be emphasized that the functions (92) depend essentially on V. By virtue of our choice, which leads to the conditions (69)–(71), we can go to the limit $V \to \infty$. Hitherto, δ has been an arbitrary positive quantity.

We now choose

$$\delta = \eta/2. \tag{94}$$

On the other hand, it is easy to see that

$$\left.\begin{aligned} &g_0(\delta,\ t-t_0) \to \Psi_0(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\exp[(\delta-i\Omega)(t-t_0)]}{Z_+(\Omega+i\delta)}d\Omega;\\ &g_1(\delta,\ t-t_0) \to \Psi_1(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\Delta_+(\Omega+i\delta)}{Z_+(\Omega+i\delta)}\\ &\qquad\times\exp[(\delta-i\Omega)(t-t_0)]\,d\Omega;\\ &f(\nu,\ \delta,\ t-t_0) \to \Phi(\nu,\ t-t_0)\\ &= \frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{\exp[(\delta-i\Omega)(t-t_0)]\,d\Omega}{(\Omega+i\delta-\nu)Z_+(\Omega+i\delta)}\quad \text{as}\quad V\to\infty. \end{aligned}\right\} \tag{95}$$

Using the identity $1/Z(\Omega+i\delta) \equiv 1/(\delta - i\Omega) + \Delta(\Omega+i\delta)/[(i\Omega-\delta)Z(\Omega+i\delta)]$ and δ fixed by (94), we can show that the convergence

$$\begin{aligned} &|f(\nu,\ \delta,\ t-t_0) - \Phi(\nu,\ t-t_0)| \to 0;\\ &\nu|f(\nu,\ \delta,\ t-t_0) - \Phi(\nu,\ t-t_0)| \to 0,\quad V\to\infty \end{aligned} \tag{96}$$

is uniform with respect to real ν when $|t-t_0| \leqslant T$, where T is a constant that does not depend on V.

We now study the behavior of the limit functions Ψ_0, Ψ_1, and Φ for $t - t_0 \to \infty$. It is appropriate to recall that the functions $\Delta_+(\Omega+i\delta)$ and $1/Z_+(\Omega+i\delta)$ are regular analytic functions of the variable Ω in the region where $\mathrm{Im}\,\Omega \geqslant -\delta-\eta = -3\delta$. Therefore, the integration performed in the expressions Ψ_0 and Ψ_1 can be shifted from the real axis to the axis $(-3i\delta-\infty, -3i\delta+\infty)$ by the change of variables $\Omega \to \Omega - 3i\delta$. Therefore

$$\Psi_1(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\Delta_+(\Omega-i\eta)}{Z_+(\Omega-i\eta)}\exp[-i\Omega(t-t_0)]\,d\Omega\exp[-\eta(t-t_0)];$$

$$\Psi_0(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\exp[-i\Omega(t-t_0)]}{Z_+(\Omega-i\eta)}d\Omega\exp[-\eta(t-t_0)]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\Delta_+(\Omega-i\eta)}{(\eta+i\Omega)Z_+(\Omega-i\eta)}\exp[-i\Omega(t-t_0)]\,d\Omega\exp[-\eta(t-t_0)],$$

since

$$\int_{-\infty}^{+\infty}\frac{\exp[-i\Omega(t-t_0)]}{\eta+i\Omega}d\Omega = 0,\quad \text{for}\quad t > t_0.$$

Using the previously formulated inequalities we obtain

$$\begin{aligned} &|\Psi_1(t-t_0)| \leqslant K_1\exp[-\eta(t-t_0)];\\ &|\Psi_0(t-t_0)| \leqslant K_0\exp[-\eta(t-t_0)];\quad t > t_0, \end{aligned} \tag{97}$$

where K_0 and K_1 are constants. We apply a similar procedure to the expression $\Phi(\nu, t-t_0)$. We need only note that in the region $\mathrm{Im}\,\Omega + \delta \geqslant -\eta$ the function in the integrand of (95) has a pole at $\Omega = \nu - i\delta$. Hence

$$\Phi(\nu,\ t-t_0) = \frac{\exp[-i\nu(t-t_0)]}{Z_+(\nu)}$$

$$+\frac{i}{2\pi}\exp[-\eta(t-t_0)]\int_{-\infty}^{+\infty}\frac{\exp[-i\Omega(t-t_0)]\,d\Omega}{(\Omega+i\delta-\nu)Z_+(\Omega-i\eta)}$$

$$= \frac{\exp[-i\nu(t-t_0)]}{Z_+(\nu)} + \frac{i}{2\pi}\exp[-\eta(t-t_0)]$$

$$\times\int_{-\infty}^{+\infty}\frac{\Delta_+(\Omega-i\eta)\exp[-i\Omega(t-t_0)]\,d\Omega}{(\Omega-i\eta-\nu)(i\Omega+\eta)Z_+(\Omega-i\eta)}, \tag{98}$$

since

$$\int_{-\infty}^{+\infty}\frac{\exp[-i\Omega(t-t_0)]}{(\Omega-i\eta-\nu)(\Omega-i\eta)}d\Omega = 0,\quad t > t_0.$$

The expressions (98) lead to the inequalities

$$\left.\begin{aligned} &\left|\Phi(\nu,\ t-t_0) - \frac{\exp[-i\nu(t-t_0)]}{Z_+(\nu)}\right| \leqslant K_2\exp[-\eta(t-t_0)];\\ &\left|\nu\Phi(\nu,\ t-t_0) - \nu\frac{\exp[-i\nu(t-t_0)]}{Z_+(\nu)}\right| \leqslant K_3\exp[-\eta(t-t_0)], \end{aligned}\right\} \tag{99}$$

where K_2 and K_3 are constants.

We can now turn to the calculation of the expressions (44) in our model based on the Hamiltonian H_L. We have

$$\Phi_k^{(a)}(t,\ \tau,\ t_0) = \mathrm{Sp}\exp[i\overline{kr}(\tau)]\exp[-i\overline{kr}(t)\mathcal{D}_{t_0}. \tag{100}$$

Here, the superscript (a) indicates the use of an approximation—the replacement of the function $\overline{r}(t)$ determined by the exact equations of motion with the given Hamiltonian by the function $\overline{r}(t)$ determined by Eqs. (58), which correspond to the model Hamiltonian H_L.

We formulate the approximate equation that we intend to solve instead of the exact relation (45) in the form

$$\frac{d\langle\overline{p}\rangle_t}{dt}+\overline{E}(t)=-\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}}\frac{1}{(2\pi)^3}\int d\overline{k}\,\frac{\overline{k}\mathscr{L}^2(k)}{2\omega_k[1-\exp(-\beta\hbar\omega_k)]}$$
$$\times\int\limits_{-\infty}^{t}d\tau\exp[-\varepsilon(t-\tau)]\{\exp[i\omega_k(t-\tau)]+\exp[-i\omega_k(t-\tau)]$$
$$\times\exp[-\beta\hbar\omega_k]\}\lim_{t_0\to-\infty,}\lim_{V\to\infty}\Phi_k^{(a)}(t,\tau,t_0)$$
$$-\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}}\frac{1}{(2\pi)^3}\int d\overline{k}\,\frac{\overline{k}\mathscr{L}^2(k)}{2\omega_k[1-\exp(-\beta\hbar\omega_k)]}\tag{101}$$
$$\times\int\limits_{-\infty}^{t}d\tau\exp[-\varepsilon(t-\tau)]\{\exp[-i\omega_k(t-\tau)]$$
$$+\exp[i\omega_k(t-\tau)]\exp[-\beta\hbar\omega_k]\}\lim_{t_0\to-\infty}\lim_{V\to\infty}\Phi_k^{*(a)}(t,\tau,t_0).$$

Note that this equation is obtained from (45) by replacing Φ_k by $\Phi_{(k)}^{(a)}$ and then going to limits as follows: 1) $V\to\infty$, 2) $t_0\to-\infty$, 3) $\varepsilon\to 0$.

To give the approximate equation (101) explicitly, we expand the expression (100) for $\Phi_{(k)}^{(a)}$. We note first the circumstance that the considered "model equations" (58) are linear, by virtue of which the commutators $[\overline{r}_j(t), \overline{r}_{j'}(t)]$, for $j, j'=1, 2, 3$, are C numbers. Therefore

$$\exp[i\overline{kr}(\tau)]\exp[-i\overline{kr}(t)]$$
$$=\exp\{[\overline{kr}(\tau),\ \overline{kr}(t)]/2\}\exp[-i\overline{k}(\overline{r}(t)-\overline{r}(\tau)]$$

and

$$\Phi_k^{(a)}(t,\tau,t_0)=\exp\{[\overline{kr}(\tau),\ \overline{kr}(t)/2\}$$
$$\times\mathop{\mathrm{Sp}}_{(S,\Sigma)}\exp\Big[-ik\int\limits_{\tau}^{t}p(s)\,ds/m^*\Big]\mathscr{D}_{t_0}.\tag{102}$$

