Exact cylindrically symmetric solutions to the classical equations of gauge theories for arbitrary compact Lie groups

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The main results in the field of exact cylindrically symmetric solutions to the classical equations of gauge theories for arbitrary compact Lie groups are presented.

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INTRODUCTION

In recent years, significant progress has been achieved in the investigation of exact solutions to the classical equations of gauge theories. This applies to general treatments and existence theorems, as well as the explicit construction of special solutions in the framework of particular assumptions. These solutions are of great interest in connection with the frequently expressed hope that they could provide the basis for the construction of a realistic quantum theory; they also represent an independent important mathematical problem, especially for nonlinear equations, in view of the absence of regular methods of integration of such equations. Exact solutions (especially for completely integrable equations) also undoubtedly retain their importance even if some of the physical ideas underlying a particular theory are shown to be incorrect. Currently, the most topical field of application of this many-sided problem is in the theory of Yang-Mills gauge fields. Although the connections are not always immediately obvious, there are here interwoven many problems and aspects of physics and mathematics (group theory and group representations, algebraic and differential geometry, topology, the inverse scattering problem, Bäcklund transformations, etc.; field theory, statistical physics, solid-state physics, and so forth). Here, the combined efforts of physicists and mathematicians have yielded important results such as the description of general self-dual solutions to the classical Yang-Mills equations[1] on a four-dimensional sphere for compact classical Lie groups,[2] a new understanding of the anomalies of the axial-vector current on the basis of the Atiyah-Singer index theorem,[3] and more. There has by now been published a huge number of papers on individual aspects of the problem of exact solutions to the classical equations of gauge theories on the basis of different mathematical methods. Besides the general description of the solutions to the duality equations, there have been obtained numerous exact solutions that use particular restrictions or additional symmetries (see, for example, the reviews of Ref. 4). The most frequently studied are cylindrically symmetric configurations, whose physical content relates to the problem of monopoles,[4-6] merons,[4,7] and pseudoparticle systems, which reflect a number of basic properties of the complete theory (infinite set of topologically inequivalent classical vacuums, tunneling effect, and other characteristic manifestations of topological charge; see, for example, Refs. 4 and 8).

The present review aims at a compact presentation of a unified method for constructing exact cylindrically symmetric solutions to the classical equations of gauge theories on the basis of the group-theoretical approach and a gauge-invariant formulation of the basic entities of the theory. In the framework of this approach one can explicitly and completely integrate the system of duality equations for cylindrically symmetric fields and describe instanton and monopole configurations for an arbitrary compact gauge group. It appears to us expedient to divide the Introduction into two parts. In the first, we briefly describe the part played by exact classical solutions in gauge theories, and we give some of the main results in this field. These equations have been fairly fully treated in the reviews of Refs. 9 and 10 in recent years. In the second part of the Introduction, we dwell in more detail on the basic definitions and results relating directly to the subject of the review—exact solutions for cylindrically symmetric configurations.

On the part played by exact solutions to the classical equations of Gauge Theories; some basic results in this field. The importance of exact solutions to the classical Yang-Mills equations is to a large degree connected with problems of quantization of gauge theories. It is currently widely accepted that non-Abelian gauge theories have a direct relationship to a correct description of the interactions of elementary particles. Despite the renormalizability of such theories with spontaneously broken symmetry, the use of perturbative methods in the physics of strongly interacting particles seems hardly justified, since one then assumes analyticity of the matrix elements at zero value of the coupling constant. There has been put forward the alternative view (see, for example, Refs. 11 and 12) in the form of the assumption that these theories have a nonanalytic dependence on the coupling constant. The realization of this program is based on the assumption that the basic features of quantum theory can be described reasonably fully by taking into account quantum fluctuations near definite classical paths that arise from exact solutions to the nonlinear differential equations for the classical fields. The need to use exact solutions is usually attributed to the fact that the basic physical effects (for example, tunneling) associated inseparably with the nonlinearity cannot, as a rule, be reproduced by approximate methods. The problem of describing monopole configurations, i.e., static solutions with finite energy (see, for example, Ref. 5), also leads to exact solutions to the classical equations of gauge theories in the presence of Higgs fields. At large distances, these solutions realize a model of a Dirac monopole, whereas at short distances the solutions are modified in such a way as to ensure a finite energy. In this connection, we should also mention multimeron systems (see, for example, Ref. 7).

We list some of the main achievements and approaches used to investigate exact solutions to the classical Yang-Mills equations. We begin with those that are not related to simplifications (or restrictions) due to the presence of an additional invariance under some subgroup of the symmetry group of the system.

All the currently known nontrivial solutions to the classical equations of gauge theories in the Euclidean space R_4 corresponding to finite action are self-dual (or anti-self-dual) and are called instantons (anti-instantons) in accordance with the adopted terminology. These solutions ensure a minimum of the Euclidean action of the Yang-Mills field for fixed value of the topological charge. (More exact definitions of these quantities will be given subsequently.) The first and simplest example of a single-instanton solution for the group $SU(2)$ was obtained in Ref. 13. There were then constructed self-dual solutions with arbitrary integral value (k) of the topological charge described by $5k+4$ independent parameters in the framework of the gradient ansatz[14,15] for $SU(2)$ gauge fields.[16,17] Note that this ansatz reduces the classical Yang-Mills equations to the equation for the scalar φ^4 theory[17,18] (up to a set of singular points of the field φ) and for the dual subclass to the Laplace equation. Various formulations have been used to study such solutions, including the spinor formulation, a five-dimensional formalism, and others (see, for example, Ref. 19).

On the basis of the index theorem and in the framework of the method of infinitesimal deformations,[20,21] it has been shown that the most general self-dual solution for the group $SU(2)$ in R_4 is characterized by $8k-3$ independent parameters (or degrees of freedom). This result has been generalized for an arbitrary compact single group in Ref. 22.

The following investigations were directed toward the search for an ansatz describing the general $(8k-3)$-parameter self-dual solutions in R_4 on the basis of the approaches of Yang, Wu and Yang, Atiyah and Ward, Belavin and Zakharov, and others (Refs. 23–27), various mathematical schemes being employed. In particular, this problem was reduced in Ref. 25 by the methods of algebraic geometry to a hierarchy of A_l, $l=1,2,\ldots$ ansatzes, each of these being associated with the components of massless self-dual linear fields of spin $l-1$, while in Ref. 26 the duality equations were regarded as the consistency conditions for two linear equations in the framework of the inverse scattering problem. The definitive mathematical solution to the problem of describing all instantons for an arbitrary compact classical Lie group was obtained in the papers of Atiyah, Hitchin, Drinfeld, and Manin[2] (see also the subsequent Refs. 28–30). Their construction consists of reducing the nonlinear partial differential equations of duality to a nonlinear equation for a finite-dimensional matrix and leads to an elegant compact ansatz for all instantons. It should, however, be noted that for many concrete applications in the physics of instantons it would be desirable to have the corresponding solutions in a form that admits a perspicuous physical interpretation of the parameters as the degrees of freedom of a system of pseudoparticles. So far, solutions in such a form for the complete number of parameters are known only for the group $SU(2)$ for $k\leqslant 3$.

The question of whether instantons exhaust the complete set of finite-action solutions (without the duality restriction) to the Yang-Mills equations in R_4 remains open. In this connection, we mention Ref. 31, which gives a new formulation (in twistor space) of field equations in Minkowski space and R_4 on the basis of the results of Ref. 26.

The results of Ref. 2 were used to obtain compact expressions for the propagator of a spinless field belonging to the fundamental vector representation of the gauge group and solutions of the massless Dirac equation.[32] The tensor product of instantons and their Green's functions have been calculated in the framework of the matrix formalism[33] and on the basis of a description of instantons as fields over a single-point superspace.[34]

The investigation of the Yang-Mills equations without the duality restriction is of considerable interest even if one does not consider the part played by infinite-action solutions (merons and other irregular solutions) in the quantum theory of gauge fields, including the problem of quark confinement (see, for example, Ref. 35), and one makes the assumption that there are no solutions other than the self-dual ones which are regular everywhere in R_4. Study of the group structure of these equations in R_4 has elucidated the symmetry basis of the self-dual solutions and the nature of the duality condition. The point is that this reveals a jump in the symmetry at the "point" of duality (see Sec. 2), and one can attempt to formulate the duality condition as the critical "point" of such a jump in the symmetry of the solutions.

Cylindrically symmetric configurations of gauge fields. We begin by introducing the basic definitions and notation from the theory of gauge fields needed for what follows.

A Yang-Mills field over the Euclidean space R^4 with a compact simple gauge Lie group G of rank r is defined by potentials $A_\mu(x)$, $1\leqslant\mu\leqslant 4$, which take values in the algebra $\mathfrak{g}$ of the group $G\,[A_\mu(x)=e\sum_a A_\mu^a(x)\tau^a,\ \tau^a\in\mathfrak{g}\,]$ and are differentiable functions of $x\in R_4$; e is the coupling constant. The field $A'_\mu(x)$ is said to be gauge equivalent to the field

$A_\mu(x)$ if there exists a differentiable function $g(x)$ with values in G such that $A'_\mu = g^{-1}(\partial_\mu + A_\mu)g$. The field tensor $F_{\mu\nu}(\equiv e\sum_a F^a_{\mu\nu}\tau^a)$ is defined by the formula $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} + [A_\mu, A_\nu], A_{\mu,\nu} \equiv \partial A_\mu/\partial x_\nu$. The requirement of stationarity of the Euclidean action $\mathscr{S} = -1/2e^2 \int d^4x \mathrm{Sp} F_{\mu\nu} \cdot F_{\mu\nu}$ yields the equations of motion $F_{\mu\nu,\mu} + [A_\mu, F_{\mu\nu}] = 0$. Using the Bianchi identity, it is easy to show that these equations are satisfied if the field is self-dual or anti-self-dual, i.e., $\pm F_{\mu\nu} = {}^*F_{\mu\nu}(\equiv (\frac{1}{2})\tau_{\mu\nu\rho\tau} F_{\rho\tau})$. Here, $\varepsilon_{\mu\nu\rho\tau}$ is the completely antisymmetric (with respect to all indices) tensor, $\varepsilon_{1234} \equiv 1$. A self-dual field $A_\mu(x)$ which is regular at all points $x \in R_4$ (including infinity) is an instanton field. It corresponds to finite values of the action $\mathscr{S}$, which is proportional in this case to a topological characteristic, the charge $Q = 1/16\pi^2 \int d^4x \mathrm{Sp} {}^*F_{\mu\nu} \cdot F_{\mu\nu}$, which takes integral values. (A detailed discussion of the integral nature of the characteristic is given in the reviews of Ref. 8.) Anti-instantons are obtained from instantons by changing the orientation of space.

Yang-Mills theory has the important property of invariance under the conformal group of coordinate transformations; these include not only translations $x_\mu \xrightarrow{P} x_\mu + a_\mu$ and four-dimensional rotations $x_\mu \xrightarrow{M} O^\nu_\mu \cdot x_\nu$, but also the scale transformations $x_\mu \xrightarrow{D} \lambda^\varepsilon x_\mu$ and the special conformal transformations

$$x_\mu \xrightarrow{K} \frac{x_\mu + c_\mu x^2}{1 + 2(cx) + c^2 x^2}.$$

The 15 generators $\{P_\mu, M_{\mu\nu}, D, K_\mu\}$ form the algebra of the conformal group, which is isomorphic to $O(5,1)$ in R_4. Under infinitesimal transformations $\delta x_\mu = X\mu(x)\ (\equiv a_\mu + \omega^\nu_\mu x_\nu + \varepsilon x_\mu - c_\mu x^2,\ \omega^{\mu\nu} \equiv -\omega^{\nu\mu})$ the gauge field transforms in accordance with $\delta A_\mu = X^\nu A_{\mu,\nu} + (X^\nu A_\nu)_{,\mu}$. The complete symmetry group of the theory is the direct product of the conformal group and the gauge group. However, the Lorentz group [or $O(4)$ in R_4] may be contained in several, in general, nonisomorphic ways in the direct product. From the point of view of physics, this last circumstance means that the transformation properties of the entities occurring in the theory are not uniquely determined with respect to transformations in the Lorentz group; in particular, this leads to the possible existence of different types of solutions that are completely symmetric with respect to this group.

The existence of a symmetry group of the classical Yang-Mills equations makes it possible to classify their solutions in accordance with the irreducible representations of the corresponding group. Naturally, solutions which are symmetric with respect to transformations in some subgroup have the simplest properties. As we have already noted, numerous investigations associated with analysis of the quantum theory of gauge fields have been based on a reduction of the total number of degrees of freedom of a physical system to ones invariant under a subgroup of the conformal group. One then obtains quantum systems that are significantly simpler than the complete Yang-Mills theory but still preserve a number of its essential properties. Even a very simplified scheme containing only $O(4)$-invariant degrees of freedom leads to a quantum model containing features of the complete theory such as classical degeneracy, vacuum tunneling, and the bound-state spectrum of an anharmonic oscillator.[4] Much richer in their physical content are the approaches based on cylindrical symmetry in R_4 and spherical symmetry in Minkowski space. However, before we turn to their description, we introduce definitions, needed in what follows, of the diagonal group $\mathscr{L}$ and the group of gauge transformations of two-dimensional space S (or the invariance subgroup).

In the algebra $\mathfrak{g}$ of the gauge group G we separate the operators T_i that satisfy the commutation relations of the algebra $su(2)$ and construct the operators $L_i \equiv T_i + M_i$, where M_i are the generators of spatial rotations (including spin, if necessary); $M_i = -i\varepsilon_{ijk} x_j \partial/\partial x_k$ + (spin part); $\nabla \equiv \partial/\partial \mathbf{x} = (\mathbf{x}/r)\partial/\partial r + (1/r^2)\mathbf{x} \times \mathbf{x} \times \nabla$. The operators M_i form the algebra $su(2)$ and commute with T_j. Then the operators L_i interchange internal and spatial indices and are the generators of the so-called diagonal ($\mathscr{L}$) group algebra $su(2)$. We shall say that solutions to the equations are cylindrically symmetric if they are annihilated by the operators L_i, i.e., they are singlets under the group $\mathscr{L}$ (i.e., the corresponding gauge fields transform as scalars A_0 or vectors $\mathbf{A}$ under transformations in $\mathscr{L}$, and the structure functions parametrizing the fields A_μ depend only on $r \equiv \sqrt{\mathbf{x}^2}$ and t). The spherically symmetric solutions do not depend on the time. (For more general definitions of such systems, see, for example, Refs. 36 and 37.) We now consider the vector space of the algebra $\mathfrak{g}$ spanned by the $\mathscr{L}$-invariant elements (i.e., they commute with L_i) which depend on the point $\mathbf{x}$ of a three-dimensional subspace of R_4 and transform cylindrically symmetric configurations into other cylindrically symmetric configurations. These elements form a subalgebra of the algebra of the gauge group G which generates the group S of gauge transformations of the two-dimensional space (r, t) in the cylindrically symmetric case. The main simplification that arises in the study of cylindrically symmetric systems is that the structure of the gauge group of the two-dimensional space is much simpler than that of the original group G. In addition, in the two-dimensional space only one tensor component of the field corresponds to each index of the gauge group.

Naturally, in the cases when the transformation properties of the system are rigidly fixed with respect to the transformations M_i of the three-dimensional rotation group, the cylindrically symmetric solutions are unique. In gauge theories, because of the presence of transformations in G additional to $SU(2)$, the transformation properties with respect to spatial rotations can be specified in different ways. The number of different possibilities is exactly equal to the number of inequivalent embeddings of the subgroup $SU(2)$ generated by the operators T_i in the gauge group G, and to each such embedding there corresponds an invariance subgroup S. The simplest case is when S decomposes into the direct product of $U(1)$ subgroups whose number is exactly equal to the rank of the original gauge group G. We shall investigate this case in detail.

It should be mentioned that the problem of decomposing the algebras $\mathfrak{g}$ of simple Lie groups G with respect to the irreducible representations of a subalgebra $\mathfrak{g}_0 \mathscr{C} \mathfrak{g}$ is frequently encountered in physical applications. In particular, in nuclear physics, elementary-particle physics, gauge field theories, etc., such a subalgebra $\mathfrak{g}_0$ is represented by $su(2)$ (angular momentum, isotopic spin, etc.), and the need arises to classify the generators of $\mathfrak{g}$ with respect to irreducible representations of $su(2)$. At the mathematical level, the problem of embedding a semisimple subalgebra $\mathfrak{g}_0$ in a simple algebra $\mathfrak{g}$ was solved in Ref. 38 (see also Ref. 39). We shall return to this question in more detail in Sec. 1.

After it had been established that nonsingular magnetic monopoles exist in the $SO(3)$ gauge theory,[40,41] numerous investigations were made into the possibility of generalizing this result to larger gauge groups and the finding of special solutions describing spherically symmetric monopole configurations for various unitary groups. An exposition of the main results in this field up to the end of 1977 is given in the reviews of Refs. 5 and 6.

General questions of the description of spherically symmetric configurations and the establishment of conditions for the existence of extremals corresponding to finite-energy solutions for arbitrary types of embedding of $SU(2)$ in G were considered in detail in Refs. 42 and 43. Because the subject of the present review is the constructive part of this problem, we shall not dwell on this question, and we list briefly the main results relating to explicit solutions for cylindrically (spherically) symmetric configurations.

