# Field-theoretical description of deep inelastic scattering

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**The most important theoretical notions concerning deep inelastic scattering are reviewed. Topics discussed are the model-independent approach, which is based on the general principles of quantum field theory, the application of quantum chromodynamics to deep inelastic scattering, approaches based on the quark-parton model, the light cone algebra, and conformal invariance, and also investigations in the framework of perturbation theory.**

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## INTRODUCTION

The investigation of deep inelastic lepton–hadron scattering at high energies is one of the methods of studying the structure of hadrons and the dynamics of their interaction.

Because the electromagnetic or weak coupling constants are small, the scattering of leptons on hadrons can be treated in the first order of perturbation theory. In this approximation, the scattering amplitude $T_{fi}$ splits up into the readily calculated leptonic part $l_\mu(k, k')$ and the investigated hadronic part

$$h_\mu(p, p_x) = \langle X | j_\mu(0) | p, s\rangle.$$

Here, $k$ and $k'$ are the momenta of the initial and final leptons, and $p$ and $p_x$ are the momenta of the ingoing and outgoing hadrons. The hadronic part $h_\mu(p, p_x)$ corresponds to scattering of a virtual particle (for example a virtual photon) on a hadron. There is no significant difference between electron and neutrino scattering by hadrons. In both cases, the amplitude $T_{fi}$ contains the matrix elements of the hadronic currents, namely, in the first case those of the electromagnetic current $j_\mu^{em}(x)$ and in the second case those of the weak currents $j_\mu^{\pm}(x)$.

By deep inelastic scattering one understands the elastic scattering of particles at high energies with a large value of the square $q^2$ of the mass[1] of the virtual intermediate particle.

Besides $q^2$, we shall in what follows use the independent variable $\nu = 2(pq)$; in the laboratory system,

$$q^2 = -4EE' \sin^2(\theta/2); \quad \nu = 2M(E - E'),$$

where $M$ is the nucleon mass, $\theta = \theta_{\text{lab}}$ is the scattering angle in the laboratory system, and $E$ and $E'$ are the energies of the initial and final leptons.

The basic entities for describing deep inelastic scattering are the *invariant* amplitudes $T_i$, which follow from the kinematic expansion of the virtual amplitudes $T_{\mu\nu}$, and the *structure* functions $W_i$, which arise from the kinematic expansion of the absorptive parts $W_{\mu\nu}$ of the virtual amplitudes. The connections between $T_i$ and $T_{\mu\nu}$, and also between $W_i$ and $W_{\mu\nu}$, are given by the expressions $T_{\mu\nu} = \sum_i K^i_{\mu\nu} T_i$ and $W_{\mu\nu} = \sum_i K^i_{\mu\nu} W_i$.

Individually, for *eN scattering* we have[2]

$$T = (e^2/q^2)\bar{u}(k')\gamma^\mu u(k)\langle X | j_\mu^{em}(0) | p, s\rangle; \tag{1}$$

$$d^2\sigma/d|q^2|d\nu = (e^4/8\pi M^2 q^4)(E'/E)\{\cos^2(\theta/2) M^2 W_2 + 2\sin^2(\theta/2) W_1\}; \tag{2}$$

$$W_{\mu\nu} = \frac{1}{8\pi}\sum_{s=1,2}\int d^4x \exp(iqx)\langle p, s | [j_\mu^{em}(x), j_\nu^{em}(0)] | p, s\rangle$$
$$= (-g_{\mu\nu} + q_\mu q_\nu/q^2) W_1 + (p_\mu - pq q_\mu/q^2)(p_\nu - pq q_\nu/q^2) W_2; \tag{3}$$

for $\nu N$ ($\bar{\nu}N$) scattering,

$$\left.\begin{aligned} &T^{\nu(\bar{\nu})} = (G/\sqrt{2})\, l_\mu^{\mp}(k, k')\langle X | j_\mu^{\pm}(0) | p, s\rangle;\\ &l_\mu^- = \bar{u}(k')\gamma_\mu(1+\gamma_5)u(k);\\ &l_\mu^+ = \bar{v}(k)\gamma_\mu(1-\gamma_5)v(k'); \end{aligned}\right\} \tag{4}$$

$$d^2\sigma^{\nu(\bar{\nu})}/d|q^2|d\nu = (G^2/4\pi M^2)(E'/E)\{\cos^2(\theta/2) M^2 W^{\nu(\bar{\nu})}$$
$$+ 2\sin^2(\theta/2) W_1^{\nu(\bar{\nu})} \mp M(E+E') W_3^{\nu(\bar{\nu})}\sin^2(\theta/2)\} + O(m_\mu^2) W_{4,5}^{\nu(\bar{\nu})}; \tag{5}$$

$$W_{\mu\nu}^{\nu(\bar{\nu})} = \frac{1}{8\pi}\sum_{s=1,2}\int d^4x \exp(iqx)\langle p, s | [j_\mu^{\mp}(x), j_\nu^{\pm}(0)] | p, s\rangle$$
$$= (-g_{\mu\nu} + q_\mu q_\nu/q^2) W_1^{\nu(\bar{\nu})} + (p_\mu - pq q_\mu/q^2)(p_\nu - pq q_\nu/q^2) W_2^{\nu(\bar{\nu})}$$
$$- (i/2)\varepsilon_{\mu\nu\rho\sigma} p_\rho q_\sigma W_3^{\nu(\bar{\nu})} + q_\mu q_\nu W_4^{\nu(\bar{\nu})} + (p_\mu q_\nu + p_\nu q_\mu) W_5^{\nu(\bar{\nu})}. \tag{6}$$

Estimates of the influence of higher orders on these quantities are basically model-independent,[3] since they do not refer to the radiative corrections of the leptonic parts, and they will therefore not be considered in what follows.

The scaling principle enables one to make a number of assertions about the asymptotic behavior of the structure functions.[1,2] This principle requires that the functions that are encountered depend only on dimensionless combinations of the independent variables.

For reasons of convenience, we shall in what follows express all dimensions in terms of the dimensions of $\nu$. Since the dimension of $W_1$ is 0, and the dimensions of $W_2$, $W_3$, and $\sigma$ are 2, the quantity $W_1$ must depend on only a single dimensionless variable, which it is convenient to take to be

$$\xi = -q^2/\nu, \tag{7}$$

and $W_2$ and $W_3$ must behave as $\nu^{-1}$. Thus, the structure functions $W_i$ behave asymptotically as

$$W_1(p, q) = F_1(\xi);$$
$$W_2(p, q) = \nu^{-1} F_2(\xi);$$
$$W_3(p, q) = \nu^{-1} F_3(\xi). \tag{8}$$

The functions $F_i(\xi)$ are called scaling functions.

For the experimental verification of the asymptotic behavior (8), it is expedient to investigate deep inelastic

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scattering in the *Bjorken region*:

$$\nu\to\infty;\quad q^2\to-\infty;\quad \xi=-q^2/\nu_{\text{fixed}}\,. \tag{9}$$

The experimental data[4] agree in practice with the theoretically predicted behavior (8).

## 1. MODEL-INDEPENDENT DESCRIPTION OF DEEP INELASTIC LEPTON–HADRON SCATTERING

*Dyson–Jost–Lehmann Representation of the Structure Functions and Invariant Amplitudes.* As we say in the Introduction, the description of deep inelastic lepton–nucleon scattering in the first order in the electromagnetic (or weak) coupling constant leads to the investigation of the virtual amplitudes and their absorptive parts:

$$T_{\mu\nu}=\frac{i}{4\pi}\sum_{s=1,\,2}\int d^4x\exp(iqx)\langle p,\ s|Tj_\mu(x)\,j_\nu(0)|p,\ s\rangle; \tag{10}$$

$$W_{\mu\nu}=\frac{1}{8\pi}\sum_{s=1,\,2}\int d^4x\exp(iqx)\langle p,\ s|[j_\mu(x),\ j_\nu(0)]|p,\ s\rangle. \tag{11}$$

In this section, we shall restrict ourselves to deep inelastic electron–nucleon scattering. We consider the virtual Compton amplitude and its absorptive part. The kinematic expansion of $W_{\mu\nu}$ gives in this case

$$W_{\mu\nu}=(-g_{\mu\nu}+q_\mu q_\nu/q^2)\,W_1+(p_\mu-pqq_\mu/q^2)\,(p_\nu-pqq_\nu/q^2)\,W_2 \tag{12}$$

or also

$$W_{\mu\nu}=(-g_{\mu\nu}q^2+q_\mu q_\nu)\,V_1+[p_\mu p_\nu q^2-(p_\mu q_\nu+p_\nu q_\mu)\,pq+g_{\mu\nu}(pq)^2]\,V_2. \tag{13}$$

The invariant functions $W_i$ (or $V_i$), $i=1,2$, which occur here can be determined directly from the effective cross section (2). Note that all the results can be transferred in essence to neutrino–nucleon scattering as well.

Model-independent investigations can obviously be based on only those properties of $T_{\mu\nu}$ and $W_{\mu\nu}$ that follow from the general principles of quantum field theory such as Lorentz invariance, causality, and the spectral condition.[5] The subject of the present investigation is provided by the invariant structure functions $W_i$ and the corresponding invariant amplitudes $T_i$. In what follows, they will be required in different functional dependences, and we shall always denote them by symbols such as $W_i(p,q)=W_i(\nu,q^2)=W_i(\nu,\xi)=W_i(\xi,q^2)$ and $\widetilde{W}_i(p,x)=\widetilde{W}(px,x^2)$. On $W_i$ we impose the following conditions:

covariance:

$$W_i(p,\ q)=W_i(\nu,\ q^2); \tag{14}$$

symmetry:

$$W_i(\nu,\ q^2)=-W_i(-\nu,\ q^2); \tag{15}$$

Hermiticity:

$$W_i^*(\nu,\ q^2)=W_i(\nu,\ q^2); \tag{16}$$

the spectral condition:

$$W_i(\nu,\ q^2)=0,\quad -q^2/|\nu|>1; \tag{17}$$

causality:

$$\widetilde{W}(px,\ x^2)=\frac{1}{(2\pi)^4}\int d^4q\exp(-iqx)\,W_i(p,\ q)=0,\ x^2<0 \tag{18}$$

(and similarly for $T_i$).

By virtue of the nonlocal nature of the kinematic tensors occurring in the expansion (12), the causality property of the structure functions $\widetilde{W}_i(p,x)$ must of course be proved specially.[6]

Functions that satisfy the conditions (14)–(18) can be represented in a form that was determined by Dyson, Jost, and Lehmann.[7] This Dyson–Jost–Lehmann representation in the nucleon rest frame can be written in the form

$$W_i(\nu,\ q^2)=\varepsilon(q_0)\int d^3\mathbf{u}\int d\lambda^2\delta(q_0^2-(\mathbf{k}-\mathbf{q})^2-\lambda^2)\,\psi_i(\mathbf{u},\ \lambda^2), \tag{19}$$

the support of the spectral function $\psi_i(\mathbf{u},\lambda^2)$ lying in the region

$$|\mathbf{u}|\leqslant M;\quad \lambda^2\geqslant(M-\sqrt{M^2-u^2})^2. \tag{20}$$

The corresponding representation for the Fourier transform is written in the form

$$\widetilde{W}_i(x,\ p)=-\frac{i}{2\pi}\int_0^\infty d\lambda^2\Delta(x,\ \lambda^2)\,\widetilde{\psi}_i(x,\ \lambda^2) \tag{21}$$

with

$$\Delta(x,\ \lambda^2)=\frac{i}{(2\pi)^3}\int d^3q\exp(-iqx)\,\varepsilon(q_0)\,\delta(q^2-\lambda^2)$$
$$=(\varepsilon(x_0)/2\pi)\,(\partial/\partial x^2)\,\{\theta(x^2)\,J_0(\lambda\sqrt{x^2})\} \tag{22}$$

and

$$\widetilde{\psi}_i(x,\ \lambda^2)=\int d^3\mathbf{u}\exp(iux)\,\psi_i(\mathbf{u},\ \lambda^2). \tag{23}$$

*Relations between Asymptotic Quantities.* Intuitive arguments based on extending the principle of scale invariance to deep inelastic scattering suggest that particular attention must be paid to the behavior of the structure functions[1] in the Bjorken region (9). In Refs. 6 and 9, it was shown by means of the Dyson–Jost–Lehmann representation that the neighborhood of the light cone[8] in $X$ space corresponds to the Bjorken region (9).

**Quasilimit of generalized functions; theoretical formulation of the Bjorken limit and asymptotic behavior on the light cone.** The amplitudes (10) and their Fourier transforms are generalized functions. We introduce the concept of a quasilimit[9] (denoted by the symbol $q$-lim), which determines the asymptotic behavior of these functions.

*DEFINITION.* A generalized function $f(x)\in M_+$ behaves in the sense of a quasilimit when $x\to+\infty$ as $x^k$ or, more precisely,

$$q-\lim_{x\to+\infty}f(x)=cx_+^k/\Gamma(k+1), \tag{24}$$

if

$$\lim_{t\to\infty}t^{-k}(f(tx),\ \Phi(x))=\left(c\frac{x_+^k}{\Gamma(k+1)},\ \Phi(x)\right)\forall\Phi\in\varphi(R^1). \tag{25}$$

A generalized function $f(x)\in\bar{M}_+$ behaves in the sense of a quasilimit when $x\to+0$ as $x^k$ or, more precisely,

$$q-\lim_{x\to+0}f(x)=cx_+^k/\Gamma(k+1), \tag{26}$$

if

$$\lim_{t\to\infty}(t^k(f(x/t),\ \Phi(x))$$
$$=(cx_+^k/\Gamma(k+1),\ \Phi(x))\,\forall\Phi\in\varphi(R^1). \tag{27}$$

Here, $M_+\in\varphi'(R^1)$ is the set of distributions in $\varphi'(R^1)$ that in the limit $x\to-\infty$ behave as test functions, and $\bar{M}_+$ is the set of tempered distributions whose Fourier transforms belong to $M_+$. In what follows, we shall use the space $\varphi'$ of tempered distributions with support on the positive axis, for which it is sufficient to use test functions with support on $R^1_+$. In the elucidation of the part played by the light cone in deep inelastic scattering, the representation (19) for the structure functions and the representation (21) for their Fourier transforms are important; they contain important information that follows from the general principles of quantum field theory. The spectral functions $\psi_i$ here play the part of averaged quantities.

In accordance with (24), the quasilimit of the spectral function $\psi_i(\mathbf{u}, \lambda^2)$ can be written in the form

$$q-\lim_{\lambda^2\to\infty} \psi(\mathbf{u}, \lambda^2) = \psi_0(\mathbf{u})(\lambda^2)^k_+/\Gamma(k+1), \tag{28}$$

i.e.,

$$\lim_{t\to\infty} t^{-k}(\psi(\mathbf{u}, t\lambda^2), \Phi(\mathbf{u}, \lambda^2))$$
$$= \left(\psi_0(\mathbf{u})\frac{(\lambda^2)^k_+}{\Gamma(k+1)}, \Phi(\mathbf{u}, \lambda^2)\right) \forall\Phi\in\varphi_3\times\varphi_+. \tag{29}$$

To smear the boundary value of $\bar{W}_i(x, p)$, we go over to the symmetric commutator $\tilde{W}(x, p)$ extended with respect to $x_0$; this is defined by the relation

$$(\tilde{W}(x, p), \Phi(x_0\mathbf{x})) = (\bar{W}(x^2, \mathbf{x}), (1/2)(x^2+\mathbf{x}^2)^{-1/2}$$
$$\times[\Phi(\sqrt{x^2+\mathbf{x}^2}, \mathbf{x}) - \Phi(-\sqrt{x^2+\mathbf{x}^2}, \mathbf{x})], \tag{30}$$

in which $\Phi(x_0, \mathbf{x})$ is a test function in $\varphi(R^4)$, and, thus,

$$(x^2+\mathbf{x}^2)^{-1/2}[\Phi\sqrt{x^2+\mathbf{x}^2}, \mathbf{x}) - \Phi(-\sqrt{x^2+\mathbf{x}^2}, \mathbf{x})]$$

is a test function in $\varphi(R^3)\times\varphi_+$. Therefore, $\bar{W}$ is a generalized function of $x^2$ and $\mathbf{x}$.