Substituting here (93) and noting that

$$\mathscr{D}_{t0}=\rho(S)\mathscr{D}_L(S);\quad \mathscr{D}(\Sigma)=\mathrm{const}\exp[-\beta\sum_{(k)}\hbar\nu(k)\,b_k^+b_k];$$
$$\mathop{\mathrm{Sp}}_{(S)}\rho(S)=1;\quad \mathop{\mathrm{Sp}}_{(\Sigma)}\mathscr{D}(\Sigma)=1,\tag{103}$$

we obtain from (102)

$$\Phi_k^{(a)}(t,\tau,t_0)=\Phi_k^{(1)}(t,\tau,t_0)\,\Phi_k^{(2)}(t,\tau,t_0),\tag{104}$$

where

$$\Phi_k^{(1)}(t,\tau,t_0)=\exp\{[\overline{kr}(\tau),\ \overline{kr}(t)]/2\}\exp\Big[-i\frac{\overline{k}}{m^*}\int\limits_{\tau}^{t}\overline{p}^{(E)}(s)\,ds\Big]$$
$$\times\mathop{\mathrm{Sp}}_{(\Sigma)}\exp\Big[-i\frac{\overline{k}}{m^*}\int\limits_{\tau}^{t}\overline{p}^{(\Sigma)}(s)\,ds\Big]\mathscr{D}_L(\Sigma);$$
$$\Phi_k^{(2)}(t,\tau,t_0)=\mathrm{Sp}\exp\Big[-i\frac{\overline{k}}{m^*}\int\limits_{\tau}^{t}\overline{p}^{(s)}(s)\,ds\Big]p(s).$$

Further,

$$\left.\begin{aligned}
&[\overline{kr}(\tau),\ \overline{kr}(t)]=\frac{1}{m^*}\Big[\overline{kr}(\tau);\ \int\limits_{\tau}^{t}\overline{kp}(s)\,ds\Big];\\
&\overline{r}(\tau)=\overline{r}(t)-\int\limits_{\tau}^{t}\frac{\overline{p}(s)}{m}\,ds\\
&[\overline{kr}(\tau),\ \overline{kp}(s)]=[\overline{k}\,(\overline{r}(\tau)-\overline{r}(s);\overline{kp}(s)]+i\hbar k^2\\
&\qquad=-\frac{1}{m^*}\Big[\int\limits_{\tau}^{s}\overline{kp}(\sigma)\,d\sigma,\ \overline{kp}(s)\Big]+i\hbar k^2.
\end{aligned}\right\}\tag{105}$$

Thus,

$$[\overline{kr}(\tau);\ \overline{kr}(t)]=\frac{i\hbar k^2}{m^*}(t-\tau)-\Big(\frac{1}{m^*}\Big)^2\int\limits_{\tau}^{t}ds\int\limits_{\tau}^{s}d\sigma\,[\overline{kp}(\sigma),\ \overline{kp}(s)]$$
$$=i\hbar\frac{k^2}{m^*}(t-\tau)+\Big(\frac{1}{m_k}\Big)^2\int\limits_{\tau}^{t}ds\int\limits_{s}^{t}d\sigma\,[\overline{kp}(\sigma),\ \overline{kp}(s)],\tag{106}$$

since

$$\int\limits_{\tau}^{t}ds\int\limits_{\tau}^{t}d\tau\ [\overline{kp}(\sigma),\ \overline{kp}(s)]=\Big[\int\limits_{\tau}^{t}\overline{kp}(\xi)\,d\xi,\ \int\limits_{\tau}^{t}\overline{kp}(\xi)\,d\xi\Big]=0$$

and

$$\int\limits_{\tau}^{t}\int\limits_{\tau}^{s}A+\int\limits_{\tau}^{t}\int\limits_{s}^{t}A=\int\limits_{\tau}^{t}\int\limits_{\tau}^{t}A=0.$$

Thus, on the basis of (93) and noting that the components $\overline{p}(t)=\overline{p}^{(S)}(t)+\overline{p}^{(\Sigma)}(t)+\overline{p}^{(E)}(t)$ commute with one another, we find

$$[\overline{kr}(\tau),\ \overline{kr}(t)]=i\hbar\frac{k^2}{m^*}(t-\tau)+\Big(\frac{1}{m^*}\Big)^2\int\limits_{\tau}^{t}ds\int\limits_{s}^{t}d\sigma\,[\overline{kp}^{(S)}(\sigma);$$
$$\overline{kp}^{(S)}(s)]+\Big(\frac{1}{m^*}\Big)^2\int\limits_{\tau}^{t}ds\int\limits_{s}^{t}d\sigma\,[\overline{kp}^{(\Sigma)}(\sigma),\ \overline{kp}^{(\Sigma)}(s)].\tag{107}$$

Here, having in mind (93),

$$\Big(\frac{1}{m^*}\Big)^2[\overline{kp}^{(S)}(\sigma);\ \overline{kp}^{(S)}(s)]=i\hbar k^2\{g_0(\delta,\sigma-t_0)\,g_1(\delta,s-t_0)$$
$$-g_0(\delta,s-t_0)\,g_1(\delta,\sigma-t_0)\}\tag{108}$$

and

$$\Big(\frac{1}{m^*}\Big)^2[\overline{kp}^{(\Sigma)}(\sigma),\ \overline{kp}^{(\Sigma)}(s)]=k^2\{F(\sigma,s,t_0)-F(s,\sigma,t_0)\},\tag{109}$$

where

$$F(\sigma,s,t_0)=\frac{1}{V}\sum_{(k)}\frac{\hbar}{6\nu(k)}\frac{L^2(k)\,k^2}{\{1-\exp[-\beta\hbar\nu(k)]\}}\{f(\nu(k),\delta,\sigma-t_0)$$
$$\times f(-\nu(k),\delta,s-t_0)$$
$$+\exp[-\beta\hbar\nu(k)]\,f(-\nu(k),\delta,\sigma-t_0)\,f(\nu(k),\delta,s-t_0)\}$$

or

$$F(\sigma,s,t_0)$$
$$=\int\limits_{0}^{\infty}d\nu I(\nu)\frac{\hbar\nu}{1-\exp(-\beta\hbar\nu)}\{f(\nu,\delta,\sigma-t_0)\,f(-\nu,\delta,s-t_0)$$
$$+\exp(-\beta\hbar\nu)\,f(-\nu,\delta,\sigma-t_0)\,f(\nu,\delta,s-t_0)\}.\tag{110}$$

We note also, bearing in mind the form (103) of the operator $\mathscr{D}_L(\Sigma)$ and the linearity of $\overline{p}^{(E)}(s)$ in the Bose operators, we can write[3)]

$$\mathop{\mathrm{Sp}}_{(\Sigma)}\exp\Big\{-i\frac{\overline{k}}{m}\int\limits_{\tau}^{t}\overline{p}^{(\Sigma)}(s)\,ds\Big]\mathscr{D}_L(\Sigma)$$
$$=\exp\Big\{-\frac{1}{2m^2}\mathrm{Sp}\Big(\int\limits_{\tau}^{t}\overline{kp}^{(\Sigma)}(s)\,ds\Big)^2\mathscr{D}_L(\Sigma)\Big\}$$
$$=\exp\Big\{-\frac{1}{2m^2}\int\limits_{\tau}^{t}ds\int\limits_{\tau}^{t}d\sigma\mathop{\mathrm{Sp}}_{(\Sigma)}(\overline{kp}^{(\Sigma)}(s))(\overline{kp}^{(\Sigma)}(\sigma))\,\mathscr{D}_L(\Sigma)\Big\}$$
$$=\exp\Big\{-\frac{k^2}{2}\int\limits_{\tau}^{t}ds\int\limits_{\tau}^{t}d\sigma F(s,\sigma,t_0)\Big\}.\tag{111}$$

We also recall that

$$\frac{1}{m^*}\overline{p}^E(t)=-\sum_{(\omega)}\overline{E}_\omega f(\omega,\delta,t-t_0)\exp(-i\omega t_0).\tag{112}$$

Taking into account (69)–(71), (95), and (96), we ob-

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3)Here, we have used the formula $\langle e^A\rangle=e^{1/2}\langle A\rangle^2$, where A is a linear form in the Bose operators, and the averaging is over a quadratic Hamiltonian of the form $\sum_\omega E_\omega b_\omega^+ b_\omega$.

tain

$$[\overline{kr}(\tau),\ \overline{kr}(t)] \xrightarrow[V\to\infty]{} i\hbar\frac{k^2}{m^*}(t-\tau)$$

$$+i\hbar k^2\int_\tau^t ds\int_s^t d\sigma\{\Psi_0(\sigma-t_0)\Psi_1(s-t_0)-\Psi_0(s-t_0)\Psi_1(\sigma-t_0)\}$$

$$+k^2\int_\tau^t ds\int_s^t d\sigma\{F_\infty(\sigma,\ s,\ t_0)-F_\infty(s,\ \sigma,\ t_0)\}. \qquad (113)$$

Here, using (110), we obtain

$$F_\infty(\sigma,\ s,\ t_0)=\lim_{V\to\infty}F(\sigma,\ s,\ t_0)$$

$$=\int_0^\infty J(\nu)\frac{\hbar\nu}{1-\exp(-\beta\hbar\nu)}\{\Phi(\nu,\ \sigma-t_0)\Phi(-\nu,\ s-t_0)$$

$$+\exp(-\beta\hbar\nu)\Phi(\nu,\ s-t_0)\Phi(-\nu,\ \sigma-t_0)\}d\nu. \qquad (114)$$

From (112), we also have

$$\frac{1}{m^*}\overline{p}^{(E)}(t)\xrightarrow[V\to\infty]{}-\sum_{(\omega)}\overline{E}_\omega\Phi(\omega,\ t-t_0)\exp(-i\omega t_0). \qquad (115)$$

Taking into account (113), (114), and (115), we can now write [see (104)]

$$\Phi_k^{(1)}(t,\ \tau,\ t_0)\xrightarrow[V\to\infty]{}\exp\{1/2\lim_{V\to\infty}[\overline{kr}(\tau),\ \overline{kr}(t)]\}$$

$$\times\exp\Big[-i\frac{\overline{k}}{m^*}\int_\tau^t\lim_{(V\to\infty)}\overline{p}^{(E)}(s)\,ds\Big]$$

$$\times\exp\Big\{-\frac{k^2}{2}\int_\tau^t ds\int_\tau^t d\sigma F_\infty(s,\ \sigma,\ t_0). \qquad (116)$$