In the case of the group $SU(2)$ and in the framework of the cylindrically symmetric ansatz for the Yang-Mills fields in R_4, which reduces the system of self-duality equations to the Liouville equation for a function that is gauge invariant under $S \equiv U(1)$, general solutions have been obtained for an arbitrary number (k) of instantons described by $2k$ parameters[44] (see also Ref. 45). In Ref. 46, on the basis of the same ansatz, the system of classical Yang-Mills equations was explicitly constructed and its symmetry properties investigated, this resulting in a significant simplification of these equations (and of the expressions for the densities of the topological charge and the action), which are formulated in terms of two gauge-invariant quantities. The resulting equations contain as special cases the self-dual[44] and meron[47] subclasses. It was also shown that in the presence of additional $O(2)$ symmetry Witten's solutions are the matrix elements $t^{4,k}|_{00}$ of a representation of class 1 of the Lorentz group, and operators which raise and lower the topological charge were constructed explicitly.[46]

Further investigations of constructive nature in this direction are associated with the construction of exact cylindrically (spherically) symmetric solutions to the classical equations for compact gauge groups. In Ref. 48, equations are explicitly constructed that describe cylindrically symmetric configurations of Yang-Mills gauge fields for the embedding of $SU(2)$ in an arbitrary compact semisimple group G of rank r, the invariance subgroup being $S = \Pi_1^r \otimes U(1)$. The final equations and expressions for the densities of the action and the topological charge can be expressed *solely* in terms of $2r$ entities that are gauge invariant under S, the entire dependence of the equations on the group structure being *completely* concentrated in the Cartan matrix of the corresponding algebra. For the subclass of self-dual fields there arises a system of r equations that generalizes in a natural manner to an arbitrary compact gauge group Liouville equation, which describes in the special case of the group $SU(2)$ self-dual configurations. The scheme admits inclusion of a Higgs field that transforms in accordance with the adjoint representation of G. Further, after very special but subsequently very helpful and instructive theoretical experiments with the groups $SU(3)$ and $O(5)$ (Ref. 49), it proved possible to find explicitly general solutions to the duality equations for groups of second rank ($SU(3)$, $\mathrm{Sp}(4) \cong O(5)$, and G_2) (Ref. 50), these depending on the single variable $z+\bar{z}$ (or $z\bar{z}$), where $2z \equiv r+it$, to separate instanton configurations corresponding to finite values of the action and to calculate the topological charges for these groups. This served as the basis for a generalization of the listed results to the case of arbitrary compact simple Lie groups, for which, in the framework of the root technique, *general* $2r$-parameter solutions were constructed[51] for the system of self-duality equations of Liouville type, the solutions depending on a single variable. Realization of the boundary conditions at infinity and at short distances ensuring finite action of the Yang-Mills fields made is possible to describe instanton configurations and calculate the topological charge, which depends on r additional "quantum" numbers, with respect to which its values are degenerate. This leads to the existence of a discrete series of solutions with a fixed value of the charge. Similar treatments have been given for monopole spherically symmetric systems,[52] for which one can also separate a subclass of finite-energy solutions parametrized by r quantum numbers and find explicitly the matrices of the magnetic charge and masses of the monopoles. Essential use is made of a symmetry of a certain kind between the cylindrically symmetric self-dual static configurations of Yang-Mills fields in R_4 and spherically symmetric monopoles in Minkowski space (with a Higgs field in the adjoint representation of the corresponding gauge group) in the Bogomol'nyi-Prasad-Sommerfield limit.[53,54]

Similar investigations were made in the series of papers by Bais, Weldon, and Wilkinson.[55,56] In them, on the basis of purely computational methods that do not employ the root technique, special cylindrically symmetric instanton solutions were obtained[55] for the unitary group $SU(n+1)$, these corresponding to the choice $m_1 = 2, m_\alpha = 1, 2 \leqslant \alpha \leqslant n$, in the general solutions (see Sec. 4), and general r-parameter spherically symmetric monopole solutions,[56] which are identical with (68) (see Sec. 3) for a unitary group.

All the previously obtained exact solutions for cylindrically symmetric instantons and spherically symmetric monopoles arise naturally as special cases of our general scheme. The only exceptions are some pointlike solutions for unitary groups corresponding to other types of embedding of $SU(2)$ in G (see, for example, Ref. 57). The problem of constructing the explicit form of the general solutions for all types of embeddings of this part of the review is not considered.

Finally, in Ref. 58 there are constructed general solutions to the system of r second-order partial differential equations for cylindrically symmetric self-dual systems, the solutions depending on $2r$ arbitrary functions. This shows that the system of self-duality equations of Liouville type is completely integrable for an arbitrary compact gauge group G of rank r.

We now outline the plan of the present review, which gives an exposition of the main results of Refs. 48, 50–52, 58, and 59.

In Sec. 1, we construct a mathematical basis for parametrizing cylindrically symmetric gauge fields on the basis of an explicit realization of the algebra $\mathfrak{g}$ of the gauge group G. The operators F_m^{ls} of the algebra $\mathfrak{g}$ are classified in accordance with the irreducible representations of its $su(2)$ subalgebra and are labeled by the indices of the angular momentum l, its projection m, and the multiplicity s of the embedding. We consider in detail an embedding, called the minimal embedding, that is universal for all simple Lie algebras and in which the number of $su(2)$ multiplets F_m^l is equal to the rank of $\mathfrak{g}$. It is on this embedding, which corresponds to the Abelian invariance subgroup $S=\Pi_1^r\otimes U(1)$, that all our subsequent constructions are based. The main characteristics of the minimal embedding are given in Table II.

Section 2 is devoted to the explicit construction of classical equations for cylindrically symmetric configurations of gauge fields in Euclidean space in the framework of the minimal embedding of $SU(2)$ in an arbitrary compact group G. We give a *general* parametrization of Yang-Mills fields by means of the generators of the gauge group S of two-dimensional space, and we investigate the transformation properties of the structure functions under transformations in its symmetry group, which is isomorphic to the direct product of the group S and the subgroup $O(1,2)$ of the conformal group. To separate unmixed combinations of contributions of different l multiplets to the parametrization of the gauge field, a transition is made to a diagonal representation of the generators of S. On the basis of the properties of the structure functions that parametrize fields in the diagonal representation with respect to the action of the generators of the group S, we construct $2r$ gauge-invariant (with respect to S) quantities.

In explicit form, we obtain general equations of motion and expressions for the action of gauge fields and the topological charge. These fundamental entities of the theory are formulated in noninvariant form in terms of $4r$ structure functions, and also in terms of $2r$ gauge-invariant entities using the equations of motion. The *entire* information concerning the properties of the gauge group G is completely concentrated in the corresponding Cartan matrix. The field equations admit an obvious transition to hyperbolic type by replacement of the real (in the case of R_4) time components by imaginary components. The spectrum of pointlike solutions is calculated on the basis of the obtained equations. We identify the self-dual (anti-self-dual) subclass of the general equations, this consisting of the Cauchy-Riemann analyticity conditions for r complex scalar fields l_α and r algebraic equations connecting $2r$ gauge-invariant (with respect to S) quantities. The latter equations reduce to a system of r coupled equations of Liouville type,

$$\partial^2\rho_\alpha/\partial z\,\partial\bar{z}=\sum_{\beta=1}^{r}\delta_\beta\cdot k_{\alpha\beta}\exp\rho_\beta,\quad \delta_\alpha\equiv 2\sum_{\gamma}k_{\alpha\gamma}^{-1},$$

where k is the Cartan matrix of the group G and $2z=r+it$; this generalizes naturally Witten's well-known result[44] for the group $SU(2)$. For actual calculations, a convenient expression is obtained for the topological charge of the instanton configurations in the form of the sum of the contributions of the topological charges of r two-dimensional Abelian gauge fields. The subclass of general equations describing multimeron system is given. We show how a Higgs field that transforms in accordance with the adjoint representation of the gauge group can be included in the scheme, and we describe spherically symmetric monopole configurations in Minkowski space. We construct the energy functional and the equations of motion in generally adopted notation for the $2r$ structure functions that parametrize the vector part of the potential of the Yang-Mills field $W(W_0=0)$ and the Higgs field φ. We also discuss the connection between the duality conditions in R_4 and the Bogomol'nyĭ-Prasad-Sommerfield limit[53,54] in Minkowski space.

In Sec. 3, we construct general solutions to the system of equations

$$\partial^2\rho_\alpha/\partial z\,\partial\bar{z}=\sum_{\beta}\delta_\beta\cdot k_{\alpha\beta}\exp\rho_\beta.$$

We investigate the symmetry properties of this system and propose a reduction scheme that makes it possible to reduce the problem of constructing solutions to a partial differential equation of order $2r$ for a single unknown function. On the basis of the invariant root technique, we construct general solutions to this system that depend on $2r$ arbitrary functions. We give the $2r$-parameter subclass of general solutions to the system that depend on a single variable $z\bar{z}$ (or $z+\bar{z}$) and are used in what follows to describe instanton and monopole configurations.

Section 4 contains a classification of the instanton and monopole solutions and a calculation of the corresponding values of the topological charge of the instantons and the matrices of the magnetic charge and masses of the monopoles.

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1. EMBEDDINGS OF THREE-ELEMENT SUBALGEBRA IN AN ARBITRARY SIMPLE LIE ALGEBRA; THE MINIMAL EMBEDDING

In the case of cylindrical symmetry, the problem of complete classification of the generators of the algebra of a simple Lie group G with respect to irreducible representations of $SU(2)$ can be solved on the basis of a description of all inequivalent embeddings of $SU(2)$ in G. Each embedding is, in its turn, determined by specifying an embedding vector whose components can be expressed in an obvious manner in terms of the expansion coefficients of the Cartan element of the corresponding $su(2)$ subalgebra (of the diagonal group) with respect to the generators of the Cartan subalgebra of the group G (Refs. 38 and 60). Among all possible embeddings of $SU(2)$ in G there exists an embedding that is universal for all simple groups; in it, the number of resulting $SU(2)$ multiplets is exactly equal to the rank of G, which automatically leads to the

$$S=\underbrace{U(1)\otimes U(1)\otimes\cdots\otimes U(1)}_{r}$$

gauge group of two-dimensional space. This embedding was first described in Ref. 38 (see also Ref. 39) and was realized in explicit form in Ref. 59. We shall call it the minimal embedding, bearing in mind that the number of multiplets when $SU(2)$ is embedded in G in other ways is always greater than the rank of G. (This embedding is known by other names such as the maximal or principal one, etc., depending on the characteristics of it that are taken as basic. The expression "minimal" appears preferable to us.) In addition, this embedding is distinguished by the fact that it is closest to the $SU(2)$ case. The Liouville equation, which realizes the duality condition for the group $SU(2)$ (Ref. 44), is then generalized for an arbitrary compact group G to a system of Liouville-type equations that is completely integrable (see Sec. 3). In the framework of the minimal embedding, the generators F^l_m of the algebra $\mathfrak{g}$ are labeled by the pair of indices l and m of the angular momentum, where l are the values of the angular momentum [the index of the irreducible representation of $SU(2)$], whose spectrum is fixed for every simple Lie algebra, and $m\in[-l,l]$ is the projection of the angular momentum (the index of the basis vector) within each multiplet. Thus, the total number of generators of $\mathfrak{g}$ is $\sum_{\alpha=1}^{r} c_\alpha(2l_\alpha+1)$, where c_α is the multiplicity of the multiplet l_α in $\mathfrak{g}$, and $c_\alpha>1(=2)$ only for the algebras D_n for even n.

We illustrate what we have said above by the example of the groups of second rank $SU(3)$, $O(5)\cong \mathrm{Sp}(4)$, and G_2 in the form of Table I.

To parametrize the generators of $\mathfrak{g}$, one usually employs either the root technique, which applies for all simple Lie algebras but is rather cumbersome in actual calculations and inadequately known in the physics literature, or tensor notation, whose use is restricted to the classical algebras. Our classifica-

TABLE I.

G	$SU(3)$	$O(5)\cong\mathrm{Sp}(4)$	G_2
Dimension $\Sigma(2l_\alpha+1)$	$8=3+5$	$10=3+7$	$14=3+11$
l_1, l_2	1, 2	1, 3	1, 5

tion of the generators of a simple algebra $\mathfrak{g}$ occupies in a certain sense an intermediate position, since in its framework the generality of the methods of the root technique is augmented by the perspicuity of the multiplet structure, which is familiar and convenient for physicists.

Before we turn to the explicit realization of the algebras for the minimal embedding, we give some notation and definitions needed for what follows (all additional results from the theory of simple algebras and Lie groups can be found, for example, in the monographs of Ref. 61; we follow the notation adopted by Bourbaki): G is an arbitrary simple Lie group of rank r and dimension ζ; $\mathfrak{g}$ is the Lie algebra of G; $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$; R is the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$, the number of roots being equal to $\zeta-r$; $X_{\pm i}$ are the elements of the root space of root i, $i\in R$; h_α are the generators of $\mathfrak{h}$ corresponding to simple roots π_α, $1\leq\alpha\leq r$; k is the Cartan matrix of $\mathfrak{g}$ with elements $k_{\alpha\beta}$, which realizes a complete identification of $\mathfrak{g}$ (up to isomorphism); ω_ν are the fundamental weights of $\mathfrak{g}$, $\omega_\nu\equiv(k^{-1}\pi)_\nu$.

We give the explicit form of the Cartan matrices for the groups of second rank:

$$k_{SU(3)}=\begin{vmatrix}2&-1\\-1&2\end{vmatrix};\quad k_{O(5)}=\begin{vmatrix}2&-2\\-1&2\end{vmatrix};$$
$$k_{\mathrm{Sp}(4)}=\begin{vmatrix}2&-1\\-2&2\end{vmatrix};\quad k_{G_2}=\begin{vmatrix}2&-1\\-3&2\end{vmatrix}.$$

In the Cartan-Weyl basis, the elements $X_{\pm i}$ and h_α satisfy the relations

$$[X_i,X_j]_-=\left\{\begin{array}{l}N_{ij}X_{i+j},\quad i+j\in R,\quad [h_\alpha,h_\beta]_-=0;\\ 0,\ i+j\neq 0,\ i+j\notin R,\\ h_i,\ i+j=0,\end{array}\right\}\ [h_\alpha,X_{\pm j}]_-=\pm j(h_\alpha)X_{\pm j},$$

and for compact roots $X_i^\dagger=X_{-i}$, $h_\alpha^\dagger=h_\alpha$, where $\dagger$ and $*$ are, respectively, the symbols of Hermitian and complex conjugation (T denotes the transpose). The elements $X_{\pm i}$ and h_α, which form a complete set, can be normalized by the relations $\mathrm{Sp}X_iX_j=\delta_{i+j,0}$, $\mathrm{Sp}h_\alpha h_\beta=\delta_{\alpha\beta}$, $\mathrm{Sp}X_ih_\alpha=0$.

We represent each positive (or negative) root α_i as a linear combination of simple roots: $\alpha_i=\sum_{\nu=1}^{r}t_i^\nu\pi_\nu$ with integral coefficients t_i^ν. The quantity $\sum_\nu t_i^\nu$ is called the height of root α_i; each simple root π_ν is encountered $2\sum_\alpha k^{-1}_{\alpha\nu}$ times in the system R_+ of positive roots. The height of the fundamental weight ω_ν is $\sum_\alpha k^{-1}_{\nu\alpha}\equiv(\frac{1}{2})\delta_\nu$. With each simple root π_ν (or fundamental weight ω_ν) of $\mathfrak{g}$ one can associate the variable y_ν and define $\alpha_i(y)$ by the expression $\alpha_i(y)=\sum t_i^\nu y_\nu$, where the set t_i^ν corresponds to the decomposition of the root α_i.

We now turn to the explicit realization of the algebra $\mathfrak{g}$ for the minimal embedding defined above. For this, we consider the element $H^{(1)} = \sum_\alpha H^1_\alpha h_\alpha$ in $\mathfrak{q}$ (the significance of the superscript will become clear in what follows), which takes unit value on all simple roots of $\mathfrak{g}$, i.e.,

$$[H^{(1)}, X_{\pm\alpha}]_- = \pm X_{\pm\alpha}, \quad \sum_\beta k_{\alpha\beta} H^1_\beta = 1 \forall 1 \leqslant \alpha \leqslant r, \tag{1}$$

from which, since the Cartan matrix is nondegenerate, we obtain $H^1_\alpha = \sum_\alpha k^{-1}_{\beta\alpha}$. For all simple algebras $H^1_\alpha > 0$, which makes it possible to introduce the elements $Z_\pm = \sum_\alpha (H^1_\alpha)^{1/2} X_{\pm\alpha}$ of the three-element algebra $su(2)$, in which $H^{(1)}$ plays the part of a Cartan generator. (In what follows, the operators $Z_\pm$ and $H^{(1)}$ will be used as the generators T_i in the diagonal group defined in the Introduction.) Hence, using the relation $[X_\alpha, X_{-\beta}]_- = 2\delta_{\alpha\beta} h_\alpha$, we obtain $[H^{(1)}, Z_\pm]_- = \pm Z_\pm$, $[Z_+, Z_-]_- = 2H^{(1)}$. With respect to the algebra $su(2)$, $\{Z_\pm, H^{(1)}\}$, the generators of $\mathfrak{g}$ are decomposed into a system of multiplets, whose structure can be studied by means of the following relation, which is obvious by virtue of (1):

$$[H^{(1)}, X_{\pm i}]_- = \pm m(i) X_{\pm i}, \tag{2}$$

where $m(i)$ is the order of root i, i.e., the number of simple roots (with allowance for multiplicity) from which the given root is composed. Thus, to all the roots of $\mathfrak{g}$ of a given order there corresponds the same quantum number m (basis index), and definite linear combinations of them belong to different multiplets. The index of the maximal multiplet is equal to the order of the maximal root of the algebra.

For complete recovery of the multiplet, it is sufficient to know just one member of it, since all the others can be found by means of the raising and lowering operators $Z_\pm$. We denote the element of the l-th

TABLE II.