We then find the *asymptotic behavior on the light cone* as follows. Using the definition (26), we conclude that on the light cone, i.e., as $x^2\to 0$, $\bar{W}(x^2, \mathbf{x})$ behaves as $(x^2)^k$. Thus, for real $k$

$$\lim t^k(\bar{W}(x^2/t, \mathbf{x}), \Phi(x^2))$$
$$= (G(x)(x^2)^k_+/\Gamma(k+1), \Phi(x^2)) \forall\Phi\in\varphi_+. \tag{31}$$

As will be shown later [see the expression (51)], $\bar{W}(x^2, \mathbf{x})$ is an entire analytic function in $\mathbf{x}$, so that the use of a test function with respect to these variables is completely justified. In contrast to this, integration in (29) with a test function in the variable $\mathbf{u}$ is impossible.

Strictly speaking, the structure functions $W_i(\nu, q^2)$ are generalized functions with respect to $q_0$ and $q$. Of course, the positivity property strongly restricts the admissible class of functions. It follows from the Dyson–Jost–Lehmann representation (19) that the functional

$$\mathcal{F}_W(\nu) = \int d\xi W(\nu, \xi)\Phi(\xi), \quad \Phi\in D, \tag{32}$$

is a smooth function with respect to $\nu$ [this is obtained from the value of the integral (40); see below]. One can therefore define the Bjorken limit of the structure functions in terms of the classical asymptotic behavior of the functional (32) by

$$B_j-\lim_{\nu\to\infty} \nu^{-k}W(\nu, q^2) = W_0(\xi), \tag{33}$$

if

$$\lim_{\nu\to\infty} \nu^{-k}\int d\xi W(\nu, \xi)\Phi(\xi) = \int d\xi W_0(\xi)\Phi(\xi) \forall\Phi\in D. \tag{34}$$

This definition is fairly general; the boundary value is obtained as a result of integration with a δ function.

Equivalence of the Bjorken limit and the light-cone asymptotic behavior. The existence of boundary values of $W(\nu, q^2)$, $\bar{W}(x^2, \mathbf{x})$, or $\psi(\mathbf{u}, \lambda^2)$ cannot be proved without recourse to a definite field-theory model. Rather, the problem is to prove the equivalence of these boundary values. To obtain quantitative relations between the boundary values, it is expedient to assume in advance[6,9] the existence of a quasilimit of $\psi$ and of the Bjorken limit (33), and also to prove the existence, which follows from the validity of Eq. (28), of the light-cone asymptotic behavior (31). Let us dwell on this question in more detail.

We begin by proving the existence of the light-cone asymptotic behavior of $\bar{W}(x^2, \mathbf{x})$, for which we investigate the functional

$$(\bar{W}(x^2, \mathbf{x}), \Phi(x^2)) \forall\Phi\in\varphi_+.$$

For $\bar{W}$, we use the Dyson–Jost–Lehmann representation

$$(\bar{W}(x^2, \mathbf{x}), \Phi(x^2)) = -\frac{i}{2\pi}\int dx^2\Phi(x^2)\frac{1}{2\pi}$$
$$\times\frac{\partial}{\partial x^2}\left\{\theta(x^2)\int_0^\infty d\lambda^2 J_0(\lambda\sqrt{x^2})\,\tilde{\psi}(\mathbf{x}, \lambda^2)\right\}$$
$$= \frac{i}{4\pi^2}\int d\lambda^2\tilde{\psi}(\mathbf{x}, \lambda^2)\int dx^2\frac{\partial\Phi}{\partial x^2}J_0(\lambda\sqrt{x^2})\,\theta(x^2)$$
$$= -\frac{i}{4\pi^2}(\tilde{\psi}(\mathbf{x}, \lambda^2), \Phi_B(\lambda^2)). \tag{35}$$

We obtain the function $\Phi_B(\lambda^2)$ by means of the transformation

$$B:\ \Phi(x^2)\to\Phi_B(\lambda^2) = -\int dx^2\,\theta(x^2)\,J_0(\lambda\sqrt{x^2})\frac{\partial\Phi}{\partial x^2}, \tag{36}$$

$\varphi_+$ being mapped to $\varphi_+$ (Ref. 9). If we also proceed from

$$(\bar{W}(x^2/t, \mathbf{x}), \Phi(x^2)) = -(i/4\pi^2)t^2(\tilde{\psi}(\mathbf{x}, t\lambda^2), \Phi_B(\lambda^2)),$$

it then follows from Eq. (28) and the definition of a quasilimit that

$$\lim_{t\to\infty} t^{-k-2}(\bar{W}(x^2/t, \mathbf{x}), \Phi(x^2))$$
$$= -(i/4\pi^2)\lim_{t\to\infty} t^{-k}(\tilde{\psi}(\mathbf{x}, t\lambda^2), \Phi_B(\lambda^2))$$
$$= -(i/4\pi^2)(\psi_0(\mathbf{x})(\lambda^2)^k_+/\Gamma(k+1), \Phi_B(\lambda^2))$$
$$= -\frac{i}{4\pi^2}\tilde{\psi}_0(\mathbf{x})\int dx^2\,\Phi(x^2)\frac{\partial}{\partial x^2}\left\{\theta(x^2)\int d\lambda^2 J_0(\lambda\sqrt{x^2})\frac{(\lambda^2)^k_+}{\Gamma(k+1)}\right\}.$$

Therefore, the existence of the quasilimit of $\bar{W}(x^2, \mathbf{x})$ is explained by the existence of the quasilimit of $\psi(\mathbf{u}, \lambda^2)$. If, in addition, we use the relation

$$\frac{\partial}{\partial x^2}\left\{\theta(x^2)\int d\lambda^2 J_0(\lambda\sqrt{x^2})\frac{(\lambda^2)^k_+}{\Gamma(k+1)}\right\}$$
$$= -4^{k+1}(x^2)^{-k-2}_+/\Gamma(-k-1), \tag{37}$$

we obtain

$$q-\lim_{x^2\to 0}\bar{W}(x^2, \mathbf{x}) = (i/4\pi^2)\tilde{\psi}_0(\mathbf{x})[4^{k+1}/\Gamma(-k-1)](x^2)^{-k-2}_+. \tag{38}$$

Sometimes one writes formally

$$\bar{W}(x^2, \mathbf{x}) = G(x_0)(x^2)^{-k-2}_+/\Gamma(-k-1), \tag{39}$$

and, noting that $\tilde{\psi}_0(x) = \tilde{\psi}_0(\mathbf{x}^2)$ and $x_0^2 = \mathbf{x}^2$, one obtains

$$G(x_0) = (i4^k/\pi^2)\psi_0(\mathbf{x}).$$

Similarly, we conclude that the existence of the quasilimit (38) on the light cone implies the existence of the quasilimit (28) of the spectral function. The proof is very lengthy, and we therefore omit it.

We now show that the existence of a Bjorken limit of the structure function $W(\nu, q^2)$ follows from the existence of the quasilimit (28) of the spectral function $\psi(\mathbf{u}, \lambda^2)$.

We investigate the behavior of the functional

$$(W(\nu, \xi), \Phi(\xi)) = \varepsilon(q_0) \int d\xi \int d^3\mathbf{u} \int d\lambda^2 \times \psi(\mathbf{u}, \lambda^2)\, \delta(q_0^2 - (\mathbf{q}-\mathbf{k})^2 - \lambda^2)\, \Phi(\xi), \quad \Phi \in D. \tag{40}$$

We transform this expression to make $\Phi$ become a test function with respect to u and $\lambda^2$:

$$(W(\xi, \nu), \Phi(\xi)) = \frac{\varepsilon(q_0)}{\nu} \int d^3\mathbf{u} \int d\lambda^2 \psi(\mathbf{u}, \lambda^2) \times \int d\xi \delta\left(\xi + \frac{\mathbf{u}^2+\lambda^2}{\nu} - \frac{\mathbf{ue}}{M}\sqrt{M^2 + \frac{4\xi M}{\nu}}\right) \Phi(\xi)$$
$$= \frac{\varepsilon(q_0)}{\nu} \int d^3\mathbf{u} \int d\lambda^2 \psi(\mathbf{u}, \lambda^2)\, \Phi(-\lambda^2/\nu + \mathbf{ue}/M + \chi_1/\nu)(1 + \chi_2/\nu) \tag{41}$$

with $\mathbf{e} = \mathbf{q}/|\mathbf{q}|$. The functions $\chi_i(\mathbf{u}, \nu, \lambda^2/\nu), i = 1, 2$, together with all their derivatives in the domain of integration are bounded if $\nu$ is sufficiently large. Therefore, the functions

$$\Phi_\nu(\mathbf{u}, \tau^2) = \Phi(-\tau^2 + \mathbf{ue}/M + \chi_1(\mathbf{u}, \nu, \tau^2)/\nu)(1 + \chi_2(\mathbf{u}, \nu, \tau^2)/\nu)$$

in the limit $\nu \to \infty$ converge in the region $\tau^2 = \lambda^2/\nu \geqslant 0$, $|\mathbf{u}| \leqslant M$. Since $\lim_{\nu\to\infty} \Phi_\nu = \Phi(-\tau^2 + \mathbf{ue}/M)$ is a test function, we can use the definition (28). We obtain

$$\lim_{\nu\to\infty} \nu^{-k} \int d\xi W(\xi, \nu)\, \Phi(\xi)$$
$$= \lim_{\nu\to\infty} \nu^{-k} \int d^3\mathbf{u} \int d\tau^2 \psi(\mathbf{u}, \nu\tau^2)\, \Phi_\nu(\mathbf{u}, \tau^2)$$
$$= \int d^3\mathbf{u}\psi_0(\mathbf{u}) \int d\tau^2 \frac{(\tau^2)_+^k}{\Gamma(k+1)} \Phi(-\tau^2 + \mathbf{ue}/M)$$
$$= \int d\xi \Phi(\xi) \int d^3\mathbf{u}\psi_0(\mathbf{u}) \frac{(\mathbf{ue}/M - \xi)_+^k}{\Gamma(k+1)}.$$

From this there follows the existence of the Bjorken limit (33) as a consequence of the existence of the quasilimit (28) of the spectral function, i.e.,

$$B_j - \lim_{\nu\to\infty} \nu^{-k} W(\nu, q^2) = \int d^3\mathbf{u}\psi_0(\mathbf{u}) \frac{(\mathbf{ue}/M - \xi)_+^k}{\Gamma(k+1)} \equiv W_0(\xi). \tag{42}$$

From Eq. (42) for the scaling function $W_0(\xi)$ we obtain the following properties:

1. $W_0(\xi)$ is a generalized function and therefore cannot be defined at every point.

2. The support of $W_0(\xi)$ lies in the interval $(-\infty, +1]$; since $k$ is not integral, the support extends to $-\infty$.

3. For $k = -1, -2, \ldots$ the support is bounded by the interval $[-1, +1]$, and

$$W_0(\xi) = (-1)^{k+1} W_0(-\xi); \tag{43}$$

for neutrino–nucleon scattering, we obtain

$$W_0^{\nu}(\xi) \equiv (-1)^{k+1} W_0^{\bar{\nu}}(-\xi). \tag{44}$$

4. For $k = 0, 1, 2, \ldots, W_0(\xi)$ in the region $\xi < -1$ is a polynomial of degree $k+1$.

5. The following connection can be established between the coefficient functions of the relations (42) and (39):

$$G(x_0) = (4^k/i\pi^2) \exp[i\pi(k+1)/2](x_0 - i\varepsilon)^{k+1} \times \int_{-\infty}^{1} d\xi \exp(ix_0\xi)\, W_0(\xi). \tag{45}$$

The proof of the existence of the quasilimit (28) of the spectral function from the relation (42) is fairly lengthy, and we therefore merely refer to the literature.[9] Thus, the asymptotic behavior of the structure functions and, therefore, the absorptive part of the virtual Compton amplitude in the Bjorken region is uniquely related to the behavior of the current commutator on the light cone:

$$B_j - \lim_{\nu\to\infty} W_i(\nu, q^2) \Longleftrightarrow q - \lim_{\lambda^2\to+\infty} \psi_i(\mathbf{u}, \lambda^2)$$
$$\Longleftrightarrow q - \lim_{x^2\to 0} \overline{W}_i(x^2, \mathbf{x}).$$

The scaling function $W_0(\xi)$ must be known as a generalized function over its support. Note that from the asymptotic behavior of this function in the physically accessible region $0 < \xi < 1$ one cannot draw a conclusion about the dominant singularity on the light cone, when $W_0(\xi)$ defined by Eq. (11) vanishes.[10,11]

**Characterization of the Regge limit.** We begin with the difference between the Bjorken limit (33) and the Regge limit, i.e., the limit on the mass shell. The Regge limit reflects the asymptotic behavior of the scattering amplitude as $\nu \to \infty$ for a fixed value of $q^2$, which corresponds formally to the Bjorken limit at the point $\xi = 0$. It is well known that in many cases the experiments indicate different behaviors of this amplitude in the Bjorken and Regge regions. Let us take, for example, the structure function $V_2(\nu, q^2)$, which is related to the total effective cross section of real photons by

$$V_2(\nu, q^2 = 0) = -(2/\pi e^2)\, \nu^{-1} \sigma_t(\nu).$$

If the total cross section $\sigma_t^{(\nu)}$ tends asymptotically, i.e., as $\nu \to \infty$, to a constant, then

$$V_2(\nu, q^2 = 0) \sim \nu^{-1},$$

whereas in the Bjorken limit one expects the asymptotic behavior

$$V_2(\nu, q^2) \sim \nu^{-2}.$$

It should be said that such behavior of the function $V_2$ is determined by different mechanisms and different properties of the structure functions. The general definition of the Regge limit in the framework of the Dyson–Jost–Lehmann representation is given by means of the quasilimit of the functional $\int dq_0 W(q_0, q^2) \Phi(q_0)$:

$$\int dq_0 W(q_0, q^2)\, \Phi(q_0) = 2y \int d\pi \chi(y)\, H(y), \tag{46}$$

where

$$H(y) = \int dq_0^2 \frac{\Phi(q_0) - \Phi(-q_0)\, \theta(q_0^2 - q^2 - y)}{rq_0 \sqrt{q_0^2 - q^2}} \tag{47}$$

is a test function with respect to $y$; $\chi(y)$ can be expressed in terms of the spectral function

$$\chi(y) = \int d\rho \rho^2 \psi(\rho, 2py - \rho^2 + q^2). \tag{48}$$

The equation

$$(W(tq_0, q^2), \Phi(q_0)) = 2\pi(\chi(ty), H(y))$$

can also be understood as a quasilimit of $\chi(y)$, this determining the Regge limit of $W(q_0, q^2)$.

In contrast to the Regge limit, the Bjorken limit of the structure function is determined by means of the quasilimit of the spectral function as $\lambda^2 \to \infty$. Therefore, there are no correlations between the Regge limit and the singularities on the light cone of the current commutator.

Note that the behavior $V_2(\nu, q^2) \sim \nu^{-2}$ at $q^2 = 0$ is open to great doubt.[10] On the one hand, the singularity that appears in the function $W_0(\xi)$ at $\xi=0$ leads to different behaviors of the structure functions in the Bjorken and Regge limits. On the other hand, regular behavior of $W_0(\xi)$ at $\xi=0$ does not prevent the structure functions from having terms that are negligible in the Bjorken limit but important in the Regge limit.

*Moments of the Structure Functions and Singularities on the Light Cone.* Of all the numerous studies of deep inelastic scattering, the greatest success has been achieved by dynamical theories based on non-Abelian gauge field theories,[12] which we consider in Sec. 2. They make essential use of a light-cone expansion[13] for the current commutator:

$$[j(x), j(0)]_{x^2=0} \approx \sum_{n=0} c_n(x^2)\, x^{\mu_1} \ldots x^{\mu_n} O^n_{\mu_1\ldots\mu_n}(0). \quad (49)$$

It is shown that the operators $O^n_{\mu_1\ldots\mu_n}(x)$ with singularities on the light cone may be operators of different degrees.

In this section, we shall investigate the connection between the light-cone expansion (49) and the observable quantities, the moments of the structure functions. We again consider the matrix element

$$\sum_s \langle p, s\,|\,[j_\mu(x), j_\nu(0)]\,|\,p, s\rangle,$$

whose structure functions can be expressed by means of the Dyson–Jost–Lehmann representation, and for the invariant function $\overline{W}(x^2, x)$ we use the everywhere convergent Taylor expansion, which in the case of existence is identical with the expansion (49). We then determine the moments $\mu_n(q^2)$ of the structure functions $W(\nu, q^2)$ and investigate their asymptotic connection with the coefficients $f_n(x^2)$ of this Taylor expansion. It can then be shown that the relations between the Bjorken limit and the light-cone asymptotic behavior presented in the previous subsection can be extended under certain restrictions to asymptotic relations between $\mu_n(q^2)$ and $f_n(x^2)$.