We now consider the situation when $t_0\to-\infty$. Using (99) and (115), we note that

$$\Big|\lim_{V\to\infty}\frac{1}{m^*}\overline{p}^E(t)-\overline{v}(t)\Big|\leqslant\sum_{(\omega)}|\overline{E}_\omega|\,k_2\exp[-\eta(t-t_0)], \qquad (117)$$

where

$$\overline{v}(t)=-\sum_{(\omega)}\frac{\overline{E}_\omega}{Z_+(\omega)}\exp(-i\omega t). \qquad (118)$$

Further, using (99) to estimate the expression (114), we find

$$|F_\infty(\sigma,\ s,\ t_0)-F(\sigma-s)|\leqslant\widetilde{K}\{\exp[-\eta(\sigma-t_0)]$$
$$+\exp[-\eta(s-t_0)]\},\quad \widetilde{K}=\text{const}, \qquad (119)$$

where

$$F(\sigma-s)=\int_0^\infty d\nu J(\nu)\frac{\hbar\nu}{1-\exp(-\beta\hbar\nu)}$$

$$\times\frac{\exp[-i\nu(\sigma-s)]+\exp(-\beta\hbar\nu)\exp[i\nu(\sigma-s)]}{Z_+(\nu)Z_+(-\nu)}$$

or, since $J(\nu)=J(-\nu)$,

$$F(\sigma-s)=\int_{-\infty}^{+\infty}d\nu J(\nu)\frac{\hbar\nu}{[1-\exp(-\beta\hbar\nu)]}\frac{\exp[-i\nu(\sigma-s)]}{Z_+(\nu)Z_+(-\nu)}. \qquad (120)$$

We transform this formula somewhat. Since

$$2\pi J(\nu)=\Delta_+(\nu)-\Delta_-(\nu)=Z_+(\nu)-Z_-(\nu)=Z_+(\nu)+Z_+(-\nu),$$

we have

$$\frac{1}{Z_+(\nu)Z_+(-\nu)}J(\nu)=\frac{1}{2\pi}\frac{Z_+(\nu)+Z_+(-\nu)}{Z_+(\nu)Z_+(-\nu)}$$

$$=\frac{1}{2\pi}\Big\{\frac{1}{Z_+(\nu)}+\frac{1}{Z_+(-\nu)}\Big\}=\frac{1}{2\pi}\Big\{\frac{1}{Z_+(\nu)}-\frac{1}{Z_-(\nu)}\Big\}.$$

Further, since $J(\nu)$ is real,

$$\operatorname{Im}Z_+(\nu)=-\operatorname{Im}Z_+(-\nu),$$

but by definition

$$Z_+(\nu)=i\int_{-\infty}^{+\infty}\frac{J(\Omega)}{\Omega-\nu+i0}d\Omega,$$

and therefore

$$\operatorname{Re}Z_+(\nu)=\pi J(\nu)=\operatorname{Re}Z_+(-\nu).$$

Thus,

$$Z_+(-\nu)=Z_+^*(\nu).$$

Therefore, we can rewrite (120) in the form

$$\left.\begin{aligned}&F(\sigma-s)=\int_{-\infty}^{+\infty}G(\nu)\frac{\hbar\nu}{1-\exp(-\beta\hbar\nu)}\exp[-i\nu(\sigma-s)]\,d\nu;\\&G(\nu)=\frac{1}{2\pi}\Big\{\frac{1}{Z_+(\nu)}-\frac{1}{Z_-(\nu)}\Big\}=\frac{J(\nu)}{|Z_+(\nu)|^2}.\end{aligned}\right\} \qquad (121)$$

Thus, using (105), (113), (116), (117), (119), and (97), we obtain

$$\lim_{t_0\to-\infty,\,V\to\infty}\lim\Phi_k^{(1)}(t,\ \tau,\ t_0)=\exp\Big[-i\int_\tau^t(\overline{kv}(s))\,ds\Big]A(k^2,\ t-\tau), \qquad (122)$$

where

$$A(k^2,\ t-\tau)=\exp k^2\Big\{\frac{i\hbar}{2m^*}(t-\tau)+\frac{1}{2}\int_\tau^t ds\int_s^t d\sigma(F(\sigma-s)$$

$$-F(s-\sigma))-\frac{1}{2}\int_\tau^t ds\int_\tau^t d\sigma F(s-\sigma)\Big\}. \qquad (123)$$

But with allowance for (121)

$$\frac{1}{2}\int_\tau^t ds\int_\tau^t d\sigma F(s-\sigma)=\int_{-\infty}^{+\infty}G(\nu)\frac{\{1-\cos\nu(t-\tau)\}}{\nu^2}\frac{\hbar\nu}{1-\exp(-\beta\hbar\nu)}d\nu;$$

$$F(\sigma-s)-F(s-\sigma)=2i\int_{-\infty}^{+\infty}G(\nu)\frac{\hbar\nu}{1-\exp(-\beta\hbar\nu)}\sin\nu(s-\sigma)\,d\nu$$

$$=2i\int_{-\infty}^{+\infty}G(\nu)\frac{\hbar\nu}{1-\exp(\beta\hbar\nu)}\sin\nu(s-\sigma)\,d\nu$$

$$=-2i\int_{-\infty}^{+\infty}G(\nu)\frac{\exp(-\beta\hbar\nu)\hbar\nu}{1-\exp(-\beta\hbar\nu)}\sin\nu(s-\sigma)\,d\nu$$

$$=i\int_{-\infty}^{+\infty}G(\nu)\hbar\nu\sin\nu(s-\sigma)\,d\nu;$$

$$\int_\tau^t ds\int_s^t d\sigma\{F(\sigma-s)-F(s-\sigma)\}=i\hbar\int_{-\infty}^{+\infty}G(\nu)\Big\{\frac{\sin\nu(t-\tau)}{\nu}-(t-\tau)\Big\}d\nu.$$

Thus, taking into account (123), we write

$$A(k^2,\ t,\ \tau)=A(k^2,\ t-\tau)$$

$$=\exp k^2\Big\{\frac{i\hbar}{2m^*}(t-\tau)+\frac{i\hbar}{2}\int_{-\infty}^{+\infty}G(\nu)\Big\{\frac{\sin\nu(t-\tau)}{\nu}-(t-\tau)\Big\}d\nu$$

$$-\frac{1}{2}\int_{-\infty}^{+\infty}G(\nu)\frac{\hbar(1-\cos\nu(t-\tau))}{\nu[1-\exp(-\beta\hbar\nu)]}d\nu. \qquad (124)$$

We now consider the expression

$$\Phi_k^{(2)}(t,\ \tau,\ t_0)=\operatorname*{Sp}_{(S)}\exp\Big[-\frac{i}{m^*}\int_\tau^t\overline{kp}^{(s)}(\sigma)\,d\sigma\Big]p(s).$$

Here

$$\frac{1}{m^*}\int_\tau^t\overline{kp}^{(s)}(\sigma)\,d\sigma=\overline{kp}\int_\tau^t g_0(\delta,\ \sigma-t_0)\,d\sigma-\overline{kr}\int_\tau^t g_1(\delta,\ \sigma-t_0)\,d\sigma.$$

In accordance with the above results,

$$\left.\begin{aligned}&\int_\tau^t g_0(\delta,\ \sigma-t_0)\,d\sigma\to\int_\tau^t\psi_0(\sigma-t_0)\,d\sigma;\\&\int_\tau^t g_1(\delta,\ \sigma-t_0)\,d\sigma\to\int_\tau^t\psi_1(\sigma-t_0)\,d\sigma;\\&(V\to\infty),\end{aligned}\right| \qquad (125)$$

and also

$$\left|\int_\tau^t \psi_0(\sigma-t_0)\,d\sigma\right|\leqslant K_0\int_\tau^t \exp\left[-\eta(\sigma-t_0)\right]d\sigma;$$

$$\left|\int_\tau^t \psi_1(\sigma-t_0)\,d\sigma\right|\leqslant K_1\int_\tau^t \exp\left[-\eta(\sigma-t_0)\right]d\sigma. \qquad (126)$$

Hence, it is natural to expect that

$$\Phi_k^{(2)}(t,\ \tau,\ t_0)=\underset{(S)}{\mathrm{Sp}}\,\rho(s)\exp\Big\{-i\overline{kp}\int_\tau^t g_0(\delta,\ \sigma-\tau_0)\,d\sigma$$

$$+i\,\overline{kr}\int_\tau^t g_1(\delta,\ \sigma-t_0)d\sigma\Big\}\to\underset{(S)}{\mathrm{Sp}}\,\rho(s)\exp\Big\{-i\overline{kp}\int_\tau^t\psi_0(\sigma-t_0)\,d\sigma$$

$$+i\,\overline{kr}\int_\tau^t\psi_1(\sigma-t_0)\,d\sigma\Big\}\quad(V\to\infty); \qquad (127)$$

$$\mathrm{Sp}\,\rho(s)\exp\Big\{-i\,\overline{kp}\int_\tau^t\psi_0(\sigma-t_0)\,d\sigma$$

$$+i\,\overline{kr}\int_\tau^t\psi_1(\sigma-t_0)\,d\sigma\Big\}\to\underset{(S)}{\mathrm{Sp}}\,\rho(s)=1\quad(V\to\infty). \qquad (128)$$

However, despite the relations (125) and (126), serious difficulties arise in the proof of (127) and (128) due to the fact that the operators $\overline{p}$ and $\overline{r}$ *are not bounded*.

Nevertheless, the validity of (127) and (128) can be established[4] when $\rho(s)$ depends neither on V nor on t_0.