$\mathfrak{g}$	l spectrum	δ_α	$\varkappa_\alpha H^1_\alpha$	H^l_α, $l \neq 1$
A_n $[su(n+1)]$	m, $1 \leqslant m \leqslant n$	$\delta_\alpha = \alpha(n-\alpha+1)$	1	$H^l_\alpha = \Delta_{\alpha-1} / \prod_1^{\alpha-1} \delta_\beta$, $2 \leqslant \alpha \leqslant n$; $H^l_1 = 1$
C_n $[\mathrm{Sp}(2n)]$	$2m-1$, $1 \leqslant m \leqslant n$	$\delta_\alpha = \alpha(2n-\alpha)$	1, 1, ..., 1, 1/2	$H^l_\alpha = \Delta_{\alpha-1} / \prod_1^{\alpha-1} \delta_\beta$
B_n $[O(2n+1)]$	$2m-1$, $1 \leqslant m \leqslant n$	$\delta_\alpha = \alpha(2n-\alpha+1)$, $1 \leqslant \alpha \leqslant n-1$, $\delta_n = n(n+1)/2$	1, 1, ..., 1, 2	$H^l_\alpha = \Delta_{\alpha-1} / \prod_1^{\alpha-1} \delta_\beta$, $1 \leqslant \alpha \leqslant n-1$; $H^l_n = (1/2)\, \Delta_{n-1} / \prod_1^{n-1} \delta_\beta$
D_n $[O(2n)]$	$2m-1$, $1 \leqslant m \leqslant n-1$; $n-1$	$\delta_\alpha = \alpha(2n-\alpha-1)$, $1 \leqslant \alpha \leqslant n-2$; $\delta_{n-1} = \delta_n = n(n-1)/2$	1	$l \neq n-1$ $(2\delta_n - l(l+1) \equiv n(n-1) - l(l+1) \neq 0)$ $H^l_\alpha = \Delta_{\alpha-1} / \prod_1^{\alpha-1} \delta_\beta$, $1 \leqslant \alpha \leqslant n-2$; $H^l_\alpha = \Delta_{n-3} / [2\delta_n + y_l] \prod_1^{n-3} \delta_\beta$, $\alpha = n-1, n$
				$l = n-1$, n — odd $H^{n-1}_\alpha = 0$, $1 \leqslant \alpha \leqslant n-2$; $H^{n-1}_n = -H^{n-1}_{n-1} = 1$
				$l = n - 1$, n even. The moment $(n - 1)$ is doubly degenerate and the second vector with $l = n - 1$ is $\overline{H}^{n-1}_\alpha = \Delta_{\alpha-1} / \prod_1^{\alpha-1} \delta_\beta$, $1 \leqslant \alpha \leqslant n-2$;
				$\overline{H}^{n-1}_{n-1} = -\Delta_{n-4} / \prod_1^{n-4} \delta_\beta$; $\overline{H}^{n-1}_n = 0$
E_6	1, 4, 5, 7, 8, 11	16, 22, 30, 42, 30, 16	1	$H^l_1 = 1$, $H^l_2 = \frac{\delta_2}{2\delta_2 + y_l} \frac{y_l^2 + 2(\delta_1+\delta_3) y_l + 3\delta_1\delta_3}{\delta_1\delta_3}$; $H^l_3 = \frac{2\delta_1 + y_l}{\delta_1}$, $H^l_4 = \frac{y_l^2 + 2(\delta_1+\delta_3) y_l + 3\delta_1\delta_3}{\delta_1\delta_3}$; $H^l_5 = \Delta_4/\delta_5$; $H^l_6 = \Delta_5/\delta_6$
E_7	1, 5, 7, 9, 11, 13, 17	34, 49, 66, 96, 75, 52, 27	1	are determined by the corresponding formulas for E_6: $H^l_5 = \frac{\Delta_4}{\delta_4}$; $H^l_6 = \frac{\Delta_5}{\delta_4\delta_5}$; $H^l_7 = \frac{\Delta_6}{\delta_4\delta_5\delta_6}$
E_8	1, 7, 11, 13, 17, 19, 23, 29	2·(46, 68, 91, 135, 110, 84, 57, 29)	1	are determined by the corresponding formulas for E_6: $H^l_5 = \frac{\Delta_4}{\delta_4}$; $H^l_6 = \frac{\Delta_5}{\delta_4\delta_5}$; $H^l_7 = \frac{\Delta_6}{\delta_4\delta_5\delta_6}$; $H^l_8 = \frac{\Delta_7}{\delta_4\delta_5\delta_6\delta_7}$
F_4	1, 5, 7, 11	2·(11, 21, 15, 8)	1/2, 1/2, 1, 1	$H^l_\alpha = \Delta_{\alpha-1} / \prod_1^{\alpha-1} \delta_\beta$
G_2	1, 5	2·(3,5)	3, 1	$H^l_1 = 1$; $H^l_2 = \frac{2\delta_1 + y_l}{3\delta_1}$

Note. Δ_α are the principal minors of order α of the matrix $R \equiv P + y_l I$, where $y_l \equiv -l(l+1)$, $(I)_{\alpha\beta} \equiv \delta_{\alpha\beta}$.

multiplet with zero basis index by $H^{(l)}=\sum_\alpha H^{(l)}_\alpha h_\alpha$. The Casimir operator of $su(2)$ must take on this element the value $l(l+1)$, i.e.,

$$[Z_+[Z_-, H^{(l)}]_-]_- = l(l+1)\,H^{(l)} \quad \text{or} \quad 2\sum_\alpha k_{\beta\alpha}H^l_\beta H^l_\alpha = l(l+1)\,H^l_\beta. \qquad (3)$$

Thus, the multiplet structure of the embedding is completely determined by the eigenvalues $l(l+1)$ of the matrix $P, P_{\alpha\beta}\equiv\delta_\alpha k_{\alpha\beta}, \delta_\alpha\equiv 2H^1_\alpha$, in Eq. (3), and the values of l are equal to the indices of the corresponding algebra.[61] In general, the matrix P is nonsymmetric, and its eigenvectors can be labeled by means of a matrix W such that $WP=P^TW$. For all simple algebras, the matrix W is diagonal. For algebras $\mathfrak{g}$ with a symmetric Cartan matrix $(A_n, D_n, E_{6,7,8})$, the diagonal elements $\varkappa_\alpha$ of the matrix W are equal to $c\delta_\alpha^{-1}$, where c is an arbitrary constant. Note that among solutions of the system (3) multiple eigenvalues are encountered only for even n for the algebras D_n. The eigenvectors H^l_α of the matrix P can be conveniently normalized by the relation $\sum_\alpha H^l_\alpha\varkappa_\alpha H^{l'}_\alpha=[l(l+1)]^{-2}\delta_{ll'}$. The vectors of the orthonormal basis are determined by the expression $\tilde H^l_\alpha=\varkappa_\alpha^{1/2}l(l+1)H^l_\alpha$, the relation $k_{\alpha\beta}=\delta_\alpha^{-1}(\varkappa_\alpha/\varkappa_\beta)^{-1/2}\sum_l l(l+1)\tilde H^l_\alpha\tilde H^l_\beta$ holding. The remaining generators of the l-th multiplet can be obtained from $H^{(l)}$ by applying the raising and lowering operators:

$$F^l_{\pm m}=\left[\frac{(l-m)!}{(l+m)!}\right]^{1/2}[Z_\pm[Z_\pm\ldots[Z_\pm, H^{(l)}]\ldots]], \quad F^l_0\equiv H^{(l)}. \qquad (4)$$

In the basis (4) the commutation relations have the form

$$[F^{l_1}_m, F^{l_2}_n]=\sum_L a(l_1, l_2, L)\,c(l_1, l_2, L;\, m, n)\,F^L_{m+n}, \qquad (5)$$

where $c(l_1, l_2, L; m, n)$ are Clebsch-Gordan coefficients of the group $SU(2)$; $a(l_1, l_2, L)$ is the set of structure constants, which do not depend on the indices m of the basis vectors; the summation in (5) is over the complete multiplet spectrum of $\mathfrak{g}$.

To find explicit expressions for $a(l_1, l_2, L)$, we take $m=1$ and $n=0$ in (5):

$$[F^{l_1}_1, F^{l_2}_0]_- = \frac{1}{\sqrt{l_1(l_1+1)}}[[Z_+, H^{(l_1)}]\,H^{(l_2)}]$$

$$=\sum_L a(l_1, l_2, L)\,c(l_1, l_2, L;\, 1, 0)\frac{[Z_+, H^{(L)}]}{\sqrt{L(L+1)}},$$

whence

$$\sqrt{l_1(l_1+1)\,l_2(l_2+1)}\;H^{l_1}_m H^{l_2}_m\delta_m^{-1}$$

$$=\sum_L\sqrt{L(L+1)}\,a(l_1, l_2, L)\,c(l_1, l_2, L;\, 1, 0)\,H^L_m.$$

Using the normalization relation for H^l_m, we find

$$a(l_1, l_2, L)$$

$$=\sqrt{L(L+1)/l_1(l_1+1)}\,c^{-1}(l_1, l_2, L;\, 1, 0)\sum_m\varkappa_m^{-1/2}\delta_m^{-1}\tilde H^{l_1}_m\tilde H^{l_2}_m\tilde H^L_m. \qquad (6)$$

Thus, formulas (5) with the structure functions (6) solve the problem of explicit realization of the simple algebra $\mathfrak{g}$ in the basis (4). In Table II, we give H^l_m, δ_m, $\varkappa_m H^1_m$ and the spectrum of l values (which are determined directly by the Cartan matrix of the corresponding algebra), which we shall need for actual calculations in the framework of the minimal embedding, for all types of simple Lie algebras (A_n, B_n, C_n, D_n, $E_{6,7,8}$, F_4, G_2).

2. EQUATIONS FOR CYLINDRICALLY SYMMETRIC GAUGE FIELDS

Parametrization of Gauge Fields and their symmetry properties. In accordance with the definition of cylindrically symmetric configurations given in the Introduction, the Yang-Mills fields are parametrized in this case by means of the generators of the group S, which commute with the elements of the diagonal group $\mathscr{L}$ isomorphic to $SU(2)$, i.e., they are invariant with respect to the total angular momentum $\mathbf{L}=\mathbf{T}+\mathbf{M}$ of the system. Here, $\mathbf{M}$ are the generators of the three-dimensional group of spatial rotations, and $\mathbf{T}$ are the generators of $SU(2)$ when this group is embedded in a definite manner in the gauge group G. [In particular, for the minimal embedding this $SU(2)$ subgroup is generated by the operators $\{Z_\pm, H^{(1)}\}$ introduced in Sec. 1.] The problem therefore arises of the explicit realization of the group algebra of S. For an arbitrary method of embedding, the decomposition of the generators of $\mathfrak{g}$ with respect to the multiplets F^l_m of the irreducible representations of $SU(2)$ is, in general, such that the multiplicity of the l-th multiplet may be greater than unity, and the spectrum of l values contains half-integral values. [For example, for the group $SU(4)$ there are embeddings with the following spectra of l values: $\{1, 2, 3\}$, $\{0, 1, 1, 1, 2\}$, $\{0, 0, 0, 1, 1, 1, 1\}$, $\{0, 0, 0, 0, 1/2, 1/2, 1/2, 1/2, 1\}$, these corresponding to the following groups S: $U(1)\otimes U(1)\otimes U(1)$, $SU(2)\otimes U(1)\otimes U(1)$, $U(1)\otimes SU(2)\otimes SU(2)$, $U(1)\otimes U(1)\otimes SU(2)$.] In the framework of our method of construction, the generators of S are obviously $\mathscr{L}$-invariant operators $W^{ls}=\sum_m F^{ls}_m\bar Y^l_m(\mathbf{n})$, where $Y^l_m(\mathbf{n})$ are spherical functions of the argument $\mathbf{n}\equiv\mathbf{x}/r$ (in what follows, we shall omit the index s, which takes into account the possibility of the presence of multiple values of the moment l). Moreover, it is only for the minimal embedding that the operators W^l form an Abelian group

$$S(\equiv\underbrace{U(1)\otimes\cdots\otimes U(1)}_{r})$$

of rank r of the original gauge group. (In the matrix basis, this method of constructing the invariant operators corresponds to combinations of $\hat F$: $\hat F_i n_i$, $\hat F_{ij}n_in_j$, $\hat F_{ij}=\hat F_{ji}$, $\mathrm{Sp}\hat F_{ii}=0,\ldots$ for $l=0, 1, 2, \ldots$, respectively.)

From the components of each irreducible multiplet W^l and the unit vector $\mathbf{n}$, which transforms in accordance with the vector representation ($l=1$) of $SU(2)$, we can in accordance with the rule for adding angular momenta construct three types of operator vector structures, namely $\mathbf{W}^l_{1,2,3}=\{\mathbf{n}W^l, \mathbf{M}W^l, \mathbf{n}\times\mathbf{M}W^l\}$, these parametrizing the vector part of the potential. [For $SU(2)$, the part of $Y^l_m(\mathbf{n})$, W^l, $\mathbf{n}W^l$, $\mathbf{M}W^l$ and $\mathbf{n}\times\mathbf{M}W^l$ is played by $\mathbf{n}$, $(\mathbf{n}\cdot\sigma)$, $\mathbf{n}(\mathbf{n}\cdot\sigma)$, $\mathbf{n}\times\sigma$ and $\mathbf{n}\times\mathbf{n}\times\sigma$, respectively, where σ are the Pauli matrices.] By means of the operator structure introduced above, the components of the scalar and vector parts of the Yang-Mills potential (or field) can be parametrized as follows:

$$A_0=\sum_{\{l\}}\varphi^l_0W^l;\quad \mathbf{A}=\sum_{\{l\}}\sum_{i=1}^3\varphi^l_i\mathbf{W}^l_i,\quad \varphi^l\equiv\varphi^l(r, t), \qquad (7)$$

where the summation is over the complete spectrum of eigenvalues of the angular momentum operator for the corresponding embedding of $SU(2)$ in G. Note that the expressions (7) are a direct generalization of Witten's $SU(2)$ ansatz[44] to the case of an arbitrary compact gauge group G.

We now turn to the symmetry properties of the ansatz (7), whose symmetry group is the direct product of the three-dimensional subgroup $O(2,1)$ of the conformal group and the subgroup S of the gauge group with generators W^l (Ref. 45). There are also the discrete transformations of space-time reflection (changes in the orientation of the space R_4), which, in particular, permit the transition from instantons to anti-instantons for the dual subclass of fields. With respect to transformations in the group $O(2,1)$, the structures $\varphi^l(r,t)$ transform as scalar functions. Because the group S is, in general, a direct product of the subgroups S_M, $S=\Pi_{M=1}^{k}\otimes S_M$, $k\leq r$ [one of which, generated by the operators $T_i n_i$, is $U(1)$], the set of operators W^l is decomposed in accordance with the structure of S into subsets W_M^l, which form the algebras of S_M. The functions $\varphi(M)_0^l$ and $\varphi(M)_1^l$ play the part of the potentials of the fields of the gauge group S_M in two-dimensional space [for the subgroup $U(1)\subset S$, the corresponding structure functions are the electromagnetic potentials], and φ_2^l and φ_3^l transform under the action of the operators W_M^l of the algebra s_M in accordance with a reducible representation of it.

The treatment that follows will be for the special case of the minimal embedding, in which the structure of the gauge group of two-dimensional space is simplest, $S=\Pi_1^r\otimes U(1)$, which makes it possible to carry through all the constructions explicitly to the final expressions for the solutions to the classical Yang-Mills equations.

Under gauge transformations with the generators W^L, the contributions of the different l multiplets [with respect to $U(1)\subset S=\Pi_1^r\otimes U(1)$] to the expansions (7) are, as in the expressions for the field intensities, mixed up, and to separate "pure" (unmixed) gauge-invariant combinations we must diagonalize the r mutually commuting generators W^L. Under a gauge transformation generated by the operators W^L in S, the vector structures behave as follows:

$$[W^L, \mathbf{M}W^l]_- = -\mathrm{i}\sum_{l'} c_{ll'}^L \mathbf{n}\times\mathbf{M}W^{l'};\quad [W^L, \mathbf{n}\times\mathbf{M}W^l]_- = \mathrm{i}\sum_{l'} c_{ll'}^L \mathbf{M}W^{l'}. \tag{8}$$

Thus, we are faced with diagonalizing a matrix c with elements $c_{ll'}^L$, corresponding to the gauge transformation with the generators W^L. Setting $\mathbf{n}=(0,0,1)$ in (7) and (8) and noting that $W^l=H^{(l)}$ for this choice of $\mathbf{n}$, we obtain from (8)

$$[H^{(L)}, [Z_\pm, H^{(l)}]] = \pm\sum_{l'} c_{ll'}^L [Z_\pm, H^{(l')}]$$

or

$$l(l+1)L(L+1)H_\alpha^l H_\alpha^L = \sum_{l'} l'(l'+1)c_{ll'}^L \delta_\alpha H_\alpha^{l'}. \tag{9}$$

Going over in (9) to the orthonormalized basis $\tilde{H}_\alpha^l$, we find

$$c_{ll'}^L = \sum_\alpha \delta_\alpha^{-1}\varkappa_\alpha^{-\frac{1}{2}}\tilde{H}_\alpha^l\tilde{H}_\alpha^L\tilde{H}_\alpha^{l'};\quad \sum_{L,l,l'} c_{ll'}^L\tilde{H}_\alpha^L\tilde{H}_\beta^l\tilde{H}_\gamma^{l'} = \delta_{\alpha\beta}\delta_{\beta\gamma}\delta_\alpha^{-1}\varkappa_\alpha^{-1/2}, \tag{10}$$

from which it follows that the matrix $\tilde{H}_\alpha^l$ diagonalizes the generators of the gauge transformations. The diagonalized operators[1)]

$$T_\alpha = \sum_l \tilde{h}_\alpha^l W^l,\ \mathbf{T}_\alpha^i = \sum_l \tilde{h}_\alpha^l \mathbf{W}_i^l,$$
$$\tilde{h}_\alpha^l \equiv \tilde{H}_\alpha^l(\delta_\alpha v_\alpha)^{1/2},\ v_\alpha \equiv \varkappa_\alpha\delta_\alpha$$

satisfy the commutation relations

$$\left.\begin{aligned}
&[T_\alpha, T_\beta]_- = 0;\quad [T_\alpha, \mathbf{M}T_\beta]_- = -\mathrm{i}\delta_{\alpha\beta}\mathbf{n}\times\mathbf{M}T_\alpha;\\
&[T_\alpha, \mathbf{n}\times\mathbf{M}T_\beta]_- = \mathrm{i}\delta_{\alpha\beta}\mathbf{M}T_\beta;\\
&\varepsilon_{kij}[M_iT_\alpha, M_jT_\beta]_- = \varepsilon_{kij}[(\mathbf{n}\times\mathbf{M})_i T_\alpha, (\mathbf{n}\times\mathbf{M})_j T_\beta]_-\\
&= \mathrm{i}n_k\delta_{\alpha\beta}T_\gamma\delta_\alpha k_{\gamma\alpha}.
\end{aligned}\right\} \tag{11}$$