**Taylor expansion of the asymptotic commutator.** We consider first the Taylor expansion for the symmetric commutator $\overline{W}(x^2, \mathbf{x})$ (Ref. 14). For this, we write Eq. (35) in the form

$$(\overline{W}(x^2, x),\ \Phi(x^2)) = \frac{1}{4i\pi^2}\int d^3\mathbf{u}\, \exp(i\mathbf{ux})\,(\psi(\mathbf{u}, \lambda^2), \Phi_B(\lambda^2)). \quad (50)$$

The bracketed quantity on the right-hand side is a generalized function with compact support with respect to $\mathbf{u}$, so that its Fourier transform is an entire analytic function with respect to $x^2$ with Taylor expansion

$$(\overline{W}(x^2, \mathbf{x}), \Phi) = \frac{1}{4i\pi^2}\sum_{n=0}^{\infty}\frac{i^{2n}}{(2n)!}(\mathbf{x}^2)^n \frac{1}{2n+1} \times \int d^3\mathbf{u}\,(u^2)^n (\psi(\mathbf{u}, \lambda^2),\ \Phi_B(\lambda^2))$$
$$= \frac{1}{4i\pi^2}\sum_{n=0}^{\infty}\frac{i^{2n}(x^2)^n}{(2n)!\,(2n+1)}\int d^3\mathbf{u}\,(\mathbf{u}^2)^n \int dx^2\,\Phi(x^2) \times \frac{\partial}{\partial x^2}\left\{\theta(x^2)\int d\lambda^2 J_0(\lambda\sqrt{x^2})\,\psi(\mathbf{u}, \lambda^2)\right\}.$$

We have here used (36). Thus, for $W(x^2, \mathbf{x})$ we obtain the representation

$$W(x^2, \mathbf{x}) = \frac{1}{4i\pi^2}\sum_{n=0}^{\infty}\frac{i^{2n}}{(2n)!}(\mathbf{x}^2)^n f_{2n}(x^2), \quad (51)$$

where

$$f_{2n}(x^2) = \frac{\partial}{\partial x^2}\left\{\theta(x^2)\int d\lambda^2 J_0(\lambda\sqrt{x^2})\,h_{2n}(\lambda^2)\right\}; \quad (52)$$

and

$$h_{2n}(\lambda^2) = \frac{1}{2n+1}\int d^3\mathbf{u}\,(\mathbf{u}^2)^n \psi(\mathbf{u}, \lambda^2). \quad (53)$$

The series (51), as can be seen from its derivation, in a global representation whose convergence is not restricted to only the region of the light cone.

On the one hand, such a far-reaching assertion cannot of course be made for the light-cone expansion (49) in operator form; on the other hand, if it exists in some sense, then its single-particle matrix element near the light cone coincides with the expansion (51).

We make some comments concerning the light-cone singularities that appear in Eq. (51).

We must distinguish two cases:

*Case 1.* All the coefficients $f_{2n}(x^2)$ have the same quasilimit on the light cone. Then the polynomial coefficients in the quasilimit are summed to the entire function $G(x_0)$ which appears in Eq. (39).

*Case 2.* The majority of the coefficients $f_{2n}(x^2)$ have the same quasilimit on the light cone. Then $G(x_0)$ is reduced to a polynomial. Further, one can readily show that the contribution made by each individual term of the expression (51) to the structure function has in the momentum space a support that is not spacelike, i.e., the experimentally accessible region is enlarged. This follows from the fact that each term of the structure $(x_i)^l f_{2n}(x^2)$ can be represented in terms of the spectral function $(i\partial/\partial u_i)^l \delta(\mathbf{u}) h_{2n}(\lambda^2)$, which is convergent for $\mathbf{u}=0$ and which can then be substituted in the Dyson–Jost–Lehmann representation (19). But in the general case $W(\nu, q^2)$ may have spacelike support. Later, we shall consider an example when such behavior is compatible with causality and the spectral condition.

**Connection between the coefficients of the Taylor expansion and the moments of the structure functions.** We consider quantities that are accessible to experimental study and correspond to the terms of the light-cone expansion (51). The *Cornwall–Norton* moments are defined as follows[15,16]:

$$\mu_{2n}(q^2) = \int_0^1 d\xi\, \xi^{2n-1} W(\xi, q^2). \quad (54)$$

For the following investigation, which is based on the three-dimensional Dyson–Jost–Lehmann representation, it is expedient to introduce other modified moments:

$$\hat{\mu}_{2n}(q^2) = \int_0^1 d\eta\, \eta^{2n-1} W(\eta, q^2). \quad (55)$$

Here, the variable $\eta$ in $W(\eta, q^2)$ appears as a function of $q^2$ and $\nu$:

$$\eta = -q^2/2M\,|\mathbf{q}| = -(q^2/\nu)\,(1-4q^2M^2/\nu^2)^{-1/2}$$
$$\equiv \xi\,(1-4\xi^2M^2/q^2)^{-1/2}. \quad (56)$$

As follows from (56), the moments (54) and (55) are equal in the limit $q^2 \to -\infty$.

On the basis of the Dyson–Jost–Lehmann representation for the structure function $W(\nu, q^2)$ and using the variables $\eta$ and $Q^2=-q^2$ instead of $\nu$ and $q^2$, it is necessary to obtain only integral representations for the moment $\hat{\mu}_{2n}(q^2)$ (Ref. 14). To show that $W(\eta, Q^2)$ for $Q^2>0$ is a generalized function with respect to $\eta$, we investigate by means of the test function $\Phi(\eta)$ the functional

$$(W(\eta, Q^2), \Phi(\eta)) = 2\pi Q^2 \int_0^M d\rho\rho^2 \int_{-1} dz \times \int d\lambda^2 \frac{\rho z \Psi(\rho, \lambda^2)}{(Q^2+\rho^2+z^2)^2} \Phi\left(\frac{\rho z Q^2}{M(Q^2+\rho^2+\lambda^2)}\right), \quad (57)$$

which is defined if the integration over $d\lambda^2$ converges. The differentiability of $\Phi$ with respect to $\eta$ is, naturally, transferred to the variables $\rho$ and $\lambda^2$. Note that convergence as $\lambda^2 \to \infty$ is no longer satisfied. The reason for this lies in the unrealistic choice of the variables; for example, the functional $\mathscr{F}_W(\nu) = \int d\xi W(\nu, \xi)\Phi(\xi)$ is always defined, whereas in the case of the functional

$$\mathscr{F}(q^2) = \int d\xi W(\nu = -q^2/\xi, q^2)\Phi(\xi)$$

we encounter convergence difficulties like those in the expression (57). If we recall that the moments $\mu_{2n}(Q^2)$ and $\hat{\mu}_{2n}(Q^2)$ (generalized) can be defined as the partial derivatives of the amplitude,

$$\mu_{2n}(Q^2) = \frac{\pi}{2}\frac{(Q^2)^{2n}}{(2n)!}\left(\frac{\partial}{\partial\nu}\right)^{2n} T(\nu, Q^2)|_{Q^2_{\text{fixed}},\ \nu=0}, \quad (58)$$

and accordingly

$$\hat{\mu}_{2n}(Q^2) = \frac{\pi}{2^{2n}}\frac{(Q^2)^{2n}}{(2n)!}\Delta_q^n T(\nu, Q^2)|_{Q^2_{\text{fixed}},\ q=0},$$

the connection between the difficulties mentioned above with the convergence of the integral (57) and the subtractional problem of the Dyson–Jost–Lehmann representation for $T(\nu, Q^2)$ becomes obvious.

We return to the expression (57) and replace $\Phi(\eta)$ by $\eta^{2n-1}$, and we then obtain the moments in (55) in the form

$$\hat{\mu}_{2n}(Q^2) = (Q^2)^{2n} \int_0^\infty d\lambda^2 \frac{\hat{h}_{2n}(\lambda^2)}{(Q^2+\lambda^2)^{2n+1}}, \quad (59)$$

where

$$\hat{h}_{2n}(\lambda^2) = \frac{1}{2n+1}\int d^3\mathbf{u}(\mathbf{u}^2)^n \psi(\mathbf{u}, \lambda^2 - \mathbf{u}^2). \quad (60)$$

Depending on the nature of the behavior of $\psi(\mathbf{u}, \lambda^2)$, which increases with respect to $\lambda^2$, several moments remain undetermined. From the expressions (52), (53), (59), and (60) there follow a number of relations that connect the functions $f_{2n}(x^2)$ and $\hat{\mu}_{2n}(Q^2)$ in which we are interested by means of the spectral function:

$$f_{2n}(x^2) = \frac{\partial}{\partial x^2}\left\{\theta(x^2)\int_0^\infty d\lambda^2 J_0(\lambda\sqrt{x^2})\, h_{2n}(\lambda^2)\right\}; \quad (61)$$

$$h_{2n}(\lambda^2) = \frac{1}{2n+1}\int d^3\mathbf{u}(\mathbf{u}^2)^n \psi(\mathbf{u}, \lambda^2); \quad (62)$$

$$\hat{h}_{2n}(\lambda^2) = \frac{1}{2n+1}\int d^3\mathbf{u}(\mathbf{u}^2)^n \psi(\mathbf{u}, \lambda^2 - \mathbf{u}^2); \quad (63)$$

$$\hat{\mu}_{2n}(Q^2) = (Q^2)^{2n}\int d\lambda^2 (Q^2+\lambda^2)^{-2n-1}\hat{h}_{2n}(\lambda^2). \quad (64)$$

From these relations there follows a connection between $f_{2n}(x^2)$ and $\hat{\mu}_{2n}(Q^2)$.

Taking into account the results of the dynamical theory of Ref. 12 and experimental data, one can explain the logarithmic departure from scaling behavior by introducing the concept of a modified quasilimit. We give the following *definition*: A generalized function $f(x)$ has a modified quasilimit of order $k$ as $x\to\infty$ if

$$\lim_{t\to\infty} \frac{1}{t^k L(t)}(f(tx),\ \Phi(x)) = (f_\infty(x),\ \Phi(x)), \quad (65)$$

where $L(t)$ is a suitably chosen, weakly increasing function with the property

$$\lim\,[L(at)/L(t)] = 1. \quad (66)$$

As $L(t)$ one can, for example, take the function $\ln t$ or $\ln\ln t$. In the general case, the condition (66) characterizes an asymptotic power-law behavior which is somewhat modified, depending on the choice of the weakly increasing function $L(t)$.

By means of this generalization we can define the asymptotic behavior of the functions $f_{2n}(x^2)$, $h_{2n}(\lambda^2)$, and $\hat{h}_{2n}(\lambda^2)$ by using this concept of a modified quasilimit.

The existence of a quasilimit for $\hat{h}_{2n}(\lambda^2)$ implies the existence of a quasilimit for $\hat{\mu}_{2n}(Q^2)$ as an analytic function in the $Q^2$ plane cut along the negative axis. Therefore, the limit of the modifed moment $\mu_{2n}(Q^2)$ can be measured experimentally.

To be able to establish relations between the singularities on the light cone of the coefficients $f_{2n}(x^2)$ and the asymptotic behavior of the moments $\hat{\mu}_{2n}(Q^2)$, it is necessary to establish the correspondence of the boundary values of $q-\lim_{x^2\to 0} f_{2n}(x^2)$, $q-\lim_{\lambda^2\to\infty} h_{2n}(\lambda^2)$, $q-\lim_{\lambda^2\to\infty}\hat{h}_{2n}(\lambda^2)$, and $\lim_{Q^2\to\infty}\hat{\mu}_{2n}(Q^2)$, with one another. Without going into details, we give the results obtained in Ref. 14. From the relation

$$\lim_{t\to\infty}\frac{t^{-2-k_n}}{L_n(t)}(f_{2n}(x^2/t),\ \Phi(x^2)) = (f^0_{2n}(x^2),\ \Phi(x^2)) \quad (67)$$

there follows

$$\lim_{Q^2\to\infty}\frac{(Q^2)^{k_n}}{L_n(Q^2)}\hat{\mu}_{2n}(Q^2) = \text{const.} \quad (68)$$

In general, the converse is not true; it is only under certain restrictions imposed on the spectral functions that the validity of (67) follows from fulfillment of Eq. (68). We shall not dwell on this in detail, but merely mention that these restrictions are in reality equivalent to the fact that the functions $\hat{\mu}_{2n}(Q^2)$ have in all directions $\arg Q^2 \neq \pm\pi$ the same asymptotic behavior as the left-hand side of (68).

In this connection, it is interesting to make a comparison with Ref. 18. In Ref. 18 the general principles of quantum field theory are used to derive restrictions on the asymptotic behavior in the Bjorken limit, these being related in a known manner to the conditions imposed here on the individual spectral functions.

These conditions are satisfied when the spectral functions $\hat{h}_{2n}$ and $\hat{\mu}_{2n}$ have a positive classical antiderivative function of order $N$, so that one can have recourse to Eq. (67). In the proof of the equivalence of

$$q-\lim_{\lambda^2\to\infty} h_{2n}(\lambda^2) \text{ and } q-\lim_{\lambda^2\to\infty}\hat{h}_{2n}(\lambda^2)$$

needed here, one uses the positivity of $W(\nu, q^2)$; first of all, it is necessary to assume that the difference between the orders of the quasilimits of $f_{2n}$ and $f_{2n+1}$ is less than unity. These restrictions are satisfied in non-Abelian gauge field theories, which agree with the experiments (cf. Sec. 2).

In conclusion, we demonstrate an example of how different behavior of the singularities on the light cone is compatible with causality and the spectral condition.

Suppose the spectral function is given by

$$\psi(\mathbf{u}, \lambda^2) = \theta(\lambda^2 - 1)(\lambda^2)_+^{k(\rho)}$$

with

$$k(\rho) = k_0 - c\rho^r; \quad c > 0; \quad k_0 > 0,$$

and $\rho = |\mathbf{u}|$. There then follows the relation

$$\lim_{t\to\infty} t^{-2-k_0} (l_n t)^{(n+1)/r} (f_{2n}(x^2/t), \Phi(x^2)) = \text{const},$$

whereas the Bjorken limit of the structure functions is

$$B_j - \lim_{\nu\to\infty} \nu^{-k_0} (\ln \nu)^{1/r} W(\nu, q^2) = \text{const}\,(-\xi)_+^{k_0}.$$

Although the scaling function vanishes in a spacelike region, all the moments $\hat{\mu}_{2n}$ are nonzero, and therefore the support of $W(\nu, q^2)$ extends into the spacelike region.

Our results can also be applied to the vacuum expectation value of the current commutator $\langle 0|[j_\mu(x), j_\nu(0)]|0\rangle$. The connection between the invariant commutator $C(x)$ and the spectral function $\rho(s)$ of the Källén–Lehmann representation

$$C(x) = \int_{s_0}^{\infty} ds \rho(s) \Delta(x, s)$$

corresponds to (61), whereas the connection between the vacuum polarization operator $\Pi(q^2)$ and $\rho(s)$

$$Q^2\Pi(Q^2) = \frac{1}{\pi} \int_{s_0}^{\infty} ds\rho(s) [s(s+Q^2)]^{-1}$$

corresponds to (64).

Just as the connection is established between the functions $f_{2n}(x^2)$ and $\hat{\mu}_{2n}(Q^2)$, one can establish the asymptotic connection between $c(x)$, $\Pi(Q^2)$, and $\rho(s)$, which must be understood in the sense of a quasilimit.[17,19]

*Virtual Compton Amplitude, Singularities on the Light Cone, and the Electromagnetic Mass Difference of Nucleons.* To use the investigations made so far to describe the total amplitude (10) of the virtual Compton effect, it is necessary to proceed from the following. First, as was shown, to determine the singularities on the light cone knowledge of the structure functions alone, i.e., the absorptive part of the scattering amplitude, is inadequate. Second, in the investigation of the total virtual Compton amplitude it is necessary to take into account the connection between deep inelastic scattering and the electromagnetic mass corrections of the nucleons.