In this case, therefore,

$$\lim_{t_0\to-\infty,\ V\to\infty}\ \lim\ \Phi_k^{(2)}(t,\ \tau,\ t_0)=1 \qquad (129)$$

and therefore, on the basis of (104) and (122),

$$\lim_{t_0\to-\infty,\ V\to\infty}\ \lim\ \Phi_k^{(a)}(t,\ \tau,\ t_0)=\exp\Big[-i\int_\tau^t\overline{kv}(\sigma)\,d\sigma\Big]A(k^2,\ t-\tau). \qquad (130)$$

Substituting this expression in the approximate equation (101), we obtain

$$d\langle\overline{p}\rangle_t/dt+\overline{E}(t)=-\lim_{\substack{\varepsilon>0\\ \varepsilon\to0}}\frac{1}{(2\pi)^3}\int d\overline{k}\,\frac{\mathscr{L}^2(k)\,\overline{k}}{2\omega_k\left[1-\exp(-\beta\hbar\omega_k)\right]}$$

$$\times\int_\tau^t d\tau\exp\left[-\varepsilon(t-\tau)\right]\{\exp\left[i\omega_k(t-\tau)\right]$$

$$+\exp\left[-i\omega_k(t-\tau)\right]\exp(-\beta\hbar\omega_k)\}$$

$$\times\exp\Big[-i\int_\tau^t\overline{kv}(\sigma)\,d\sigma\Big]A(k^2,\ t-\tau)$$

$$-\lim_{\substack{\varepsilon\to0\\ \varepsilon>0}}\frac{1}{(2\pi)^3}\int d\overline{k}\,\frac{\mathscr{L}^2(k)\,\overline{k}}{2\omega_k\left[1-\exp(-\beta\hbar\omega_k)\right]}$$

$$\times\int_{-\infty}^t d\tau\exp\left[-\varepsilon(t-\tau)\right]\{\exp\left[-i\omega_k(t-\tau)\right]$$

$$+\exp\left[i\omega_k(t-\tau)\right]\exp(-\beta\hbar\omega_k)\}$$

$$\times\exp\Big[i\int_\tau^t\overline{kv}(\sigma)\,d\sigma\Big]A^*(k^2,\ t-\tau). \qquad (131)$$

Thus, we have obtained a general approximate equation from which the results of Refs. 2 and 3 follow. Note that in them we have $m^*=m$, and the function (86) is used in the limit $\gamma\to0$.

We consider in particular the case of a weak external field, when one can use a linear approximation for the expression of the mean velocity as a function of $\overline{E}$. In (131), we then set

$$\exp\Big[\mp i\int_\tau^t(\overline{kv}(\sigma))\,d\sigma\Big]=1\mp i\int_\tau^t(\overline{kv}(\sigma))\,d\sigma.$$

Using the radial symmetry, we find

$$d\langle\overline{p}\rangle_t/dt+\overline{E}(t)=-\lim_{\substack{\varepsilon>0\\ \varepsilon\to0}}\frac{1}{(2\pi)^3}\int d\overline{k}\,\frac{k^2\mathscr{L}^2(k)}{6\omega_k\left[1-\exp(-\beta\hbar\omega_k)\right]}$$

$$\times\int_{-\infty}^t d\tau\exp\left[-\varepsilon(t-\tau)\right]\{\exp\left[i\omega_k(t-\tau)\right]$$

$$+\exp\left[-i\omega_k(t-\tau)\right]\exp(-\beta\hbar\omega_k)\}\,i\int_\tau^t\overline{v}(\sigma)\,d\sigma A(k^2,\ t-\tau)$$

$$-\lim_{\substack{\varepsilon>0\\ \varepsilon\to0}}\frac{1}{(2\pi)^3}\int d\overline{k}\,\frac{k^2\mathscr{L}^2(k)}{6\omega_k\left[1-\exp(-\beta\hbar\omega_k)\right]}$$

$$\times\int_{-\infty}^t d\tau\exp\left[-\varepsilon(t-\tau)\right]\{\exp\left[-i\omega_k(t-\tau)\right]$$

$$+\exp\left[i\omega_k(t-\tau)\right]\exp\left[-\beta\hbar\omega_k\right]\}\,i\int_\tau^t\overline{v}(\sigma)\,d\sigma A^*(k^2,\ t-\tau). \qquad (132)$$

Here

$$\overline{v}(t)=-\sum_\omega\frac{E_\omega}{Z_+(\omega)}\exp(-i\omega t)$$

is the stationary mean velocity excited in the model system by the external field

$$E(t)=\sum_{(\omega)}E_\omega\exp(-i\omega t).$$

Note that in the considered case of a sufficiently weak field the stationary mean velocity for the real system can also be represented in the analogous form

$$\langle\overline{p}_t\rangle/m=-\sum_{(\omega)}\frac{R_\omega}{z_+(\omega)}\exp(-i\omega t)$$

though, of course, with different coefficients, $z_+(\omega)$ corresponding to the impedance of the real system.

To obtain an approximate "self-consistent" equation for determining this impedance, we choose

$$Z_+(\omega)=z_+(\omega) \qquad (133)$$

and use the approximate equation (132).

We have

$$-i\int_\tau^t\overline{v}(\sigma)\,d\sigma=-\sum_{(\omega)}\frac{E_\omega\exp(-i\omega t)}{Z_+(\omega)}\Big\{\frac{1-\exp\left[i\omega(t-\tau)\right]}{\omega}\Big\}.$$

We introduce the variable of integration $t-\tau=s$. Then we find

$$m\sum_{(\omega)}\frac{E_\omega i\omega\exp(-i\omega t)}{z_+(\omega)}+\sum_{(\omega)}E_\omega\exp(-i\omega t)$$

$$=\lim_{\substack{\varepsilon>0\\ \varepsilon\to0}}\sum_{(\omega)}\frac{E_\omega\exp(-i\omega t)}{z_+(\omega)}\Big\{\frac{1}{(2\pi)^3}\int d\overline{k}\,\frac{k^2\mathscr{L}^2(k)}{6\omega_k\left[1-\exp(-\beta\hbar\omega_k)\right]}$$

$$\times\int_0^\infty ds\exp(-\varepsilon s)\left[\exp(-i\omega_k s)\right.$$

$$\left.+\exp(i\omega_k s)\exp(-\beta\hbar\omega_k)\right]\Big[\frac{\exp(i\omega s)-1}{\omega}\Big]A^*(k^2,\ s)$$

$$-\frac{1}{(2\pi)^3}\int d\overline{k}\,\frac{k^2\mathscr{L}^2(k)}{6\omega_k\left[1-\exp(-\beta\hbar\omega_k)\right]}\int_0^\infty ds\exp(-\varepsilon s)\left[\exp(i\omega_k s)\right.$$

$$\left.+\exp(-i\omega_k s)\exp(-\beta\hbar\omega_k)\right]\Big[\frac{\exp(i\omega s)-1}{\omega}\Big]A(k^2,\ s)\Big\}.$$

From this we obtain a "self-consistent" approximate equation for determining the impedance:

$$z_+(\omega)=-im\omega$$

$$+\lim_{\substack{\varepsilon>0\\ \varepsilon\to0}}\frac{1}{(2\pi)^3}\int d\overline{k}\,\frac{k^2\mathscr{L}^2(k)}{6\omega_k\left[1-\exp(-\beta\hbar\omega_k)\right]}\int_0^\infty ds\exp(-\varepsilon s)\Big\{\left[\exp(-i\omega_k s)\right.$$

$$\left.+\exp(i\omega_k s)\exp(-\beta\hbar\omega_k)\right]\Big[\frac{\exp(i\omega s)-1}{\omega}\Big]A^*(k^2,\ s)$$

$$-\left[\exp(i\omega_k s)+\exp(-i\omega_k s)\exp(-\beta\hbar\omega_k)\right]$$

$$\times\Big[\frac{\exp(i\omega s)-1}{\omega}\Big]A(k^2,\ s)\Big\}. \qquad (134)$$

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[4]See Appendix 2.

We also consider, as an example, Eq. (131) for a constant field, when

$$\overline{E} = \text{const}, \quad \overline{v} = \text{const}. \tag{135}$$

Since with allowance for (123)

$$A^*(k^2, s) = A(k^2, -s),$$

making the substitution $\overline{k} \to -\overline{k}$ in the terms containing $\exp(\pm i\omega_k s)\exp(-\beta\omega_k)$, we can write

$$-\overline{E} = \lim_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} \frac{1}{(2\pi)^3} \int d\overline{k} \frac{\mathscr{L}^2(k)\overline{k}}{2\omega_k} \int_{-\infty}^{+\infty} ds \exp(-\varepsilon|s|)$$
$$\times \left\{ \frac{\exp[i(\omega_k - \overline{kv})s]}{1-\exp(-\beta\hbar\omega_k)} - \frac{\exp[-i(\omega_k - \overline{kv})s]}{\exp(\beta\hbar\omega_k)-1} \right\} A(k^2, s). \tag{136}$$

For a weak interaction, we obtain directly from here Eq. (42) by means of the replacement of $A(k^2, s)$ by its "zeroth approximation," ignoring the interaction and setting $s = \hbar\xi$.

Finally, we make a number of comments on the structure of the stationary probability density distribution of the momenta for the particle s in the model system with the Hamiltonian H_L.