With these operators and the new structure functions $f^\alpha=\sum_l \tilde{\tilde{h}}_\alpha^l\varphi^l$, $\tilde{\tilde{h}}_\alpha^l=\tilde{\tilde{H}}_\alpha^l(\delta_\alpha v_\alpha)^{-1/2}$, the expansion (7) can rewritten in the form

$$A_0 = \sum_\alpha T_\alpha f_0^\alpha;\quad \mathbf{A} = \sum_\alpha\sum_{i=1}^{3} T_\alpha^i f_i^\alpha. \tag{12}$$

(The use of $\tilde{h}$ and $\tilde{\tilde{h}}$ instead of the orthogonal matrix $\tilde{H}$ simplifies the final expressions.) Under gauge transformations with the generators T_α of the subgroup $U(1)\subset S$ and the phases $\phi^\alpha(r,t)$, which are arbitrary functions of r and t, the structure functions f^α behave as follows:

$$\left.\begin{aligned}
&-\mathrm{i}\hat{\delta}_\alpha f_0^\beta = \delta_{\alpha\beta}\phi^\alpha_{,t};\quad \mathrm{i}\hat{\delta}_\alpha f_1^\beta = \delta_{\alpha\beta}\phi^\alpha_{,r};\\
&-\mathrm{i}\hat{\delta}_\alpha f_2^\beta = \delta_{\alpha\beta}\phi^\alpha f_3^\alpha;\quad -\mathrm{i}\hat{\delta}_\alpha f_3^\beta = -\delta_{\alpha\beta}\phi^\alpha(f_2^\alpha + r^{-1}).
\end{aligned}\right\} \tag{13}$$

Here, the symbol $\phi^\alpha_{,x}$ denotes the derivative of the function φ^α with respect to the argument x. It follows from the relations (13) that the α component of the field tensor of the two-dimensional space, $f^\alpha_{1,t}-f^\alpha_{0,r}$, and the quantities $J_\alpha=|W_\alpha|^2$, $W_\alpha\equiv(rf_2^\alpha+1)+irf_3^\alpha$, are invariant under gauge transformations in S. It is for these $2r$ quantities that one can obtain a system of $2r$ equations describing cylindrically symmetric solutions to the gauge theory in the framework of the minimal embedding of $SU(2)$ in G.

General equations of motion; action and topological charge (noninvariant and invariant formulations); Pointlike Solutions. Proceeding from the parametrization (12) and using the commutation relations (11), we find expressions for the Yang-Mills field intensities $E_i\equiv F_{i0}$ and $B_k\equiv(1/2)\varepsilon_{ijk}F_{ij}$:

$$\left.\begin{aligned}
&-\mathbf{E} = \mathbf{n}T_\alpha(f^\alpha_{1,t}-f^\alpha_{0,r}) + \mathbf{M}T_\alpha(f^\alpha_{2,t}-f_0^\alpha f_3^\alpha)\\
&+\mathbf{n}\times\mathbf{M}T_\alpha(f^\alpha_{3,t}+r^{-1}f_0^\alpha+f_0^\alpha f_2^\alpha);\\
&\mathbf{B} = (1/2r^2)\,\mathbf{n}\delta_\alpha k_{\beta\alpha}G_\alpha T_\beta + \mathbf{M}T_\alpha(f^\alpha_{3,r}+r^{-1}f_1^\alpha+r^{-1}f_3^\alpha+f_1^\alpha f_2^\alpha)\\
&+\mathbf{n}\times\mathbf{M}T_\alpha(-f^\alpha_{2,r}-r^{-1}f_2^\alpha+f_1^\alpha f_3^\alpha),\quad G_\alpha\equiv J_\alpha-1.
\end{aligned}\right\} \tag{14}$$

To calculate the densities of the Euclidean action and the topological charge, namely $r^2\,\mathrm{Sp}(\mathbf{E}^2+\mathbf{B}^2)$ and $r^2\,\mathrm{Sp}\mathbf{E}\cdot\mathbf{B}$, we need the traces of the mutual products of the operators T_α, $\mathbf{M}T_\beta$, and $\mathbf{n}\times\mathbf{M}T_\gamma$, whose nonzero values are given by

$$\mathrm{Sp}\,T_\alpha\cdot T_\beta = k_{\beta\alpha}^{-1}v_\beta;\quad \mathrm{Sp}\,(\mathbf{M}T_\alpha)^2 = \mathrm{Sp}\,(\mathbf{n}\times\mathbf{M}T_\alpha)^2 = \delta_\alpha v_\alpha \equiv \mathscr{B}_\alpha. \tag{15}$$

Using this, we obtain

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1)Note that the diagonal matrix V with elements v_α satisfies the relation $Vk=k^TV$.

$$r^2\,\mathrm{Sp}\,(\mathbf{E}^2+\mathbf{B}^2)=\mathscr{B}_\alpha\{(1/4r^2)(S_\alpha\delta_\alpha k_{\beta\alpha}S_\beta+G_\alpha\delta_\alpha k_{\beta\alpha}G_\beta)+|D_t^\alpha W_\alpha|^2+|D_r^\alpha W_\alpha|^2\},\tag{16}$$

$$r^2\,\mathrm{Sp}\,\mathbf{E}\cdot\mathbf{B}=(-1/2)\,\mathscr{B}_\alpha\{(1/2r^2)\,G_\alpha\delta_\alpha k_{\beta\alpha}S_\beta+\mathrm{i}\,[(D_t^\alpha W_\alpha)(D_r^\alpha W_\alpha)^*-(D_t^\alpha W_\alpha)^*(D_r^\alpha W_\alpha)]\},\tag{17}$$

where $S_\alpha\equiv 2r^2k_{\beta\alpha}^{-1}\delta_\beta^{-1}(f_{1,t}^\beta-f_{0,r}^\beta)$; $D_t^\alpha=\partial/\partial t+if_0^\alpha$, $D_r^\alpha=\partial/\partial r+if_1^\alpha$ are covariant derivatives in the two-dimensional space.

The density of the topological charge can be rewritten in a somewhat different form that is more convenient for actual calculations by introducing the notation

$$V_0^\alpha\equiv\mathrm{Re}\,W_\alpha=rf_2^\alpha+1;\quad V_1^\alpha\equiv\mathrm{Im}\,W_\alpha=rf_3^\alpha;\quad \hat{D}_\mu^\alpha V_\lambda^\alpha=V_{\lambda,\mu}^\alpha-\varepsilon_{\lambda\tau}f_\mu^\alpha V_\tau^\alpha;$$
$$f_{\mu\nu}^\alpha\equiv f_{\nu,\mu}^\alpha-f_{\mu,\nu}^\alpha;\quad \varepsilon_{\mu\nu}=(-1)^\mu\,\delta_{\mu+\nu,1},\quad \partial_0\equiv\partial/\partial t,\ \partial_1\equiv\partial/\partial r;\tag{18}$$

the indices μ,ν,λ,τ take the values 0 and 1. Then

$$r^2\,\mathrm{Sp}\,\mathbf{E}\cdot\mathbf{B}=(-1/2)\,\mathscr{B}_\alpha\{\partial_\mu[\varepsilon_{\mu\nu}\varepsilon_{\lambda\tau}V_\lambda^\alpha\hat{D}_\nu^\alpha V_\tau^\alpha]-(1/2)\,\varepsilon_{\mu\nu}f_{\mu\nu}^\alpha\}.\tag{19}$$

Varying the action density (16) with respect to the structure functions f^α, we arrive at the following system of $4r$ equations of motion:

$$2f_0^\alpha=-J_\alpha^{-1}S_{\alpha,r}+\mathrm{i}\Psi_{\alpha,t};\quad 2f_1^\alpha=J_\alpha^{-1}S_{\alpha,t}+\mathrm{i}\Psi_{\alpha,r};\tag{20}$$

$$[(D_t^\alpha)^2+(D_r^\alpha)^2]\,W_\alpha=W_\alpha\,[(1/2)\,\delta_\beta k_\beta k_{\alpha\beta}J_\beta-1],\tag{21}$$

where $\Psi_\alpha=\ln W_\alpha/W_\alpha^*$. This system exactly coincides with the equations obtained by direct substitution of (12) in the original Yang-Mills equations $F_{\mu\nu,\mu}+[A_\mu,F_{\mu\nu}]=0$ with the traces subsequently taken by means of formulas (15). Expressing the functions f^α in (20) and (21) in terms of the invariants J_α and S_α, we obtain the system of $2r$ equations

$$\Delta J_\alpha=\frac{1}{2}J_\alpha^{-1}[(\nabla J_\alpha)^2+(\nabla S_\alpha)^2]+2r^{-2}J_\alpha\Big(\frac{1}{2}\sum_\beta\delta_\beta k_{\alpha\beta}J_\beta-1\Big);$$
$$\Delta S_\alpha=J_\alpha^{-1}(\nabla J_\alpha)(\nabla S_\alpha)+r^{-2}J_\alpha\delta_\alpha\sum_\beta k_{\beta\alpha}S_\beta,\tag{22}$$

where $\Delta\equiv\partial^2/\partial r^2+\partial^2/\partial t^2$, $\nabla\equiv(\partial/\partial t,\ \partial/\partial r)$.

Note that, as usual, the transition to Minkowski space is made by replacing the real (in the case of R_4) time components by imaginary components, i.e., $t\to it$, $S_\alpha\to iS_\alpha$.

The system (22) can be rewritten in a more symmetric form in terms of the invariants $R_\alpha^{(\mu)}=(1/2)[G_\alpha+(-1)^{\mu-1}S_\alpha]$, $\mu=1,2$:

$$\Delta R_\alpha^{(\mu)}=(R_\alpha^{(1)}+R_\alpha^{(2)}+1)^{-1}(\nabla R_\alpha^{(\mu)})^2+r^{-2}\sum_\beta\delta_\beta k_{\alpha\beta}R_\beta^{(\mu)}(R_\alpha^{(1)}+R_\alpha^{(2)}+1).\tag{23}$$

The expressions for the densities of the action and the topological charge in terms of the structures $R_\alpha^{(\mu)}$, which satisfy the system (23), are

$$\frac{r^2}{2}\,\mathrm{Sp}\,F_{\mu\nu}F_{\mu\nu}=\sum_{\alpha,\mu}\mathscr{B}_\alpha\Big[(R_\alpha^{(1)}+R_\alpha^{(2)}+1)^{-1}(\nabla R_\alpha^{(\mu)})^2+r^{-2}\delta_\alpha\sum_\beta k_{\beta\alpha}R_\alpha^{(\mu)}R_\beta^{(\mu)}\Big];\tag{24}$$

$$\frac{r^2}{2}\,\mathrm{Sp}^*F_{\mu\nu}F_{\mu\nu}=\sum_{\alpha,\mu}\mathscr{B}_\alpha(-1)^\mu\Big[-\Delta R_\alpha^{(\mu)}+r^{-2}\sum_\beta\delta_\beta k_{\alpha\beta}R_\beta^{(\mu)}\Big].\tag{25}$$

Equations (23) generalize in a natural manner to an arbitrary compact gauge group the equations obtained in Ref. 46 for the case of the group $SU(2)$, and the Lagrangian (24) generalizes the corresponding expression of Ref. 44. Equations (22) and (23) cannot be obtained by varying (24).

We emphasize once more that all the basic entities of the theory (23)–(25) have been formulated in a completely gauge-invariant form, and all noninvariant quantities have been eliminated.

It can be seen from (23)–(25) that the entire information about the properties of the gauge group G is concentrated in the matrix $(1/2)\delta_\beta k_{\alpha\beta}\equiv k_{\alpha\beta}\sum_\gamma k_{\beta\gamma}^{-1}$. As an illustration, we give explicit expressions for the matrices $(1/2)\delta_\beta k_{\alpha\beta}$ in the case of the groups of second rank:

$$\begin{array}{cccc} SU(3) & O(5) & \mathrm{Sp}(4) & G_2\\ \begin{vmatrix}2&-1\\-1&2\end{vmatrix} & \begin{vmatrix}4&-3\\-2&3\end{vmatrix} & \begin{vmatrix}3&-2\\-3&4\end{vmatrix} & \begin{vmatrix}6&-5\\-9&10\end{vmatrix}. \end{array}$$

On the basis of Eqs. (22) [or (23)] we can immediately draw some general conclusions; in particular, we can obtain the spectrum of pointlike solutions, which are described in accordance with (22) by an algebraic system of the form $J_\alpha\delta_\alpha k_{\beta\alpha}S_\beta=0$; $J_\alpha[(1/2)\delta_\beta k_{\alpha\beta}J_\beta-1]=0$. It follows from this that either $J_\alpha=0$ and S_α are arbitrary for all $1\leqslant\alpha\leqslant r$, or $S_\alpha=0$ and $J_\alpha=1$ for all $1\leqslant\alpha\leqslant r$.

Equations (23) admit a natural classification of subclasses of instanton, anti-instanton, and meron type depending on the choice of the invariant structures $R_\alpha^{(\mu)}$ ($R_\alpha^{(2)}=0$, $R_\alpha^{(1)}=0$ and $R_\alpha^{(1)}=R_\alpha^{(2)}$, respectively). At the same time, (23)–(25) are obviously invariant under the substitution $R_\alpha^{(1)}\rightleftarrows R_\alpha^{(2)}$ up to the sign in (25).

Realization of the duality condition: Equations and contour representation of the topological charge. Proceeding from the expressions (14) for the intensities **E** and **B** of the gauge field, we can obtain equations of self-duality (anti-self-duality) by setting $\mathbf{E}=\mathbf{B}$ (respectively, $\mathbf{E}=-\mathbf{B}$). To make the derivation as clear as possible, we shall use the notation (18), in which the self-duality equations have the form

$$f_{0,1}^\alpha=r^{-2}[1-(1/2)\,\delta_\beta k_{\alpha\beta}V_\mu^\beta V_\mu^\beta];\tag{26}$$

$$V_{0,0}^\alpha-f_0^\alpha V_1^\alpha=-V_{1,1}^\alpha-f_1^\alpha V_0^\alpha;\quad V_{1,0}^\alpha+f_0^\alpha V_0^\alpha=V_{0,1}^\alpha-f_1^\alpha V_1^\alpha.\tag{27}$$

These equations are a direct generalization of Witten's system[44] to the case of an arbitrary simple compact group. In the Lorentz gauge, $\partial_\mu f_\mu^\alpha=0$, we have $f_\mu^\alpha=\varepsilon_{\mu\nu}\psi_{,\nu}^\alpha$. For the new functions $\chi_\mu^\alpha=\exp(-\psi^\alpha)V_\mu^\alpha$, Eqs. (27) can be written in the form

$$\chi_{0,0}^\alpha=-\chi_{1,1}^\alpha;\quad \chi_{1,0}^\alpha=\chi_{0,1}^\alpha,$$

which are Cauchy-Riemann conditions, from which it follows that $l^\alpha=\chi_1^\alpha-\mathrm{i}\chi_0^\alpha=-\mathrm{i}\exp(-\psi^\alpha)W_\alpha$ are analytic functions of $z=(r+it)/2$. Equations (26) for the tensor field f_{01}^α in the two-dimensional space in the new notation

$$\Delta\psi^\alpha=-r^{-2}\Big(1-\frac{1}{2}\sum_\beta\delta_\beta k_{\alpha\beta}\exp(2\psi^\beta)\,|l^\beta|^2\Big)$$

can be conveniently expressed in terms of functions ρ_α defined by $\psi^\alpha=\ln r-(1/2)\ln|l^\alpha|^2+\rho_\alpha/2$, namely,

$$\Delta\rho_\alpha=\sum_{\beta=1}^{r}\delta_\beta k_{\alpha\beta}\exp\rho_\beta,\quad 1\leqslant\alpha\leqslant r.\tag{28}$$

Note that with $\rho_\alpha\equiv\rho$ for all $1\leqslant\alpha\leqslant r$, the system (28) goes over into the well-known Liouville equation, which realizes the duality condition for the group $SU(2)$ (Ref. 44); this follows in a natural manner from (28) for $r \equiv 1$ ($k_{\alpha\beta} \equiv k_{11} \equiv 2$).

The equations of self-duality (anti-self-duality) (28), written down for the gauge-invariant ρ_α, follow directly from the general system (22) [or (23)] for the special choice $S_\alpha = \pm (J_\alpha - 1)$ or $R_\alpha^{(1/2)} = 0$. These algebraic relations are actually the first of the self-duality equations (26), and they lead automatically to a system of r equations for the functions $\rho_\alpha \equiv \ln(J_\alpha/r^2)$ in the form (28). The form of the system (28) indicates that there is an abrupt raising of the symmetry of the original general system (23) at the duality "point." The system satisfies the conditions of conformal covariance [in contrast to (23)], which makes it possible to obtain a number of nontrivial solutions of the system on the basis of certain special $\rho_\alpha^0(z)$ (in particular, pointlike), $\rho_\alpha(z) = \rho_\alpha^0(g(z)) + \ln|dg/dz|^2$, where $g(z)$ is an arbitrary analytic function, and one can completely integrate the system by expressing the general solutions in terms of $2r$ arbitrary functions (see Sec. 3).

The existence of the sudden jump of the symmetry poses the interesting problem of constructing an extended symmetry group of the cylindrically symmetric classical equations of gauge theories at the duality "point" and the possible formulation of the duality condition itself as a critical point of abrupt increase in symmetry.