**Determination of the asymptotic behavior on the light cone from the total amplitude.** In what follows, we shall consider the virtual Compton amplitude, not restricting ourselves to the forward direction:

$$T_{\mu\nu}(p, Q, \Delta) = \frac{i}{4\pi} \int d^4x \exp(iQx) \times \sum_{s_1, s_2} \langle p_1, s_1 | Tj_\mu(x/2) j_\nu(-x/2) | p_2, s_2 \rangle. \qquad (69)$$

The variables are defined as follows:

$$p = (p_1 + p_2)/2; \quad Q = (q_1 + q_2)/2;$$
$$\Delta = p_1 - p_2.$$

The Bjorken region of this process is characterized by the variables

$$\nu = 2pQ \underset{Q^2\to\infty}{\to} +\infty; \quad \{\xi = -Q^2/\nu, \mathbf{u} = Q/|Q|, \Delta\}_{\text{fixed}}.$$

Note that for a real photon the kinematic conditions $q_1^2 = 0$, $q_2^2 > 0$ restrict the variable $\xi$ to the interval $[-1, 0]$.

The amplitude $T_{\mu\nu}(p, Q, \Delta)$, like (12), can be decomposed into invariant amplitudes $T_i$, $i = 1, \ldots, 5$, which satisfy the causality condition.[20] For $T_i$ we write down, like (19), a subtracted Dyson–Jost–Lehmann representation in the Breit system,

$$T(Q, \mathbf{p}) = \mathscr{P}_{N-1}(Q^2, pQ, Q\Delta) - \frac{1}{\pi} \int [d^3\mathbf{u}[Q_0^2 - (\mathbf{Q}-\mathbf{u})^2 + M^2]^N \times \int \frac{d\lambda^2}{(\lambda^2+M^2)^N} \frac{\psi(\mathbf{u}, \mathbf{p}, \lambda^2)}{[Q_0^2 - (\mathbf{Q}-\mathbf{u})^2 - \lambda^2 + i\varepsilon]}, \qquad (70)$$

where $\mathscr{P}_{N-1}$ is a subtraction polynomial of degree $N-1$ in the variables $p = (E_p, 0)$, $\Delta = (0, 2\mathbf{p})$, $\nu = 2E_p Q_0$, $E_p = \sqrt{\mathbf{p}^2 + M^2}$, and the support of $\psi(\mathbf{u}, \mathbf{p}, \lambda^2)$ lies in the region

$$|\mathbf{u}| \leqslant E_p, \quad \lambda \geqslant \max\{0, E_p - \sqrt{E_p^2 - \mathbf{u}^2}\}. \qquad (71)$$

The representation (70) is written down in subtracted form to ensure convergence of the integration over $d\lambda^2$. The question arises of when such subtractions are necessary. As was shown above, the increasing behavior of $\psi(\mathbf{u}, \mathbf{p}, \lambda^2)$ with respect to $\lambda^2$ is due to the definition of the quasilimit [see (28)]. One can show[21] that a definite number $N$ of subtractions is sufficient if $\psi$ has a quasilimit of order $k$ under the condition $-1 \leqslant k - N < 0$. It follows from this inequality that subtractions are not necessary when $k < 0$. It follows from the earlier results of this section that the number of subtractions is determined by the degree of the singularity on the light cone.

We investigate the Bjorken limit of $T_i$. We assume that $\psi$ has a quasilimit of order $k$. In accordance with (33), we define

$$B_j - \lim_{\nu\to\infty} \nu^{-k} T(Q, \mathbf{p}) = T_0(\xi, \mathbf{p}, \mathbf{e}) \qquad (72)$$

subject to the condition

$$\lim_{\nu\to\infty} \nu^{-k} \int d\xi T(\xi, \mathbf{p}, \mathbf{e}, \nu) \Phi(\xi) = \int d\xi T_0(\xi, \mathbf{p}, \mathbf{e}) \Phi(\xi); \quad \Phi \in D. \qquad (73)$$

We then obtain the results[22]

$$B_j - \lim_{\nu\to\infty} \nu^{-k} T(Q, \mathbf{p}) = \Gamma(-k)/\pi \int d^3\mathbf{u}\psi_0(\mathbf{u}, \mathbf{p}) \left(\xi - \frac{\mathbf{u}\mathbf{e}}{E_p} - i\xi\right)^k \qquad (74)$$

for $k \neq 0, 1, 2, \ldots$ and

$$B_j - \lim_{\nu\to\infty} \frac{\nu^{-k}}{\ln \nu} T(Q, \mathbf{p}) = \frac{-1}{\pi\Gamma(k+1)} \int_{|\mathbf{u}|\leqslant E_p} d^3\mathbf{u}\psi_0(\mathbf{u}, \mathbf{p}) \left(\frac{\mathbf{u}\mathbf{e}}{E_p} - \xi\right)^k \qquad (75)$$

for $k = 0, 1, 2, \ldots$.

In deriving these last relations, we have made repeated integrations with respect to the variables $\lambda^2$. The spectral function was then transformed into a function continuous with respect to $\lambda^2$, which, on the basis of the assumed existence of the quasilimit, has a power-law behavior. We then calculated the classical integral.

We now turn to a discussion of Eqs. (74) and (75). Note that the undetermined subtractional polynomial of the representation (70) does not contribute to the

Bjorken limit. For a positive integral value of $k$, the amplitude behaves as $\nu^k \ln\nu$, whereas its absorptive part behaves as $\nu^k$. This behavior agrees with the circumstance that the scaling function $T_0(\xi, \mathbf{p}, \mathbf{e})$ does not have an absorptive part. It is a polynomial in $\xi$ [cf. (75)]; $T_0(\xi, \mathbf{p}, \mathbf{e})$ is a function analytic in the $\xi$ plane with continuation

$$\mathrm{disc}_\xi T_0(\xi, \mathbf{p}, \mathbf{e}) = \frac{1}{\Gamma(k+1)} \int\limits_{|\mathbf{u}| \leqslant E_p} d^3\mathbf{u}\,\psi_0(\mathbf{u}, \mathbf{p})\,(\mathbf{ue}/E_p) - \xi)_+^k. \qquad (76)$$

The scaling function $T_0$, in contrast to $W_0$, as an analytic function is nonzero up to discrete positions of zeros. Because of this, the virtual Compton amplitude can be measured in the region of small $\xi$. The experimental data indicate its asymptotic behavior as $\nu^k$, which makes it possible to determine the singularity on the light cone of the current commutator $\overline{W}(x^2, \mathbf{x})$.

The singularity on the light cone of the invariant $T$ product of the currents [see (69)] arises from the representation corresponding to the expression (21), in which the $\Delta$ function is replaced by the causal function $\Delta^c(x, \lambda^2)$; for forward scattering,

$$\widetilde{T}(x, p) \underset{x^2 \to 0}{\approx} \begin{cases} G(x_0)(x^2)^{-k-2} \ln(-x^2 + i0), & k = -1, -2, \ldots \\ G(x_0)(x^2 - i0)^{-k-2}, & k \text{ — nonintegral}; \\ G(x_0)(x^2 - i0)_{\mathrm{FP}}^{-k-2}, & k = 0, 1, 2, \ldots \end{cases}$$

Here, FP denotes the finite part of the generalized function $(x^2 - i0)^\alpha$ for $\alpha \to -k-2$ (Ref. 23). We recall that the absorptive part of the current commutator $\overline{W}(x^2, \mathbf{x})$ as $x^2 \to 0$ is uniquely characterized by the form $G(x_0)(x^2)_+^{-k-2}/\Gamma(-k-1)$ [see (39)].

**Convergence condition for the electromagnetic mass difference of nucleons.** We now turn to the connection between deep inelastic scattering and the electromagnetic mass corrections. In the single-photon approximation for the electromagnetic mass shift (or self-energy) of the hadrons, we have in accordance with Cottingham's formula[24]

$$\delta_m = -\frac{e^2}{4i} \int d^4q \frac{g_{\mu\nu}}{q^2 + i0} T_{\mu\nu}(q, p). \qquad (77)$$

Although $\delta_m$ itself may diverge, the difference $\delta_m^p - \delta_m^n$ between the proton and neutron mass shifts is a finite and observable quantity. The convergence of $\delta_m^p - \delta_m^n$ is a criterion of the electromagnetic origin of the proton–neutron mass difference. It can be seen from (77) that the behavior of $T_{\mu\nu}(q, p)$ at large $q_\mu$ is important here. Investigation of the amplitude $T_{\mu\nu}(q, p)$ by means of the Dyson–Jost–Lehmann representation shows that the Bjorken limit or the asymptotic behavior on the light cone ensure convergence of $\delta_m^p - \delta_m^n$.

For what follows, we assume the existence of canonical singularities on the light cone. By canonical singularities, we must understand behavior of the commutator $[j_\mu(x), j_\nu(0)]$ on the light cone such as is obtained in the framework of light-cone algebra by means of the construction of currents of free fields with spin 0 and $\frac{1}{2}$. In accordance with Ref. 25,

$$V_1(\nu, q^2) \underset{\nu \to \infty}{\approx} \nu^{-1} h_0(\xi) + \nu^{-2} h_1(\xi);$$

$$V_2(\nu, q^2) \underset{\nu \to \infty}{\approx} \nu^{-2} h_2(\xi),$$

where $V_1$ and $V_2$ are the structure functions from the expansion (13). The moments of the structure functions $W_i$ behave in the limit $Q^2 \to \infty$ as follows:

$$\mu_n^1(Q^2) = \int d\xi\,\xi^{n-1} W_1(\xi, Q^2) \leqslant \text{const}; \qquad (78)$$

$$\mu_n^2(Q^2) = \int d\xi\,\xi^{n-1} W_2(\xi, Q^2) \leqslant Q^{-2}\,\text{const}. \qquad (79)$$

As a condition of finiteness of $\delta_m^p - \delta_m^n$ we obtain by means of the Dyson–Jost–Lehmann representation the relations

$$\left.\begin{aligned} &\int_0^1 d\xi\,[h_0(\xi)]^{p-n} = 0; \\ &\int_0^1 d\xi \left[\xi^2 h_0(\xi) + \frac{1}{2}\xi h_2(\xi) - \xi h_1(\xi)\right]^{p-n} = 0. \end{aligned}\right\} \qquad (80)$$

These, in general, true sum rules cannot nevertheless, for certain circumstances, be always used, since they contain generalized functions that cannot be completely determined from deep inelastic scattering experiments (the behavior at $\xi = 0$ here again plays a particular part). For the moments (78) and (79), the following convergence condition can be derived[14]:

$$\delta_m^p - \delta_m^n \sim \int_0^\infty dQ^2\,[\mu_0^2(Q^2) + 4Q^{-2}\mu_0^1(Q^2) - 2\pi T_0(Q^2)]^{p-n} < \infty, \qquad (81)$$

where $T_0(Q^2)$ is a subtraction constant that also cannot be determined experimentally by means of deep inelastic scattering. In some way $T_0(Q^2)$ compensates for the nonexistent zeroth moment of $W_1$. In addition, there is an intimate connection with the so-called fixed pole of $T_1$, i.e., the asymptotically constant contribution to the real part of $T_1(\nu, Q^2)$. A common property of the condition (80) and (81) is obviously that they contain quantities that cannot be determined from deep inelastic scattering data.[26]

## 2. APPLICATION OF QUANTUM CHROMODYNAMICS TO DEEP INELASTIC LEPTON–HADRON SCATTERING

*Non-Abelian Gauge Field Theory of the Strong Interaction.* In view of the success of the quark–parton model and light-cone current algebra, the dynamical description of deep inelastic scattering must evidently be based on the quantum theory of quark fields. The interaction of the quarks must be described by the introduction of an additional gluon field. Since the theory must give scaling behavior of the structure functions, it can be a *non-Abelian gauge field theory*[31] *of quarks and gluons*, which has the property of asymptotic freedom.[12]

Quantum chromodynamics includes the quark–parton model and the light-cone algebra of quark currents.[28] In quantum chromodynamics, one introduces $3N$ fundamental *quark fields* $\Psi_{Aa}(x)$, which transform with respect to the Poincaré group as Dirac spinors, and with respect to the flavor group $SU(N)$ and the color group $SU(3)$ as the fundamental representations of these groups.[32] Quantum chromodynamics preserves the nature of a gauge field theory because the color transformations are understood as local, i.e., they depend on a space-time symmetry transformation (in this connection, the flavor group plays a secondary role). The massless vector fields $B_\mu^a(x)$ (Yang–Mills fields), which transform in accordance with the adjoint, i.e., octet, representation of the color group, are interpreted as *gluon* fields. In order not to modify significantly the description achieved in the quark–parton model, the gluons cannot participate in the electromagnetic or weak interaction, and they behave as singlets with respect to the flavor group.

The gluon fields can be understood as a decomposition

$$(B_\mu)_{bc} = B^a_\mu (t_a)_{bc}$$

of a Lorentz vector field (in the sense of transformations in the Poincaré group) $B_\mu$ that is a tensor under the color group with respect to the generators $t^a$ of the fundamental representation of $SU(3)$. These generators satisfy the relations

$$\left.\begin{array}{l} [t^a, t^b] = i f^{abc} t^c; \\ \{t^a, t^b\} = d^{abc} t^c; \\ \mathrm{Sp}\,(t^a t^b) = \delta_{ab}/2. \end{array}\right\} \tag{82}$$

We denote $\hat{X} = \partial \mu X_\mu$; then the classical Lagrangian density in quantum chromodynamics is

$$L_{\mathrm{cl}} = \overline{\Psi}\,(i\hat{D} - M)\,\Psi - \mathrm{Sp}\,(F_{\mu\nu}F^{\mu\nu})/2, \tag{83}$$

where the trace is taken in the color space, and

$$D_\mu = \partial_\mu + igB_\mu; \tag{84}$$

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + ig\,[B_\mu, B_\nu]. \tag{85}$$

The mass matrix $M$ is assumed to be a square unit matrix with respect to the color group.

In this theory, the quantization on the basis of the gauge symmetry is complicated and requires fixing of the gauge and the introduction of a (complex) ghost field $\Phi(x) = \Phi^a(x) t_a$. The point of departure of traditional perturbation theory is the effective Lagrangian[31,33]

$$L_{\mathrm{eff}} = L_{\mathrm{cl}} - (1/\alpha)\,\mathrm{Sp}\,(B^\mu_{,\mu})^2 + 2\mathrm{Sp}\,(\Phi^*_{,\mu}\Phi'^\mu + ig\Phi^*_{,\mu}[B_\mu, \Phi]). \tag{86}$$

Application of quantum chromodynamics to deep inelastic scattering encounters a number of difficulties. For the rigorous description of the virtual Compton amplitude (respectively, the weak current–current amplitude), it is necessary to solve the bound-state problem and the problem of the infrared singularities that occur in gauge theories.[34] If these difficulties are eliminated, the use of a light-cone expansion of the products of the current operators and the renormalization-group method enables one to establish the asymptotic properties of the moments of the structure functions.

The light-cone expansion of the products of current operators[13] is a direct generalization of the formulation of light-cone current algebra. All attempts at proof are restricted to perturbation theory [Ref. 35 (first reference)]. In the framework of axiomatic quantum field theory, one obtains a light-cone expansion only when additional assumptions are made [Ref. 35 (second reference)]. It is an expansion of the considered operator product with respect to local nonsingular operators $O^n_{\mu_1 \cdots \mu_n}(\dot{x})$ and singular (with respect to the light cone) $C$-number factors $C_n(x^2)$. The expansion, which entails an extensive calculation of the kinematic expansion (3) or, respectively, (6), can be written in the form

$$\begin{aligned} j_\mu(x)\, j_\nu(0) \underset{x^2 \to 0}{\approx} & \left[(g_{\mu\nu}\Box - \partial_\mu\partial_\nu)\frac{1}{x^2}\right] \\ & \times \sum_{n,i} C^1_{n,i}(x^2, g)\, x^{\mu_1} \cdots \ldots x^{\mu_n} O^{n,i}_{[\mu]}(0) \\ & + \frac{1}{x^2} \sum_{n,i} C^2_{n+2,i}(x^2, g)\, x^{\mu_1} \ldots x^{\mu_n} O^{n+2,i}_{\mu\nu[\mu]}(0) \\ & + \sum_{n,i} \left\{\partial_\mu \frac{1}{x^2} x^{\mu_1} \ldots x^{\mu_n} O^{n+1,i}_{\nu[\mu]}(0) + (\mu \Leftrightarrow \nu\right\} C^3_{n+1,i}(x^2, g) \\ & + \partial_\mu\partial_\nu \frac{1}{x^2} \sum_{n,i} C^4_{n,i}(x^2, g)\, x^{\mu_1} \ldots x^{\mu_n} O^{n,i}_{[\mu]}(0) \\ & + \frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}\partial^\rho \frac{1}{x^2} \sum C^5_{n+1,i}(x^2, g)\, x^{\mu_1} \ldots x^{\mu_n} g^{\sigma\tau} O^{n+1,i}_{\tau[\mu]}(0). \end{aligned} \tag{87}$$

Canonical singularities on the light cone follow from the given sums, so that current conservation is taken into account partly; the functions on the light cone have exceptionally *anomalous dimensions*, which arise from the renormalization of the basis operators $O^n_{[\mu]}(x)$. It is assumed further that the operators have the following properties.