Denoting this probability density distribution by $w_t(\overline{p})$, we have

$$\int \exp(-i\overline{\lambda p}) w_t(\overline{p}) d\overline{p} = \underset{S,\Sigma}{\mathrm{Sp}} \exp[-i\overline{\lambda}\overline{p}(t)] \mathscr{D}_{t0}$$
$$= \exp[-i\lambda\overline{p^{(E)}}(t)] \underset{(S)}{\mathrm{Sp}} \exp[-i\overline{\lambda p^{(S)}}(t)] \rho(s)$$
$$\times \underset{(\Sigma)}{\mathrm{Sp}} \exp[-i\overline{\lambda p^{(\Sigma)}}(t)] \mathscr{D}_L(\Sigma). \tag{137}$$

Here, as before,

$$\mathrm{Sp} \exp[-i\overline{\lambda p^{(\Sigma)}}(t)] \mathscr{D}_L(\Sigma) = \exp\left\{-\frac{\lambda^2 m^{*2}}{2} F(t,t,t_0)\right\}. \tag{138}$$

We now recall that

$$\left.\begin{aligned} &\overline{p}^{(E)}(t) \underset{V\to\infty}{\longrightarrow} \overline{p}_\infty^{(E)}(t) = -m^* \sum_{(\omega)} \overline{E}_\omega \Phi(\omega, t-t_0) \exp(-i\omega t_0); \\ &\overline{p}_\infty^{(E)}(t) - m^*\overline{v}(t) \underset{t\to+\infty}{\longrightarrow} 0; \\ &\overline{p}^{(S)}(t) \underset{(V\to\infty)}{\longrightarrow} p_\infty(t) = m^*\overline{p}\psi_0(t-t_0) - m^*\overline{r}\psi_1(t-t_0) \end{aligned}\right\} \tag{139}$$

and

$$\left.\begin{aligned} &F(t,t,t_0) \underset{(V\to\infty)}{\longrightarrow} F_\infty(t_1, t, t_0); \\ &|F_\infty(t,t,t_0) - F(0)| < 2\widetilde{K}\exp[-\eta(t-t_0)] \underset{t\to\infty}{\longrightarrow} 0. \end{aligned}\right\} \tag{140}$$

Here

$$F(0) = \int_{-\infty}^{+\infty} \frac{J(\nu)}{|Z_+(\nu)|^2} \frac{\hbar\nu}{[1-\exp(-\beta\hbar\nu)]} d\nu > 0. \tag{141}$$

Repeating the arguments used previously to study the function $\Phi_k^{(a)}$, we find on the basis of (134)

$$\int \exp(-i\overline{\lambda p}) w_t(\overline{p}) d\overline{p} \underset{(V\to\infty)}{\longrightarrow} \exp[-i\overline{\lambda p_\infty^{(E)}}(t)]$$
$$\times \underset{(S)}{\mathrm{Sp}} \exp[-i\overline{\lambda p_\infty^{(S)}}(t)] \rho(s) \exp\left\{-\frac{\lambda^2 m^{*2}}{2} F_\infty(t,t,t_0)\right\} \tag{142}$$

and

$$\lim_{(V\to\infty)} \left\{ \int \exp(-i\overline{\lambda p}) w_t(\overline{p}) d\overline{p} \right.$$
$$\left. - \exp[-im^*\overline{\lambda v}(t)] \exp\left[-\frac{\lambda^2 m^{*2}}{2} F(0)\right] \right\} \to 0. \tag{143}$$

We now consider the distribution function with respect to the momenta $\overline{p}$:

$$w_t(\overline{p}) = \frac{1}{(2\pi)^3} \int d\overline{\lambda} \exp(i\overline{\lambda p}) \left\{ \int \exp(-i\overline{\lambda p}) w_t(\overline{p}) d\overline{p} \right\}. \tag{144}$$

Note that

$$\left| \exp(i\overline{\lambda p}) \int \exp(-i\overline{\lambda p}) w_t(\overline{p}) d\overline{p} \right|$$
$$\leqslant \exp\left[-\frac{\lambda^2 m^{*2}}{2} F(t,t,t_0)\right]. \tag{145}$$

But, using (140) and (142), we readily see that

$$F_\infty(t,t,t_0) > F(0)/2 > 0 \tag{146}$$

for a sufficiently large difference $t - t_0$.

We also fix t and t_0. Since for fixed t and t_0

$$F(t,t,t_0) \to F_\infty(t,t,t_0), \quad (V\to\infty),$$

we see that for sufficiently large V

$$F(t,t,t_0) > F(0)/4 > 0$$

and

$$\left| \exp(i\overline{\lambda p}) \left\{ \int \exp(-i\overline{\lambda p}) w_t(p) d\overline{p} \right\} \right|$$
$$\leqslant \exp[-(\lambda^2 m^{*2}/8) F(0)].$$

Therefore, the transition to the limit in (144) for $V \to \infty$ can be made in the integrand of the integral over $\overline{\lambda}$ on the basis of (142).

Thus, we obtain

$$\lim_{V\to\infty} w_t(\overline{p})$$
$$= \frac{1}{(2\pi)^3} \int d\overline{\lambda} \exp[i\overline{\lambda}(\overline{p} - \overline{p_\infty^{(E)}}(t))] \{\underset{(S)}{\mathrm{Sp}} \exp[-i\overline{\lambda p_\infty^{(S)}}(t)] \rho(S)\}$$
$$\times \exp\left\{-\frac{\lambda^2 m^{*2}}{2} F_\infty(t,t,t_0)\right\}. \tag{147}$$

Using (146), we see that the integrand in (147) is in modulus smaller than $\exp[-(\lambda^2 m^{*2}/4)F(0)]$ for a sufficiently large difference $t - t_0$.

We can, therefore, again go to the limit $t \to \infty$ in the integrand of the integral over $\overline{\lambda}$ in (147) and obtain in accordance with (143)

$$\lim \left\{ w_t(\overline{p}) - \frac{1}{(2\pi)^3} \int d\overline{\lambda} \exp\left[i\overline{\lambda}(\overline{p} - m^*\overline{v}(t)) - \frac{\lambda^2 m^{*2}}{2} F(0)\right] \right\} \to 0. \tag{148}$$

Taking the Gaussian integral, we find

$$\frac{1}{(2\pi)^3} \int d\overline{\lambda} \exp\left[i\overline{\lambda}(\overline{p} - m^*\overline{v}(t)) - \frac{\lambda^2 m^{*2}}{2} F(0)\right]$$
$$= \left[\frac{2\pi}{m^{*2}F(0)}\right]^{3/2} \exp\left\{-\frac{[\overline{p} - m^*\overline{v}(t)]^2}{2m^{*2}F(0)}\right\}.$$

Thus, if the initial statistical operator for the model system has the form

$$\mathscr{D}_{t0} = \rho(S) \mathscr{D}_L(\Sigma),$$

and $\rho(S)$ depends neither on V nor on t_0, the corresponding distribution function of the momenta $\overline{p}$ in the limit $V \to \infty$,

$$\lim_{V\to\infty} w_t(\overline{p}) = \lim_{V\to\infty} \mathscr{Z}_t,$$

approaches the stationary distribution function:

$$\lim_{(V\to\infty)} \{ w_t(\overline{p}) -$$
$$- \left(\frac{1}{m^*}\right)^3 \left(\frac{2\pi}{F(0)}\right)^{3/2} \exp\left[-\frac{(\overline{p} - m^*\overline{v}(t))^2}{2m^{*2}F(0)}\right] \} \to 0, \quad t \to \infty. \tag{149}$$

It can be seen that this stationary distribution function with respect to the momenta $\overline{p}$ is a "shifted" Maxwellian function.

Thus, the use of $\overline{H}_L$ as an approximating Hamiltonian involves the assumption that a shifted Maxwellian distribution can serve as the original approximation.

APPENDIX 1

LEMMA. The expectation value of the product of the operator $\tilde{b}_k(t)$ and an operator $\mathfrak{A}(S, \Sigma)$ satisfies the relation

$$\underset{(S,\,\Sigma)}{\mathrm{Sp}}\ \tilde{b}_k(t)\,\mathfrak{A}(S,\ \Sigma)\,\mathscr{D}_{t0}=\frac{1}{1-\exp(-\beta\hbar\omega_k)}\ \underset{(S,\,\Sigma)}{\mathrm{Sp}}\ \{\tilde{b}_k(t)\,\mathfrak{A}(S,\Sigma)-\mathfrak{A}(S,\ \Sigma)\,\tilde{b}_k(t)\}\,\mathscr{D}_{t0}.$$

Proof. We note that the Bose operators b_k commute with arbitrary operators of the electron subsystem $\Phi(S)$. We form an operator expression averaged with respect to the statistical operator of the complete system, i.e., $\mathscr{D}_{t0}=\rho(S)\mathscr{D}(\Sigma)$:

$$\underset{(S,\,\Sigma)}{\mathrm{Sp}}\ \tilde{b}_k(t)\,\mathfrak{A}(S,\ \Sigma)\,\mathscr{D}_{t0}=\underset{(S,\,\Sigma)}{\mathrm{Sp}}\ \tilde{b}_k(t)\,\mathfrak{A}(S,\ \Sigma)\,\rho(S)\,\mathscr{D}(\Sigma)$$
$$=\underset{(\Sigma)}{\mathrm{Sp}}\,\tilde{b}_k(t)\,\{\underset{S}{\mathrm{Sp}}\ \mathfrak{A}(S,\ \Sigma)\,\rho(S)\}\,\mathscr{D}(\Sigma);$$
$$\underset{(S,\,\Sigma)}{\mathrm{Sp}}\ \mathfrak{A}(S,\ \Sigma)\,\tilde{b}_k(t)\,\mathscr{D}_{t0}=\underset{(S,\,\Sigma)}{\mathrm{Sp}}\ \mathfrak{A}(S,\ \Sigma)\,\tilde{b}_k(t)\,\rho(S)\,\mathscr{D}(\Sigma)$$
$$=\underset{(\Sigma)}{\mathrm{Sp}}\,\{\underset{(S)}{\mathrm{Sp}}\ \mathfrak{A}(S,\ \Sigma)\,\rho(S)\}\,\tilde{b}_k(t)\,\mathscr{D}(\Sigma).$$

We denote $\underset{(S)}{\mathrm{Sp}}\mathfrak{A}(S,\Sigma)\rho(S)=\mathfrak{B}(\Sigma)$; then

$$\left.\begin{aligned}&\underset{(S,\,\Sigma)}{\mathrm{Sp}}\,\tilde{b}_k(t)\,\mathfrak{A}(S,\ \Sigma)\,\mathscr{D}_{t0}=\underset{(\Sigma)}{\mathrm{Sp}}\,b_k(t)\,\mathfrak{B}(\Sigma)\,\mathscr{D}(\Sigma);\\&\underset{(S,\,\Sigma)}{\mathrm{Sp}}\ \mathfrak{A}(S,\ \Sigma)\,\tilde{b}_k(t)\,\mathscr{D}_{t0}=\underset{(\Sigma)}{\mathrm{Sp}}\,\mathfrak{B}(\Sigma)\,\tilde{b}_k(t)\,\mathscr{D}(\Sigma).\end{aligned}\right\}\qquad\text{(A.1)}$$

We recall here an important property of equilibrium mean values in statistical mechanics. We consider an isolated dynamical system characterized by some Hamiltonian H that does not depend on the time, and two dynamical variables A and B corresponding to this system that do not depend explicitly on the time t.