Note that for the self-dual (anti-self-dual) case the action density (24) can be rewritten in the form

$$(r^2/2)\,\mathrm{Sp}\, F_{\mu\nu}F_{\mu\nu} = \sum_\alpha \mathscr{B}_\alpha \Delta(\exp\varphi_\alpha - \varphi_\alpha), \quad \varphi_\alpha \equiv \rho_\alpha + 2\ln r$$

[cf. the corresponding expression of Ref. 45 for the group $SU(2)$].

For actual calculations of the values of the topological charge of self-dual configurations (instantons) it is convenient to use the expression (19). Because the functions V_r^α corresponding to the instanton solutions constructed in Sec. 4 satisfy at infinity the relations $\hat{D}_\mu^\alpha V_r^\alpha|_\infty = 0$, the first term in the integrand of the charge $Q = 1/\pi \int d^2x r^2\,\mathrm{Sp}\mathbf{E}\cdot\mathbf{B}$ with density in the form (19) is absent. This makes it possible to rewrite the formula for the charge by means of Stokes's theorem in the form of a linear integral over an infinite contour:

$$Q = \frac{1}{\pi}\sum_\alpha \mathscr{B}_\alpha \oint dx_\mu f_\mu^\alpha.$$

Further, by virtue of the relation $f_\mu^\alpha = -\mathrm{i}(\ln W_\alpha)_{,\mu}$, which follows from Eqs. (27) when the equation $\hat{D}_\mu^\alpha W_\alpha|_\infty = 0$ is used, the last expression takes the form

$$Q = \frac{1}{\pi \mathrm{i}}\sum_\alpha \mathscr{B}_\alpha \oint dx_\mu \partial_\mu \ln W_\alpha. \tag{29}$$

Thus, the topological charge Q can be represented as a sum of the contributions of the topological charges of r two-dimensional field configurations of Abelian

$$\underbrace{U(1)\otimes\cdots\otimes U(1)}_{r}$$

theories, each of which is determined by the change in the phase of the complex scalar fields W_α around the contour at infinity.

Equations for multimeron configurations. By virtue of its symmetry, the system (23) also admits another obvious subclass corresponding to the choice $R_\alpha^{(1)} = R_\alpha^{(2)} \equiv R_\alpha$, which contains, in particular, meron configurations. In this case, the system (23) for the functions $f_\alpha \equiv (2R_\alpha + 1)^{1/2}$ can be written in the form

$$\Delta f_\alpha = \frac{1}{2r^2}\sum_\beta \delta_\beta k_{\alpha\beta} f_\alpha (f_\beta^2 - 1); \tag{30}$$

for the group $SU(2)$, it is identical with the basic equation that describes multimeron configurations.[47] The corresponding solutions lead to infinite values of the action [see (24)] and give the correct value for the topological charge if we remember that $R_\alpha^{(\mu)}$ are proportional to r^2 in accordance with their definition and that the metric in the two-dimensional space also contains r^2. It should, however, be noted that the properties of the generalized functions used here require the well-known stipulations concerning the rules for operation with them.

Inclusion of a Higgs field; equations for spherically symmetric monopoles; energy functional. In the scheme we have developed, one can readily include a Higgs field $\varphi = \sum_\alpha \varphi^\alpha T_\alpha$ transforming in accordance with the adjoint representation of the gauge group G. Leaving aside a discussion of questions relating to the presence of additional (compared with the Yang-Mills fields) gauge invariance of the Higgs fields and the potential term in the total Lagrangian,[5] we consider only the kinetic part

$$L_{\mathrm{H}} = (\check{D}_\mu \varphi^\alpha)^2 \equiv \mathrm{Sp}\,(\check{D}_\mu\varphi\check{D}_\mu\varphi); \quad \check{D}_\mu\varphi = \partial_\mu\varphi + \mathrm{i}^{-1}[A_\mu, \varphi], \tag{31}$$

written down already in the diagonal representation. Using the commutation relations (11), we obtain

$$r^2 L_{\mathrm{H}} = r^2 k_{\beta\alpha}^{-1} v_\beta (\nabla\varphi^\alpha)(\nabla\varphi^\beta) + \mathscr{B}_\alpha (\varphi^\alpha)^2 J_\alpha, \tag{32}$$

from which it follows that in the presence of the Higgs field the only modification is in the subsystem (21) with the addition on the right-hand side of the term $-W_\alpha(\varphi^\alpha)^2$, and a new subsystem of equations arises for the structures $\hat{\varphi}^\alpha \equiv \mathscr{B}_\alpha \varphi^\alpha$:

$$\Delta\hat{\varphi}^\alpha = r^{-2}\delta_\alpha k_{\beta\alpha}\hat{\varphi}^\beta J_\beta + (\text{potential term}). \tag{33}$$

Thus, there arises a modified system of $3r$ equations for the $3r$ quantities $\hat{\varphi}_\alpha$ and $R_\alpha^{(\mu)}$.

In the framework of the developed approach, the description of spherically symmetric monopole configurations for an arbitrary compact gauge group with Higgs field φ in the adjoint representation is completely analogous to the description given above. The Hamiltonian density $\mathscr{H}$ in Minkowski space for purely magnetic, time-independent solutions is given by the usual (up to normalization of the trace of the generators of the algebra $\mathfrak{g}$ of the group G) expression

$$\begin{aligned}\mathscr{H} = -L &= (1/4)\,\mathrm{Sp}\, F_{\mu\nu}F_{\mu\nu} + (1/2)\,\mathrm{Sp}\,(\check{D}_\mu\varphi)^2 + \eta V(\varphi)\\ &= (1/2)\,\mathrm{Sp}\,(\mathbf{B} \mp \check{\mathbf{D}}\varphi)^2 \pm \mathrm{Sp}\,(\mathbf{B}\cdot\check{\mathbf{D}}\varphi) + \eta V(\varphi).\end{aligned} \tag{34}$$

In the Bogomol'nyi-Prasad-Sommerfield limit, any solution of the differential equations $\mathbf{B} = \check{\mathbf{D}}\varphi$ ($\mathbf{B} = -\check{\mathbf{D}}\varphi$) realizes a minimum of the energy

$$E = \int d^3x \mathscr{H} \geqslant \int d^3x\,\mathrm{Sp}\,(\mathbf{B}\cdot\check{\mathbf{D}}\varphi), \tag{35}$$

which is directly related to the topological charge and therefore does not actually depend on the detailed structure of the solutions. [Note that in this limit the parameters of the Higgs potential $V(\varphi)$ tend to zero for a fixed ratio of their values.] With allowance for the relation $\check{D}\mathbf{B}=0$, the integral (35) can, after integration by parts, be written in the form

$$\lim_{r\to\infty} r^2 \int d\Omega\,\mathrm{Sp}\,(B_r\varphi). \tag{36}$$

As usual, the matrix of the magnetic charge g is defined by the asymptotic behavior of the magnetic field $\mathbf{B}$:

$$\mathbf{B}\cdot\mathbf{n} \Rightarrow g/4\pi r^2. \tag{37}$$

Following the generally adopted notation[5] for the structure functions that parametrize the vector part of the potential $\mathbf{W}$ of the Yang-Mills field ($W_0=0$) and the Higgs field φ, we write the diagonal representation (12) in the static limit in the form

$$\varphi(r) = \sum_\alpha r^{-1} H_\alpha(r)\, T_\alpha; \quad \mathbf{W}(r) = \sum_\alpha r^{-1}[K_\alpha(r)-1]\,\mathbf{M}T_\alpha. \tag{38}$$

Then the energy functional in these structures is

$$E = -4\pi \int_0^\infty dr \sum_\alpha \mathscr{B}_\alpha r^{-2}\Big[r^2\dot{K}_\alpha^2 + K_\alpha^2 H_\alpha^2 + \frac{1}{4}\sum_\beta \delta_\beta k_{\alpha\beta}(K_\alpha^2-1)(K_\beta^2-1) + \delta_\alpha^{-1}\sum_\beta k_{\alpha\beta}^{-1}(r\dot{H}_\alpha - H_\alpha)(r\dot{H}_\beta - H_\beta)\Big], \tag{39}$$

and variation of it with respect to the functions K_α and H_α leads to the system of $2r$ equations

$$r^2\ddot{K}_\alpha = K_\alpha\Big(1/2\sum_\beta \delta_\beta k_{\alpha\beta}K_\beta^2 + H_\alpha^2 - 1\Big);$$

$$r^2\ddot{H}_\alpha = \sum_\beta \delta_\beta k_{\alpha\beta} K_\beta^2 H_\beta. \tag{40}$$

Here, the dot above the functions stands for differentiation with respect to r. The same system naturally arises by direct substitution of (38) in the equations $\mathbf{B}=\check{D}\varphi$. It should be noted that if we were to consider purely static self-dual Yang-Mills fields A_0, $\mathbf{A}$ in Euclidean space (without the Higgs field) and were to use for them the ansatz

$$A_0 = \sum_\alpha r^{-1}\tilde{H}_\alpha(r)\,T_\alpha;$$

$$\mathbf{A} = \sum_\alpha r^{-1}[K_\alpha(r)-1]\,\mathbf{M}T_\alpha,$$

which is analogous to (38), we should arrive at a system whose only difference from (40) is in the sign of the term H_α^2. This circumstance indicates a symmetry between the self-duality condition in R_4 and the Bogomol'nyĭ-Prasad-Sommerfield limit in Minkowski space. Namely, the pre-images $\mathbf{W}=\mathbf{A}$ and $\varphi=\pm A_0$ of static self-dual fields A_0, $\mathbf{A}$ in R_4 are, as was noted in Ref. 54 for the group $SU(2)$, monopole solutions in $R_{3,1}$.

The system (40) is a natural generalization of the well-known equations (describing monopole configurations for unitary gauge groups of low rank; see, for example, Ref. 5) for an arbitrary gauge group G. It is obvious that dyon solutions can be obtained from $\mathbf{W}$ and φ by the trivial substitution

$$\mathbf{W}'=\mathbf{W}; \quad W_0' = \mathrm{sh}\,\theta\varphi; \quad \varphi' = \mathrm{ch}\,\theta\varphi,$$

where θ is an arbitrary constant. We shall consider monopole configurations in more detail in Sec. 4, where we shall give an explicit construction of the corresponding solutions (40) and find a subclass of them satisfying the boundary conditions necessary to ensure finiteness of the energy (39).

3. EXACT SOLUTIONS FOR CYLINDRICALLY SYMMETRIC SELF-DUAL GAUGE FIELDS

Before we turn to the direct construction of solutions of the nonlinear system (28) describing cylindrically symmetric self-dual configurations, we consider the symmetry properties of these equations, which are due, in their turn, to the symmetry of the corresponding root systems. Unfortunately, we have not so far succeeded in finding a unified formulation for all simple Lie groups, and we are therefore forced to construct solutions for each type of group ($A_n; B_n; C_n; D_n; E_r, r=6,7,8; F_4, G_2$) separately in terms of their root spaces. For a number of groups, there is a symmetry of a definite kind that makes it possible to obtain solutions for a group of one type by equating the corresponding functions x_α that are solutions for a group of a different type. (In particular, $A_{2k} \rightleftharpoons B_k$, $A_{2k-1} \rightleftharpoons C_k$, $E_6 \rightleftharpoons F_4$, $B_3 \rightleftharpoons G_2$, where $\rightleftharpoons$ is the symbol of such "restriction.") To demonstrate the difference between the structure of the equations for the different simple Lie groups, we realize for them in explicit form a general reduction construction, this leading in the case of the series A_n to the complete solution of the system (28). For the remaining types of the considered groups, this procedure is more helpful heuristically in seeking and verifying solutions.

Symmetry properties of the duality equations and the reduction procedure. In Eqs. (28), it is convenient to go over to the formal complex variable $2z=r+it$, $2\bar{z}=r-it$ and introduce new functions x_α, which are related to ρ_β by $\rho_\alpha=\sum_{\beta=1}^r k_{\alpha\beta}x_\beta - \ln\delta_\alpha$. Then (28) can be rewritten in the form

$$\frac{\partial^2 x_\alpha}{\partial z\,\partial\bar{z}} \equiv x_{\alpha,\,z\bar{z}} = \exp\Big(\sum_{\beta=1}^r k_{\alpha\beta}x_\beta\Big). \tag{41}$$

We note immediately that the proposed method of solution of the system (41) is equally applicable to hyperbolic equations

$$(2z=r+t,\ 2\bar{z}=r-t,\ \partial^2/\partial z\,\partial\bar{z} = \partial^2/\partial r^2 - \partial^2/\partial t^2),$$

implementing thereby the possibility noted earlier of going over in Eqs. (23) from R_4 to Minkowski space.

We begin by considering the classical series A_n, B_n, C_n, D_n, for which the Cartan matrices have a unified structure with the Cartan matrix of A_n, $k_{\alpha\beta} = 2\delta_{\alpha\beta} - \delta_{\alpha\beta+1} - \delta_{\alpha+1\beta}$, $1\leq\alpha,\ \beta\leq n$, except for the elements of the right-hand lowest 3×3 block $\hat{k}$, which has, respectively, the form (see, for example, Ref. 61):

$$\hat{k}_{A_n} = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix}; \quad \hat{k}_{B_n} = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{vmatrix};$$

$$\hat{k}_{C_n} = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{vmatrix}; \quad \hat{k}_{D_n} = \begin{vmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{vmatrix}.$$

Substituting in (41) the Cartan matrices of these series in explicit form, we obtain the equations

$$\left.\begin{array}{l} x_{1,z\bar z}=\exp(2x_1-x_2);\\ x_{2,z\bar z}=\exp(-x_1+2x_2-x_3);\\ \cdots\cdots\cdots\cdots\\ x_{\alpha,z\bar z}=\exp(-x_{\alpha-1}+2x_\alpha-x_{\alpha+1});\\ \cdots\cdots\cdots\cdots\\ x_{n-3,z\bar z}=\exp(-x_{n-4}+2x_{n-3}-x_{n-2}); \end{array}\right\} \tag{42}$$

$$x_{n-2,z\bar z}=\begin{cases}\exp(-x_{n-3}+2x_{n-2}-x_{n-1}) & \text{for } A_n,\ B_n,\ C_n;\\ \exp(-x_{n-3}+2x_{n-2}-x_{n-1}-x_n) & \text{for } D_n;\end{cases} \tag{43}$$

$$x_{n-1,z\bar z}=\left\{\begin{array}{ll}\exp(-x_{n-2}+2x_{n-1}-x_n) & \text{for } A_n,\ C_n;\\ \exp(-x_{n-2}+2x_{n-1}-2x_n) & \text{for } B_n;\\ \exp(-x_{n-2}+2x_{n-1}) & \text{for } D_n;\end{array}\right\} \tag{44}$$

$$x_{n,z\bar z}=\left\{\begin{array}{ll}\exp(-x_{n-1}+2x_n) & \text{for } A_n,\ B_n;\\ \exp(-2x_{n-1}+2x_n) & \text{for } C_n;\\ \exp(-x_{n-2}+2x_n) & \text{for } D_n,\end{array}\right\} \tag{45}$$

which enable us to express the unknown functions $\exp(-x_\alpha)$, $2\leqslant\alpha\leqslant n$, in terms of the single unknown function $X\equiv\exp(-x_1)$. Indeed, the first equation in (42) gives

$$\exp(-x_2)=X_{,z}X_{,\bar z}-XX_{,z\bar z}=-\det\begin{pmatrix}X & X_{,z}\\ X_{,\bar z} & X_{,z\bar z}\end{pmatrix}\equiv-\Delta_2(X).$$

From the second equation, we obtain

$$\exp(-x_3)=-\det\begin{pmatrix}X & X_{,z} & X_{,zz}\\ X_{,\bar z} & X_{,\bar z z} & X_{,\bar z zz}\\ X_{,\bar z\bar z} & X_{,\bar z\bar z z} & X_{,\bar z\bar z zz}\end{pmatrix}\equiv-\Delta_3(X).$$

Continuing the reduction process to the $(n-3)$-th step, we obtain for all series

$$\exp(-x_\alpha)=(-1)^{\alpha(\alpha-1)/2}\Delta_\alpha(X),\quad 2\leqslant\alpha\leqslant n-2, \tag{46}$$

where $\Delta_\alpha(X)$ are the principal minors of order α of the matrix $(\partial^{i-1}/\partial z^{i-1})(\partial^{j-1}/\partial\bar z^{j-1})X$.[2)] The following stages for the classical series A_n, B_n, C_n, D_n are different by virtue of (43)–(45). From (43) and (44), we have

$$(-1)^{\frac{(n-1)(n-2)}{2}}\Delta_{n-1}(X)=\begin{cases}\exp(-x_{n-1}) & \text{for } A_n,\ B_n,\ C_n;\\ \exp(-x_{n-1}-x_n) & \text{for } D_n,\end{cases} \tag{47}$$

$$(-1)^{\frac{n(n-1)}{2}}\Delta_n(X)=\begin{cases}\exp(-x_n) & \text{for } A_n,\ C_n;\\ \exp(-2x_n) & \text{for } B_n;\\ \exp(-2x_n)+\exp(-2x_{n-1}) & \text{for } D_n.\end{cases} \tag{48}$$

The last n-th step for the series A_n gives the relation

$$\Delta_{n+1}(X)=(-1)^{n(n+1)/2}, \tag{49}$$

which is a nonlinear equation of order $2n$ for determining the single unknown function X, in terms of which the remaining unknown functions $\exp(-x_\alpha)$, $2\leqslant\alpha\leqslant n$, can be expressed by means of (46)–(48). For the classical series B_n and C_n, the expressions for the determinant $\Delta_{n+1}(X)$ have the form

$\underline{B_n}$: $\Delta_{n+1}=2(-1)^n\Delta_n$; $\underline{C_n}$: $\Delta_{n+1}=-\Delta_{n-1}$.