1. They are completely symmetric with respect to the indices $\mu_1 \ldots \mu_n$ and have vanishing trace, i.e.,

$$g^{\mu_i\mu_j} O^n_{\mu_1 \ldots \mu_i \ldots \mu_j \ldots \mu_n} = 0.$$

2. The *twist* of these operators, i.e., $\dim O^n - n$, is minimal, so that the singularities of the functions corresponding to them are maximal.

The *basis operators* with minimal twist can be[12]

(A) $SU(N)$-singlet operators

$$O^{\mathrm{v}}_{\mu_1 \ldots \mu_n} = 2i^{n-2}\varphi\,\mathrm{Sp}\,(F_{\mu_1\alpha}D_{\mu_2} \ldots D_{\mu_{n-1}}F^\alpha_{\mu_n}) - \text{traces}; \tag{88}$$

$$O^{F,0}_{\mu_1 \ldots \mu_n} = i^{n-1}\varphi\,(\overline{\Psi}\gamma_{\mu_1}D_{\mu_2} \ldots D_{\mu_n}\Psi) - \text{traces}; \tag{89}$$

(B) $SU(N)$-octet operators

$$O^{F,a}_{\mu_1 \ldots \mu_n} = i^{n-1}\varphi\left[\overline{\Psi}\gamma_{\mu_1}D_{\mu_2} \ldots D_{\mu_n}\frac{1}{2}\lambda^a\Psi\right] - \text{traces} \tag{90}$$

($\varphi$ denotes complete symmetrization of the Lorentz indices). The light-cone expansion may include operators constructed from ghost fields or ones that do not have a gauge-invariant structure.[36] The anomalous dimensions of these operators are calculated in the single-loop approximation, which is sufficient to derive the asymptotic behavior. Even in this approximation, they do not appear as contour terms in the renormalization of the singlet operators, and all singlet operators with the same symmetry and tensor structure occur as gauge-noninvariant and ghost operators (mixing operators). Detailed investigations show that such gauge-noninvariant and ghost operators need not be taken into account in the calculation of the anomalous dimensions in the light-cone expansion, since the matrix elements are constructed exclusively between physical states.[36]

If only the gauge-invariant operators indicated in (88)–(90) are taken into account, then with allowance for the operator mixing the following renormalization scheme arises:

$$(O^{nF}, O^{nV})_{\mathrm{ren}} = (O^{nF}, O^{nV})_{\mathrm{nonren}} \begin{pmatrix} z_{FF} z_{FV} \\ z_{VF} z_{VV} \end{pmatrix}, \tag{91}$$

and similarly for the octet operators:

$$O^{nF}_{\mathrm{ren}} = z^{-1}_{FF} O^{nF}_{\mathrm{nonren}}. \tag{92}$$

The existing anomalous dimensions

$$^{n}\gamma^{i}_{jj}=\mu\frac{\partial}{\partial\mu}z_{jl}$$

are given by the expressions[12]

$$\left.\begin{aligned}
&^{n}\gamma^{F}_{FF}=\frac{g^2}{8\pi^2}C_N\Big[1-\frac{2}{n(n+1)}+4\sum_{j=2}^{n}\frac{1}{j}\Big];\\
&^{n}\gamma^{V}_{VV}=\frac{g^2}{8\pi^2}\Big[C_A\Big(\frac{1}{3}-\frac{4}{n(n-1)}-\frac{4}{(n-1)(n-2)}\\
&+\sum_{j=2}^{n}\frac{1}{j}\Big)+\frac{4}{3}NT;\\
&^{n}\gamma^{F}_{VV}=-(g^2/8\pi^2)(4(n^2+n+2)/n(n+1)(n+2))NT;\\
&^{n}\gamma^{V}_{FF}=-(g^2/8\pi^2)(2(n^2+n+2)/n(n^2-1))C_N.
\end{aligned}\right\}\qquad(93)$$

The quark masses are zero, since a detailed treatment shows that in the quantum chromodynamics of massive quarks the asymptotic properties of the moments and the light-cone singularities are the same as in the massless theory. Therefore, the anomalous dimensions (93) are also valid for the chiral operators $O^{nF,\pm}$ obtained from the operators $O^{nF}$ by the inclusion of $(1\pm\gamma_5)/2$. The constants $C_A$, $C_N$, and $T$ have a group-theoretical origin. In the case of color SU$(n)$

$$C_A=n;\quad C_N=(n^2-1)/2n;\quad T=1/2.$$

The anomalous singularities of the coefficients on the light cone are calculated by means of the renormalization group. Applying the renormalization-group equation[7,37] to the matrix element of the light-cone expansion, we arrive at the corresponding equation for the coefficients on the light cone[38]:

$$\{\mu\partial/\partial\mu+\beta\partial/\partial g-{}^{n}\gamma\}C_n(x^2,g,\mu)=0. \qquad (94)$$

Here, $\mu^2$ is the renormalization point (in the momentum space); $\beta(g,\mu)$ is determined by the relation

$$\mu\partial g/\partial\mu=\beta(g,\mu); \qquad (95)$$

$C_n(x^2)$ are found as the solution of Eq. (3.10):

$$C_n^{(i)}\Big(\frac{x^2}{\lambda^2},g,m,\mu\Big)=\Big[\mathscr{T}\exp\int_{\mu}^{\mu/\lambda}\frac{d\mu'}{\mu'}{}^{n}\gamma\Big]\times C_n^{(i)}(x^2,\bar g(\lambda),m/\lambda,\mu) \qquad (96)$$

($\mathscr{T}$ denotes the order in which the terms are written down with respect to $\mu'$ if the anomalous dimensions $^{n}\gamma$ of the basis operators $O^n$ for the operator mixing are given by matrices.) When $C_n^i(x^2)$ have canonical dimensions [in contrast to the expansion (87)], the additional factor $\lambda^{2d_j-(d_{O^n}-n)}$ appears on the right-hand side of this equation. The *effective coupling constant* $\bar g(\lambda,g)$ is defined as the solution of the differential equation

$$\lambda\partial\bar g/\partial\lambda=\beta(\bar g) \qquad (97)$$

with the initial value

$$\bar g(1,g)=g. \qquad (98)$$

In the single-loop approximation

$$\beta=-bg^3;\quad b=(M-2N/3)/16\pi^2, \qquad (99)$$

so that we obtain[1)]

$$\bar g(\lambda,g)=g^2/(1+g^2b\ln\lambda^2). \qquad (100)$$

With allowance for (100), the light-cone behavior of the coefficient function $({}^{n}\gamma=C_ng^2+\cdots)$ is given by

$$C^{(i)}_{n,\,\mathrm{pr}}(x^2/\lambda^2,g)\approx(b\ln\lambda^2)^{-c_n/2b}C^{(i)}_{n,\,\mathrm{pr}}(x^2,\bar g). \qquad (101)$$

For the operator mixing, Eq. (96) is estimated in such a way that the eigenvalues of the matrix $^{n}\gamma$ are determined; in the asymptotic region, the dominant term is the one with the smallest eigenvalue of the anomalous dimension, so that $C_n$ is determined from the expression

$$^{n}\lambda_{\mathrm{eigen}}=C_ng^2+\dots\,.$$

Knowledge of these singularities on the light cone is as necessary as knowledge of the asymptotic behavior of the moments of the structure functions. We point out that without solution of the bound-state problem the theoretical predictions of quantum chromodynamics are limited to the asymptotic properties of the moments. The numerical coefficients of the asymptotic expressions, which arise from the single-nucleon matrix elements of the operators $O^n_{[\mu]}(0)$, remain undetermined. This infinite set of undetermined coefficients reflects in some measure our unsatisfactory knowledge of the exact wave function of the nucleon.

*Asymptotic Behavior of the Moments.* We have shown earlier that the light-cone singularities determine the asymptotic behavior of the moments as follows. Generalizing the connection between the coefficient functions $f_n(x^2)$ and the Taylor expansion (51) of the symmetric commutator to the case $x^2=x_0^2-\mathbf{x}^2=0$, we obtain

$$\overline{W}(x_0,x^2)=\frac{1}{4\pi i}\sum_{n=0}^{\infty}\frac{i^n}{n!}x_0^nf_n(x^2). \qquad (102)$$

If at the same time the coefficient functions $f_n(x^2)$ have a singularity on the light cone,

$$f_n(x^2/\lambda^2)\underset{\lambda\to\infty}{\approx}(\lambda^2)^{\alpha_n}(\ln\lambda^2)^{\beta_n}, \qquad (103)$$

then for the asymptotic behavior of the moments of the structure functions we have

$$\mu_n(Q^2)\underset{Q^2\to\infty}{\approx}(Q^2)^{\alpha_n-2}\ln(Q^2/\mu^2)^{\beta_n}. \qquad (104)$$

(The renormalization parameter $\mu$ has been introduced to ensure that the logarithm of a dimensionless quantity is taken.)

By means of the light-cone expansion (102) of $\overline{W}(x_0,x^2)$ one determines either the even-integral or odd-integral moments. For $eN$ scattering, the relation $W_i(\nu,q^2)=-W_i(-\nu,q^2)$ has the consequence that the index in Eq. (87) must be only even-integral. And for $\nu N$ scattering from the relation $W_i^{\nu p}(\nu,q^2)=-W_i^{\nu n}(-\nu,q^2)$ one determines either the even-integral moments for the combination $W_i^{\nu p}+W_i^{\nu n}$ or the odd-integral moments for the combination $W_i^{\nu p}-W_i^{\nu n}$. The singularities of the coefficient functions $f_n(x^2)$ are determined from (103) by means of (87) by forming the single-nucleon matrix elements from the corresponding anomalous singular function $C_n(x^2)$ and the canonical singularities belonging to them. In what follows, we shall consider only asymptotic relations. Because of the asymptotic equality of the different sealing parameters, the moments $\mu_n(Q^2)$ in (104) can be understood either as the Cornwall–Norton moments (54), or as the modified moments (55), or as Nachtmann moments.[16]

The moments usually formed by means of the structure functions $W_i$ can be expressed in terms of the ones formed with the scaling functions:

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1) It follows from Eq. (100) that $\bar g$ tends to zero as $\lambda\to\infty$. This means that $\bar g=0$ there is a $uv$-stable fixed point. This behavior indicates asymptotic freedom.

$$F_1 = W_1; \quad (105)$$
$$F_2 = pqW_2 = \nu W_2/2; \quad (106)$$
$$F_3 = pqW_3 = \nu W_3/2. \quad (107)$$

Such functions are constant for scaling, and the deviations from scaling are logarithmic.

For the nonsinglet ($NS$) contributions, we obtain the relations

$$\int_0^1 d\xi \xi^{n-1} F_1^{NS}(\xi, Q^2) \sim \ln (Q^2/\mu^2)^{-A_n^{NS}}; \quad (108)$$

$$\int_0^1 d\xi \xi^n F^{NS}(\xi, Q^2) \sim \ln (Q^2/\mu^2)^{-A_{n+2}^{NS}}; \quad (109)$$

$$\int_0^1 d\xi \xi^n F_3^{NS}(\xi, Q^2) \sim \ln (Q^2/\mu^2)^{-A_{n+1}^{NS}}. \quad (110)$$

The value of $n$ for $eN$ scattering and the symmetric combination $\sigma^{\nu p} + \sigma^{\nu n}$ is even, and for the asymmetric combination $\sigma^{\nu p} - \sigma^{\nu n}$ it is odd. In deriving these equations, we have used the circumstance that the $n$-th moment of the structure functions $W_i$ corresponds to the coefficients of the expansion with respect to $(x_0)^n$ in Eq. (102). For the moments of $F_2$, the first three terms in the expansion (87) are completely sufficient.

From Eqs. (102)–(104), as from (93), (99), and (101), we find for the flavor group $SU(N)$

$$A_n^{NS} = \frac{C_N}{11 - 2N/3} \left\{1 - \frac{2}{n(n+1)} + 4 \sum_{j=2}^{n} 1/j\right\}; \quad (111)$$

as an approximation for any $n$, we can with good accuracy use the expression[12]

$$A_n^{NS} \approx [18/(11 - 2N/3)] (0.296 \ln n - 0.051). \quad (112)$$

In describing the singlet ($S$) contributions, one takes into account the mixing problem and in accordance with (101) determines the smallest eigenvalue of the ${}^n\gamma$ matrix:

$${}^n\gamma = \begin{pmatrix} {}^n\gamma_{FF}^F & {}^n\gamma_{FF}^V \\ {}^n\gamma_{VV}^F & {}^n\gamma_{VV}^V \end{pmatrix}; \quad (113)$$

it is given by the expression

$${}^n\gamma_\mp = (1/2) \{{}^n\gamma_{VV}^V + {}^n\gamma_{FF}^F \mp \sqrt{({}^n\gamma_{VV}^V - {}^n\gamma_{FF}^F)^2 + 4\,{}^n\gamma_{VV}^F\,{}^n\gamma_{FF}^V}\}. \quad (114)$$

From this one obtains relations analogous to (108)–(110), and the exponent

$$A_n^S = [8\pi^2/(11 - 2N/3)] ({}^n\gamma_-/g^2) \quad (115)$$

or

$$A_n^S \approx A_n^{NS} - O(1/n^2 \ln n). \quad (116)$$

Note that quantum chromodynamics gives positive anomalous dimensions for the $NS$ operators, and also for the eigenvalues of the matrix ${}^n\gamma$. This is in agreement with the general propositions of quantum field theory, and the total two-point functions $\langle 0 | O^n(x) O^n(0) | 0 \rangle$ must have enhanced singularities on the light cone as the corresponding free two-point functions.

A consequence of the positivity of the structure functions is an order relation for the moments; this follows directly from their definition (54):

$$\mu_{n+2} \leqslant \mu_n.$$

In addition, one obtains relations (albeit somewhat complicated) between different pairs of moments.[16] The results of quantum chromodynamics agree with these relations; they also mean that the moments have different asymptotic behavior as $Q^2 \to \infty$. This has the consequence that the structure functions $W(\xi, Q^2)$ are concentrated with increasing $Q^2$ near $\xi = 0$ (Ref. 4) (see Sec. 1).

Hence, in accordance with the canonical dimension of the currents, we can follow the canonical scaling behavior in the mathematical sense when the scaling function is concentrated around $\xi = 0$.

As we have already said, there is also one moment that can be experimentally measured but for which there is no relation analogous to (108)–(110). It is

$$-\int_0^1 d\xi \xi (F_3^{\nu p}(\xi, Q^2) + F_3^{\nu n}(\xi, Q^2)) = 3\pi (\sigma^\nu - \sigma^{\bar\nu})/2G^2ME.$$

This quantity is interesting from the point of view of the quark–parton model. The ratio

$$-\frac{\int d\xi \xi (F_3^{\nu p} + F_3^{\nu n})}{\int d\xi (F_2^{\nu p} + F_2^{\nu n})} = \frac{\int d\xi \xi [q(\xi) - \bar q(\xi)]}{\int d\xi \xi [q(\xi) + \bar q(\xi)]}$$

shows the relative contribution of the quarks and antiquarks[28]; here $q = U_p + D_p$. In the asymptotic region, quantum chromodynamics predicts a constant for the denominator of the left-hand side, but no prediction has been made as yet for the numerator.

We note that similar theoretical investigations of deep inelastic scattering were also made for a polarized target[53]; in this case, it is necessary to take into account additional operators in the light-cone expansion and their anomalous dimensions must be determined.