Then for the equilibrium mean values

$$\langle A(t)\,B\rangle_{\mathrm{eq}}=\mathrm{Sp}\,A(t)\,B\mathscr{D}_{\mathrm{eq}};\quad\langle BA(t)\rangle_{\mathrm{eq}}=\mathrm{Sp}\,BA(t)\,\mathscr{D}_{\mathrm{eq}},$$

in which

$$A(t)=\exp(\mathrm{i}Ht/\hbar)\,A(0)\exp(-\mathrm{i}Ht/\hbar),$$

we have

$$\langle A(t)\,B\rangle_{\mathrm{eq}}=\int\limits_{-\infty}^{+\infty}J(\omega)\exp(-\mathrm{i}\omega t)\,d\omega;$$
$$\langle BA(t)\rangle_{\mathrm{eq}}=\int\limits_{-\infty}^{+\infty}\exp(-\beta\hbar\omega)\,J(\omega)\exp(-\mathrm{i}\omega t)\,d\omega.$$

We write these relations in the form

$$\left.\begin{aligned}&\mathrm{Sp}\,\{\exp[\mathrm{i}H(t-t_0)/\hbar]\,A\exp[-\mathrm{i}(t-t_0)/\hbar]\,B\mathscr{D}_{\mathrm{eq}}\}\\&=\int\limits_{-\infty}^{+\infty}J(\omega)\exp[-\mathrm{i}\omega(t-t_0)]\,d\omega;\\&\mathrm{Sp}\,\{B\exp[\mathrm{i}(t-t_0)/\hbar]\,A\exp[-\mathrm{i}(t-t_0)/\hbar]\,\mathscr{D}_{\mathrm{eq}}\}\\&=\int\limits_{-\infty}^{+\infty}\exp(-\beta\hbar\omega)\,J(\omega)\exp[-\mathrm{i}\omega(t-t_0)]\,d\omega.\end{aligned}\right\}\qquad\text{(A.2)}$$

We now take our system Σ as this system and set

$$H=H(\Sigma);\quad\mathscr{D}_{\mathrm{eq}}=\mathscr{D}(\Sigma);\quad A=b_k,\quad B=\mathfrak{B}(\Sigma).$$

We note also that in this case

$$\tilde{b}_k(t)=\exp[-\mathrm{i}\omega_k(t-t_0)]\,b_k$$
$$=\exp[\mathrm{i}(t-t_0(H/\hbar]\,b_k\exp[-\mathrm{i}(t-t_0)\,H/\hbar].$$

Thus, Eq. (A.2) is reduced to the form

$$\left.\begin{aligned}&\underset{(\Sigma)}{\mathrm{Sp}}\,\tilde{b}_k\mathfrak{B}(\Sigma)\,\mathscr{D}(\Sigma)=\exp[-\mathrm{i}\omega_k(t-t_0)]\,\underset{(\Sigma)}{\mathrm{Sp}}\,b_k\mathfrak{B}(\Sigma)\,\mathscr{D}(\Sigma)\\&=\int\limits_{-\infty}^{+\infty}J_k(\omega)\exp[-\mathrm{i}\omega(t-t_0)]\,d\omega;\\&\underset{(\Sigma)}{\mathrm{Sp}}\,\mathfrak{B}(\Sigma)\,\tilde{b}_k(t)\,\mathscr{D}(\Sigma)=\exp[-\mathrm{i}\omega_k(t-t_0)]\,\underset{(\Sigma)}{\mathrm{Sp}}\,\mathfrak{B}(\Sigma)\,b_k\mathscr{D}(\Sigma)\\&=\int\limits_{-\infty}^{+\infty}\exp(-\beta\hbar\omega)\,J_k(\omega)\exp[-\mathrm{i}\omega(t-t_0)]\,d\omega.\end{aligned}\right\}\qquad\text{(A.3)}$$

These relations show that $J_k(\omega)$ is proportional to $\delta(\omega-\omega_k)$:

$$J_k(\omega)=I_k\delta(\omega-\omega_k)$$

and hence

$$\exp(-\beta\hbar\omega)\,J_k(\omega)=\exp(-\beta\hbar\omega_k)\,J_k(\omega).$$

Therefore, from (A.3) we obtain

$$\underset{(\Sigma)}{\mathrm{Sp}}\,\mathfrak{B}(\Sigma)\,\tilde{b}_k(t)\,\mathscr{D}(\Sigma)$$
$$=\exp(-\beta\hbar\omega_k)\,\underset{(\Sigma)}{\mathrm{Sp}}\,\tilde{b}_k(t)\,\mathfrak{B}(\Sigma)\,\mathscr{D}(\Sigma).$$

We use (A.1):

$$\underset{(S,\,\Sigma)}{\mathrm{Sp}}\ \mathfrak{A}(S,\Sigma)\,\tilde{b}_k(t)\,\mathscr{D}_{t0}$$
$$=\exp(-\beta\hbar\omega_k)\ \underset{(S,\,\Sigma)}{\mathrm{Sp}}\ \tilde{b}_k(t)\,\mathfrak{A}(S,\ \Sigma)\,\mathscr{D}_{t0},$$

which gives

$$\underset{(S,\,\Sigma)}{\mathrm{Sp}}\ \{\tilde{b}_k(t)\,\mathfrak{A}(S,\ \Sigma)-\mathfrak{A}(S,\ \Sigma)\,\tilde{b}_k(t)\}\,\mathscr{D}_{t0}$$
$$=[1-\exp(-\beta\hbar\omega_k)]\ \underset{(S,\,\Sigma)}{\mathrm{Sp}}\ \tilde{b}_k(t)\,\mathfrak{A}(S,\ \Sigma)\,\mathscr{D}_{t0}.$$

We now see that

$$\left.\begin{aligned}&\underset{(S,\,\Sigma)}{\mathrm{Sp}}\ \tilde{b}_k(t)\,\mathfrak{A}(S,\Sigma)\,\mathscr{D}_{t0}\\&=\frac{1}{1-\exp(-\beta\hbar\omega_k)}\ \underset{(S,\,\Sigma)}{\mathrm{Sp}}\ \{\tilde{b}_k(t)\,\mathfrak{A}(S,\ \Sigma)-\mathfrak{A}(S,\ \Sigma)\,\tilde{b}_k(t)\}\,\mathscr{D}_{t0};\\&\underset{(S,\,\Sigma)}{\mathrm{Sp}}\ \mathfrak{A}(S,\ \Sigma)\,\tilde{b}_k(t)\,\mathscr{D}_{t0}\\&=\frac{\exp(-\beta\hbar\omega_k)}{1-\exp(-\beta\hbar\omega_k)}\ \underset{(S,\,\Sigma)}{\mathrm{Sp}}\ \{\tilde{b}_k(t)\,\mathfrak{A}(S,\ \Sigma)-\mathfrak{A}(S,\ \Sigma)\,b_k(t)\}\,\mathscr{D}_{t0}.\end{aligned}\right\}\qquad\text{(A.4)}$$

Thus, we have proved the lemma.

We denote $\exp(-\beta\hbar\omega_k)/[1-\exp(-\beta\hbar\omega_k)]=N_k$. Then the relation (A.4) can be expressed in terms of the population numbers $b_k^{+}b_k$:

$$\underset{(S,\,\Sigma)}{\mathrm{Sp}}\ \tilde{b}_k(t)\,\mathfrak{A}(S,\Sigma)\,\mathscr{D}_{t0}=(1+N_k)\ \underset{(S,\,\Sigma)}{\mathrm{Sp}}\ \{\tilde{b}_k(t)\,\mathfrak{A}(S,\ \Sigma)-\mathfrak{A}(S,\ \Sigma)\,\tilde{b}_k(t)\}\,\mathscr{D}_{t0};$$
$$\underset{(S,\,\Sigma)}{\mathrm{Sp}}\ \mathfrak{A}(S,\ \Sigma)\,\tilde{b}_k(t)\,\mathscr{D}_{t0}=N_k\ \underset{(S,\,\Sigma)}{\mathrm{Sp}}\ \{\tilde{b}_k(t)\,\mathfrak{A}(S,\ \Sigma)-\mathfrak{A}(S,\ \Sigma)\,\tilde{b}_k(t)\}\,\mathscr{D}_{t0}.$$

Remark. The proof of the lemma might give the impression that the operator $\mathfrak{A}(S,\Sigma)$ must not depend explicitly on the time t. It is, however, readily seen that the validity of the lemma for operators that do not depend explicitly on the time leads directly to its validity as well for the general case of operators that do depend explicitly on t.