A similar situation obtains for the exceptional Cartan groups E_6, E_7, E_8, F_4, G_2, on the basic stages of the reduction on which we shall briefly dwell.

From the system (41), substituting the explicit form of the Cartan matrices of the exceptional groups, we obtain the reduction formulas

$$\left.\begin{array}{l}\underline{E_r\ (r=6,7,8)}:\\ \quad\exp(-x_2)=(-1)^{(r-4)(r-5)/2}\Delta_4(X)/\Delta_{r-4}(Y);\\ \quad\exp(-x_3)=-\Delta_2(X);\\ \quad\exp(-x_\alpha)=(-1)^{(r-\alpha)(r-\alpha+1)/2}\Delta_{r-\alpha+1}(Y);\\ \quad\Delta_3(X)=(-1)^{(r-3)(r-4)/2}\Delta_{r-3}(Y),\\ \quad 4\leqslant\alpha\leqslant r.\end{array}\right\} \tag{50}$$

Unused remains the equation

$$x_{2,z\bar z}=\exp(2x_2-x_4),$$

which leads to complicated relationships between the minors

$\underline{F_4}$:

$$\exp(-x_\alpha)=(-1)^{\alpha(\alpha-1)/2}\Delta_{5-\alpha}(Y),\quad 2\leqslant\alpha\leqslant 4;$$
$$\Delta_2(X)=\Delta_3(Y). \tag{51}$$

The remaining equation $x_{2,z\bar z}=\exp(-x_1+2x_2-2x_3)$ gives $\Delta_3(X)=-\Delta_2^2(Y)$:

$\underline{G_2}$:

$$\exp(-x_2)=-\Delta_2(X), \tag{52}$$

and it follows from the second equation of the system (41) that $\Delta_3(X)=-X^2$.

Thus, the above reduction scheme gives an algorithm for calculating the solutions of the system (41) in the form of explicit expressions for $\exp(-x_\alpha)$, $2\leqslant\alpha\leqslant r$ ($2\leqslant\alpha\leqslant r-1$ for the groups E_r and F_4), which are expressed in terms of one (two) unknown functions X (X and Y). The final step of the reduction, which determines the function X (respectively, X and Y), is constructive only for the series A_n, whereas for the remaining simple Lie groups the explicit construction of the remaining unknown functions requires a more detailed analysis of the structure of the root systems of the corresponding groups. The series B_n and C_n (and also the groups F_4 and G_2) occupy in this respect a distinguished position; for them one can obtain systems of equations by a certain limiting process from the corresponding equations of other groups.

The system (41) for the series A_n is symmetric under the substitution $x_\alpha\rightleftarrows x_{n-\alpha+1}$, $1\leqslant\alpha\leqslant n$. For even $n=2k$, the system goes over as a result of the equating $x_\alpha=x_{n-\alpha+1}$ for A_{2k} $[SU(2k+1)]$ into the corresponding system of the series B_k $[SO(2k+1)]$ with a necessary additional replacement of x_k by $2x_k$ (due to the doubling of the nondiagonal element of the penultimate row of the Cartan matrix of the group B_k compared with A_k). For odd $n=2k-1$ $[SU(2k)]$ we thus arrive at the system (41) for the series C_k $[\mathrm{Sp}(2k)]$. A similar situation obtains for the groups F_4 and G_2. Namely, equating in Eqs. (41) for the group E_6 the functions $x_1=x_6$, $x_3=x_5$ and making the substitution $x_1\to x_4$, $x_2\to x_1$, $x_3\to x_3$, $x_4\to x_2$, we obtain a system corresponding to the group F_4; the equation $x_1=x_3$ for the group B_3 leads to the system for the group G_2.

Complete integrability of the system of equations and construction of their general solutions. We now turn to the construction of general solutions of the system (41), by which we shall mean solutions that depend on $2r$ arbitrary functions. The question of the extent to which such solutions are the most general ones re-

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2)Note that the minors $\Delta_\alpha(X)$ satisfy the relation

$$\Delta_{\alpha-1}\Delta_{\alpha+1}/\Delta_\alpha=(\ln\Delta_\alpha)_{,\bar z z}.$$

mains open because of the absence of a complete theory of nonlinear differential equations.

We begin with the unitary series A_n, in which the reduction scheme makes it possible to reduce the considered problem to the finding of solutions to the nonlinear equation (49) of order $2n$ for the single unknown function $X\ (\equiv \exp(-x_1))$, in terms of which the remaining functions x_α, $2 \leqslant \alpha \leqslant n$, can be expressed in accordance with formulas (46)–(48). We seek the solution of the nonlinear equation (49) in the form

$$X = \sum_{a=0}^{n} P^a(z)\, Q^a(\bar{z}). \tag{53}$$

This choice of the ansatz for X is suggested, on the one hand, by the well-known general solution of the Liouville equation[62]

$$x_{,z\bar{z}} = \exp(2x), \tag{54}$$

which is a single-component special case of our system corresponding to the algebra $A_1 (k \equiv k_{11} = 2)$, and, on the other hand, by the polynomial form of the special solutions of the system (41), which depend on $z + \bar{z}$ (or $z\bar{z}$).[51] Substitution of (53) into (48) leads to factorization of the dependence on z and $\bar{z}$, namely, to a product of two determinants of order $n+1$ of the matrices P and Q with elements

$$P^a_b \equiv P^a_{\underbrace{z,\ldots,z}_{b}},\quad Q^a_b \equiv Q^a_{\underbrace{\bar{z}\bar{z},\ldots,\bar{z}}_{b}},$$

i.e.,

$$\det P \det Q = (-1)^{n(n+1)/2}. \tag{55}$$

Thus, the system of n second-order partial differential equations (41) is ultimately reduced by virtue of the fact that P and Q do not depend on $\bar{z}$ (z) to two ordinary differential equations of order n, these determining the unknown functions $P^n(z)$ and $Q^n(\bar{z})$ from the known functions $P^a(z)$ and $Q^a(\bar{z})$, $0 \leqslant a \leqslant n-1$, respectively. Further, without loss of generality, Eq. (55) can be rewritten in the form

$$\det P = 1, \quad \det Q = (-1)^{n(n+1)/2}. \tag{56}$$

For the group $SU(2)$ ($n=1$), the system (41) reduces to the Liouville equation (54), and Eqs. (56)

$$\left.\begin{aligned} &P^0 P^1_{,z} - P^0_{,z} P^1 = 1;\\ &Q^0 Q^1_{,\bar{z}} - Q^0_{,\bar{z}} Q^1 = -1 \end{aligned}\right\} \tag{57}$$

have the special solutions

$$P^0 = z;\quad P^1 = -1;\quad Q^0 = \bar{z};\quad Q^1 = 1.$$

It follows from the conformal covariance of (54) that the function X with generators

$$\left.\begin{aligned} &P^0 = u(z)\, u_{,z}^{-1/2};\quad P^1 = -u_{,z}^{-1/2};\\ &Q^0 = \bar{u}(\bar{z})\, \bar{u}_{,\bar{z}}^{-1/2};\quad Q^1 = \bar{u}_{,\bar{z}}^{-1/2}, \end{aligned}\right\} \tag{58}$$

where $u(z)$ and $\bar{u}(\bar{z})$ are arbitrary functions of their arguments, satisfies this equation, which can also be readily seen by direct substitution of the expressions (58) in (57). The resulting solution

$$\exp x = u_{,z}^{1/2}\bar{u}_{,\bar{z}}^{1/2} / u(z)\,\bar{u}(\bar{z}) - 1$$

is identical with the well-known general solution of the Liouville equation.

Note that Eqs. (57) can be solved in an obvious manner directly [without use of the conformal covariance of (54)] by transforming them to the form $(P^1/P^0)_{,z} = (P^0)^{-2}$, whence

$$P^1 = \varphi_{-1/2}(z) \int^{z} \varphi(z')\, dz',\quad P^0 = \varphi^{-1/2}(z).$$

We now consider the general case of the series A_n, following an inductive scheme. Suppose we know functions $P^a_{(n-1)}$ and $Q^a_{(n-1)}$, $0 \leqslant a \leqslant n-1$, satisfying Eqs. (55) for the algebra A_{n-1}. Then we have the following obvious special solution (56) for A_n:

$$\begin{aligned} &P^a_{(n)} = \int^{z} dv P^a_{(n-1)}(v),\quad P^n_{(n)} = 1;\\ &Q^a_{(n)} = \int^{\bar{z}} d\bar{v} Q^a_{(n-1)}(\bar{v}),\quad Q^n_{(n)} = -1,\\ &\qquad 0 \leqslant a \leqslant n-1. \end{aligned}$$

The functions $P^a_{(n-1)}$ depend functionally through quadratures on $n-1$ arbitrary functions. (A similar assertion holds for $Q^a_{(n-1)}$.) It follows from the conformal covariance of the system (41) that the functions

$$\left.\begin{aligned} &P^a_{(n)} = \int^{u(z)} dv P^a_{(n-1)}(v)\, u_{,z}^{-n/2};\\ &Q^a_{(n)} = \int^{\bar{u}(\bar{z})} d\bar{v} Q^a_{(n-1)}(\bar{v})\, \bar{u}_{,\bar{z}}^{-n/2};\\ &P^n_{(n)} = u_{,z}^{-n/2};\quad Q^n_{(n)} = -\bar{u}_{,\bar{z}}^{-n/2}, \end{aligned}\right\} \tag{59}$$

satisfy Eqs. (55) and depend on $2n$ arbitrary functions, ensuring thereby a general solution of the system (41) for the series A_n. As follows from the reduction procedure, the explicit expressions for the functions $P^a\ (\equiv P^a_{(n)})$ and $Q^a\ (\equiv Q^a_{(n)})$ have the form

$$\left.\begin{aligned} &P^a = \varphi_0(z_0) \prod_{s=1}^{n-a} \int^{z_{s-1}} \varphi_s(z_s)\, dz_s;\\ &Q^a = (-1)^a\, \bar{\varphi}_0(\bar{z}_0) \prod_{s=1}^{n-a} \int^{\bar{z}_{s-1}} \varphi_s(\bar{z}_s)\, d\bar{z}_s, \end{aligned}\right\} \tag{60}$$

where $z_0 \equiv z$, $\bar{z}_0 \equiv \bar{z}$, the product of integrals for $a=n$ is absent, and $\Pi_1^0 \equiv 1$. The factor $(-1)^a$ in the expression for Q^a ensures the correct sign on the right-hand side of Eq. (55). Between the $n+1$ arbitrary functions $\varphi_0, \ldots, \varphi_n$ there is the single relationship $\varphi_0^{-1} = \Pi_{s=1}^n \varphi_s^{n-s+1/n+1}$, which is a consequence of Eq. (56). (The functions $\bar{\varphi}_s$ satisfy an analogous relationship.)

The obtained solutions can be related to the properties of the group algebra if the functions φ_s, $1 \leqslant s \leqslant n$, are associated with the simple roots of the series A_n. Then the multiple integral of the product of the functions $\varphi_s(z_s)$ in (60) is associated with a multiple root of this series that includes the simple root π_1. To the function φ_0^{-1} there corresponds the first fundamental weight of the algebra of A_n, and the absence of integration in the expression for P^n means that the system of positive roots containing π_1 must be augmented by the zero element. Thus, the expression for X in the case of the series A_n is completely determined by specifying the following sequence of "roots":

$$0,\ \pi_1,\ \pi_1 + \pi_2,\ \ldots,\ \pi_1 + \ldots + \pi_n. \tag{61}$$

In accordance with (46), the expression for $\exp(-x_\alpha)$ is equal to the sum of the mutual products of the minors of order α constructed from the elements of the first α rows of the matrices P and Q. The function $\exp(-x_n)$ is determined by the formula

$$\exp(-x_n) = \sum_{a=0}^{n} P^{(a)}(z) Q^a(\bar{z}), \tag{62}$$

where P^a and Q^a are minors of order n; they are obtained from P^a and Q^a of the form (60) by replacement of φ_{n+1-a} by φ_a. [A similar connection holds for the functions $\exp(-x_s)$ and $\exp(-x_{n-s+1})$.] Note that any permutation of the functions P^a with simultaneous permutation of Q^a in which the sign on the right-hand side of Eq. (55) is preserved also leads to a complete solution of the system (41) and reflects the invariance of the algebra of $SU(n+1)$ with respect to transformations of the Weyl group [or rather, of the Weyl group $GL(n+1, C)$, if conditions of reality of the solutions of the system (41) are not imposed].

The invariance of the system (41) noted in the previous section for A_n under the substitution $x_\alpha \rightleftarrows x_{n+1-\alpha}$ enables us to obtain solutions in the case of the series B_n and C_n directly from the form of the solutions (60) by setting $\varphi_{n+1-\alpha} = \varphi_\alpha$. For even $n=2k$, the system (41) for A_n goes over into the corresponding system (41) of the series B_k, whereas for odd $n=2k-1$ we arrive at solutions for the series C_n. For the series C_n, the group meaning of the functions P^a and Q^a as systems of multiple roots containing the simple root π_1 remains unchanged. In the case of the series B_n, the system of its multiple roots must be augmented by not only the zero element but also twice the first fundamental weight $2\omega_1 = 2(\pi_1 + \cdots \pi_n)$. Finally, the augmented systems of roots, which determine the function X for the series B_n and C_n, have the form

$$\begin{aligned}\underline{B_n}:\ & 0,\ \pi_1,\ \pi_1+\pi_2,\ \ldots,\ \pi_1+\ldots+\pi_n,\\ & \pi_1+\ldots+\pi_{n-1}+2\pi_n,\\ & \pi_1+\ldots+\pi_{n-2}+2(\pi_{n-1}+\pi_n),\ \ldots,\ \pi_1\\ & +2(\pi_2+\ldots+\pi_n),\ 2(\pi_1+\ldots+\pi_n);\end{aligned} \tag{63}$$

$$\begin{aligned}\underline{C_n}:\ & 0,\ \pi_1,\ \pi_1+\pi_2,\ \ldots,\ \pi_1+\ldots+\pi_n,\\ & \pi_1+\ldots+\pi_{n-2}+2\pi_{n-1}+\pi_n,\ \ldots,\ \pi_1+2(\pi_2+\ldots\\ & \ldots+\pi_{n-1})+\pi_n,\ 2(\pi_1+\ldots+\pi_{n-1})+\pi_n.\end{aligned} \tag{64}$$

For the orthogonal groups $O(2n)$, the augmented system of roots of D_n, which determines the function X,

$$\begin{aligned}& 0,\ \pi_1,\ \pi_1+\pi_2,\ \ldots,\ \pi_1+\ldots+\pi_{n-2},\ \pi_1+\ldots+\pi_{n-2}+\pi_{n-1},\\ & \pi_1+\ldots+\pi_{n-2}+\pi_n,\ \pi_1+\ldots+\pi_n,\ \pi_1+\ldots+\pi_{n-3}+2\pi_{n-2}+\\ & +\pi_{n-1}+\pi_n,\ \pi_1+\ldots+\pi_{n-4}+2(\pi_{n-3}+\pi_{n-2})+\pi_{n-1}+\pi_n,\ \ldots\\ & \ldots,\ \pi_1+2(\pi_2+\ldots+\pi_{n-2})+\pi_{n-1}+\pi_n,\\ & 2(\pi_1+\ldots+\pi_{n-2})+\pi_{n-1}+\pi_n,\end{aligned} \tag{65}$$

contains, as in the case of the series B_n, twice the first fundamental weight $2\omega_1 = 2(\pi_1 + \cdots + \pi_{n-2}) + \pi_{n-1} + \pi_n$. In the expressions corresponding to multiple roots in which the simple roots π_{n-1} and π_n are simultaneously encountered, symmetrization with respect to them must be performed, i.e., to the corresponding multiple integral one must add another one, but with the contributions from π_{n-1} and π_n interchanged.

The root systems corresponding to the solutions $\exp(-x_{n-1})$ and $\exp(-x_n)$ associated with the simple roots π_{n-1} and π_n must be augmented by (in addition to zero) $\omega_{n-1}+\omega_n$ for odd n and $2\omega_{n-1}$ and $2\omega_n$, respectively, for even n. After this, the basic expressions for $\exp(-x_{n-1})$ and $\exp(-x_n)$ can be described in accordance with the rules given above. The expressions for $\exp(-x_\alpha)$, $2 \leq \alpha \leq n-2$, are given by the corresponding formulas (46). Note that the expressions for the solutions of the series B_{n-1} are obtained from the corresponding formulas of the series D_n by the ansatz $\varphi_{n-1} = \varphi_n$, which corresponds to equating the corresponding roots in the system (65).