*Justification of Perturbation Theory with Respect to the Effective Coupling Constant and its Physical Consequences*. The conclusions now presented from quark chromodynamics are based on the fact that, by virtue of the asymptotically free nature of the behavior in a number of known cases, as, for example, in the determination of the functions $C_n(x^2, g)$ on the light cone, perturbative methods can be used. This is possible when the light-cone expansion (87) with the substitution $x \to x/\lambda$ (only for scalar currents) can be written in the form

$$j\left(\frac{x}{\lambda}\right) j(0) \underset{\lambda\to\infty}{\approx} \sum C_n\left(\frac{x^2}{\lambda^2}, g\right) \frac{1}{\lambda^n} x^{\mu_1} \ldots x^{\mu_n} O^n_{[\mu]}(0, g).$$

If we use the asymptotic expression for $C_n(x^2)$ given by Eq. (101), we obtain

$$j\left(\frac{x}{\lambda}\right) j(0) \underset{\lambda\to\infty}{\approx} \sum C_n\left(\frac{x^2}{\lambda^2}, \bar g(\lambda)\right) \lambda^{-n} (\ln \lambda)^{-A_n} x^{\mu_1} \ldots x^{\mu_n} O^n_{[\mu]}(0, g). \quad (117)$$

Thus, the light-cone expansion of the product of current operators will now be determined by the coefficients $C_n(x^2, g)$, in which the substitution $g \to \bar g$ has been made. By virtue of the asymptotic decrease of the effective coupling constant, $\bar g(\lambda) \underset{\lambda\to\infty}{\to} 0$, the coefficients $C_n(x^2, \bar g(\lambda))$ for the principal and nonprincipal terms of the expansion can be found by perturbation theory.

C u r r e n t  a l g e b r a. We now show how the validity of current algebra follows from quantum chromodynamics.

The current commutator is defined for equal times, i.e., one calculates the commutator $[j_\mu^a(x), j_\nu^b(0)]$ for $x_0 = 0$ or at the apex of the light cone. For the commutator

$[j_0^a(x) j_0^b(0)]$ in the expansion (87), augmented by flavor indices, the following types of singularity arise for $x_0 = 0$:

$$\Delta\varepsilon(x_0)\,\delta(x^2)|_{x_0=0} = 0;$$
$$\partial_0\varepsilon(x_0)\,\delta(x^2)|_{x_0=0} = 2\pi\delta(x);$$
$$\partial_0^2\varepsilon(x_0)\,\delta(x^2)|_{x_0=0} = 0,$$

which by virtue of the negative powers of $\ln x^2$ vanish.[39]

Thus, in the expansion (3.6) there remain only the terms

$$[j_0^a(x) j_0^b(0)]|_{x_0=0} = 4\pi\delta(x) \sum C_{1,ic}^{3ab}(x^2, g)\, 0_0^{1,ic}(0). \tag{118}$$

Here we have used the circumstance that $O_0^{1,i}(x)$, as a (current) operator, with a conservation law does not have an anomalous dimension, i.e., the corresponding coefficient $C_{1,ic}^{3ab}$ does not contain a logarithmic singularity. If we remember that $C_{1,ic}^{3ab}(x^2, g)$ can be calculated by perturbation theory, in the first order we obtain the results of free field theory: The functions on the light cone are constants and the gluon operators are eliminated because the corresponding coefficients tend to zero. On this basis, the validity of current algebra follows in the first order. The current-algebra results for the other commutators are obtained similarly.

Therefore, one must take into account the circumstance that the validity of current algebra presupposes the existence of a *uv*-stable fixed point, which, by virtue of the requirement that perturbation theory be applicable, must exist at $\bar{g} = 0$.

**Canonical singularities on the light cone.** Quantum chromodynamics differs essentially from free quantum field theory in that different functions on the light cone have different singularities. The principal singularity on the light cone is a canonical one, which can be studied using the expansion (87).

The coefficients $C_n^i$ contain negative powers of $\ln x^2$ in increasing order, which lead to a constant weakening of the singularity. Therefore, only the first terms of the series of the expansion (87) which belong to operators with a conservation law and also to the currents of the energy–momentum tensor have a canonical singularity on the light cone:

$$\begin{aligned} j_\mu(x)\, j_\nu(0) \underset{x^2=0}{\approx} & [(g_{\mu\nu}\Box - \partial_\mu\partial_\nu)(1/x^2)]\, C_2^1(x^2, g) \\ & \times x^\alpha x^\beta \theta_{\alpha\beta}(0) + (1/x^2)\, C_2^2(x^2, g)\, \theta_{\mu\nu}(0) + [[\partial_\mu(1/x^2)]\, j_\nu(0) \\ & + [\partial_\nu(1/x^2)]\, j_\mu(0)]\, C_1^3(x^2, g) + [\partial_\mu\partial_\nu(1/x^2)]\, C_2^4(x^2, g)\, x^\alpha x^\beta \theta_{\alpha\beta}(0) \\ & + (i/2)\, \varepsilon_{\mu\nu k\lambda}\, [\partial^\lambda(1/x^2)]\, C_1^5(x^2, g)\, g^{k\alpha} j_\alpha(0). \end{aligned} \tag{119}$$

The scaling functions corresponding to this expression are concentrated at $\xi = 0$, since there appear singular terms only of the type $1/x^2$. This can be seen by comparing the expansion (119) with the relations (39) and (45).

**Higher orders of perturbation theory.** Hitherto, we have considered only the first nonvanishing order of perturbation theory; however, we must consider the higher orders as well. For example, it is important to know how the function $\beta(g)$ behaves in the higher approximations in order to estimate the occurrence of (finite) infrared stable fixed points; also important are the corrections to the sum rules of the quark–parton model. To determine the corrections of higher order in the coupling constant to the asymptotic behavior of the moments, we must return to considering the coefficients on the light cone.

Application of the renormalization group to $C_{n,i}$ gave [cf. (96)]

$$C_n\left(\frac{x^2}{\lambda^2}, g\right) = C_n(x^2, \bar{g}(\lambda)) \exp\left\{\int_\mu^{\mu/\lambda} \frac{d\mu'}{\mu'}\, {}^n\gamma_-(g)\right\}.$$

The corrections apply to both $C_n(x^2, \bar{g})$ and ${}^n\gamma_-$ and $\bar{g}$. The two-loop corrections[40,41] are given by the expressions

$$\beta(g) = -\beta_0 g^3/16\pi^2 - \beta_1 g^5/(16\pi^2)^2 + \ldots;$$
$${}^n\gamma(g) = {}^n\gamma_0 g^2/16\pi^2 + {}^n\gamma_1 g^4/(16\pi^2)^2 + \ldots;\ \beta_1 = 17N/3 - 102;$$

the corrections to $C_n$ are

$$C_n(g) = 1 + H^n g^2/12\pi^2 + \ldots;$$

then, by approximate integration, for the effective coupling constant we obtain[41]

$$g^2(Q^2) \underset{Q^2\to\infty}{\approx} \bar{g}_0^2(Q^2) - (\beta_1/\beta_0)\,(\bar{g}_0^4/16\pi^2)\ln\ln(Q^2/\mu^2)$$

and the corresponding moment is

$$\mu_n(Q^2) \approx A_n\{1 + (\bar{g}_0^2/12\pi^2)\,[H^n + P^n + L^n(Q^2)]\,[\ln(Q^2/\mu^2)^{-{}^n\gamma_0/(2\beta_0)}]\},$$

where

$$P^n = (3/8)\,({}^n\gamma_1/\beta_0 - {}^n\gamma_0\beta_1/\beta_0^2);$$

$$L^n(Q^2) = -(3/8)\,(\beta_1/\beta_0^2)^n\,\gamma_0 \ln\ln(Q^2/\mu^2).$$

Note that the sum $H^n + P^n$ does not depend on the chosen renormalization conditions.[41] A systematic calculation in perturbation theory shows that all the correction terms, especially $\beta_1$ and ${}^n\gamma_1$, are appreciable only in the form factor of the two-loop approximation. As before, the principal asymptotic behavior is determined by the first order.

*Sum Rules.* To verify quantum chromodynamics in the framework of the light-cone algebra, and also the quark–parton model, the sum rules given below have decisive importance, since they can be verified experimentally. The sum rules are as follows.

Callan–Gross:

$$F_2 = 2\xi F_1; \tag{120}$$

Llewellyn Smith:

$$6\,(F_2^{ep} - F_2^{en}) = \xi\,(F_3^{\nu p} - F_3^{\nu n}); \tag{121}$$

Adler:

$$\int_0^1 d\xi\, \xi^{-1}\,(F_2^{\bar{\nu}n} - F_2^{\bar{\nu}p}) = 2; \tag{122}$$

Gross–Llewellyn Smith:

$$\int_0^1 d\xi\,[F_3^{\bar{\nu}p} + F_3^{\bar{\nu}n}] = 6; \tag{123}$$

momentum balance:

$$\int_0^1 d\xi\left[\frac{9}{2}(F_2^{ep} + F_2^{en}) + \frac{3}{4}(F_2^{\bar{\nu}p} + F_2^{\bar{\nu}n})\right] = 1. \tag{124}$$

The derivation of these sum rules in quantum chromodynamics differs from the treatment of the moments in that, on the one hand, the constants on the right-hand sides of (108)–(110) can be determined by means of perturbation theory in $\bar{g}$ and, on the other, in the treatment can be given for a conveniently chosen linear combination of the structure functions.

Quite generally, one can distinguish two types of sum rules: local, (120) and (121), and global, (122)–(124). The global sum rules in quantum chromodynamics can be used unchanged up to corrections calculated in accordance with perturbation theory. The local sum rules appear in a form that depends on the conditions imposed on their moments.

We consider first the global sum rules and examine the derivation of the Gross–Llewellyn Smith rule. In quantum chromodynamics, its derivation is simple. The point of departure is the relation (110) for $n=0$ for a symmetric combination of the proton and neutron matrix elements, this holding for the singlet and nonsinglet contributions. The determining operators are vector operators, for which there is no mixing problem and for which the anomalous dimension vanishes: $A_1^{NS}=A_1^S=0$. We calculate the single-nucleon matrix elements in the rest frame, so that we take into account only the zeroth components of the vector operators. Because of its transformation behavior under spatial reflection, the axial-vector part in the single-particle matrix element can be ignored. The remaining operators of the quark number $\bar{\Psi}\gamma_0\Psi$ and the hypercharge $\bar{\Psi}\gamma_0(1/\sqrt{3})\lambda^8\Psi$ have well-defined matrix elements, which give the numerical value of the sum rule. The light-cone functions $C_1^3(x^2,g)$ and the terms in (110) that arise in accordance with perturbation theory are equal to the values of the quantities obtained from light-cone current algebra, so that the sum rule is given in its original form (123). The second-order corrections in $\bar{g}$ are proportional to an inverse power of a logarithm:

$$\bar{g}^2\sim 1/\ln(Q^2/\mu^2).$$

Then in the second order in $\bar{g}$ we have[42]

$$\int_0^1 d\xi\left[F_3^{\nu p}+F_3^{\nu n}\right]=-6+\frac{15}{16\pi^2}\bar{g}^2C_N. \qquad (125)$$

The *Adler sum rule* is obtained similarly. The point of departure is the relation (109) for $F_2^{\nu p}-F_2^{\nu n}$ with $n=1$. Again, only vector operators are important. It is fairly difficult to calculate the corrections to the Adler sum rule; implicitly, they are contained in Calvo's results.[42]

We now discuss the sum rule (124) corresponding to the *momentum balance*. In deriving this rule, one starts from (110) with $n=0$ for the symmetric combination of the proton and neutron matrix elements, this behaving as a singlet with respect to the flavor group. In contrast to the global sum rules hitherto considered, the the mixing problem here plays an important part. The dominant operators can be expressed in terms of the fermion and gluon parts of the energy–momentum tensor. The derivation of (124) in quantum chromodynamics differs somewhat, because of the appearance of the gluon field, from the derivation of the previous sum rules. The difference is manifested in the appearance of an algebraic factor which arises from diagonalization of the matrix $^n\gamma$. The structure of the expansion of the current operator product (87) on the light cone is given asymptotically by the expression

$$C_F(x^2,g)\,\theta_{\mu\nu}^F+C_V(x^2,g)\,\theta_{\mu\nu}^V.$$

Here we have taken into account the circumstance that $C_V(x^2,g)$ does not give a contribution in the zeroth approximation in perturbation theory. The fermion part $\theta_{\mu\nu}^F$ must be expressed in terms of diagonal combinations of operators belonging to the eigenvalues $\gamma$. The matrix of anomalous dimension for $n=2$ can be written in the form [cf. (93)]

$$^2\gamma=\frac{g^2}{8\pi^2}\begin{pmatrix}(8/3)\,C_N & (-8/3)\,C_N\\ (-4/3)\,NT & (4/3)\,NT\end{pmatrix}.$$

The operator whose eigenvalue is zero is the total energy–momentum tensor $\theta_{\mu\nu}^F+\theta_{\mu\nu}^V$, since the vector $\binom{1}{1}$ vanishes by virtue of $^2\gamma$. The fermion part $\theta_{\mu\nu}^F$ can be expressed from here in terms of $\theta_{\mu\nu}$ and the other operator $\theta'_{\mu\nu}$ belonging to eigenvalue $\gamma>0$:

$$\theta_{\mu\nu}^F=NT\,(NT+2C_N)^{-1}\,\theta_{\mu\nu}+(NT+2C_N)^{-1}\,\theta'_{\mu\nu}.$$

Actually, only the first term, the asymptotic behavior of the dominant operator, is of interest, so that the left-hand side of the sum rule is replaced by the factor[12]

$$r=\frac{NT}{NT+2C_N}=\begin{cases}9/25 & \text{for } SU(3)\text{-flavor};\\ 3/7 & \text{for } SU(4)\text{-flavor}\end{cases}$$

We conclude from this that with increasing number of flavor degrees of freedom the experimental value approaches 0.5.

We now return to local sum rules. To derive the *Callan–Gross rule* it is expedient to proceed from the structure function $W_L$ and its scaling behavior:

$$\nu W_L=\nu W_2+(\nu/2\xi)\,(F_2-2\xi F_1), \qquad (126)$$

it being possible to determine $W_L$ from the current commutator:

$$W_L=\frac{1}{8\pi\,(1+\nu/4\xi)}\int d^4x\exp(iqx)\sum\langle p\mid[j_\mu(x),j_\nu(0)]\mid p\rangle\,p^\mu p^\nu. \qquad (127)$$

If in this expression we substitute the light-cone expansion (87), we can obtain the relations derived earlier between the light-cone functions $C_{n,i}(x^2,g)$ and the moments $\nu W_L/4\xi$:

$$\frac{\nu W_L}{4\xi}=\frac{1}{8\pi}\int d^4x\exp(iqx)\left\{\left[p^2\,\square-(p\partial)^2\frac{1}{x^2}\right]\right.$$
$$\times\sum C_{n,i}^1\langle p\mid x^{[\mu]}O_{[\mu]}^{n,i}(0)\mid p\rangle+\frac{1}{x^2}\sum C_{n+2,i}^2$$
$$\times\,p^\mu p^\nu\langle p\mid x^{[\mu]}A_{\mu\nu[\mu]}^{n+2,i}(0)\mid p\rangle$$
$$+2\left(p\partial\frac{1}{x^2}\right)\sum C_{n+1,i}^3\,p^\lambda\langle p\mid x^{[\mu]}O_{\lambda[\mu]}^{n+1,i}(0)\mid p\rangle$$
$$\left.+\left[(p\partial)^2\frac{1}{x^2}\right]\sum C_{n,i}^4\langle p\mid x^{[\mu]}O_{[\mu]}^{n,i}(0)\mid p\rangle\right\}$$

($x^{[\mu]}$ denotes the product $x^{\mu_1}\ldots x^{\mu_n}$). The principal singularity on the light cone arises from the first and last terms of this sum and takes the form $[(p\partial)^2(1/x^2)]C_{n,i}^L$, with

$$C_{n,i}^L(x^2,g)=-C_{n,i}^1(x^2,g)+C_{n,i}^4(x^2,g).$$

Use of the renormalization group for these functions gives

$$C_{n,i}^L(x^2/\lambda^2,g)\underset{\lambda\to\infty}{\approx}(\ln\lambda^2)^{-A_n}C_{n,i}^L(x^2,\bar{g}(\lambda)).$$

A perturbation-theory calculation in accordance with the light-cone algebra of the free quark fields leads to vanishing of $C_{n,i}^L(x^2,\bar{g})$ in the zeroth order. The perturbation-theory expansion begins with the term of second order, so that we obtain

$$C^L_{n,i}(x^2/\lambda^2, g) \underset{\lambda\to\infty}{\approx} \bar{g}^2(\lambda)(\ln\lambda^2)^{-A_n}.$$

Therefore, the moments $\nu W_L/4\xi$ satisfy

$$\int d\xi\xi^{n-1}\frac{\nu W_L}{4\xi} \sim \bar{g}^2 Q^2 [\ln Q^2]^{-A_{n-2}},$$

where we have used the fact that the factors $x_0^2$ that arise from differentiation shift the relation of the moments by −2. Hence, for the combination

$$F_L = F_2 - 2\xi F_1 = 2\xi(W_L - W_2) \qquad (128)$$

we obtain

$$\int_0^1 d\xi\xi^n F_L(\xi, Q) \sim \bar{g}^2(\ln Q^2)^{-A_{n+2}} \qquad (129)$$

(the contribution from $W_2$ can be ignored, being weaker than the one from $\nu W_L$ by an integral power). The behavior of this moment is also determined by the anomalous dimension of the operator $O^{n+2,i}_{[\mu]}$ of twist 2.