Indeed, let us consider the operator $\mathfrak{A}(t,S,\Sigma)$, fixing $t=t_1$. Then the operator $\mathfrak{A}(t_1,S,\Sigma)$ does not formally depend on the time and the relation (A.4) is obtained for it. Since t_1 can be fixed arbitrarily, in the relations (A.4) we can set $t_1=t$ and prove the validity of our assertion.

APPENDIX 2

Our assertion concerning the validity of (127) and (128) will be proved if we can prove the following lemma.

LEMMA. Let $\mathbf{A}_N$ and $\mathbf{B}_N$ be sequences of real three-vectors that tend to finite limits as $N\to\infty$:

$$\mathbf{A}_N\longrightarrow\mathbf{A},\quad\mathbf{B}_N\longrightarrow\mathbf{B}\quad\text{as}\quad N\longrightarrow\infty.\qquad\text{(A.5)}$$

Then, if our statistical operator $\rho(S)$ does not depend on N,

$$\underset{(S)}{\mathrm{Sp}}\exp[\mathrm{i}(\mathbf{A}_N\mathbf{r}+\mathbf{B}_N\mathbf{p})]\,\rho(S)\longrightarrow\underset{(S)}{\mathrm{Sp}}\exp[\mathrm{i}(\mathbf{A}\mathbf{r}+\mathbf{B}\mathbf{p})]\,\rho(S).\qquad\text{(A.6)}$$

Indeed, for the relation (127) it is only necessary to set in this lemma

$$N=V,\ \mathrm{B}_N=-\mathrm{k}\int_{\tau}^{t} g_0(\delta,\ \sigma-t_0)\,d\sigma;\ \mathrm{A}_N=\mathrm{k}\int_{\tau}^{t} g_1(\delta,\ \sigma-t_0)\,d\sigma;$$

$$\mathrm{B}=-\mathrm{k}\int_{\tau}^{t}\psi_0(\sigma-t_0)\,d\sigma;\ \mathrm{A}=\mathrm{k}\int_{\tau}^{t}\psi_1(\sigma-t_0)\,d\sigma.$$

In the case of the limit relation (128), we can take $N=t-t_0$, with τ fixed, and

$$\mathrm{B}_N=-\mathrm{k}\int_{\tau}^{t}\psi_0(\sigma-t_0)\,d\sigma;\ \mathrm{A}_N=\mathrm{k}\int_{\tau}^{t}\psi_1(\sigma-t_0)\,d\sigma;$$

$$\mathrm{B}=0;\ \mathrm{A}=0.$$

We therefore turn to the proof of the lemma.

Proof. Since the commutator of the components of the vectors p and r are C numbers, $p_\alpha r_\beta - r_\beta p_\alpha = -i\hbar\delta_{\alpha\beta}$, it follows by the well-known identity

$$\exp[\mathrm{i}(\mathrm{C}_1\mathbf{r}+\mathrm{C}_2\mathbf{p})]\equiv\exp[\mathrm{i}\hbar\mathrm{C}_1\mathrm{C}_2/2]\exp(\mathrm{i}\mathrm{C}_1\mathbf{r}_1)\exp(\mathrm{i}\mathrm{C}_2\mathbf{p}_2)$$

that we can write

$$\begin{aligned}&\exp[\mathrm{i}(\overline{A}_N\bar{r}+\overline{\mathfrak{B}}_N\bar{p})-\exp[\mathrm{i}(\overline{A}\bar{r}+\overline{\mathfrak{B}}\bar{p})]\\&=\exp(\mathrm{i}\hbar\overline{A}_N\overline{B}_N/2)\exp(\mathrm{i}\overline{A}_N\bar{r})\exp(\mathrm{i}\overline{\mathfrak{B}}_N\bar{p})\\&\quad-\exp(\mathrm{i}\hbar\overline{A}\,\overline{B}/2)\exp(\mathrm{i}\overline{A}\,\bar{r})\exp(\mathrm{i}\overline{B}\bar{p})\\&=[\exp(\mathrm{i}\hbar\overline{A}_N\overline{B}_N/2)-\exp(\mathrm{i}\hbar\overline{A}\,\overline{B}/2)]\exp(\mathrm{i}\overline{A}_N\bar{r})\exp(\mathrm{i}\overline{B}_N\bar{p})\\&\quad+\exp(\mathrm{i}\hbar\overline{A}\,\overline{B}/2)\{\exp(\mathrm{i}\overline{A}_N\bar{r})[\exp(\mathrm{i}\overline{B}_N\bar{r})-\exp(\mathrm{i}\overline{B}\,\bar{p})]\\&\quad+\exp(\mathrm{i}\overline{A}_N\bar{r})-\exp(\mathrm{i}\overline{A}\bar{r})]\exp(\mathrm{i}\overline{B}\,\bar{p})\}.\end{aligned}$$

But, since the operator $\exp(i\overline{A}_N\bar{r})\exp(i\overline{B}_N\bar{p})$ is unitary because A_N and B_N are real,

$$|\mathop{\mathrm{Sp}}_{(S)}\exp(\mathrm{i}\overline{A}_N\bar{r})\exp(\mathrm{i}\overline{B}_N\bar{p})\,\rho(S)|\leqslant 1$$

and therefore

$$\begin{aligned}&|\mathop{\mathrm{Sp}}_{(S)}(\exp[\mathrm{i}(\overline{A}_N\bar{r}+\overline{B}_N\bar{p})]\rho(S))-\mathop{\mathrm{Sp}}_{(S)}(\exp[\mathrm{i}(\overline{A}\,\bar{r}+\overline{B}\,\bar{p})]\rho(S))|\\&\qquad\leqslant|\exp[\mathrm{i}\hbar(\overline{A}_N\overline{B}_N-\overline{A}\,\overline{B})/2]-1|\\&\qquad+|\mathop{\mathrm{Sp}}_{(S)}\exp(\mathrm{i}\overline{A}_N\bar{r})[\exp(\mathrm{i}\overline{B}_N\bar{p})-\exp(\mathrm{i}\overline{B}\,\bar{p})]\rho(S)|\\&\qquad+|\mathop{\mathrm{Sp}}_{(S)}[\exp(\mathrm{i}\overline{A}_N\bar{r})-\exp(-\mathrm{i}\overline{A}\bar{r})]\exp(\mathrm{i}\overline{B}\,\bar{p})\rho(S)|.\end{aligned}\tag{A.7}$$

Bearing in mind that $\rho(S)$ is non-negative, we have the general inequality (see Appendix 3)

$$|\mathop{\mathrm{Sp}}_{(S)}UV\rho(S)|^2\leqslant\mathop{\mathrm{Sp}}_{(S)}U\overset{+}{U}\rho(S)\mathop{\mathrm{Sp}}_{(S)}\overset{+}{V}V\rho(S).\tag{A.8}$$

We use this inequality to estimate the first and second expression with the trace symbol $\mathop{\mathrm{Sp}}_{(S)}$ on the right-hand side of (A.7), setting

$$U_{\mathrm{I}}=\exp(\mathrm{i}\overline{A}_N\bar{r});\quad V_{\mathrm{I}}=[\exp(\mathrm{i}\overline{B}_N\bar{p})-\exp(\mathrm{i}\overline{B}\bar{p})];$$

$$U_{\mathrm{II}}=[\exp(\mathrm{i}\overline{A}_N\bar{r})-\exp(\mathrm{i}\overline{A}\bar{r})];\quad V_{\mathrm{II}}=\exp(\mathrm{i}\overline{B}\bar{r}).$$

It can be seen that V_{II} and U_{I} are unitary and, since $\mathop{\mathrm{Sp}}_{(S)}\rho(S)=1$, we have

$$\mathop{\mathrm{Sp}}_{(S)}U_{\mathrm{I}}\overset{+}{U}_{\mathrm{I}}\rho(S)=\mathop{\mathrm{Sp}}_{(S)}\overset{+}{V}_{\mathrm{II}}V_{\mathrm{II}}\rho(S)=1.$$

Further, since the components of the vector $\bar{r}$ commute with one another, like the components of $\bar{p}$, we obtain

$$U_{\mathrm{II}}U^{+}_{\mathrm{II}}=2(1-\cos(\overline{A}_N-\overline{A})\bar{r});\quad V^{+}_{\mathrm{I}}V_{\mathrm{I}}=2(1-\cos(\overline{B}_N-\overline{B})\bar{p}).$$

Thus, from (A.7), we find

$$\begin{aligned}&|\mathop{\mathrm{Sp}}_{(S)}\{\exp[\mathrm{i}(\overline{A}_N\bar{r}+\overline{B}_N\bar{p})]\rho(S)\}-\mathop{\mathrm{Sp}}_{(S)}\{\exp[\mathrm{i}(\overline{A}\bar{r}+\overline{B}\bar{p})]\rho(S)\}|\\&\leqslant\sqrt{2\{1-\cos[\hbar(\overline{A}_N\overline{B}_N-AB)/2]\}}+\sqrt{\mathop{\mathrm{Sp}}_{(S)}2(1-\cos(\overline{B}_N-\overline{B})\bar{p})\rho(S)}\\&\qquad+\sqrt{\mathop{\mathrm{Sp}}_{(S)}2(1-\cos(\overline{A}_N-A)\bar{r})\rho(S)}.\end{aligned}\tag{A.9}$$