We now consider the exceptional Cartan series. For the algebra of E_6, the augmented system of roots determining the function X has the form

$$\begin{aligned}& 0,\ \pi_1,\ \pi_1+\pi_3,\ \pi_1+\pi_3+\pi_4,\ \pi_1+\pi_3+\pi_4+\pi_5,\ \pi_1+\pi_2+\pi_3+\pi_4,\\ & \pi_1+\pi_3+\pi_4+\pi_5+\pi_6,\ \pi_1+\pi_2+\pi_3+\pi_4+\pi_5,\ \pi_1+\pi_2+\pi_3+\pi_4\\ & +\pi_5+\pi_6,\ \pi_1+\pi_2+\pi_3+2\pi_4+\pi_5,\ \pi_1+\pi_2+\pi_3+2\pi_4+\pi_5+\pi_6,\\ & \pi_1+\pi_2+2\pi_3+2\pi_4+\pi_5;\ \ \pi_1+\pi_2+\pi_3+2\pi_4+2\pi_5+\pi_6,\ \pi_1+\pi_2\\ & +2\pi_3+2\pi_4+\pi_5+\pi_6,\ \ 2\pi_1+\pi_2+2\pi_3+2\pi_4+\pi_5+\pi_6,\ 2\pi_1+\pi_2\\ & +2\pi_3+2\pi_4+2\pi_5+\pi_6;\ 2\pi_1+\pi_2+2\pi_3+3\pi_4+2\pi_5+\pi_6,\ \ 2\pi_1+\pi_2\\ & +3\pi_3+3\pi_4+2\pi_5+\pi_6,\ 2\pi_1+2\pi_2+2\pi_3+3\pi_4+2\pi_5+\pi_6,\\ & 2\pi_1+2\pi_2+3\pi_3+3\pi_4+2\pi_5+\pi_6,\ 2\pi_1+2\pi_2+3\pi_3+4\pi_4+2\pi_5+\pi_6,\\ & 2\pi_1+2\pi_2+3\pi_3+4\pi_4+3\pi_5+\pi_6;\\ & 2\pi_1+2\pi_2+3\pi_3+4\pi_4+3\pi_5+2\pi_6.\end{aligned} \tag{66}$$

Besides the zero element and the positive roots of the algebra $R^+_{\pi_1}$ with height ≤ 8 containing the simple root π_1, this system also includes the sum of the first and the sixth fundamental weight, $\tilde{\omega} \equiv \omega_1 + \omega_6$, and all possible combinations $\tilde{\omega} - \alpha$:

$$\alpha \in R^+_{\left\{\substack{\pi_1 \rightleftarrows \pi_6 \\ \pi_3 \rightleftarrows \pi_5}\right.},\quad \{\tilde{\omega}-\alpha\} \cap R^+_{\pi_6} = \varnothing,$$

which are absent for the classical series. The functions $\exp(-x_6)$ and $\exp(-x_5)$ can be obtained from $X \equiv \exp(-x_1)$ and $\exp(-x_3)$, respectively, by the substitution $\varphi_6 \rightleftarrows \varphi_1, \varphi_5 \rightleftarrows \varphi_3$ [the remaining functions are recovered in accordance with the expressions (50)]. As in the case of the series D_n, symmetrization must be carried out in the corresponding expressions for P^a and Q^a.

The solutions for the algebra of F_4 are obtained from the corresponding E_6 solutions by setting $\varphi_6 = \varphi_1$ and $\varphi_5 = \varphi_3$, whereas the functions $\exp(-x_1)$ and $\exp(-x_2)$ for the algebra of G_2 arise from the solutions for B_3 ($O(7)$) by equating φ_1 and $2\varphi_3$.

We shall not write out here the very cumbersome augmented root systems for E_7 and E_8.

In conclusion we note that although we have succeeded in constructing the explicit form of the solutions for all the classical series and the exceptional Cartan algebras, we have each time had to use the concrete form of the corresponding Cartan matrices, i.e., we have had to particularize the system (41) for each type of simple algebra. It would be extremely interesting to find the solutions of the system (41) in an invariant uniform form for all types of considered algebras with recourse to only the general properties of the Cartan matrix.

One-dimensional 2r-parameter subclass of solutions. In actual applications in physics associated with instanton and monopole configurations, one requires solutions that depend on a single variable $z+\bar{z}$ (or $z\bar{z}$), which can be readily obtained from our solutions of the system (41) by the substitution $\varphi_s = c_s \exp(zm_s)$, $\bar{\varphi}_s = c_s^* \exp(\bar{z}m_s)(\varphi_s = c_s z^{m_s}, \bar{\varphi}_s = c_s^* \bar{z}^{m_s})$, where c_s and m_s are arbitrary parameters. As a result of this ansatz, the general solutions, which depend on $2r$ arbitrary functions, are transformed into $2r$-parameter solutions. It was solutions of this type that served as the point of departure in the construction of solutions of general form. Anticipating, we note that the description of instantons and monopoles requires (to ensure that the action and energy, respectively, are finite) an additional restriction of the class of solutions, which are characterized by r "quantum" numbers $m_1, \ldots, m_r$, with respect to which there is degeneracy.

It appears to us helpful to give here a number of basic formulas of the general case for solutions that depend only on $z+\bar{z}=r$, which will be needed in what follows in actual calculations for instantons and monopoles and also for the purpose of greater clarity of the group interpretation in the framework of the root technique. For the functions x_α, which depend on $z+\bar{z}$, the system (41) has the form[3)]

$$\ddot{x}_\alpha = \exp\left(\sum_{\beta=1}^{r} k_{\alpha\beta} x_\beta\right). \tag{67}$$

As before, we consider first the simplest and symmetric case of the series A_n. Then, following the notation introduced in Sec. 1 and setting $\varphi_s = c_s \exp(zm_s)$ in (60), we arrive at the following expression for the solutions of the system (67):

$$\begin{aligned} &\exp(-x_\nu) = (-1)^{\nu(\nu-1)/2} \exp[\omega_\nu (c - rm)] \\ &\times \sum_{i>j>\ldots>s} \{\exp[\alpha_i^1 (rm-c) + \alpha_j^2 (rm-c) + \ldots + \alpha_s^\nu (rm-c)] \\ &\times \prod_{p\neq i, \ldots, q\neq s} f^{-1}[\alpha_i^1(m) + \ldots + \alpha_s^\nu(m) - \alpha_p^1(m) - \ldots - \alpha_q^\nu(m)]\}, \end{aligned} \tag{68}$$

where $\alpha_i^\lambda (\equiv \sum_\mu t_i^\mu \pi_\mu)$ ranges over the subset $R_{\pi\lambda}^+ \subset R^+$ of all positive roots of the series A_n containing the simple root π_λ augmented by the zero element $\alpha_0^\lambda \equiv 0$ and ordered by height, i.e., i takes the values $0, \pi_\lambda, \pi_\lambda + \pi_{\lambda+1}, \ldots, \pi_\lambda + \cdots + \pi_n$. (At the same time $\alpha_0^\lambda(m) = 0$, $\alpha_{\pi_\lambda}^\lambda(m) = m_\lambda, \ldots, \alpha_{\pi_\lambda}^\lambda + \cdots + \pi_n = \sum_{\mu=\lambda}^n \pi_\mu$.) The factor f is equal to the square of its argument if the latter coincides with a negative root of A_n and is equal to unity otherwise.

The obvious presence of the symmetry relations $x_\alpha \rightleftarrows x_{n-\alpha+1}$ under replacement of the arbitrary parameters m_α and c_α by $m_{n+1-\alpha}$ and $c_{n+1-\alpha}$ enables us, as before, to realize directly at the level of the solutions the possibility noted in the previous sections of going over from the unitary series A_n to the orthogonal series B and the symplectic series C. This is achieved automatically by equating in (68) the parameters m_α, c_α and $m_{n-\alpha+1}$, $c_{n-\alpha+1}$ for even ($n=2k$) and odd ($n=2k-1$) values of n. An important circumstance is the fact that X for the series C_n preserves in this procedure its root structure, but in terms of the root space. We have not yet succeeded in obtaining the factors f for the B and C algebras, which are determined by setting $m_\alpha = m_{n+1-\alpha}$ and $c_\alpha = c_{n+1-\alpha}$ in the corresponding expressions for the unitary series, in terms of their root spaces.

Omitting consideration of the simple algebras considered earlier, we give for illustration expressions for the functions $\exp(-x_1)$ and $\exp(-x_4)$ in the algebra D_4 and the explicit form of the solutions of the system (67) for the second-rank algebras of $SU(3)$, $\mathrm{Sp}(4) \cong O(5)$, and G_2, transforming them for the sake of brevity to a form that depends on $R = 2(z\bar{z})^{1/2}$:

$\underline{D_4}$:

$$\begin{aligned} &\exp(-x_1) = \tilde{c}_1^{-1} \tilde{c}_2^{-1} \tilde{c}_3^{-1/2} \tilde{c}_4^{-1/2} \\ &\times \exp\{-r[m_1+m_2+(m_3+m_4)/2]\} \Big\{ \exp[(2m_1+2m_2+m_3+m_4)r] \\ &- (\tilde{c}_1/m_1^2) \exp[(m_1+2m_2+m_3+m_4)r] \\ &+ \frac{\tilde{c}_1\tilde{c}_2 \exp[(m_1+m_2+m_3+m_4)r]}{(m_1+m_2)^2 m_2^2} - \frac{\tilde{c}_1\tilde{c}_2\tilde{c}_4 \exp[(m_1+m_2+m_3)r]}{(m_1+m_2+m_4)^2 (m_2+m_4)^2 m_4^2} \\ &- \frac{\tilde{c}_1\tilde{c}_2\tilde{c}_3 \exp[(m_1+m_2+m_4)r]}{(m_1+m_2+m_3)^2 (m_2+m_3)^2 m_3^2} \\ &+ \frac{\tilde{c}_1\tilde{c}_2\tilde{c}_3\tilde{c}_4 \exp[(m_1+m_2)r]}{(m_1+m_2+m_3+m_4)^2 (m_2+m_3+m_4)^2 m_3^2 m_4^2} \\ &- \frac{\tilde{c}_1\tilde{c}_2^2\tilde{c}_3\tilde{c}_4 \exp(m_1 r)}{(m_1+2m_2+m_3+m_4)^2 (m_2+m_3+m_4)^2 (m_2+m_3)^2 (m_2+m_4)^2 m_2^2} \\ &+ \frac{\tilde{c}_1^2\tilde{c}_2^2\tilde{c}_3\tilde{c}_4}{(m_1+2m_2+m_3+m_4)^2 (m_1+m_2+m_3+m_4)^2 (m_1+m_2+m_3)^2 \times} \Big\}; \\ &\times (m_1+m_2+m_4)^2 (m_1+m_2)^2 m_1^2 \end{aligned} \tag{69}$$

$$[\tilde{c}_i \equiv \exp(-c_i)];$$

$$\begin{aligned} &\exp(-x_4) = \tilde{c}_1^{-1/2} \tilde{c}_2^{-1} \tilde{c}_3^{-1/2} \tilde{c}_4^{-1} \exp\{-r[m_2+m_4+(m_1+m_3)/2]\} \\ &\times \Big\{ \exp[(m_1+2m_2+m_3+2m_4)r] \\ &- \frac{\tilde{c}_4 \exp[(m_1+2m_2+m_3+m_4)r]}{m_4^2} + \frac{\tilde{c}_2\tilde{c}_4 \exp[(m_1+m_2+m_3+m_4)r]}{(m_2+m_4)^2 m_2^2} \\ &- \frac{\tilde{c}_2\tilde{c}_3\tilde{c}_4 \exp[(m_1+m_2+m_4)r]}{(m_2+m_3+m_4)^2 (m_2+m_3)^2 m_3^2} - \frac{\tilde{c}_1\tilde{c}_2\tilde{c}_4 \exp[(m_2+m_3+m_4)r]}{(m_1+m_2+m_4)^2 (m_1+m_2)^2 m_1^2} \\ &+ \frac{\tilde{c}_1\tilde{c}_2\tilde{c}_3\tilde{c}_4 \exp[(m_2+m_4)r]}{(m_1+m_2+m_3+m_4)^2 (m_1+m_2+m_3)^2 m_1^2 m_3^2} \\ &- \frac{\tilde{c}_1\tilde{c}_2^2\tilde{c}_3\tilde{c}_4 \exp(m_4 r)}{(m_1+2m_2+m_3+m_4)^2 (m_1+m_2+m_3)^2 (m_1+m_2)^2 (m_2+m_3)^2 m_2^2} \\ &+ \frac{\tilde{c}_1\tilde{c}_2^2\tilde{c}_3\tilde{c}_4^2}{(m_1+2m_2+m_3+m_4)^2 (m_1+m_2+m_3+m_4)^2 (m_1+m_2+m_4)^2} \Big\}; \\ &\times (m_2+m_3+m_4)^2 (m_2+m_4)^2 m_4^2 \end{aligned} \tag{70}$$

$$\begin{aligned} &\underline{SU}(3):\quad \exp(-x_1) = \left(\frac{c_1}{2}\right)^{-1/3} \left(\frac{c_2}{2}\right)^{-2/3} R^{(-2m_1-m_2+6)/3} \\ &\times \left(R^{m_1+m_2} - \frac{c_2}{m_2^2} R^{m_1} + \frac{c_1 c_2}{m_1^2 (m_1+m_2)^2} \right); \\ &\exp(-x_2) = \left(\frac{c_1}{2}\right)^{-2/3} \left(\frac{c_2}{2}\right)^{-1/3} R^{(-m_1-2m_2+6)/3} \\ &\times \left(R^{m_1+m_2} - \frac{c_1}{m_1^2} R^{m_2} + \frac{c_1 c_2}{m_2^2 (m_1+m_2)^2} \right); \end{aligned} \tag{71}$$

[3)]The system (67) [see also (28)] for the group $SU(n+1)$ is identical with the Toda lattice[72] (with boundary conditions $\rho_0 = \rho_{n+1} = 0$) and, thus, the formulas given below solve the problem of its complete integration. Evidently, the complete integration of the Korteweg–de Vries equation[64] can be achieved by a "passage to the limit" in the expression (68): $c_\alpha \to c(\alpha)$, $m_\alpha \to m(\alpha)$.

$$
\left.
\begin{aligned}
&\underline{O(5)}:\ \exp(-x_1)=\left(\frac{c_1}{3}\right)^{-1}\left(\frac{c_2}{4}\right)^{-1/2}R^{(-2m_1-m_2+6)/2}\\
&\times\Big(R^{2m_1+m_2}-\frac{c_1}{m_1^2}R^{m_1+m_2}+\frac{c_1c_2}{m_2^2(m_1+m_2)^2}R^{m_1}\\
&-\frac{c_1^2c_2}{m_1^2(m_1+m_2)^2(2m_1+m_2)^2}\Big);\\
&\exp(-x_2)=\left(\frac{c_1}{3}\right)^{-1}\left(\frac{c_2}{4}\right)^{-1}R^{-m_1-m_2+4}\\
&\times\Big(R^{2m_1+2m_2}-\frac{c_2}{m_2^2}R^{2m_1+m_2}+\frac{2c_1c_2}{m_1^2(m_1+m_2)^2}R^{m_1+m_2}\\
&-\frac{c_1^2c_2}{m_1^4(2m_1+m_2)^2}R^{m_2}+\frac{c_1^2c_2^2}{m_2^2(m_1+m_2)^4(2m_1+m_2)^2}\Big);
\end{aligned}
\right\}\quad(72)
$$

$$
\left.
\begin{aligned}
&\underline{G_2}:\ \exp(-x_1)=\left(\frac{c_1}{6}\right)^{-2}\left(\frac{c_2}{10}\right)^{-1}R^{-2m_1-m_2+6}\\
&\times\Big(R^{4m_1+2m_2}-\frac{c_1}{m_1^2}R^{3m_1+2m_2}+\frac{c_1c_2}{m_2^2(m_1+m_2)^2}\\
&\times R^{3m_1+m_2}-\frac{2c_1^2c_2}{(2m_1+m_2)^2(m_1+m_2)^2m_1^2}R^{2m_1+m_2}\\
&+\frac{c_1^3c_2}{(3m_1+m_2)^2(2m_1+m_2)^2m_1^4}R^{m_1+m_2}\\
&-\frac{c_1^3c_2^2}{(3m_1+2m_2)^2(m_1+m_2)^4(2m_1+m_2)^2m_2^2}R^{m_1}\\
&+\frac{c_1^4c_2^2}{(2m_1+m_2)^4(3m_1+2m_2)^2(3m_1+m_2)^2(m_1+m_2)^2m_1^2}\Big);\\
&\exp(-x_2)=\left(\frac{c_1}{6}\right)^{-3}\left(\frac{c_2}{10}\right)^{-2}R^{-3m_1-2m_2+10}\\
&\times\Big(R^{6m_1+4m_2}-\frac{c_2}{m_2^2}R^{6m_1+3m_2}+\frac{3c_1c_2}{m_1^2(m_1+m_2)^2}R^{5m_1+3m_2}\\
&-\frac{3c_1^2c_2}{m_1^4(2m_1+m_2)^2}R^{4m_1+3m_2}+\frac{3c_1^2c_2^2}{m_2^2(m_1+m_2)^4(2m_1+m_2)^2}\\
&\times R^{4m_1+2m_2}+\frac{c_1^3c_2}{m_1^6(3m_1+m_2)^2}R^{3m_1+3m_2}\\
&-\frac{24c_1^3c_2^2(3m_1^2+3m_1m_2+m_2^2)}{m_1^2m_2^2(m_1+m_2)^2(2m_1+m_2)^2(3m_1+m_2)^2(3m_1+2m_2)^2}\\
&\times R^{3m_1+2m_2}+\frac{c_1^3c_2^3}{m_2^4(m_1+m_2)^6(3m_1+2m_2)^2}R^{3m_1+m_2}\\
&+\frac{3c_1^4c_2^2}{m_1^4(2m_1+m_2)^4(3m_1+m_2)^2(m_1+m_2)^2}R^{2m_1+2m_2}\\
&-\frac{3c_1^4c_2^3}{m_2^2(m_1+m_2)^4(2m_1+m_2)^4m_1^2(3m_1+2m_2)^2}R^{2m_1+m_2}\\
&+\frac{3c_1^5c_2^3}{m_1^4(m_1+m_2)^4(2m_1+m_2)^4(3m_1+m_2)^2(3m_1+2m_2)^2}R^{m_1+m_2}\\
&-\frac{c_1^6c_2^3}{m_1^2(2m_1+m_2)^6(3m_1+m_2)^4(3m_1+2m_2)^2}R^{m_2}\\
&+\frac{c_1^6c_2^4}{m_2^2(m_1+m_2)^6(2m_1+m_2)^6(3m_1+2m_2)^4(3m_1+m_2)^2}\Big).
\end{aligned}
\right\}\quad(73)
$$

As can be seen from these explicit expressions, the exponents of the terms in the polynomials exactly reproduce the elements of the augmented system of roots of the corresponding algebras and ensure a perspicuous group interpretation of the structure of the solutions.