To free ourselves from the anomalous dimensions in (129), we divide it by the corresponding relation for $F_2$ [see (104)–(106) and (115)–(116)]. We obtain

$$\int d\xi\xi^n F_L \Big/ \int d\xi\xi^n F_2 \approx a_n \bar{g}^2(Q^2/\mu^2). \qquad (130)$$

In contrast to the canonical case of the quark–parton model or the light-cone algebra, $F_L$ decreases much more weakly, namely

$$\bar{g}^2(Q^2/\mu^2) = g^2[1 + bg^2\ln(Q^2/\mu^2)]^{-1}$$

[cf. (3.14)]. The coefficient $a_n$ is determined by calculating $C^L_{n,i}(x^2, g)$ in perturbation theory; it is given by[43]

$$a_n = (1/16\pi^2)\,C_N 4/(n+3). \qquad (131)$$

*The Llewellyn Smith rule* is similarly transformed into a relation between corresponding moments. To second order, it is represented by the equation[48]

$$\frac{\int d\xi\xi^n(F_2^{ep} - F_2^{en})}{\int d\xi\xi^{n+1}(F_3^{\nu p} - F_3^{\nu n})} = -\left[1 + \frac{\bar{g}^2}{4\pi^2}C_N\ln^2(n+2)\right]. \qquad (132)$$

We conclude our study of the sum rules with a last remark. As is shown by the analysis just made, quantum chromodynamics leads to the well-known sum rules, albeit in the first order of perturbation theory. The higher orders give corrections that depend on the group-theoretical structure; it can be concluded from this that experimental investigations of the sum rules will make it possible to draw a conclusion about the number of quark degrees of freedom (flavors).

*Group-Theoretical Structure of the Light-Cone Expansion.* Here, we consider the group-theoretical aspect of the expansion of a product of operators. As we have already said, the resulting structures do not depend on the operator form of the light-cone expansion but merely reflect the Lorentz covariance of the operators $O^n_{[\mu]}(x)$.

The point of departure of the calculations which follow is the circumstance that the operators $O^n_{[\mu]}(x)$ in the expansion

$$j(x)j(0) \underset{x^2=0}{\approx} \sum C_n(x^2, g)\,x^{\mu_1}\ldots x^{\mu_n}O^n_{\mu_1\ldots\mu_n}(0)$$

are tensors with a definite Lorentz spin, so that in the group-theoretical sense they are irreducible tensors. In addition, one introduces, in the first place in Euclidean space, $(n+1)^2$ linearly independent orthonormalized homogeneous polynomials[44] $H_{nm}(x)$ of degree $n$ with $m = 1, 2, \ldots, (n+1)^2$, these being related to the Gegenbauer polynomials $C^1_n(\eta)$:

$$\frac{2\pi^2}{n+1}\sum_m H_{nm}(\mathbf{x})H_{nm}(\mathbf{x}') = |\mathbf{x}|^n|\mathbf{x}'|^n C^1_n\left(\frac{\mathbf{x}\mathbf{x}'}{|\mathbf{x}||\mathbf{x}'|}\right)$$

($|\mathbf{x}|$ is the modulus of the Euclidean four-vector $\mathbf{x}$). If we go over to Minkowski space by means of analytic continuation ($H_{nm}(\mathbf{x}) \to H_{nm}(x)$), introduce spherical components $O_{nm}(0)$ of the operators $O^n_{[\mu]}(0)$,

$$\sum_m \frac{2\pi^2}{n+1}H_{nm}(x)\,O_{nm}(0) = (-2)^n x^{\mu_1}\ldots x^{\mu_n}O^n_{\mu_1\ldots\mu_n}(0)$$

and remember that the matrix element $\langle p|O_{nm}(0)|p\rangle$ must be proportional to $H_{nm}(p)$, we obtain[16]

$$\langle p|x^{\mu_1}\ldots x^{\mu_n}O^n_{\mu_1\ldots\mu_n}(0)|p\rangle = \frac{2\pi^2}{2^n(n+1)}M_n\sum H_{nm}(x)H_{nm}(p).$$

Here, $M_n = \langle p\|O^n\|p\rangle$ denotes the reduced matrix element. In accordance with this, the light-cone expansion of the matrix element of the currents $j(x)j(0)$ appears as an expansion in Gegenbauer polynomials:

$$\langle p|j(x)j(0)|p\rangle \underset{x^2=0}{\approx} \sum_n C_n(x^2, g)\left(\frac{i\sqrt{x^2}}{2}\right)M_n C^1_n(ipx/\sqrt{x^2}). \qquad (133)$$

This expansion can be considered from the following point of view. As is well known, the Taylor expansion of the scattering amplitude in accordance with (58) corresponds to the Cornwall–Norton moments:

$$T(p, q) = \frac{2}{\pi}\sum_{n=0}^{\infty}\left(\frac{2pq}{Q^2}\right)^n\mu_n(Q^2). \qquad (134)$$

Here, the group-theoretical structure of $T(p, q)$ remains hidden. It can be brought out by termwise application of the Fourier transformation to the light-cone expansion of the matrix element $\langle p|Tj(x)j(0)|p\rangle$. Using the relation

$$H_{nm}(-i\partial/\partial q)f(Q^2) = H_{nm}(q)(2i\partial/\partial Q^2)^n f(Q^2),$$

we can expand the amplitude $T(p, q)$ in Gegenbauer polynomials:

$$T(p, q) = \frac{i}{4\pi}\sum_{n=0}^{\infty}C^1_n\left(\frac{ipq}{\sqrt{Q^2}}\right)(\sqrt{Q^2})^{-n}M_n(Q^2)\left(\frac{\partial}{\partial Q^2}\right)^n\tilde{C}_n(Q^2); \qquad (135)$$

the series (135) converges outside the spectral region, i.e., for $|2pq| < Q^2 = -q^2$. This expansion is a regrouping of the Taylor expansion (134).

The structure of Eqs. (134) and (135) enables one to introduce a general expansion of the scattering amplitude in Gegenbauer polynomials:

$$T(p, q) = \frac{2}{\pi}\sum_{n=0}^{\infty}\left(\frac{4i}{\sqrt{Q^2}}\right)^n\mu^\xi_n(Q^2)\,C^1_n\left(\frac{ipq}{\sqrt{Q^2}}\right). \qquad (136)$$

Using the orthogonalization relation of the Gegenbauer polynomials, we can represent the expansion coefficients $\mu^\xi_n(Q^2)$ as projections of $T(p, q)$. Using dispersion relations, we can obtain the special moments

$$\mu^\zeta_n(Q^2) = \int d\zeta\zeta^{n-1}(1 + \zeta^2/Q^2)\,W(p, q), \qquad (137)$$

where

$$\zeta = Q^2[pq + \sqrt{(pq)^2 + p^2Q^2}]^{-1} \qquad (138)$$

are the so-called Nachtmann moments. It is obvious that at large $Q^2$ the $\mu^\xi_n(Q^2)$ are equal to the moments introduced by means of (54) and (55). Comparing the relations (135) and (136), we obtain

$$\mu_n^{\zeta}(Q^2)=\frac{2i^{n+1}}{4^{2(n+1)}}(Q^2)^n\left(\frac{\partial}{\partial Q^2}\right)^n\tilde{C}_n(Q^2)M_n. \tag{139}$$

Such an analysis is valid for vector currents,[45] it being convenient to proceed from an expansion modified compared with (87), which for the electromagnetic currents leads to the following expression for the single-nucleon matrix element:

$$T_{\mu\nu}(p,x)=\sum_n\{C_n^1(x^2,g)\,g_{\mu\nu}P^{[\mu]}(x)$$
$$\times\langle p\,|\,O^n_{[\mu]}(0)\,|\,p\rangle+C_n^2(x^2,g)\,g_{\mu k}g_{\nu\lambda}P^{k\lambda\,[\mu]}(x)$$
$$\times\langle p\,|\,O^n_{[\mu]}(0)\,|\,p\rangle+C_n^3(x^2,g)\,P^{[\mu]}(x)$$
$$\times\langle p\,|\,O^n_{\nu\mu_1\ldots\mu_{n-1}}(0)\,|\,p\rangle+g_{\nu k}P^{k\,[\mu]}(x)$$
$$\times\langle p\,|\,O^n_{\mu\mu_1\ldots\mu_{n-1}}(0)\,|\,p\rangle\}. \tag{140}$$

Here, $P^{\mu_1\cdots\mu_n}(x)$ is a completely symmetric and traceless tensor constructed from $x^\mu$. Using the relations

$$\left.\begin{array}{l}\langle p\,|\,O^n_{\mu_1\ldots\mu_n}(0)\,|\,p\rangle=M_nP_{\mu_1\ldots\mu_n}(p)\\ G_n^j(q^2)=M_n\left(-2i\frac{\partial}{\partial q^2}\right)^n\int d^4x\exp(i\,qx)\,C_n^i(x^2),\\ i=1,3,4;\\ G_n^2(q^2)=M_n\left(-2i\frac{\partial}{\partial q^2}\right)^{n+2}\int d^4x\exp(i\,qx)\,C_n^2(x^2),\end{array}\right\} \tag{141}$$

we obtain a result similar to the one that for scalar currents led to (139), i.e., we obtain the relations

$$\frac{1}{2(n+2)^2}\int_{Q^2}^{\infty}\frac{d\nu}{Q^2}\zeta^{n+1}\left\{\left[(n+4)+\frac{3n+2}{2}\frac{\zeta}{2}-\frac{n+2}{2}\frac{2}{\zeta}\right]M^2W_2-2(2n+3)(n+2)W_1\right\}=\frac{\pi}{2^n}(Q^2)^nG_n^1(-Q^2);$$

$$\frac{1}{2}\int_{Q^2}^{\infty}\frac{d\nu}{Q^2}\zeta^{n+1}\left\{\left[\frac{2n}{n+2}+\frac{n^2+n+2}{(n+2)(n+3)}\frac{\zeta}{2}+\frac{z}{\zeta}\right]M^2W_2-4W_1\right\}=\frac{\pi}{2^n}(Q^2)^{n+1}G_n^2(-Q^2);$$

$$\frac{n-1}{2(n+1)}\int_{Q^2}^{\infty}\frac{d\nu}{Q^2}\zeta^{n+1}\left\{\left[\frac{n+2}{2n}+\frac{1}{4}\frac{\zeta}{2}+\frac{n^2+3n+4}{4n(n-1)}\frac{2}{\zeta}\right]M^2W_2-W_1\right\}=\frac{\pi}{2^n}(Q^2)^{n-1}G_n^3(-Q^2);$$

$$\frac{1}{2(n+2)^2}\int_{Q^2}^{\infty}\frac{d\nu}{Q^2}\zeta^{n+1}\left\{\left[2(n^2+2n-4)+n(n-2)\frac{\zeta}{2}+(n^2+6n+8)\frac{2}{\zeta}\right]M^2W_2+4n(n+2)W_1\right\}$$
$$=(\pi/2^n)(Q^2)^nG_n^4(-Q^2),$$

where $M$ is the mass of the target, $\zeta$ is the expression (138), and

$$z=Q^2[-pq+\sqrt{(pq)^2+p^2Q^2}]^{-1}.$$

In such an approach, one does not take into account conservation of the currents, so that additional relations arise between moments of the same order.

Note that relations of the form (139) are in no way related to the validity of the expansion of an operator product.[46] On the basis of the Jost–Lehmann representation (19), one can show that the symmetric commutator $\overline{W}(p,x)$ of the causal Lorentz-invariant structure function $W(p,q)$ has a globally convergent series expansion in Gegenbauer polynomials:

$$\overline{W}(x,p)=\frac{1}{4i\pi^2}\sum_n\frac{(2i)^n}{n!}f_n^{\zeta}(x^2)\,C_n^1\left(\frac{xp}{\sqrt{x^2}}\right)(\sqrt{x^2})^n, \tag{142}$$

this differing from the expansion (51). We can now establish an analogous chain of connections between (61)–(64), from which we obtain the relation

$$\mu_n^{\zeta}(Q^2)=(Q^2)^n(\partial/\partial Q^2)^n\tilde{f}_n^{\zeta}(Q^2)/32\pi^2n!, \tag{143}$$

which is valid for all $Q^2>0$. This equation, which holds for any integral $n$, is thus based not on the expansion of an operator product but, in the general case, on Lorentz invariance. Thus, Eq. (139), which is only asymptotically true for the moments $\mu_n(Q^2)$, is an exact relation for the moments $\mu_n^{\zeta}(Q^2)$.

Note that in a discussion of the experimental results the variable $\zeta$ is more convenient than $\xi$.

*Reconstruction of the Structure Functions; Field-Theoretical Justification of the Quark-Parton Model.* The approach to deep inelastic scattering so far presented yields asymptotically true assertions about the moments of the structure functions. But it is the structure functions themselves that have the greatest interest. Although, as we have shown in Sec. 1, it is impossible, given the asymptotic behavior of the moments, to draw an unambiguous conclusion concerning the asymptotic behavior of the singulariteis on the light cone and, finally, concerning the structure functions, this nevertheless appears to be possible for large values of $Q^2$.

It is convenient to proceed from the moments $\mu_n^{\zeta}(Q^2)$ considered above and to use (137) as the point of departure for reconstructing the structure functions. It is convenient to determine the moments by means of a Mellin transformation. However, in such an approach the existence of the first moment is problematic. Difficulties can arise both from the Regge limit ($\zeta=0$) and from the subtractions of the Dyson–Jost–Lehmann representation. Use of a current operator expansion on the light cone cannot give the result. If a definite moment does not exist, then from the light-cone expansion there follows information about the subtraction term (see the discussion on this in Sec. 1). In what follows, we assume that all moments exist.

Formally, the inversion of Eq. (137) is given by

$$W(q,p)=(1+\zeta^2/Q^2)^{-1}\frac{1}{2\pi i}\int_{n_0-i\infty}^{n_0+i\infty}dn\,\zeta^{-n-1}\mu_n^{\zeta}(Q^2), \tag{144}$$

the necessary analytic properties of $\mu_n^{\zeta}(Q^2)$ following from the representation (137). It is true that this inversion does not yet have a physical interpretation. If we use Eq. (139) and the Mellin transformation

$$\left.\begin{array}{l}\bar{C}_n(Q^2,g)=\frac{2i^{n+1}}{4^{2(n+1)}}(Q^2)^n\left(\frac{\partial}{\partial Q^2}\right)^n\tilde{C}_n(Q^2,g)\\ =\int_0^1x^{n-1}E(x,Q^2)\,dx;\\ M_n=\int_0^1dx\,x^{n-1}G(x),\end{array}\right\} \tag{145}$$

then (144) takes the form[47]

$$W(q,p)=(1+\zeta^2/Q^2)^{-1}\int_0^1\frac{dx}{x}E\left(\frac{\zeta}{x},Q^2\right)G(x). \tag{146}$$

In this formula, $G(x)$, the Mellin transform of the matrix element $M_n$, is interpreted as the distribution function of the partons in the corresponding target, the remaining part of the formula reflecting the parton interaction.