We consider the matrix elements of $\rho(S)$ in the $\bar{r}$ representation, $\langle\bar{r}|\rho(S)|\bar{r}'\rangle$, and in the $\bar{p}$ representation, $\langle\bar{p}|\rho(S)|\bar{p}'\rangle$. Then

$$\mathop{\mathrm{Sp}}_{(S)}(1-\cos(\overline{B}_N-\overline{B})\bar{p})\rho(S)=\int[1-\cos(\overline{B}_N-\overline{B})\bar{p}]\langle\bar{p}|\rho(S)|\bar{p}\rangle\,d\bar{p};$$

$$\mathop{\mathrm{Sp}}_{(S)}[1-\cos(\overline{A}_N-\overline{A})\bar{r}]\rho(S)=\int[1-\cos(\overline{A}_N-\overline{A})\bar{r}]\langle\bar{r}|\rho(S)|\bar{r}\rangle\,d\bar{r}.$$

But the diagonal elements of a non-negative operator are not negative,

$$\langle\bar{p}|\rho(S)|\bar{p}\rangle\geqslant 0;\quad\langle\bar{r}|\rho(S)|\bar{r}\rangle\geqslant 0$$

and, since $\mathrm{Sp}\rho(S)=1$,

$$\int\langle\bar{p}|\rho(S)|\bar{p}\rangle\,d\bar{p}=1;\quad\int\langle\bar{r}|\rho(S)|\bar{r}\rangle\,d\bar{r}=1.$$

Noting that $\rho(S)$ *does not depend on* N, $(1-\cos x)\leqslant 2$, $[1-\cos(\overline{A}_N-\overline{A})\bar{r}]\to 0$ for bounded $\bar{r}$ and $[1-\cos(\overline{B}_N-\overline{B})\bar{p}]\to 0$ for bounded $\bar{p}$ as $N\to\infty$, we see that

$$\mathop{\mathrm{Sp}}_{(S)}(1-\cos(\overline{A}_N-\overline{A})\bar{r})\rho(S)\underset{N\to\infty}{\to}0;$$

$$\mathop{\mathrm{Sp}}_{(S)}(1-\cos(\overline{B}_N-\overline{B})\bar{p})\rho(S)\underset{N\to\infty}{\to}0.$$

Thus, on the basis of (A.9) we conclude that the lemma is proved.

It can be seen that the inequality (A.9) is valid irrespective of whether $\rho(S)$ depends on N or not.

In addition, it is clear that $2(1-\cos x)<x^2$, and therefore

$$\begin{aligned}&|\mathop{\mathrm{Sp}}_{(S)}\{\exp[\mathrm{i}(\overline{A}_N\bar{r}+\overline{B}_N\bar{p})]\rho(S)\}-\mathop{\mathrm{Sp}}_{(S)}\{\exp[\mathrm{i}(\overline{A}r+\overline{B}p)]\rho(S)\}|\\&\leqslant\hbar|\overline{A}_N\overline{B}_N-\overline{A}\overline{B}|/2+\sqrt{\mathop{\mathrm{Sp}}_{(S)}|\bar{p}|^2\rho(S)}\,|B_N-B|+|A_N-A|\sqrt{\mathop{\mathrm{Sp}}_{(S)}|\bar{r}|^2\rho(S)}.\end{aligned}$$

Therefore, if $\rho(S)$ does depend on N but is such that

$$\mathop{\mathrm{Sp}}_{(S)}|\bar{p}|^2\rho(S)\leqslant k_1^2;\quad\mathop{\mathrm{Sp}}_{(S)}|\bar{r}|^2\rho(S)\leqslant k_2^2,$$

where k_1 and k_2 *do not depend on* N, the relation (A.6) is valid.

Thus, if in Sec. 3 $\rho(S)$ depends on t_0 and V, but such that

$$\langle\bar{p}^2\rangle_{t_0}=\mathop{\mathrm{Sp}}_{(S)}|\bar{p}|^2\rho(S);\quad\langle\bar{r}^2\rangle_{t_0}=\mathop{\mathrm{Sp}}_{(S)}|\bar{r}|^2\rho(S)$$

are bounded quantities that depend neither on V nor on t_0, all the arguments of Sec. 3 remain valid.

APPENDIX 3

We consider the expectation value of the product $\langle AB\rangle$ of two operators as a bilinear form in A and B (linear with respect to each of these operators).

Let $Z(A,B)$ be an arbitrary bilinear form in A and B with the properties

$$Z(\overset{+}{A},\ A)\geqslant 0;\tag{A.10}$$

$$\{Z(A,\ B)\}^*=Z(\overset{+}{B},\ \overset{+}{A}).\tag{A.11}$$

We show that we always have

$$|Z(A,\ B)|^2\leqslant Z(A,\ \overset{+}{A})\,Z(\overset{+}{B},\ B).\tag{A.12}$$

Setting here $A=U\sqrt{\rho(S)}$ and $B=V\sqrt{\rho(S)}$, we arrive at the inequality (A.8).

The proof of this inequality is contained in Ref. 4. We give this proof. We note that on the basis of (A.10)

$$Z(xA+\overset{*+}{y}B,\ \overset{*}{x}A^{+}+yB)\geqslant 0,\tag{A.13}$$

where x and y are arbitrary numbers.

Hence, expanding this expression, we obtain

$$x\overset{*}{x}Z(A, \overset{+}{A}) + xyZ(A, B) + \overset{**}{y}\overset{*}{x}Z(\overset{+}{B}, \overset{+}{A}) + \overset{*}{y}yZ(\overset{+}{B}, B) \geqslant 0.$$

We take the numbers $x, y, \overset{*}{x}, \overset{*}{y}$ to be

$$\overset{*}{x} = -Z(A, B); \quad x = -\{Z(A, B)\}^* = -Z(\overset{+}{B}, \overset{+}{A}); \quad y = \overset{*}{y} = Z(A, \overset{+}{A}).$$

Then

$$-|Z(A, B)|^2 Z(A, \overset{+}{A}) + \{Z(A, \overset{+}{A})\}^2 Z(\overset{+}{B}, B) \geqslant 0.$$

Hence, if $Z(A, \overset{+}{A}) \neq 0$, we obtain the inequality (A.12).

It only remains to prove that if

$$Z(A, \overset{+}{A}) = 0,$$

then also

$$Z(A, B) = 0. \tag{A.14}$$

For this, we set in (A.13)

$$x^* = -Z(A, B)R; \quad x = -Z(\overset{+}{B}, \overset{+}{A})R, \quad y = y^* = 1,$$

where R is an arbitrary positive number. We find

$$-2R|Z(A, B)|^2 + Z(\overset{+}{B}, B) \geqslant 0. \tag{A.15}$$

Suppose $R \to \infty$; then if (A.14) is false, we see that the left-hand side must tend to $-\infty$, and this is impossible.

APPENDIX 4

Let Γ be a quadratic positive-definite form in the Bose operators b_α^+ and b_α.

We denote by $Z = \mathrm{Sp}\exp(-\beta\Gamma)$ the partition function and consider the linear forms $A_1, A_2, \ldots, A_s$ constructed from the Bose operators b_α^+ and b_α, and the expectation value of the product of the operators $A_1 \ldots A_s$:

$$\langle A_1 A_2 \ldots A_s \rangle = Z^{-1}\,\mathrm{Sp}\exp(-\beta\Gamma) A_1 \ldots A_s. \tag{A.16}$$

We apply to (A.16) the well-known Bloch–De Dominicis theorem, which generalizes Wick's theorem. Introducing the pairings $\overline{A_j A_l} = \langle A_j A_l \rangle_\Gamma$, we see that (A.16) is equal to the sum of all possible pairings.

For example,

$$\langle A_1A_2A_3A_4\rangle_\Gamma = \langle \overline{A_1 \overline{A_2 A_3} A_4}\rangle + \langle \overline{A_1 A_2 A_3} A_4\rangle + \langle \overline{A_1 A_2}\,\overline{A_3 A_4}\rangle$$
$$= \langle A_1A_4\rangle\langle A_2A_3\rangle + \langle A_1A_3\rangle\langle A_2A_4\rangle + \langle A_1A_2\rangle\langle A_3A_4\rangle.$$

Of course, the expression (A.16) vanishes if s is odd, since then one of the products of the operators $A_1, \ldots, A_s$ remains unpaired and $\langle A_j\rangle_\Gamma = 0$, since A_j is a linear form in b_α and b_α^+; Γ is a quadratic form.

We now use this well-known technique to calculate the expression $\langle e^A\rangle_\Gamma$ (A is a linear form in the considered Bose operators). The upshot is

$$\langle e^A\rangle_\Gamma = \sum_{n=0}^{\infty}\frac{1}{n!}\langle A^n\rangle_\Gamma = \sum_{k=0}^{\infty}\frac{1}{(2k)!}\langle A^{2k}\rangle_\Gamma = 1 + \sum_{k=1}^{\infty}\frac{1}{(2k)!}\langle A^{2k}\rangle_\Gamma. \tag{A.17}$$

By virtue of the Bloch–De Dominicis theorem, $\langle A^{2k}\rangle_\Gamma = G(k)\langle A^2\rangle_\Gamma^k$, where $G(k)$ is the number of all possible ways of pairing in the expression $\langle \underbrace{A_1 \ldots A_s}_{2k} \rangle_\Gamma$.

We see that

$$G(1) = 1;$$
$$G(2) = 3;$$
$$\ldots\ldots$$
$$G(k+1) = (2k+1)\,G(k).$$

Thus, $G(k) = 1 \times 3 \ldots (2k-1)$ and

$$\frac{G(k)}{(2k)!} = \frac{1\cdot 3 \ldots (2k-1)}{1\cdot 2\cdot 3\cdot 4 \ldots (2k)} = \frac{1}{2^k k!}.$$

Hence

$$\langle e^A\rangle = 1 + \sum_{k=1}^{\infty}\frac{1}{k!}\left\langle \frac{1}{2}A^2\right\rangle_\Gamma^k = e^{\langle A^2\rangle_\Gamma/2}.$$

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