4. DESCRIPTION OF CYLINDRICALLY SYMMETRIC INSTANTON CONFIGURATIONS AND SPHERICALLY SYMMETRIC MONOPOLE CONFIGURATIONS

Instanton solutions and values of the topological charge. The cylindrically symmetric instantons form the subclass of $2r$-parameter solutions constructed in Sec. 3 that are regular at all points of space (including infinity). The arbitrary parameters c_α and m_α must be chosen to ensure that the corresponding solutions are nonsingular and the values of the action are finite or, which is the same thing, the topological charge is finite. As we have already noted in Sec. 2, one can go over by means of a conformal transformation from the solutions $\overset{\circ}{\rho}_\alpha=\sum_{\beta=1}^{r}k_{\alpha\beta}x_\beta$, which depend on $z+\bar{z}$ (or $z\bar{z}$), to $\rho_\alpha(z)=\rho_\alpha^0(g(z))+\ln|dg/dz|^2$, which are also solutions of the system (41) for an arbitrary analytic function $g(z)$. If the structure functions $\psi^\alpha=(1/2)\ln[r^2\exp(\rho_\alpha)/l^\alpha|^2]$ in the expression for the density of the topological charge are to be finite for $r\equiv z+\bar{z}=0$, it is necessary to compensate the double zero at this point by poles of corresponding order in the function $\exp\overset{\circ}{\rho}_\alpha$, i.e., to impose boundary conditions on the behavior of the functions $\exp(-x_\alpha)$ that we have constructed at short "distances." It is easy to show that to ensure these conditions it is sufficient to require that the function $X(\equiv\exp(-x_1))$ have a root of order δ_1 at $r=0$. Then the system (41) automatically guarantees the appearance of roots of zeroth order ($=\delta_\alpha$) at $r=0$ for the remaining functions $\exp(-x_\alpha)$, $2\leqslant\alpha\leqslant r$. The realization of this requirement in the form of the set of equations $\partial^\alpha X/\partial r^\alpha|_{r=0}=0$, $0\leqslant\alpha\leqslant\delta_1-1$, leads to complete determination of the parameters c_α, $1\leqslant\alpha\leqslant r$, in terms of the parameters m_β. As an illustration, we give the solutions for groups of rank 2, which ensure the necessary boundary conditions at $r=0$:

$$
\left.
\begin{aligned}
&\underline{SU(3)}:\ \exp(-x_1)=R^{(-2m_1-m_2+6)/3}(R-1)^2\pi_1^1;\\
&c_1=m_1^2(m_1+m_2)/m_2;\\
&\exp(-x_2)=R^{(-m_1-2m_2+6)/3}(R-1)^2\pi_2^1;\\
&c_2=m_2^2(m_1+m_2)/m_1;
\end{aligned}
\right\}\quad(74)
$$

$$
\left.
\begin{aligned}
&\underline{O(5)}:\ \exp(-x_1)=R^{(-2m_1-m_2+6)/2}(R-1)^3\pi_1^2;\\
&c_1=m_1^2(2m_1+m_2)/m_2;\\
&\exp(-x_2)=R^{-m_1-m_2+4}(R-1)^4\pi_2^2;\\
&c_2=m_2^2(m_1+m_2)^2/m_1^2;
\end{aligned}
\right\}\quad(75)
$$

$$
\left.
\begin{aligned}
&G_2:\ \exp(-x_1)=R^{-2m_1-m_2+6}(R-1)^6\pi_1^3;\\
&c_1=m_1^2(2m_1+m_2)(3m_1+m_2)/(m_1+m_2)m_2;\\
&\exp(-x_2)=R^{-3m_1-2m_2+10}(R-1)^{10}\pi_2^3;\\
&c_2=(m_1+m_2)^3(3m_1+2m_2)m_2^2/m_1^3(3m_1+m_2).
\end{aligned}
\right\}\quad(76)
$$

These expressions are obtained from the solutions (71)–(73) by equating to zero the corresponding number of derivatives. Here, we can assume that $R\equiv 2(g(z))^{1/2}(\overline{g(z)})^{1/2}$ becomes equal to unity for an appropriate choice of the functions $g(z)$. We emphasize that for positive values of R the finite polynomials π_α^b are positive definite and vanish nowhere. So far, the values of the parameters m_α have not been fixed in any way. However, to ensure the necessary analyticity properties of the corresponding solutions we must, in general, require them to be integral.

We now elucidate the conditions imposed on the function $g(z)$ to ensure nonsingularity of the functions ψ^α and the corresponding solutions of the self-duality equations. As we have already said, it is necessary that the modulus $|g(z)|$ become equal to unity at $r=0$. Further, we must require that the functions in $\exp(2\psi^\alpha)$ which remain after the canceling of r^2 by the pole factors in $\exp(x_\alpha)$ have neither poles nor zeros in the right-hand half-plane of the variable z. As in the example of the second-rank groups considered above, the polynomial part of the solutions $\exp(-x_\alpha)$ after elimination of the root factors at the origin vanishes nowhere. Therefore, it is sufficient to take the functions l^α in such a way that they compensate the contribution of $|dg/dz|$ and of the pre-polynomial factor in $\exp(-x_\alpha)$, i.e., with allowance for the equation $k_{\alpha\beta}\omega_\beta(m)\equiv m_\alpha$ to set $|l^\alpha|^2=|dg/dz|^2|g|^{m_\alpha-2}$. We now recall that because of the Cauchy-Riemann conditions the functions l^α must be analytic as subsystems of the self-duality equations (27). Therefore, to ensure that the exponent $(m_\alpha - 2)/2$ is integral, we must require $g(z)$ to be the square of some analytic function $G(z), g(z) = G^2(z)$, whence $l^\alpha = 2G^{m_\alpha - 1} dG/dz$. It is then obvious that the solutions ψ^α have the necessary analytic properties if $|G| = 1$ at $r = 0$ and $|G| < 1$ for $r > 0$. The most general analytic function satisfying these conditions is the Blaschke function

$$G(z) = \prod_{i=1}^{l} \frac{a_i - z}{a_i^* + z},$$

where a_i are arbitrary complex numbers with positive real part. This function has l zeros and no pole on the right-hand side of the half-plane of the variable z.

To calculate the topological charge, it is convenient to use formula (29), taking $W_\alpha = i \exp(\psi^\alpha) l^\alpha$ in it. Since the functions ψ^α are single-valued, they do not contribute to the contour integral, and therefore

$$Q = \frac{1}{4\pi i} \sum_{\alpha=1}^{r} \mathscr{B}_\alpha \oint dx_\mu \, \partial_\mu \ln G^{m_\alpha - 1} \, dG/dz. \tag{77}$$

Bearing in mind that $G^{m_\alpha - 1} dG/dz$ has $m_\alpha l - 1$ zeros in the right-hand half-plane of z, each of them making the contribution $2\pi i$ to the integral (77), we obtain for the topological charge the expression

$$Q = \frac{1}{2} \sum_{\alpha=1}^{r} \mathscr{B}_\alpha (l m_\alpha - 1). \tag{78}$$

Thus, the instanton solutions we have constructed correspond to the topological charge (78) and are characterized by the l arbitrary complex parameters a_i, $\mathrm{Re}\, a_i > 0$, of the Blaschke function and also by the r additional "quantum" numbers m_α. The presence of the latter corresponds to the existence of discrete series of solutions with a fixed value of the charge and can be interpreted as internal degrees of freedom of the instantons.

Nonsingular monopole solutions; matrices of the magnetic charge and masses of the monopoles. The method of integration of the nonlinear systems developed in the previous sections is fully applicable to the construction of exact nonsingular solutions to Eqs. (40), which describe spherically symmetric monopole configurations in the Bogomol'nyĭ-Prasad-Sommerfield limit for an arbitrary compact gauge group G with Higgs field in the adjoint representation. It is easy to show that as a result of the ansatz

$$K_\alpha = r \exp \rho_\alpha/2; \quad H_\alpha = 1 + (1/2) r \dot{\rho}_\alpha, \tag{79}$$

Eqs. (40) reduce to the system

$$\ddot{\rho}_\alpha = \sum_{\beta=1}^{r} \delta_\beta k_{\alpha\beta} \exp \rho_\beta, \tag{80}$$

which describes in accordance with (67) cylindrically symmetric self-dual configurations in the static limit. Therefore, the relations (79) realize the correspondence noted in Sec. 2 between the cylindrically symmetric static self-dual fields in R_4 and the spherically symmetric monopoles in Minkowski space. This circumstance makes it possible to reconstruct directly exact monopole solutions on the basis of the solutions we have constructed to the system (80), which are parametrized by $2r$ arbitrary constants. For this, it is sufficient to substitute the expression for the solutions $\rho_\alpha = \sum_\beta k_{\alpha\beta} x_\beta$ in Sec. 3 into Eq. (79).

The constructed solutions do not have singularities in a finite interval of r values. To ensure that the Hamiltonian density of the system is finite, it is necessary to satisfy appropriate boundary conditions at infinity and at short distances. The procedure for separating the subclass of solutions that have the correct behavior as $r \to 0$ is analogous to that carried through in the instanton case, as a result of which the monopole solutions are parametrized in terms of the structures K_α and H_α by the set of r parameters m_α.

To calculate the energy of the monopole configuration defined in the Bogomol'nyĭ-Prasad-Sommerfield limit by the expression (36) and the matrix of the magnetic charge (37), we require the explicit form of the asymptotic behaviors as $r \to \infty$ of the magnetic field $\mathbf{B}$, or rather $B_r = \mathbf{n} \cdot \mathbf{B}$, and the Higgs field φ,[4)] which for our solutions have the form

$$\left.\begin{aligned} B_r &= -\frac{1}{2r^2} \sum_\alpha \delta_\alpha h_\alpha (K_\alpha^2 - 1); \\ \varphi &= \sum_{\alpha,\beta} k_{\beta\alpha}^{-1} h_\beta \frac{d}{dr} \ln K_\alpha. \end{aligned}\right\} \tag{81}$$

We recall that here h_α are the generators of the Cartan subalgebra $\mathfrak{g}$ in $\mathfrak{g}$ corresponding to the simple roots of G. Formula (81) obviously follows, in view of the correspondence noted above between $(A_0, \mathbf{A})$ and $(\varphi, \mathbf{W})$, from the expressions (36) and (37) when the relation $H_\alpha = r\, d/dr \ln K_\alpha$ [see (79)] is used. One must also use the replacement of the operators T_α by $\sum_\beta k_{\beta\alpha}^{-1} h_\beta$, whose validity can be readily verified by directing the unit vector $\mathbf{n}$ along the third axis, $\mathbf{n} = (0, 0, 1)$, when $W^l \Rightarrow H^{(l)}$, and the following chain of equations holds:

$$\begin{aligned} T_\alpha &= \sum_l \delta_\alpha \varkappa_\alpha^{1/2} W^l \tilde{H}_\alpha^l \Rightarrow \sum_l \delta_\alpha \varkappa_\alpha^{1/2} H^{(l)} \tilde{H}_\alpha^l = \sum_{l,\beta} \delta_\alpha \varkappa_\alpha^{1/2} \tilde{H}_\alpha^l H_\beta^l h_\beta \\ &= \sum_{\beta, l} \delta_\alpha \varkappa_\alpha^{1/2} \tilde{H}_\alpha^l h_\beta \varkappa_\beta^{-1/2} \frac{1}{l(l+1)} \tilde{H}_\beta^l = \sum_\beta k_{\beta\alpha}^{-1} h_\beta. \end{aligned}$$

Because of the boundary conditions at infinity, the contribution of the Higgs field is completely determined by the arguments of the pre-polynomial exponentials of the corresponding solutions $\exp(-x_\alpha)$, i.e., $\omega_\alpha(-rm) = -r \sum_\beta k_{\alpha\beta}^{-1} m_\beta$. Indeed, from the boundary conditions at infinity $K_\alpha \Rightarrow 0$ as $r \to \infty$, whereas

$$\lim_{r\to\infty} \frac{d}{dr} \ln K_\alpha^2 = -k_{\alpha\beta} \lim_{r\to\infty} \frac{d}{dr} \ln \exp(-x_\beta) = k_{\alpha\beta} \omega_\beta(m) = \frac{1}{2} m_\alpha.$$

Therefore, (36) and (37) lead to the following expressions for the matrices of the magnetic charge g and the masses M of the monopoles:

$$g = 2\pi \sum_\alpha \delta_\alpha h_\alpha; \quad M = \pi \sum_{\alpha,\beta} \delta_\alpha k_{\alpha\beta}^{-1} m_\beta. \tag{82}$$

Note that the derivation of (82) for the matrix g does not depend on the explicit form of the solution and is completely determined by the boundary conditions.

---

4)As before, the coupling constant e of the gauge field is omitted.

Omitting the derivation of the formula for the mass matrix M_W of the gauge field, we give only its final expression in the adopted normalization,

$$M_W = \frac{1}{4} \sum_{\alpha, \beta} \delta_\alpha k^{-1}_{\alpha\beta} m_\beta,$$

which can be obtained by repeating the corresponding arguments of Ref. 63.

A striking property of our solutions and the mass formula is the existence of discrete series of monopole solutions labeled by the "quantum" numbers $m_1, \ldots, m_r$ and degenerate with respect to them for fixed values of the elements $(\delta k^{-1} m)_{\alpha\beta}$ of the matrix of the monopole masses. This circumstance can evidently be regarded as an indication of the possibility of the existence of structural characteristics of monopole configurations.

All the previously known monopole solutions (see, for example, Refs. 5, 54, 56, and 57) which correspond to the lowest admissible nonzero values of the magnetic charges and masses of nonsingular monopoles are special cases of the expressions obtained above. In particular, the solutions of Ref. 56 for the unitary group are obtained from our solutions by taking $m_1 = 2$ and $m_\alpha = 1$, $2 \leqslant \alpha \leqslant n$, in (68).

CONCLUSIONS

Let us summarize briefly the main results presented in the review, and draw attention to a number of promising problems in this field.

The main result is the constructive proof of the complete integrability of the system of essentially nonlinear second-order partial differential equations

$$x_{\alpha, z\bar{z}} = \exp \sum_{\beta=1}^{r} k_{\alpha\beta} x_\beta, \qquad 1 \leqslant \alpha \leqslant r, \tag{83}$$

where k is the Cartan matrix of the simple Lie algebra $\mathfrak{g}$ of rank r, through the construction of general solutions of this system depending on $2r$ arbitrary functions. The system (83) is a realization of the self-duality condition of cylindrically symmetric Yang-Mills fields for the minimal embedding of $SU(2)$ in an arbitrary compact gauge group G, for which the invariance subgroup is Abelian, $S = \Pi_1^r \otimes U(1)$. This has enabled us to describe spherically symmetric instanton and monopole configurations in the framework of a unified method as r-parameter subclasses of solutions of the system (83) and to calculate explicitly the values of the topological charge of the instantons and the matrices of the magnetic charge and masses of the monopoles.

The possibility of complete integration of the system (83) in the case when k is the Cartan matrix of $\mathfrak{g}$ poses numerous interesting problems of both mathematical and physical nature that require further investigation. Let us briefly list some of them.

1. The connection between the fact of complete integrability of the system (83) and the properties of simple Lie algebras and the possible formulation of a criterion of simplicity of finite-dimensional Lie algebras in the form of conditions of (finite) polynomiality of the corresponding general solutions.

2. The construction of a uniform method of integrating the system (83) using only the general properties of Cartan matrices and not the root spaces of each type of Lie algebra separately.

3. The construction of systems of cylindrically symmetric duality equations of gauge theories for all types of embedding of $SU(2)$ in G and the solution of the problem of their complete integrability. This problem is intimately related to the possibility of formulating the corresponding equations solely in terms of entities that are gauge invariant with respect to S.

4. The connection between the approach presented in this review to the integration of the system (83) and the inverse scattering technique.[64]

5. The finding of Bäcklund transformations[65] for the system (83) relating its solutions not only to solutions of the same system but also to r free Laplace equations $(x_{\alpha, z\bar{z}} = 0)$. It is well known that for the group $SU(2)$ such transformations exist.[66] Further, as is well known, the Liouville equation is intimately related to the sine-Gordan equation. If this connection also holds in the multi-component case, it would be possible on the basis of the multi-component sine-Gordon equation to construct a model of a quantum theory with an exact S matrix, as in the single-component case.[67] The quantum numbers of the soliton states of such models would form multi-dimensional manifolds.

6. The connection of Bäcklund transformations of equations of gauge theories to the Weyl automorphism, which distinguishes different real forms of complex groups, and the reality condition of the solutions. There are certain indications of the possible existence of such relations based on Bäcklund transformations connecting solutions of the Yang equations[23] for the groups $SU(p, q)$ and $SU(p-1, q+1)$ (Ref. 68).

7. The integrability of the system (83) for other nontrivial matrices k different from the Cartan matrix. A positive solution to this problem would evidently enable one to obtain a number of important physical consequences in other fields of theoretical physics in which equations of the Liouville type are encountered; we have here, for example, possible generalizations of the Ginzburg-Landau equation,[69] the Debye-Hückel equation,[70] etc. In particular, for the degenerate matrices $k = \begin{vmatrix} 2 & -2 \\ -2 & 2 \end{vmatrix}$, which correspond in accordance with Ref. 71 to infinite-dimensional groups, the corresponding system (83) leads to the sine-Gordon equation.[5)]

8. The complete integrability of the duality equations $(F_{\mu\nu} = {}^*F_{\mu\nu})$ without the assumption of cylindrical symmetry.

9. The construction of solutions to the general Yang-Mills equations $(\partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] = 0)$ without the assumption of duality but in the cylindrically sym-

[5)] The matrix $\begin{vmatrix} 2 & -1 \\ -4 & 2 \end{vmatrix}$ leads to a different interesting equation with right-hand side $\exp(2x) - 2\exp(-x)$.

metric case [see (23)].

10. Interpretation of the "quantum" numbers $m_1, \ldots, m_r$, which determine the instanton and monopole configurations that we have constructed, as internal structural characteristics of these objects.

11. The formulation of the duality condition as a critical "point" of sudden raising of the symmetry of the solutions of the corresponding system.

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