A specific feature of Eq. (146) is that such a representation for the structure function remains valid irrespective of the particular assumptions made concerning the interaction dynamics. All that is assumed are the analytic properties with respect to $n$ needed for Eq. (145), and these are probably satisfied in the framework of quantum chromodynamics.

Because of the analyticity of $\mu_n^{\xi}(Q^2)$, which follows from (137), it is sufficient to prove analyticity of the function $\overline{C}_n$, which through the relation

$$\overline{C}_n(Q^2)=\frac{2i^{n+1}}{4^{2(n+1)}}(Q^2)^n\int dt\,\frac{(t+Q^2)^{-n-1}}{\Gamma(-n)}\,\tilde{C}_n(t)$$

is related to the analyticity of $\tilde{C}_n(Q^2)$. We obtain the analyticity of $\tilde{C}_n(Q^2)$ by solving the renormalization-group equation (101):

$$\tilde{C}_n(Q^2, g)\approx[\ln(Q^2/\mu^2)]^{-c_n/2b}\,\tilde{C}_n(Q^2, \bar{g}^2(Q^2/\mu^2)).$$

The anomalous dimensions $C_n$ are analytic in the right-hand half-plane [see Eq. (93)], and the expression for $\tilde{C}_n(Q^2,\bar{g}^2)$ are independent of $n$, since they are determined by perturbation theory in the limit $Q^2\to\infty$.

The relation (146) has been the subject of many interesting discussions. In Ref. 48, the total kinematic structure without neglect of the terms of order $M^2/Q^2$ is taken into account. An expansion of the product of current operators is taken as the basis, this containing fermion operators and operators that follow from free field theory; a restriction is imposed on operators of twist 2. A representation of the structure functions is obtained in terms of the functions $f_i$, which depend weakly on $Q^2$. This representation is written in the form

$$\left.\begin{aligned}
&W_1(q, p)=\frac{\xi}{2(1+4\xi^2M^2/Q^2)^{1/2}}f_1(\zeta, Q^2)\\
&+\frac{M^2}{Q^2}\frac{\xi^2}{(1+4\xi^2M^2/Q^2)}\int_\xi^1 d\zeta' f_1(\zeta', Q^2)\\
&+\frac{2M^4}{Q^4}\frac{\xi^3}{(1+4\xi^2M^2/Q^2)^{3/2}}\int_\xi^1 d\zeta'\int_{\zeta'}^1 d\zeta'' f_1(\zeta'', Q^2);\\
&(pq)\,W_2(p, q)=\frac{\xi^2}{(1+4\xi^2M^2/Q^2)^2}\int_\xi^1 d\zeta' f_2(\zeta', Q^2)\\
&+\frac{12M^4}{Q^4}\frac{\xi^4}{(1+4\xi^2M^2/Q^2)^{5/2}}\int_\xi^1 d\zeta'\int_{\zeta'}^1 d\zeta'' f_2(\zeta'', Q^2).
\end{aligned}\right\}\quad(147)$$

In such a representation of the structure functions the variable $\zeta$ appears automatically, as in (138), and this ensures the symmetry property of the expansion on the light cone. The functions $f_i(\zeta, Q^2)$ are determined by means of the asymptotically valid relations for their moments; with allowance for the perturbation-theory corrections we obtain for the coefficients on the light cone

$$\int_0^1 d\zeta\zeta^n f_i(\zeta, Q^2)=\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right]^{-A_n^{NS}}\left\{1+\frac{\bar{g}^2(Q^2)}{12\pi^2}f_{i,n}\right\},\quad(148)$$

where

$$f_{i,n}=4\sum_{k=1}^{n}\frac{1}{k}-2-8\sum_{k=1}^{n}\frac{1}{k^2}+(2+4i)/(n+1)+4/n^2-4/(n+1)^2.\quad(149)$$

The solution for $f_i(\zeta, Q^2)$, i.e., the inversion by means of a Mellin transformation, can be given in the form of an integro-differential equation.[49] In principle, such an equation enables one to calculate the structure functions $W_i(\xi, Q^2)$ from $W_i(\xi, Q_0^2)$.

The relation (146) can be estimated by using the decomposition of the product of the current operators and their structure with respect to the flavor group. The decomposition contains three contributions to Eq. (146) (Ref. 50), these corresponding to fermion $NS$, fermion $S$, and gluon operators:

$$\begin{aligned}
W(p, q)\approx\int_0^1\frac{dx}{x}\Big\{&E^{NS}(\zeta/x, Q^2)\sum_{p=q,\bar{q}}(Q_p^2-\langle Q^2\rangle)\,G^{Np}(x)\\
&+E^S(\zeta/x, Q^2)\langle Q^2\rangle\sum_{p=q,\bar{q}}G^{Np}(x)+E^G(\zeta/x, Q^2)\langle Q^2\rangle G^{NG}(x)\Big\},
\end{aligned}\quad(150)$$

where $G^{Np}$ and $G^{NG}$ are the distribution functions of the quarks and gluons in the nucleon $N$, and the summation is over the corresponding quarks and antiquarks; $Q_p$ are the quark charges. In the derivation of this relation, the fermion operators are approximated by free-field operators, i.e.,

$$\langle N\,|\,OF^a\,|\,N\rangle=\sum_{p=q,\bar{q}}(Q_p^2-\langle Q^2\rangle)\,M_n^{Np};$$

$$\langle N\,|\,O^nF^0\,|\,N\rangle=\langle Q^2\rangle\sum_{p=q,\bar{q}}M_n^{Np}$$

[cf. (145)].

The representation (150) is reminiscent of the formulas of the quark–parton model, gluons being here taken into account. Note that all the distribution functions depend weakly on $Q^2$.

These representations can be given a further quantitative estimate, in which the distribution functions for different values are connected with one another.[51] For this, one proceeds from the following considerations.

1. The moments of the structure functions, for example, of $F_2$, are made up of $NS$ contributions $\mu_n^a[a=3, 8, 15$ for $SU(4)]$ and two $S$ contributions $\mu_n^{\pm}$, which follow from the eigenvalues of the mixing matrix (113). For these moments, one obtains the $Q^2$ dependence that follows from quantum chromodynamics:

$$\mu_n^{\alpha}(Q^2)=\mu_n^{\alpha}(Q_0^2)\,[\ln(Q^2/\Lambda^2)\ln(Q_0^2/\Lambda^2)]^{-A_n^{\alpha}},\quad(151)$$

with $\alpha=(a,+,-)$ and $A_n^a=A_n^{NS}$ [cf. (108)–(110)].

2. Following the parton model,[27] we can express the structure functions for each value of $Q^2$ in terms of the quark distribution functions. They can be conveniently expressed in terms of linear combinations of the contributions:

$$\left.\begin{aligned}&F^3=U_V-D_V;\quad F^8=U_V+D_V;\\&F^{15}=U_V+D_V+6(s-s')\end{aligned}\right\}\quad(152)$$

and the fermion singlet contributions

$$F^0=U_V+D_V+6s+2s'.$$

We have here used the relations

$$U=U_V+s;\quad\overline{U}=\overline{D}=S=\overline{S}=s;$$

$$D=D_V+s;\quad C=\overline{C}=s',$$

where $s$ and $s'$ are the distribution functions of the sea quarks, and the index $V$ denotes the valence quarks. In addition, for each value of $Q^2$ the gluon distribution function is determined, so that the momentum balance is satisfied; the corresponding moments are determined as usual.

3. The moments $\mu_n^{\pm}$ in (151) correspond to eigenvectors of the mixing matrix with eigenvalues $\gamma_{\pm}$:

$$\mu_n^+ = \frac{1}{\gamma_- - \gamma_+} \{(\gamma_- - \gamma_{FF}^F)\, \mu_n^0 - \gamma_{VV}^F \mu_n^G\};$$

$$\mu_n^- = \frac{1}{\gamma_- - \gamma_+} \{(\gamma_{FF}^F - \gamma_+)\, \mu_n^0 + \gamma_{VV}^F \mu_n^G\}$$

[cf. (114)].

If the distribution functions at $Q_0^2$ are determined experimentally, then through these relations we can determine the corresponding distribution functions at a different value of $Q^2$. Using such a method, one can conclude that with increasing $Q^2$ the valence quarks contribute, whereas the contribution of the sea quarks and the gluons increases at $\xi = 0$. Compared with Eq. (150), which follows from quantum chromodynamics, we here have recourse also to the results of the naive quark model, in particular Eq. (152).

In this connection, we mention the interesting approach of Ref. 54, in which the point of departure is provided by the quark and gluon distribution functions, for whose moments the behavior (108)–(110), which follows from quantum chromodynamics, is postulated. The equation that results from the Mellin transformation is interpreted as a master equation. The application of statistical methods to quark and gluon scattering processes gives a simple method for calculating the exponents in (108)–(110). This method is based on simple processes in the framework of quantum chromodynamics and requires, by virtue of the light-cone expansion, an additional renormalization procedure.

*Influence of the Mass Parameters*. As we have already seen from the discussion in the preceding section, the influence of the mass parameters must be taken into account in the pre-asymptotic energy region. We begin by discussing the influence of the mass parameters on the effective coupling constant. From Eqs. (99) and (100), we obtain

$$\bar{g}^2(Q^2) = g^2 [1 + (g^2/16\pi^2)(11 - 2N/3) \ln (Q^2/Q_0^2)]^{-1},$$

where we replace $\lambda^2$ by $Q^2/Q_0^2$; $Q_0$ is any relative momentum. This formula can be rewritten as

$$\bar{g}^2(Q^2) = 16\pi^2/(11 - 2N/3) \ln (Q^2/\Lambda^2)$$

(with a conveniently chosen parameter $\Lambda$). Thus, the effective coupling constant depends on a single parameter with the dimensions of mass.

There are physical arguments[49] that permit one to estimate this parameter. Since scaling behavior of the structure functions is manifested at $Q^2 \approx 1\ \mathrm{GeV}^2$, the effective coupling constant $\bar{g}(Q^2)$ must be small, but for momenta that correspond to the reciprocal proton radius $(r_p^2)^{-1} \approx 0.243\ \mathrm{GeV}^2$ this coupling constant must be large, and one therefore takes $0.2\ \mathrm{GeV} \leqslant \Lambda \leqslant 0.5\ \mathrm{GeV}$.

In addition, in a quantum field-theoretical description the theory contains the quark masses, which are contained in the Lagrangian formalism, and the target masses, which occur in the matrix elements of the expansion of the operator products. Hitherto, this dependence has not been investigated, since it is not known for the asymptotic behavior of the structure functions.

In principle, the target masses occur in the reduced matrix elements $M_n = \langle p \| O^n \| p \rangle$ in implicit form. These masses are also contained in the different definitions of the scaled variables and, thus, simply in the expressions for the moments of the structure functions. They appear above all when one is converting the expressions for the moments to other variables. For example, from Eqs. (137) and (54) we obtain the connection

$$\mu_n^{\xi}(Q^2) = \int_0^1 d\xi\, \xi^{n-1} W(\xi, Q^2)\, [1 - (n+1)\, \xi^2 M^2/Q^2]$$
$$+ O(M^4/Q^4)\, \mu_n^{\xi} = \mu_n^{\xi} - (n+1)(M^2/Q^2)\, \mu_{n+2}^{\xi} + O(M^4/Q^4)\, \mu_n^{\xi}.$$

Since the part played by the physical quark masses is not yet known, they are regarded as coupling constants which are renormalized multiplicatively. The coefficients of such a generalized renormalization-group equation

$$(\mu\partial/\partial\mu + \beta\partial/\partial g + \delta\partial/\partial\alpha + \eta m \partial/m - n_F\gamma_F - n_V\gamma_V)\, \Gamma^{(n_F;\, n_V)} = 0$$

are expressed with allowance for the mass dependence in the form[48]

$$\beta = \frac{-g^3}{16\pi^2} \Big\{ 11 - \frac{2}{3} \sum_{\text{flavors}} \Big[ 1 - \frac{6m_i^2}{\mu^2}$$
$$+ \frac{12 m_i^4/\mu^4}{(1 + 4m_i^2/\mu^2)^{1/2}} \ln \frac{(1 + 4m_i^2/\mu^2)^{1/2} + 1}{(1 + 4m_i^2/\mu^2)^{1/2} - 1} \Big] \Big\}; \tag{153}$$

$$\eta_i = (-8g^2/16\pi^2)\, \{1 - (m_i^2/\mu^2) \ln (1 + \mu^2/m_i^2)\}. \tag{154}$$

The differential equations for determining the effective coupling constant $\bar{g}^2(Q^2)$ and the effective masses $m_i(Q^2)$ were solved numerically. It is obvious that the solutions contain the initial values $g(\mu)$ and $m_i(\mu)$, which are either found experimentally or determined from the quark model.

Finally, there is one further source for manifestation of parameters that have the dimensions of mass. They are related to the hitherto ignored operators of higher twist, or light-cone functions, whose canonical singularities are reduced by an integral order of $x^2$ compared with the singularities of the minimal-twist operators. In the field theory of free quarks, such powers appeared in the form $m_q^2 x^2$ (Ref. 48). Instead of them, one introduces the (undetermined) mass parameter $M_0$, which occurs in the asymptotic behavior of the moments in the form

$$\mu_n^{\xi} = A_n(Q^2) + \sum_{k=1}^{\infty} \Big\{ \frac{n M_0^2}{Q^2} \Big\}^k B_{nk}(Q^2),$$

where $A_n(Q^2)$ is the right-hand side of Eq. (139).

From this we can say in summary that the nonasymptotic behavior of the moments can be described only when allowance has been made for the influence of the quark masses, the operators with higher twist (and therefore $M_0$), and the previously given higher approximations in the effective coupling constant.

## CONCLUSIONS

We approach the completion of the theoretical description of deep inelastic scattering and therefore, in a certain sense, of quantum chromodynamics as well.

The general conclusion could be that deep inelastic lepton–hadron scattering is an excellent method for investigating hadron structure, enabling one to obtain new physical ideas about strong interactions, i.e., to give a concrete theory.

To establish a connection between the effective scattering cross sections, i.e., the structure functions associated with them, which can be measured experimentally, and the field-theoretical concepts (light-cone singularities), it is necessary to make a model-independent analysis based on the general principles of quantum field theory. Such an analysis, based on the Dyson–Jost–Lehmann representation, has been set forth in some detail in the second section.

In our opinion, the most attractive theory, which describes both deep inelastic lepton–hadron scattering and other processes with large momentum transfer as well as low-energy hadron–hadron processes, is the quark-parton model. Requiring asymptotic freedom, we arrive from this model at quantum chromodynamics. However, in quantum chromodynamics it is necessary to take into account the problem of infrared divergences if one does not introduce supersymmetric gauge fields.[52] We note that the requirement of asymptotic freedom is subject to much discussion. The restricted accuracy of the experimental data at the present time does not permit one to make a distinction between asymptotic freedom (logarithmically modified asymptotic behavior of the structure functions) and the existence of a fixed point with small nonvanishing anomalous dimensions (weak power-law asymptotic behavior of the structure functions). The differences between the Callan–Gross and Llewellyn Smith sum rules are a consequence of asymptotic freedom.

The results of quantum chromodynamics relating to deep inelastic lepton–hadron scattering are at present unfortunately restricted: There exists assertions which, strictly speaking, have a special nature, for example, the logarithmic nature of the decrease of the moments and the concentration of the structure functions near $\xi = 0$, or conditional assertions such as, for example, the relations between the moments for two different values of $Q^2$ and $Q_0^2$. Only the sum rules can be regarded as numerical results.

All the results of quantum chromodynamics obtained for deep inelastic lepton–hadron scattering are based on an expansion of a product of current operators. We note that the description of deep inelastic scattering in field theory can also be based on a modified method of canonical quantization and Feynman diagrams. However, in contrast to deep inelastic lepton–hadron scattering, other topical problems such as large-angle scattering and the calculation of electromagnetic form factors cannot be described by the method of expanding proproducts of current operators. With regard to the other theoretical approaches (for example, the quasiclassical approximations and instanton solutions),[55] the question of their validity naturally arises.

We draw attention to the fact that hitherto quantum chromodynamics has given definite assertions only for deep inelastic scattering. There is therefore undoubted interest in new experiments in this field, including electron–positron annihilation, which could confirm the validity of quantum chromodynamics.


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